EXAM NO:

GOVT. PUBLIC EXAM (MAY 2022)

STD: XII MATHEMATICS MARKS: 90

Type A TIME: 3.00 hr

PART – A

Choose the correct answer **20 x 1 = 20**

1. If $f(x) = \begin{cases} 2x & 0 \le x \le a \\ 0 & \text{otherwise} \end{cases}$ is a probability density function of a random variable, then the value of a is
   (a) 3 **(b) 1** (c) 4 (d) 2
2. Which one of the following is not true in the case discrete random variable X?
   (a) $\lim_{x \to \infty} F(x) = F(\infty) = 1$ (b) $0 \le F(x) \le 1$ for all $x \in R$
   (c) F(x) is real valued decreasing function (d) $\lim_{x \to -\infty} F(x) = F(-\infty) = 0$
3. If $f(x) = \frac{x}{x+1}$, then its differential is given by
   (a) $\frac{1}{x+1}dx$ (b) $\frac{-1}{(x+1)^2}dx$ (c) $\frac{-1}{x+1}dx$ **(d) $\frac{1}{(x+1)^2}dx$**
4. The value of $\int_0^1 x(1-x)^{49}\,dx$ is:
   (a) $\frac{1}{10010}$ (b) $\frac{1}{11000}$ (c) $\frac{1}{10001}$ **(d) $\frac{1}{10100}$**
5. The principal value of $\cos^{-1}\left(\frac{\sqrt{3}}{2}\right)$ is?
   (a) $\frac{\pi}{2}$ (b) $\frac{\pi}{3}$ (c) $\frac{5\pi}{6}$ **(d) $\frac{\pi}{6}$**
6. If $A = \begin{bmatrix} 2 & 3 \\ 5 & -2 \end{bmatrix}$ be such that $\lambda A^{-1} = A$, then λ is:
   (a) 19 (b) 17 (c) 21 (d) 14
7. If α , β and γ are the roots of $x^3 + px^2 + qx + r$, then $\sum \frac{1}{\alpha}$ is
   (a) $\frac{q}{r}$ **(b) $-\frac{q}{r}$** (c) $-\frac{q}{p}$ (d) $-\frac{p}{r}$
8. If $(1+i)(1+2i)(1+3i) \dots (1+ni) = x + iy$, then $2.5.10 \dots (1+n^2)$ is
   (a) $x^2 + y^2$ (b) 1 (c) $1 + n^2$ (d) i
9. The minimum value of the function $|3 - x| + 9$ is
   (a) 6 (b) 0 **(c) 9** (d) 3
10. The value of $\sum_{n=1}^{12} i^n$ is:
    (a) 0 (b) 1 (c) −1 (d) i
11. If the vectors $2\hat{i} - \hat{j} + 3\hat{k}, 3\hat{i} + 2\hat{j} + \hat{k}, \hat{i} + m\hat{j} + 4\hat{k}$ are coplanar, then the value of m is:
    (a) 2 (b) 3 (c) −2 **(d) −3**
12. The general equation of a circle with centre (−3, −4) and radius 3 units is :
    (a) $x^2 + y^2 - 6x + 8y - 16 = 0$ (b) $x^2 + y^2 - 6x - 8y + 16 = 0$
    (c) $x^2 + y^2 + 6x - 8y + 16 = 0$ **(d) $x^2 + y^2 + 6x + 8y + 16 = 0$**

13. The solution of $\frac{dy}{dx} + p(x)y = 0$ is:

(a) $x = ce^{-\int pdy}$ (b) $y = ce^{\int pdx}$ (c) $x = ce^{\int pdy}$ **(d) $y = ce^{-\int pdx}$**

14. The value of $\int_0^\infty e^{-3x}x^2\,dx$ is :

(a) $\frac{4}{27}$ (b) $\frac{7}{27}$ **(c) $\frac{2}{27}$** (d) $\frac{5}{27}$

15. The point of inflection of the curve $y = (x-1)^3$ is :

(a) $(1, 0)$ (b) $(0,0)$ (c) $(1,1)$ (d) $(0,1)$

16. The angle between the lines $\frac{x-4}{2} = \frac{y}{1} = \frac{z+1}{-2}$ and $\frac{x-1}{4} = \frac{y+1}{-4} = \frac{z-2}{2}$ is :

(a) $\frac{\pi}{2}$ (b) $\frac{\pi}{4}$ (c) $\frac{2\pi}{3}$ (d) $\frac{\pi}{3}$

17. Which one of the following is a binary operation on N ?

(a) Multiplication (b) Division (c) Subtraction (d) All the above

18. Which one of the following is incorrect?

(a) If A is a sqqunre matrix of order n, and λ is a scalar, then $\text{Adj}(\lambda A) = \lambda^n(\text{Adj}A)$

(b) Adjoint of a symmetric matrix is also a symmetric matrix.

(c) $A(\text{Adj }A) = (\text{Adj }A)A = |A|I$.

(d) Adjoint of a diagonal matrix is also a diagonal matrix.

19. If $\sin x$ is the integrating factor of the linear differential equation $\frac{dy}{dx} + Py = Q$, then P is :

(a) $\tan x$ (b) $\log \sin x$ **(c) $\cot x$** (d) $\cos x$

20. The length of the latus rectum of the parabola $x^2 = 24y$ is :

(a) 8 **(b) 24** (c) 6 (d) 12

PART - B

Answer any SEVEN questions

Question number 30 is compulsory **7 x 2 = 14**

21. Prove the following properties : $\text{Re}(z) = \frac{z+\bar{z}}{2}$ and $\text{Im}(z) = \frac{z-\bar{z}}{2i}$.

Solution:

$$\text{Let } z = x + iy,\ \bar{z} = x - iy \Rightarrow \text{Re}(z) = x \text{ and } \text{Im}(z) = y$$

$$z + \bar{z} = 2x \Rightarrow x = \frac{z+\bar{z}}{2} \Rightarrow \text{Re}(z) = \frac{z+\bar{z}}{2}$$

$$z - \bar{z} = 2iy \Rightarrow y = \frac{z-\bar{z}}{2i} \Rightarrow \text{Im}(z) = \frac{z-\bar{z}}{2i}$$

22. Find a polynomial equation of minimum degree with rational coefficients, having $2 - \sqrt{3}$ as a root.

Solution:

G.T. $2-\sqrt{3}$ is a root $\Rightarrow$ $2+\sqrt{3}$ is also a root

$\text{Sum} = 2-\sqrt{3}+2+\sqrt{3} = 4$

$\text{Product} = (2-\sqrt{3})(2+\sqrt{3}) = 2^2 - (\sqrt{3})^2 = 4-3 = 1$

W.K.T. $x^2 - (\text{sum})x + \text{product} = 0$

i.e., $x^2 - 4x + 1 = 0$ is a minimum degree polynomial.

23. Find the principal value of $\tan^{-1}(\sqrt{3})$.

Solution:

$$\text{Let } y = \tan^{-1}(\sqrt{3})$$
$$\tan y = \sqrt{3}$$
$$\tan y = \tan\frac{\pi}{3} \qquad \because \frac{\pi}{3} \in \left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$$
$$y = \frac{\pi}{3}$$
$$\tan^{-1}(\sqrt{3}) = \frac{\pi}{3}$$

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24. Find the points on the curve $y = x^3 - 3x^2 + x - 2$ at which the tangent is parallel to the line $y = x$.

Solution:

$$\text{Given curve } y = x^3 - 3x^2 + x - 2 \Rightarrow \frac{dy}{dx} = 3x^2 - 6x + 1 = m_1$$
$$\text{Slope of } y = x \Rightarrow \ x - y = 0 \text{ is } \frac{-x \text{ coefficient}}{y \text{ cofficient}} = \frac{-1}{-1} = 1 = m_2$$
$$\text{Since they are parallel, then } m_1 = m_2$$
$$\Rightarrow 3x^2 - 6x + 1 = 1$$
$$\Rightarrow 3x^2 - 6x = 0 \Rightarrow 3x(x-2) = 0$$
$$\Rightarrow x = 0, \ \ x = 2$$
$$\text{When } x = 0, \ \ y = 0 - 0 + 0 - 2 = -2$$
$$\text{When } x = 2, \ \ y = 2^3 - 3(2^2) + 2 - 2 = -4$$
$$\boxed{\therefore \text{The required points are } (0,-2), (2,-4)}$$

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25. Find df for $f(x) = x^2 + 3x$ and evaluate it for $x = 2$ and $dx = 0.1$.

Solution:

$$\text{G.T } \ f(x) = x^2 + 3x$$
$$f'(x) = 2x + 3$$
$$\text{W.k.T. } \quad df = f'(x)dx$$
$$df = (2x+3)dx$$
$$x = 2 \text{ and } dx = 0.1$$
$$df = (2(2)+3)(0.1)$$
$$df = (7)(0.1)$$
$$\therefore df = 0.7$$

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26. Show that the differential equation of the family of curves $y = Ae^x + Be^{-x}$, where A and B are arbitrary constants, is $\frac{d^2y}{dx^2} - y = 0$.

Solution:

$$y = Ae^x + Be^{-x}$$
$$\frac{dy}{dx} = Ae^x + Be^{-x}(-1)$$
$$\frac{dy}{dx} = Ae^x - Be^{-x}$$

$$\frac{d^2y}{dx^2} = Ae^x - Be^{-x}(-1)$$
$$\frac{d^2y}{dx^2} = Ae^x + Be^{-x}$$
$$\frac{d^2y}{dx^2} - y = 0$$

Hence, $y = Ae^x + Be^{-x}$ is a solution of $\frac{d^2y}{dx^2} - y = 0$

27. Solve : $\frac{dy}{dx} = \frac{\sqrt{1-y^2}}{\sqrt{1-x^2}}$.

Solution:

$$\frac{dy}{dx} = \frac{\sqrt{1-y^2}}{\sqrt{1-x^2}}$$
$$\frac{1}{\sqrt{1-y^2}}dy = \frac{1}{\sqrt{1-x^2}}dx$$
$$\int \frac{1}{\sqrt{1-y^2}}dy = \int \frac{1}{\sqrt{1-x^2}}dx$$
$$\sin^{-1} y = \sin^{-1} x + c$$
$$\sin^{-1} y - \sin^{-1} x = c$$
$$\sin^{-1}\left[y\sqrt{1-x^2} - x\sqrt{1-y^2}\right] = c$$
$$y\sqrt{1-x^2} - x\sqrt{1-y^2} = \sin c = a(\text{say})$$
$$y\sqrt{1-x^2} - x\sqrt{1-y^2} = a$$

28. A random variable X has the following probability mass function.

x	1	2	3	4	5	6
f(x)	k	2k	6k	5k	6k	10k

Find k.

Solution:

Given probability mass function

$$\Rightarrow \sum_x f(x) = 1$$
$$k + 2k + 6k + 5k + 6k + 10k = 1$$
$$30k = 1$$
$$k = \frac{1}{30}$$

29. X is the number of tails occurred when three fair coins are tossed simultaneously. Find the values of the random variable X and number af points in its reverse images.

Solution:

$$S = \{H, T\} \times \{H, T\} \times \{H, T\}$$
$$S = \{HHH, HHT, HTH, THH, HTT, THT, TTH, TTT\}$$

Let $X : S \longrightarrow \mathbb{R}$ be the number of tails, then

$X(HHH) = 0$ (0 Tails) & $X(HHT) = 1$ (1 Tails)

$X(HTH) = 1$ (1 Tails) & $X(THH) = 1$ (1 Tails)

$X(HTT) = 2$ (2 Tails) & $X(THT) = 2$ (2 Tails)

$X(TTH) = 2$ (2 Tails) & $X(TTT) = 3$ (3 Tails)

The X be a random variables that takes the values 0, 1, 2 & 3

Number of elements in inverse images

Values of the Random Variable	0	1	2	3	Total
Number of elements in inverse image	1	3	3	1	8

30. Show that the distance from the origin to the plane $3x + 6y + 2x + 7 = 0$ is 1.

Solution:

Here $(x_1, y_1, z_1) = (0, 0, 0)$

$$3x + 6y + 2x + 7 = 0$$

$$\Longrightarrow a = 3\,;\, b = 6\,;\, c = 2\,;\, d = 7$$

$$\text{distance } (\delta) = \frac{|ax_1 + by_1 + cz_1 + d|}{\sqrt{a^2 + b^2 + c^2}}$$

$$= \frac{|3(0) + 6(0) + 2(0) + 7|}{\sqrt{(3)^2 + (6)^2 + (2)^2}} = \frac{|7|}{\sqrt{49}} = \frac{7}{7} = 1 \text{ units}$$

PART – C

Answer any SEVEN questions

Question number 40 is compulsory **7 x 3 = 21**

31. Show that the rank of the matrix $\begin{bmatrix} 1 & 2 & -1 \\ 3 & -1 & 2 \\ 1 & -2 & 3 \\ 1 & -1 & 1 \end{bmatrix}$ is 3.

Solution:

$$\text{Let } A = \begin{bmatrix} 1 & 2 & -1 \\ 3 & -1 & 2 \\ 1 & -2 & 3 \\ 1 & -1 & 1 \end{bmatrix}$$

$$\xrightarrow[R_4 \to R_4 - R_1]{\substack{R_2 \to R_2 - 3R_1 \\ R_3 \to R_3 - R_1}} \begin{bmatrix} 1 & 2 & -1 \\ 0 & -7 & 5 \\ 0 & -4 & 4 \\ 0 & -3 & 2 \end{bmatrix}$$

$$\xrightarrow[R_4 \to 4R_4 - 3R_3]{R_3 \to 7R_3 - 4R_2} \begin{bmatrix} 1 & 2 & -1 \\ 0 & -7 & 5 \\ 0 & 0 & 8 \\ 0 & 0 & -4 \end{bmatrix}$$

$$\xrightarrow{R_4 \to 2R_4 + R_3} \begin{bmatrix} 1 & 2 & -1 \\ 0 & -7 & 5 \\ 0 & 0 & 8 \\ 0 & 0 & 0 \end{bmatrix}$$

It is row – echelon form

The number of non – zero rows is 3

$\therefore \boxed{\rho(A) = 3}$

32. Solve the following system of linear equations, using matrix inversion method: $5x + 2 = 3, 3x + 2y = 5$.

Solution:

The matrix form of the system is $AX = B$

Where $A = \begin{bmatrix} 5 & 2 \\ 3 & 2 \end{bmatrix}$; $X = \begin{bmatrix} x \\ y \end{bmatrix}$; $B = \begin{bmatrix} 3 \\ 5 \end{bmatrix}$

$|A| = 10 - 6 = 4 \neq 0$

A^{-1} exists

$$A^{-1} = \frac{1}{|A|} \text{adj } (A)$$

$$A^{-1} = \frac{1}{4}\begin{bmatrix} 2 & -2 \\ -3 & 5 \end{bmatrix}$$

The solution is,

$$X = A^{-1}B$$

$$\begin{bmatrix} x \\ y \end{bmatrix} = \frac{1}{4}\begin{bmatrix} 2 & -2 \\ -3 & 5 \end{bmatrix}\begin{bmatrix} 3 \\ 5 \end{bmatrix} = \frac{1}{4}\begin{bmatrix} 6 - 10 \\ -9 + 25 \end{bmatrix} = \frac{1}{4}\begin{bmatrix} -4 \\ 16 \end{bmatrix} = \begin{bmatrix} -1 \\ 4 \end{bmatrix}$$

$\therefore$ The solution is $x = -1, \quad y = 4$

33. Which one of the points $10 - 8i, 11 + 6i$ is closest to $1 + i$

Solution:

The distance from the point $10 - 8i$, $11 + 6i$ to $1 + i$

$$|(1 + i) - (10 - 8i)| = |-9 + 9i| = 9|-1 + i| = 9\sqrt{(-1)^2 + 1^2} = 9\sqrt{2}$$

$$|(1 + i) - (11 + 6i)| = |-10 - 5i| = 5|-2 - i| = 5\sqrt{(-2)^2 + (-1)^2} = 5\sqrt{5}$$

Since $5\sqrt{5} < 9\sqrt{2}$, the closest point to $1 + i$ is $11 + 6i$

34. Solve the equation $2x^3 - 9x^2 + 10x = 3$, if 1 is a root, find the other roots.

Solution:

G. T. $\quad 2x^3 - 9x^2 + 10x = 3 \Longrightarrow 2x^3 - 9x^2 + 10x - 3 = 0$

G. T. $x = 1$ is a root

	2	−9	10	−3
1	0	2	−7	3
	2	−7	3	0

$$\therefore 2x^2 - 7x + 3 = 0$$

$$\left(x - \frac{6}{2}\right)\left(x - \frac{1}{2}\right) = 0$$

$$(x - 3)\left(x - \frac{1}{2}\right) = 0$$

$$x = 3 ; x = \frac{1}{2}$$

Hence, The roots are $1, 3$ and $\frac{1}{2}$

35. Find the magnitude and the direction cosines of the torque about the point $(2, 0, -1)$ of a force $2\hat{i} + \hat{j} - \hat{k}$, whose line of action passes through the origin.

Solution:

$$\vec{F} = 2\hat{i} + \hat{j} - \hat{k}$$

$$\vec{r} = \text{at (or) through} - \text{about} = (0\hat{i} + 0\hat{j} + 0\hat{k}) - (2\hat{i} + 0\hat{j} - \hat{k})$$

$$\vec{r} = -2\hat{i} + 0\hat{j} + \hat{k}$$

$$\text{Torque}(\vec{\tau}) = \vec{r} \times \vec{F} = \begin{vmatrix} \hat{\imath} & \hat{\jmath} & \hat{k} \\ -2 & 0 & 1 \\ 2 & 1 & -1 \end{vmatrix}$$

$$= \hat{\imath}(0 - 1) - \hat{\jmath}(2 - 2) + \hat{k}(-2 - 0)$$

$$\vec{\tau} = -\hat{\imath} + 0\hat{\jmath} - 2\hat{k}$$

$$\text{Magnitude torque} = |\vec{\tau}| = \sqrt{(-1)^2 + 0^2 + (-2)^2} = \sqrt{5}$$

$$\text{direction cosines of the torque} = -\frac{1}{\sqrt{5}}, 0, -\frac{2}{\sqrt{5}}$$

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36. Evaluate: $\lim\limits_{x \to \infty} \frac{2x^2-3}{x^2-5x+3}$

Solution:

$$\lim_{x \to \infty} \frac{2x^2 - 3}{x^2 - 5x + 3} \qquad \left(\frac{\infty}{\infty} \text{form}\right)$$

Appling l'Hôpital Rule, we get

$$= \lim_{x \to \infty} \frac{4x}{2x - 5} \qquad \left(\frac{\infty}{\infty} \text{form}\right)$$

Appling l'Hôpital Rule, we get

$$= \lim_{x \to \infty} \frac{4}{2} = 2$$

$$\therefore \lim_{x \to \infty} \frac{2x^2 - 3}{x^2 - 5x + 3} = 2$$

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37. Assume that the cross section of the artery of human is circular, A drug is given to a patient to dilate his arteries. If the radius of an artery is increased from 2 mm to 2.1 mm, how much is eross-sectional area increased approximately.

Solution:

$$\text{G.T.} \;\; r = 2 \text{ mm} \;;\;\; dr = \Delta r = 2.1 - 2 = 0.1$$

$$\text{W.K.T.} \;\; \text{Area of the circle } (A) = \pi r^2$$

$$A'(r) = 2\pi r$$

$$\text{Increasing area} = dA = A'(r)\, dr$$

$$dA = 2\pi r\, dr = 2\pi(2)(0.1) = 0.4\pi \text{ mm}^2$$

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38. Show that $\int_0^{\frac{\pi}{3}} \frac{\sec x \tan x}{1+\sec^2 x} dx = \tan^{-1}(2) - \frac{\pi}{4}$.

Solution:

$$\text{Let } I = \int_0^{\frac{\pi}{3}} \frac{\sec x \tan x}{1 + \sec^2 x} dx$$

Put $t = \sec x$

$dt = \sec x \tan x \, dx$

x	0	$\pi/3$
t	1	2

$$\therefore I = \int_1^2 \frac{dt}{1+t^2} = [\tan^{-1} t]_1^2$$

$$= \tan^{-1} 2 - \tan^{-1} 1$$

$$= \tan^{-1} 2 - \frac{\pi}{4}$$

$$\therefore \int_0^{\frac{\pi}{3}} \frac{\sec x \tan x}{1+\sec^2 x} dx = \tan^{-1} 2 - \frac{\pi}{4}$$

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39. Let * be defined on **R** by $(a * b) = a + b + ab - 7$, is * binary on **R** ? If so, find $3^* \left(\frac{-7}{15}\right)$.

Solution:

$$a * b = a + b + ab - 7$$

$$\forall\, a, b \in \mathbb{R}$$

since $-, +, \times$ are binary operation on $\mathbb{R}$

$$\Rightarrow a + b \in \mathbb{R}, \text{ and } ab \in \mathbb{R}$$

$$\Rightarrow a + b + ab - 7 \in \mathbb{R}$$

$$\Rightarrow a * b = a + b + ab - 7 \in \mathbb{R} \Rightarrow a * b \in \mathbb{R}$$

Hence, * is a binary operation on $\mathbb{R}$.

$$3 * \left(\frac{-7}{15}\right) = 3 + \left(\frac{-7}{15}\right) + (3)\left(\frac{-7}{15}\right) - 7 = \frac{45 - 7 - 21 - 105}{15} = -\frac{88}{15}$$

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40. Prove that the general equation of the circle whose diameter is the line segment joining the paints $(-4, -2)$ and $(-1, -1)$, is $x^2 + y^2 + 5x + 3y + 6 = 0$.

Solution:

Let $A(-4, -2)$ and $B(-1, -1)$

Equation of circle is

$$(x - x_1)(x - x_2) + (y - y_1)(y - y_2) = 0$$

$$(x + 4)(x + 1) + (y + 2)(y + 1) = 0$$

$$x^2 + 4x + x + 4 + y^2 + 2y + y + 2 = 0$$

$$x^2 + y^2 + 5x + 3y + 6 = 0$$

PART – D

Answer ALL questions **7 x 5 = 35**

41. (a) Cramer's rule is not applicable to solve the system $3x + y + z = 2$, $x - 3y + 2z = 1$, $7x - y + 4z = 5$. Why?

Solution:

$$\Delta = \begin{vmatrix} 3 & 1 & 1 \\ 1 & -3 & 2 \\ 7 & -1 & 4 \end{vmatrix} = 3(-12 + 2) - 1(4 - 14) + 1(-1 + 21)$$

$$= -30 + 10 + 20 = 0$$

$$\Delta = 0 \Rightarrow \text{Cramer's rule is not applicable}$$

OR

(b) Prove that the local minimum values for the function $f(x) = 4x^6 - 6x^4$ attain at -1 and 1.

Solution:

$$f(x) = 4x^6 - 6x^4$$
$$f'(x) = 24x^5 - 24x^3$$
$$f''(x) = 120x^4 - 72x^2$$
$$f'(x) = 0$$
$$24x^5 - 24x^3 = 0$$
$$24x^3(x^2 - 1) = 0$$
$$24x^3 = 0 \quad ; \quad x^2 - 1 = 0$$
$$x^3 = 0 \quad ; \quad x^2 = 1$$
$$x = 0 \quad ; \quad x = \pm 1$$

Sub $x = 0$ in $f''(x)$
$$f''(0) = 120(0)^4 - 72(0)^2 = 0$$
Sub $x = -1$ in $f''(x)$
$$f''(-1) = 120(-1)^4 - 72(-1)^2 = 120 - 72 = 48 > 0$$
Sub $x = 1$ in $f''(x)$
$$f''(1) = 120(1)^4 - 72(1)^2 = 120 - 72 = 48 > 0$$
$\therefore$ $f(x)$ has local minimum at $x = \pm 1$

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42. (a) Show that the locus of $z = x + iy$ if $|z + i| = |x - 1|$, is $x + y = 0$.

Solution:

G. T. $z = x + iy$
$$|z + i| = |z - 1|$$
$$\Rightarrow |x + iy + i| = |x + iy - 1|$$
$$\Rightarrow |x + i(y + 1)| = |(x - 1) + iy|$$
$$\Rightarrow \sqrt{x^2 + (y + 1)^2} = \sqrt{(x - 1)^2 + y^2}$$
Square on both sides,
$$\Rightarrow x^2 + (y + 1)^2 = (x - 1)^2 + y^2$$
$$\Rightarrow x^2 + y^2 + 2y + 1 = x^2 - 2x + 1 + y^2$$
$$\Rightarrow 2x + 2y = 0$$
$\Rightarrow x + y = 0$, the locus of z in Cartesian form

OR

(b) Show that $\int_0^a \frac{f(x)}{f(x)+f(a-x)}\,dx = \frac{a}{2}$.

Solution:

$$\text{Let } I = \int_0^a \frac{f(x)}{f(x) + f(a - x)}\,dx \rightarrow ①$$

$$I = \int_0^a \frac{f(a - x)}{f(a - x) + f(a - (a - x))}\,dx \qquad \because \int_0^a f(x)\,dx = \int_0^a f(a - x)\,dx$$

$$I = \int_0^a \frac{f(a-x)}{f(a-x)+f(x)}dx \quad \rightarrow \quad ②$$

① + ② ⇒

$$2I = \int_0^a \frac{f(x)+f(a-x)}{f(x)+f(a-x)}dx$$

$$2I = \int_0^a dx = [x]_0^a = a - 0 = a \Rightarrow I = \frac{a}{2}$$

$$\therefore \int_0^a \frac{f(x)}{f(x)+f(a-x)}dx = \frac{a}{2}$$

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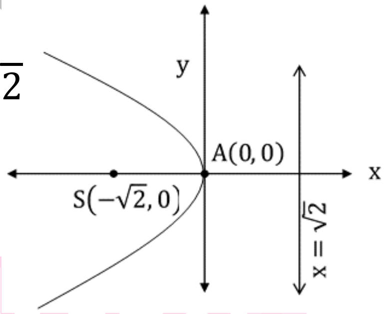
43. (a) Shaw that the equation of the parabola with focus $(-\sqrt{2}, 0)$ and directrix $x = \sqrt{2}$ is $y^2 = -4\sqrt{2}x$

Solution:

$$\text{focus} = S(-a, 0) = S(-\sqrt{2}, 0) \Longrightarrow a = \sqrt{2}$$

$$\text{directrix } x = \sqrt{2}$$



Parabola is open left and axis of symmetry as x – axis and vertex (0, 0)

Then the equation of parabola is

$$(y-k)^2 = -4a(x-h)$$

$$(y-0)^2 = -4\sqrt{2}(x-0)$$

$$y^2 = -4\sqrt{2}x$$

OR

(b) Find the value of $\cot^{-1}(1) + \sin^{-1}\left(-\frac{\sqrt{3}}{2}\right) - \sec^{-1}(-\sqrt{2})$.

Solution:

$$\cot^{-1}(1) + \sin^{-1}\left(-\frac{\sqrt{3}}{2}\right) - \sec^{-1}(-\sqrt{2})$$

$$= \cot^{-1}\left(\cot\left(\frac{\pi}{4}\right)\right) + \sin^{-1}\left(-\sin\left(\frac{\pi}{3}\right)\right) - \sec^{-1}\left(-\sec\left(\frac{\pi}{4}\right)\right)$$

$$= \cot^{-1}\left(\cot\left(\frac{\pi}{4}\right)\right) + \sin^{-1}\left(\sin\left(-\frac{\pi}{3}\right)\right) - \sec^{-1}\left(\sec\left(\pi - \frac{\pi}{4}\right)\right)$$

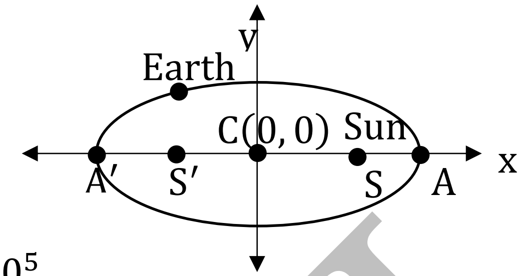
$$= \cot^{-1}\left(\cot\left(\frac{\pi}{4}\right)\right) + \sin^{-1}\left(\sin\left(-\frac{\pi}{3}\right)\right) - \sec^{-1}\left(\sec\left(\frac{3\pi}{4}\right)\right)$$

$$\because \frac{\pi}{4} \in (0, \pi), \quad -\frac{\pi}{3} \in \left(-\frac{\pi}{2}, \frac{\pi}{2}\right) \;\&\; \frac{3\pi}{4} \in [0, \pi] - \left\{\frac{\pi}{2}\right\}$$

$$= \frac{\pi}{4} - \frac{\pi}{3} - \frac{3\pi}{4}$$

$$= \frac{3\pi - 4\pi - 9\pi}{12} = -\frac{10\pi}{12} = -\frac{5\pi}{6}$$

44. (a) The maximum and minimum distances of the Earth from the Sun respectively are 152×10^6 km and 94.5×10^6 km. The Sun is at one focus of the elliptical orbit. Show that the distance from the Sun to the other focus is 575×10^5 km



Solution:

$SA' = a + c = 152 \times 10^6 \rightarrow (1)$

$SA = a - c = 94.5 \times 10^6 \rightarrow (2)$

$(1) - (2) \Rightarrow$

$2c = 57.5 \times 10^6 = 575 \times 10^5$

$\therefore$ The distance from the Sun to the other focus is 575×10^5 km

OR

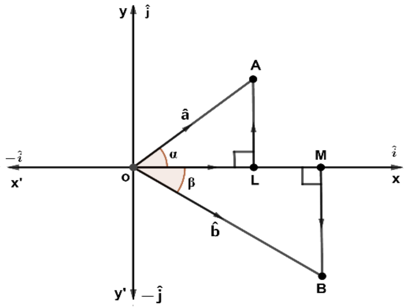
(b) Prove by vector method $\sin(A + B) = \sin A . \cos B + \cos A . \sin B$.

Solution:

Let $\hat{a} = \overrightarrow{OA}$ and $\hat{b} = \overrightarrow{OB}$ be the unit vectors and which makes angle α and β with positive $x-$axis respectively.

$\therefore$ The angle between $\hat{a}$ and $\hat{b}$ is $\alpha + \beta$

From the diagram



$$\cos\alpha = \frac{OL}{OA} = \frac{OL}{1} = OL \qquad \cos\beta = \frac{OM}{OB} = \frac{OM}{1} = OM$$

$$\sin\alpha = \frac{LA}{OA} = \frac{LA}{1} = LA \qquad \sin\beta = \frac{MB}{OB} = \frac{MB}{1} = MB$$

$$\hat{a} = \overrightarrow{OA} = \overrightarrow{OL} + \overrightarrow{LA} = OL\,\hat{\imath} + LA\,\hat{\jmath}$$

$$\hat{a} = \cos\alpha\,\hat{\imath} + \sin\alpha\,\hat{\jmath}$$

$$\hat{b} = \overrightarrow{OB} = \overrightarrow{OM} + \overrightarrow{MB} = OM\,\hat{\imath} + MB(-\hat{\jmath})$$

$$\hat{b} = \cos\beta\,\hat{\imath} - \sin\beta\,\hat{\jmath}$$

$$\hat{b} \times \hat{a} = \begin{vmatrix} \hat{\imath} & \hat{\jmath} & \hat{k} \\ \cos\beta & -\sin\beta & 0 \\ \cos\alpha & \sin\alpha & 0 \end{vmatrix}$$

$$\hat{b} \times \hat{a} = \hat{k}(\sin\alpha\cos\beta + \cos\alpha\sin\beta) \rightarrow (1)$$

$$\hat{b} \times \hat{a} = |\hat{b}||\hat{a}|\sin(\alpha+\beta)\,\hat{k} = (1)(1)\sin(\alpha+\beta)\,\hat{k} = \hat{k}\sin(\alpha+\beta) \rightarrow (2)$$

From (1) and (2), we get

$$\hat{k}\sin(\alpha+\beta) = \hat{k}(\sin\alpha\cos\beta + \cos\alpha\sin\beta)$$

$$\sin(\alpha+\beta) = \sin\alpha\cos\beta + \cos\alpha\sin\beta$$

similarly we can prove

$$\sin(A+B) = \sin A\cos B + \cos A\sin B$$

45. (a) Find the vector equation (any form) or Cartesian equation of a plane passing through the points (2,2,1), (9,3,6) and perpendicular to the plane $2x + 6y + z = 9$.

Solution:

The requied equation of plane passing through a points $(2, 2, 1), (9, 3, 6)$ and parallel to vector $2\hat{\imath} + 6\hat{\jmath} + 6\hat{k}$.

$$\text{Let} \quad \left.\begin{aligned}\vec{a} &= 2\hat{i}+2\hat{j}+\hat{k}\\ \vec{b} &= 9\hat{i}+3\hat{j}+6\hat{k}\end{aligned}\right\} \Rightarrow \vec{b}-\vec{a} = 7\hat{i}+\hat{j}+5\hat{k}$$

$$\vec{c} = 2\hat{i}+6\hat{j}+6\hat{k}$$

Vector equation:

$$\vec{r} = \vec{a} + s(\vec{b}-\vec{a}) + t\vec{c} \quad \text{where } s,t \in \mathbb{R}$$

$$\boxed{\vec{r} = (2\hat{i}+2\hat{j}+\hat{k}) + s(7\hat{i}+\hat{j}+5\hat{k}) + t(2\hat{i}+6\hat{j}+6\hat{k})} \text{ where } s,t \in \mathbb{R}$$

(or) Non – parametric vector equation:

$$(\vec{r}-\vec{a}).\left((\vec{b}-\vec{a})\times\vec{c}\right) = 0$$

$$(\vec{b}-\vec{a})\times\vec{c} = \begin{vmatrix}\hat{\imath} & \hat{j} & \hat{k}\\ 7 & 1 & 5\\ 2 & 6 & 6\end{vmatrix} = \hat{\imath}(6-30) - \hat{j}(42-10) + \hat{k}(42-2)$$

$$(\vec{b}-\vec{a})\times\vec{c} = -24\hat{\imath} - 32\hat{j} + 40\hat{k}$$

$$\therefore \left(\vec{r} - (2\hat{i}+2\hat{j}+\hat{k})\right).(-24\hat{\imath}-32\hat{j}+40\hat{k}) = 0$$

$$\vec{r}.(-24\hat{\imath}-32\hat{j}+40\hat{k}) - (2\hat{i}+2\hat{j}+\hat{k}).(-24\hat{\imath}-32\hat{j}+40\hat{k}) = 0$$

$$\vec{r}.(-24\hat{\imath}-32\hat{j}+40\hat{k}) - (-48-64+40) = 0$$

$$\vec{r}.(-24\hat{\imath}-32\hat{j}+40\hat{k}) = -72$$

$$\boxed{\vec{r}.(3\hat{\imath}+4\hat{j}-5\hat{k}) = 9}$$

Cartesian equation:

$$\text{W.K.T.} \quad \vec{r} = x\hat{\imath}+y\hat{j}+z\hat{k}$$

$$(x\hat{\imath}+y\hat{j}+z\hat{k}).(3\hat{\imath}+4\hat{j}-5\hat{k}) = 9$$

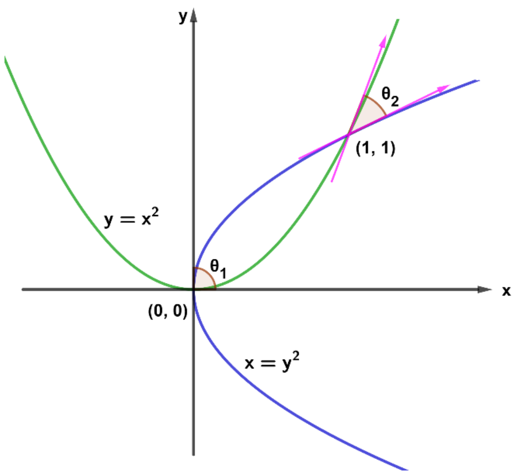
$$\boxed{3x+4y-5z-9=0}$$

OR

(b) Show that the angle between the curves $y = x^2$ and $x = y^2$ at (1,1)is $\tan^{-1}\left(\frac{3}{4}\right)$.

Solution:

$y = x^2$	$x = y^2$
$\frac{dy}{dx} = 2x$	$1 = 2y\frac{dy}{dx}$
$\frac{dy}{dx} = 2x$	$\frac{dy}{dx} = \frac{1}{2y}$



$$\text{At } (1,1), \quad m_1 = 2 \quad ; \quad m_2 = \frac{1}{2}$$

$$\tan\theta = \left|\frac{m_1 - m_2}{1+m_1m_2}\right|$$

$$= \left|\frac{2-\frac{1}{2}}{1+(2)\left(\frac{1}{2}\right)}\right| = \left|\frac{\frac{3}{2}}{1+1}\right| = \frac{3}{4}$$

$$\tan\theta = \frac{3}{4} \Rightarrow \boxed{\theta = \tan^{-1}\left(\frac{3}{4}\right)}$$

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46. (a) The distribution function of a continuous random variable X is:

$$F(x)=\begin{cases}0, & x<1\\ \dfrac{x-1}{4}, & 1\le x\le 5\\ 1, & x>5\end{cases}$$

Find (i) $P(X<3)$ (ii) $P(2<X<4)$ (iii) $P(3\le X)$

Solution:

$$\text{(i) } P(X<3)=P(X\le 3)=F(3)=\frac{3-1}{4}=\frac{2}{4}=\frac{1}{2}$$

$$\text{(ii) } P(2<X<4)=P(2\le X\le 4)=F(4)-F(2)=\frac{4-1}{4}-\frac{2-1}{4}=\frac{1}{2}$$

$$\text{(iii) } P(3\le X)=P(X\ge 3)=1-P(X<3)=1-F(3)=1-\frac{1}{2}=\frac{1}{2}$$

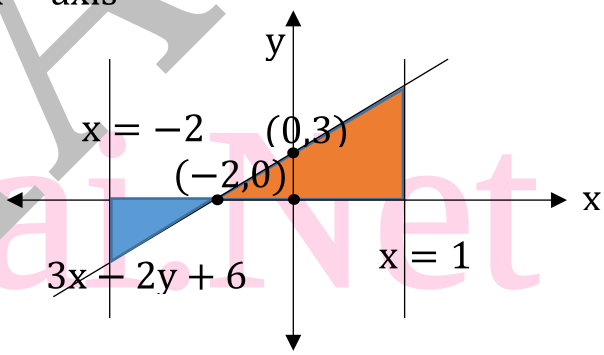
OR

(b) Show that the area of the region bounded by $3x-2y+6=0, x=-3, x=1$ and x-axis, is $\frac{15}{2}$.

Solution:

$3x-2y+6=0, x=-3, x=1$ and $x-$axis

The required area is



$$A=\int_{-3}^{-2}(-y)\,dx+\int_{-2}^{1}y\,dx$$

$$3x-2y+6=0$$

$$\Rightarrow 3x+6=2y$$

$$\Rightarrow y=\frac{3}{2}(x+2)$$

$$A=\int_{-3}^{-2}-\frac{3}{2}(x+2)\,dx+\int_{-2}^{1}\frac{3}{2}(x+2)\,dx$$

$$=-\frac{3}{2}\int_{-3}^{-2}(x+2)\,dx+\frac{3}{2}\int_{-2}^{1}(x+2)\,dx$$

$$=-\frac{3}{2}\left[\frac{(x+2)^2}{2}\right]_{-3}^{-2}+\frac{3}{2}\left[\frac{(x+2)^2}{2}\right]_{-2}^{1}$$

$$=-\frac{3}{2}\left[\frac{(-2+2)^2}{2}-\frac{(-3+2)^2}{2}\right]+\frac{3}{2}\left[\frac{(1+2)^2}{2}-\frac{(-2+2)^2}{2}\right]$$

$$=-\frac{3}{2}\left[0-\frac{1}{2}\right]+\frac{3}{2}\left[\frac{9}{2}-0\right]=\frac{3}{4}+\frac{27}{4}=\frac{30}{4}=\frac{15}{2}$$

47. (a) Show that the solution of the differential equation $(1+x^2)\frac{dy}{dx}=1+y^2$ is $\tan^{-1}y=\tan^{-1}x+C$ (or) $\tan^{-1}x=\tan^{-1}y+C$.

Solution:

$$(1+x^2)\frac{dy}{dx}=1+y^2$$

$$\frac{1}{1+y^2}dy = \frac{1}{1+x^2}dx$$

$$\int \frac{1}{1+y^2}dy = \int \frac{1}{1+x^2}dx$$

$$\tan^{-1} y = \tan^{-1} x + c$$

$$\tan^{-1} y - \tan^{-1} x = c$$

OR

(b) Prove $p \rightarrow (q \rightarrow r) = (p \wedge q) \rightarrow r$ using truth table.

Solution:

p	q	r	$q \rightarrow r$	$p \rightarrow (q \rightarrow r)$	$p \wedge q$	$(p \wedge q) \rightarrow r$
T	T	T	T	T	T	T
T	T	F	F	F	T	F
T	F	T	T	T	F	T
T	F	F	T	T	F	T
F	T	T	T	T	F	T
F	T	F	F	T	F	T
F	F	T	T	T	F	T
F	F	F	T	T	F	T
				(1)		(2)

From (1) and (2), we get

$\therefore p \rightarrow (q \rightarrow r) = (p \wedge q) \rightarrow r$

PREPARED BY
M.SANKAR M.SC., B.ED.,
PGT MATHEMATICS
PHONE NO: 9047952772
EMAIL ID: shark.uk1986@gmail.com
You Tube channel: https://www.youtube.com/c/MSMathC