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It gives me great pride and pleasure in bringing to you **Sura's Mathematics Guide Volume I and Volume II** for 11th Standard. A deep understanding of the text and exercises is rudimentary to have an insight into Mathematics. The students and teachers have to carefully understand the topics and exercises.

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Though these salient features are available in our Sura's Mathematics Guide 11th Standard, I cannot negate the indispensable role of the teachers in assisting the student to understand the Mathematics.

I sincerely believe this guide satisfies the needs of the students and bolsters the teaching methodologies of the teachers.

I pray the almighty to bless the students for consummate success in their examinations.

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01

SETS, RELATIONS AND FUNCTIONS

MUST KNOW DEFINITIONS

A set is a collection of well defined objects.

Type of sets

Empty set	:	A set containing no element.
Finite set	:	The number of elements in the set is finite.
Infinite set	:	The number of elements in the set is not finite.
Singleton set	:	A set containing only one element.
Equivalent set	:	Two sets having same number of elements.
Equal sets	:	Two sets exactly having the same elements.
Subset	:	A set X is a subset of Y if every element of X is also an element of Y. $(X \subseteq Y)$
Proper subset	:	X is a proper subset of Y if $X \subset Y$ and $X \neq Y$.
Power set	:	The set of all subsets of A is the power set of A.
Universal set	:	The set contains all the elements under consideration

Algebra of sets

Union	:	The union of two sets A and B is the set of elements which are either in A or in B $(A \cup B)$
Intersection	:	The intersection of two sets A and B is the set of all elements common to both A and B $(A \cap B)$.
Complement of a set	:	The complement of a set is the set of all elements of U (Universal set) that are not elements of A. (A') Set different$(A \setminus B)$ or $(A - B)$
Difference of two sets	:	The difference of the two sets A and B is the set of all elements belonging to A but not to B. Set different$(A \setminus B)$ or $(A - B)$
Disjoint sets	:	Two sets A and B are said to be disjoint if there is no element common to both A and B.
Open interval	:	The set $\{x: a < x < b\}$ is called an open interval and denoted by (a, b)
Closed interval	:	The set $\{x: a \leq x \leq b\}$ is called a closed interval and denoted by $[a, b]$
Neighbourhood of a point	:	Let a be any real number. Let $\in > 0$ be arbitrarily small real number. Then $(a - \in, a + \in)$ is called an "$\in$" neighbourhood of the point a and denoted by $N_{a,\in}$

Cartesian product of sets	The set of all ordered pairs (a, b) such that $a \in$ A and $b \in$ B is called the cartesian product of A and B and is denoted by A × B.

Types of relation

Reflexive	A relation R on a set A is said to be reflexive if every element of A is related to itself.
Symmetric	A relation R on a set A is said to be symmetric if $(a, b) \in R \Rightarrow (b, a) \in R$ for all $a, b \in$ A.
Transitive	A relation R on a set A is said to be transitive if $(a, b) \in$ R and $(b, c) \in$ R $\Rightarrow (a, c) \in$ R for all $a, b, c \in$ A.
Equivalent	A relation R on a set A is said to be equivalence relation if it is reflexive, symmetric and transitive.
Function	A function f from a set A to a set B is a rule which assigns to each element of A, a unique element of B. If f: A → B, then A is the domain, B is the co-domain.

Types of algebraic functions

Identity function	A function that associates each real number to itself.
Absolute value function	The function $f(x)$ defined by $f(x) = \lvert x \rvert = \begin{cases} x, & x \geq 0 \\ -x, & x < 0 \end{cases}$
Constant function	A function $f(x)$ defined by $f(x) = k$ where k is a real number.
Greatest integer function	The greatest integer function $f: \mathbb{R} \to \mathbb{R}$ defined by $f(x) = \lfloor x \rfloor$ for all $x \in$ R.
Smallest integer function	The smallest integer function $f: \mathbb{R} \to \mathbb{R}$ defined by $f(x) = \lceil x \rceil$ for all $x \in$ R.
Signum function	The function f defined by $f(x) = \begin{cases} \frac{\lvert x \rvert}{x}, & x \neq 0 \\ 0, & x = 0 \end{cases}$
Polynomial function	The function $f: \mathbb{R} \to \mathbb{R}$ defined by $f(x) = a_0x^n + a_1x^{n-1} + a_2x^{n-2} + \ldots + a_n$ where $a_0, a_1, \ldots, a_n$ are constants.
Rational function	The function $f: \mathbb{R} \to \mathbb{R}$ defined by $f(x) = \frac{p(x)}{q(x)}$, $q(x) \neq 0$ and $p(x)$, $q(x)$ are polynomial.

Algebra of functions

Addition	If f: $D_1 \to$ R and g: $D_2 \to$ R, then their sum $f + g$: $D_1 \cap D_2 \to$ R such that $(f + g)(x) = f(x) \;\; g(x)$ for all $x \in D_1 \cap D_2$.
Subtraction	If f_1: $D_1 \to$ R and g: $D_2 \to$ R, then their difference $f - g$: $D_1 \cap D_2 \to$ R such that $(f - g)(x) = f(x) - g(x)$ for all $x \in D_1 \cap D_2$.
Product	If f_1: $D_1 \to$ R and g: $D_2 \to$ R, then their product $f \cdot g$: $D_1 \cap D_2 \to$ R such that $(fg)(x) = f(x) . g(x)$ for all $x \in D_1 \cap D_2$.
Quotient	If f_1: $D_1 \to$ R and g: $D_2 \to$ R, then their quotient $\frac{f}{g} : D_1 \cap D_2 - \{x : g(x) = 0\} \to$ R such that $\left(\frac{f}{g}\right)(x) = \frac{f(x)}{g(x)}$ such that for all $x \in D_1 \cap D_2 - \{x : g(x) = 0\}$.
Composition of functions	If f: A → B and g: B → C then gof : A → C defined by gof$(x) = g[f(x)]$ for all $x \in$ A.

Kinds of functions

One-one	:	A function f: A $\to$ B is said to be a one-one function (injection) if different elements of A have different images in B.
Onto	:	A function f: A $\to$ B is said to be an onto (surjection) function if every element of B is the image of some element of A.
Bijection	:	A function f: A $\to$ B is a bijection if one-one as well as onto.
Inverse of a function	:	Let f: A $\to$ B be a bijection. Then g: B $\to$ A which associates each element $y \in$ B to a unique element $x \in$ A such that $g(y) = x$ is called the inverse of f, and it denoted as f'.

Formulae to remember

Demorgan's laws	:	1. $(A \cup B)' = A' \cap B'$ 2. $(A \cap B) = A' \cup B'$ 3. $A \backslash (B \cup C) = (A \backslash B) \cap (A \backslash C)$ 4. $A \backslash (B \cap C) = (A \backslash B) \cup (A \backslash C)$
Reflexive	:	aRa for all $a \in$ A
Symmetric	:	aR$b \Rightarrow b$Ra for all $a, b \in$ A
Transitive	:	aRb, bR$c \Rightarrow a$Rc for all $a, b, c \in$ A
Antisymmetric	:	aRb and bR$a \Rightarrow a = b$ for all $a, b \in$ A $\quad A \Delta B = (A \backslash B) \cup (B \backslash A)$
One-one function	:	If f: A $\to$ A then $f(x_1) = f(x_2) \Rightarrow x_1 = x_2$ for all $x_1, x_2 \in$ A
Onto function	:	Co-domain = Range.
		If a set has n elements, then total number of subsets is 2^n.

TEXTUAL QUESTIONS

EXERCISE 1.1

1. Write the following in roster form.

(i) $\{x \in \mathbb{N} : x^2 < 121$ and x is a prime$\}$.

(ii) the set of all positive roots of the equation $(x-1)(x+1)(x^2-1) = 0$.

(iii) $\{x \in \mathbb{N} : 4x + 9 < 52\}$.

(iv) $\left\{x : \dfrac{x-4}{x+2} = 3, x \in \mathbb{R} - \{-2\}\right\}$

Sol : **(i) $\{x \in \mathbb{N} : x^2 < 121$ and x is a prime$\}$.**

Let A = $\{x \in \mathbb{N}: x^2 < 121$, and x is a prime$\}$

A = $\{2, 3, 5, 7\}$.

(ii) the set of all positive roots of the equation $(x-1)(x+1)(x^2-1) = 0$.

Let B = {the set of positive roots of the equation $(x-1)(x+1)(x^2-1) = 0$}

$$(x-1)(x+1)(x-1)(x+1) = 0$$
$$(x+1)^2 (x-1)^2 = 0$$
$$(x+1)^2 = 0 \text{ or } (x-1)^2 = 0$$
$$x+1 = 0 \text{ or } x-1 = 0$$
$$x = -1 \text{ or } x = 1$$
$$\Rightarrow \quad x = 1, -1$$
$$B = \{1\}.$$

(iii) $\{x \in \mathbb{N} : 4x + 9 < 52\}$.

Let C = $\{x \in \mathbb{N}: 4x + 9 < 52\}$

$\Rightarrow$ C = $\{x \in \mathbb{N}: 4x < 52 - 9\}$

$\Rightarrow$ C = $\{x \in \mathbb{N}: 4x < 43\}$

$\Rightarrow$ C = $\left\{x \in \mathbb{N}: x < \dfrac{43}{4}\right\}$

$\Rightarrow$ C = $\{x \in \mathbb{N}: x < 10.75\}$

$\Rightarrow$ C = $\{1, 2, 3, 4, 5, 6, 7, 8, 9, 10\}$.

(iv) $\left\{x : \dfrac{x-4}{x+2} = 3, x \in \lceil - \{-2\}\right\}$.

Let D = $\left\{x : \dfrac{x-4}{x+2} = 3, x \in \mathbb{R} - \{-2\}\right\}$

$\Rightarrow$ D = $\{x: x - 4 = 3x + 6, x \in \mathbb{R}\}$

$\Rightarrow$ D = $\{x: -4 - 6 = 3x - x, x \in \mathbb{R}\}$

$\Rightarrow$ D = $\{x: 2x = -10, x \in \mathbb{R}\}$

$\Rightarrow$ D = $\{x: x = -5, x \in \mathbb{R}\}$

$\Rightarrow$ D = $\{-5\}$

2. Write the set $\{-1, 1\}$ in set builder form.

Sol : Let P = $\{-1, 1\}$

$\Rightarrow$ P = $\{ x : x$ is a root of $x^2 - 1 = 0\}$

$\Rightarrow$ P = $\{ x : x^2 - 1 = 0, x \in \mathbb{R}\}$

3. State whether the following sets are finite or infinite.

(i) $\{x \in \mathbb{N} : x$ **is an even prime number**$\}$

(ii) $\{x \in \mathbb{N} : x$ **is an odd prime number**$\}$

(iii) $\{x \in \mathbb{Z} : x$ **is even and less than 10**$\}$

(iv) $\{x \in \mathbb{R} : x$ **is a rational number**$\}$

(v) $\{x \in \mathbb{N} : x$ **is a rational number**$\}$

Sol : **(i)** $\{x \in \mathbb{N} : x$ **is an even prime number**$\}$

Let A = $\{x \in \mathbb{N}: x$ is an even prime number$\}$

$\Rightarrow$ A = $\{2\}$ $\Rightarrow$ A is a finite set.

(ii) $\{x \in \mathbb{N} : x$ **is an odd prime number**$\}$

Let B = $\{x \in \mathbb{N}: x$ is an odd prime number$\}$

$\Rightarrow$ B = $\{1, 3, 5, 7, 11, ...\}$

$\Rightarrow$ B is an infinite set.

(iii) $\{x \in \mathbb{Z} : x$ **is even and less than 10**$\}$

Let C = $\{x \in \mathbb{Z}: x$ is even and $< 10\}$

$\Rightarrow$ C $=\{...-4, -2, 0, 2, 4, 6, 8\}$. C is a infinite set.

(iv) $\{x \in \mathbb{R} : x$ **is a rational number**$\}$

Let D = $\{x \in \mathbb{R}: x$ is a rational number$\}$

$\Rightarrow$ D = {set of all rational number}

$\Rightarrow$ D is an infinite set.

(v) $\{x \in \mathbb{N} : x$ **is a rational number**$\}$

Let $\mathbb{N}$ = $\{x \in \mathbb{N}: x$ is a rational number$\}$

$$\Rightarrow \mathbb{N} = \left\{\frac{1}{1}, \frac{2}{1}, \frac{3}{1}, \frac{4}{1}, ...\infty\right\}$$

$\Rightarrow \mathbb{N}$ is an infinite set.

4. By taking suitable sets A, B, C, verify the following results:

(i) $\mathbf{A \times (B \cap C) = (A \times B) \cap (A \times C)}$

(ii) $\mathbf{A \times (B \cup C) = (A \times B) \cup (A \times C)}$

(iii) $\mathbf{(A \times B) \cap (B \times A) = (A \cap B) \times (B \cap A)}$

(iv) $\mathbf{C - (B - A) = (C \cap A) \cup (C \cap B')}$

(v) $\mathbf{(B - A) \cap C = (B \cap C) - A = B \cap (C - A)}$

(vi) $\mathbf{(B - A) \cup C = (B \cup C) - (A - C)}$

Sol : **(i)** $\mathbf{A \times (B \cap C) = (A \times B) \cap (A \times C)}$

Let A = $\{1, 2, 3\}$, B = $(4, 5, 6, 7\}$

C = $\{3,4,5,9\}$

and U = $\{1, 2, 3, 4, 5, 6, 7, 9\}$

LHS = $A \times (B \cap C)$

= $A \times \{4, 5\}$ $[\because B \cap C = \{4,5\}]$

= $\{1, 2, 3\} \times \{4, 5\}$

= $\{(1, 4) (1, 5) (2, 4) (2, 5) (3, 4) (3, 5)\}$ ...(1)

$A \times B$ = $\{1, 2, 3\} \times \{4, 5, 6, 7\}$

= $\{(1, 4) (1, 5) (1, 6) (1, 7) (2, 4) (2, 5) (2, 6) (2, 7) (3, 4) (3, 5) (3, 6) (3, 7)\}$

$A \times C$ = $\{1, 2, 3\} \times \{3, 4, 5, 9\}$

= $\{(1, 3) (1, 4) (1, 5) (1, 9) (2, 3) (2, 4) (2, 5) (2, 9) (3, 3) (3, 4) (3, 5) (3, 9)\}$

RHS = $(A \times B) \cap (A \times C)$

= $\{(1, 4) (1, 5) (2, 4) (2, 5) (3, 4) (3, 5)\}$ ...(2)

From (1) and (2), LHS = RHS. Hence verified.

(ii) $\mathbf{A \times (B \cup C) = (A \times B) \cup (A \times C)}$

$(B \cup C)$ = $\{3, 4, 5, 6, 7, 9\}$

Now, $A \times (B \cup C)$ = $\{1, 2, 3\} \times \{3, 4, 5, 6, 7, 9\}$

= $\{(1, 3) (1, 4) (1, 5) (1, 6) (1, 7) (1, 9) (2, 3) (2, 4) (2, 5) (2, 6) (2, 7) (2, 9) (3, 3) (3, 4) (3, 5) (3, 6) (3, 7) (3, 9)\}$ ...(1)

Now $A \times B$ = $\{1, 2, 3\} \times \{4, 5, 6, 7\}$

= $\{(1, 4) (1, 5) (1, 6) (1, 7) (2, 4) (2, 5) (2, 6) (2, 7) (3, 4) (3, 5) (3, 6) (3, 7)\}$

$A \times C$ = $\{1, 2, 3\} \times \{3, 4, 5, 9\}$

= $\{(1, 3) (1, 4) (1, 5) (1, 9) (2, 3) (2, 4) (2, 5) (2, 9) (3, 3) (3, 4) (3, 5) (3, 9)\}$

RHS $(A \times B) \cup (A \times C)$

= $\{(1, 3) (1, 4) (1, 5) (1, 6) (1, 7) (1, 9) (2, 3) (2, 4) (2, 5) (2, 6) (2, 7) (2, 9) (3, 3) (3, 4) (3, 5) (3, 6) (3, 7) (3,9)\}$ ...(2)

From (1) & (2), LHS = RHS Hence verified

(iii) $\mathbf{(A \times B) \cap (B \times A) = (A \cap B) \times (B \cap A)}$

$(A \times B)$ = $\{(1, 4) (1, 5) (1, 6) (1, 7) (2, 4) (2, 5) (2, 6) (2, 7) (3, 4) (3, 5) (3, 6) (3, 7)\}$

$(B \times A)$ = $\{(4, 1) (4, 2) (4, 3) (5, 1) (5, 2) (5, 3) (6, 1) (6, 2) (6, 3) (7, 1) (7, 2) (7, 3)\}$

LHS = $(A \times B) \cap (B \times A) = \{\ \}$ ...(1)

$(A \cap B)$ = $\{\ \}$, $(B \cap A) = \{\ \}$

$\therefore$ RHS = $(A \cap B) \times (B \cap A) = \{\ \}$...(2)

From (1) and (2), LHS = RHS

(iv) $\mathbf{C - (B - A) = (C \cap A) \cup (C \cap B')}$

$B - A$ = $\{4, 5, 6, 7\}$

LHS = $C - (B - A) = \{3, 9\}$ ...(1)

$C \cap A$ = $\{3\}$

B' = $\{1, 2, 3, 9\}$

$C \cap B'$ = $\{3, 9\}$

RHS = $(C \cap A) \cup (C \cap B')$

= $\{3, 9\}$ ...(2)

From (1) and (2), LHS = RHS

(v) $\mathbf{(B - A) \cap C = (B \cap C) - A = B \cap (C - A)}$

$B - A$ = $\{4, 5, 6, 7\}$

$(B - A) \cap C$ = $\{4, 5\}$ ...(1)

$B \cap C$ = $\{4, 5\}$

$(B \cap C) - A$ = $\{4, 5\}$ ...(2)

$$\begin{aligned} C-A &= \{4, 5, 9\} \\ B\cap(C-A) &= 4, 5\} \qquad ...(3) \end{aligned}$$

From (1), (2) and (3),

$$(B-A)\cap C = (B\cap C)-A = B\cap(C-A).$$

(vi) $(B - A) \cup C = (B \cup C) - (A - C)$

$$\begin{aligned} B-A &= \{4, 5, 6, 7\} \\ (B-A)\cup C &= \{3, 4, 5, 6, 7, 9\} \qquad ...(1) \\ B\cup C &= \{3, 4, 5, 6, 7, 9\} \\ A-C &= \{1, 2\} \\ (B\cup C)-(A-C) &= \{3, 4, 5, 6, 7, 9\} \qquad ...(2) \end{aligned}$$

From (1) and (2), $(B-A)\cup C = (B\cup C)-(A-C)$

Hence verified.

5. Justify the trueness of the statement "*An element of a set can never be a subset of itself*".

Sol : Let $P = \{a, b, c, d\}$.

Each and every element of the set P can be a subset of the set itself

Eg : $\{a\}, \{b\}, \{c\}, \{d\}$.

Hence, the given statement is not true.

6. If $n(p(A)) = 1024$, $n(A \cup B) = 15$ and $n(P(B)) = 32$, then find $n(A \cap B)$.

Sol : $[\therefore$ If $n(A) = n$, then $n(P(A)) = 2^n]$

$$\begin{aligned} \text{Given } n(P(A)) &= 1024 = 2^{10} \\ \Rightarrow n(A) &= 10 \\ n(P(B)) &= 32 = 2^5 \\ \Rightarrow n(B) &= 5. \end{aligned}$$

We know that,

$$\begin{aligned} n(A\cup B) &= n(A)+n(B)-n(A\cap B) \\ \Rightarrow 15 &= 10+5-n(A\cap B) \\ \Rightarrow n(A\cap B) &= 0. \end{aligned}$$

7. If $n(A \cap B) = 3$ and $n(A \cup B) = 10$, then find $n(P(A \Delta B))$ **[Qy - 2018]**

Sol : We know that $n(A\cup B) = n(A-B) + n(B-A) + n(A\cap B)$ if A and B are not disjoint.

$$\begin{aligned} \Rightarrow n(A-B)+n(B-A) &= n(A\cup B)-n(A\cap B) \\ \Rightarrow n(A\Delta B) &= 10-3 \\ \Rightarrow \therefore n(A\Delta B) &= 7 \\ \therefore n[P(A\Delta B)] &= 2^7 = 128. \end{aligned}$$

8. For a set A, A × A contains 16 elements and two of its elements are (1, 3) and (0, 2). Find the elements of A.

Sol : Since A × A contains 16 elements, then A must have 4 elements

$\Rightarrow n(A) = 4.$

The elements of A × A are (1, 3) and (0, 2)

$\therefore$ The possibilities of elements of A are $\{0, 1, 2, 3\}$

9. Let A and B be two sets such that $n(A) = 3$ and $n(B) = 2$. If $(x, 1)$ $(y, 2)$ $(z, 1)$ are in A × B, find A and B, where x, y, z are distinct elements. **[Hy - 2018]**

Sol : Given $A\times B = \{(x, 1)\,(y, 2)\,(z, 1)\}$

Since $n(A) = 3$ and $n(B) = 2$,

A × B will have 6 elements.

The remaining elements of A × B will be $(x, 2)\,(y, 1)\,(z, 2)$

$$\begin{aligned} \therefore A\times B &= \{(x, 1)\,(y, 2)\,(z, 1)\,(x, 2)(y, 1)\,(z, 2)\} \\ \therefore A &= \{x, y, z\} \text{ and } B = \{1, 2\} \end{aligned}$$

10. If A × A has 16 elements, $S = \{(a, b) \in A \times A: a < b\}$; (–1, 2) and (0, 1) are two elements of S, then find the remaining elements of S. **[Qy - 2018; June - 2019]**

Sol :

$$\begin{aligned} n(A\times A) &= 16 \Rightarrow n(A) = 4. \\ \text{Given } S &= \{(a, b)\in A\times A: a<b\} \\ \therefore A &= \{-1,0, 1, 2\}. \\ A\times A &= \{(-1, -1)\,(-1, 0)\,(-1, 1) \\ &\quad (-1, 2)(0, -1),(0,0)\,(0,1) \\ &\quad (0,2)\,(1, -1)\ (1, 0)(1, 1) \\ &\quad (1, 2)\,(2, -1)(2, 0)\,(2, 1) \\ &\quad (2, 2)\} \\ \text{Now, } S &= \{(-1, 0)\,(-1, 1)\,(-1, 2)\,(0, 1) \\ &\quad (0, 2)\,(1,2)\} \end{aligned}$$

$\therefore$ The remaining elements of S are (–1, 0) (–1, 1) (0, 2) (1,2)

EXERCISE 1.2

1. Discuss the following relations for reflexivity, Symmetric and Transitive :

- **(i) The relation R defined on the set of all positive integers by "*m*R*n* if *m* divides *n*".**
- **(ii) Let *P* denote the set of all straight lines in a plane. The relation *R* defined by "*l*R*m* if *l* is perpendicular to *m*".** **[Qy - 2019]**
- **(iii) Let A be the set consisting of all the members of a family. The relation R defined by "*a*R*b* if *a* is not a sister of *b*".**
- **(iv) Let A be the set consisting of all the female members of a family. The relation R defined by "*a*R*b* if *a* is not a sister of *b*".**
- **(v) On the set of natural numbers the relation R defined by "*x*R*y* if $x + 2y = 1$".**

Sol : **(i) The relation R defined on the set of all positive integers by "*m*R*n*" if m divides *n*".**

Given relation is "*m*R*n* if *m* divides *n*".

Reflexive : *m*R*m* since *m* divides *m* for all positive integers *m*.

$\therefore$ R is reflexive.

Sol :

Given $f(x) = 3x - 4$

Let $y = 3x - 4 \Rightarrow y + 4 = 3x$

$$\Rightarrow \quad x = \frac{y+4}{3}$$

Let $g(y) = \frac{y+4}{3}$.

Now $gof(x) = g(f(x)) = g(3x-4)$

$$= \frac{3x - \cancel{4} + \cancel{4}}{3} = \frac{\cancel{3}x}{\cancel{3}} = x$$

and $fog(y) = f(g(y)) = f\left(\frac{y+4}{3}\right)$

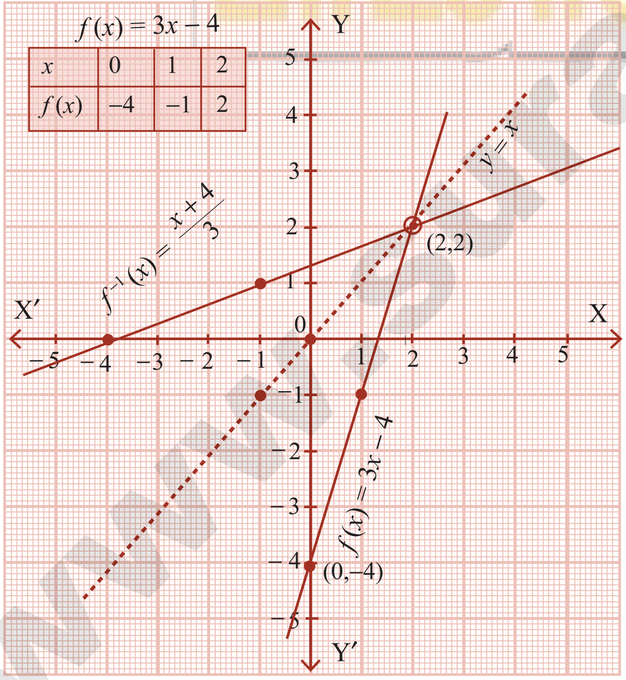
$$= \cancel{3}\left(\frac{y+4}{\cancel{3}}\right) - 4 = y + \cancel{4} - \cancel{4} = y$$

Thus, $gof(x) = I_x$ and fog $(y) = I_y$.

This implies that f and g are bijections and inverses to each other.

Hence f is bijection and $f^{-1}(y) = \frac{y+4}{3}$

Replacing y by x, we get $f^{-1}(x) = \boxed{\frac{x+4}{3}}$.



Hence, the graph of $y = f^{-1}(x)$ is the reflection of the graph of f in $y = x$

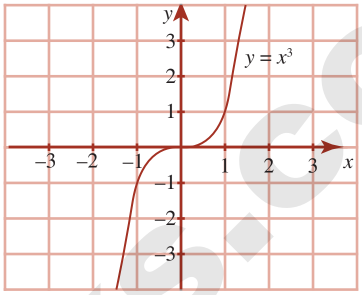
EXERCISE 1.4

1. For the curve $y = x^3$ given in figure draw,

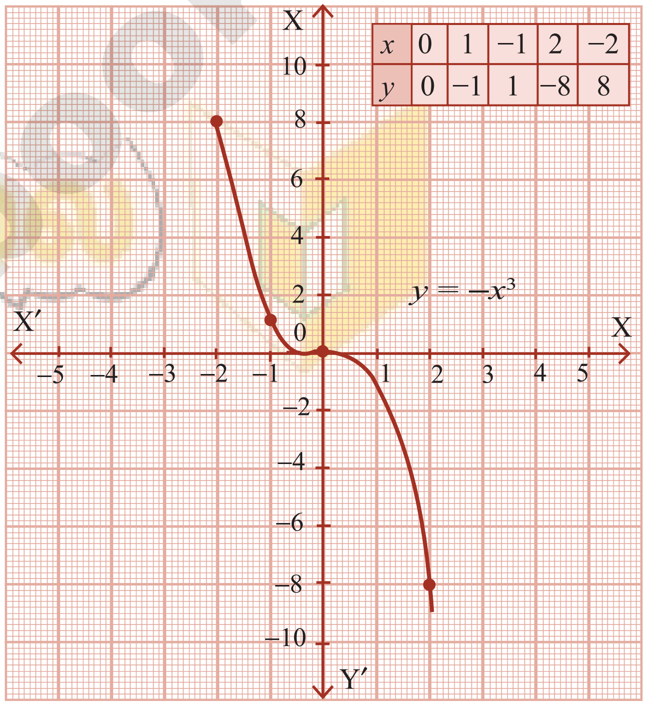
(i) $y = -x^3$ [Qy - 2019] (ii) $y = x^3 + 1$

(iii) $y = x^3 - 1$ (iv) $y = (x + 1)^3$

with the same scale.



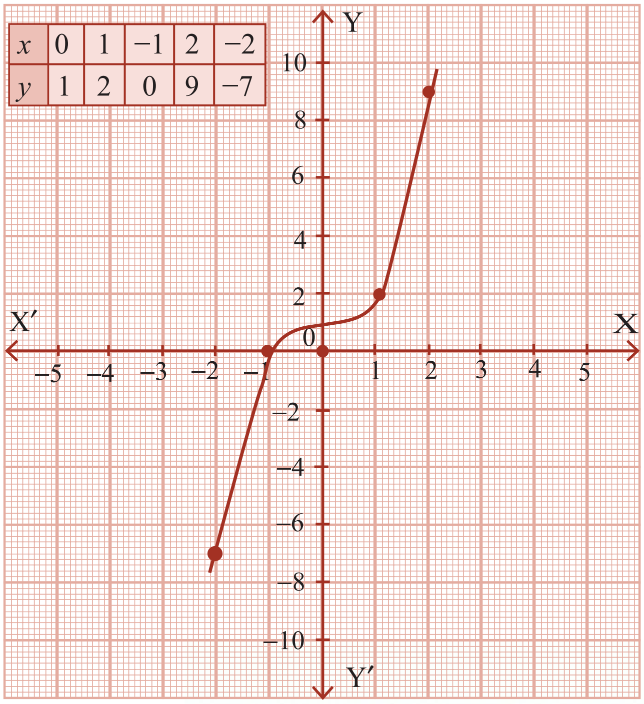
Sol : **(i) $y = -x^3$**



Let $f(x) = x^3$

Since $y = -f(x)$, this is the reflection of the graph of f about the x-axis

(ii) $y = x^3 + 1$ **[Qy - 2019]**

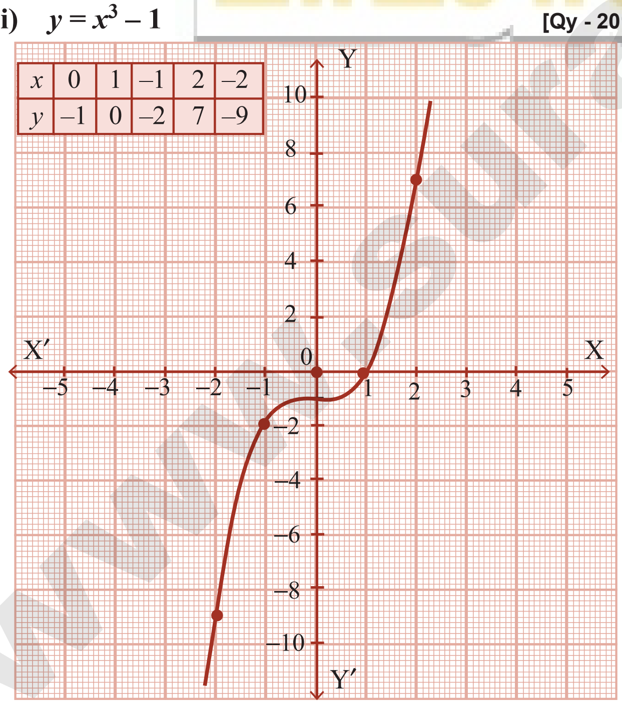


x	0	1	−1	2	−2
y	1	2	0	9	−7

Let $f(x) = x^3$

Since $y = f(x) + 1$, this is the graph of $f(x)$ shifts to the upward for one unit

(iii) $y = x^3 - 1$ **[Qy - 2019]**

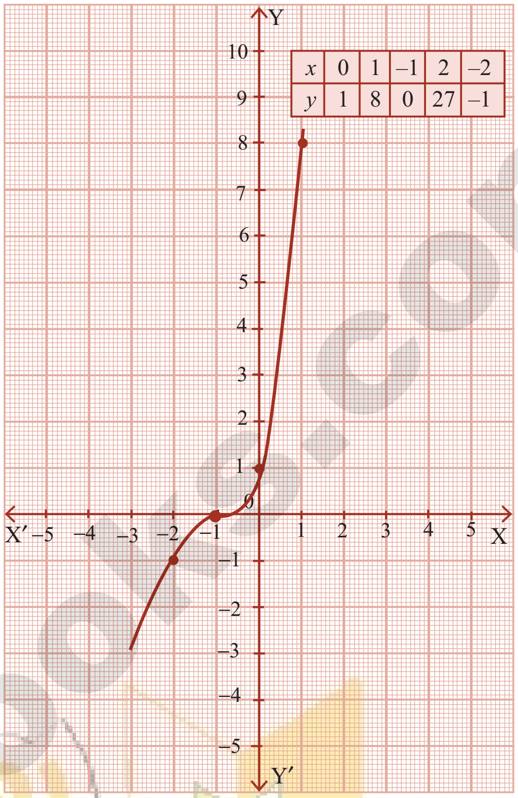


x	0	1	−1	2	−2
y	−1	0	−2	7	−9

Let $f(x) = x^3$

Since $y = f(x) - 1$, this is the graph of $f(x)$ shifts to the downward for one unit.

(iv) $y = (x + 1)^3$ **[Qy - 2019; Govt. MQP - 2018]**



x	0	1	−1	2	−2
y	1	8	0	27	−1

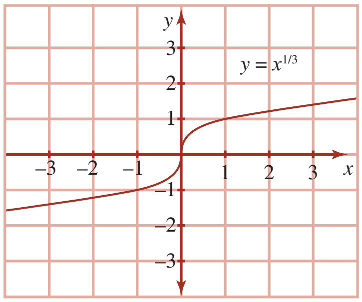
Let $f(x) = x^3$

$y = (x + 1)^3$, causes the graph of $f(x)$ shifts to the left for one unit.

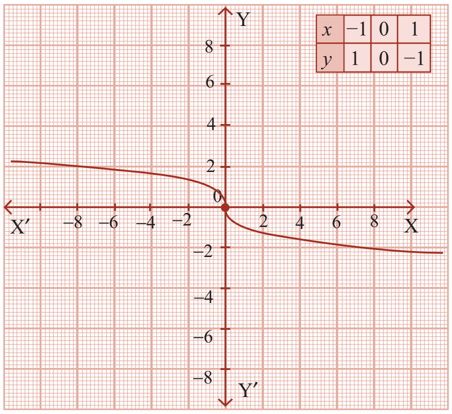
2. **For the curve,** $y = x^{\left(\frac{1}{3}\right)}$ **given in figure draw.**

(i) $y = -x^{\left(\frac{1}{3}\right)}$ **(ii)** $y = x^{\left(\frac{1}{3}\right)} + 1$

(iii) $y = x^{\left(\frac{1}{3}\right)} - 1$ **(iv)** $(x+1)^{\left(\frac{1}{3}\right)}$

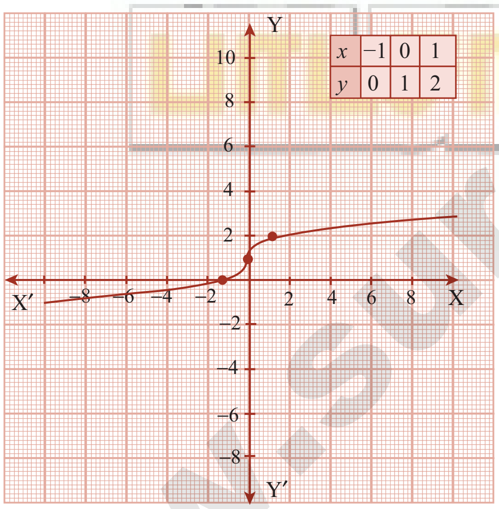


Sol : (i) $y=-x^{\frac{1}{3}}$



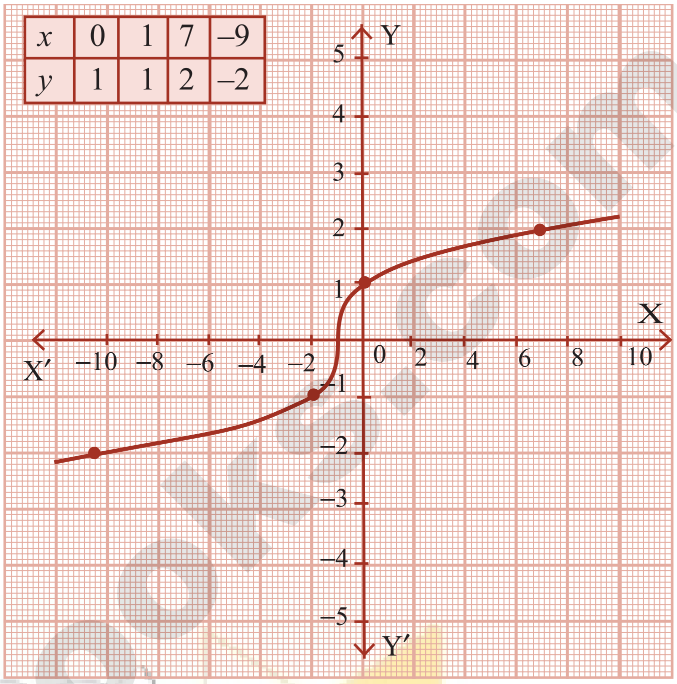
Then $y=-x^{\frac{1}{3}}$ is the reflection of the graph of $y=x^{\frac{1}{3}}$ about the x-axis.

(ii) $y=x^{\frac{1}{3}}+1$.



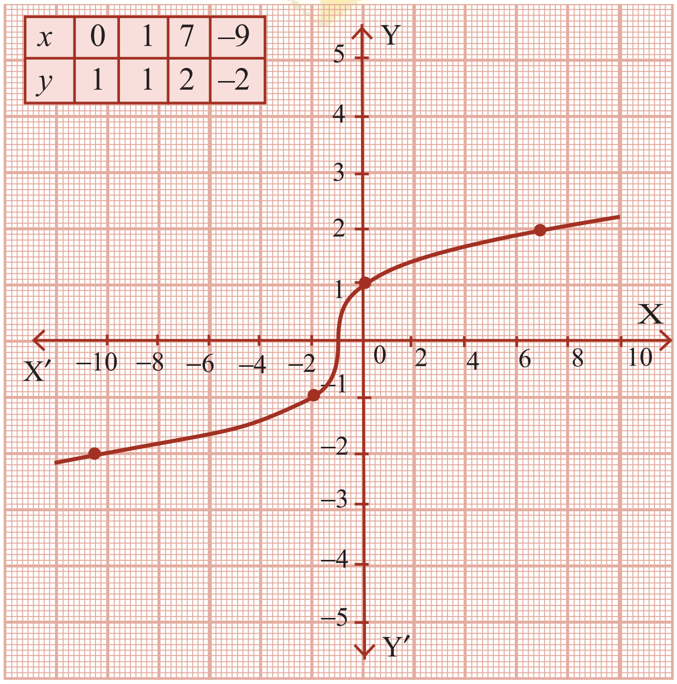
Then $y=x^{\frac{1}{3}}+1$ is the x graph of $y=x^{\frac{1}{3}}$ shifts to the upward for one unit.

(iii) $y=x^{\frac{1}{3}}-1$.



Then $y=x^{\frac{1}{3}}-1$ is the graph of $x^{\frac{1}{3}}$ shifts to the downward for one unit.

(iv) $y=(x+1)^{\frac{1}{3}}$



$y=(x+1)^{\frac{1}{3}}$, it causes the graph of $x^{\frac{1}{3}}$, shifts to the left for one unit.

02

BASIC ALGEBRA

MUST KNOW DEFINITIONS

Rational numbers : Any number of the form $\frac{p}{q}$, where $q \neq 0$ is called a rational number where $p, q \in z$.

Irrational numbers : A number that cannot be expressed as a ratio between two integers and is not an imaginary number.

Intervals :

- If a, b are real numbers such that $a < b$, then the set $\{x: a < x < b\}$ is called the open interval from a to b i.e. (a, b).
- The set $\{x: a \leq x \leq b\}$ is called the closed interval from a to b and is written as $[a, b]$
- If a is any real number, then the sets of the type $\{x: x < a\}$, $\{x: x \leq a\}$, $\{x: x > a\}$ and $\{x: x \geq a\}$ are called infinite intervals and are respectively written as $(-\infty, a)$, $(-\infty, a]$, (a, ∞) and $[a, \infty)$. These are semi-open and semi-closed intervals.

Absolute value of *x* : Absolute value of $x = |x|$ is defined as:

$$|x| = \begin{cases} x & \text{if} \quad x \geq 0 \\ -x & \text{if} \quad x < 0 \end{cases}$$

Radical (Surd) : If 'a' is a positive rational number and n is a positive integer such that $\sqrt[n]{a}$ is an irrational number then $\sqrt[n]{a}$ is called a surd or a radical.

Mixed surd : A surd is called a mixed if its rational co-efficient is other than unity. If the product of two irrational numbers is rational, then each one is called the rationalizing.

Pure surd : A surd is a pure surd if its rational co-efficient is unity.

Polynomial : An expression of more than two algebraic terms, especially the sum of several terms that contain different powers of the same variable

Identity : An identity is a statement of equality between two expressions which is true for all values of the variable involved.

Equation : An equation is a statement of equality between two expressions which is not true for all values of the variable involved.

Root of an equation : A value of the variable for which an equation is satisfied is called a root of an equation.

Logarithm : If $a > 0$, $a \neq 1$ and $a^x = y$, then we define the logarithm of y to the base a as x and written as $\log_a y = x$.

Common logarithm : The logarithm to the base '10' are called **Common logarithm.**

Disjoint set : Two sets A and B are said to be disjoint if there is no element common to both A and B.

Characteristic : The integral part of the common logarithm of a number., is characteristic

It may be either positive or zero or negative.

Mantissa : The positive decimal part of the common logarithm of a number is Mantissa

It may be either positive or zero.

Partial fractions :

(i) For Linear factors: Rational expression of the form "$\frac{p}{q}$ where q is the non-repeated product of linear factors like $(ax + b)$ $(cx + d)$ can be written as

$$\frac{\mathrm{M}}{ax+b}+\frac{\mathrm{N}}{cx+d}.$$

(ii) $$\frac{p}{(ax+b)^n}=\frac{\mathrm{A}_1}{ax+b}+\frac{\mathrm{A}_2}{(ax+b)^2}+\cdots+\frac{\mathrm{A}_n}{(ax+b)^n}$$

(iii) $$\frac{p}{ax^2+bx+c}=\frac{\mathrm{A}x+\mathrm{B}}{ax^2+bx+c}$$ where p is a rational expression of degree less than the denominator

Formulae to Remember :

Laws of Radicals : For positive integers m, n and positive rational numbers a, b we have

(i) $\left(\sqrt[n]{a}\right)^n = a = \sqrt[n]{a^n}$ (ii) $\sqrt[n]{a}\times\sqrt[n]{b} = \sqrt[n]{ab}$

(iii) $\sqrt[m]{\sqrt[n]{a}} = \sqrt[mn]{a}$ (iv) $\frac{\sqrt[n]{a}}{\sqrt[n]{b}} = \sqrt[n]{\frac{a}{b}}$

Inequality : If $a > b$, then

(i) $a + c > b + c$ for any $c \in \mathrm{R}$

(ii) $\mathrm{a} - \mathrm{c} > \mathrm{b} - \mathrm{c}$ for any $c \in \mathrm{R}$

(iii) $-a < -b$

(iv) $ac > bc$, $\frac{a}{c} > \frac{b}{c}$ for any positive real number c.

(v) $ac < bc$, $\frac{a}{c} < \frac{b}{c}$ for any negative real number c.

Identities :
(i) $(x + a)(x + b) = x^2 + x(a + b) + ab$
(ii) $(a + b)^2 = a^2 + 2ab + b^2$
(iii) $(a - b)^2 = a^2 - 2ab + b^2$
(iv) $(a + b)^3 = a^3 + 3a^2b + 3ab^2 + b^3$
(v) $(a - b)^3 = a^3 - 3a^2b + 3ab^2 - b^3$
(vi) $(a + b)(a - b) = a^2 - b^2$
(vii) $a^3 + b^3 = (a + b)^3 - 3ab(a + b)$ (OR) $(a + b)(a^2 - ab + b^2)$
(viii) $a^3 - b^3 = (a - b)^3 + 3ab(a - b)$ (OR) $(a - b)(a^2 + ab + b^2)$
(ix) $x^n - 1 = (x - 1)(x^{n-1} + x^{n-2} + \ldots + 1)$

Quadratic equation : The roots of $ax^2 + bx + c = 0$ is $x = \dfrac{-b \pm \sqrt{b^2 - 4ac}}{2a}$

Laws of logarithm :
(i) **Product rule:** $\log_a mn = \log_a m + \log_a n$
(ii) **Quotient rule:** $\log_a (m/n) = \log_a m - \log_a n$
(iii) **Power rule:** $\log_a nm = n \cdot \log_a m$
(iv) **Change of base rule:** $\log_a m = \log_b m \times \log_a b$
Also $\log_a b \times \log_a b = 1$

Basic Results :
(v) If $a > 0$, $a \neq 1$, and $a^x = y$, then $\log_a y = x$
(vi) $\log_a x_1 = \log_a x_2 \Rightarrow x_1 = x_2$
(vii) $\log_a 1 = 0$ and $\log_a a = 1$

Absolute value : If a is a positive real number, then
(i) $|x| < a \Leftrightarrow x \in (-a, a)$
(ii) $|x| \leq a \Leftrightarrow x \in [-a, a]$
(iii) $|x| > a \Leftrightarrow x < -a$ or $x > a$
(iv) $|x| > a \Leftrightarrow x < -a$ or $x \geq a$

TEXTUAL QUESTIONS

EXERCISE 2.1

1. Classify each element of $\left\{\sqrt{7}, \dfrac{-1}{4}, 0, 3.14, 4, \dfrac{22}{7}\right\}$ as a member of $\mathbb{N}, \mathbb{Q}, \mathbb{R} - \mathbb{Q}$ or $\mathbb{Z}$.

Sol : Since $\sqrt{7}$ is an irrational number, $\sqrt{7} \in \mathbb{R} - \mathbb{Q}$.

Since $\dfrac{-1}{4}$ is a negative rational number, $\dfrac{-1}{4} \in \mathbb{Q}$

0 is an integer and $0 \in \mathbb{Z}, \mathbb{Q}$

$3.14 = \pi$ is a non-recurring and non terminating decimal.

$\therefore$ 3.14 is an irrational number $\Rightarrow 3.14 \in \mathbb{Q}$

4 is a positive integer $\Rightarrow 4 \in \mathbb{N}, \mathbb{Z}, \mathbb{Q}$.

$\dfrac{22}{7} \in \mathbb{Q}$ Which is an irrational number.

2. Prove that $\sqrt{3}$ is an irrational number. (Hint: Follow the method that we have used to prove $\sqrt{2} \notin \mathbb{Q}$.) **[First Mid - 2018]**

Sol : Suppose $\sqrt{3}$ is a rational number.

Then $\sqrt{3}$ can be written as $\sqrt{3} = \dfrac{m}{n}$

Where m and n are positive integers with no common factors other than 1.

Squaring both sides we get,

$$3 = \frac{m^2}{n^2} \Rightarrow 3n^2 = m^2$$

Multiplying by 2 we get,

$$6n^2 = 2m^2 \Rightarrow 3(2n^2) = 2m^2 \quad \ldots(1)$$

Since $2n^2$ is divisible by 2, m^2 is also an even number

$\Rightarrow m$ must be even

$\Rightarrow \quad m = 2k$ for some natural number k

$\Rightarrow \quad 3n^2 = (2k)^2$

$\Rightarrow \quad 3n^2 = 4k^2$ [From (1)]

$\Rightarrow n$ is also an even number.

Thus both m and n are even numbers having a common factor 2.

This contradicts our initial assumption that m and n do not have a common factor.

Hence $\sqrt{3}$ cannot be a rational number.

$\Rightarrow \sqrt{3}$ is an irrational number.

Hence proved.

3. Are there two distinct irrational numbers such that their difference is a rational number? Justify.

Sol : Let the two distinct irrational numbers be

$\left(2+\sqrt{3}\right)$ and $\left(4+\sqrt{3}\right)$.

Their difference is $\left(2+\sqrt{3}\right)-\left(4+\sqrt{3}\right)$

$= 2 + \sqrt{3} - 4 - \sqrt{3} = 2 - 4 = -2$ which is rational.

4. Find two irrational numbers such that their sum is a rational number. Can you find two irrational numbers whose product is a rational number.

Sol : Let the two irrational numbers be

$5+\sqrt{7}$ and $7-\sqrt{7}$.

Their sum $= \left(5+\sqrt{7}\right)+\left(7-\sqrt{7}\right) = 5+\sqrt{7}+7-\sqrt{7}$

$= 5 + 7 = 12$ which is a rational number.

Consider the two irrational numbers

$4+\sqrt{6}$ and $4-\sqrt{6}$.

Their product $= \left(4+\sqrt{6}\right)+\left(4-\sqrt{6}\right) = 4^2 - \left(\sqrt{6}\right)^2$

$= 16 - 6 = 10$ which is a rational number.

5. Find a positive number smaller than $\frac{1}{2^{1000}}$. Justify.

Sol : Given number is $\frac{1}{2^{1000}}$.

We know $1000 < 1001$

$\Rightarrow 2^{1000} < 2^{1001} \Rightarrow \frac{1}{2^{1000}} > \frac{1}{2^{1001}}$

$\therefore$ A positive number smaller than $\frac{1}{2^{1000}}$ is $\frac{1}{2^{1001}}$.

EXERCISE 2.2

1. Solve for x

(i) $|3 - x| < 7$ **(ii)** $|4x - 5| \geq -2$

(iii) $\left|3 - \frac{3}{4}x\right| \leq \frac{1}{4}$ **(iv)** $|x| - 10 < -3$

Sol : (i) $|3 - x| < 7$

Given $|3 - x| < 7$

This means $-7 < 3 - x < 7$

$\Rightarrow -7 - 3 < -x < 7 - 3 \qquad \Rightarrow -10 < -x < 4$

$\Rightarrow 10 > x > -4$ $[a < b \Rightarrow ay > by$ for all $y < 0]$

$\Rightarrow -4 < x < 10$ Here $y = -1$.

(ii) $|4x - 5| \geq -2$

$\Rightarrow 4\left|x-\frac{5}{4}\right| \geq -2 \Rightarrow \left|x-\frac{5}{4}\right| \geq -\frac{2}{4}$

Any $x \in$ R will satisfy this inequality.

(iii) $\left|3 - \frac{3}{4}x\right| \leq \frac{1}{4}$

This means $-\frac{1}{4} \leq 3 - \frac{3}{4}x \leq \frac{1}{4}$.

$\Rightarrow -\frac{1}{4} - 3 \leq -\frac{3}{4}x \leq \frac{1}{4} - 3$

$\Rightarrow -\frac{13}{4} \leq -\frac{3}{4}x \leq -\frac{11}{4}$

$\Rightarrow 13 \geq 3x \geq 11 \Rightarrow \frac{11}{3} \leq x \leq \frac{13}{3}$

(iv) $|x| - 10 < -3$

Given $|x| - 10 < -3$

$\Rightarrow |x| < -3 + 10$

$\Rightarrow |x| < 7$

This means $-7 < x < 7$.

2. Solve $\frac{1}{|2x-1|} < 6$ and express the solution using the interval notation.

Sol : Given $\frac{1}{|2x-1|} < 6$

Multiplying the numerator and denominator by $|2x - 1|$ we get, $\frac{|2x-1|}{|2x-1|^2} < 6$

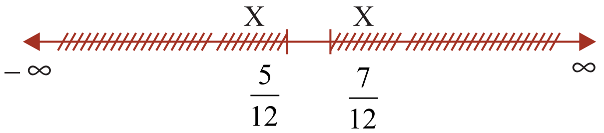
$\Rightarrow \quad 1 < 6\,|2x - 1|$

$\Rightarrow \quad 0 < 6\,|2x - 1| - 1$

$\Rightarrow \quad 6\,|2x - 1| - 1 > 0$

$\Rightarrow \quad \pm 6\,(2x - 1) - 1 > 0$

$6(2x-1)-1>0$	$-6(2x-1)-1>0$
$12x-6-1>0$	$-12x+6-1>0$
$12x-7>0$	$-12x+5>0$
$12x>7$	$-12x>-5$
$x>\frac{7}{12}$	$x<\frac{5}{12}$



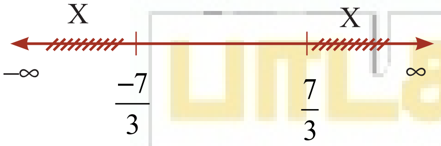
$\therefore$ The solution set is $\left(-\infty,\frac{5}{12}\right)\cup\left(\frac{7}{12},\infty\right)$

3. Solve –3 |x| + 5 ≤ –2 and graph the solution set in a number line.

Sol : Given $-3\,|x|+5\le-2 \Rightarrow -3\,|x|\le-2-5 \Rightarrow -3\,|x|\le-7$

$\Rightarrow |x|\ge\frac{7}{3}$ [Dividing by –3]

This means $-\frac{7}{3}\ge|x|\ge\frac{7}{3}$



$\therefore$ The solution set is $\left(-\infty,\frac{-7}{3}\right]\cup\left[\frac{7}{3},\infty\right)$.

4. Solve 2|x + 1| – 6 ≤ 7 and graph the solution set in a number line. **[Hy - 2018]**

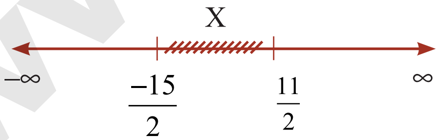
Sol : Given $2\,|x+1|-6\le7$

$\Rightarrow\ 2\,|x+1|\le7+6 \Rightarrow 2\,|x+1|\le13$

$\Rightarrow\ |x+1|\le\frac{13}{2}$.

This means $\frac{-13}{2}\le x+1\le\frac{13}{2}$

$\Rightarrow\ \frac{-13}{2}-1\le x\le\frac{13}{2}-1 \Rightarrow \frac{-15}{2}\le x\le\frac{11}{2}$.



$\therefore$ The solution set is $\left[\frac{-15}{2},\frac{11}{2}\right]$.

5. Solve: $\frac{1}{5}|10x-2|<1$.

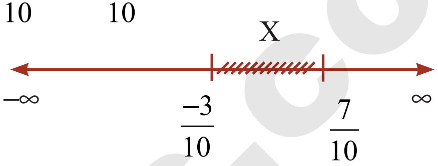
Sol : Given inequality is $\frac{1}{5}|10x-2|<1$.

$\Rightarrow\ |10x-2|<5$

This means, $-5<10x-2<5$.

$\Rightarrow\ -5+2<10x<5+2\ \ \Rightarrow -3<10x<7$

$\Rightarrow\ -\frac{3}{10}<x<\frac{7}{10}$.



$\therefore$ Solution set is $\left(\frac{-3}{10},\frac{7}{10}\right)$.

6. Solve: |5x – 12| < –2.

Sol : $-(-2)<5x-12<-2$.

$+2+12<5x<-2+12$

$14<5x<10$

$\frac{14}{5}<x<2$

$2.8<x<2$, which is not possible.

Hence no solution.

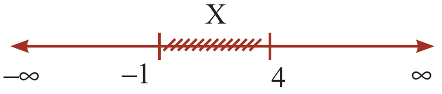
EXERCISE 2.3

1. Represent the following inequalities in the interval notation:

(i) $x\ge-1$ and $x<4$ **(ii) $x\le5$ and $x\ge-3$**

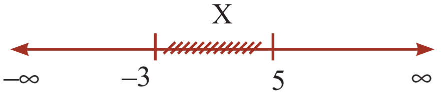
(iii) $x<-1$ or $x<3$ **(iv) $-2x>0$ or $3x-4<11$**

Sol : (i) **$x\ge-1$ and $x<4$.**



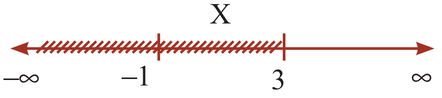
$\Rightarrow x\in[-1,4)$

(ii) **$x\le5$ and $x\ge-3$**



$\Rightarrow x\in[-3,5]$

(iii) **$x<-1$ or $x<3$**



$\Rightarrow x\in(-\infty,3)$

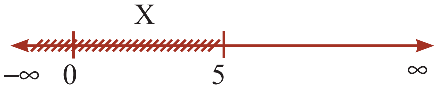
(iv) $-2x > 0$ **or** $3x - 4 < 11$

$\Rightarrow \quad -x > 0 \text{ or } 3x < 11 + 4$

$\Rightarrow \quad x < 0 \text{ or } 3x < 15 \qquad [a > b \Rightarrow -a < -b]$

$\Rightarrow \quad x < 0 \text{ or } x < \dfrac{15}{3} \Rightarrow x < 0 \text{ or } x < 5$

$\Rightarrow \quad x \in (-\infty, 5)$



2. **Solve $23x < 100$ when**
(i) x is a natural number (ii) x is an integer.

Sol : Given $23x < 100$.

(i) when x is a natural number $23x < 100$

$\Rightarrow x < \dfrac{100}{23} \Rightarrow x < 4.348$

$\Rightarrow x = \{1, 2, 3, 4\}$

(ii) when x is an integer $x < 4.348$

$\Rightarrow x = \{\cdots\cdots -3, -2, -1, 0\ 1, 2, 3, 4\}$

Hence solution set is $\{\cdots\cdots -3, -2, -1, 0\ 1, 2, 3, 4\}$.

3. **Solve $-2x \geq 9$ when**
(i) x is a real number (ii) x is an integer
(iii) x is a natural number.

Sol : Given $-2x \geq 9 \Rightarrow -x \geq \dfrac{9}{2} \quad \Rightarrow x \leq \dfrac{-9}{2}$

(i) when x is a real number $x \in \left(-\infty, \dfrac{-9}{2}\right]$

(ii) when x is an integer $x \in \{\cdots, -7, -6, -5\}$

(iii) x is natural number

$x = \{\ \}$. Since there is no solution.

4. **Solve : (i) $\dfrac{3(x-2)}{5} \leq \dfrac{5(2-x)}{3}$ (ii) $\dfrac{5-x}{3} < \dfrac{x}{2} - 4$.**

Sol : (i) $\dfrac{3(x-2)}{5} \leq \dfrac{5(2-x)}{3}$

Given : $\dfrac{3(x-2)}{5} \leq \dfrac{5(2-x)}{3}$

$\Rightarrow \quad \dfrac{3x-6}{5} \leq \dfrac{10-5x}{3}$

$\Rightarrow \quad 3(3x-6) \leq 5(10-5x)$

$\Rightarrow \quad 9x - 18 \leq 50 - 25x$

$\Rightarrow \quad 9x + 25x \leq 50 + 18 \quad \Rightarrow 34x \leq 68$

$\Rightarrow \quad x \leq 2$

$\therefore$ Solution set is $(-\infty, 2]$.

(ii) $\dfrac{5-x}{3} < \dfrac{x}{2} - 4$

Multiplying by 3, throughout,

$5 - x < \dfrac{3x}{2} - 12$

Multiplying by 2, we get,

$10 - 2x < 3x - 24.$

$\Rightarrow \quad 10 + 24 < 3x + 2x$

$34 < 5x \Rightarrow 5x > 34 \Rightarrow x > \dfrac{34}{5}$

$\Rightarrow x > 6.8$

$\therefore$ Solution set is $(6.8, \infty)$

5. **To secure A grade one must obtain an average of 90 marks or more in 5 subjects each of maximum 100 marks. If one scored 84, 87, 95, 91 in first four subjects, what is the minimum mark one scored in the fifth subject to get A grade in the course?**

[First Mid - 2018]

Sol : Let the person obtain x marks in the fifth examination.

Then $\dfrac{84+87+95+91+x}{5} \geq 90 \Rightarrow \dfrac{357+x}{5} \geq 90$

Multiplying both sides by 5 we get,

$357 + x \geq 450$

Subtracting 357 from both sides, we get,

$x \geq 450 - 357. \Rightarrow x \geq 93$

Thus, the person must obtain a minimum of 93 marks to get A grade in the Course.

6. **A manufacturer has 600 litres of a 12 percent solution of acid. How many litres of a 30 percent acid solution must be added to it so that the acid content in the resulting mixture will be more than 15 percent but less than 18 percent?**

Sol : Let x be the number of litres of 30% acid solution.

$\therefore$ Total mixture $= (600 + x)$ litres

30% of x + 12% of 600 > 15% of $(600 + x)$

$\Rightarrow \quad \dfrac{30x}{100} + \dfrac{12}{100} \times 600 > \dfrac{15}{100}(600 + x)$

$\Rightarrow \quad 30x + 7200 > 9000 + 15x$

[Multiplying by 100]

$\Rightarrow \quad 30x + 7200 - 15x > 9000 \quad$ [Subtracting $15x$]

$\Rightarrow \quad 15x + 7200 > 9000 \Rightarrow 15x > 9000 - 7200$

$\Rightarrow \quad 15x > 1800 \quad \Rightarrow \quad x > 120 \qquad \ldots(1)$

Also, 30% of x + 12% of 600 < 18% of $(600 + x)$

$\Rightarrow \quad \dfrac{30x}{100} + \dfrac{12}{100} \times 600 < \dfrac{18}{100}(600 + x)$

$\Rightarrow \quad 30x + 7200 < 18\,(600 + x)$ [Multiplying by 100]

$\Rightarrow \quad 30x + 7200 < 10,800 + 18x$

$\Rightarrow \quad 12x + 7200 < 10,800$ [Subtracting $18x$]

$\Rightarrow \quad 12x < 10,800 - 7200$ [Subtracting 7200]

$\Rightarrow \quad 12x < 3600$

$\Rightarrow \quad x < \dfrac{3600}{12} \quad \Rightarrow \quad x < 300 \quad ...(2)$

From (1) and (2), $120 < x < 300$.

Thus, the number of litres of the 30% acid solution will have to be greater than 120 litres and less than 300 litres.

7. Find all pairs of consecutive odd natural numbers both of which are larger than 10 and their sum is less than 40.

Sol : Let x be the smaller of two positive odd integers, so that other one is $x + 2$

Given $x > 10$, and $x + 2 > 10$. ...(1)

$\Rightarrow x > 10 - 2 \quad \Rightarrow x > 8$ ...(2)

And $(x) + (x + 2) < 40$ ...(3)

From (1) and (2) we get, $x > 10$ ...(4)

From (3) we get

$$2x + 2 < 40$$

$$\Rightarrow \quad 2x < 40 - 2 \quad \Rightarrow \quad 2x < 38$$

$$\Rightarrow \quad x < \frac{38}{2} \quad \Rightarrow \quad x < 19 \quad ...(5)$$

From (4) and (5) we get, $10 < x < 19$.

Since x is an odd natural number, x can take the values 11, 13, 15, 17.

Hence the required possible consecutive pairs will be (11, 13), (13, 15), (15, 17) (17, 19)

8. A model rocket is launched from the ground. The height 'h' reached by the rocket after t seconds from lift off is given by $h(t) = -5t^2 + 100t, 0 \leq t \leq 20$. At what time the rocket is 495 feet above the ground?

Sol : $h(t) = -5t^2 + 100t, 0 \leq t \leq 20$

Let the time be 't' sec, when the rocket is 495 feet above the ground

$$h(t) = -5t^2 + 100t = 495$$

$$\Rightarrow -5t^2 + 100t - 495 = 0$$

$$\Rightarrow \quad t^2 - 20t + 99 = 0 \quad \text{[Divided by } -5]$$

$$\Rightarrow \quad (t - 11)(t - 9) = 0 \quad \Rightarrow t = 11 \text{ or } 9.$$

$\therefore$ At 11 or 9 sec, the rocket is 495 feet above the ground.

9. A plumber can be paid according to the following schemes: In the first scheme he will be paid rupees 500 plus rupees 70 per hour, and in the second scheme he will paid rupees 120 per hour. If he works x hours, then for what value of x does the first scheme give better wages?

Sol : Let the number of hours to complete the job is x.

Wages from the first scheme = ₹$(500 + 70x)$

Wages from the second scheme = ₹$120x$

Given $500 + 70x > 120x$

$$\Rightarrow \quad 500 > 120x - 70x$$

$$\Rightarrow \quad 500 > 50x \quad \Rightarrow \quad \frac{\overset{10}{\cancel{500}}}{\cancel{50}} > x$$

$$\Rightarrow \quad 10 > x \quad \Rightarrow x < 10$$

$\therefore$ Number of hours should be less than ten hours.

10. A and B are working on similar jobs but their annual salaries differ by more than ₹ 6000. If B earns rupees 27000 per month, then what are the possibilities of A's salary per month?

Sol : Let A's salary be x. B's salary is ₹27,000

Given their difference in salary is more than ₹6000.

Assume A's salary is more than B's salary.

$$x - 27,000 > 6,000$$

$$\Rightarrow \quad x > 6,000 + 27,000$$

$$\Rightarrow \quad x > 33,000 \quad ...\ (1)$$

Assume B's salary is more than A's salary

$$\therefore ₹\ 27,000 - x > 6,000$$

$$\therefore ₹\ 27,000 - 6,000 > x$$

$$\Rightarrow \quad x < 21,000 \quad ...\ (2)$$

From (1) and (2),

The possibilities of A's salary are greater than ₹ 33,000 or less than ₹ 21,000.

EXERCISE 2.4

1. Construct a quadratic equation with roots 7 and –3.

Sol : Given roots are 7 and –3

Sum of the roots $\alpha + \beta = 7 + (-3) = 4$

Product of the roots $\alpha\,\beta = 7\,(-3) = -21$.

The quadratic equation is x^2 – (Sum of the roots) x + Product of the roots = 0

$$\Rightarrow \quad x^2 - 4x - 21 = 0$$

Hence, the required quadratic equation is

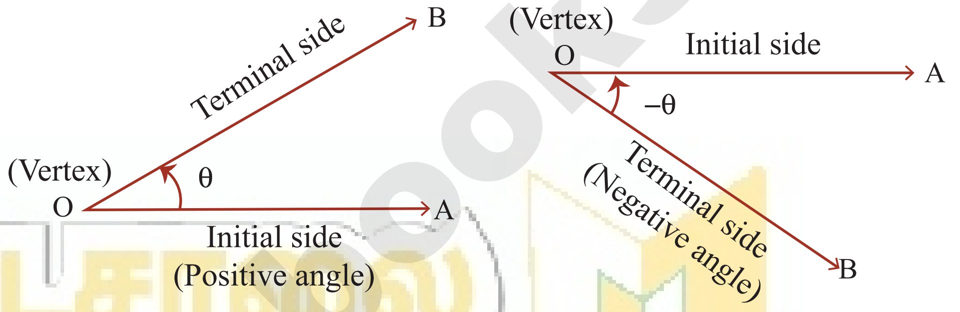
$$x^2 - 4x - 21 = 0$$

03

TRIGONOMETRY

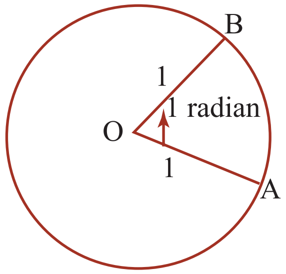
MUST KNOW DEFINITIONS

Angles : Angle is a measure of rotation of a given ray about its initial point.



Degree measure : If a rotation from the initial side to terminal side is $\left(\frac{1}{360}\right)^{th}$ of a revolution, the angle is one degree.

Radian measure : Angle subtended at the center by an arc of length 1 unit of a unit circle is 1 radian.



Trigonometric equations : Equations involving trigonometric functions of a variable are called **trigonometric equations.**

Principal solutions : Among all solutions, the solution which is in $\left[\frac{-\pi}{2}, \frac{\pi}{2}\right]$ for sine ratio, $[0, \pi]$ for cosine ratio, $\left(\frac{-\pi}{2}, \frac{\pi}{2}\right)$ for tan ratio is the principal solution.

General solutions : The expression involving integer '*n*' which gives all solutions of a trigonometric equation is called the general solution.

Inverse trigonometrical functions : The quantities $\sin^{-1}x$, $\cos^{-1}x$, $\tan^{-1}x$, are called inverse circular functions. $\sin^{-1}x$ is an angle θ, whose sine is *x*.

Even functions : A function $f(x)$ is said to be an even function if $f(-x) = f(x)$ for all *x* in its domain.

Odd function : A function $f(x)$ is said to be an odd function if $f(-x) = -f(x)$ for all *x* in its domain.

FORMULAE TO REMEMBER

- 1 radian $= \left(\frac{180}{\pi}\right)^{\circ} = 57^{\circ}\ 16'$ (app)
- $1^{\circ} = \frac{\pi}{180}$ radian = 0.01746 radian (app)
- $\sin(-x) = -\sin x$, $\cos(-x) = \cos x$, $\tan(-x) = -\tan x$
- $\sin^2 x + \cos^2 x = 1$, $1 + \tan^2 x = \sec^2 x$, $1 + \cot^2 x = \text{cosec}^2 x$
- Sign of trigonometric functions

	I	II	III	IV
$\sin x$	+	+	−	−
$\cos x$	+	−	−	+
$\tan x$	+	−	+	−
$\text{cosec}\ x$	+	+	−	−
$\sec x$	+	−	−	+
$\cot x$	+	−	+	−

- **Domain and range of trigonometric functions**

1. The domain of $y = \sin x$, $y = \cos x$ is the set of all real number and the range is $[-1, 1]$.

2. Domain of $y = \tan x$ is $\{x : x \in \text{R}, x \neq (2n+1)\frac{\pi}{2}, n \in \mathbb{Z}\}$ and the range is all real numbers.

3. Domain of $y = \text{cosec}\ x$ is $\{x : x \in \text{R and } x \neq n\pi, n \in \mathbb{Z}\}$ and Range is $\{y : y \in \text{R}, y \geq 1 \text{ or } y \leq -1\}$

4. Domain of $y = \sec x$ is $\{x : x \in \text{R and } x \neq (2n+1)\frac{\pi}{2}, n \in \text{R}\}$ and Range is $\{y : y \in \text{R}, y \leq -1, y \geq 1\}$

5. Domain of $y = \cot x$ is $\{x : x \in \text{R and } x \neq n\pi, n \in \mathbb{Z}\}$ and the range is the set of all real numbers.

- **Sum and difference of trigonometric functions**

1. $\sin(x + y) = \sin x \cos y + \cos x \sin y$

2. $\sin(x - y) = \sin x \cos y - \cos x \sin y$

3. $\cos(x + y) = \cos x \cos y - \sin x \sin y$

4. $\cos(x - y) = \cos x \cos y + \sin x \sin y$

5. $\tan(x + y) = \frac{\tan x + \tan y}{1 - \tan x\ \tan y}$

6. $\tan(x - y) = \frac{\tan x - \tan y}{1 + \tan x\ \tan y}$

Multiple angle formulae:

1. $\sin 2x = 2\sin x\cos x = \dfrac{2\tan x}{1+\tan^2 x}$

2. $\cos 2x = \cos^2 x - \sin^2 x = 2\cos^2 x - 1 = 1 - 2\sin^2 x = \dfrac{1-\tan^2 x}{1+\tan^2 x}$

3. $\tan 2x = \dfrac{2\tan x}{1-\tan^2 x}$

4. $\sin 3x = 3\sin x - 4\sin^3 x$

5. $\cos 3x = 4\cos^3 x - 3\cos x$

6. $\tan 3x = \dfrac{3\tan x - \tan^3 x}{1-3\tan^2 x}$

Transformation of a product into sum or difference

1. $\cos x + \cos y = 2\cos\left(\dfrac{x+y}{2}\right)\cos\left(\dfrac{x-y}{2}\right)$

2. $\cos x - \cos y = -2\sin\left(\dfrac{x+y}{2}\right)\sin\left(\dfrac{x-y}{2}\right)$

3. $\sin x + \sin y = 2\sin\left(\dfrac{x+y}{2}\right)\cos\left(\dfrac{x-y}{2}\right)$

4. $\sin x - \sin y = 2\cos\left(\dfrac{x+y}{2}\right)\sin\left(\dfrac{x-y}{2}\right)$

5. $2\cos x\cos y = \cos(x+y) + \cos(x-y)$

6. $-2\sin x\sin y = \cos(x+y) - \cos(x-y)$

7. $2\sin x\cos y = \sin(x+y) + \sin(x-y)$

8. $2\cos x\sin y = \sin(x+y) - \sin(x-y)$.

General solutions:

1. $\sin\theta = 0 \Rightarrow \theta = n\pi,\ n \in \mathbb{Z}$

2. $\cos\theta = 0 \Rightarrow \theta = (2n+1)\dfrac{\pi}{2},\ n \in \mathbb{Z}$

3. $\tan\theta = 0 \Rightarrow \theta = n\pi,\ n \in \mathbb{Z}$

4. $\sin\theta = \sin\alpha \Rightarrow \theta = n\pi + (-1)^n\alpha,\ n \in \mathbb{Z}$

5. $\cos\theta = \cos\alpha \Rightarrow \theta = 2n\pi \pm \alpha,\ n \in \mathbb{Z}$

6. $\tan\theta = \tan\alpha \Rightarrow \theta = n\pi + \alpha,\ n \in \mathbb{Z}$

Properties of triangles:

1. **Sine formula:** $\dfrac{a}{\sin A} = \dfrac{b}{\sin B} = \dfrac{c}{\sin C} = 2R$

Where R is the circumradius of the triangle.

2. **Napier's formulae:** In any triangle ABC,

(i) $\tan\left(\dfrac{A-B}{2}\right) = \dfrac{a-b}{a+b}\cot\left(\dfrac{C}{2}\right)$ (ii) $\tan\left(\dfrac{B-C}{2}\right) = \dfrac{b-c}{b+c}\cot\dfrac{A}{2}$

(iii) $\tan\left(\dfrac{C-A}{2}\right) = \dfrac{c-a}{c+a}\cot\dfrac{B}{2}$

TEXTUAL QUESTIONS

EXERCISE 3.1

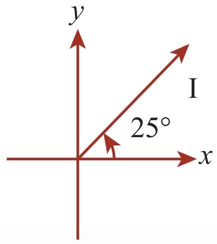
1. Identify the quadrant in which an angle of each given measure lies:

(i) 25° (ii) 825° (iii) –55° (iv) 328°

(v) –230°

Sol : **(i) 25°**

Since 25°, is an acute angle, 25° lies in the I quadrant.

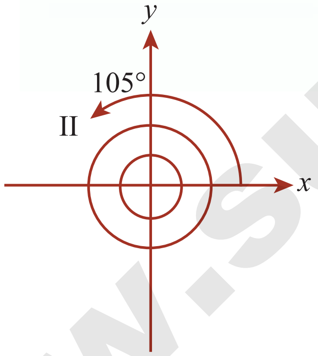


(ii) 825°

825° = 2 × 360 + <u>105°</u>

After two complete rounds the angle is 105° which lies between 90° and 180°

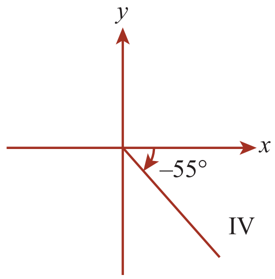
∴ 825° lies in the II quadrant



(iii) –55°

Since the given angle is negative, it moves in the clockwise direction.

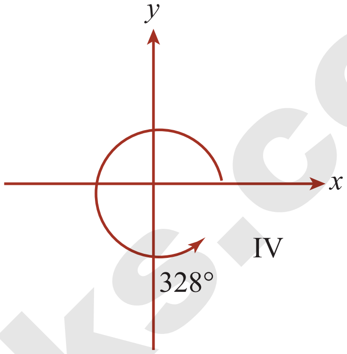
∴ – 55° lies in the IV quadrant



(iv) 328°

328° = 270° + 58°.

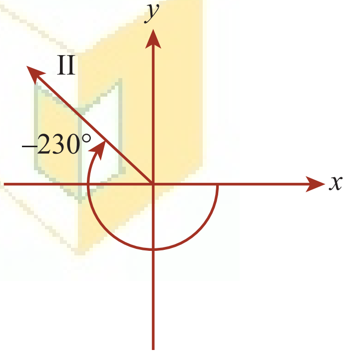
∴ 328° lies in the IV quadrant



(v) –230°

–230° = –180° + (–50°)

∴ –230° lies in the II quadrant.



2. For each given angle, find a co-terminal angle with measure of θ such that 0° ≤ θ ≤ 360°

(i) 395° (ii) 525° (iii) 1150° (iv) –270°

(v) –450°

Sol : **(i) 395°**

$$\begin{aligned} 395° &= 360° + 35° \\ \Rightarrow \quad 395 - 35° &= 360° \end{aligned}$$

∴ Co-terminal angle for 395° is 35°.

(ii) 525°

$$\begin{aligned} 525° &= 360 + 165° \\ \Rightarrow \quad 525° - 165° &= 360° \end{aligned}$$

∴ Co-terminal angle of 525° is 165°

(iii) 1150°

$$\begin{aligned} 1150° &= 360 + 360 + 360° + 70° \\ &= 3 \times 360° + 70° \\ \Rightarrow 1150° - 70° &= 3 \times 360° \end{aligned}$$

∴ Co-terminal angle of 1150° is 70°

(iv) –270°

$$-270° = -360° + 90°$$

$$\Rightarrow \quad -270° - 90° = -360°$$

$\therefore$ co-terminal angle of (–270°) is 90°

(v) –450°

$$-450° = 2 \times (-360°) + 270°$$

$$\Rightarrow \quad -450° - 270° = -720°$$

$\therefore$ Co-terminal angle of (–450°) is 270°

3. If $a\cos\theta - b\sin\theta = c$, show that $a\sin\theta + b\cos\theta = \pm\sqrt{a^2+b^2-c^2}$

Sol : Given $a\cos\theta - b\sin\theta = c$

Squaring both sides we get,

$(a\cos\theta - b\sin\theta)^2 = c^2$

$\Rightarrow \quad a^2\cos^2\theta + b^2\sin^2\theta - 2ab\sin\theta\cos\theta = c^2$

$\Rightarrow \quad a^2(1-\sin^2\theta) + b^2(1-\cos^2\theta) - 2ab\sin\theta\cos\theta = c^2$

$\Rightarrow \quad a^2 - a^2\sin^2\theta + b^2 - b^2\cos^2\theta - 2ab\sin\theta\cos\theta = c^2$

$\Rightarrow \quad -a^2\sin^2\theta - b^2\cos^2\theta - 2ab\sin\theta\cos\theta = c^2 - a^2 - b^2$

$\Rightarrow \quad a^2\sin^2\theta + b^2\cos^2\theta + 2ab\sin\theta\cos\theta = a^2 + b^2 - c^2$

$\Rightarrow \quad (a\sin\theta + b\cos\theta)^2 = a^2 + b^2 - c^2$

$\Rightarrow \quad a\sin\theta + b\cos\theta = \pm\sqrt{a^2+b^2-c^2}$.

4. If $\sin\theta + \cos\theta = m$, show that $\cos^6\theta + \sin^6\theta = \dfrac{4-3(m^2-1)^2}{4}$, where $m^2 \le 2$.

Sol : Given $\sin\theta + \cos\theta = m$

$$\begin{aligned}
\text{LHS} &= \cos^6\theta + \sin^6\theta \\
&= (\cos^2\theta)^3 + (\sin^2\theta)^3 \\
&\qquad [\because (a^3+b^3) = (a+b)(a^2-ab+b^2)] \\
&= (\cos^2\theta + \sin^2\theta) \\
&\qquad (\cos^4\theta - \cos^2\theta\sin^2\theta + \sin^4\theta) \\
&= 1(\cos^4\theta - \cos^2\theta\sin^2\theta + \sin^4\theta) \\
&\qquad [\because \cos^2\theta + \sin^2\theta = 1] \\
&= (\cos^2\theta)^2 + (\sin^2\theta)^2 - \cos^2\theta\sin^2\theta \\
&= (\cos^2\theta + \sin^2\theta)^2 \\
&\qquad - 2\sin^2\theta\cos^2\theta - \cos^2\theta\sin^2\theta \\
&\qquad [\because a^2 + b^2 = (a+b)^2 - 2ab] \\
&= 1 - 3\sin^2\theta\cos^2\theta \qquad \ldots(1)
\end{aligned}$$

$$\begin{aligned}
\text{RHS} &= \frac{4-3(m^2-1)^2}{4} \\
&= \frac{4-3[(\sin\theta+\cos\theta)^2-1]^2}{4} \\
&= \frac{4-3[\sin^2\theta+\cos^2\theta+2\sin\theta\cos\theta-1]^2}{4} \\
&= \frac{4-3(2\sin\theta\cos\theta)^2}{4} = \frac{4-3(4\sin^2\theta\cos^2\theta)}{4} \\
&= \frac{4-12\sin^2\theta\cos^2\theta}{4} = \frac{4}{4} - \frac{12}{4}\sin^2\theta\cos^2\theta \\
&= 1 - 3\sin^2\theta\cos^2\theta \qquad \ldots(2)
\end{aligned}$$

From (1) and (2), LHS = RHS.

5. If $\dfrac{\cos^4\alpha}{\cos^2\beta} + \dfrac{\sin^4\alpha}{\sin^2\beta} = 1$, prove that

(i) $\sin^4\alpha + \sin^4\beta = 2\sin^2\alpha\sin^2\beta$

(ii) $\dfrac{\cos^4\beta}{\cos^2\alpha} + \dfrac{\sin^4\beta}{\sin^2\alpha} = 1$

Sol : Given $\dfrac{\cos^4\alpha}{\cos^2\beta} + \dfrac{\sin^4\alpha}{\sin^2\beta} = 1$

$\Rightarrow \quad \cos^4\alpha\sin^2\beta + \sin^4\alpha\cos^2\beta = \cos^2\beta\sin^2\beta$

$\Rightarrow \quad \cos^4\alpha(1-\cos^2\beta) + \cos^2\beta(1-\cos^2\alpha)^2 = \cos^2\beta(1-\cos^2\beta)$

$\Rightarrow \quad \cos^4\alpha - \cancel{\cos^4\alpha\cos^2\beta} + \cos^2\beta - 2\cos^2\alpha\cos^2\beta + \cancel{\cos^4\alpha\cos^2\beta} = \cos^2\beta - \cos^4\beta$

$\Rightarrow \quad \cos^4\alpha - 2\cos^2\alpha\cos^2\beta + \cos^4\beta = 0$

$\Rightarrow \quad (\cos^2\alpha - \cos^2\beta)^2 = 0 \Rightarrow \cos^2\alpha - \cos^2\beta = 0$

$\Rightarrow \quad \cos^2\alpha = \cos^2\beta \qquad \ldots(1)$

$\Rightarrow \quad \cancel{1} - \sin^2\alpha = \cancel{1} - \sin^2\beta$

$\Rightarrow \quad \sin^2\alpha = \sin^2\beta \qquad \ldots(2)$

(i) $\sin^4\alpha + \sin^4\beta = 2\sin^2\alpha\sin^2\beta$

$$\begin{aligned}
\text{LHS} &= \sin^4\alpha + \sin^4\beta = (\sin^2\alpha - \sin^2\beta)^2 + 2\sin^2\alpha\sin^2\beta \\
&= 2\sin^2\alpha\sin^2\beta
\end{aligned}$$

$[\because \sin^2\alpha = \sin^2\beta$, We have,$(\sin^2\alpha - \sin^2\beta)^2 = 0$ from (2)]

= RHS. Hence proved

(ii) $\dfrac{\cos^4\beta}{\cos^2\alpha} + \dfrac{\sin^4\beta}{\sin^2\alpha} = 1$

$$\begin{aligned}
\text{LHS} &= \frac{\cos^4\beta}{\cos^2\alpha} + \frac{\sin^4\beta}{\sin^2\alpha} \\
&= \frac{\cos^2\beta\cdot\cos^2\beta}{\cos^2\alpha} + \frac{\sin^2\beta\cdot\sin^2\beta}{\sin^2\alpha} \\
&= \frac{\cos^2\beta\cdot\cancel{\cos^2\alpha}}{\cancel{\cos^2\alpha}} + \frac{\sin^2\beta\cdot\cancel{\sin^2\alpha}}{\cancel{\sin^2\alpha}} \\
&\qquad \text{[using (1) and (2)]} \\
&= \cos^2\beta + \sin^2\beta = 1 = \text{RHS}.
\end{aligned}$$

Hence proved.

05

BINOMIAL THEOREM, SEQUENCES AND SERIES

MUST KNOW DEFINITIONS

Binomial theorem for positive integral index : If x and a are real numbers, then for all $n \in \mathbb{N}$,

$$(x+a)^n = nC_0x^na^0 + nC_1x^{n-1}a^1 + nC_2x^{n-2}a^2 + \ldots + nC_rx^{n-r}a^r + \ldots + a^n.$$

Sequence : A sequence is a function whose domain is the set N of natural numbers.

Series : If $a_1, a_2, \ldots a_n$ is a sequence, then the expression $a_1 + a_2 + a_3 + \ldots + a_n$ is a series.

A.P : A sequence is called an arithmetic progression (A.P) if the difference of a term and the previous term is always same.

G.P : A sequence of non-Zero numbers is called a Geometric progression (G.P) if the ratio of a term and the term proceeding to it is always a constant.

H. P: : The reciprocals of the terms of an, A.P form a H.P.

Arithmetico – geometric progression (AGP) : An AGP is a progression in which each term can be represented as the product of the terms of an AP and a G.P.

FORMULAE TO REMEMBER

- $(x+a)^n = x^n + nC_1x^{n-1}a^1 + nC_2x^{n-2}a^2 + \ldots + nC_rx^{n-r}a^r + \ldots + a^n$.
- $(x-a)^n = \sum_{r=0}^{n}(-1)^r\ nC_rx^{n-r}a^r$
- Middle terms in binomial expansion:
- If n is even, then $\left(\frac{n}{2}+1\right)^{\text{th}}$ term
- If n is odd, then $\left(\frac{n+1}{2}\right)^{\text{th}}$ and $\left(\frac{n+3}{2}\right)^{\text{th}}$ term are the middle terms.
- **A.P**
- n^{th} term of A.P., $t_n = a + (n-1)\,d$
- Sum to n terms of an A.P.

$$S_n = \frac{n}{2}\left[2a+(n-1)d\right] \text{ or } S_n = \frac{n}{2}(a+l) \text{ where } l \text{ is the last term of an A.P.}$$

Properties of A. P.

1. If a constant is added or subtracted from each term of an A.P., then the resulting sequence is also an A.P with the same common difference
2. If each term of an A.P is multiplied or divided by a non-zero constant K, then the resulting sequence is an A.P. with the common difference Kd or $\frac{d}{k}$
3. In a finite A.P, the sum of the terms equidistant from the beginning and the end is always same and is equal to the sum of first and last term.
4. Three numbers a, b, c are in A.P if $2b = a + c$.
5. **G. P**
   - nth term of a G. P is $a\,r^{n-1}$
   - nth term from the end = $l\left(\frac{1}{r}\right)^{n-1}$
   - Sum to n terms of a G.P

$$S_n = a\left(\frac{r^n-1}{r-1}\right) \text{ if } r>1 \text{ (OR) } S_n = a\left(\frac{1-r^n}{1-r}\right) \text{ if } r<1.$$

11th Standard

MATHEMATICS

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07

MATRICES AND DETERMINANTS

MUST KNOW DEFINITIONS

Matrix	:	A matrix is a rectangular array or arrangement of entries or elements displayed in rows and columns put within a square bracket [ ].
Order of Matrix	:	If a matrix A has m rows and n columns then the order or size of the matrix A is defined to be $m \times n$.
Column Matrix	:	A matrix having only one column is called a column matrix.
Row matrix	:	A matrix having only one row is called a row matrix.
Square matrix	:	A matrix in which number of rows is equal to the number of columns, is called a square matrix.
Diagonal matrix	:	A square matrix A = $[a_{ij}]_{m\times n}$ is called a diagonal matrix. If a_{ij} = 0 whenever $i \neq j$
Scalar matrix	:	A diagonal matrix whose entries along the principal diagonal are equal is called a scalar matrix.
Unit matrix	:	A square matrix in which all the diagonal entries are 1 and the rest are all zero is called a unit matrix.
Triangular matrix	:	A square matrix which is either upper triangular or lower triangular is called a triangular matrix.
Singular and Non - Singular Matrix	:	A square matrix A is said to be singular if $\|A\| = 0$. A square matrix A is said to be non-singular if $\|A\| \neq 0$.

Properties of Determinants :

1. The value of the determinant remains unchanged if its rows and columns are interchanged.
2. If any two rows (or columns) of a determinant are interchanged, then sign of determinant changes.
3. If any two rows (or columns) of a determinant are identical, then the value of the determinant is zero.
4. If each element of a row (or column) of a determinant is multiplied by a constant k, then its value gets multiplied by k.
5. If some or all elements of a row or column of a determinant are expressed as sum of two (or more) terms, then the determinant can be expressed as sum of two (or more) determinants.
6. The value of the determinant remain same if we apply the operation. $R_i \rightarrow R_i + kR_j$ or $C_i \rightarrow C_i + kC_j$

Minor of an element

- The concept of determinant can be extended to the case of square matrix or order n, $n \geq 4$. Let $A = [a_{ij}]_{m \times n}$, $n \geq 4$.
- If we delete the i^{th} row and j^{th} column from the matrix of $A = [a_{ij}]_{n \times m}$, we obtain a determinant of order $(n - 1)$, which is called the minor of the element a_{ij}.

Adjoint

- Adjoint of a square matrix $A = [a_{ij}]_{n \times n}$ is defined as the transpose of the matrix $[A_{ij}]_{n \times n}$ where A_{ij} is the co-factor of the element a_{ij}.

Solving linear equations by Gaussian Elimination method

- Gaussian elimination is a method of solving a linear system by bringing the augmented matrix to an upper triangular form.

FORMULAE TO REMEMBER

- $kA = [a_{ij}]_{m \times n}$ $[ka_{ij}]_{m \times n}$ where k is a scalar.
- $-A = (-1)A$, $A - B = A + (-1)B$
- $A + B = B + A$, (Commutative property for addition)
- $(A + B) + C = A + (B + C)$, (Associative property for addition)
- $k(A + B) = kA + kB$ where A, B are of same order, k is a constant.
- $(k + l) A = kA + lA$ where k and l are constants.
- $A (BC) = (AB) C$, $A(B + C) = AB + AC$, $(A + B) C = AC + BC$. (Distributive law)
- If $A = (a_{ij})_{m \times n}$, then $A^T = (a_{ji})_{n \times m}$
- Elementary operations of a matrix are as follows
  (i) $R_i \leftrightarrow R_j$ or $C_i \leftrightarrow C_j$ (ii) $R_i \to kR_i$ or $C_i \to kC_j$ (iii) $R_i \to R_i + kR_j$ or $C_i \to C_i + kC_j$
- Evaluation of determinant $A = [a_{11}]_{1 \times 1} = |A| = a_{11}$
- Evaluation of determinant $A = \begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix} = a_{11}a_{22} - a_{21}a_{12}$
- Evaluation of determinant $A = \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}$ is $|A| = a_1 \begin{vmatrix} b_2 & c_3 \\ b_3 & c_3 \end{vmatrix} - b_1 \begin{vmatrix} a_2 & c_2 \\ a_3 & c_3 \end{vmatrix} + c_1 \begin{vmatrix} a_2 & b_2 \\ a_3 & b_3 \end{vmatrix}$
- If $A = [a_{ij}]_{3 \times 3}$, then $|k \cdot A| = k^3 |A|$.
- $A (adj\ A) = (adj\ A) A = |A|. I$ where A is a square matrix of order n.
- A square matrix A is said to be singular or non-singular according as $|A| = 0$ or $|A| \neq 0$.
- **Transpose of a matrix:** $(A^T)^T = A$, $(kA)^T = kA^T$. $(A + B)^T = A^T + B^T$, $(AB)^T = B^T A^T$.
- Co-factor of a_{ij} of $A_{ij} = (-1)^{i+j} m_{ij}$ where m_{ij} is the minor of a_{ij}.
- $|AB| = |A| \cdot |B|$ where A and B are square matrices of same order.

TEXTUAL QUESTIONS

EXERCISE 7.1

1. Construct an $m \times n$ matrix A = $[a_{ij}]$, where a_{ij} is given by

(i) $a_{ij} = \frac{(i-2j)^2}{2}$ **with $m = 2, n = 3$** **[Sep. - 2021]**

(ii) $a_{ij} = \frac{|3i-4j|}{4}$ **with $m = 3, n = 4$**

Sol : (i) Given $a_{ij} = \frac{(i-2j)^2}{2}$ with $m = 2, n = 3$

we need to construct a 2 × 3 matrix.

$$\therefore a_{11} = \frac{(1-2(1))^2}{2} = \frac{(-1)^2}{2} = \frac{1}{2}$$

$$a_{12} = \frac{(1-2(2))^2}{2} = \frac{(-3)^2}{2} = \frac{9}{2}$$

$$a_{13} = \frac{(1-2(3))^2}{2} = \frac{(-5)^2}{2} = \frac{25}{2}$$

$$a_{21} = \frac{(2-2(1))^2}{2} = \frac{0}{2} = 0$$

$$a_{22} = \frac{(2-2(2))^2}{2} = \frac{(-2)^2}{2} = \frac{4}{2}$$

$$a_{23} = \frac{(2-2(3))^2}{2} = \frac{(-4)^2}{2} = \frac{16}{2}$$

$$\therefore A = \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \end{pmatrix}$$

$$= \begin{pmatrix} \frac{1}{2} & \frac{9}{2} & \frac{25}{2} \\ 0 & \frac{4}{2} & \frac{16}{2} \end{pmatrix} = \frac{1}{2}\begin{pmatrix} 1 & 9 & 25 \\ 0 & 4 & 16 \end{pmatrix}$$

(ii) Given $a_{ij} = \frac{|3i-4j|}{4}$ with $m = 3, n = 4$.

Let B be a 3×4 matrix with entries as

$$B = \begin{pmatrix} a_{11} & a_{12} & a_{13} & a_{14} \\ a_{21} & a_{22} & a_{23} & a_{24} \\ a_{31} & a_{32} & a_{33} & a_{34} \end{pmatrix}$$

$$a_{ij} = \frac{|3i-4j|}{4}$$

$$a_{11} = \frac{|3-4|}{4} = \frac{|-1|}{4} = \frac{1}{4}$$

$$a_{12} = \frac{|3-8|}{4} = \frac{|-5|}{4} = \frac{5}{4}$$

$$a_{13} = \frac{|3-12|}{4} = \frac{|-9|}{4} = \frac{9}{4}$$

$$a_{14} = \frac{|3-16|}{4} = \frac{|-13|}{4} = \frac{13}{4}$$

$$a_{21} = \frac{|3(2)-4(1)|}{4} = \frac{|6-4|}{4} = \frac{2}{4}$$

$$a_{22} = \frac{|3(2)-4(2)|}{4} = \frac{|6-8|}{4} = \frac{2}{4}$$

$$a_{23} = \frac{|3(2)-4(3)|}{4} = \frac{|6-12|}{4} = \frac{6}{4}$$

$$a_{24} = \frac{|3(2)-4(4)|}{4} = \frac{|6-16|}{4} = \frac{10}{4}$$

$$a_{31} = \frac{|3(3)-4(1)|}{4} = \frac{|9-4|}{4} = \frac{5}{4}$$

$$a_{32} = \frac{|3(3)-4(2)|}{4} = \frac{|9-8|}{4} = \frac{1}{4}$$

$$a_{33} = \frac{|3(3)-4(3)|}{4} = \frac{|9-12|}{4} = \frac{3}{4}$$

$$a_{34} = \frac{|3(3)-4(4)|}{4} = \frac{|9-16|}{4} = \frac{7}{4}$$

$$\therefore B = \begin{bmatrix} {}^{1}\!/_{4} & {}^{5}\!/_{4} & {}^{9}\!/_{4} & {}^{13}\!/_{4} \\ {}^{2}\!/_{4} & {}^{2}\!/_{4} & {}^{6}\!/_{4} & {}^{10}\!/_{4} \\ {}^{5}\!/_{4} & {}^{1}\!/_{4} & {}^{3}\!/_{4} & {}^{7}\!/_{4} \end{bmatrix} = \frac{1}{4}\begin{bmatrix} 1 & 5 & 9 & 13 \\ 2 & 2 & 6 & 10 \\ 5 & 1 & 3 & 7 \end{bmatrix}$$

2. Find the values of p, q, r, and s if

$$\begin{bmatrix} p^2-1 & 0 & -31-q^3 \\ 7 & r+1 & 9 \\ -2 & 8 & s-1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & -4 \\ 7 & \frac{3}{2} & 9 \\ -2 & 8 & -\pi \end{bmatrix}$$

Sol : Given $\begin{bmatrix} p^2-1 & 0 & -31-q^3 \\ 7 & r+1 & 9 \\ -2 & 8 & s-1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & -4 \\ 7 & {}^{3}\!/_{2} & 9 \\ -2 & 8 & -\pi \end{bmatrix}$

Since the matrices are equal, the corresponding entries on both sides are equal.

$\therefore p^2 - 1 = 1 \Rightarrow p^2 = 2 \Rightarrow p = \pm\sqrt{2}$ [Equating a_{11}]

$$-31 - q^3 = -4 \Rightarrow -q^3 = -4 + 31 \text{ [Equating } a_{13}\text{]}$$

$$\Rightarrow \quad -q^3 = 27$$

$$q^3 = -27 = (-3)^3$$

$$\Rightarrow \quad q = -3$$

$$\text{Also } r + 1 = {}^3\!/_2 \Rightarrow r = {}^3\!/_2 - 1 = \frac{3-2}{2} = {}^1\!/_2$$

[Equating a_{22}]

$$s - 1 = -\pi \qquad \Rightarrow s = 1 - \pi$$

[Equating a_{33}]

$$p = \pm\sqrt{2},\ q = -3,\ r = 1/2,\ s = 1 - \pi.$$

3. Determine the value of $x + y$ if $\begin{bmatrix} 2x+y & 4x \\ 5x-7 & 4x \end{bmatrix} = \begin{bmatrix} 7 & 7y-13 \\ y & x+6 \end{bmatrix}$

Sol : Given $\begin{bmatrix} 2x+y & 4x \\ 5x-7 & 4x \end{bmatrix} = \begin{bmatrix} 7 & 7y-13 \\ y & x+6 \end{bmatrix}$

Equating the corresponding entries on both sides we get,

$$2x + y = 7 \text{ [Equating } a_{11}\text{]} \qquad \ldots (1)$$

$$4x = x + 6 \text{ [Equating } a_{22}\text{]} \qquad \ldots (2)$$

From (2), $4x - x = 6 \Rightarrow 3x = 6 \Rightarrow x = \frac{\overset{2}{\cancel{6}}}{\cancel{3}} \Rightarrow x = 2$

Substituting $x = 2$ in (1) we get,

$$4 + y = 7 \Rightarrow y = 7 - 4 \Rightarrow y = 3$$

$$\therefore x + y = 2 + 3 = 5$$

4. Determine the matrices A and B if they satisfy $2A - B + \begin{bmatrix} 6 & -6 & 0 \\ -4 & 2 & 1 \end{bmatrix} = 0$ and $A - 2B = \begin{bmatrix} 3 & 2 & 8 \\ -2 & 1 & -7 \end{bmatrix}$

Sol : Given $2A - B + \begin{bmatrix} 6 & -6 & 0 \\ -4 & 2 & 1 \end{bmatrix} = 0$

$$\Rightarrow \quad 2A - B = \begin{bmatrix} -6 & 6 & 0 \\ 4 & -2 & -1 \end{bmatrix} \qquad \ldots (1)$$

$$\text{Also given } A - 2B = \begin{bmatrix} 3 & 2 & 8 \\ -2 & 1 & -7 \end{bmatrix} \qquad \ldots (2)$$

$$(1) \times 2 \Rightarrow 4A - 2B = \begin{bmatrix} -12 & 12 & 0 \\ 8 & -4 & -2 \end{bmatrix}$$

$$(2) \Rightarrow \overset{(-)}{A} - \overset{(+)}{2B} = \overset{(-)}{\begin{bmatrix} 3 & 2 & 8 \\ -2 & 1 & -7 \end{bmatrix}}$$

$$\text{Subtracting, } 3A = \begin{bmatrix} -15 & 10 & -8 \\ 10 & -5 & 5 \end{bmatrix}$$

$$\Rightarrow \quad A = \frac{1}{3}\begin{bmatrix} -15 & 10 & -8 \\ 10 & -5 & 5 \end{bmatrix}$$

Substituting the matrix A in (1) we get,

$$\frac{1}{3}\begin{bmatrix} -30 & 20 & -16 \\ 20 & -10 & +10 \end{bmatrix} - B = \begin{bmatrix} -6 & 6 & 0 \\ 4 & -2 & -1 \end{bmatrix}$$

$$\Rightarrow \frac{1}{3}\begin{bmatrix} -30 & 20 & -16 \\ 20 & -10 & +10 \end{bmatrix} - \begin{bmatrix} -6 & 6 & 0 \\ 4 & -2 & -1 \end{bmatrix} = B$$

$$\Rightarrow B = \begin{bmatrix} -10+6 & \frac{20}{3} - 6 & \frac{-16}{3} - 0 \\ \frac{20}{3} - 4 & \frac{-10}{3} + 2 & \frac{10}{3} + 1 \end{bmatrix}$$

$$= \begin{bmatrix} -4 & \frac{20-18}{3} & \frac{-16}{3} \\ \frac{20-12}{3} & \frac{-10+6}{3} & \frac{10+3}{3} \end{bmatrix}$$

$$\Rightarrow B = \begin{bmatrix} -4 & {}^2\!/_3 & {}^{-16}\!/_3 \\ {}^8\!/_3 & {}^{-4}\!/_3 & {}^{13}\!/_3 \end{bmatrix} \therefore B = \frac{1}{3}\begin{bmatrix} -12 & 2 & -16 \\ 8 & -4 & 13 \end{bmatrix}$$

5. If $A = \begin{bmatrix} 1 & a \\ 0 & 1 \end{bmatrix}$, then compute A^4.

Sol : Given $A = \begin{bmatrix} 1 & a \\ 0 & 1 \end{bmatrix} \Rightarrow A^2 = \begin{bmatrix} 1 & a \\ 0 & 1 \end{bmatrix}\begin{bmatrix} 1 & a \\ 0 & 1 \end{bmatrix}$

$$= \begin{bmatrix} 1+0 & a+a \\ 0+0 & 0+1 \end{bmatrix} = \begin{bmatrix} 1 & 2a \\ 0 & 1 \end{bmatrix}$$

$$A^4 = A^2.A^2 = \begin{bmatrix} 1 & 2a \\ 0 & 1 \end{bmatrix}\begin{bmatrix} 1 & 2a \\ 0 & 1 \end{bmatrix}$$

$$A^4 = \begin{bmatrix} 1+0 & 2a+2a \\ 0+0 & 0+1 \end{bmatrix};\ A^4 = \begin{bmatrix} 1 & 4a \\ 0 & 1 \end{bmatrix}$$

6. Consider the matrix $A_\alpha = \begin{bmatrix} \cos\alpha & -\sin\alpha \\ \sin\alpha & \cos\alpha \end{bmatrix}$

(i) Show that $A_\alpha A_\beta = A_{(\alpha+\beta)}$

(ii) Find all possible real values of α satisfying the condition $A_\alpha + A_\alpha^T = I$.

Sol : Given $A_\alpha = \begin{bmatrix} \cos\alpha & -\sin\alpha \\ \sin\alpha & \cos\alpha \end{bmatrix}$

$$A_\beta = \begin{bmatrix} \cos\beta & -\sin\beta \\ \sin\beta & \cos\beta \end{bmatrix}$$

(i) $$\therefore A_\alpha A_\beta = \begin{bmatrix}\cos\alpha & -\sin\alpha\\ \sin\alpha & \cos\alpha\end{bmatrix}\begin{bmatrix}\cos\beta & -\sin\beta\\ \sin\beta & \cos\beta\end{bmatrix}$$

$$= \begin{bmatrix}\cos\alpha\cos\beta-\sin\alpha\sin\beta & -\cos\alpha\sin\beta-\sin\alpha\cos\beta\\ \sin\alpha\cos\beta+\cos\alpha\sin\beta & -\sin\alpha\sin\beta+\cos\alpha\cos\beta\end{bmatrix}$$

$$= \begin{bmatrix}\cos(\alpha+\beta) & -\sin(\alpha+\beta)\\ \sin(\alpha+\beta) & \cos(\alpha+\beta)\end{bmatrix}$$

$$\begin{bmatrix}\text{since } \sin(\alpha+\beta)=\sin\alpha\cos\beta+\cos\alpha\sin\beta\\ \cos(\alpha+\beta)=\cos\alpha\cos\beta-\sin\alpha\sin\beta\end{bmatrix}$$

$$A_\alpha . A_\beta = A_{\alpha+\beta}$$

Hence proved.

(ii) $$\text{Given } A_\alpha = \begin{bmatrix}\cos\alpha & -\sin\alpha\\ \sin\alpha & \cos\alpha\end{bmatrix}$$

$$A_\alpha{}^T = \begin{bmatrix}\cos\alpha & \sin\alpha\\ -\sin\alpha & \cos\alpha\end{bmatrix}$$

Also, it is given that $A_\alpha + A_\alpha{}^T = I$

$$\Rightarrow \begin{bmatrix}\cos\alpha & -\sin\alpha\\ \sin\alpha & \cos\alpha\end{bmatrix}+\begin{bmatrix}\cos\alpha & \sin\alpha\\ -\sin\alpha & \cos\alpha\end{bmatrix}=\begin{bmatrix}1 & 0\\ 0 & 1\end{bmatrix}$$

$$\Rightarrow \begin{bmatrix}2\cos\alpha & 0\\ 0 & 2\cos\alpha\end{bmatrix}=\begin{bmatrix}1 & 0\\ 0 & 1\end{bmatrix}$$

Equating the corresponding entries on both sides, we get

$$2\cos\alpha = 1 \Rightarrow \cos\alpha = \frac{1}{2} = \cos\frac{\pi}{3}.$$

$$\Rightarrow \quad \alpha = 2n\pi \pm \frac{\pi}{3},\ n\in\mathbb{Z}$$

$$[\because \cos\alpha = \cos\theta \Rightarrow \alpha = 2n\pi \pm \theta,\ n\in\mathbb{Z}\,]$$

$$\therefore \alpha = 2n\pi \pm \frac{\pi}{3},\ n\in\mathbb{Z}$$

7. If $A = \begin{bmatrix}4 & 2\\ -1 & x\end{bmatrix}$ and such that $(A-2I)(A-3I) = 0$, find the value of x.

Sol : $$\text{Given } A = \begin{bmatrix}4 & 2\\ -1 & x\end{bmatrix}$$

Also, $(A-2I)(A-3I) = 0$

$$\therefore A-2I = \begin{bmatrix}4 & 2\\ -1 & x\end{bmatrix}-2\begin{bmatrix}1 & 0\\ 0 & 1\end{bmatrix}$$

$$= \begin{bmatrix}4-2 & 2-0\\ -1-0 & x-2\end{bmatrix}$$

$$= \begin{bmatrix}2 & 2\\ -1 & x-2\end{bmatrix}$$

$$A-3I = \begin{bmatrix}4 & 2\\ -1 & x\end{bmatrix}-3\begin{bmatrix}1 & 0\\ 0 & 1\end{bmatrix}$$

$$= \begin{bmatrix}4-3 & 2-0\\ -1-0 & x-3\end{bmatrix}$$

$$= \begin{bmatrix}1 & 2\\ -1 & x-3\end{bmatrix}$$

$$\therefore (A-2I)(A-3I) = \begin{bmatrix}2 & 2\\ -1 & x-2\end{bmatrix}\begin{bmatrix}1 & 2\\ -1 & x-3\end{bmatrix}$$

$$= \begin{bmatrix}0 & 0\\ 0 & 0\end{bmatrix}$$

$$\Rightarrow \begin{bmatrix}2-2 & 4+2(x-3)\\ -1-1(x-2) & -2+(x-2)(x-3)\end{bmatrix}=\begin{bmatrix}0 & 0\\ 0 & 0\end{bmatrix}$$

$$\Rightarrow \begin{bmatrix}0 & 4+2x-6\\ -1-x+2 & -2+x^2-5x+6\end{bmatrix}=\begin{bmatrix}0 & 0\\ 0 & 0\end{bmatrix}$$

$$\Rightarrow \begin{bmatrix}0 & 2x-2\\ -x+1 & x^2-5x+4\end{bmatrix}=\begin{bmatrix}0 & 0\\ 0 & 0\end{bmatrix}$$

Equating the corresponding entries we get,

$$x^2-5x+4 = 0$$

$$\Rightarrow (x-4)(x-1) = 0$$

$$\Rightarrow x = 1,\ x = 4$$

$$(x = 4 \text{ not possible})$$

$$2x-2 = 0 \Rightarrow 2x = 2 \Rightarrow x = 1$$

$$-x+1 = 0 \Rightarrow -x = -1 \Rightarrow x = 1$$

Since $x = 1$ alone satisfies the equation $(A-2I)(A-3I) = 0$, we get $x = 1$.

8. If $A = \begin{bmatrix}1 & 0 & 0\\ 0 & 1 & 0\\ a & b & -1\end{bmatrix}$, show that A^2 is a unit matrix.

Sol : $$\text{Given } A = \begin{bmatrix}1 & 0 & 0\\ 0 & 1 & 0\\ a & b & -1\end{bmatrix}$$

$$A^2 = A.A = \begin{bmatrix}1 & 0 & 0\\ 0 & 1 & 0\\ a & b & -1\end{bmatrix}\begin{bmatrix}1 & 0 & 0\\ 0 & 1 & 0\\ a & b & -1\end{bmatrix}$$

$$= \begin{bmatrix}1+0+0 & 0+0+0 & 0+0+0\\ 0+0+0 & 0+1+0 & 0+0+0\\ a+0-a & 0+b-b & 0+0+1\end{bmatrix}=\begin{bmatrix}1 & 0 & 0\\ 0 & 1 & 0\\ 0 & 0 & 1\end{bmatrix}$$

$\therefore A^2$ is a unit matrix.

10

DIFFERENTIAL CALCULUS - DIFFERENTIABILITY AND METHODS OF DIFFERENTIATION

MUST KNOW DEFINITIONS

- **Tangent line with slope *m* :** Let f be defined on an open interval containing x_0 and if the limit $\lim\limits_{\Delta x\to 0}\frac{\Delta y}{\Delta x} = \lim\limits_{\Delta x\to 0}\frac{f(x_0+\Delta x)-f(x)}{\Delta x} = m_{\tan}$ exists, then the line passing through $(x_0, f(x_0))$ with slope m is the tangent line to the graph of f at the point $(x_0, f(x_0))$

- **Position functions :** Suppose an object moves along a straight line according to an equation of motion $s = f(t)$ where s is the displacement (directed distance) of the object from the origin at time t. The function f that describes the motion is called the position function, of the object.

- **Differentiation :** The process of finding the derivative of a function is called differentiation.

- **Leibnitz symbol :** The notation $\frac{dy}{dx}$ is read as "derivative of y with respect to x" or simply "dy-dx", or we should rather read it as "Dee y Dee x" or "Dee Dee x of y". But it is cautioned that we should not regard $\frac{dy}{dx}$ as the quotient $dy \div dx$ and should not refer it as "dy by dx". The symbol $\frac{dy}{dx}$ is know as Leibnitz symbol.

- **Derivatives from first Principle :** The process of finding the derivative of a function using the conditions stated in the definition of derivatives is known as derivatives from first principle.

- **Intermediate Argument :** Thus, to differentiate a function $y = f(g(x))$, we have to take the derivative of the outer function f regarding the argument $g(x) = u$, and multiply the derivative of the inner function $g(x)$ with respect to the independent variable x. The variable u is known as intermediate argument.

- **Logarithmic differentiation:** The operation consists of first taking the logarithm of the function $f(x)$ (to base e) then differentiating is called logarithmic differentiation.

- **Parameter :** If two variables x and y are defined separately as a function of an intermediating (auxiliary) variable t, then the specification of a functional relationship between x and y is described as parametric and the auxiliary variable is known as parameter.

FORMULAE TO REMEMBER

☐ $(u \pm v)' = u' \pm v'$ where u and v are functions of x.

☐ $(uv)' = u'v + uv'$ (Product Rule)

☐ $\left(\frac{u}{v}\right)' = \frac{u'v - uv'}{v^2}, v \neq 0$ (Quotient Rule)

☐ Let $y = f(u)$ be function of u and let $u = g(x)$ be a function of x so that $y = f(g(x)) = f \, o \, g(x)$

Then $\frac{d}{dx}(f(g(x)) = f'(g(x))g'(x)$ [Chain Rule/composite function rule]

☐ The first derivative of y with respect to 'x' is $\frac{dy}{dx}$

☐ The second derivative of y with respect to 'x' is $\frac{d^2y}{dx^2}$

☐ The third derivative of y with respect to 'x' is $\frac{d^3y}{dx^3}$

Following are some of the standard derivatives:

☐ $\frac{d}{dx}(x^n) = nx^{n-1}$

☐ $\frac{d}{dx}(e^x) = e^x \Rightarrow \frac{d}{dx}(e^{ax+b}) = a \cdot e^{ax+b}$

☐ $\frac{d}{dx}(\log x) = \frac{1}{x} \Rightarrow \frac{d}{dx}(\log(x+a)) = \frac{1}{x+a}$

☐ $\frac{d}{dx}(\sin x) = \cos x$

☐ $\frac{d}{dx}(\cos x) = -\sin x$

☐ $\frac{d}{dx}(\tan x) = \sec^2 x$

☐ $\frac{d}{dx}\left(\frac{1}{x}\right) = -\frac{1}{x^2}$

☐ $\frac{d}{dx}\left(\sin^{-1} x\right) = \frac{1}{\sqrt{1-x^2}}$

☐ $\frac{d}{dx}\left(\cos^{-1} x\right) = \frac{-1}{\sqrt{1-x^2}}$

☐ $\frac{d}{dx}\left(\tan^{-1} x\right) = \frac{1}{1+x^2}$

☐ $\frac{d}{dx}\left(\cot^{-1} x\right) = \frac{-1}{1+x^2}$

☐ $\frac{d}{dx}\left(\sec^{-1} x\right) = \frac{1}{x\sqrt{x^2-1}}$

☐ $\frac{d}{dx}\left(\text{cosec}^{-1} x\right) = \frac{-1}{x\sqrt{x^2-1}}$

☐ $\frac{d}{dx}(c) = 0$ where c is a constant.

☐ $\frac{d}{dx}(a^x) = a^x \cdot \log a.$

TEXTUAL QUESTIONS

EXERCISE 10.1

1. Find the derivatives of the following functions using first principle.

(i) $f(x) = 6$ (ii) $f(x) = -4x + 7$

(iii) $f(x) = -x^2 + 2$

Sol : (i) Given $f(x) = 6$ ... (1)

$$f'(x) = \lim_{\Delta x \to 0} \frac{f(x+\Delta x) - f(x)}{\Delta x}$$

Given $f(x) = 6$

$\Rightarrow$ $f(x + \Delta x) = 6$ ... (2)

$$\therefore f'(x) = \lim_{\Delta x \to 0} \frac{6-6}{\Delta x} = \lim_{\Delta x \to 0} \frac{0}{\Delta x} = 0$$

[Using (1) and (2)]

$\therefore f'(x) = 0$

(ii) Given $f(x) = -4x + 7$ ... (1)

$\therefore f(x + \Delta x) = -4(x + \Delta x) + 7$

$= -4x - 4\Delta x + 7$ ... (2)

$$\therefore \lim_{\Delta x \to 0} \frac{f(x+\Delta x) - f(x)}{\Delta x} = \frac{-4x - 4\Delta x + 7 - (-4x+7)}{\Delta x}$$

[Using (1) and (2)]

$$= \frac{-\cancel{4x} - 4\Delta x + \cancel{7} + \cancel{4x} - \cancel{7}}{\Delta x} = \frac{-4\cancel{\Delta x}}{\cancel{\Delta x}} = -4$$

$\therefore f'(x) = -4$

(iii) Given $f(x) = -x^2 + 2$ ... (1)

$f(x + \Delta x) = -(x + \Delta x)^2 + 2$

$= -(x^2 + 2x\,\Delta x + (\Delta x)^2) + 2$

$f(x + \Delta x) = -x^2 - 2x\,\Delta x - (\Delta x)^2 + 2$ ... (2)

$$(2)-(1) \Rightarrow f(x+\Delta x) - f(x) = -\cancel{x^2} - 2x\Delta x - (\Delta x)^2 + \cancel{2} + \cancel{x^2} - \cancel{2}$$

$= -2x\,\Delta x - (\Delta x)^2 = \Delta x(-2x - \Delta x)$

$$\therefore \lim_{\Delta x \to 0} \frac{f(x+\Delta x) - f(x)}{\Delta x} = \lim_{\Delta x \to 0} \frac{\cancel{\Delta x}(-2x - \Delta x)}{\cancel{\Delta x}}$$

$$= \lim_{\Delta x \to 0} -2x - \Delta x = -2x - (0) = -2x$$

$\therefore f'(x) = -2x$

2. Find the derivatives from the left and from the right at $x = 1$ (if they exist) of the following functions. Are the functions differentiable at $x = 1$?

(i) $f(x) = |x - 1|$ (ii) $f(x) = \sqrt{1-x^2}$

(iii) $$f(x) = \begin{cases} x, & x \le 1 \\ x^2, & x > 1 \end{cases}$$

Sol :

(i) Given $f(x) = |x - 1|$

$$f'(1^-) = \lim_{x \to 1^-} \frac{f(x) - f(1)}{x-1} = \lim_{x \to 1^-} \frac{|x-1| - 0}{x-1}$$

$$= \lim_{x \to 1^-} \frac{|x-1|}{x-1} = \lim_{x \to 1^-} -\frac{\cancel{(x-1)}}{\cancel{x-1}} = -1$$

$\therefore f'(1^-) = -1$

$$\therefore f'(1^+) = \lim_{x \to 1^+} \frac{f(x) - f(1)}{x-1}$$

$$= \lim_{x \to 1^+} \frac{|x-1| - 0}{x-1} = \lim_{x \to 1^+} \frac{|x-1|}{x-1}$$

$$= \lim_{x \to 1^+} \frac{\cancel{x-1}}{\cancel{x-1}} = 1$$

$\therefore f'(1^+) = 1$

Since the one sided derivatives $f'(1^-)$ and $f'(1^+)$ are not equal, $f'(1)$ does not exist.

$\therefore f$ is not differentiable at $x = 1$.

(ii) Given $f(x) = \sqrt{1-x^2}$

$f(1) = \sqrt{1-1} = 0$

$$\therefore f'(1^-) = \lim_{x \to 1^-} \frac{\sqrt{1-x^2} - 0}{x-1} \qquad [\because f(1) = 0]$$

$$= \lim_{x \to 1^-} \frac{\sqrt{1-x^2}}{x-1} = \lim_{x \to 1^-} \frac{\sqrt{1+x}.\sqrt{1-x}}{-(1-x)}$$

$$= \lim_{x \to 1^-} \frac{\sqrt{1+x}.\cancel{\sqrt{1-x}}}{-\left(\sqrt{1-x}\right).\cancel{\sqrt{1-x}}} = \lim_{x \to 1^-} -\frac{\sqrt{1+x}}{\sqrt{1-x}}$$

$$= \lim_{x \to 1^-} -\sqrt{\frac{1+x}{1-x}} = -\sqrt{\frac{2}{0}} = -\infty \qquad ...\ (1)$$

Since $f'(1^-) = -\infty$, we can say that f is not differentiable at $x = 1$.

(iii) $f(x) = \begin{cases} x, & x \le 1 \\ x^2, & x > 1 \end{cases}$

$$\therefore f'(1^-) = \lim_{x \to 1^-} = \frac{f(x) - f(1)}{x-1} = \lim_{x \to 1^-} \frac{\cancel{x-1}}{\cancel{x-1}}$$

$$= \lim_{x \to 1^-} 1 = 1 \qquad \text{... (1)}$$

$$\therefore f'(1^+) = \lim_{x \to 1^+} \frac{f(x) - f(1)}{x-1}$$

$$= \lim_{x \to 1^+} \frac{x^2 - 1}{x-1} = \lim_{x \to 1^+} \frac{(x+1)\cancel{(x-1)}}{\cancel{(x-1)}}$$

$$= \lim_{x \to 1^+} x + 1 = 1 + 1 = 2$$

Since $f'(1^-) \ne f'(1^+)$, $f(x)$ is not differentiable at $x = 1$.

3. Determine whether the following function is differentiable at the indicated values.

(i) $f(x) = x\,|\,x\,|$ at $x = 0$

(ii) $f(x) = |\,x^2 - 1|$ at $x = 1$

(iii) $f(x) = |\,x\,| + |x - 1|$ at $x = 0, 1$

(iv) $f(x) = \sin|\,x\,|$ at $x = 0$

Sol : (i) Given $f(x) = x\,|\,x\,| = \begin{cases} x^2, & x \ge 0 \\ -x^2, & x < 0 \end{cases}$

$$f'(0^-) = \lim_{x \to 0^-} \frac{f(x) - f(0)}{x-0} = \lim_{x \to 0^-} \frac{-x^2 - 0}{x - 0}$$

$$= \lim_{x \to 0^-} \frac{-x^2}{x} = \lim_{x \to 0^-} (-x) = 0 \quad \text{... (1)}$$

$$\therefore f'(0^+) = \lim_{x \to 0^+} \frac{f(x) - f(0)}{x-0} = \lim_{x \to 0^+} \frac{x^2 - 0}{x - 0}$$

$$= \lim_{x \to 0^+} \frac{x^2}{x} = \lim_{x \to 0^+} x = 0 \qquad \text{... (2)}$$

From (1) and (2), $f'(0^-) = f'(0^+)$

Hence, $f(x) = x\,|\,x\,|$ is differentiable at $x = 0$.

(ii) Given $f(x) = |\,x^2 - 1|$

$$f(x) = \begin{cases} -(x^2 - 1), & x < 1 \\ +(x^2 - 1), & x \ge 1 \end{cases}$$

$$\therefore f'(1^-) = \lim_{x \to 1^-} \frac{f(x) - f(1)}{x-1} = \lim_{x \to 1^-} \frac{-(x^2-1) - 0}{x-1}$$

$$= \lim_{x \to 1^-} \frac{1 - x^2}{-(1-x)} = \lim_{x \to 1^-} \frac{(1+x)\cancel{(1-x)}}{-\cancel{(1-x)}}$$

$$= \lim_{x \to 1^-} -(1+x) = -2 \qquad \text{... (1)}$$

$$\therefore f'(1^+) = \lim_{x \to 1^+} \frac{f(x) - f(1)}{x-1} = \lim_{x \to 1^+} \frac{x^2 - 1 - 0}{x-1}$$

$$= \lim_{x \to 1^+} \frac{x^2 - 1}{x-1} = \lim_{x \to 1^+} \frac{(x+1)\cancel{(x-1)}}{\cancel{(x-1)}}$$

$$= \lim_{x \to 1^+} x + 1 = 2 \qquad \text{... (2)}$$

From (1) and (2), $f'(1^-) \ne f'(1^+)$

Hence $f(x)$ is not differentiable at $x = 1$.

(iii) Given $f(x) = |\,x\,| + |\,x - 1\,|$

$$f(x) = \begin{cases} -x-(x-1) & = & -2x+1 & \text{if } x < 0 \\ x-(x-1) & = & 1 & \text{if } 0 \le x < 1 \\ x+x-1 & = & 2x-1 & \text{if } x \ge 1 \end{cases}$$

$$\therefore f'(0^-) = \lim_{x \to 0^-} \frac{f(x) - f(0)}{x-0} = \lim_{x \to 0^-} \frac{-2x + 1 - (1)}{x-0}$$

$$= \lim_{x \to 0^-} \frac{-2\cancel{x}}{\cancel{x}} = -2$$

$$\therefore f'(0^+) = \lim_{x \to 0^+} \frac{f(x) - f(0)}{x-0} = \lim_{x \to 0^+} \frac{1-1}{x} = 0$$

$\therefore$ $f(x)$ is not differentiable at $x = 0$

$$f'(1^-) = \lim_{x \to 1^-} \frac{f(x) - f(1)}{x-1}$$

$$= \lim_{x \to 1^-} \frac{0}{x-1} = 0$$

$$f'(1^+) = \lim_{x \to 1^+} \frac{f(x) - f(1)}{x-1} = \lim_{x \to 1^+} \frac{(2x-1) - 1}{x-1}$$

$$= \lim_{x \to 1^+} \frac{2x-2}{x-1} = \lim_{x \to 1^+} \frac{2(x-1)}{x-1} = 2$$

$\therefore$ $f(x)$ is not differentiable at $x = 1$

(iv) Given $f(x) = \sin|\,x\,| = \begin{cases} \sin x \text{ if} & x \ge 0 \\ -\sin x \text{ if} & x < 0 \end{cases}$

$$\therefore f'(0^-) = \lim_{x \to 0^-} \frac{f(x) - f(0)}{x-0} = \lim_{x \to 0^-} \frac{-\sin x - \sin 0}{x - 0}$$

$$= \lim_{x \to 0^-} \frac{-\sin x}{x} = -1 \quad \left[\because \lim_{x \to 0} \frac{\sin x}{x} = 1\right]$$

$$\therefore f'(0^+) = \lim_{x \to 0^+} \frac{f(x) - f(0)}{x-0}$$

$$= \lim_{x \to 0^+} \frac{\sin x - \sin 0}{x - 0} = \lim_{x \to 0^+} \frac{\sin x}{x} = 1$$

$$f'(0^-) \ne f'(0^+)$$

$\therefore$ $f(x)$ is not differentiable at $x = 0$.

4. Show that the following functions are not differentiable at the indicated value of x.

(i) $f(x)=\begin{cases}-x+2, & x\le 2\\ 2x-4, & x>2\end{cases};\ x=2$

(ii) $f(x)=\begin{cases}3x, & x<0\\ -4x, & x\ge 0\end{cases};\ x=0$

Sol : (i) $f(x) = \begin{cases}-x+2 & x\le 2\\ 2x-4 & x>2\end{cases};\ x=2$

$$f'(2^-) = \lim_{x\to 2^-}\frac{f(x)-f(2)}{x-2}$$

$$= \lim_{x\to 2^-}\frac{-x+2-(0)}{x-2} \quad \begin{bmatrix}\because f(x)=-x+2\\ \Rightarrow f(2)=-2+2=0\end{bmatrix}$$

$$= \lim_{x\to 2^-}\frac{-x+2}{x-2} = \lim_{x\to 2^-}\frac{-\cancel{(x-2)}}{\cancel{x-2}} = -1 \quad \dots (1)$$

$$\therefore f'(2^+) = \lim_{x\to 2^+}\frac{f(x)-f(2)}{x-2}$$

$$= \lim_{x\to 2^+}\frac{(2x-4)-0}{x-2} \quad \begin{bmatrix}\because f(x)=2x-4\\ f(2)=4-4=0\end{bmatrix}$$

$$= \lim_{x\to 2^+}\frac{2\cancel{(x-2)}}{\cancel{x-2}} = 2 \quad \dots (2)$$

From (1) and (2), $f'(2^-) \neq f'(2^+)$

Hence, $f(x)$ is not differentiable at $x = 2$.

(ii) $f(x) = \begin{cases}3x, & x<0\\ -4x, & x\ge 0\end{cases};\ x=0$

$$\therefore f'(0^-) = \lim_{x\to 0^-}\frac{f(x)-f(0)}{x-0} = \lim_{x\to 0^-}\frac{3x-0}{x}$$

$$= \lim_{x\to 0^-}\frac{3\cancel{x}}{\cancel{x}} = 3 \quad [f(x) = 3x] \quad \dots (1)$$

$$\therefore f'(0^+) = \lim_{x\to 0^+}\frac{f(x)-f(0)}{x-0} = \lim_{x\to 0^+}\frac{-4x-0}{x}$$

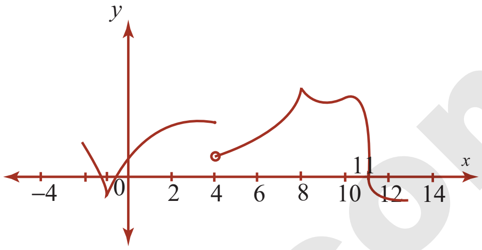
$$\begin{bmatrix}\because f(x)=-4x\\ f(0)=4(0)=0\end{bmatrix}$$

$$= \lim_{x\to 0^+}\frac{-4\cancel{x}}{\cancel{x}} = -4 \quad \dots (2)$$

From (1) and (2), $f'(0^-) \neq f'(0^+)$

Hence $f(x)$ is not differentiable at $x = 0$.

5. The graph of f is shown below. State with reasons that x values (the numbers), at which f is not differentiable.



Sol : (i) From the graph it is clear that at $x = -1$, the graph has a sharp edge.

$\therefore$ It is not differentiable at $x = -1$.

(ii) At $x = 4$, it is discontinuous.

$\therefore$ f is not differentiable at $x = 4$.

(iii) At $x = 8$, it has a sharp peak.

$\therefore$ f is not differentiable at $x = 8$.

(iv) At $x = 11$, the tangent is perpendicular

$\Rightarrow$ At $x = 11$, it has a vertical tangent

$\therefore$ f is also not differentiable at $x = 11$.

6. If $f(x) = |x + 100| + x^2$, test whether $f'(-100)$ exists.

Sol : Given $f(x) = |x + 100| + x^2$

$$\therefore f'(-100^-) = \lim_{x\to -100^-}\frac{f(x)-f(-100)}{x-(-100)}$$

$$= \lim_{x\to -100^-}\frac{|x+100|+x^2-(100)^2}{x+100}$$

$$\begin{bmatrix}\because f(x)=|x+100|+x^2\\ f(-100)=|-100+100|+(-100)^2=100^2\end{bmatrix}$$

$$= \lim_{x\to -100^-}\frac{|x+100|+x^2-100^2}{x+100}$$

$$= \lim_{x\to -100^-}\frac{-(x+100)+x^2-100^2}{x+100} = \lim_{x\to -100^-}\frac{-x-100+x^2-100^2}{x+100}$$

$$= \lim_{x\to -100^-}\frac{-(x+100)+(x+100)(x-100)}{x+100} = \lim_{x\to -100^-}\frac{\cancel{(x+100)}\,(-1+x-100)}{\cancel{x+100}}$$

$$= \lim_{x\to -100^-}(x-101) = -201 \quad \dots (1)$$

$$\therefore f'(-100^+) = \lim_{x\to -100^+}\frac{f(x)-f(-100)}{x-(-100)}$$

$$= \lim_{x\to -100^+}\frac{|x+100|+x^2-(100)^2}{x+100}$$

$$= \lim_{x\to-100^+} \frac{(x+100)+x^2-100^2}{x+100}$$

$$= \lim_{x\to-100^+} \frac{(x+100)+(x-100)(x+100)}{x+100}$$

$$= \lim_{x\to-100^+} \frac{\cancel{(x+100)}\,[1+x-100]}{\cancel{x+100}}$$

$$= \lim_{x\to-100^+} [1+x-100] = -199 \quad \text{... (2)}$$

From (1) and (2), $f(x)$ is not differentiable at $x = -100$

$\Rightarrow f'(-100)$ does not exist.

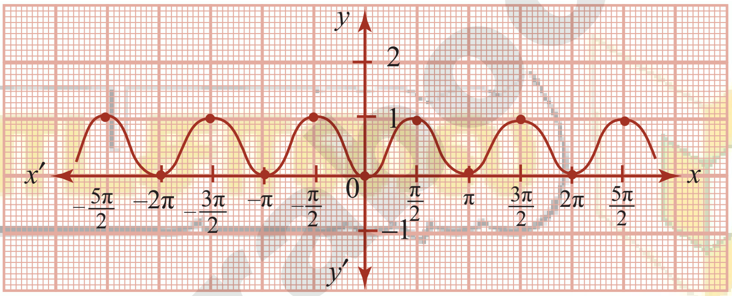
7. Examine the differentiability of functions in $\mathbb{R}$ by drawing the diagrams.

(i) | sin *x* | (ii) | cos *x* | .

Sol :

(i) Let $f(x) = |\sin x|$ when $x = 0$, $f(x) = |\sin 0| = 0$

x	0	$\frac{\pi}{2}$	$-\frac{\pi}{2}$	π	$-\pi$	$\frac{3\pi}{2}$	$-\frac{3\pi}{2}$	$\frac{5\pi}{2}$	$-\frac{5\pi}{2}$	2π	-2π
$f(x)$	0	1	1	0	0	1	1	1	1	0	0

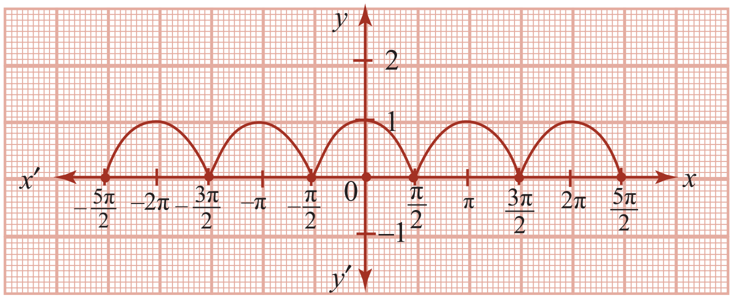


The curve $f(x) = |\sin x|$ has got vertical tangents at $x = \pi, -\pi, 2\pi, -2\pi$ etc.

$\therefore f(x) = |\sin x|$ is not differentiable at $x = n\pi$, $n \in \mathbb{Z}$.

(ii) Let $f(x) = |\cos x|$ when $x = 0$, $f(0) = |\cos 0| = |1| = 1$

x	0	$\frac{\pi}{2}$	$-\frac{\pi}{2}$	π	$-\pi$	$\frac{3\pi}{2}$	$-\frac{3\pi}{2}$	$\frac{5\pi}{2}$	$-\frac{5\pi}{2}$	2π
$f(x)$	1	0	0	1	1	0	0	0	0	1



$f(x) = |\cos x|$ has got vertical tangents at $x = \frac{\pi}{2}, -\frac{\pi}{2}, \frac{3\pi}{2}, -\frac{3\pi}{2}, \frac{5\pi}{2}, -\frac{5\pi}{2}$ etc.

$\therefore f(x) = |\cos x|$ is not differentiable at $x = (2n+1)\frac{\pi}{2}$, $n \in \mathbb{Z}$.