3. TRIGONOMETRY

Basic Trigonometric ratios using right triangle.

$$\sin\theta = \frac{\text{opp. side}}{\text{hypotenuse}}$$

$$\cos\theta = \frac{\text{adjacent side}}{\text{hypotenuse}}$$

$$\tan\theta = \frac{\text{opp. side}}{\text{adjacent side}}$$

$\operatorname{cosec}\theta = \frac{1}{\sin\theta}$ $\sec\theta = \frac{1}{\cos\theta}$ $\cot\theta = \frac{1}{\tan\theta}$

$\sin\theta \cdot \operatorname{cosec}\theta = 1$ $\operatorname{cosec}\theta \cdot \sec\theta = 1$ $\tan\theta \cot\theta = 1$

$\tan\theta = \frac{\sin\theta}{\cos\theta}$ $\cot\theta = \frac{\cos\theta}{\sin\theta}$

Basic Trigonometric Identities:

$$\sin^2\theta + \cos^2\theta = 1$$
$$\cos^2\theta = 1 - \sin^2\theta$$
$$\sin^2\theta = 1 - \cos^2\theta$$

$$\operatorname{cosec}^2\theta - \cot^2\theta = 1$$
$$\operatorname{cosec}^2\theta = 1 + \cot^2\theta$$
$$\cot^2\theta = \operatorname{cosec}^2\theta - 1$$

$$\sec^2\theta - \tan^2\theta = 1$$
$$\sec^2\theta = 1 + \tan^2\theta$$
$$\tan^2\theta = \sec^2\theta - 1.$$

Exact values of trigonometric functions of widely used angles:

θ	$0°$	$30°$	$45°$	$60°$	$90°$
$\sin\theta$	0	$\frac{1}{2}$	$\frac{1}{\sqrt{2}}$	$\frac{\sqrt{3}}{2}$	1
$\cos\theta$	1	$\frac{\sqrt{3}}{2}$	$\frac{1}{\sqrt{2}}$	$\frac{1}{2}$	0
$\tan\theta$	0	$\frac{1}{\sqrt{3}}$	1	$\sqrt{3}$	∞

3.1 Prove that $\frac{\tan\theta+\sec\theta-1}{\tan\theta-\sec\theta+1} = \frac{1+\sin\theta}{\cos\theta}$

$$L.H.S = \frac{\tan\theta+\sec\theta-1}{\tan\theta-\sec\theta+1} = \frac{\tan\theta+\sec\theta-(\sec^2\theta-\tan^2\theta)}{\tan\theta-\sec\theta+1}$$

$$= \frac{(\tan\theta+\sec\theta)-(\sec\theta+\tan\theta)(\sec\theta-\tan\theta)}{(\tan\theta-\sec\theta+1)}$$

$$= \frac{(\tan\theta+\sec\theta)\{\cancel{1-\sec\theta+\tan\theta}\}}{(\cancel{\tan\theta-\sec\theta+1})}$$

$$= \tan\theta+\sec\theta = \frac{\sin\theta}{\cos\theta}+\frac{1}{\cos\theta} = \frac{1+\sin\theta}{\cos\theta} = R.H.S$$

3.2 Prove that $(\sec A-\operatorname{cosec} A)(1+\tan A+\cot A) = \tan A\sec A-\cot A\cdot\operatorname{cosec} A$

$$L.H.S = (\sec A-\operatorname{cosec} A)(1+\tan A+\cot A)$$

$$= \left(\frac{1}{\cos A}-\frac{1}{\sin A}\right)\left(1+\frac{\sin A}{\cos A}+\frac{\cos A}{\sin A}\right)$$

$$= \frac{\sin A-\cos A}{\sin A\cos A}\left[\frac{\sin A\cos A+\sin^2 A+\cos^2 A}{\sin A\cos A}\right]$$

$$= \frac{(\sin A-\cos A)(\sin^2 A+\sin A\cos A+\cos^2 A)}{\sin^2 A\cos^2 A}$$

$$= \frac{\sin^3 A-\cos^3 A}{\sin^2 A\cos^2 A} \quad \text{------ (1)}$$

$$R.H.S = \tan A\sec A-\cot A\cdot\operatorname{cosec} A$$

$$= \frac{\sin A}{\cos A}\cdot\frac{1}{\cos A}-\frac{\cos A}{\sin A}\cdot\frac{1}{\sin A}$$

$$= \frac{\sin A}{\cos^2 A}-\frac{\cos A}{\sin^2 A}$$

$$= \frac{\sin^3 A-\cos^3 A}{\sin^2 A\cos^2 A} \quad \text{------ (2)}$$

From (1) & (2)

L.H.S = R.H.S

Hence Proved.

Ex : 3.1

3. If $a\cos\theta - b\sin\theta = c$. show that $a\sin\theta + b\cos\theta = \pm\sqrt{a^2+b^2-c^2}$

Proof :- $a\cos\theta - b\sin\theta = c$

squaring on both sides,

$(a\cos\theta - b\sin\theta)^2 = c^2$

$a^2\cos^2\theta + b^2\sin^2\theta - 2ab\sin\theta\cos\theta = c^2$

$a^2(1-\sin^2\theta) + b^2(1-\cos^2\theta) - 2ab\sin\theta\cos\theta = c^2$

$a^2 - a^2\sin^2\theta + b^2 - b^2\cos^2\theta - 2ab\sin\theta\cos\theta = c^2$

$-a^2\sin^2\theta - b^2\cos^2\theta - 2ab\sin\theta\cos\theta = c^2 - a^2 - b^2$

$a^2\sin^2\theta + b^2\cos^2\theta + 2ab\sin\theta\cos\theta = a^2+b^2-c^2$

$(a\sin\theta + b\cos\theta)^2 = a^2+b^2-c^2$

$a\sin\theta + b\cos\theta = \pm\sqrt{a^2+b^2-c^2}$

Hence Proved.

4. If $\sin\theta + \cos\theta = m$ show that $\cos^6\theta + \sin^6\theta = \dfrac{4-3(m^2-1)^2}{4}$ where $m^2 \le 2$

$\sin\theta + \cos\theta = m$

squaring on both sides,

$\sin^2\theta + \cos^2\theta + 2\sin\theta\cos\theta = m^2$

$1 + 2\sin\theta\cos\theta = m^2$

$2\sin\theta\cos\theta = m^2-1$

$\sin\theta\cos\theta = \dfrac{m^2-1}{2}$

$\cos^6\theta + \sin^6\theta = (\cos^2\theta)^3 + (\sin^2\theta)^3$

$= (\cos^2\theta + \sin^2\theta)^3 - 3\sin^2\theta\cos^2\theta(\sin^2\theta + \cos^2\theta)$

$= 1 - 3\sin^2\theta\cos^2\theta$

$\left\{ a^3+b^3 = (a+b)^3 - 3ab(a+b) \right\}$

$= 1 - 3\left[\dfrac{m^2-1}{2}\right]^2$

$= 1 - \dfrac{3(m^2-1)}{4}$

$= \dfrac{4-3(m^2-1)}{4}$

5. If $\frac{\cos^4\alpha}{\cos^2\beta} + \frac{\sin^4\alpha}{\sin^2\beta} = 1$ then Prove that

(i) $\sin^4\alpha + \sin^4\beta = 2\sin^2\alpha\sin^2\beta$

(ii) $\frac{\cos^4\beta}{\cos^2\alpha} + \frac{\sin^4\beta}{\sin^2\alpha} = 1$.

Solution :-

$$\frac{\cos^4\alpha}{\cos^2\beta} + \frac{\sin^4\alpha}{\sin^2\beta} = 1$$

$$\cos^4\alpha\sin^2\beta + \sin^4\alpha\cos^2\beta = \cos^2\beta\sin^2\beta$$

$$\cos^4\alpha(1-\cos^2\beta) + (1-\cos^2\alpha)^2\cos^2\beta = \cos^2\beta(1-\cos^2\beta)$$

$$\cos^4\alpha - \cos^4\alpha\cos^2\beta + (1+\cos^4\alpha - 2\cos^2\alpha)\cos^2\beta = \cos^2\beta - \cos^4\beta$$

$$\cos^4\alpha - \cancel{\cos^4\alpha\cos^2\beta} + \cos^2\beta + \cancel{\cos^4\alpha\cos^2\beta} - 2\cos^2\alpha\cos^2\beta = \cos^2\beta - \cos^4\beta$$

$$\cos^4\alpha + \cancel{\cos^2\beta} - 2\cos^2\alpha\cos^2\beta = \cancel{\cos^2\beta} - \cos^4\beta$$

$$\cos^4\alpha + \cos^4\beta - 2\cos^2\alpha\cos^2\beta = 0$$

$$(\cos^2\alpha - \cos^2\beta)^2 = 0$$

$$\cos^2\alpha - \cos^2\beta = 0$$

$$\boxed{\cos^2\alpha = \cos^2\beta}$$

$$1 - \sin^2\alpha = 1 - \sin^2\beta$$

$$\boxed{\sin^2\alpha = \sin^2\beta}$$

(i) $\sin^4\alpha + \sin^4\beta = 2\sin^2\alpha\sin^2\beta$

L.H.S $= \sin^4\alpha + \sin^4\beta$ $\left\{ a^2+b^2 = (a-b)^2 + 2ab \right\}$

$= (\sin^2\alpha - \sin^2\beta)^2 + 2\sin^2\alpha\sin^2\beta$

$= 0 + 2\sin^2\alpha\sin^2\beta$

$= 2\sin^2\alpha\sin^2\beta$

$=$ R.H.S

(ii) $\frac{\cos^4\beta}{\cos^2\alpha} + \frac{\sin^4\beta}{\sin^2\alpha} = 1$

L.H.S $= \frac{\cos^4\beta}{\cos^2\alpha} + \frac{\sin^4\beta}{\sin^2\alpha} = \frac{(\cos^2\beta)^2}{\cos^2\alpha} + \frac{(\sin^2\beta)^2}{\sin^2\alpha}$

$= \frac{(\cos^2\alpha)^2}{\cos^2\alpha} + \frac{(\sin^2\alpha)^2}{\sin^2\alpha} = \cos^2\alpha + \sin^2\alpha$

$= 1$

$=$ R.H.S

$\left\{ \because \begin{array}{l} \sin^2\alpha = \sin^2\beta \\ \cos^2\alpha = \cos^2\beta \end{array} \right\}$

6. If $y = \dfrac{2\sin\alpha}{1+\cos\alpha+\sin\alpha}$ then Prove that $\dfrac{1-\cos\alpha+\sin\alpha}{1+\sin\alpha} = y$

$$y = \frac{2\sin\alpha}{1+\cos\alpha+\sin\alpha} \times \frac{1+\sin\alpha-\cos\alpha}{1+\sin\alpha-\cos\alpha}$$

$$= \frac{2\sin\alpha(1+\sin\alpha-\cos\alpha)}{(1+\sin\alpha)^2-\cos^2\alpha} = \frac{2\sin\alpha(1+\sin\alpha-\cos\alpha)}{1+\sin^2\alpha+2\sin\alpha-\cos^2\alpha}$$

$$= \frac{2\sin\alpha(1+\sin\alpha-\cos\alpha)}{\sin^2\alpha+2\sin\alpha+\sin^2\alpha} = \frac{2\cancel{\sin\alpha}(1+\sin\alpha-\cos\alpha)}{2\cancel{\sin\alpha}(1+\sin\alpha)}$$

$$y = \frac{1-\cos\alpha+\sin\alpha}{1+\sin\alpha}$$

Hence Proved.

7. If $x=\sum_{n=0}^{\infty}\cos^{2n}\theta$ $y=\sum_{n=0}^{\infty}\sin^{2n}\theta$ and $z=\sum_{n=0}^{\infty}\cos^{2n}\theta.\sin^{2n}\theta$ $0<\theta<\pi/2$, then show that $xyz = x+y+z$.

Solution :-

$$x = \sum_{n=0}^{\infty}\cos^{2n}\theta = 1+\cos^2\theta+\cos^4\theta+\cos^6\theta+\cdots\infty$$

$$x = \frac{1}{1-\cos^2\theta} = \frac{1}{\sin^2\theta}$$

$$y = \sum_{n=0}^{\infty}\sin^{2n}\theta = 1+\sin^2\theta+\sin^4\theta+\sin^6\theta+\cdots\infty$$

$$y = \frac{1}{1-\sin^2\theta} = \frac{1}{\cos^2\theta}$$

$$z = \sum_{n=0}^{\infty}\cos^{2n}\theta.\sin^{2n}\theta = 1+\sin^2\theta\cos^2\theta+\sin^4\theta\cos^4\theta+\cdots\infty$$

$$z = \frac{1}{1-\sin^2\theta.\cos^2\theta}$$

$$L.H.S = xyz = \frac{1}{\sin^2\theta}\cdot\frac{1}{\cos^2\theta}\cdot\frac{1}{1-\sin^2\theta\cos^2\theta}$$

$$= \frac{1}{\sin^2\theta.\cos^2\theta(1-\sin^2\theta\cos^2\theta)} \quad \text{——— (1)}$$

$$R.H.S = x+y+z = \frac{1}{\sin^2\theta}+\frac{1}{\cos^2\theta}+\frac{1}{1-\sin^2\theta\cos^2\theta}$$

$$= \frac{\sin^2\theta+\cos^2\theta}{\sin^2\theta.\cos^2\theta}+\frac{1}{1-\sin^2\theta\cos^2\theta} = \frac{1}{\sin^2\theta\cos^2\theta}+\frac{1}{1-\sin^2\theta\cos^2\theta}$$

$$= \frac{1-\cancel{\sin^2\theta\cos^2\theta}+\cancel{\sin^2\theta\cos^2\theta}}{\sin^2\theta.\cos^2\theta(1-\sin^2\theta\cos^2\theta)}$$

$$= \frac{1}{\sin^2\theta\cos^2\theta(1-\sin^2\theta\cos^2\theta)} \quad \text{——— (2)}$$

from (1) & (2) L.H.S = R.H.S

8. If $\tan^2\theta = 1-k^2$, Show that $\sec\theta + \tan^3\theta \csc\theta = (2-k^2)^{3/2}$ Also find the values of k for which this result holds.

$$\tan^2\theta = 1-k^2$$

$$L.H.S = \sec\theta + \tan^3\theta . \csc\theta$$
$$= \sec\theta\left[1+\tan^3\theta . \frac{\csc\theta}{\sec\theta}\right]$$
$$= \sec\theta\left[1+\tan^2\theta\right]$$
$$= \sqrt{1+\tan^2\theta}\,(1+\tan^2\theta)$$
$$= (1+\tan^2\theta)^{3/2}$$
$$= [1+(1-k^2)]^{3/2} = (2-k^2)^{3/2} = R.H.S.$$

$\tan^2\theta \geq 0 \;\forall\; \theta$

$\Rightarrow 1-a^2 \geq 0$

$a^2-1 \leq 0 \Rightarrow a^2 \leq 1$

$|a| \leq 1$

$\Rightarrow \boxed{-1 \leq a \leq 1}$ ——— ①

Since L.H.S of $\sec\theta + \tan^3\theta \csc\theta = (2-a^2)^{3/2}$ is real for all $\theta \in R$. So RHS must also be real.

$2-a^2 \geq 0$

$a^2-2 \leq 0$

$a^2 \leq 2$

$|a| \leq \sqrt{2}$

$\boxed{-\sqrt{2} \leq a \leq a}$ ——— ②

from ① & ②

The given relation holds true for all $\theta \in [-1, 1]$.

9. If $\sec\theta + \tan\theta = P$, obtain the values of $\sec\theta$, $\tan\theta$ and $\sin\theta$ in terms of P.

$$P = \sec\theta + \tan\theta$$
$$\frac{1}{P} = \frac{1}{\sec\theta+\tan\theta} \times \frac{\sec\theta-\tan\theta}{\sec\theta-\tan\theta}$$
$$= \frac{\sec\theta-\tan\theta}{\sec^2\theta-\tan^2\theta} = \sec\theta-\tan\theta.$$

$P = \sec\theta + \tan\theta \qquad \frac{1}{P} = \sec\theta - \tan\theta$

$P + \frac{1}{P} = \sec\theta + \tan\theta + \sec\theta - \tan\theta$

$$2\sec\theta = P + \frac{1}{P}$$

$$\sec\theta = \frac{1}{2}\left(P + \frac{1}{P}\right)$$

$P - \frac{1}{P} = \cancel{\sec\theta} + \tan\theta - \cancel{\sec\theta} + \tan\theta$

$$2\tan\theta = P - \frac{1}{P}$$

$$\tan\theta = \frac{1}{2}\left(P - \frac{1}{P}\right)$$

$$\frac{P - \frac{1}{P}}{P + \frac{1}{P}} = \frac{\cancel{2}\tan\theta}{\cancel{2}\sec\theta}$$

$$\frac{P^2-1}{P^2+1} = \sin\theta.$$

$$\therefore \sin\theta = \frac{P^2-1}{P^2+1}.$$

10. If $\cot\theta(1+\sin\theta) = 4m$ and $\cot\theta(1-\sin\theta) = 4n$ then Prove that $(m^2-n^2)^2 = mn$.

$\cot\theta(1+\sin\theta) = 4m \qquad \cot\theta(1-\sin\theta) = 4n$

$\cot\theta + \cos\theta = 4m \qquad \cot\theta - \cos\theta = 4n$

$(\cot\theta+\cos\theta)^2 - (\cot\theta-\cos\theta)^2 = (4m)^2 - (4n)^2$

$$\{(a+b)^2 - (a-b)^2 = 4ab\}$$

$$4\cot\theta\cos\theta = 16(m^2-n^2)$$

$$4\frac{\cos^2\theta}{\sin\theta} = 16(m^2-n^2)$$

$$\frac{\cos^2\theta}{\sin\theta} = 4(m^2-n^2)$$

Squaring on both sides

$$\frac{\cos^4\theta}{\sin^2\theta} = 16(m^2-n^2)^2 \quad \text{———} \quad (1)$$

$(\cot\theta + \cos\theta)(\cot\theta - \cos\theta) = 4m.4n$

$$\cot^2\theta - \cos^2\theta = 16mn$$

$$\cos^2\theta\left(\frac{1}{\sin^2\theta} - 1\right) = 16mn$$

$$\frac{\cos^4\theta}{\sin^2\theta} = 16mn \quad \text{———} \quad (2)$$

from (1) & (2)

$$16(m^2-n^2)^2 = 16mn$$

$$(m^2-n^2)^2 = mn$$

Hence Proved

11. If $\operatorname{cosec}\theta - \sin\theta = a^3$ and $\sec\theta - \cos\theta = b^3$ then Prove that $a^2b^2(a^2+b^2)=1$.

$\operatorname{cosec}\theta - \sin\theta = a^3$

$\frac{1}{\sin\theta} - \sin\theta = a^3$

$\frac{1-\sin^2\theta}{\sin\theta} = a^3$

$\frac{\cos^2\theta}{\sin\theta} = a^3$

$a = \frac{\cos^{2/3}\theta}{\sin^{1/3}\theta}$

$\sec\theta - \cos\theta = b^3$

$\frac{1}{\cos\theta} - \cos\theta = b^3$

$\frac{1-\cos^2\theta}{\cos\theta} = b^3$

$\frac{\sin^2\theta}{\cos\theta} = b^3$

$b = \frac{\sin^{2/3}\theta}{\cos^{1/3}\theta}$

To Prove : $a^2b^2(a^2+b^2)=1$.

$L.H.S = a^2b^2(a^2+b^2) = \frac{(\cos^{4/3}\theta)}{\sin^{2/3}\theta} \cdot \frac{\sin^{4/3}\theta}{\cos^{2/3}\theta}\left[\frac{\cos^{4/3}\theta}{\sin^{2/3}\theta} + \frac{\sin^{4/3}\theta}{\cos^{2/3}\theta}\right]$

$= \cancel{\cos^{2/3}\theta} \cdot \cancel{\sin^{2/3}\theta}\left[\frac{\cos^2\theta+\sin^2\theta}{\cancel{\sin^{2/3}\theta}\cdot\cancel{\cos^{2/3}\theta}}\right]$

$= 1$

$= R.H.S.$

12. Eliminate 'θ' from the equations $a\sec\theta - c\tan\theta = b$ and $b\sec\theta + d\tan\theta = c$

$a\sec\theta - c\tan\theta - b = 0$

$b\sec\theta + d\tan\theta - c = 0$

$\sec\theta$		$\tan\theta$		1	
$-c$	$-b$		a		$-c$
d	$-c$		b		d

$\frac{\sec\theta}{c^2+bd} = \frac{\tan\theta}{-b^2+ac} = \frac{1}{ad+bc}$

$\sec\theta = \frac{c^2+bd}{ad+bc}$ $\tan\theta = \frac{ac-b^2}{ad+bc}$

W.K.T

$\sec^2\theta - \tan^2\theta = 1$

$\left(\frac{c^2+bd}{ad+bc}\right)^2 - \left(\frac{ac-b^2}{ad+bc}\right)^2 = 1$

$(c^2+bd)^2 - (ac-b^2)^2 = (ad+bc)^2$

$(c^2+bd)^2 = (ad+bc)^2 + (ac-b^2)^2$

3.3 Eliminate θ from $a\cos\theta = b$ and $c\sin\theta = d$ where a, b, c, d are constants.

$a\cos\theta = b$ $\qquad$ $c\sin\theta = d$

$\cos\theta = \frac{b}{a}$ $\qquad$ $\sin\theta = \frac{d}{c}$

W.K.T $\sin^2\theta + \cos^2\theta = 1$

$$\frac{d^2}{c^2} + \frac{b^2}{a^2} = 1$$

$$a^2d^2 + b^2c^2 = a^2c^2$$

3.3. Radian Measure :

The radian measure of an angle is the ratio of the arc length it subtends, to the radius of the circle in which it is the central angle.

Consider a circle of radius r. let s be the arc length subtending an angle θ at the centre.

$$\boxed{\begin{aligned} \theta &= \frac{\text{arc length}}{\text{radius}} = \frac{s}{r} \text{ radians.} \\ \theta &= \frac{s}{r} \\ s &= r\theta \end{aligned}}$$

Relationship between Degree and Radian Measures :

In a unit circle, a full rotation corresponds to 360° whereas, a full rotation is related to 2π radians., the circumference of the unit circle.

$$2\pi = 360^\circ \qquad \pi = 180^\circ$$

$$1 \text{ radian} = \left(\frac{180}{\pi}\right)^\circ \text{ (or) } 1^\circ = \frac{\pi}{180} \text{ radians.}$$

$$x \text{ radians} = \left(\frac{180x}{\pi}\right)^\circ \text{ (or) } x^\circ = \frac{\pi x}{180} \text{ radians.}$$

Remarks :

* Arc length = $\begin{cases} \frac{\theta}{360} \times 2\pi r & \theta \text{ in deg} \\ r\theta & \theta \text{ in rad} \end{cases}$

* Area of the Sector $= \begin{cases} \frac{\theta}{360} \times \pi r^2 \text{ (or) } \frac{sr}{2} & \theta \text{ in deg} \\ \frac{\theta}{2} r^2 & \theta \text{ in rad} \end{cases}$

* Area of the segment $= \frac{\theta}{360}\pi r^2 - \frac{1}{2}r^2\sin\theta$.

3.4 Convert (i) 18° to radians
(ii) -108° to radians.

(i) $18^\circ = 18^\circ \times \frac{\pi}{180} = \frac{\pi}{10}$ radians.

(ii) $-108^\circ = -108 \times \frac{\pi}{180} = -\frac{3\pi}{5}$ radians.

3.5. Convert : (i) $\frac{\pi}{5}$ radians to degree.
(ii) 6 radians to degree.

(i) $\frac{\pi}{5} = \frac{\pi}{5} \times \frac{180}{\pi} = 36^\circ$

(ii) $6 = 6 \times \frac{180}{\pi}$

$= 6 \times 180 \times \frac{7}{22}$

$= 3 \times 180 \times \frac{7}{11} = \frac{3780}{11}$

$= \left(343\frac{7}{11}\right)^\circ$

$$\begin{array}{r} 343 \\ 11\overline{)3780} \\ 33 \\ \hline 48 \\ 44 \\ \hline 40 \\ 33 \\ \hline 7 \end{array}$$

Ex : 3.2

1 Express each of the following angles in radian measure.

(i) $30^\circ = 30 \times \frac{\pi}{180} = \frac{\pi}{6}$ radians.

(ii) $135^\circ = 135 \times \frac{\pi}{180} = \frac{3\pi}{4}$ radians.

(iii) $-205^\circ = -205 \times \frac{\pi}{180} = -\frac{41\pi}{36}$ radians.

(iv) $150^\circ = 150 \times \frac{\pi}{180} = \frac{5\pi}{6}$ radians.

(v) $330^\circ = 330 \times \frac{\pi}{180} = \frac{11\pi}{6}$ radians.

2. Find the degree measure corresponding to the following radian measures.

(i) $\frac{\pi}{3} = \frac{\pi}{3} \times \frac{180}{\pi} = 60°$

(ii) $\frac{\pi}{9} = \frac{\pi}{9} \times \frac{180}{\pi} = 20°$

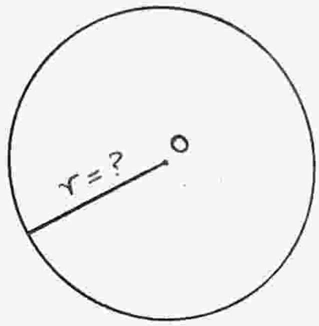
(iii) $\frac{2\pi}{5} = \frac{2\pi}{5} \times \frac{180}{\pi} = 72°$

(iv) $\frac{7\pi}{3} = \frac{7\pi}{3} \times \frac{180}{\pi} = 420°$

(v) $\frac{10\pi}{9} = \frac{10\pi}{9} \times \frac{180}{\pi} = 200°$

3. What must be the radius of a circular running path, around which an athlete must run 5 times in order to describe 1 km?

Distance travelled in 5 rounds $= 1\text{ km} = 1000\text{m}$



Distance travelled in 1 round $= \frac{1000}{5} = 200\text{m}.$

Perimeter $= 200\text{m}$

$2\pi r = 200$

$2 \times \frac{22}{7} \times r = 200$

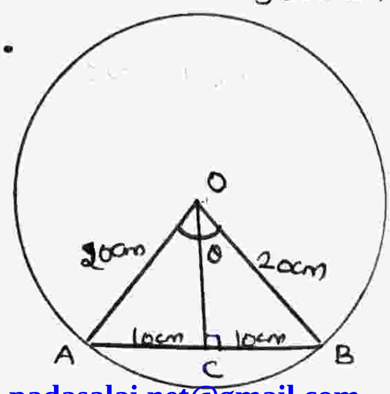
$r = \frac{200 \times 7}{2 \times 22} = \frac{350}{11}$

$= 31.82\text{ m}.$

$\therefore$ The radius of the circular Path $= 31.82\text{m}.$

4. In a circle of diameter 40cm, a chord is of length 20cm. Find the length of the minor arc of the chord.

O be the centre of the circle.



$OA = OB = 20\text{cm}$

$AB = 20\text{ cm}$

$AC = BC = 10\text{cm}$

$\angle AOB = \theta \qquad \angle AOC = \theta/2$

Consider the ΔAOC

$\sin\theta/2 = \frac{AC}{OA} = \frac{10}{20} = \frac{1}{2}$

$\theta/2 = 30°$

$\theta = 60° = \frac{\pi}{3}$ radians.

arc length $= r\theta$

$= 20 \times \frac{\pi}{3} = \frac{20\pi}{3}$ cm

$= \frac{20}{3} \times \frac{22}{7} = 20.95$ cm.

$\therefore$ The length of the minor arc is 20.95 cm.

5. Find the degree measure of the angle subtended at the centre of circle of radius 100 cm by an arc of length 22 cm.

radius $r = 100$ cm

arc length $= 22$ cm.

$s = r\theta = 22$

$100 \times \theta = 22$

$\theta = \frac{22}{100}$ rad

$= \frac{22}{100} \times \frac{180}{\pi}$ deg

$= \frac{\cancel{22}}{\cancel{100}_{5}} \times \cancel{180}^{9} \times \frac{7}{\cancel{22}} = \frac{63}{5}$

$\theta = 12° 36'$

6. What is the length of the arc intercepted by a central angle of measure 41° in a circle of radius 10 ft?

$\theta = 41° = 41 \times \frac{\pi}{180}$ rad $\qquad$ radius = 10 ft

arc length $s = r\theta$

$= 1\cancel{0} \times 41 \times \frac{\cancel{22}^{11}}{7} \times \frac{1}{\cancel{180}_{9}} = \frac{451}{63}$

$= 7.1587$

$= 7.16$ ft.

$\therefore$ The length of the arc is 7.16 ft.

7 If in two circles, arcs of the same length subtend angles 60° and 75° at the centre, find the ratio of their radii

Let r_1 and r_2 be the two radii of the circles.

$\theta_1 = 60^\circ = \frac{\pi}{3}$ rad

$\theta_2 = 75^\circ = 75 \times \frac{\pi}{180} = \frac{5\pi}{12}$

$S_1 = S_2 \Rightarrow r_1\theta_1 = r_2\theta_2$

$\Rightarrow r_1 \frac{\cancel{\pi}}{3} = r_2 \cdot \frac{5\cancel{\pi}}{12}$

$\frac{r_1}{\cancel{3}} = \frac{5 r_2}{\cancel{12}_4}$

$\frac{r_1}{r_2} = \frac{5}{4}$

$r_1 : r_2 = 5 : 4.$

8. The Perimeter of a certain sector of a circle is equal to the length of the arc of a semicircle having the same radius. Express the angle of the sector in degrees, min, and sec.

Let r be the radius of the sector.

Given that

Perimeter of the sector $= \frac{1}{2}(2\pi r)$

$2r + r\theta = \pi r$

$\cancel{r}(2+\theta) = \pi \cancel{r}$

$\theta = (\pi - 2)$ rad

$\theta = (\pi - 2) \times \frac{180}{\pi}$ deg

$= 180 - \frac{360}{\pi}$

$= 180 - 360 \times \frac{7}{22} = 180^\circ - \frac{1260}{11}$

$= 180^\circ - 114^\circ 32' 43''$

$\theta = 65^\circ 27' 17''$

$11\overline{)1260}$ = 114, remainder 6; $6 \times 60 = 360$; $11\overline{)360}$ = 32, remainder 8; $8 \times 60 = 480$; $11\overline{)480}$ = 43, remainder 7

$179^\circ 59' 60''$
$- 114^\circ 32' 43''$
$= 65^\circ 27' 17''$

9. An airplane Propeller rotates 1000 times per minute. Find the number of degrees that a Point on the edge of the Propeller will rotate in 1 second.

No of rotations in 1min = 1000

No of rotations in 1sec = $\frac{1000}{60} = \frac{50}{3}$.

The angle rotated in 1 rotation = 360°

∴ The angle rotated in $\frac{50}{3}$ rotations

$= \frac{50}{3} \times 360 = 6000^\circ$

10. A train is moving on a circular track of 1500m radius at the rate of 66km/h. what angle will it turn in 20 seconds?

radius = 1500 m.

Speed of the train = 66 km/hr

$= \frac{66000}{60 \times 60}$ m/sec

$= \frac{110}{6}$ m/sec.

Train will travel in 20 sec = $20 \times \frac{110}{6}$

$= \frac{1100}{3}$ m.

arc length = $\frac{1100}{3}$ m

$r\theta = \frac{1100}{3}$

$15\cancel{00} \times \theta = \frac{11\cancel{00}}{3}$

$\theta = \frac{11}{3} \times \frac{1}{15}$ rad

$= \frac{11}{3} \times \frac{1}{15} \times \frac{180}{\pi}$ deg

$= \frac{\cancel{11}}{\cancel{3}} \times \frac{1}{\cancel{15}} \times \cancel{180} \times \frac{7}{\cancel{22}}$

$= 14^\circ$

$\boxed{\theta = 14^\circ}$

11. A circular metallic plate of radius 8cm and thickness 6mm is melted and molded into a pie of radius 16cm and thickness 4mm. Find the angle of the sector.

Circular metallic plate: (cylinder)

radius = 8cm

height = 6mm = $\frac{6}{10}$ cm.

Pie

radius = 16cm

thickness (height) = 4mm = $\frac{4}{10}$ cm.

Given that

Vol. of Pie = Vol. of circular Plate

$$\frac{\theta}{360} \times \pi R^2 H = \pi r^2 h$$

$$\frac{\theta}{360} \times \frac{4}{10} \times 16 \times 16 = 8 \times 8 \times \frac{6}{10}$$

$$\frac{\theta}{45} = 3$$

$$\theta = 3(45)$$

$$\boxed{\theta = 135^\circ} \text{ (or) } \theta = \frac{3\pi}{4}.$$

3.6 Find the length of an arc of a circle of radius 5cm subtending a central angle measuring 15°.

radius = 5cm

$\theta = 15^\circ = 15 \times \frac{\pi}{180} = \frac{\pi}{12}$ rad

arc length $= r\theta$

$= 5 \times \frac{\pi}{12} = \frac{5\pi}{12}$ cm.

3.7 If the arcs of same length in two circles subtend central angles 30° and 80°. Find the ratio of their radii

Let r_1, r_2 be the radii of two given circles.

$\theta_1 = 30^\circ = \pi/6$ $\quad \theta_2 = 80^\circ = \frac{4\pi}{9}$ rad.

Given that $S_1 = S_2$

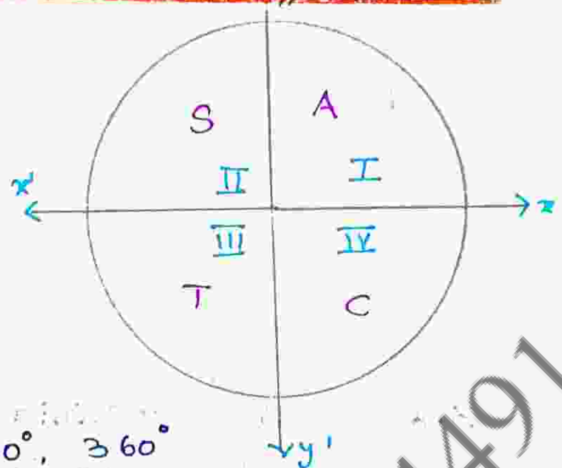
$$r_1\theta_1 = r_2\theta_2$$

$$r_1 \frac{\pi}{6} = r_2 \cdot \frac{4\pi}{9} \Rightarrow \frac{r_1}{r_2} = \frac{8}{3}$$

$\boxed{r_1 : r_2 = 8 : 3}$

Trigonometric functions and their Properties:

$\sin(-\theta) = -\sin\theta$
$\cos(-\theta) = \cos\theta$
$\tan(-\theta) = -\tan\theta$
$\operatorname{cosec}(-\theta) = -\operatorname{cosec}\theta$
$\sec(-\theta) = \sec\theta$
$\cot(-\theta) = -\cot\theta$



$90°, 270°$

$\sin\theta \rightarrow \cos\theta$
$\cos\theta \rightarrow \sin\theta$
$\tan\theta \rightarrow \cot\theta$
$\operatorname{cosec}\theta \rightarrow \sec\theta$
$\sec\theta \rightarrow \operatorname{cosec}\theta$
$\cot\theta \rightarrow \tan\theta$

$180°, 360°$

$\sin\theta \rightarrow \sin\theta$
$\cos\theta \rightarrow \cos\theta$
$\tan\theta \rightarrow \tan\theta$
$\operatorname{cosec}\theta \rightarrow \operatorname{cosec}\theta$
$\sec\theta \rightarrow \sec\theta$
$\cot\theta \rightarrow \cot\theta$

$90-\theta$	$90+\theta$
$\sin(90-\theta) = \cos\theta$	$\sin(90+\theta) = \cos\theta$
$\cos(90-\theta) = \sin\theta$	$\cos(90+\theta) = -\sin\theta$
$\tan(90-\theta) = \cot\theta$	$\tan(90+\theta) = -\cot\theta$
$\operatorname{cosec}(90-\theta) = \sec\theta$	$\operatorname{cosec}(90+\theta) = \sec\theta$
$\sec(90-\theta) = \operatorname{cosec}\theta$	$\sec(90+\theta) = -\operatorname{cosec}\theta$
$\cot(90-\theta) = \tan\theta$	$\cot(90+\theta) = -\tan\theta$
$180-\theta$	$180+\theta$
$\sin(180-\theta) = \sin\theta$	$\sin(180+\theta) = -\sin\theta$
$\cos(180-\theta) = -\cos\theta$	$\cos(180+\theta) = -\cos\theta$
$\tan(180-\theta) = -\tan\theta$	$\tan(180+\theta) = \tan\theta$
$\operatorname{cosec}(180-\theta) = \operatorname{cosec}\theta$	$\operatorname{cosec}(180+\theta) = -\operatorname{cosec}\theta$
$\sec(180-\theta) = -\sec\theta$	$\sec(180+\theta) = -\sec\theta$
$\cot(180-\theta) = -\cot\theta$	$\cot(180+\theta) = \cot\theta$
$270-\theta$	$270+\theta$
$\sin(270-\theta) = -\cos\theta$	$\sin(270+\theta) = -\cos\theta$
$\cos(270-\theta) = -\sin\theta$	$\cos(270+\theta) = \sin\theta$
$\tan(270-\theta) = \cot\theta$	$\tan(270+\theta) = -\cot\theta$
$\operatorname{cosec}(270-\theta) = -\sec\theta$	$\operatorname{cosec}(270+\theta) = -\sec\theta$
$\sec(270-\theta) = -\operatorname{cosec}\theta$	$\sec(270+\theta) = \operatorname{cosec}\theta$
$\cot(270-\theta) = \tan\theta$	$\cot(270+\theta) = -\tan\theta$

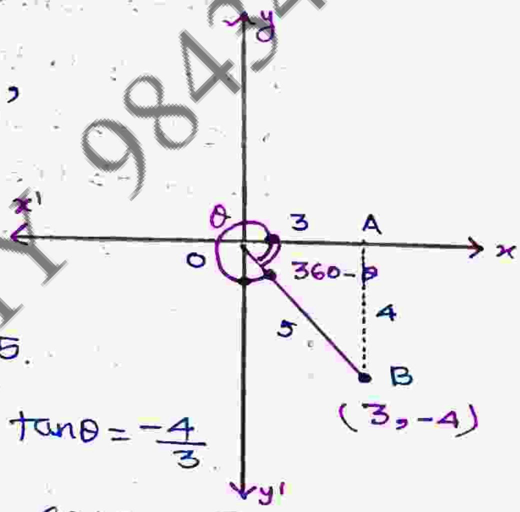
360 − θ	360 + θ
$\sin(360-\theta) = -\sin\theta$	$\sin(360+\theta) = \sin\theta$
$\cos(360-\theta) = \cos\theta$	$\cos(360+\theta) = \cos\theta$
$\tan(360-\theta) = -\tan\theta$	$\tan(360+\theta) = \tan\theta$
$\operatorname{cosec}(360-\theta) = -\operatorname{cosec}\theta$	$\operatorname{cosec}(360+\theta) = \operatorname{cosec}\theta$
$\sec(360-\theta) = \sec\theta$	$\sec(360+\theta) = \sec\theta$
$\cot(360-\theta) = -\cot\theta$	$\cot(360+\theta) = \cot\theta$

3.8 The terminal side of an angle θ in standard position passes through the point $(3,-4)$. Find the six trigonometric function values at an angle θ.

By the given data,

θ lies in IV quadrant

In IVth quad $\cos\theta$ and $\sec\theta$ only positive.



$$OB = \sqrt{3^2+(-4)^2} = \sqrt{9+16} = 5.$$

$\sin\theta = \frac{-4}{5}$ $\cos\theta = \frac{3}{5}$ $\tan\theta = \frac{-4}{3}$

$\operatorname{cosec}\theta = \frac{-5}{4}$ $\sec\theta = \frac{5}{3}$ $\cot\theta = -\frac{3}{4}$.

3.9 If $\sin\theta = \frac{3}{5}$ and the angle θ is in the second quadrant, then find the values of other five trigonometric functions:

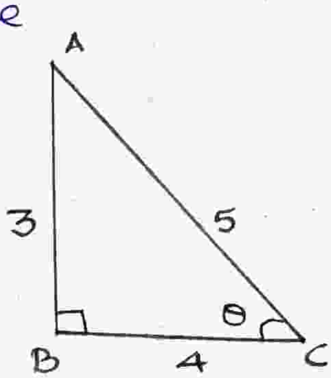
θ lies in II quadrant, In II quadrant $\sin\theta$ and $\operatorname{cosec}\theta$ only positive

$$\sin\theta = \frac{3}{5}$$

$$AC^2 = AB^2 + BC^2$$

$$25 = 9 + BC^2$$

$$BC^2 = 16 \quad \boxed{BC = 4}$$



$\cos\theta = \frac{-4}{5}$ $\tan\theta = \frac{-3}{4}$

$\operatorname{cosec}\theta = \frac{5}{3}$ $\sec\theta = \frac{-5}{4}$ $\cot\theta = -\frac{4}{3}$

3.10 Find the values of

(i) $\sin(-45^\circ)$ (ii) $\cos(-45^\circ)$ (iii) $\cot(-45^\circ)$

(i) $\sin(-\theta) = -\sin\theta$

$\sin(-45^\circ) = -\sin 45^\circ = \frac{-1}{\sqrt{2}}$

(ii) $\cos(-\theta) = \cos\theta$

$\cos(-45^\circ) = \cos 45^\circ = \frac{1}{\sqrt{2}}$

(iii) $\cot(-\theta) = -\cot\theta$

$\cot(-45^\circ) = -\cot 45^\circ = -1.$

3.11 Find the value of (i) $\sin 150^\circ$ (ii) $\cos 135^\circ$ (iii) $\tan 120^\circ$

(i) $\sin 150^\circ = \sin(180 - 30^\circ) = \sin 30^\circ = \frac{1}{2}.$

(ii) $\cos 135^\circ = \cos(180 - 45^\circ) = -\cos 45^\circ = \frac{-1}{\sqrt{2}}$

(iii) $\tan 120^\circ = \tan(180 - 60^\circ) = -\tan 60^\circ = -\sqrt{3}.$

3.12 Find the value of

(i) $\sin 765^\circ$ (ii) $\operatorname{cosec}(-1410^\circ)$ (iii) $\cot\left(-\frac{15\pi}{4}\right)$

(i) $\sin 765^\circ = \sin(2\times 360^\circ + 45^\circ) = \sin 45^\circ = \frac{1}{\sqrt{2}}$

(ii) $\operatorname{cosec}(-1410^\circ) = -\operatorname{cosec} 1410$

$= -\operatorname{cosec}[4\times 360 - 30^\circ]$

$= \operatorname{cosec} 30^\circ = 2.$

(iii) $\cot\left(-\frac{15\pi}{4}\right) = -\cot\frac{15\pi}{4} = -\cot\left(4\pi - \frac{\pi}{4}\right)$

$= \cot\frac{\pi}{4} = 1..$

3.13 Prove that $\tan 315^\circ . \cot(-405^\circ) + \cot 495^\circ \tan(-585^\circ) = 2.$

$\tan 315^\circ = \tan(360 - 45^\circ) = -\tan 45^\circ = -1$

$\cot(-405^\circ) = -\cot 405^\circ = -\cot(360 + 45^\circ) = -\cot 45^\circ = -1.$

$\cot(495^\circ) = \cot(360 + 135^\circ) = \cot 135^\circ$

$= \cot(180 - 45^\circ) = -\cot 45^\circ = -1$

$\tan(-585^\circ) = -\tan(585^\circ) = -\tan(360 + 225^\circ)$

$= -\tan 225^\circ = -\tan(180 + 45^\circ) = -\tan 45^\circ = -1$

$L.H.S = (-1)(-1) + (-1)(-1) = 1 + 1 = 2$

Ex : 3.3.

1. Find the values of

(i) $\sin 480^\circ = \sin(360+120^\circ) = \sin 120^\circ$
$= \sin(180-60^\circ) = \sin 60^\circ = \sqrt{3}/2$

(ii) $\sin(-1110^\circ) = -\sin(1110^\circ) = -\sin(3\times 360+30^\circ)$
$= -\sin 30^\circ = -\frac{1}{2}$

(iii) $\cos 300^\circ = \cos(360-60^\circ)$
$= \cos 60^\circ = \frac{1}{2}$

(iv) $\tan 1050^\circ = \tan(3\times 360-30^\circ)$
$= -\tan 30^\circ = \frac{-1}{\sqrt{3}}$

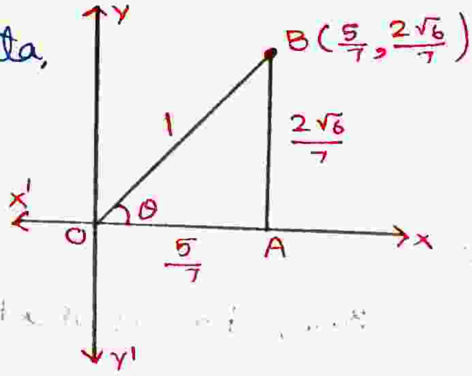
v) $\cot 660^\circ = \cot(2\times 360-60^\circ)$
$= -\cot 60^\circ = -1/\sqrt{3}$

vi) $\tan \frac{19\pi}{3} = \tan\left(6\pi+\frac{\pi}{3}\right) = \tan\frac{\pi}{3} = \sqrt{3}$

(vii) $\sin\left(-\frac{11\pi}{3}\right) = -\sin\frac{11\pi}{3} = -\sin\left(4\pi-\frac{\pi}{3}\right)$
$= +\sin\frac{\pi}{3} = \frac{\sqrt{3}}{2}$

2. $\left(\frac{5}{7}, \frac{2\sqrt{6}}{7}\right)$ is a Point on the terminal side of an angle θ in Standard Position. Determine the trigonometric function values of angle 'θ'

From the given data, θ lies in I quadrant.



$OB^2 = OA^2 + AB^2$
$= \frac{25}{49} + \frac{24}{49} = \frac{49}{49} = 1$
$\boxed{OB = 1}$

$\sin\theta = \frac{AB}{OB} = \frac{2\sqrt{6}}{7}$ $\cos\theta = \frac{OA}{OB} = \frac{5}{7}$ $\tan\theta = \frac{2\sqrt{6}}{5}$

$\operatorname{cosec}\theta = \frac{7}{2\sqrt{6}}$ $\sec\theta = \frac{7}{5}$ $\cot\theta = \frac{5}{2\sqrt{6}}$

3. Find the values of other five trigonometric functions for the following:

(i) $\cos\theta = -\frac{1}{2}$, θ lies in the III quadrant.

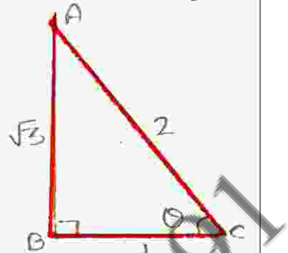
In III quadrant $\tan\theta$ and $\cot\theta$ are only Positive.

$AC^2 = AB^2 + BC^2$

$4 = AB^2 + 1$

$AB^2 = 3$

$AB = \sqrt{3}$



$\cos\theta = -\frac{1}{2}$

$\sin\theta = -\frac{\sqrt{3}}{2}$

$\tan\theta = \sqrt{3}$

$\operatorname{cosec}\theta = \frac{-2}{\sqrt{3}}$ $\quad \sec\theta = -2$ $\quad \cot\theta = \frac{1}{\sqrt{3}}$.

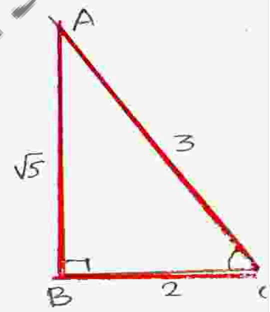
(ii) $\cos\theta = \frac{2}{3}$, θ lies in I quadrant.

In I quadrant all functions are Positive.

$AC^2 = AB^2 + BC^2$

$9 = AB^2 + 4 \Rightarrow AB^2 = 5$

$AB = \sqrt{5}$



$\sin\theta = \frac{\sqrt{5}}{3}$ $\quad \cos\theta = \frac{2}{3}$ $\quad \tan\theta = \frac{\sqrt{5}}{2}$

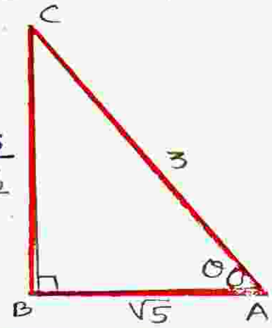
$\operatorname{cosec}\theta = \frac{3}{\sqrt{5}}$ $\quad \sec\theta = \frac{3}{2}$ $\quad \cot\theta = \frac{2}{\sqrt{5}}$.

(iii) $\sin\theta = -\frac{2}{3}$ θ lies in IV quadrant.

In IV quadrant $\cos\theta$, $\sec\theta$ are Positive.

$AC^2 = AB^2 + BC^2 \Rightarrow 9 = AB^2 + 4$

$AB^2 = 5$ $\quad \boxed{AB = \sqrt{5}}$



(i) $\sin\theta = -\frac{2}{3}$ $\quad \cos\theta = \frac{\sqrt{5}}{3}$ $\quad \tan\theta = \frac{-2}{\sqrt{5}}$

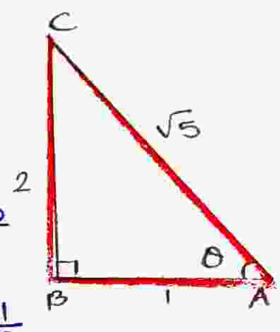
$\operatorname{cosec}\theta = -\frac{3}{2}$ $\quad \sec\theta = \frac{3}{\sqrt{5}}$ $\quad \cot\theta = -\frac{\sqrt{5}}{2}$

(iv) $\tan\theta = -2$; θ lies in the II quadrant.

In II quadrant, $\sin\theta$, $\operatorname{cosec}\theta$ only Positive.

$AC^2 = AB^2 + BC^2 = 4 + 1 = 5$

$AC = \sqrt{5}$

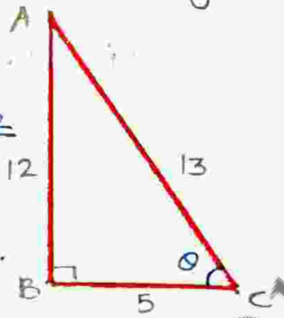


(i) $\sin\theta = \frac{2}{\sqrt{5}}$ $\quad \cos\theta = -\frac{1}{\sqrt{5}}$ $\quad \tan\theta = -2$

$\operatorname{cosec}\theta = \frac{\sqrt{5}}{2}$ $\quad \sec\theta = -\sqrt{5}$ $\quad \cot\theta = -\frac{1}{2}$

(v) $\sec\theta = \frac{13}{5}$, θ lies in the IV quadrant.

In IV quadrant, $\cos\theta$ and $\sec\theta$ only positive.



i) $\sin\theta = -\frac{12}{13}$ $\quad \cos\theta = \frac{5}{3}$ $\quad \tan\theta = -\frac{12}{5}$

$\operatorname{cosec}\theta = -\frac{13}{12}$ $\quad \sec\theta = \frac{13}{5}$ $\quad \cot\theta = \frac{-5}{12}$.

4. Prove that $\frac{\cot(180°+\theta)\sin(90-\theta)\cos(-\theta)}{\sin(270+\theta)\tan(-\theta)\operatorname{cosec}(360+\theta)} = \cos^2\theta\cot\theta$

$$L.H.S = \frac{\cot(180+\theta)\sin(90-\theta)\cos(-\theta)}{\sin(270+\theta)\tan(-\theta)\operatorname{cosec}(360+\theta)}$$

$$= \frac{\cot\theta \cdot \cos\theta \cdot \cos\theta}{(-\cos\theta)(-\tan\theta)\cdot\operatorname{cosec}\theta}$$

$$= \frac{\cot\theta \cdot \cos^2\theta}{\tan\theta\cdot\cot\theta} = \cot\theta\cdot\cos^2\theta = R.H.S$$

5. Find all the angles between $0°$ and $360°$ which satisfy the equation $\sin^2\theta = \frac{3}{4}$

$$\sin^2\theta = \frac{3}{4}$$

$$\sin\theta = \pm\frac{\sqrt{3}}{2}$$

$\sin\theta = \frac{\sqrt{3}}{2}$ $\qquad$ $\sin\theta = -\frac{\sqrt{3}}{2} = 240°$

$\theta = 60°, 120°$ $\qquad$ $\sin\theta = \sin(-60°)$

$= \sin(360-300°)$

$\theta = 300°$

$\therefore \theta = 60°, 120°, 240°, 300°$

6. Show that $\sin^2\frac{\pi}{18} + \sin^2\frac{\pi}{9} + \sin^2\frac{7\pi}{18} + \sin^2\frac{4\pi}{9} = 2$

$$L.H.S = \sin^2\frac{\pi}{18} + \sin^2\frac{\pi}{9} + \sin^2\frac{7\pi}{18} + \sin^2\frac{4\pi}{9}$$

$$= \sin^2 10° + \sin^2 20° + \sin^2 70° + \sin^2 80°$$

$$= \sin^2 10° + \sin^2 20° + \sin^2(90-20) + \sin^2(90-10)$$

$$= \sin^2 10° + \sin^2 20° + \cos^2 20° + \cos^2 10°$$

$$= (\sin^2 10° + \cos^2 10°) + (\sin^2 20° + \cos^2 20°)$$

$$= 1+1 = 2 = R.H.S.$$

Periodicity of Trigonometric Functions:

A function f is said to be a Periodic function with Period P. if there exists a Smallest Positive number p such that $f(x+P) = f(x)$ for all x in the domain.

$\sin x$, $\cos x$, $\operatorname{cosec} x$ and $\sec x$ are Periodic functions with Period 2π.

$\tan x$ and $\cot x$ are Periodic functions with Period π.

odd and even trigonometric functions:

A real valued function $f(x)$ is an even function if it satisfies $f(-x) = f(x)$ for all real number x and an odd function if it satisfies $f(-x) = -f(x)$ for all real values of x.

$\cos x$ and $\sec x$ are even functions

$\cos(-x) = \cos x$ $\qquad$ $\sec(-x) = \sec x$.

$\sin x$, $\tan x$, $\operatorname{cosec} x$ and $\cot x$ are odd functions.

$\sin(-x) = -\sin x$ $\qquad$ $\tan(-x) = -\tan x$

$\operatorname{cosec}(-x) = -\operatorname{cosec} x$ $\qquad$ $\cot(-x) = -\cot x$.

3.14 Determine whether the following functions are even, odd or neither.

(i) $\sin^2 x - 2\cos^2 x - \cos x$

$$f(x) = \sin^2 x - 2\cos^2 x - \cos x$$

$$f(-x) = [\sin(-x)]^2 - 2[\cos(-x)]^2 - \cos(-x)$$

$$= (-\sin x)^2 - 2\cos^2 x - \cos x$$

$$= \sin^2 x - 2\cos^2 x - \cos x$$

$$f(-x) = f(x)$$

$\therefore f(x)$ is an even function.

(ii) $f(x) = \sin \cos x$

$f(-x) = \sin(\cos(-x))$

$= \sin \cos x = f(x)$

$f(-x) = f(x)$

$\therefore f(x)$ is an even function.

(iii) $\cos(\sin x)$

$f(x) = \cos \sin x$

$f(-x) = \cos \sin(-x)$

$= \cos[-\sin x]$

$= \cos(\sin x)$

$f(-x) = f(x)$

$\therefore f(x)$ is an even function.

(iv) $\sin x + \cos x$

$f(x) = \sin x + \cos x$

$f(-x) = \sin(-x) + \cos(-x)$

$= -\sin x + \cos x$

$f(-x) \neq f(x)$, $f(-x) \neq -f(x)$

$\therefore f(x)$ is neither an odd nor an even function.

Trigonometric Identities :

Compound angle identities:

$\sin(\alpha+\beta) = \sin\alpha\cos\beta + \cos\alpha\sin\beta$

$\sin(\alpha-\beta) = \sin\alpha\cos\beta - \cos\alpha\sin\beta$

$\cos(\alpha+\beta) = \cos\alpha\cos\beta - \sin\alpha\sin\beta$

$\cos(\alpha-\beta) = \cos\alpha\cos\beta + \sin\alpha\sin\beta$

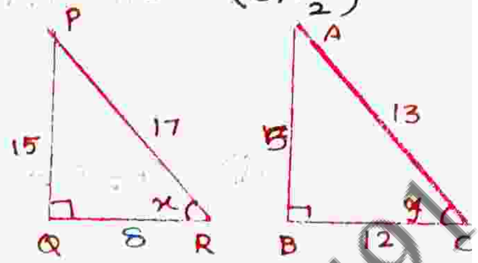
$$\tan(\alpha+\beta) = \frac{\tan\alpha + \tan\beta}{1 - \tan\alpha\tan\beta}$$

$$\tan(\alpha-\beta) = \frac{\tan\alpha - \tan\beta}{1 + \tan\alpha\tan\beta}$$

Ex : 3.4

1. If $\sin x = \frac{15}{17}$ and $\cos y = \frac{12}{13}$ $x \in (0, \frac{\pi}{2})$

$y \in (0, \frac{\pi}{2})$



From ΔPQR

$PR^2 = PQ^2 + QR^2$

$17^2 = 15^2 + QR^2$

$QR^2 = 289 - 225 = 64$

$\boxed{QR = 8}$

From ΔABC, $AC^2 = AB^2 + BC^2$

$13^2 = AB^2 + 12^2$

$AB^2 = 169 - 144 = 25$

$\boxed{AB = 5}$

$\sin x = \frac{15}{17}$ $\sin y = \frac{5}{13}$

$\cos x = \frac{8}{17}$ $\cos y = \frac{12}{13}$

$\tan x = \frac{15}{8}$ $\tan y = \frac{5}{12}$

(i) $\sin(x+y) = \sin x \cos y + \cos x \sin y$

$= \frac{15}{17} \cdot \frac{12}{13} + \frac{8}{17} \cdot \frac{5}{13} = \frac{180}{221} + \frac{40}{221} = \frac{220}{221}$

(ii) $\cos(x-y) = \cos x \cos y + \sin x \sin y$

$= \frac{8}{17} \cdot \frac{12}{13} + \frac{15}{17} \cdot \frac{5}{13} = \frac{96}{221} + \frac{75}{221} = \frac{171}{221}$

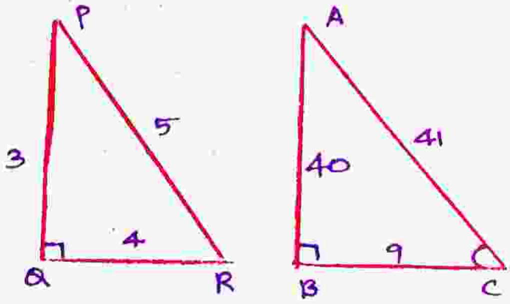
(iii) $\tan(x+y) = \frac{\tan x + \tan y}{1 - \tan x \tan y} = \frac{\frac{15}{8} + \frac{5}{12}}{1 - \frac{15}{8} \cdot \frac{5}{12}} = \frac{180+40}{96-75}$

$= \frac{220}{21}$

---

2. If $\sin A = \frac{3}{5}$ and $\cos B = \frac{9}{41}$, $A, B \in (0, \pi/2)$

find the value of (i) $\sin(A+B)$ (ii) $\cos(A-B)$



From the ΔPQR

$PR^2 = PQ^2 + QR^2$

$25 = 9 + QR^2 \Rightarrow QR^2 = 16$

$\boxed{QR = 4}$

From the ΔABC

$AC^2 = AB^2 + BC^2$

$41^2 = AB^2 + 9^2 \Rightarrow 1681 = AB^2 + 81$

$AB^2 = 1600$

$\boxed{AB = 40}$

$\sin A = \frac{3}{5}$ $\quad$ $\sin B = \frac{40}{41}$

$\cos A = \frac{4}{5}$ $\quad$ $\cos B = \frac{9}{41}$ $\Big\}$ A, B lies in I quad

(i) $\sin(A+B) = \sin A \cos B + \cos A \sin B$

$$= \frac{3}{5}\cdot\frac{9}{41} + \frac{4}{5}\cdot\frac{40}{41} = \frac{27}{205} + \frac{160}{205} = \frac{187}{205}$$

(ii) $\cos(A-B) = \cos A \cos B + \sin A \sin B$

$$= \frac{4}{5}\cdot\frac{9}{41} + \frac{3}{5}\cdot\frac{40}{41}$$

$$= \frac{36}{205} + \frac{120}{205} = \frac{156}{205}.$$

---

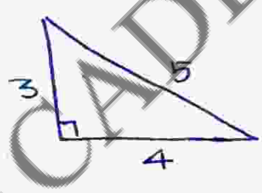
3. Find $\cos(x-y)$ given that $\cos x = -\frac{4}{5}$ with $x \in (\pi, \frac{3\pi}{2})$ and $\sin y = -\frac{24}{25}$ with $y \in (\pi, \frac{3\pi}{2})$

Since x and y both are lies in the third quadrant.

In third quadrant $\tan x$ only Positive.

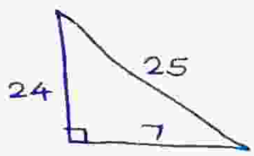
$\cos x = -\frac{4}{5}$

$\sin x = -\frac{3}{5}$



$\sin y = \frac{-24}{25}$

$\cos y = \frac{-7}{25}$



$\cos(x-y) = \cos x \cos y + \sin x \sin y$

$$= \left(\frac{-4}{5}\right)\left(\frac{-7}{25}\right) + \left(\frac{-3}{5}\right)\left(\frac{-24}{25}\right)$$

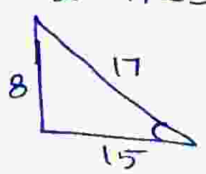
$$= \frac{28}{125} + \frac{72}{125} = \frac{100}{125} = \frac{4}{5}$$

---

4. Find $\sin(x-y)$ given that $\sin x = \frac{8}{17}$ $x \in (0, \frac{\pi}{2})$ and $\cos y = \frac{-24}{25}$ with $y \in (\pi, \frac{3\pi}{2})$

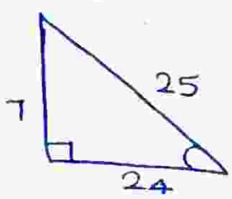
$\sin x = \frac{8}{17}$ $\quad$ x lies in I quadrant

$\cos x = \frac{15}{17}$



$\cos y = \frac{-24}{25}$

$\sin y = \frac{-7}{25}$



y lies in III quad

$$\sin(x-y) = \sin x \cos y - \cos x \sin y$$
$$= \frac{8}{17}\cdot\left(\frac{-24}{25}\right) - \frac{15}{17}\cdot\left(\frac{-7}{25}\right)$$
$$= \frac{-192}{425} + \frac{105}{425} = \frac{-87}{425}$$

5. Find the value of (i) $\cos 105^\circ$ (ii) $\sin 105^\circ$ (iii) $\tan\frac{7\pi}{12}$

(i) $\cos 105^\circ = \cos(60+45^\circ)$

$\cos(A+B) = \cos A\cos B - \sin A \sin B$

$\cos 105^\circ = \cos 60^\circ \cos 45^\circ - \sin 60^\circ \sin 45^\circ$

$= \frac{1}{2}\cdot\frac{1}{\sqrt{2}} - \frac{\sqrt{3}}{2}\cdot\frac{1}{\sqrt{2}} = \frac{1-\sqrt{3}}{2\sqrt{2}}.$

(ii) $\sin 105^\circ = \sin(60+45^\circ)$

$\sin(A+B) = \sin A\cos B + \cos A \sin B$

$\sin 105^\circ = \sin 60^\circ \cos 45^\circ + \cos 60^\circ \sin 45^\circ$

$= \frac{\sqrt{3}}{2}\cdot\frac{1}{\sqrt{2}} + \frac{1}{2}\cdot\frac{1}{\sqrt{2}} = \frac{\sqrt{3}+1}{2\sqrt{2}}.$

(iii) $\tan\frac{7\pi}{12} = \tan 105^\circ = \tan(60+45^\circ)$

$\tan(A+B) = \frac{\tan A + \tan B}{1-\tan A \tan B}$

$\tan 105^\circ = \frac{\tan 60^\circ + \tan 45^\circ}{1-\tan 60^\circ \tan 45^\circ} = \frac{\sqrt{3}+1}{1-\sqrt{3}}$

$= \frac{1+\sqrt{3}}{1-\sqrt{3}} \times \frac{1+\sqrt{3}}{1+\sqrt{3}} = \frac{4+2\sqrt{3}}{-2}$

$= -(2+\sqrt{3}).$

$\sin 15^\circ = \frac{\sqrt{3}-1}{2\sqrt{2}}$	$\cos 15^\circ = \frac{\sqrt{3}+1}{2\sqrt{2}}$	$\tan 15^\circ = \frac{\sqrt{3}-1}{\sqrt{3}+1} = 2-\sqrt{3}$
$\sin 75^\circ = \frac{\sqrt{3}+1}{2\sqrt{2}}$	$\cos 75^\circ = \frac{\sqrt{3}-1}{2\sqrt{2}}$	$\tan 75^\circ = 2+\sqrt{3}$
$\sin 105^\circ = \frac{\sqrt{3}+1}{2\sqrt{2}}$	$\cos 105^\circ = -\frac{(\sqrt{3}-1)}{2\sqrt{2}}$	$\tan 105^\circ = -(2+\sqrt{3})$

6. Prove that (i) $\cos(30+x) = \frac{\sqrt{3}\cos x - \sin x}{2}$
(ii) $\cos(\pi+\theta) = -\cos\theta$ (iii) $\sin(\pi+\theta) = -\sin\theta$

(i) $\cos(30+x) = \cos 30^\circ \cos x - \sin 30^\circ \sin x$

$= \frac{\sqrt{3}}{2}\cos x - \frac{1}{2}\sin x$

$= \frac{1}{2}(\sqrt{3}\cos x - \sin x)$

$=$ R.H.S.

(ii) $\cos(\pi+\theta) = \cos\pi\cos\theta - \sin\pi\sin\theta$

$= (-1)\cos\theta - 0\cdot\sin\theta$

$= -\cos\theta$

$=$ R.H.S

(iii) $\sin(\pi+\theta) = \sin\pi\cos\theta + \cos\pi\sin\theta$

$= 0\cdot\cos\theta + (-1)\sin\theta)$

$= -\sin\theta$

$=$ R.H.S

$\sin n\pi = 0$ $\cos n\pi = (-1)^n$

$\left.\begin{array}{l}\sin\pi\\2\pi\\3\pi\\\vdots\end{array}\right\} = 0$ $\left.\begin{array}{l}\cos\pi\\3\pi\\5\pi\\\vdots\end{array}\right\} = -1$ $\left.\begin{array}{l}\cos 2\pi\\4\pi\\6\pi\\\vdots\end{array}\right\} = 1$

7. Find a quadratic equation whose roots are $\sin 15^\circ$ and $\cos 15^\circ$

We know that

$\sin 15^\circ = \frac{\sqrt{3}-1}{2\sqrt{2}}$ $\cos 15^\circ = \frac{\sqrt{3}+1}{2\sqrt{2}}$

Sum of the roots $= \alpha+\beta = \frac{\sqrt{3}-1}{2\sqrt{2}} + \frac{\sqrt{3}+1}{2\sqrt{2}}$

$= \frac{\sqrt{3}-1+\sqrt{3}+1}{2\sqrt{2}} = \frac{2\sqrt{3}}{2\sqrt{2}} = \frac{\sqrt{3}}{\sqrt{2}}$

Product of the roots $= \alpha\beta = \left(\frac{\sqrt{3}-1}{2\sqrt{2}}\right)\left(\frac{\sqrt{3}+1}{2\sqrt{2}}\right)$

$= \frac{3-1}{8} = \frac{2}{8} = \frac{1}{4}.$

$\therefore$ The Required equation is

$x^2 - (\alpha+\beta)x + \alpha\beta = 0$

$$x^2 - \frac{\sqrt{3}}{\sqrt{2}}x + \frac{1}{4} = 0$$

$$\boxed{4\sqrt{2}x^2 - 4\sqrt{3}x + \sqrt{2} = 0}$$

(or)

$$4x^2 - 2\sqrt{6}x + 1 = 0$$

8. Expand $\cos(A+B+C)$. Hence Prove that $\cos A \cos B \cos C = \sin A \sin B \cos C + \sin B \sin C \cos A + \sin C \sin A \cos B$. if $A+B+C = \pi/2$.

$$\cos(A+B+C) = \cos(A+\overline{B+C})$$

$$= \cos A \cos(B+C) - \sin A \sin(B+C)$$

$$= \cos A \{\cos B \cos C - \sin B \sin C\} - \sin A \{\sin B \cos C + \cos B \sin C\}$$

$$= \cos A \cos B \cos C - \cos A \sin B \sin C - \sin A \sin B \cos C - \sin A \cos B \sin C.$$

If $A+B+C = \frac{\pi}{2}$

$$\cos(A+B+C) = \cos A \cos B \cos C - \cos A \sin B \sin C - \sin A \sin B \cos C - \sin A \cos B \sin C$$

$$\cos(\pi/2) = \cos A \cos B \cos C - \cos A \sin B \sin C - \sin A \sin B \cos C - \sin A \cos B \sin C$$

$$0 = \cos A \cos B \cos C - \cos A \sin B \sin C - \sin A \sin B \cos C - \sin A \cos B \sin C$$

$$\therefore \cos A \cos B \cos C = \cos A \sin B \sin C + \sin A \sin B \cos C + \sin A \cos B \sin C$$

Hence Proved.

9. Prove that

(i) $\sin(45+\theta) - \sin(45-\theta) = \sqrt{2}\sin\theta$

(ii) $\sin(30+\theta) + \cos(60+\theta) = \cos\theta$

(i) $\sin(A+B) + \sin(A-B) = 2\sin A \cos B$

$\sin(A+B) - \sin(A-B) = 2\cos A \sin B$

$$\sin(45+\theta) - \sin(45-\theta) = 2\cos 45^\circ \cdot \sin\theta$$

$$= 2 \cdot \frac{1}{\sqrt{2}} \cdot \sin\theta$$

$$= \sqrt{2}\sin\theta$$

$$= R.H.S$$

(ii) $\sin(30+\theta) + \cos(60+\theta)$

$$= \{\sin 30^\circ \cos\theta + \cos 30^\circ \sin\theta\} + \{\cos 60^\circ \cos\theta - \sin 60^\circ \sin\theta\}$$

$$= \frac{1}{2}\cos\theta + \cancel{\frac{\sqrt{3}}{2}\sin\theta} + \frac{1}{2}\cos\theta - \cancel{\frac{\sqrt{3}}{2}\sin\theta}$$

$$= \frac{1}{2}\cos\theta + \frac{1}{2}\cos\theta = \cos\theta = R.H.S$$

10. If $a\cos(x+y) = b\cos(x-y)$, show that $(a+b)\tan x = (a-b)\cot y$.

$$\boxed{\begin{aligned}\cos(A-B)+\cos(A+B) &= 2\cos A\cos B\\ \cos(A-B)-\cos(A+B) &= 2\sin A\sin B\end{aligned}}$$

$$a\cos(x+y) = b\cos(x-y)$$

$$\frac{a}{b} = \frac{\cos(x-y)}{\cos(x+y)}$$

$$\frac{a+b}{a-b} = \frac{\cos(x-y)+\cos(x+y)}{\cos(x-y)-\cos(x+y)}$$

$$\frac{a+b}{a-b} = \frac{\cancel{2}\cos x\cos y}{\cancel{2}\sin x\sin y}$$

$$\frac{a+b}{a-b} = \cot x \cdot \cot y$$

$$(a+b)\tan x = (a-b)\cot y$$

Hence Proved.

11. Prove that $\sin 105^\circ + \cos 105^\circ = \cos 45^\circ$

we know that

$$\sin 105^\circ = \frac{\sqrt{3}+1}{2\sqrt{2}} \qquad \cos 105^\circ = -\frac{(\sqrt{3}-1)}{2\sqrt{2}}$$

$$\sin 105^\circ + \cos 105^\circ = \frac{\sqrt{3}+1}{2\sqrt{2}} + \frac{1-\sqrt{3}}{2\sqrt{2}}$$

$$= \frac{\sqrt{3}+1+1-\sqrt{3}}{2\sqrt{2}} = \frac{2}{2\sqrt{2}} = \frac{1}{\sqrt{2}}$$

$$= \frac{1}{\sqrt{2}}$$

$$= \cos 45^\circ$$

$$= R.H.S.$$

12. Prove that $\sin 75^\circ - \sin 15^\circ = \cos 105^\circ + \cos 15^\circ$

$R.H.S = \cos 105^\circ + \cos 15^\circ$

$= \cos(90+15^\circ) + \cos(90-75^\circ)$

$= -\sin 15^\circ + \sin 75^\circ$

$= \sin 75^\circ - \sin 15^\circ$

$= L.H.S.$

13. Show that $\tan 75^\circ + \cot 75^\circ = 4$

we know that

$\tan 75^\circ = 2+\sqrt{3}$ (or) $\frac{\sqrt{3}+1}{\sqrt{3}-1}$

$\cot 75^\circ = \tan 15^\circ = 2-\sqrt{3}$ (or) $\frac{\sqrt{3}-1}{\sqrt{3}+1}$

$\tan 75^\circ + \cot 75^\circ = 2+\sqrt{3}+2-\sqrt{3} = 4 = R.H.S.$

14. Prove that $\cos(A+B)\cdot\cos C - \cos(B+C)\cos A = \sin B \sin(C-A)$

$L.H.S = \cos(A+B)\cdot\cos C - \cos(B+C)\cdot\cos A$

$= (\cos A\cos B - \sin A\sin B)\cdot\cos C - [\cos B\cos C - \sin B\sin C]\cos A$

$= \cancel{\cos A\cos B\cos C} - \sin A\sin B\cos C - \cancel{\cos A\cos B\cos C} + \cos A\sin B\sin C$

$= \cos A\sin B\sin C - \sin A\sin B\cos C$

$= \sin B\{\sin C\cos A - \cos C.\sin A\}$

$= \sin B\ \sin(C-A) = R.H.S$

15. Prove that $\sin(n+1)\theta\ \sin(n-1)\theta + \cos(n+1)\theta\ \cos(n-1)\theta = \cos 2\theta,\ n\in Z.$

$\boxed{\sin A\sin B + \cos A\cos B = \cos(A-B)}$

$L.H.S = \sin(n+1)\theta\ \sin(n-1)\theta + \cos(n+1)\theta.\cos(n-1)\theta$

$= \cos\{(n+1)\theta - (n-1)\theta\}$

$= \cos\{(\cancel{n}+1-\cancel{n}+1)\theta\} = \cos 2\theta \quad n\in Z.$

$= R.H.S$

16. If $x\cos\theta = y\cos\left(\theta+\frac{2\pi}{3}\right) = z\cos\left(\theta+\frac{4\pi}{3}\right)$ find the value of $xy+yz+zx$

$$x\cos\theta = y\cos\left(\theta+\frac{2\pi}{3}\right) = z\cos\left(\theta+\frac{4\pi}{3}\right) = k$$

$x\cos\theta = k \qquad y\cos\left(\theta+\frac{2\pi}{3}\right) = k \qquad z\cos\left(\theta+\frac{4\pi}{3}\right) = k$

$\frac{k}{x} = \cos\theta \qquad \frac{k}{y} = \cos\left(\theta+\frac{2\pi}{3}\right) \qquad \frac{k}{z} = \cos\left(\theta+\frac{4\pi}{3}\right)$

$$\frac{k}{x}+\frac{k}{y}+\frac{k}{z} = \cos\theta+\cos\left(\theta+\frac{2\pi}{3}\right)+\cos\left(\theta+\frac{4\pi}{3}\right)$$

$$= \cos\theta+\cos\theta\cos\frac{2\pi}{3}-\sin\theta\sin\frac{2\pi}{3}+\cos\theta\cos\frac{4\pi}{3}-\sin\theta\sin\frac{4\pi}{3} \quad \text{—— (1)}$$

$\sin 2\pi/3 = \sin 120^\circ = \sin(90+30^\circ) = \cos 30^\circ = \frac{\sqrt{3}}{2}$

$\cos\frac{2\pi}{3} = \cos 120^\circ = \cos(90+30^\circ) = -\sin 30^\circ = -\frac{1}{2}$

$\cos\frac{4\pi}{3} = \cos 240^\circ = \cos(270-30^\circ) = -\sin 30^\circ = -\frac{1}{2}$

$\sin\frac{4\pi}{3} = \sin 240^\circ = \sin(270-30^\circ) = -\cos 30^\circ = -\frac{\sqrt{3}}{2}$

sub the values in (1)

$$\frac{k}{x}+\frac{k}{y}+\frac{k}{z} = \cancel{\cos\theta}-\cancel{\frac{1}{2}\cos\theta}-\cancel{\frac{\sqrt{3}}{2}\sin\theta}-\cancel{\frac{1}{2}\cos\theta}+\cancel{\frac{\sqrt{3}}{2}\sin\theta}$$

$$= 0$$

$$\frac{1}{x}+\frac{1}{y}+\frac{1}{z} = 0$$

$$\frac{xy+yz+zx}{xyz} = 0$$

$$\boxed{xy+yz+zx = 0}$$

17. Prove that:

(i) $\sin(A+B)\sin(A-B) = \sin^2 A-\sin^2 B$

$L.H.S = (\sin A\cos B+\cos A\sin B)(\sin A\cos B-\cos A\sin B)$

$= \sin^2 A\cos^2 B-\cos^2 A\sin^2 B$

$= \sin^2 A(1-\sin^2 B)-(1-\sin^2 A)\sin^2 B$

$= \sin^2 A-\cancel{\sin^2 A\sin^2 B}-\sin^2 B+\cancel{\sin^2 A\sin^2 B}$

$= \sin^2 A-\sin^2 B = R.H.S$

$\sin(A+B)\cdot\sin(A-B) = \sin^2 A-\sin^2 B.$

(ii) $\cos(A+B)\cdot\cos(A-B) = \cos^2 A - \sin^2 B$ (or)

$= \cos^2 B - \sin^2 A$

L.H.S $= \cos(A+B)\cdot\cos(A-B)$

$= (\cos A\cos B - \sin A\sin B)(\cos A\cos B + \sin A\sin B)$

$= \cos^2 A\cos^2 B - \sin^2 A\sin^2 B$ ——— (1)

$= \cos^2 A(1-\sin^2 B) - (1-\cos^2 A)\sin^2 B$

$= \cos^2 A - \cancel{\cos^2 A\sin^2 B} - \sin^2 B + \cancel{\cos^2 A\sin^2 B}$

$= \cos^2 A - \sin^2 B$ = R.H.S

from (1)

L.H.S $= \cos^2 A\cos^2 B - \sin^2 A\sin^2 B$

$= (1-\sin^2 A)\cos^2 B - \sin^2 A(1-\cos^2 B)$

$= \cos^2 B - \cancel{\sin^2 A\cos^2 B} - \sin^2 A + \cancel{\sin^2 A\cos^2 B}$

$= \cos^2 B - \sin^2 A$ = R.H.S

Hence Proved.

$\sin(A+B)\cdot\sin(A-B) = \sin^2 A - \sin^2 B$

$= \cos^2 B - \cos^2 A$

$\cos(A+B)\cdot\cos(A-B) = \cos^2 A - \sin^2 B$

$= \cos^2 B - \sin^2 A.$

(iii) $\sin^2(A+B) - \sin^2(A-B) = \sin 2A\cdot\sin 2B$

[W.K.T $\sin^2 A - \sin^2 B = \sin(A+B)\cdot\sin(A-B)$]

L.H.S $= \sin^2(A+B) - \sin^2(A-B)$

$= \sin(A+B+A-B)\cdot\sin(A+B-\overline{A-B})$

$= \sin(2A)\cdot\sin(\cancel{A}+B-\cancel{A}+B)$

$= \sin 2A\cdot\sin 2B$

= R.H.S

(iv) $\cos 8\theta \cdot \cos 2\theta = \cos^2 5\theta - \sin^2 3\theta$

$$L.H.S = \cos 8\theta \cdot \cos 2\theta$$
$$= \cos(5\theta+3\theta) \cdot \cos(5\theta-3\theta)$$
$$= \cos^2 5\theta - \sin^2 3\theta$$
$$= R.H.S$$

18. Show that $\cos^2 A + \cos^2 B - 2\cos A \cos B \cos(A+B) = \sin^2(A+B)$.

$$L.H.S = \cos^2 A + \cos^2 B - 2\cos A \cos B \cdot \cos(A+B)$$
$$= \cos^2 A + (1-\sin^2 B) - 2\cos A \cos B \cdot \cos(A+B)$$
$$= \cos^2 A - \sin^2 B - 2\cos A \cos B \cos(A+B) + 1$$
$$= \cos(A+B) \cdot \cos(A-B) - 2\cos A \cos B \cdot \cos(A+B) + 1$$
$$= \cos(A+B) \{\cos(A-B) - 2\cos A \cos B\} + 1$$
$$= \cos(A+B) \{\cos A \cos B + \sin A \sin B - 2\cos A \cos B\} + 1$$
$$= \cos(A+B) \{-\cos A \cos B + \sin A \sin B\} + 1$$
$$= -\cos(A+B) \{\cos A \cos B - \sin A \sin B\} + 1$$
$$= -\cos(A+B) \cos(A+B) + 1$$
$$= 1 - \cos^2(A+B)$$
$$= \sin^2(A+B) = R.H.S$$

19. If $\cos(\alpha-\beta) + \cos(\beta-\gamma) + \cos(\gamma-\alpha) = -\frac{3}{2}$

Prove that $\cos\alpha + \cos\beta + \cos\gamma = \sin\alpha + \sin\beta + \sin\gamma = 0$.

$$\cos(\alpha-\beta) + \cos(\beta-\gamma) + \cos(\gamma-\alpha) = -\frac{3}{2}$$

$$\cos\alpha\cos\beta + \sin\alpha\sin\beta + \cos\beta\cos\gamma + \sin\beta\sin\gamma + \cos\gamma\cos\alpha + \sin\gamma\sin\alpha = -\frac{3}{2}$$

$$2\cos\alpha\cos\beta + 2\sin\alpha\sin\beta + 2\cos\beta\cos\gamma + 2\sin\beta\sin\gamma + 2\cos\gamma\cos\alpha + 2\sin\gamma\sin\alpha = -3$$

$$2\cos\alpha\cos\beta + 2\sin\alpha\sin\beta + 2\cos\beta\cos\gamma + 2\sin\beta\sin\gamma + 2\cos\gamma\cos\alpha + 2\sin\gamma\sin\alpha + 3 = 0$$

$$2\cos\alpha\cos\beta + 2\sin\alpha\sin\beta + 2\cos\beta\cos\gamma + 2\sin\beta\sin\gamma + 2\cos\gamma\cos\alpha + 2\sin\gamma\sin\alpha + 1 + 1 + 1 = 0$$

$$2\cos\alpha\cos\beta + 2\sin\alpha\sin\beta + 2\cos\beta\cos\gamma + 2\sin\beta\sin\gamma + 2\cos\gamma\cos\alpha + 2\sin\gamma\sin\alpha + (\sin^2\alpha + \cos^2\alpha) + (\sin^2\beta + \cos^2\beta) + (\sin^2\gamma + \cos^2\gamma) = 0$$

$$[\sin^2\alpha + \sin^2\beta + \sin^2\gamma + 2\sin\alpha\sin\beta + 2\sin\beta\sin\gamma + 2\sin\gamma\sin\alpha] + [\cos^2\alpha + \cos^2\beta + \cos^2\gamma + 2\cos\alpha\cos\beta + 2\cos\beta\cos\gamma + 2\cos\gamma\cos\alpha] = 0$$

$$[\sin\alpha + \sin\beta + \sin\gamma]^2 + [\cos\alpha + \cos\beta + \cos\gamma]^2 = 0$$

$$\sin\alpha + \sin\beta + \sin\gamma = 0 \qquad \cos\alpha + \cos\beta + \cos\gamma = 0$$

---

20. S.T

(i) $\tan(45+A) = \dfrac{1+\tan A}{1-\tan A}$

(ii) $\tan(45-A) = \dfrac{1-\tan A}{1+\tan A}$

(i) $L.H.S = \tan(45+A) = \dfrac{\tan 45 + \tan A}{1 - \tan 45 \tan A}$

$= \dfrac{1+\tan A}{1-\tan A} = R.H.S$

(ii) $L.H.S = \tan(45-A) = \dfrac{\tan 45 - \tan A}{1 + \tan 45 \tan A}$

$= \dfrac{1-\tan A}{1+\tan A} = R.H.S$

---

21. Prove that $\cot(A+B) = \dfrac{\cot A \cot B - 1}{\cot A + \cot B}$

$R.H.S = \dfrac{\cot A \cot B - 1}{\cot A + \cot B} = \dfrac{\frac{1}{\tan A}\cdot\frac{1}{\tan B} - 1}{\frac{1}{\tan A} + \frac{1}{\tan B}}$

$= \dfrac{1 - \tan A \tan B}{\tan A + \tan B} = \dfrac{1}{\tan(A+B)}$

$= \cot(A+B)$

$= L.H.S.$

22. If $\tan x = \frac{n}{n+1}$ and $\tan y = \frac{1}{2n+1}$ find $\tan(x+y)$

$$\tan(x+y) = \frac{\tan x + \tan y}{1 - \tan x \tan y}$$

$$= \frac{\frac{n}{n+1} + \frac{1}{2n+1}}{1 - \frac{n}{n+1} \cdot \frac{1}{2n+1}} = \frac{n(2n+1) + (n+1)}{(n+1)(2n+1) - n}$$

$$= \frac{2n^2+n+n+1}{2n^2+n+2n+1-n} = \frac{2n^2+2n+1}{2n^2+2n+1} = 1.$$

$$\tan(x+y) = 1.$$

23. Prove that $\tan\left(\frac{\pi}{4}+\theta\right) \cdot \tan\left(\frac{3\pi}{4}+\theta\right) = -1$

$$\tan\left(\frac{\pi}{4}+\theta\right) = \frac{1+\tan\theta}{1-\tan\theta}$$

$$\tan\left(\frac{3\pi}{4}+\theta\right) = \tan\left(\pi - \frac{\pi}{4} - \theta\right)$$

$$= -\tan\left(\frac{\pi}{4}-\theta\right) = -\left(\frac{1-\tan\theta}{1+\tan\theta}\right)$$

$$\tan\left(\frac{\pi}{4}+\theta\right) \tan\left(\frac{3\pi}{4}+\theta\right) = -\left(\frac{1+\tan\theta}{1-\tan\theta}\right)\left(\frac{1-\tan\theta}{1+\tan\theta}\right)$$

$$= -1$$

Hence Proved.

24. Find the values of $\tan(\alpha+\beta)$, given that $\cot\alpha = \frac{1}{2}$ $\alpha \in \left(\pi, \frac{3\pi}{2}\right)$ and $\sec\beta = -\frac{5}{3}$ $\beta \in \left(\frac{\pi}{2}, \pi\right)$

$\cot\alpha = \frac{1}{2}$

$\tan\alpha = 2$

$\sec\beta = -\frac{5}{3}$

$\sec^2\beta = \frac{25}{9}$

$\sec^2\beta - 1 = \frac{16}{9}$

$\tan^2\beta = \frac{16}{9}$

$\tan\beta = -\frac{4}{3}$ β lies in II quad

$$\tan(\alpha+\beta) = \frac{\tan\alpha + \tan\beta}{1 - \tan\alpha \tan\beta}$$

$$= \frac{2-\frac{4}{3}}{1+2(4/3)} = \frac{6-4}{3+8} = \frac{2}{11}$$

$$\tan(\alpha+\beta) = \frac{2}{11}.$$

25. If $\theta+\varphi=\alpha$ and $\tan\theta = k\tan\phi$ then Prove that $\sin(\theta-\varphi) = \frac{k-1}{k+1}\cdot\sin\alpha$

$\theta+\phi=\alpha$

$$\tan\theta = k\tan\varphi$$

$$\frac{\tan\theta}{\tan\varphi} = \frac{k}{1}$$

$$\frac{\tan\theta+\tan\varphi}{\tan\theta-\tan\varphi} = \frac{k+1}{k-1}$$

$$\frac{\frac{\sin\theta}{\cos\theta}+\frac{\sin\varphi}{\cos\varphi}}{\frac{\sin\theta}{\cos\theta}-\frac{\sin\varphi}{\cos\varphi}} = \frac{k+1}{k-1}$$

$$\frac{\sin\theta\cos\varphi+\cos\theta\cdot\sin\varphi}{\sin\theta\cos\varphi-\cos\theta\cdot\sin\varphi} = \frac{k+1}{k-1}$$

$$\frac{\sin(\theta+\varphi)}{\sin(\theta-\varphi)} = \frac{k+1}{k-1} \qquad [\theta+\varphi=\alpha]$$

$$\frac{\sin\alpha}{\sin(\theta-\varphi)} = \frac{k+1}{k-1}$$

$$\therefore \sin(\theta-\varphi) = \frac{k-1}{k+1}\cdot\sin\alpha$$

Hence Proved

3.15 Find the values of (i) $\cos 15^\circ$ (ii) $\tan 165^\circ$

(i) $\cos 15^\circ = \cos(45-30^\circ)$

$$= \cos 45^\circ\cos 30^\circ + \sin 45\cdot\sin 30^\circ$$

$$= \frac{1}{\sqrt{2}}\cdot\frac{\sqrt{3}}{2} + \frac{1}{\sqrt{2}}\cdot\frac{1}{2} = \frac{\sqrt{3}+1}{2\sqrt{2}}$$

(ii) $\tan 165^\circ = \tan(180-15^\circ) = -\tan 15^\circ$

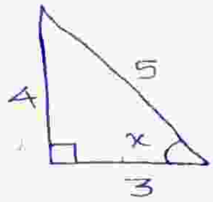
$$= -\tan(45-30^\circ) = -\left\{\frac{\tan 45-\tan 30^\circ}{1+\tan 45\tan 30^\circ}\right.$$

$$= -\left\{\frac{1-\frac{1}{\sqrt{3}}}{1+\frac{1}{\sqrt{3}}}\right\} = -\left\{\frac{\sqrt{3}-1}{\sqrt{3}+1}\right\} = \frac{1-\sqrt{3}}{1+\sqrt{3}}.$$

3.16. If $\sin x = \frac{4}{5}$ $x \in (0, \frac{\pi}{2})$ and $\cos y = \frac{-12}{13}$ $y \in (\frac{\pi}{2}, \pi)$ then find (i) $\sin(x-y)$ (ii) $\cos(x-y)$

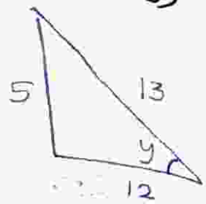
$\sin x = \frac{4}{5}$

$\cos x = \frac{3}{5}$



$\cos y = -\frac{12}{13}$

$\sin y = \frac{5}{13}$



(i) $\sin(x-y) = \sin x \cos y + \cos x \sin y$

$= \frac{4}{5} \cdot \left(\frac{-12}{13}\right) - \frac{3}{5}\left(\frac{5}{13}\right)$

$= \frac{-48}{65} - \frac{15}{65} = \frac{-63}{65}.$

(ii) $\cos(x-y) = \cos x \cos y + \sin x \sin y$

$= \frac{3}{5}\left(\frac{-12}{13}\right) + \frac{4}{5}\left(\frac{5}{13}\right)$

$= -\frac{36}{65} + \frac{20}{65} = \frac{-16}{65}.$

3.17. Prove that $\cos\left(\frac{3\pi}{4}+x\right) - \cos\left(\frac{3\pi}{4}-x\right) = -\sqrt{2}\sin x$

$$\boxed{\cos(A+B) - \cos(A-B) = -2\sin A \sin B}$$

$\cos\left(\frac{3\pi}{4}+x\right) - \cos\left(\frac{3\pi}{4}-x\right) = -2\sin\frac{3\pi}{4} \cdot \sin x$

$= -2 \cdot \sin\left(\pi - \frac{\pi}{4}\right) \cdot \sin x$

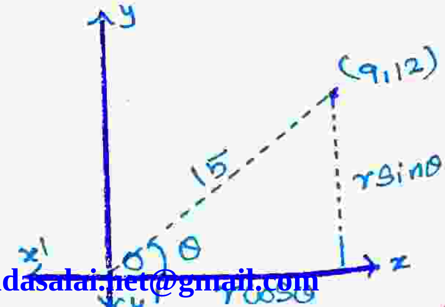
$= -2 \sin\frac{\pi}{4} \sin x$

$= -2 \cdot \frac{1}{\sqrt{2}} \sin x = -\sqrt{2}\sin x$

$= R.H.S.$

3.18 Point (9, 12) rotates around the origin in a plane through 60° in the anticlockwise direction to a new position B.

Find the co-ordinates of the point B.



Let $A(9,12) = A(r\cos\theta, r\sin\theta)$

$r = \sqrt{9^2+12^2} = \sqrt{81+144} = \sqrt{225} = 15$ units

$\therefore A(15\cos\theta, 15\sin\theta)$

Now the Point B is given by

$B(15\cos(\theta+60^\circ), 15\sin(\theta+60^\circ))$

$15\cos(\theta+60^\circ) = 15\{\cos\theta\cos60^\circ - \sin\theta\sin60^\circ\}$

$= 15\cos\theta \cdot \cos60^\circ - 15\sin\theta \cdot \sin60^\circ$

$= 9 \cdot \frac{1}{2} - 12 \cdot \frac{\sqrt{3}}{2} = \frac{9}{2} - 6\sqrt{3}$

$= \frac{3}{2}(3-4\sqrt{3})$

$15\sin(\theta+60^\circ) = 15(\sin\theta\cos60^\circ + \cos\theta\sin60^\circ)$

$= 15\sin\theta \cdot \cos60^\circ + 15\cos\theta \cdot \sin60^\circ$

$= 12 \cdot \frac{1}{2} + 9 \cdot \frac{\sqrt{3}}{2} = 6 + \frac{9\sqrt{3}}{2}$

$= \frac{3}{2}(4+3\sqrt{3})$

$\therefore$ The new Position B is $\left(\frac{3}{2}(3-4\sqrt{3}), \frac{3}{2}(4+\sqrt{3})\right)$

3.19 A ripple tank demonstrates the effect of two water waves being added together. The two waves are described by $h = 8\cos t$ and $h = 6\sin t$ where $t \in [0, 2\pi)$ is in seconds and h is the height in millimeters above still water. Find the maximum height of the resultant wave and the value of 't' at which it occurs.

Let H be the height of the resultant wave at any time 't'

$\therefore H = 8\cos t + 6\sin t = k\cos(t-\alpha)$

$= k[\cos t\cos\alpha + \sin t\sin\alpha]$

$k\cos\alpha = 8$ $\quad$ $k\sin\alpha = 6$

$k^2\cos^2\alpha = 64$ $\quad$ $k^2\sin^2\alpha = 36$

$k^2 = 64+36 = 100$ $\quad$ $\boxed{k=10}$

$\cos\alpha = \frac{8}{10} = \frac{4}{5}$ $\quad$ $\sin\alpha = \frac{6}{10} = \frac{3}{4}$

$\tan\alpha = \frac{3}{4}$

$\therefore H = 10\cos(t-\alpha)$

Thus, the maximum of $H = 10$ mm.

The maximum occurs when $t = \alpha$

where $\tan\alpha = \frac{3}{4}$.

3.20 Expand: (i) $\sin(A+B+C)$ (ii) $\tan(A+B+C)$

$\sin(A+B+C) = \sin(A + \overline{B+C})$

$= \sin A\cos(B+C) + \cos A\sin(B+C)$

$= \sin A[\cos B\cos C - \sin B\sin C] + \cos A[\sin B\cos C + \cos B\sin C]$

$= \sin A\cos B\cos C - \sin A\sin B\sin C + \cos A\sin B\cos C + \cos A\cos B\sin C$

(ii) $\tan(A+B+C) = \tan(A + \overline{B+C})$

$$= \frac{\tan A + \tan(B+C)}{1 - \tan A\tan(B+C)}$$

$$= \frac{\tan A + \left\{\frac{\tan B + \tan C}{1-\tan B\tan C}\right\}}{1 - \tan A\left\{\frac{\tan B + \tan C}{1 - \tan B\tan C}\right\}}$$

$$= \frac{\tan A - \tan A\tan B\tan C + \tan B + \tan C}{1 - \tan B\tan C - \tan A(\tan B + \tan C)}$$

$$= \frac{\tan A + \tan B + \tan C - \tan A\tan B\tan C}{1 - \tan A\tan B - \tan B\tan C - \tan C\tan A}.$$

12. Prove that $\sin 75^\circ - \sin 15^\circ = \cos 105^\circ + \cos 15^\circ$

$$R.H.S = \cos 105^\circ + \cos 15^\circ$$
$$= \cos(90+15^\circ) + \cos(90-75^\circ)$$
$$= -\sin 15^\circ + \sin 75^\circ$$
$$= \sin 75^\circ - \sin 15^\circ$$
$$= L.H.S.$$

13. Show that $\tan 75^\circ + \cot 75^\circ = 4$

we know that

$$\tan 75^\circ = 2+\sqrt{3} \text{ (or) } \frac{\sqrt{3}+1}{\sqrt{3}-1}$$
$$\cot 75^\circ = \tan 15^\circ = 2-\sqrt{3} \text{ (or) } \frac{\sqrt{3}-1}{\sqrt{3}+1}$$
$$\tan 75^\circ + \cot 75^\circ = 2+\sqrt{3}+2-\sqrt{3} = 4 = R.H.S.$$

14. Prove that $\cos(A+B)\cdot\cos C - \cos(B+C)\cos A = \sin B \sin(C-A)$

$$L.H.S = \cos(A+B)\cdot\cos C - \cos(B+C)\cdot\cos A$$
$$= (\cos A\cos B - \sin A\sin B)\cdot\cos C - [\cos B\cos C - \sin B\sin C]\cos A$$
$$= \cancel{\cos A\cos B\cos C} - \sin A\sin B\cos C - \cancel{\cos A\cos B\cos C} + \cos A\sin B\sin C$$
$$= \cos A\sin B\sin C - \sin A\sin B\cos C$$
$$= \sin B\{\sin C\cos A - \cos C\cdot\sin A\}$$
$$= \sin B\sin(C-A) = R.H.S$$

15. Prove that $\sin(n+1)\theta\sin(n-1)\theta + \cos(n+1)\theta\cos(n-1)\theta = \cos 2\theta,\ n\in Z.$

$$\boxed{\sin A\sin B + \cos A\cos B = \cos(A-B)}$$

$$L.H.S = \sin(n+1)\theta\sin(n-1)\theta + \cos(n+1)\theta\cdot\cos(n-1)\theta$$
$$= \cos\{(n+1)\theta - (n-1)\theta\}$$
$$= \cos\{(\cancel{n}+1-\cancel{n}+1)\theta\} = \cos 2\theta \quad n\in Z.$$
$$= R.H.S$$

16\. If $x\cos\theta = y\cos\left(\theta+\frac{2\pi}{3}\right) = z\cos\left(\theta+\frac{4\pi}{3}\right)$ find the value of $xy+yz+zx$

$$x\cos\theta = y\cos\left(\theta+\frac{2\pi}{3}\right) = z\cos\left(\theta+\frac{4\pi}{3}\right) = k$$

$x\cos\theta = k$ $\quad y\cos\left(\theta+\frac{2\pi}{3}\right) = k$ $\quad z\cos\left(\theta+\frac{4\pi}{3}\right) = k$

$\frac{k}{x} = \cos\theta$ $\quad \frac{k}{y} = \cos\left(\theta+\frac{2\pi}{3}\right)$ $\quad \frac{k}{z} = \cos\left(\theta+\frac{4\pi}{3}\right)$

$$\frac{k}{x}+\frac{k}{y}+\frac{k}{z} = \cos\theta+\cos\left(\theta+\frac{2\pi}{3}\right)+\cos\left(\theta+\frac{4\pi}{3}\right)$$

$$= \cos\theta+\cos\theta\cos\frac{2\pi}{3}-\sin\theta\sin\frac{2\pi}{3}+\cos\theta\cos\frac{4\pi}{3} - \sin\theta\sin\frac{4\pi}{3} \quad \text{--- (1)}$$

$\sin 2\pi/3 = \sin 120^\circ = \sin(90+30^\circ) = \cos 30^\circ = \frac{\sqrt{3}}{2}$

$\cos\frac{2\pi}{3} = \cos 120^\circ = \cos(90+30^\circ) = -\sin 30^\circ = -\frac{1}{2}$

$\cos\frac{4\pi}{3} = \cos 240^\circ = \cos(270-30^\circ) = -\sin 30^\circ = -\frac{1}{2}$

$\sin\frac{4\pi}{3} = \sin 240^\circ = \sin(270-30^\circ) = -\cos 30^\circ = -\frac{\sqrt{3}}{2}$

sub the values in (1)

$$\frac{k}{x}+\frac{k}{y}+\frac{k}{z} = \cancel{\cos\theta} - \cancel{\frac{1}{2}\cos\theta} - \cancel{\frac{\sqrt{3}}{2}\sin\theta} - \cancel{\frac{1}{2}\cos\theta} + \cancel{\frac{\sqrt{3}}{2}\sin\theta}$$

$$= 0$$

$$\frac{1}{x}+\frac{1}{y}+\frac{1}{z} = 0$$

$$\frac{xy+yz+zx}{xyz} = 0$$

$$\boxed{xy+yz+zx = 0}$$

17\. Prove that:

(i) $\sin(A+B)\sin(A-B) = \sin^2A-\sin^2B$

$$L.H.S = (\sin A\cos B+\cos A\sin B)(\sin A\cos B-\cos A\sin B)$$

$$= \sin^2A\cos^2B - \cos^2A\sin^2B$$

$$= \sin^2A(1-\sin^2B) - (1-\sin^2A)\sin^2B$$

$$= \sin^2A - \cancel{\sin^2A\sin^2B} - \sin^2B + \cancel{\sin^2A\sin^2B}$$

$$= \sin^2A - \sin^2B = R.H.S$$

$$\sin(A+B)\cdot\sin(A-B) = \sin^2A-\sin^2B.$$

(ii) $\cos(A+B)\cdot\cos(A-B) = \cos^2A - \sin^2B$ (or)

$= \cos^2B - \sin^2A$

$L.H.S = \cos(A+B)\cdot\cos(A-B)$

$= (\cos A\cos B - \sin A\sin B)(\cos A\cos B + \sin A\sin B)$

$= \cos^2A\cos^2B - \sin^2A\sin^2B$ ——— ①

$= \cos^2A(1-\sin^2B) - (1-\cos^2A)\sin^2B$

$= \cos^2A - \cancel{\cos^2A\sin^2B} - \sin^2B + \cancel{\cos^2A\sin^2B}$

$= \cos^2A - \sin^2B = R.H.S$

from ①

$L.H.S = \cos^2A\cos^2B - \sin^2A\sin^2B$

$= (1-\sin^2A)\cos^2B - \sin^2A(1-\cos^2B)$

$= \cos^2B - \cancel{\sin^2A\cos^2B} - \sin^2A + \cancel{\sin^2A\cos^2B}$

$= \cos^2B - \sin^2A = R.H.S$

Hence Proved.

$\boxed{\begin{aligned}\sin(A+B)\cdot\sin(A-B) &= \sin^2A - \sin^2B\\ &= \cos^2B - \cos^2A\\ \cos(A+B)\cdot\cos(A-B) &= \cos^2A - \sin^2B\\ &= \cos^2B - \sin^2A.\end{aligned}}$

(iii) $\sin^2(A+B) - \sin^2(A-B) = \sin 2A\cdot\sin 2B$

[W.K.T $\sin^2A - \sin^2B = \sin(A+B)\cdot\sin(A-B)$]

$L.H.S = \sin^2(A+B) - \sin^2(A-B)$

$= \sin(A+B+A-B)\cdot\sin(A+B-\overline{A-B})$

$= \sin(2A)\cdot\sin(\cancel{A}+B-\cancel{A}+B)$

$= \sin 2A\cdot\sin 2B$

$= R.H.S$

(iv) $\cos 8\theta \cdot \cos 2\theta = \cos^2 5\theta - \sin^2 3\theta$

$L.H.S = \cos 8\theta \cdot \cos 2\theta$

$= \cos(5\theta + 3\theta) \cdot \cos(5\theta - 3\theta)$

$= \cos^2 5\theta - \sin^2 3\theta$

$= R.H.S$

18. Show that $\cos^2 A + \cos^2 B - 2\cos A \cos B \cos(A+B) = \sin^2(A+B)$.

$L.H.S = \cos^2 A + \cos^2 B - 2\cos A \cos B \cdot \cos(A+B)$

$= \cos^2 A + (1 - \sin^2 B) - 2\cos A \cos B \cdot \cos(A+B)$

$= \cos^2 A - \sin^2 B - 2\cos A \cos B \cos(A+B) + 1$

$= \cos(A+B) \cdot \cos(A-B) - 2\cos A \cos B \cdot \cos(A+B) + 1$

$= \cos(A+B)\{\cos(A-B) - 2\cos A \cos B\} + 1$

$= \cos(A+B)\{\cos A \cos B + \sin A \sin B - 2\cos A \cos B\} + 1$

$= \cos(A+B)\{-\cos A \cos B + \sin A \sin B\} + 1$

$= -\cos(A+B)\{\cos A \cos B - \sin A \sin B\} + 1$

$= -\cos(A+B) \cos(A+B) + 1$

$= 1 - \cos^2(A+B)$

$= \sin^2(A+B) = R.H.S$

19. If $\cos(\alpha - \beta) + \cos(\beta - \gamma) + \cos(\gamma - \alpha) = -\frac{3}{2}$

Prove that $\cos\alpha + \cos\beta + \cos\alpha = \sin\alpha + \sin\beta + \sin\gamma = 0$

$\cos(\alpha - \beta) + \cos(\beta - \gamma) + \cos(\gamma - \alpha) = -\frac{3}{2}$

$\cos\alpha\cos\beta + \sin\alpha\sin\beta + \cos\beta\cos\gamma + \sin\beta\sin\gamma + \cos\gamma\cos\alpha + \sin\gamma\sin\alpha = -\frac{3}{2}$

$2\cos\alpha\cos\beta + 2\sin\alpha\sin\beta + 2\cos\beta\cos\gamma + 2\sin\beta\sin\gamma + 2\cos\gamma\cos\alpha + 2\sin\gamma\sin\alpha = -3$

$2\cos\alpha\cos\beta + 2\sin\alpha\sin\beta + 2\cos\beta\cos\gamma + 2\sin\beta\sin\gamma + 2\cos\gamma\cos\alpha + 2\sin\gamma\sin\alpha + 3 = 0$

$2\cos\alpha\cos\beta + 2\sin\alpha\sin\beta + 2\cos\beta\cos\gamma + 2\sin\beta\sin\gamma + 2\cos\gamma\cos\alpha + 2\sin\gamma\sin\alpha + 1 + 1 + 1 = 0$

$2\cos\alpha\cos\beta + 2\sin\alpha\sin\beta + 2\cos\beta\cos\gamma + 2\sin\beta\sin\gamma + 2\cos\gamma\cos\alpha + 2\sin\gamma\sin\alpha + (\sin^2\alpha + \cos^2\alpha) + (\sin^2\beta + \cos^2\beta) + (\sin^2\gamma + \cos^2\gamma) = 0$

$[\sin^2\alpha + \sin^2\beta + \sin^2\gamma + 2\sin\alpha\sin\beta + 2\sin\beta\sin\gamma + 2\sin\gamma\sin\alpha] + [\cos^2\alpha + \cos^2\beta + \cos^2\gamma + 2\cos\alpha\cos\beta + 2\cos\beta\cos\gamma + 2\cos\gamma\cos\alpha] = 0$

$[\sin\alpha + \sin\beta + \sin\gamma]^2 + [\cos\alpha + \cos\beta + \cos\gamma]^2 = 0$

$\sin\alpha + \sin\beta + \sin\gamma = 0 \qquad \cos\alpha + \cos\beta + \cos\gamma = 0$

20. S.T (i) $\tan(45+A) = \dfrac{1+\tan A}{1-\tan A}$

(ii) $\tan(45-A) = \dfrac{1-\tan A}{1+\tan A}$

(i) $L.H.S = \tan(45+A) = \dfrac{\tan 45 + \tan A}{1 - \tan 45 \tan A}$

$= \dfrac{1+\tan A}{1-\tan A} = R.H.S$

(ii) $L.H.S = \tan(45-A) = \dfrac{\tan 45 - \tan A}{1 + \tan 45 \tan A}$

$= \dfrac{1-\tan A}{1+\tan A} = R.H.S$

21. Prove that $\cot(A+B) = \dfrac{\cot A \cot B - 1}{\cot A + \cot B}$

$R.H.S = \dfrac{\cot A \cot B - 1}{\cot A + \cot B} = \dfrac{\frac{1}{\tan A} \cdot \frac{1}{\tan B} - 1}{\frac{1}{\tan A} + \frac{1}{\tan B}}$

$= \dfrac{1 - \tan A \tan B}{\tan A + \tan B} = \dfrac{1}{\tan(A+B)}$

$= \cot(A+B)$

$= L.H.S.$

22. If $\tan x = \frac{n}{n+1}$ and $\tan y = \frac{1}{2n+1}$ find $\tan(x+y)$

$$\tan(x+y) = \frac{\tan x + \tan y}{1 - \tan x \tan y}$$

$$= \frac{\frac{n}{n+1} + \frac{1}{2n+1}}{1 - \frac{n}{n+1}\cdot\frac{1}{2n+1}} = \frac{n(2n+1) + (n+1)}{(n+1)(2n+1) - n}$$

$$= \frac{2n^2+n+n+1}{2n^2+n+2n+1-n} = \frac{2n^2+2n+1}{2n^2+2n+1} = 1.$$

$$\tan(x+y) = 1.$$

23. Prove that $\tan\left(\frac{\pi}{4}+\theta\right)\cdot\tan\left(\frac{3\pi}{4}+\theta\right) = -1$

$$\tan\left(\frac{\pi}{4}+\theta\right) = \frac{1+\tan\theta}{1-\tan\theta}$$

$$\tan\left(\frac{3\pi}{4}+\theta\right) = \tan\left(\pi - \frac{\pi}{4} - \theta\right)$$

$$= -\tan\left(\frac{\pi}{4}-\theta\right) = -\left(\frac{1-\tan\theta}{1+\tan\theta}\right)$$

$$\tan\left(\frac{\pi}{4}+\theta\right)\cdot\tan\left(\frac{3\pi}{4}+\theta\right) = -\left(\frac{1+\tan\theta}{1-\tan\theta}\right)\left(\frac{1-\tan\theta}{1+\tan\theta}\right)$$

$$= -1$$

Hence Proved.

24. Find the values of $\tan(\alpha+\beta)$, given that $\cot\alpha = \frac{1}{2}$ $\alpha \in \left(\pi, \frac{3\pi}{2}\right)$ and $\sec\beta = -\frac{5}{3}$ $\beta \in \left(\frac{\pi}{2}, \pi\right)$

$\cot\alpha = \frac{1}{2}$

$\tan\alpha = 2$

$\sec\beta = -\frac{5}{3}$

$\sec^2\beta = \frac{25}{9}$

$\sec^2\beta - 1 = \frac{16}{9}$

$\tan^2\beta = \frac{16}{9}$

$\tan\beta = -\frac{4}{3}$ β lies in II quad

$$\tan(\alpha+\beta) = \frac{\tan\alpha + \tan\beta}{1 - \tan\alpha\tan\beta}$$

$$= \frac{2-\frac{4}{3}}{1+2(4/3)} = \frac{6-4}{3+8} = \frac{2}{11}$$

$$\tan(\alpha+\beta) = \frac{2}{11}.$$

25. If $\theta+\varphi=\alpha$ and $\tan\theta = k\tan\varphi$ then Prove that $\sin(\theta-\varphi) = \frac{k-1}{k+1}\cdot\sin\alpha$

$\theta+\varphi=\alpha$

$$\tan\theta = k\tan\varphi$$

$$\frac{\tan\theta}{\tan\varphi} = \frac{k}{1}$$

$$\frac{\tan\theta+\tan\varphi}{\tan\theta-\tan\varphi} = \frac{k+1}{k-1}$$

$$\frac{\frac{\sin\theta}{\cos\theta}+\frac{\sin\varphi}{\cos\varphi}}{\frac{\sin\theta}{\cos\theta}-\frac{\sin\varphi}{\cos\varphi}} = \frac{k+1}{k-1}$$

$$\frac{\sin\theta\cos\varphi+\cos\theta.\sin\varphi}{\sin\theta\cos\varphi-\cos\theta.\sin\varphi} = \frac{k+1}{k-1}$$

$$\frac{\sin(\theta+\varphi)}{\sin(\theta-\varphi)} = \frac{k+1}{k-1} \qquad [\theta+\varphi=\alpha]$$

$$\frac{\sin\alpha}{\sin(\theta-\varphi)} = \frac{k+1}{k-1}$$

$$\therefore \sin(\theta-\varphi) = \frac{k-1}{k+1}\cdot\sin\alpha$$

Hence Proved

3.15 Find the values of (i) $\cos 15°$ (ii) $\tan 165°$

(i) $\cos 15° = \cos(45-30°)$

$$= \cos 45°\cos 30° + \sin 45.\sin 30$$

$$= \frac{1}{\sqrt2}\cdot\frac{\sqrt3}{2} + \frac{1}{\sqrt2}\cdot\frac{1}{2} = \frac{\sqrt3+1}{2\sqrt2}$$

(ii) $\tan 165° = \tan(180-15°) = -\tan 15°$

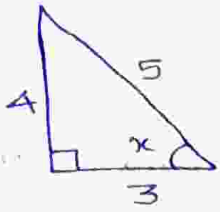
$$= -\tan(45-30°) = -\left\{\frac{\tan 45-\tan 30°}{1+\tan 45\tan 30°}\right\}$$

$$= -\left\{\frac{1-\frac{1}{\sqrt3}}{1+\frac{1}{\sqrt3}}\right\} = -\left\{\frac{\sqrt3-1}{\sqrt3+1}\right\} = \frac{1-\sqrt3}{1+\sqrt3}.$$

3.16. If $\sin x = \frac{4}{5}$ $x \in (0, \frac{\pi}{2})$ and $\cos y = \frac{-12}{13}$ $y \in (\frac{\pi}{2}, \pi)$ then find (i) $\sin(x-y)$ (ii) $\cos(x-y)$

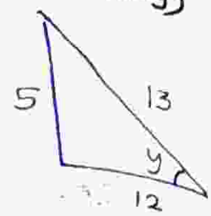
$\sin x = \frac{4}{5}$

$\cos x = \frac{3}{5}$



$\cos y = -\frac{12}{13}$

$\sin y = \frac{5}{13}$



(i) $\sin(x-y) = \sin x \cos y - \cos x \sin y$

$= \frac{4}{5} \cdot \left(\frac{-12}{13}\right) - \frac{3}{5}\left(\frac{5}{13}\right)$

$= \frac{-48}{65} - \frac{15}{65} = \frac{-63}{65}$.

(ii) $\cos(x-y) = \cos x \cos y + \sin x \sin y$

$= \frac{3}{5}\left(\frac{-12}{13}\right) + \frac{4}{5}\left(\frac{5}{13}\right)$

$= -\frac{36}{65} + \frac{20}{65} = \frac{-16}{65}$.

3.17. Prove that $\cos\left(\frac{3\pi}{4} + x\right) - \cos\left(\frac{3\pi}{4} - x\right) = -\sqrt{2}\sin x$

$\boxed{\cos(A+B) - \cos(A-B) = -2\sin A \sin B}$

$\cos\left(\frac{3\pi}{4} + x\right) - \cos\left(\frac{3\pi}{4} - x\right) = -2\sin\frac{3\pi}{4} \cdot \sin x$

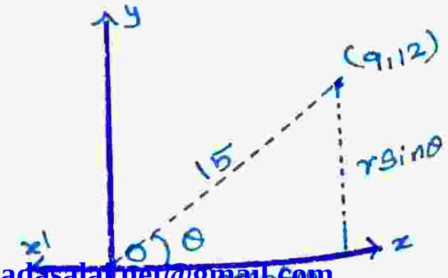
$= -2 \cdot \sin\left(\pi - \frac{\pi}{4}\right) \cdot \sin x$

$= -2\sin\frac{\pi}{4}\sin x$

$= -2 \cdot \frac{1}{\sqrt{2}}\sin x = -\sqrt{2}\sin x$

$=$ R.H.S.

3.18 Point (9, 12) rotates around the origin in a Plane through 60° in the anticlockwise direction to a new Position B. Find the Co-ordinates of the Point B.



Let $A(9,12) = A(r\cos\theta, r\sin\theta)$

$$r = \sqrt{9^2+12^2} = \sqrt{81+144} = \sqrt{225} = 15 \text{ units}$$

$\therefore A(15\cos\theta, 15\sin\theta)$

Now the Point B is given by

$$B\left(15\cos(\theta+60^\circ),\ 15\sin(\theta+60^\circ)\right)$$

$$15\cos(\theta+60^\circ) = 15\{\cos\theta\cos 60^\circ - \sin\theta\sin 60^\circ\}$$
$$= 15\cos\theta\cdot\cos 60^\circ - 15\sin\theta\cdot\sin 60^\circ$$
$$= 9\cdot\frac{1}{2} - 12\cdot\frac{\sqrt{3}}{2} = \frac{9}{2} - 6\sqrt{3}$$
$$= \frac{3}{2}(3-4\sqrt{3})$$

$$15\sin(\theta+60^\circ) = 15(\sin\theta\cos 60^\circ + \cos\theta\sin 60^\circ)$$
$$= 15\sin\theta\cdot\cos 60^\circ + 15\cos\theta\cdot\sin 60^\circ$$
$$= 12\cdot\frac{1}{2} + 9\cdot\frac{\sqrt{3}}{2} = 6+\frac{9\sqrt{3}}{2}$$
$$= \frac{3}{2}(4+3\sqrt{3})$$

$\therefore$ The new Position B is $\left(\frac{3}{2}(3-4\sqrt{3}, \frac{3}{2}(4+\sqrt{}\right.$

3.19 A ripple tank demonstates the effect of two water waves being added together. The two waves are described by $h = 8\cos t$ and $h = 6\sin t$ where $t \in [0, 2\pi)$ is in seconds and h is the height in millimeters above still water. Find the maximum height of the resultant wave and the value of 't' at which it occurs.

Let H be the height of the resultant wave at any time 't'

$$\therefore H = 8\cos t + 6\sin t = k\cos(t-\alpha)$$
$$= k[\cos t\cos\alpha + \sin t\sin\alpha]$$

$k\cos\alpha = 8$ $\qquad$ $k\sin\alpha = 6$

$k^2\cos^2\alpha = 64$ $\qquad$ $k^2\sin^2\alpha = 36$

$$k^2 = 64+36 = 100 \qquad \boxed{k=10}$$

$\cos\alpha = \frac{8}{10} = \frac{4}{5}$ $\qquad$ $\sin\alpha = \frac{6}{10} = \frac{3}{4}$

$$\tan\alpha = \frac{3}{4}$$

$\therefore H = 10\cos(t-\alpha)$

Thus, the maximum of $H = 10$ mm.

The maximum occurs when $t = \alpha$ where $\tan\alpha = \frac{3}{4}$.

3.20 Expand: (i) $\sin(A+B+C)$ (ii) $\tan(A+B+C)$

$$\sin(A+B+C) = \sin(A+\overline{B+C})$$

$$= \sin A\cos(B+C) + \cos A\sin(B+C)$$

$$= \sin A[\cos B\cos C - \sin B\sin C] + \cos A[\sin B\cos C + \cos B\sin C]$$

$$= \sin A\cos B\cos C - \sin A\sin B\sin C + \cos A\sin B\cos C + \cos A\cos B\sin C$$

(ii) $\tan(A+B+C) = \tan(A+\overline{B+C})$

$$= \frac{\tan A + \tan(B+C)}{1-\tan A\tan(B+C)}$$

$$= \frac{\tan A + \left\{\frac{\tan B+\tan C}{1-\tan B\tan C}\right\}}{1-\tan A\left\{\frac{\tan B+\tan C}{1-\tan B\tan C}\right\}}$$

$$= \frac{\tan A - \tan A\tan B\tan C + \tan B + \tan C}{1-\tan B\tan C - \tan A(\tan B+\tan C)}$$

$$= \frac{\tan A + \tan B + \tan C - \tan A\tan B\tan C}{1-\tan A\tan B - \tan B\tan C - \tan C\tan A}.$$

Multiple Angle Identities :-

$$\sin 2A = 2\sin A \cos A$$

$$\cos 2A = 2\cos^2 A - 1$$
$$= 1 - 2\sin^2 A$$
$$= \cos^2 A - \sin^2 A$$

$$\tan 2A = \frac{2\tan A}{1+\tan^2 A}$$

$$1+\cos 2A = 2\cos^2 A$$
$$\cos^2 A = \frac{1+\cos 2A}{2}$$

$$1-\cos 2A = 2\sin^2 A$$
$$\sin^2 A = \frac{1-\cos 2A}{2}$$

$$\sin 2A = \frac{2\tan A}{1+\tan^2 A}$$

$$\cos 2A = \frac{1-\tan^2 A}{1+\tan^2 A}.$$

$$\tan^2 A = \frac{1-\cos 2A}{1+\cos 2A}$$

Half Angle Identities :-

$$\sin A = 2\sin\frac{A}{2}.\cos\frac{A}{2}$$

$$\cos A = \cos^2\frac{A}{2} - \sin^2\frac{A}{2}$$
$$= 2\cos^2\frac{A}{2} - 1$$
$$= 1 - 2\sin^2 A/2$$

$$\tan A = \frac{2\tan A/2}{1-\tan^2 A/2}.$$

$$1+\cos A = 2\cos^2 A/2$$
$$\cos^2 A/2 = \frac{1+\cos A}{2}$$

$$1-\cos A = 2\sin^2 A/2$$
$$\sin^2 A/2 = \frac{1-\cos A}{2}$$

$$\sin A = \frac{2\tan A/2}{1+\tan^2 A/2}$$

$$\cos A = \frac{1-\tan^2 A/2}{1+\tan^2 A/2}$$

$$\tan^2 A/2 = \frac{1-\cos A}{1+\cos A}.$$

Triple Angle Identities:

$$\sin 3A = 3\sin A - 4\sin^3 A$$

$$\cos 3A = 4\cos^3 A - 3\cos A$$

$$\tan 3A = \frac{3\tan A - \tan^3 A}{1 - 3\tan^2 A}.$$

$$\sin^3 A = \frac{1}{4}\left[3\sin A - \sin 3A\right]$$

$$\cos^3 A = \frac{1}{4}\left[3\cos A + \cos 3A\right]$$

3.22 Find the value of $\sin 22\frac{1}{2}^\circ$

Let $\theta = 45^\circ$

We know that

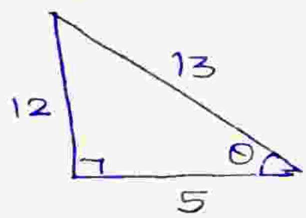
$$\sin^2 \theta/2 = \frac{1-\cos\theta}{2}$$

$$\sin^2 22\tfrac{1}{2}^\circ = \frac{1-\cos 45^\circ}{2} = \frac{1-\frac{1}{\sqrt{2}}}{2}$$

$$= \frac{\sqrt{2}-1}{2\sqrt{2}} \times \frac{\sqrt{2}}{\sqrt{2}} = \frac{2-\sqrt{2}}{4}$$

$$\sin 22\tfrac{1}{2}^\circ = \sqrt{\frac{2-\sqrt{2}}{4}} = \frac{\sqrt{2-\sqrt{2}}}{2}$$

3.23 Find the value of $\sin 2\theta$ when $\sin\theta = \frac{12}{13}$. θ lies in the first quadrant.



$$\sin\theta = \frac{12}{13}$$

$$\cos\theta = \frac{5}{13}$$

$$\sin 2\theta = 2\sin\theta\cos\theta$$

$$= 2 \cdot \frac{12}{13} \cdot \frac{5}{13} = \frac{120}{169}$$

$$\sin 2\theta = \frac{120}{169}.$$

3.24 Prove that $\sin 4A = 4\sin A\cos^3 A - 4\cos A \cdot \sin^3 A$

$$R.H.S = 4\sin A\cos^3 A - 4\cos A\sin^3 A$$
$$= 4\sin A\cos A(\cos^2 A - \sin^2 A)$$
$$= 2(2\sin A\cos A)(\cos^2 A - \sin^2 A)$$
$$= 2(\sin 2A)(\cos 2A) = \sin 2(2A)$$
$$= \sin 4A = L.H.S$$

3.25 Prove that $\sin x = 2^{10} \cdot \sin\left(\frac{x}{2^{10}}\right)\cos\left(\frac{x}{2}\right)\left(\cos\frac{x}{2^2}\right)\ldots \cos\left(\frac{x}{2^{10}}\right)$

$$L.H.S = \sin x$$
$$= 2\sin\frac{x}{2} \cdot \cos\frac{x}{2}$$
$$= 2\cos\frac{x}{2}\left\{\sin\frac{x}{2}\right\}$$
$$= 2\cos\frac{x}{2}\left\{2\sin\frac{x}{2^2} \cdot \cos\frac{x}{2^2}\right\}$$
$$= 4\cos\frac{x}{2} \cdot \cos\frac{x}{2^2}\left\{\sin\frac{x}{2^2}\right\}$$
$$= 2^2 \cdot \cos\frac{x}{2}\cos\frac{x}{2^2}\left(2\sin\frac{x}{2^3} \cdot \cos\frac{x}{2^3}\right)$$
$$= 2^3 \cdot \cos\frac{x}{2} \cdot \cos\frac{x}{2^2} \cdot \cos\frac{x}{2^3} \cdot \sin\frac{x}{2^3}$$

Applying repeatedly the half angle sine formula,

$$\sin x = 2^{10} \cdot \cos\frac{x}{2} \cdot \cos\frac{x}{2^2} \cdot \ldots \cos\frac{x}{2^{10}} \cdot \sin\frac{x}{2^{10}}$$

Hence Proved.

3.26 Prove that $\dfrac{\sin\theta + \sin 2\theta}{1+\cos\theta+\cos 2\theta} = \tan\theta$

$$L.H.S = \frac{\sin\theta + \sin 2\theta}{\cos\theta + 1 + \cos 2\theta}$$
$$= \frac{\sin\theta + 2\sin\theta\cos\theta}{\cos\theta + 2\cos^2\theta}$$
$$= \frac{\sin\theta(1+2\cos\theta)}{\cos\theta(1+2\cos\theta)} = \frac{\sin\theta}{\cos\theta}$$
$$= \tan\theta = R.H.S$$

Hence Proved.

3.27. Prove that $1-\frac{1}{2}\sin 2x = \frac{\sin^3 x+\cos^3 x}{\sin x+\cos x}$

$$R.H.S = \frac{\sin^3 x+\cos^3 x}{\sin x+\cos x} \qquad [\because a^3+b^3=(a+b)(a^2-ab+b^2)]$$

$$= \frac{\cancel{(\sin x+\cos x)}(\sin^2 x+\cos^2 x-\sin x\cos x)}{\cancel{(\sin x+\cos x)}}$$

$$= 1-\sin x\cos x$$

$$= 1-\frac{1}{2}(2\sin x\cos x)$$

$$= 1-\frac{1}{2}\sin 2x = L.H.S$$

Hence Proved.

3.28 Find x such that $-\pi \le x \le \pi$ and $\cos 2x = \sin x$

$$\cos 2x = \sin x$$

$$1-2\sin^2 x = \sin x$$

$$2\sin^2 x+\sin x-1=0$$

-2: $\frac{-1}{2\sin x}$, $\frac{2}{2\sin x}$

$$(2\sin x-1)(\sin x+1)=0$$

$\sin x = \frac{1}{2}$ $\qquad$ $\sin x = -1$

$x = \frac{\pi}{6}, \frac{5\pi}{6} \in [-\pi,\pi]$ $\qquad$ $x = -\frac{\pi}{2} \in [-\pi,\pi]$

$\therefore$ The Solutions are $-\frac{\pi}{2}, \frac{\pi}{6}, \frac{5\pi}{6}$.

3.29 Find the values of (i) $\sin 18^\circ$ (ii) $\cos 18^\circ$ (iii) $\sin 72^\circ$ (iv) $\cos 36^\circ$ (v) $\sin 54^\circ$

Let $\theta = 18^\circ$

$$5\theta = 90^\circ$$

$$2\theta+3\theta = 90^\circ$$

$$2\theta = 90-3\theta$$

$$\sin 2\theta = \sin(90-3\theta)$$

$$\sin 2\theta = \cos 3\theta$$

$$2\sin\theta\cos\theta = 4\cos^3\theta-3\cos\theta$$

$\div \cos\theta$ $\qquad$ $2\sin\theta = 4\cos^2\theta-3$

$$2\sin\theta = 4(1-\sin^2\theta)-3$$
$$2\sin\theta = 4-4\sin^2\theta-3$$
$$4\sin^2\theta+2\sin\theta-1=0$$

$a=4$, $b=2$, $c=-1$

$$\sin\theta = \frac{-b\pm\sqrt{b^2-4ac}}{2a}$$
$$= \frac{-2\pm\sqrt{4+16}}{8} = \frac{-2\pm\sqrt{20}}{8}$$
$$= \frac{-2\pm2\sqrt{5}}{8} = \frac{-1\pm\sqrt{5}}{4}$$

θ is acute angle

$$\sin\theta = \frac{\sqrt{5}-1}{4} \qquad \sin\theta \neq \frac{-1-\sqrt{5}}{4}$$

$$\therefore \sin 18^\circ = \frac{\sqrt{5}-1}{4}.$$

(i) $\sin 18^\circ = \frac{\sqrt{5}-1}{4}$

(ii) $\cos 18^\circ = \sqrt{1-\sin^2 18^\circ} = \sqrt{1-\left(\frac{\sqrt{5}-1}{4}\right)^2}$
$$= \sqrt{\frac{16-5-1+2\sqrt{5}}{16}} = \frac{\sqrt{10+2\sqrt{5}}}{4}$$

(iii) $\sin 72^\circ = \sin(90-18) = \cos 18^\circ = \frac{\sqrt{10+2\sqrt{5}}}{4}$

(iv) $\cos 36^\circ = 1-2\sin^2 18^\circ$ (or) $2\cos^2 18^\circ - 1$
$$= 2\left(\frac{10+2\sqrt{5}}{16}\right)-1 = \frac{20+4\sqrt{5}-16}{16}$$
$$= \frac{4+4\sqrt{5}}{16} = \frac{1+\sqrt{5}}{4} = \frac{\sqrt{5}+1}{4}$$

$$\cos 36^\circ = \frac{\sqrt{5}+1}{4}$$

(v) $\sin 54^\circ = \sin(90-36) = \cos 36^\circ = \frac{\sqrt{5}+1}{4}.$

$\sin 18^\circ = \frac{\sqrt{5}-1}{4}$	$\cos 18^\circ = \frac{\sqrt{10+2\sqrt{5}}}{4}$
$\sin 36^\circ = \frac{\sqrt{10-2\sqrt{5}}}{4}$	$\cos 36^\circ = \frac{\sqrt{5}+1}{4}$
$\sin 54^\circ = \frac{\sqrt{5}+1}{4}$	$\cos 54^\circ = \frac{\sqrt{10-2\sqrt{5}}}{4}$
$\sin 72^\circ = \frac{\sqrt{10+2\sqrt{5}}}{4}$	$\cos 72^\circ = \frac{\sqrt{5}-1}{4}.$

3.30 If $\tan\frac{\theta}{2} = \sqrt{\frac{1-a}{1+a}}\tan\frac{\varphi}{2}$ then Prove that $\cos\phi = \frac{\cos\theta - a}{1 - a\cos\theta}$

$$\tan\frac{\varphi}{2} = \sqrt{\frac{1+a}{1-a}}\tan\frac{\theta}{2}$$

Proof :

$$\cos\varphi = \frac{1-\tan^2\varphi/2}{1+\tan^2\frac{\varphi}{2}}$$

$$= \frac{1 - \frac{1+a}{1-a}\tan^2\frac{\theta}{2}}{1 + \frac{1+a}{1-a}\tan^2\frac{\theta}{2}}$$

$$= \frac{1-a-(1+a)\tan^2\frac{\theta}{2}}{1-a+(1+a)\tan^2\frac{\theta}{2}}$$

$$= \frac{1-a-\tan^2\frac{\theta}{2} - a\tan^2\frac{\theta}{2}}{1-a+\tan^2\frac{\theta}{2} + a\tan^2\frac{\theta}{2}}$$

$$= \frac{1-\tan^2\frac{\theta}{2} - a(1+\tan^2\theta/2)}{1+\tan^2\frac{\theta}{2} - a(1-\tan^2\theta/2)}$$

$\div (1+\tan^2\theta/2)$

$$= \frac{\frac{1-\tan^2\theta/2}{1+\tan^2\theta/2} - a}{1 - a\left(\frac{1-\tan^2\theta/2}{1+\tan^2\theta/2}\right)}$$

$$= \frac{\cos\theta - a}{1 - a\cos\theta} = \text{R.HS}$$

Hence Proved.

3.31 Find the value of $\sqrt{3}\,\text{cosec}\,20^\circ - \sec 20^\circ$

$$\sqrt{3}\,\text{cosec}\,20^\circ - \sec 20^\circ = \frac{\sqrt{3}}{\sin 20^\circ} - \frac{1}{\cos 20^\circ}$$

$$= \frac{\sqrt{3}\cos 20^\circ - \sin 20^\circ}{\sin 20^\circ \cos 20^\circ}$$

$$= \frac{2\left(\frac{\sqrt{3}}{2}\cos 20^\circ - \frac{1}{2}\sin 20^\circ\right)}{\sin 20^\circ \cos 20^\circ}$$

$$= \frac{2\cdot 2\left[\sin 60^\circ \cos 20^\circ - \cos 60^\circ \sin 20^\circ\right]}{2\sin 20^\circ \cos 20^\circ}$$

$$= \frac{4\sin(60-20^\circ)}{\sin 2(20^\circ)} = \frac{4\sin 40^\circ}{\sin 40^\circ} = 4.$$

Ex: 3.5

1. Find the value of $\cos 2A$, A lies in the first quadrant, when

(i) $\cos A = \frac{15}{17}$ (ii) $\sin A = \frac{4}{5}$ (iii) $\tan A = \frac{16}{63}$

(i) $\cos A = \frac{15}{17}$

$$\cos 2A = 2\cos^2 A - 1 = 2\left(\frac{225}{289}\right) - 1$$

$$= \frac{450 - 289}{289} = \frac{161}{289}$$

(ii) $\sin A = \frac{4}{5}$

$$\cos 2A = 1 - 2\sin^2 A = 1 - 2\left(\frac{16}{25}\right)$$

$$= \frac{25 - 32}{25} = \frac{-7}{25}$$

(iii) $\tan A = \frac{16}{63}$

$$\cos 2A = \frac{1 - \tan^2 A}{1 + \tan^2 A} = \frac{1 - \left(\frac{16}{63}\right)^2}{1 + \left(\frac{16}{63}\right)^2}$$

$$= \frac{1 - \frac{256}{3969}}{1 + \frac{256}{3969}} = \frac{3969 - 256}{3969 + 256}$$

$$= \frac{3713}{4225}$$

2. If θ is acute angle, then find,

(i) $\sin\left(\frac{\pi}{4} - \frac{\theta}{2}\right)$ when $\sin\theta = \frac{1}{25}$

(ii) $\cos\left(\frac{\pi}{4} + \frac{\theta}{2}\right)$ when $\sin\theta = \frac{8}{9}$

(i) $\sin\left(\frac{\pi}{4} - \frac{\theta}{2}\right) = \sin\frac{\pi}{4}\cos\frac{\theta}{2} - \cos\frac{\pi}{4}\sin\theta/2$

$$= \frac{1}{\sqrt{2}}\cos\frac{\theta}{2} - \frac{1}{\sqrt{2}}\sin\theta/2$$

$$= \frac{1}{\sqrt{2}}\left(\cos\theta/2 - \sin\theta/2\right)$$

$$\sin^2\left(\frac{\pi}{4} - \frac{\theta}{2}\right) = \frac{1}{2}\left[1 - 2\sin\theta/2 \cdot \cos\theta/2\right]$$

$$= \frac{1}{2}\left[1 - \sin\theta\right]$$

$$= \frac{1}{2}\left(1 - \frac{1}{25}\right) = \frac{1}{2}\left(\frac{24}{25}\right) = \frac{12}{25}$$

$$\sin\left(\frac{\pi}{4} - \frac{\theta}{2}\right) = \sqrt{\frac{12}{25}} = \frac{2\sqrt{3}}{5}.$$

(ii) $\cos\left(\frac{\pi}{4}+\frac{\theta}{2}\right) = \cos\frac{\pi}{4}\cos\frac{\theta}{2} - \sin\frac{\pi}{4}\sin\frac{\theta}{2}$

$= \frac{1}{\sqrt{2}}\left(\cos\theta/2 - \sin\theta/2\right)$

$\cos^2\left(\frac{\pi}{4}+\frac{\theta}{2}\right) = \frac{1}{2}\left[1 - 2\sin\theta/2\cos\theta/2\right]$

$= \frac{1}{2}(1-\sin\theta)$

$= \frac{1}{2}\left(1-\frac{8}{9}\right) = \frac{1}{2}\left(\frac{1}{9}\right) = \frac{1}{18}$

$\cos\left(\frac{\pi}{4}+\frac{\theta}{2}\right) = \frac{1}{3\sqrt{2}}$

3. If $\cos\theta = \frac{1}{2}\left(a+\frac{1}{a}\right)$ show that $\cos 3\theta = \frac{1}{2}\left(a^3+\frac{1}{a^3}\right)$

$$\boxed{\begin{aligned} a^3+b^3 &= (a+b)^3 - 3ab(a+b) \\ a^3+\frac{1}{a^3} &= \left(a+\frac{1}{a}\right)^3 - 3\left(a+\frac{1}{a}\right) \end{aligned}}$$

$\cos\theta = \frac{1}{2}\left(a+\frac{1}{a}\right) \Rightarrow a+\frac{1}{a} = 2\cos\theta.$

$a^3+\frac{1}{a^3} = \left(a+\frac{1}{a}\right)^3 - 3\left(a+\frac{1}{a}\right)$

$= (2\cos\theta)^3 - 3(2\cos\theta)$

$= 8\cos^3\theta - 6\cos\theta$

$= 2(4\cos^3\theta - 3\cos\theta)$

$= 2\cos 3\theta$

$\frac{1}{2}\left(a^3+\frac{1}{a^3}\right) = \cos 3\theta$

Hence Proved.

4. Prove that $\cos 5\theta = 16\cos^5\theta - 20\cos^3\theta + 5\cos\theta$

L.H.S $= \cos 5\theta = \cos(2\theta+3\theta)$

$= \cos 2\theta\cos 3\theta - \sin 2\theta\sin 3\theta$

$= (2\cos^2\theta-1)(4\cos^3\theta-3\cos\theta) - 2\sin\theta\cos\theta(3\sin\theta-4\sin^3\theta)$

$= (2\cos^2\theta-1)(4\cos^3\theta-3\cos\theta) - 2\sin^2\theta\cos\theta(3-4\sin^2\theta)$

$= (2\cos^2\theta-1)(4\cos^3\theta-3\cos\theta) - 2\cos\theta(1-\cos^2\theta)(3-4(1-\cos^2\theta))$

$= (2\cos^2\theta-1)(4\cos^3\theta-3\cos\theta) - 2\cos\theta(1-\cos^2\theta)(4\cos^2\theta-1)$

$= 8\cos^5\theta - 6\cos^3\theta - 4\cos^3\theta + 3\cos\theta - 2\cos\theta\{4\cos^2\theta - 1 - 4\cos^4\theta + \cos^2\theta\}$

$= 8\cos^5\theta - 6\cos^3\theta - 4\cos^3\theta + 3\cos\theta - 8\cos^3\theta + 2\cos\theta + 8\cos^5\theta - 2\cos^3\theta$

$= 16\cos^5\theta - 20\cos^3\theta + 5\cos\theta$

$= R.H.S.$

5. Prove that $\sin 4\alpha = 4\tan\alpha \left(\frac{1-\tan^2\alpha}{(1+\tan^2\alpha)^2}\right)$

$L.H.S = \sin 4\alpha = 2\sin 2\alpha . \cos 2\alpha$

$= 2\left(\frac{2\tan\alpha}{1+\tan^2\alpha}\right)\left(\frac{1-\tan^2\alpha}{1+\tan^2\alpha}\right)$

$= 4\tan\alpha\left(\frac{1-\tan^2\alpha}{(1+\tan^2\alpha)^2}\right) = R.H.S.$

6. If $A+B=45^\circ$ Show that $(1+\tan A)(1+\tan B)=2$

$A+B = 45^\circ$

$\tan(A+B) = \tan 45^\circ$

$\frac{\tan A + \tan B}{1-\tan A\tan B} = 1$

$\tan A + \tan B = 1 - \tan A \tan B$

$\tan A + \tan B + \tan A\tan B = 1$

$1 + \tan A + \tan B + \tan A \tan B = 1+1$

$(1+\tan A) + \tan B(1+\tan A) = 2$

$(1+\tan A)(1+\tan B) = 2$

Hence Proved.

7. Prove that $(1+\tan 1^\circ)(1+\tan 2^\circ)(1+\tan 3^\circ)\ldots(1+\tan 44^\circ)$ is a multiple of 4.

We know that

$A+B=45^\circ$

$(1+\tan A)(1+\tan B) = 2$

$(1+\tan 1^\circ)(1+\tan 44^\circ) = 2$

$(1+\tan 2^\circ)(1+\tan 43^\circ) = 2$

$\vdots$

$(1+\tan 22^\circ)(1+\tan 23^\circ) = 2$

$\therefore (1+\tan 1^\circ)(1+\tan 2^\circ)(1+\tan 3^\circ)\ldots(1+\tan 44^\circ)$

$= 2.2.2 \ldots$ up to 22 times

$= 2^{22} = (2^2)^{11} = 4^{11}$

which is multiple of 4.

8. Prove that $\tan\left(\frac{\pi}{4}+\theta\right) - \tan\left(\frac{\pi}{4}-\theta\right) = 2\tan 2\theta$

$$L.H.S = \tan\left(\frac{\pi}{4}+\theta\right) - \tan\left(\frac{\pi}{4}-\theta\right)$$

$$= \frac{1+\tan\theta}{1-\tan\theta} - \frac{1-\tan\theta}{1+\tan\theta}$$

$$= \frac{(1+\tan\theta)^2 - (1-\tan\theta)^2}{(1-\tan\theta)(1+\tan\theta)}$$

$$= \frac{4\tan\theta}{1-\tan^2\theta} \qquad \left[\because (a+b)^2-(a-b)^2 = 4ab\right]$$

$$= 2\frac{(2\tan\theta)}{1-\tan^2\theta}$$

$$= 2\tan 2\theta = R.H.S$$

9. Show that $\cot\left(7\frac{1}{2}\right)^\circ = \sqrt{2}+\sqrt{3}+\sqrt{4}+\sqrt{6}$

Let $\theta = 15^\circ$

$$\cot\theta/2 = \frac{\cos\theta/2}{\sin\theta/2} = \frac{2\cos^2\theta/2}{2\sin\theta/2\cos\theta/2}$$

$$= \frac{1+\cos\theta}{\sin\theta} = \frac{1+\cos 15^\circ}{\sin 15^\circ}$$

$$= \frac{1+\frac{\sqrt{3}+1}{2\sqrt{2}}}{\frac{\sqrt{3}-1}{2\sqrt{2}}} = \frac{2\sqrt{2}+\sqrt{3}+1}{\sqrt{3}-1} \times \frac{\sqrt{3}+1}{\sqrt{3}+1}$$

$$= \frac{2\sqrt{6}+2\sqrt{2}+3+\sqrt{3}+\sqrt{3}+1}{3-1}$$

$$= \frac{2\sqrt{6}+2\sqrt{2}+2\sqrt{3}+4}{2} = \sqrt{6}+\sqrt{2}+\sqrt{3}+2$$

$$= \sqrt{6}+\sqrt{2}+\sqrt{3}+\sqrt{4}$$

$$\therefore \cot 7\frac{1}{2}^\circ = \sqrt{2}+\sqrt{3}+\sqrt{4}+\sqrt{6}$$

Hence Proved.

11. Prove that : $32\sqrt{3}.\sin\frac{\pi}{48}.\cos\frac{\pi}{48}.\cos\frac{\pi}{24}.\cos\frac{\pi}{12}.\cos\frac{\pi}{6} = 3$

$$L.H.S = 32\sqrt{3}\sin\frac{\pi}{48}.\cos\frac{\pi}{48}.\cos\frac{\pi}{24}.\cos\frac{\pi}{12}.\cos\frac{\pi}{6}$$

$$= 16\sqrt{3}\left(2\sin\frac{\pi}{48}.\cos\frac{\pi}{48}\right).\cos\frac{\pi}{24}.\cos\frac{\pi}{12}.\cos\frac{\pi}{6}$$

$$= 16\sqrt{3}\ \sin\frac{\pi}{24}.\cos\frac{\pi}{24}.\cos\frac{\pi}{12}.\cos\frac{\pi}{6}$$

$$= 8\sqrt{3}\left[2\sin\frac{\pi}{24}.\cos\frac{\pi}{24}\right].\cos\frac{\pi}{12}.\cos\frac{\pi}{6}$$

$$= 8\sqrt{3}\ \sin\frac{\pi}{12}.\cos\frac{\pi}{12}.\cos\frac{\pi}{6}$$

$$= 4\sqrt{3}\left[2\sin\frac{\pi}{12}.\cos\frac{\pi}{12}\right].\cos\frac{\pi}{6}$$

$$= 4\sqrt{3}\ \sin\frac{\pi}{6}.\cos\frac{\pi}{6}$$

$$= 4\sqrt{3}.\ \frac{1}{2}.\frac{\sqrt{3}}{2} = 3 = R.H.S$$

3.21 A football player can kick a football from the ground level with an initial velocity of 80 ft/sec. Find the maximum horizontal distance the football travells and at what angle? [Take $g = 32$]

$u = 80$ ft/sec $\qquad g = 32$

Horizontal Range

$$\boxed{R = \frac{u^2\sin 2\alpha}{g}} = \frac{80^2\sin 2\alpha}{32}$$

$$= 10\times 20\ \sin 2\alpha$$

$$= 200\ \sin 2\alpha$$

when $\alpha = 45^{\circ}$

$$R = 200\sin 90 = 200\text{ ft}.$$

∴ The maximum distance = 200 ft.

$$\boxed{\alpha = 45^{\circ}}$$

3.32 Prove that $\cos A \cos 2A \cos 2^2A \cdot \cos 2^3A \cdots \cos 2^{n-1}A = \dfrac{\sin 2^n A}{2^n \sin A}$.

$$L.H.S = \cos A \cos 2A \cos 2^2 A \cdots \cos 2^{n-1} A$$

$$= \frac{1}{2\sin A}(2\sin A \cos A) \cdot \cos 2A \cdot \cos 2^2 A \cdots \cos 2^{n-1} A$$

$$= \frac{1}{2\sin A} \cdot \sin 2A \cdot \cos 2A \cdot \cos 2^2 A \cdots \cos 2^{n-1} A$$

$$= \frac{1}{2^2 \sin A}(2\sin 2A \cos 2A) \cdot \cos 2^2 A \cdots \cos 2^{n-1} A$$

$$= \frac{1}{2^2 \sin A} \cdot \sin 2^2 A \cdot \cos 2^2 A \cdots \cos 2^{n-1} \cdot A$$

Continuing the process, we get.

$$= \frac{\sin 2^n \cdot A}{2^n \cdot \sin A} = R.H.S.$$

---

10. Prove that $(1+\sec 2\theta)(1+\sec 4\theta)\cdots(1+\sec 2^n\theta) = \tan 2^n\theta \cdot \cot\theta$.

$$L.H.S = \left(1+\frac{1}{\cos 2\theta}\right)\left(1+\frac{1}{\cos 4\theta}\right)\cdots \frac{1}{1+\cos 2^n\theta}$$

$$= \frac{1+\cos 2\theta}{\cos 2\theta} \cdot \frac{1+\cos 2^2\theta}{\cos 2^2\theta} \cdot \frac{1+\cos 2^3\theta}{\cos 2^3\theta} \cdots \frac{1+\cos 2^n\theta}{\cos 2^n\theta}$$

$$= \frac{(2\cos^2\theta)(2\cos^2 2\theta)(2\cos^2 2^2\theta)\cdots(2\cos^2 2^{n-1}\theta)}{\cos 2\theta \cdot \cos 2^2\theta \cos 2^3\theta \cdots \cos 2^n\theta}$$

$$= \frac{2^n \cos\theta (\cos\theta \cdot \cos^2 2\theta \cdot \cos^2 2^2\theta \cdot \cos^2 2^3\theta \cdots \cos^2 2^{n-1}\theta)}{\cos 2\theta \cdot \cos 2^2\theta \cdot \cos 2^3\theta \cdots \cos 2^n\theta}$$

$$= \frac{2^n \cos\theta}{\cos 2^n\cancel{\theta}}\left[\cos\theta \cdot \cos 2\theta \cdot \cos 2^2\theta \cdot \cos^2 2^3\theta \cdots \cos 2^{n-1}\theta\right]$$

$$= \frac{\cancel{2^n}\cos\theta}{\cos 2^n\theta}\left[\frac{\sin 2^n\theta}{\cancel{2^n} \cdot \sin\theta}\right] \qquad [\text{Eg } 3.32]$$

$$= \tan 2^n\theta \cdot \cot\theta$$

$$= R.H.S.$$

Product to Sum and Sum to Product Identities

$$\sin(A+B) = \sin A\cos B + \cos A\sin B$$
$$\sin(A-B) = \sin A\cos B - \cos A\sin B$$
$$\cos(A+B) = \cos A\cos B - \sin A\sin B$$
$$\cos(A-B) = \cos A\cos B + \sin A\sin B.$$

$$\sin(A+B) + \sin(A-B) = 2\sin A\cos B$$
$$\sin(A+B) - \sin(A-B) = 2\cos A\sin B$$
$$\cos(A+B) + \cos(A-B) = 2\cos A\cos B$$
$$\cos(A+B) - \cos(A-B) = -2\sin A\sin B$$

Let $A+B = C$ $\qquad A-B = D$

$A = \frac{C+D}{2}$ $\qquad B = \frac{C-D}{2}$

$$\sin C + \sin D = 2\sin\frac{C+D}{2}\cdot\cos\frac{C-D}{2}$$
$$\sin C - \sin D = 2\cos\frac{C+D}{2}\cdot\sin\frac{C-D}{2}$$
$$\cos C + \cos D = 2\cos\frac{C+D}{2}\cdot\cos\frac{C-D}{2}$$
$$\cos C - \cos D = -2\sin\frac{C+D}{2}\cdot\sin\frac{C-D}{2}$$

$$\sin(60-A)\cdot\sin A\cdot\sin(60+A) = \frac{1}{4}\sin 3A$$
$$\cos(60-A)\cos A\cdot\cos(60+A) = \frac{1}{4}\cos 3A$$
$$\tan(60-A)\tan A\cdot\tan(60+A) = \tan 3A$$

3.33 Express each of the following Product as a Sum or difference.

(i) $\sin 40^\circ \cos 30^\circ$

$$\sin A\cos B = \frac{1}{2}\left[\sin(A+B) + \sin(A-B)\right]$$

$$\sin 40\cdot\cos 30^\circ = \frac{1}{2}\left[\sin(40+30^\circ) + \sin(40-30^\circ)\right]$$

$$= \frac{1}{2}(\sin 70^\circ + \sin 10^\circ)$$

(ii) $\cos 110^\circ \sin 55^\circ$

$$\cos A \sin B = \frac{1}{2}\left[\sin(A+B) - \sin(A-B)\right]$$

$$\cos 110^\circ \sin 55^\circ = \frac{1}{2}\left[\sin(110^\circ+55^\circ) - \sin(110-55^\circ)\right]$$

$$= \frac{1}{2}\left[\sin 165^\circ - \sin 55^\circ\right]$$

(iii) $\sin \frac{x}{2} \cdot \cos \frac{3x}{2}$

$$\sin A \cos B = \frac{1}{2}\left[\sin(A+B) + \sin(A-B)\right]$$

$$\sin \frac{x}{2} \cdot \cos \frac{3x}{2} = \frac{1}{2}\left[\sin\left(\frac{x}{2}+\frac{3x}{2}\right) + \sin\left(\frac{x}{2}-\frac{3x}{2}\right)\right]$$

$$= \frac{1}{2}\left[\sin 2x + \sin(-x)\right]$$

$$= \frac{1}{2}\left[\sin 2x - \sin x\right] \quad \left[\because \sin(-x) = -\sin x\right]$$

3.34 Express each of the following sum or difference as a Product.

(i) $\sin 50^\circ + \sin 20^\circ$

$$\sin C + \sin D = 2\sin\frac{C+D}{2} \cdot \cos\frac{C-D}{2}$$

$$\sin 50^\circ + \sin 20^\circ = 2\sin\frac{50+20}{2} \cdot \cos\frac{50-20}{2}$$

$$= 2\sin 35^\circ \cdot \cos 15^\circ$$

(ii) $\cos 6\theta + \cos 2\theta$

$$\cos C + \cos D = 2\cos\frac{C+D}{2} \cdot \cos\frac{C-D}{2}$$

$$\cos 6\theta + \cos 2\theta = 2\cos\frac{6\theta+2\theta}{2} \cdot \cos\frac{6\theta-2\theta}{2}$$

$$= 2\cos 4\theta \cdot \cos 2\theta.$$

(iii) $\cos\frac{3x}{2} - \cos\frac{9x}{2}$

$$\cos C - \cos D = -2\sin\frac{C+D}{2} \cdot \sin\frac{C-D}{2}$$

$$\cos\frac{3x}{2} - \cos\frac{9x}{2} = -2\sin\left(\frac{\frac{3x}{2}+\frac{9x}{2}}{2}\right)\sin\left(\frac{\frac{3x}{2}-\frac{9x}{2}}{2}\right)$$

$$= -2\sin 3x \sin\left(-\frac{3x}{2}\right)$$

$$= 2\sin 3x \cdot \sin\frac{3x}{2} \quad \left[\sin(-x) = -\sin x\right]$$

3.35 Find the value of $\sin 34^\circ + \cos 64^\circ - \cos 4^\circ$

$$\sin 34^\circ + \cos 64^\circ - \cos 4^\circ = \sin 34^\circ - 2\sin\frac{64+4}{2}\cdot\sin\frac{64-4}{2}$$

$$= \sin 34^\circ - 2\sin 34^\circ \cdot \sin 30^\circ$$

$$= \sin 34^\circ - \cancel{2}\sin 34^\circ \cdot \frac{1}{\cancel{2}}$$

$$= \sin 34^\circ - \sin 34^\circ = 0.$$

3.36 Show that $\cos 36^\circ \cos 72^\circ \cos 108^\circ \cdot \cos 144^\circ = \frac{1}{16}$

$$L.H.S = \cos 36^\circ \cdot \cos 72^\circ \cos 108^\circ \cdot \cos 144^\circ$$

$$= \cos 36^\circ \sin 18^\circ \cos(90+18^\circ) \cos(180-36^\circ)$$

$$= \cos 36^\circ \sin 18^\circ (-\sin 18^\circ)(-\cos 36^\circ)$$

$$\boxed{\cos 72^\circ = \sin 18^\circ}$$

$$= \sin^2 18^\circ \cdot \cos^2 36^\circ$$

$$= \left(\frac{\sqrt5-1}{4}\right)^2\left(\frac{\sqrt5+1}{4}\right)^2$$

$$\because \sin 18^\circ = \frac{\sqrt5-1}{4}, \quad \cos 36^\circ = \frac{\sqrt5+1}{4}$$

$$= \frac{\left[(\sqrt5-1)(\sqrt5+1)\right]^2}{16\times16}$$

$$= \frac{16}{16(16)} = \frac{1}{16} = R.H.S$$

3.37 Simplify : $\dfrac{\sin 75^\circ - \sin 15^\circ}{\cos 75^\circ + \cos 15^\circ}$

$$\frac{\sin 75^\circ - \sin 15^\circ}{\cos 75^\circ + \cos 15^\circ} = \frac{\cancel{2}\cos\frac{75+15}{2}\cdot\sin\frac{75-15}{2}}{\cancel{2}\cos\frac{75+15}{2}\cdot\cos\frac{75-15}{2}}$$

$$= \frac{\cancel{2}\cancel{\cos 45^\circ}\sin 30^\circ}{\cancel{2}\cancel{\cos 45^\circ}\cos 30^\circ} = \tan 30^\circ = \frac{1}{\sqrt3}.$$

3.38 Show that $\cos 10^\circ \cos 30^\circ \cos 50^\circ \cos 70^\circ = \frac{3}{16}$

$$\boxed{\cos(60-A)\cdot\cos A\cdot\cos(60+A) = \frac{1}{4}\cdot\cos 3A}$$

$$L.H.S = \cos 30\left\{\cos 50^\circ \cos 10^\circ \cos 70^\circ\right\}$$

$$\boxed{A = 10^\circ}$$

$$= \cos 30\left\{\frac{1}{4}\cos 3(10^\circ)\right\}$$

$$= \cos 30^\circ\left\{\frac{1}{4}\cos 30^\circ\right\}$$

$$= \frac{\sqrt3}{2}\cdot\frac{1}{4}\cdot\frac{\sqrt3}{2} = \frac{3}{16} = R.H.S$$

Ex : 3.6

1 Express each of the following as a sum or difference :

(i) $\sin 35^\circ \cos 28^\circ$

$$\sin A \cos B = \frac{1}{2}[\sin(A+B) + \sin(A-B)]$$

$$\sin 35^\circ \cos 28^\circ = \frac{1}{2}[\sin(35+28) + \sin(35-28)]$$

$$= \frac{1}{2}[\sin 63 + \sin 7^\circ]$$

(ii) $\sin 4x \cos 2x$

$$\sin A \cos B = \frac{1}{2}[\sin(A+B) + \sin(A-B)]$$

$$\sin 4x \cos 2x = \frac{1}{2}[\sin(4x+2x) + \sin(4x-2x)]$$

$$= \frac{1}{2}[\sin 6x + \sin 2x]$$

(iii) $2\sin 10\theta \cos 2\theta$

$$2\sin A \cos B = \frac{1}{2}[\sin(A+B) + \sin(A-B)]$$

$$2\sin 10\theta \cos 2\theta = \frac{1}{2}[\sin(10\theta+2\theta) + \sin(10\theta-2\theta)]$$

$$= \frac{1}{2}[\sin 12\theta + \sin 8\theta]$$

(iv) $\cos 5\theta \cos 2\theta$

$$\cos A \cos B = \frac{1}{2}[\cos(A+B) + \cos(A-B)]$$

$$\cos 5\theta \cos 2\theta = \frac{1}{2}[\cos(5\theta+2\theta) + \cos(5\theta-2\theta)]$$

$$= \frac{1}{2}[\cos 7\theta + \cos 3\theta]$$

(v) $\sin 5\theta \sin 4\theta$

$$\sin A \sin B = \frac{1}{2}[\cos(A-B) - \cos(A+B)]$$

$$\sin 5\theta \sin 4\theta = \frac{1}{2}[\cos(5\theta-4\theta) - \cos(5\theta+4\theta)]$$

$$= \frac{1}{2}[\cos\theta + \cos 9\theta]$$

2. Express each of the following as a Product:

(i) $\sin 75^\circ - \sin 35^\circ$

$$\sin C - \sin D = 2\cos\frac{C+D}{2}\cdot\sin\frac{C-D}{2}$$

$$\sin 75^\circ - \sin 35^\circ = 2\cos\frac{75+35}{2}\cdot\sin\frac{75-35}{2}$$

$$= 2\cos 55^\circ\cdot\sin 20^\circ$$

(ii) $\cos 65^\circ + \cos 15^\circ$

$$\cos C + \cos D = 2\cos\frac{C+D}{2}\cdot\sin\frac{C-D}{2}$$

$$\cos 65^\circ + \cos 15^\circ = 2\cos\frac{65+15}{2}\cdot\cos\frac{65-15}{2}$$

$$= 2\cos 40^\circ\cdot\cos 25^\circ$$

(iii) $\sin 50^\circ + \sin 40^\circ$

$$\sin C + \sin D = 2\sin\frac{C+D}{2}\cdot\cos\frac{C-D}{2}$$

$$\sin 50^\circ + \sin 40^\circ = 2\sin\frac{50+40}{2}\cdot\cos\frac{50-40}{2}$$

$$= 2\sin 45^\circ\cdot\cos 5^\circ$$

(iv) $\cos 35^\circ - \cos 75^\circ$

$$\cos C - \cos D = -2\sin\frac{C+D}{2}\cdot\sin\frac{C-D}{2}$$

$$\cos 35^\circ - \cos 75^\circ = -2\sin\frac{35+75}{2}\cdot\sin\frac{35-75}{2}$$

$$= -2\sin 55^\circ\cdot\sin(-20)$$

$$= 2\sin 55^\circ\sin 20^\circ.$$

3. Show that $\sin 12^\circ\cdot\sin 48^\circ\cdot\sin 54^\circ = \frac{1}{8}$

$$L.H.S = \sin 12^\circ\cdot\sin 48^\circ\ \sin 54^\circ$$

$$= \frac{1}{2}\left[\cos(48-12) - \cos(48+12)\right]\cdot\sin 54^\circ$$

$$= \frac{1}{2}\left[\cos 36^\circ - \cos 60^\circ\right]\sin 54^\circ$$

$$= \frac{1}{2}\left[\frac{\sqrt{5}+1}{4} - \frac{1}{2}\right]\frac{\sqrt{5}+1}{4}$$

$$\boxed{\sin 54^\circ = \cos 36^\circ = \frac{\sqrt{5}+1}{4}}$$

$$= \frac{1}{2}\left[\frac{\sqrt{5}+1-2}{4}\right]\left[\frac{\sqrt{5}+1}{4}\right]$$

$$= \frac{1}{2}\left[\frac{(\sqrt{5}-1)(\sqrt{5}+1)}{16}\right]$$

$$= \frac{1}{2}\left(\frac{4}{16}\right) = \frac{1}{8} = R.H.S$$

4. Show that $\cos\frac{\pi}{15}.\cos\frac{2\pi}{15}.\cos\frac{3\pi}{15}.\cos\frac{4\pi}{15}.\cos\frac{5\pi}{15}\cos\frac{6\pi}{15}$
$\cos\frac{7\pi}{15} = \frac{1}{128}$

$$\boxed{\cos(60-A)\ \cos A . \cos(60+A) = \frac{1}{4}\cos 3A}$$

$L.H.S = \cos\frac{\pi}{15}.\cos\frac{2\pi}{15}.\cos\frac{3\pi}{15}\cos\frac{4\pi}{15}\cos\frac{5\pi}{15}.\cos\frac{6\pi}{15}.\cos\frac{7\pi}{15}$

$= \cos 12^\circ \cos 24^\circ \cos 36^\circ \cos 48^\circ \cos 60^\circ \cos 72^\circ \cos 84^\circ$

$= \cos 60^\circ \{\cos 48^\circ . \cos 12^\circ \cos 72^\circ\}\{\cos 36^\circ \cos 24^\circ \cos 84^\circ\}$

$= \cos 60^\circ \{\frac{1}{4}\cos 3(12^\circ)\}\{\frac{1}{4}\cos 3(24^\circ)\}$

$= \cos 60^\circ \{\frac{1}{4}\cos 36^\circ\}\{\frac{1}{4}\cos 72^\circ\}$

$= \frac{1}{2}.\frac{1}{4}.\frac{\sqrt{5}+1}{4}.\frac{1}{4}.\frac{\sqrt{5}-1}{4}$

$= \frac{1}{2\times16\times16}(\sqrt{5}+1)(\sqrt{5}-1) = \frac{\cancel{4}}{2\times\cancel{16}_4\times16} = \frac{1}{128} = R.H.S$

5. Show that $\frac{\sin 8x.\cos x - \sin 6x.\cos 3x}{\cos 2x.\cos x - \sin 3x.\sin 4x} = \tan 2x$

$L.H.S = \frac{\sin 8x \cos x - \sin 6x.\cos 3x}{\cos 2x \cos x - \sin 3x \sin 4x}$

$= \frac{\frac{1}{2}(\sin 9x + \sin 7x) - \frac{1}{2}[\sin(9x + \sin 3x]}{\frac{1}{2}(\cos 3x + \cos x) + \frac{1}{2}(\cos 7x - \cos x)}$

$= \frac{\cancel{\sin 9x} + \sin 7x - \cancel{\sin 9x} - \sin 3x}{\cos 3x + \cancel{\cos x} + \cos 7x - \cancel{\cos x}}$

$= \frac{\sin 7x - \sin 3x}{\cos 7x + \cos 3x}$

$= \frac{\cancel{2}\cos\frac{7x+3x}{2}.\sin\left(\frac{7x-3x}{2}\right)}{\cancel{2}\cos\frac{7x+3x}{2}.\cos\left(\frac{7x-3x}{2}\right)}$

$= \frac{\cancel{\cos 5x}\sin 2x}{\cancel{\cos 5x}\cos 2x} = \tan 2x$

$= R.H.S$

6. Show that $\dfrac{(\cos\theta-\cos3\theta)(\sin8\theta+\sin2\theta)}{(\sin5\theta-\sin\theta)(\cos4\theta-\cos6\theta)}=1$

$$L.H.S = \frac{(\cos\theta-\cos3\theta)(\sin8\theta+\sin2\theta)}{(\sin5\theta-\sin\theta)(\cos4\theta-\cos6\theta)}$$

$$= \frac{-2\sin\left(\frac{\theta+3\theta}{2}\right)\sin\left(\frac{\theta-3\theta}{2}\right).\ 2\sin\left(\frac{8\theta+2\theta}{2}\right).\cos\frac{8\theta-2\theta}{2}}{2\cos\left(\frac{5\theta+\theta}{2}\right)\sin\left(\frac{5\theta-\theta}{2}\right).\left(-2\sin\left(\frac{4\theta+6\theta}{2}\right).\sin\left(\frac{4\theta-6\theta}{2}\right)\right)}$$

$$= \frac{-\cancel{4}\ \sin\cancel{2}\theta.\sin(-\theta)\ \sin\cancel{5}\theta.\cos\cancel{3}\theta}{-\cancel{4}\ \cos\cancel{3}\theta.\sin\cancel{2}\theta.\sin\cancel{5}\theta.\sin(-\theta)} = 1$$

$= R.H.S$

7. $\sin x+\sin2x+\sin3x=\sin2x(1+2\cos x)$

$L.H.S = \sin x+\sin2x+\sin3x$

$= \sin2x+(\sin x+\sin3x)$

$= \sin2x+\left[2\sin\frac{x+3x}{2}.\cos\frac{x-3x}{2}\right]$

$= \sin2x+(2\sin2x.\cos(-x)]$

$= \sin2x+2\sin2x\cos x$

$= \sin2x(1+2\cos x) = R.H.S$

8. Prove that $\dfrac{\sin4x+\sin2x}{\cos4x+\cos2x}=\tan3x$

$$L.H.S = \frac{\sin4x+\sin2x}{\cos4x+\cos2x}$$

$$= \frac{2\sin\frac{4x+2x}{2}.\cos\left(\frac{4x-2x}{2}\right)}{2\cos\left(\frac{4x+2x}{2}\right).\cos\left(\frac{4x-2x}{2}\right)}$$

$$= \frac{2\sin3x.\cos x}{2\cos3x.\cos x} = \frac{\sin3x}{\cos3x}$$

$= \tan3x = R.H.S$

9. Prove that $1+\cos 2x+\cos 4x+\cos 6x = 4\cos x\cos 2x\cos 3x$

$L.H.S = 1+\cos 2x + \cos 4x + \cos 6x$

$= 2\cos^2 x + 2\cos\left(\frac{6x+4x}{2}\right).\cos\left(\frac{6x-4x}{2}\right)$

$= 2\cos^2 x + 2\cos 5x.\cos x$

$= 2\cos x\,(\cos x + \cos 5x)$

$= 2\cos x\left\{2\cos\frac{5x+x}{2}.\cos\frac{5x-x}{2}\right\}$

$= 2\cos x\,\{2\cos 3x.\cos 2x\}$

$= 4\cos x\cos 2x\cos 3x$

$= R.H.S$

10. Prove that $\sin\frac{\theta}{2}\sin\frac{7\theta}{2}+\sin\frac{3\theta}{2}\sin\frac{11\theta}{2} = \sin 2\theta.\sin 5\theta$

$L.H.S = \sin\frac{\theta}{2}\sin\frac{7\theta}{2}+\sin\frac{3\theta}{2}\sin\frac{11\theta}{2}$

$= \frac{1}{2}\left[\cos\left(\frac{\theta}{2}-\frac{7\theta}{2}\right)-\cos\left(\frac{\theta}{2}+\frac{7\theta}{2}\right)\right]+$
$\frac{1}{2}\left[\cos\left(\frac{3\theta}{2}-\frac{11\theta}{2}\right)-\cos\left(\frac{3\theta}{2}+\frac{11\theta}{2}\right)\right]$

$= \frac{1}{2}\left[\cos(-3\theta)-\cos 4\theta\right]+\frac{1}{2}\left(\cos(-4\theta)-\cos 7\theta\right)$

$= \frac{1}{2}\left[\cos 3\theta - \cancel{\cos 4\theta} + \cancel{\cos 4\theta} - \cos 7\theta\right]$

$= \frac{1}{2}\left[\cos 3\theta - \cos 7\theta\right]$

$= \frac{1}{2}\left[-2\sin\left(\frac{3\theta+7\theta}{2}\right).\sin\left(\frac{3\theta-7\theta}{2}\right)\right]$

$= -1\sin 5\theta\sin(-2\theta)$

$= \sin 5\theta.\sin 2\theta = \sin 2\theta\sin 5\theta = R.H.S.$

11. Prove that $\cos(30-A)\cos(30+A)+\cos(45-A)\cos(45+A) = \cos 2A+\frac{1}{4}$

$L.H.S = \cos(30-A)\cos(30+A)+\cos(45-A)\cos(45+A)$

$= \frac{1}{2}\left[\cos(30-\cancel{A}+30+\cancel{A})+\cos(\cancel{30}-A-\cancel{30}-A)\right]+$
$\frac{1}{2}\left[\cos(45-\cancel{A}+45+\cancel{A})+\cos(\cancel{45}-A-\cancel{45}-A)\right]$

$= \frac{1}{2}\left[\cos 60^\circ+\cos(-2A)+\cos 90^\circ+\cos(-2A)\right]$

$= \frac{1}{2}\left[\cos 60^\circ + \cos 2A + 0 + \cos 2A\right]$

$= \frac{1}{2}\left[\frac{1}{2}+2\cos 2A\right] = \frac{1}{4}+\cos 2A$

$= R.H.S$

Another method :

$$\cos(30+A)\cos(30-A) + \cos(45+A)\cos(45-A)$$
$$= \cos^2 30^\circ - \sin^2 A + \cos^2 45^\circ - \sin^2 A$$
$$= \frac{3}{4} - \sin^2 A + \frac{1}{2} - \sin^2 A$$
$$= \frac{5}{4} - 2\sin^2 A$$
$$= \frac{1}{4} + \underbrace{1 - 2\sin^2 A} = \frac{1}{4} + \cos 2A$$
$$= R.H.S$$

$$\boxed{\cos(A+B)\cdot\cos(A-B) = \cos^2 A - \sin^2 B.}$$

12. Prove that $\dfrac{\sin x + \sin 3x + \sin 5x + \sin 7x}{\cos x + \cos 3x + \cos 5x + \cos 7x} = \tan 4x$

$$L.H.S = \frac{\sin x + \sin 3x + \sin 5x + \sin 7x}{\cos x + \cos 3x + \cos 5x + \cos 7x}$$
$$= \frac{(\sin x + \sin 5x) + (\sin 3x + \sin 7x)}{(\cos x + \cos 5x) + (\cos 3x + \cos 7x)}$$
$$= \frac{2\sin 3x \cdot \cos 2x + 2\sin 5x \cdot \cos 2x}{2\cos 3x \cdot \cos 2x + 2\cos 5x \cos 2x}$$
$$= \frac{\cancel{2\cos 2x}(\sin 3x + \sin 5x)}{\cancel{2\cos 2x}(\cos 3x + \cos 5x)}$$
$$= \frac{2\sin 4x \cdot \cos x}{2\cos 4x \cdot \cos x} = \frac{\sin 4x}{\cos 4x} = \tan 4x$$
$$= R.H.S$$

13. Prove that $\dfrac{\sin(4A-2B) + \sin(4B-2A)}{\cos(4A-2B) + \cos(4B-2A)} = \tan(A+B)$

$$L.H.S = \frac{\sin(4A-2B) + \sin(4B-2A)}{\cos(4A-2B) + \cos(4B-2A)}$$
$$= \frac{\cancel{2}\sin\left(\frac{4A-2B+4B-2A}{2}\right)\cdot\cos\left(\frac{4A-2B-4B+2A}{2}\right)}{\cancel{2}\cos\left(\frac{4A-2B+4B-2A}{2}\right)\cdot\cos\left(\frac{4A-2B-4B+2A}{2}\right)}$$
$$= \frac{\sin\left(\frac{2A+2B}{2}\right)\cancel{\cos\left(\frac{6A-6B}{2}\right)}}{\cos\left(\frac{2A+2B}{2}\right)\cancel{\cos\left(\frac{6A-6B}{2}\right)}}$$
$$= \tan(A+B)$$
$$= R.H.S$$

14. Show that $\cot(A+15^\circ) - \tan(A-15^\circ) = \frac{4\cos 2A}{1+2\sin 2A}$

$$L.H.S = \cot(A+15^\circ) - \tan(A-15^\circ)$$

$$= \frac{\cos(A+15^\circ)}{\sin(A+15^\circ)} - \frac{\sin(A-15^\circ)}{\cos(A-15^\circ)}$$

$$= \frac{\cos(A+15^\circ)\cos(A-15^\circ) - \sin(A+15^\circ)\sin(A-15^\circ)}{\sin(A+15^\circ)\cos(A-15^\circ)}$$

$$= \frac{(\cos^2 A - \sin^2 15^\circ) - (\sin^2 A - \sin^2 15^\circ)}{\frac{1}{2}\left[\sin(A+\cancel{15^\circ}+A-\cancel{15^\circ}) + \sin(\cancel{A}+15^\circ-\cancel{A}+15^\circ)\right]}$$

$$= \frac{\cos^2 A - \cancel{\sin^2 15^\circ} - \sin^2 A + \cancel{\sin^2 15^\circ}}{\frac{1}{2}(\sin 2A + \sin 30^\circ)}$$

$$= \frac{2(\cos^2 A - \sin^2 A)}{\sin 2A + \frac{1}{2}} = \frac{2\cos 2A}{2\sin 2A + 1} \times \frac{2}{1}$$

$$= \frac{4\cos 2A}{1+2\sin 2A} = R.H.S.$$

Conditional Trigonometric Identities :

3.39 If $A+B+C = \pi$. Prove the following :

(i) $\cos A + \cos B + \cos C = 1 + 4\sin\frac{A}{2}.\sin\frac{B}{2}.\sin\frac{C}{2}$

(ii) $\sin\left(\frac{A}{2}\right)\sin\left(\frac{B}{2}\right)\sin\left(\frac{C}{2}\right) \le \frac{1}{8}$

(iii) $1 < \cos A + \cos B + \cos C \le \frac{3}{2}$

$$A+B+C = \pi \qquad \frac{A+B+C}{2} = \frac{\pi}{2}$$

(i) $\cos A + \cos B + \cos C$

$$= 2\cos\frac{A+B}{2}.\cos\frac{A-B}{2} + \cos C$$

$$= 2\cos\left(\frac{\pi}{2} - \frac{C}{2}\right).\cos\frac{A-B}{2} + 1 - 2\sin^2\frac{C}{2}$$

$$= 2\sin\frac{C}{2}.\cos\frac{A-B}{2} + 1 - 2\sin^2\frac{C}{2}$$

$$= 1 + 2\sin\frac{C}{2}\left[\cos\frac{A-B}{2} - \sin\frac{C}{2}\right]$$

$$= 1 + 2\sin\frac{C}{2}\left[\cos\frac{A-B}{2} - \sin\left(\frac{\pi}{2} - \frac{A+B}{2}\right)\right]$$

$$= 1 + 2\sin\frac{C}{2}\left[\cos\frac{A-B}{2} - \cos\frac{A+B}{2}\right]$$

$= 1 + 2\sin\frac{C}{2}\left\{2\sin\frac{B}{2}.\sin\frac{A}{2}\right\}$

$= 1 + 4\sin\frac{A}{2}\sin\frac{B}{2}\sin\frac{C}{2}.$

(ii) Let $u = \sin\frac{A}{2}.\sin\frac{B}{2}.\sin\frac{C}{2}$

$= -\frac{1}{2}\left[\cos\frac{A+B}{2} - \cos\frac{A-B}{2}\right].\sin\frac{C}{2}$

$= -\frac{1}{2}\left[\cos\frac{A+B}{2} - \cos\frac{A-B}{2}\right].\sin\left(\frac{\pi}{2} - \frac{A+B}{2}\right)$

$u = -\frac{1}{2}\left[\cos\frac{A+B}{2} - \cos\frac{A-B}{2}\right]\cos\left(\frac{A+B}{2}\right)$

$2u = -\cos^2\frac{A+B}{2} + \cos\frac{A-B}{2}.\cos\frac{A+B}{2}$

$\cos^2\frac{A+B}{2} + \cos\frac{A-B}{2}.\cos\frac{A+B}{2} + 2u = 0$

It is a quadratic equation on $\cos\frac{A+B}{2}$

$\cos\frac{A+B}{2}$ is a real number

It has solution.

Discriminant ≥ 0

$b^2 - 4ac \geq 0$

$a = 1$

$b = \cos\left(\frac{A-B}{2}\right)$

$c = 2u$

$\cos^2\left(\frac{A-B}{2}\right) - 8u \geq 0$

$8u \leq \cos^2\frac{A-B}{2}$

$u \leq \frac{1}{8}\cos^2\frac{A-B}{2}$

$u \leq \frac{1}{8}$

$\therefore \sin\frac{A}{2}.\sin\frac{B}{2}.\sin\frac{C}{2} \leq \frac{1}{8}$

(iii) From (i) and (ii)

$\cos A + \cos B + \cos C > 1$

$\cos A + \cos B + \cos C \leq 1 + 4\left(\frac{1}{8}\right)$

$\cos A + \cos B + \cos C \leq \frac{3}{2}$

$\therefore 1 < \cos A + \cos B + \cos C \leq \frac{3}{2}$

Hence Proved.

3.40 Prove that $\sin\frac{A}{2}+\sin\frac{B}{2}+\sin\frac{C}{2}$

$= 1+4\sin\left(\frac{\pi-A}{4}\right)\sin\left(\frac{\pi-B}{4}\right).\sin\left(\frac{\pi-C}{4}\right)$

If $A+B+C=\pi$

$A+B+C=\pi$

$\frac{A+B+C}{2}=\frac{\pi}{2}$

$\frac{A+B}{2}=\frac{\pi}{2}-\frac{C}{2}$

$\frac{A+B}{4}=\frac{\pi}{4}-\frac{C}{4}$

L.H.S $= \sin\frac{A}{2}+\sin\frac{B}{2}+\sin\frac{C}{2}$

$= \cos\left(\frac{\pi}{2}-\frac{A}{2}\right)+\cos\left(\frac{\pi}{2}-\frac{B}{2}\right)+\cos\left(\frac{\pi}{2}-\frac{C}{2}\right)$

$= \left[2\cos\frac{\frac{\pi}{2}-\frac{A}{2}+\frac{\pi}{2}-\frac{B}{2}}{2}.\cos\left(\frac{\frac{\pi}{2}-\frac{A}{2}-\frac{\pi}{2}+\frac{B}{2}}{2}\right)\right]+\left(1-2\sin^2\left(\frac{\frac{\pi}{2}-\frac{C}{2}}{2}\right)\right)$

$= 2\cos\left(\frac{\pi}{2}-\frac{A+B}{4}\right).\cos\left(\frac{B-A}{4}\right)+1-2\sin^2\left(\frac{\pi}{4}-\frac{C}{4}\right)$

$= 2\sin\left(\frac{\pi}{4}-\frac{C}{4}\right).\cos\frac{B-A}{4}+1-2\sin^2\left(\frac{\pi}{4}-\frac{C}{4}\right)$

$= 1+2\sin\left(\frac{\pi}{4}-\frac{C}{4}\right)\left\{\cos\frac{B-A}{4}-\sin\left(\frac{\pi}{4}-\frac{C}{4}\right)\right\}$

$= 1+2\sin\left(\frac{\pi-C}{4}\right)\left\{\cos\frac{B-A}{4}-\sin\left(\frac{A+B}{4}\right)\right\}$

$= 1+2\sin\left(\frac{\pi-C}{4}\right)\left\{\cos\frac{B-A}{4}-\cos\left(\frac{\pi}{2}-\frac{A+B}{4}\right)\right\}$

$= 1+2\sin\left(\frac{\pi-C}{4}\right)\left\{2\sin\left(\frac{\frac{B-A}{4}+\frac{\pi}{2}-\frac{A+B}{4}}{2}\right)\sin\left(\frac{\frac{\pi}{2}-\frac{A+B}{4}-\frac{B-A}{4}}{2}\right)\right.$

$= 1+2\sin\frac{\pi-C}{4}\left\{2\sin\left(\frac{\pi-A}{4}\right).\sin\left(\frac{\pi-B}{4}\right)\right.$

$= 1+4\sin\frac{\pi-A}{4}.\sin\frac{\pi-B}{4}.\sin\frac{\pi-C}{4}$

= R.H.S.

3.41 If $A+B+C=\pi$ Prove that $\cos^2A+\cos^2B+\cos^2C$

$= 1-2\cos A\cos B\cos C$

$A+B+C=\pi$

$A+B=\pi-C$

L.H.S $= \cos^2A+\cos^2B+\cos^2C$

$= \frac{1+\cos 2A}{2}+\frac{1+\cos 2B}{2}+\frac{1+\cos 2C}{2}$

$= \frac{3}{2}+\frac{1}{2}\left[\cos 2A+\cos 2B+\cos 2C\right]$

$= \frac{3}{2}+\frac{1}{2}\left\{2\cos(A+B).\cos(A-B)+2\cos^2C-1\right.$

$= \frac{3}{2}+\cos(A+B)\cos(A-B)+\cos^2C-\frac{1}{2}$

$= 1+\cos(\pi-C)\cos(A-B)+\cos^2C$

$= 1-\cos C.\cos(A-B)+\cos^2C$

$= 1 - \cos C \{ \cos(A-B) - \cos C \}$

$= 1 - \cos C \{ \cos(A-B) - \cos(\pi - \overline{A+B}) \}$

$= 1 - \cos C \{ \cos(A-B) + \cos(A+B) \}$

$= 1 - \cos C \{ 2\cos A \cos B \}$

$= 1 - 2\cos A \cos B \cos C = R.H.S.$

Ex : 3.7

1 If $A+B+C = \pi$ Prove that

(i) $\sin 2A + \sin 2B + \sin 2C = 4 \sin A \sin B \sin C$

$A+B+C = \pi$
$A+B = \pi - C$
$C = \pi - \overline{A+B}$

$L.H.S = \sin 2A + \sin 2B + \sin 2C$

$= 2\sin(A+B)\cos(A-B) + 2\sin C \cos C$

$= 2\sin(\pi - C)\cos(A-B) + 2\sin C \cos C$

$= 2\sin C \cdot \cos(A-B) + 2\sin C \cos C$

$= 2\sin C \{ \cos(A-B) + \cos C \}$

$= 2\sin C \{ \cos(A-B) + \cos(\pi - \overline{A+B}) \}$

$= 2\sin C \{ \cos(A-B) - \cos(A+B) \}$

$= 2\sin C \,(2\sin A \sin B)$

$= 4\sin A \sin B \sin C = R.H.S$

(ii) $\cos A + \cos B - \cos C = -1 + 4\cos\frac{A}{2} \cdot \cos\frac{B}{2} \cdot \sin\frac{C}{2}$

$A+B+C = \pi$
$\frac{A+B}{2} = \frac{\pi}{2} - \frac{C}{2}$
$\frac{C}{2} = \frac{\pi}{2} - \frac{\overline{A+B}}{2}$

$L.H.S = \cos A + \cos B - \cos C$

$= 2\cos\frac{A+B}{2} \cdot \cos\left(\frac{A-B}{2}\right) - \cos C$

$= 2\cos\left(\frac{\pi}{2} - \frac{C}{2}\right) \cdot \cos\left(\frac{A-B}{2}\right) - \left(1 - 2\sin^2\frac{C}{2}\right)$

$= -1 + 2\sin\frac{C}{2} \cos\frac{A-B}{2} + 2\sin^2\frac{C}{2}$

$= -1 + 2\sin\frac{C}{2} \left\{ \cos\frac{A-B}{2} + \sin\frac{C}{2} \right\}$

$= -1 + 2\sin\frac{C}{2} \left\{ \cos\frac{A-B}{2} + \sin\left(\frac{\pi}{2} - \frac{A+B}{2}\right) \right\}$

$= -1 + 2\sin\frac{C}{2} \left[\cos\frac{A-B}{2} + \cos\frac{A+B}{2} \right]$

$= -1 + 2\sin\frac{C}{2} \left[2\cos\frac{A}{2} \cdot \cos\frac{B}{2} \right]$

$= -1 + 4\cos\frac{A}{2} \cdot \cos\frac{B}{2} \cdot \sin\frac{C}{2}$

$= R.H.S.$

(iii) $\sin^2 A + \sin^2 B + \sin^2 C = 2 + 2\cos A \cos B \cos C$

$$L.H.S = \sin^2 A + \sin^2 B + \sin^2 C$$

$$= \frac{1-\cos 2A}{2} + \frac{1-\cos 2B}{2} + \frac{1-\cos 2C}{2}$$

$$= \frac{3}{2} - \frac{1}{2}\left[\cos 2A + \cos 2B + \cos 2C\right]$$

$$= \frac{3}{2} - \frac{1}{2}\left\{2\cos(A+B)\cos(A-B) + 2\cos^2 C - 1\right\}$$

$$= \frac{3}{2} - \frac{1}{2}\left[2\cos(A+B)\cos A - B\right] - \cos^2 C + \frac{1}{2}$$

$$= 2 - \frac{1}{2}\left[2\cos(\pi - C)\cos(A-B)\right] - \cos^2 C$$

$$= 2 - \left[-\cos C \cdot \cos(A-B)\right] - \cos^2 C$$

$$= 2 + \cos C \cos(A-B) - \cos^2 C$$

$$= 2 + \cos C\left[\cos(A-B) - \cos C\right]$$

$$= 2 + \cos C\left[\cos(A-B) - \cos(\pi - \overline{A+B})\right]$$

$$= 2 + \cos C\left[\cos(A-B) + \cos(A+B)\right]$$

$$= 2 + \cos C\left[2\cos A \cos B\right]$$

$$= 2 + 2\cos A \cos B \cos C = R.H.S$$

(iv) $\sin^2 A + \sin^2 B - \sin^2 C = 2\sin A \sin B \cos C$

$$L.H.S = \sin^2 A + \sin^2 B - \sin^2 C$$

$$= \frac{1-\cos 2A}{2} + \frac{1-\cos 2B}{2} - \frac{1-\cos 2C}{2}$$

$$= \frac{1}{2} + \cancel{\frac{1}{2}} - \cancel{\frac{1}{2}} - \frac{1}{2}\left[\cos 2A + \cos 2B - \cos 2C\right]$$

$$= \frac{1}{2} - \frac{1}{2}\left[2\cos(A+B)\cdot\cos(A-B) - 2\cos^2 C + 1\right]$$

$$= \frac{1}{2} - \frac{1}{2}\left[2\cos(\pi - C)\cdot\cos(A-B) - 2\cos^2 C + 1\right]$$

$$= \cancel{\frac{1}{2}} - \left[\cos(\pi - C)\cdot\cos(A-B)\right] + \cos^2 C - \cancel{\frac{1}{2}}$$

$$= \cos C \cos(A-B) + \cos^2 C$$

$$= \cos C\left[\cos(A-B) + \cos C\right]$$

$$= \cos C\left[\cos(A-B) + \cos(\pi - \overline{A+B})\right]$$

$$= \cos C\left[\cos(A-B) - \cos(A+B)\right]$$

$$= \cos C\left[2\sin A \sin B\right]$$

$$= 2\sin A \sin B \cos C = R.H.S$$

(v) $\tan\frac{A}{2}\tan\frac{B}{2}+\tan\frac{B}{2}\tan\frac{C}{2}+\tan\frac{C}{2}\tan\frac{A}{2}=1.$

$A+B+C=180^\circ$ $\qquad \frac{A+B}{2}=90-\frac{C}{2}$

$$\frac{A+B}{2}=90-\frac{C}{2}$$

$$\tan\left(\frac{A+B}{2}\right)=\tan\left(90-\frac{C}{2}\right)$$

$$\frac{\tan\frac{A}{2}+\tan\frac{B}{2}}{1-\tan\frac{A}{2}.\tan\frac{B}{2}}=\cot\frac{C}{2}=\frac{1}{\tan\frac{C}{2}}$$

$$\tan\frac{C}{2}\left(\tan\frac{A}{2}+\tan\frac{B}{2}\right)=1-\tan A/2\tan\frac{B}{2}$$

$$\tan\frac{C}{2}\tan\frac{A}{2}+\tan\frac{C}{2}.\tan\frac{B}{2}=1-\tan\frac{A}{2}\tan\frac{B}{2}$$

$$\therefore \tan\frac{A}{2}\tan\frac{B}{2}+\tan\frac{B}{2}\tan\frac{C}{2}+\tan\frac{C}{2}.\tan\frac{A}{2}=1$$

Hence Proved.

(vi) $\sin A+\sin B+\sin C=4\cos\frac{A}{2}\cos\frac{B}{2}.\cos\frac{C}{2}$

$A+B+C=\pi$

$\frac{A+B}{2}=\frac{\pi}{2}-\frac{C}{2}$

$\frac{C}{2}=\frac{\pi}{2}-\frac{A+B}{2}$

$$L.H.S=\sin A+\sin B+\sin C$$

$$=2\sin\frac{A+B}{2}.\cos\frac{A-B}{2}+\sin C$$

$$=2\sin\frac{A+B}{2}.\cos\frac{A-B}{2}+2\sin\frac{C}{2}.\cos\frac{C}{2}$$

$$=2\sin\left(\frac{\pi}{2}-\frac{C}{2}\right).\cos\frac{A-B}{2}+2\sin\frac{C}{2}.\cos\frac{C}{2}$$

$$=2\cos\frac{C}{2}.\cos\frac{A-B}{2}+2\sin\frac{C}{2}.\cos\frac{C}{2}$$

$$=2\cos\frac{C}{2}\left\{\cos\frac{A-B}{2}+\sin\frac{C}{2}\right\}$$

$$=2\cos\frac{C}{2}\left\{\cos\frac{A-B}{2}+\sin\left(\frac{\pi}{2}-\frac{A+B}{2}\right)\right\}$$

$$=2\cos\frac{C}{2}\left[\cos\frac{A-B}{2}+\cos\frac{A+B}{2}\right\}$$

$$=2\cos\frac{C}{2}\left[2\cos\frac{A}{2}.\cos\frac{B}{2}\right]$$

$$=4\cos\frac{A}{2}.\cos\frac{B}{2}.\cos\frac{C}{2}$$

$$=R.H.S.$$

(vii) $\sin(B+C-A)+\sin(C+A-B)+\sin(A+B-C) = 4\sin A\sin B\sin C$

$A+B+C=\pi$

$A+B=\pi-C \qquad B+C=\pi-A \qquad C+A=\pi-B$

$$
\begin{aligned}
L.H.S &= \sin(\pi-A-A)+\sin(\pi-B-B)+\sin(\pi-C-C)\\
&= \sin(\pi-2A)+\sin(\pi-2B)+\sin(\pi-2C)\\
&= \sin 2A+\sin 2B+\sin 2C\\
&= 2\sin(A+B)\cos(A-B)+2\sin C\cos C\\
&= 2\sin(\pi-C)\cos(A-B)+2\sin C\cos C\\
&= 2\sin C\cos(A-B)+2\sin C\cos C\\
&= 2\sin C(\cos(A-B)+\cos C)\\
&= 2\sin C[\cos(A-B)+\cos(\pi-\overline{A+B})]\\
&= 2\sin C[\cos(A-B)-\cos(A+B)]\\
&= 2\sin C[2\sin A\sin B]\\
&= 4\sin A\sin B\sin C\\
&= R.H.S
\end{aligned}
$$

2. If $A+B+C=2S$ then Prove that

$\sin(S-A).\sin(S-B)+\sin S\sin(S-C)=\sin A\sin B$

$$
\begin{aligned}
L.H.S &= \sin(S-A).\sin(S-B)+\sin C.\sin(S-C)\\
&= \frac{1}{2}[\cos[(S-A)-(S-B)]-\cos(S-A+S-B)]+\\
&\qquad \frac{1}{2}[\cos(S-\overline{S-C})-\cos(S+S-C)]\\
&= \frac{1}{2}[\cos(B-A)-\cos(2S-\overline{A+B})]+\frac{1}{2}[\cos C-\cos(2S-C)]\\
&= \frac{1}{2}[\cos(A-B)-\cos(C)]+\frac{1}{2}[\cos C-\cos(A+B)]\\
&= \frac{1}{2}[\cos(A-B)-\cancel{\cos C}+\cancel{\cos C}-\cos(A+B)]\\
&= \frac{1}{2}[\cos(A-B)-\cos(A+B)]\\
&= \frac{1}{2}[2\sin A\sin B] = \sin A\sin B = R.HS
\end{aligned}
$$

$A+B+C=2S$

$C=2S-\overline{A+B} \qquad A+B=2S-C$

$\sin A\sin B=\frac{1}{2}[\cos(A-B)-\cos(A+B)]$

3. If $x+y+z = xyz$ then Prove that

$$\frac{2x}{1-x^2} + \frac{2y}{1-y^2} + \frac{2z}{1-z^2} = \frac{2x}{1-x^2} \cdot \frac{2y}{1-y^2} \cdot \frac{2z}{1-z^2}$$

Let $x = \tan A$ $\quad y = \tan B \quad z = \tan C$

$$\frac{2x}{1-x^2} = \frac{2\tan A}{1-\tan^2 A} = \tan 2A \qquad \text{Illy } \frac{2y}{1-y^2} = \tan 2B$$

$$\frac{2z}{1-z^2} = \tan 2C$$

Given that $x+y+z = xyz$

$$\boxed{\begin{array}{c}\tan A + \tan B + \tan C = \tan A \tan B \tan C \\ \Rightarrow A+B+C = 180^\circ\end{array}}$$

multiply by 2

$$2A+2B+2C = 360^\circ$$

$$2A+2B = 360 - 2C$$

$$\tan(2A+2B) = \tan(360-2C)$$

$$\frac{\tan 2A + \tan 2B}{1-\tan 2A \tan 2B} = -\tan 2C$$

$$\tan 2A + \tan 2B = -\tan 2C\,(1-\tan 2A \tan 2B)$$

$$\tan 2A + \tan 2B + \tan 2C = \tan 2A \tan 2B \tan 2C$$

$$\frac{2x}{1-x^2} + \frac{2y}{1-y^2} + \frac{2z}{1-z^2} = \frac{2x}{1-x^2} \cdot \frac{2y}{1-y^2} \cdot \frac{2z}{1-z^2}$$

Hence Proved.

4. If $A+B+C = \frac{\pi}{2}$ Prove the following:

(i) $\sin 2A + \sin 2B + \sin 2C = 4\cos A \cos B \cos C$

$$\begin{aligned}
L.H.S &= \sin 2A + \sin 2B + \sin 2C \\
&= 2\sin(A+B)\cos(A-B) + 2\sin C\cos C \\
&= 2\sin\left(\tfrac{\pi}{2}-C\right)\cos(A-B) + 2\sin C\cos C \\
&= 2\cos C \cdot \cos(A-B) + 2\sin C\cos C \\
&= 2\cos C\left[\cos(A-B) + \sin\left(\tfrac{\pi}{2} - \overline{A+B}\right)\right] \\
&= 2\cos C\left[\cos(A-B) + \cos(A+B)\right] \\
&= 2\cos C\left[2\cos A \cos B\right] \\
&= 4\cos A \cos B \cos C \\
&= R.H.S.
\end{aligned}$$

(ii) $\cos 2A + \cos 2B + \cos 2C = 1 + 4\sin A \sin B \sin C$

$A+B+C = \frac{\pi}{2}$

$A+B = \frac{\pi}{2} - \frac{c}{2}$

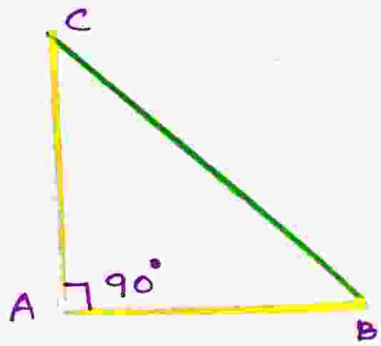
$$L.H.S = \cos 2A + \cos 2B + \cos 2C$$
$$= 2\cos(A+B)\cos(A-B) + \cos 2C$$
$$= 2\cos(\tfrac{\pi}{2} - C)\cos(A-B) + 1 - 2\sin^2 C$$
$$= 2\sin C \cos(A-B) + 1 - 2\sin^2 C$$
$$= 1 + 2\sin C[\cos(A-B) - \sin C]$$
$$= 1 + 2\sin C[\cos(A-B) - \sin(\tfrac{\pi}{2} - \overline{A+B})]$$
$$= 1 + 2\sin C[\cos(A-B) - \cos(A+B)]$$
$$= 1 + 2\sin C[2\sin A \sin B]$$
$$= 1 + 4\sin A \sin B \sin C = R.H.S.$$

5. If ΔABC is a right angle and if $\angle A = \frac{\pi}{2}$ then Prove that

(i) $\cos^2 B + \cos^2 C = 1$ (ii) $\sin^2 B + \sin^2 C = 1$

(iii) $\cos B - \cos C = -1 + 2\sqrt{2}\cos\frac{B}{2} \cdot \sin\frac{C}{2}$

(i) $\cos^2 B + \cos^2 C = 1$



$\angle A = 90°$

$\cos B = \frac{AB}{BC}$ $\cos C = \frac{AC}{BC}$

$$\cos^2 B + \cos^2 C = \frac{AB^2}{BC^2} + \frac{AC^2}{BC^2}$$
$$= \frac{AB^2 + AC^2}{BC^2} = \frac{BC^2}{BC^2} = 1$$

$\therefore \cos^2 B + \cos^2 C = 1.$

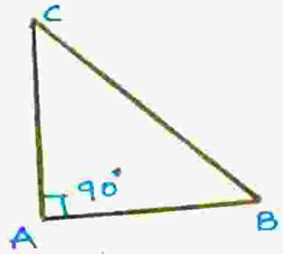
(ii) $\sin^2 B + \sin^2 C = 1$

$\sin B = \frac{AC}{BC}$ $\sin C = \frac{AB}{BC}$

$$\sin^2 B + \sin^2 C = \frac{AC^2}{BC^2} + \frac{AB^2}{BC^2} = \frac{AC^2 + AB^2}{BC^2}$$
$$= \frac{BC^2}{BC^2} = 1$$

$\therefore \sin^2 B + \sin^2 C = 1.$

(iii) $\cos B - \cos C = -1 + 2\sqrt{2}\cos\frac{B}{2}\cdot\sin\frac{C}{2}$



$B + C = 90$

$B = 90 - C$

$C = 90 - B$

$\frac{B}{2} = 45 - \frac{C}{2}$

$\frac{C}{2} = 45 - \frac{B}{2}$

$$\begin{aligned}
L.H.S &= \cos B - \cos C \\
&= \cos(90 - C) - \cos C \\
&= \sin C - \cos C \\
&= 2\sin\frac{C}{2}\cdot\cos\frac{C}{2} - \left(1 - 2\sin^2\frac{C}{2}\right) \\
&= 2\sin\frac{C}{2}\cdot\cos\frac{C}{2} - 1 + 2\sin^2\frac{C}{2} \\
&= -1 + 2\sin\frac{C}{2}\left[\cos\frac{C}{2} + \sin\frac{C}{2}\right] \\
&= -1 + 2\sin\frac{C}{2}\left[\cos\left(45 - \frac{B}{2}\right) + \sin\left(45 - \frac{B}{2}\right)\right] \\
&= -1 + 2\sin\frac{C}{2}\left\{\cos 45\cos\frac{B}{2} + \sin 45\sin\frac{B}{2} + \sin 45\cos\frac{B}{2} - \cos 45\cdot\sin\frac{B}{2}\right\} \\
&= -1 + 2\sin\frac{C}{2}\left[\frac{1}{\sqrt{2}}\cos\frac{B}{2} + \cancel{\frac{1}{\sqrt{2}}\sin\frac{B}{2}} + \frac{1}{\sqrt{2}}\cos\frac{B}{2} - \cancel{\frac{1}{\sqrt{2}}\sin\frac{B}{2}}\right] \\
&= -1 + 2\sin\frac{C}{2}\left[\frac{2}{\sqrt{2}}\cos\frac{B}{2}\right] \\
&= -1 + 2\sqrt{2}\cos\frac{B}{2}\cdot\sin\frac{C}{2} = R.H.S
\end{aligned}$$

Hence Proved.

Trigonometric Equations:

General Solutions:

$\sin\theta = 0 \Rightarrow \theta = n\pi,\ n\in Z$

$\cos\theta = 0 \Rightarrow \theta = (2n+1)\frac{\pi}{2},\ n\in Z$

$\tan\theta = 0 \Rightarrow \theta = n\pi,\ n\in Z$

$\sin\theta = \sin\alpha \Rightarrow \theta = n\pi + (-1)^n\alpha\quad n\in Z$

$\cos\theta = \cos\alpha \Rightarrow \theta = 2n\pi \pm \alpha,\ n\in Z$

$\tan\theta = \tan\alpha \Rightarrow \theta = n\pi + \alpha\quad n\in Z.$

$\sin^2\theta = \sin^2\alpha \Rightarrow x = n\pi \pm \alpha,\ n\in Z$

$\cos^2\theta = \cos^2\alpha \Rightarrow x = n\pi \pm \alpha,\ n\in Z$

$\tan^2\theta = \tan^2\alpha \Rightarrow x = n\pi \pm \alpha,\ n\in Z.$

Principal Solution :

The smallest numerical value of unknown angle satisfying the equation in the interval $[0, 2\pi]$ or $[-\pi, \pi]$ is called a Principal Solution.

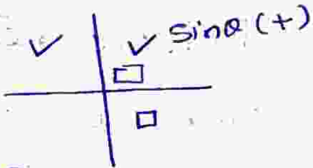
* Principal value of sine function lies in the interval $\left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$ and hence lies in I and IV quadrant.

* Principal value of cosine function lies in the interval $[0, \pi]$ and hence lies in I & II quadrant.

* Principal value of tangent function lies in the interval $\left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$ and hence lies in I & IV quadrant,

3.42 Find the Principal solution of

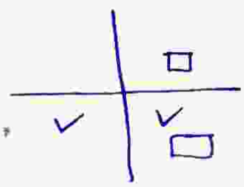
(i) $\sin\theta = \frac{1}{2}$



$\sin\theta = \frac{1}{2} = \sin\frac{\pi}{6}$

$\theta = \frac{\pi}{6} \in \left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$ [* $\sin\theta > 0$
* Principal value lies in I & II quad]

(ii) $\sin\theta = -\frac{\sqrt{3}}{2}$

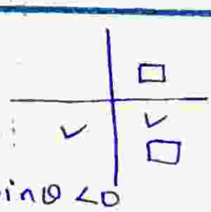


$\sin\theta = -\frac{\sqrt{3}}{2} = \sin(-\pi/3)$

$\theta = -\frac{\pi}{3} \in \left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$

[$\sin\theta < 0$
Principal vale lies in I & IV quad]

(iii) $\operatorname{cosec}\theta = -2$



$\operatorname{cosec}\theta = -2$

$\sin\theta = -\frac{1}{2}$

$\sin\theta = \sin\left(-\frac{\pi}{6}\right)$

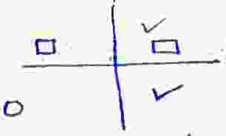
$\theta = -\frac{\pi}{6} \in \left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$

[* $\sin\theta < 0$
* Principal value lies in I & IV quad]

(iv) $\cos\theta = \frac{1}{2}$

$\cos\theta = \cos\frac{\pi}{3}$

$\theta = \frac{\pi}{3} \in [0, \pi]$



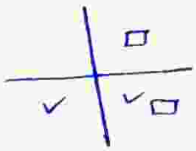
[* $\cos\theta > 0$
* Principal value lies in I and II quad]

3.43. Find the General Solution of $\sin\theta = -\frac{\sqrt{3}}{2}$

$\sin\theta = -\frac{\sqrt{3}}{2}$

$\sin\theta = \sin\left(-\frac{\pi}{3}\right)$

$\theta = -\frac{\pi}{3} \in \left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$



General solution

$\sin\theta = \sin\alpha$

$\theta = n\pi + (-1)^n \alpha, \quad n \in Z$

$\theta = n\pi + (-1)^n \left(-\frac{\pi}{3}\right) \quad n \in Z$

$$\boxed{= n\pi + (-1)^{n+1} \cdot \frac{\pi}{3} \quad n \in Z.}$$

3.44 / 129 Find the general solution of

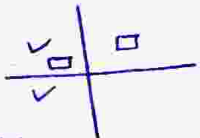
(i) $\sec\theta = -2$ (ii) $\tan\theta = \sqrt{3}$

(i) $\sec\theta = -2$

$\cos\theta = -\frac{1}{2}$

$\cos\theta = \cos\left(\pi - \frac{\pi}{3}\right)$

$= \frac{2\pi}{3} \in [0, \pi]$



[* $\cos\theta < 0$
* Principal value lies in I & II quad]

General Solution :

$\cos\theta = \cos\alpha$ $\quad \alpha = \frac{2\pi}{3}$

$\theta = 2n\pi \pm \alpha \quad n \in Z$

$$\boxed{\theta = 2n\pi \pm \frac{2\pi}{3} \quad n \in Z.}$$

(ii) $\tan\theta = \sqrt{3}$

$\tan\theta = \tan\frac{\pi}{3}$

$\theta = \frac{\pi}{3} \in \left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$

$\alpha = \frac{\pi}{3}$

General Solution :

$\tan\theta = \tan\alpha$

$\theta = n\pi + \alpha \quad n \in Z$

$$\boxed{\theta = n\pi + \frac{\pi}{3} \quad n \in Z.}$$

3.45 Solve : $3\cos^2\theta = \sin^2\theta$

$$3\cos^2\theta = \sin^2\theta$$
$$3\cos^2\theta = 1-\cos^2\theta$$
$$4\cos^2\theta = 1 \Rightarrow \cos^2\theta = \frac{1}{4}$$
$$\frac{1+\cos 2\theta}{2} = \frac{1}{4}$$
$$1+\cos 2\theta = \frac{1}{2}$$
$$\cos 2\theta = \frac{1}{2}-1 = \frac{-1}{2}$$
$$\cos 2\theta = \cos(\pi - \pi/3)$$
$$\cos 2\theta = \cos 2\pi/3$$
$$2\theta = 2\pi/3$$

$$\boxed{\begin{array}{ll} 2\theta = 2n\pi \pm \alpha & n\in Z \\ 2\theta = 2n\pi \pm 2\pi/3 & n\in Z \\ \theta = n\pi \pm \pi/3 & n\in Z \end{array}}$$

3.46 / 130 Solve : $\sin x + \sin 5x = \sin 3x$

$$\sin x + \sin 5x = \sin 3x$$
$$2\sin\left(\frac{5x+x}{2}\right)\cdot\cos\left(\frac{5x-x}{2}\right) = \sin 3x$$
$$2\sin 3x\cdot\cos 2x - \sin 3x = 0$$
$$\sin 3x(2\cos 2x - 1) = 0$$

$\sin 3x = 0$

$\sin 3x = \sin 0$

$3x = 0$

$3x = n\pi \quad n\in Z$

$$\boxed{x = \frac{n\pi}{3} \quad n\in Z}$$

$2\cos 2x - 1 = 0$

$\cos 2x = \frac{1}{2}$

$\cos 2x = \cos\frac{\pi}{3}$

$2x = 2n\pi \pm \alpha \quad n\in Z$

$= 2n\pi \pm \frac{2\pi}{6} \quad n\in Z$

$$\boxed{x = n\pi \pm \frac{\pi}{6} \quad n\in Z}$$

3.47 / 130 Solve : $\cos x + \sin x = \cos 2x + \sin 2x$

$$\cos x - \cos 2x = \sin 2x - \sin x$$
$$-2\sin\left(\frac{x+2x}{2}\right)\cdot\sin\left(\frac{x-2x}{2}\right) = 2\cos\left(\frac{x+2x}{2}\right)\cdot\sin\left(\frac{2x-x}{2}\right)$$
$$\Rightarrow -2\sin\frac{3x}{2}\cdot\sin\left(\frac{-x}{2}\right) = 2\cos\frac{3x}{2}\cdot\sin\frac{x}{2}$$
$$2\sin\frac{3x}{2}\sin\frac{x}{2} - 2\cos\frac{3x}{2}\sin\frac{x}{2} = 0$$

$$2\sin\frac{x}{2}\left(\sin\frac{3x}{2} - \cos\frac{3x}{2}\right) = 0$$

$2\sin\frac{x}{2} = 0$

$\sin\frac{x}{2} = 0$

$\sin\frac{x}{2} = \sin 0$

$\frac{x}{2} = n\pi \pm \alpha \quad n \in z$

$\frac{x}{2} = n\pi \quad n \in z$

$\boxed{x = 2n\pi \quad n \in z.}$

$\sin\frac{3x}{2} - \cos\frac{3x}{2} = 0$

$\sin\frac{3x}{2} = \cos\frac{3x}{2}$

$\tan\frac{3x}{2} = 1$

$\tan\frac{3x}{2} = \tan\frac{\pi}{4}$

$\frac{3x}{2} = n\pi + \alpha \quad n \in z$

$\frac{3x}{2} = n\pi + \frac{\pi}{4} \quad n \in z$

$3x = 2n\pi + \frac{\pi}{2} \quad n \in z$

$\boxed{x = \frac{2n\pi}{3} + \frac{\pi}{6} \quad n \in z}$

3.46 / 130 Solve the equation : $\sin 9\theta = \sin\theta$

$\sin 9\theta = \sin\theta$

$\sin 9\theta - \sin\theta = 0$

$2\cos\frac{9\theta+\theta}{2} \cdot \sin\left(\frac{9\theta-\theta}{2}\right) = 0$

$2\cos 5\theta \cdot \sin 4\theta = 0$

$2\cos 5\theta = 0$

$\cos 5\theta = 0$

$5\theta = (2n+1)\frac{\pi}{2} \quad n \in z$

$\theta = (2n+1)\frac{\pi}{10} \quad n \in z.$

$\sin 4\theta = 0$

$4\theta = n\pi \quad n \in z$

$\theta = \frac{n\pi}{4} \quad n \in z.$

3.49 / 131. Solve : $\tan 2x = -\cot\left(x + \frac{\pi}{3}\right)$

$\tan 2x = \tan\left(\frac{\pi}{2} + \left(x + \frac{\pi}{3}\right)\right)$

$\tan 2x = \tan\left(\frac{5\pi}{6} + x\right)$

$2x = n\pi + \alpha \quad n \in z$

$2x = n\pi + \frac{5\pi}{6} + x \quad n \in z$

$2x - x = n\pi + \frac{5\pi}{6}$

$x = n\pi + \frac{5\pi}{6} \quad n \in z.$

3.50 / 131. Solve : $\sin x - 3\sin 2x + \sin 3x = \cos x - 3\cos 2x + \cos 3x$

$\sin x - 3\sin 2x + \sin 3x = \cos x - 3\cos 2x + \cos 3x$

$\sin 3x + \sin x - 3\sin 2x = \cos 3x + \cos x - 3\cos 2x$

$2\sin 2x \cdot \cos x - 3\sin 2x = 2\cos 2x \cdot \cos x - 3\cos 2x$

$\sin 2x(2\cos x - 3) = \cos 2x(2\cos x - 3)$

$\sin 2x(2\cos x - 3) - \cos 2x(2\cos x - 3)$

$(\sin 2x - \cos 2x)(2\cos x - 3) = 0$

$\sin 2x - \cos 2x = 0$ $\qquad$ $2\cos x - 3 = 0$

$\tan 2x = 1 = \tan \pi/4$ $\qquad$ $2\cos x = 3$

$2x = n\pi + \alpha$ $\qquad$ $\boxed{\cos x \neq \frac{3}{2}}$

$2x = n\pi + \frac{\pi}{4}$

$\boxed{x = \frac{n\pi}{2} + \frac{\pi}{8},\ n \in Z}$

3.51 / 131 Solve : $\sin x + \cos x = 1 + \sin x \cos x$

$\sin x + \cos x = 1 + \sin x \cos x$ ——— (1)

Let $t = \sin x + \cos x$

$t^2 = (\sin x + \cos x)^2 = \sin^2 x + \cos^2 x + 2\sin x \cos x$

$t^2 = 1 + 2\sin x \cos x$

$\sin x \cos x = \frac{t^2 - 1}{2}$

(1) becomes

$t = 1 + \frac{t^2 - 1}{2} \Rightarrow 2t = 2 + t^2 - 1$

$t^2 - 2t + 1 = 0$

$(t-1)^2 = 0 \Rightarrow t = 1$

$\therefore \sin x + \cos x = 1$

$\sqrt{2}\left(\frac{1}{\sqrt{2}}\sin x + \frac{1}{\sqrt{2}}\cos x\right) = 1$

$\frac{1}{\sqrt{2}}\sin x + \frac{1}{\sqrt{2}}\cos x = \frac{1}{\sqrt{2}}$

$\cos\left(\frac{\pi}{4} - x\right) = \frac{1}{\sqrt{2}} = \cos\frac{\pi}{4}$

$\frac{\pi}{4} - x = 2n\pi \pm \pi/4 \quad n \in Z$

$x - \frac{\pi}{4} = 2n\pi \pm \pi/4 \quad n \in Z$

$\boxed{x = 2n\pi + \frac{\pi}{2} \text{ or } x = 2n\pi \quad n \in Z.}$

3.52 / 132 Solve : $2\sin^2 x + \sin^2 2x = 2$

$$2\sin^2 x + (2\sin x\cos x)^2 = 2$$

$$\sin^2 x + 2\sin^2 x\cos^2 x = 1$$

$$\cancel{1} - \cos^2 x + 2\sin^2 x\cos^2 x = \cancel{1}$$

$$\cos^2 x(2\sin^2 x - 1) = 0$$

$\cos^2 x = 0$

$\cos x = 0$

$$\boxed{x = (2n+1)\frac{\pi}{2} \quad n \in Z}$$

$2\sin^2 x - 1 = 0$

$2\sin^2 x = 1$

$\sin^2 x = \frac{1}{2}$

$\sin^2 x = \sin^2\frac{\pi}{4}$

$x = n\pi \pm \alpha \quad n \in Z$

$$\boxed{x = n\pi \pm \frac{\pi}{4} \quad n \in Z.}$$

3.53 Prove that for any a and b $-\sqrt{a^2+b^2} \le a\sin\theta + b\cos\theta \le \sqrt{a^2+b^2}$

Proof :-

$$a\sin\theta + b\cos\theta = \sqrt{a^2+b^2}\left\{\frac{a}{\sqrt{a^2+b^2}}\sin\theta + \frac{b}{\sqrt{a^2+b^2}}\cos\theta\right\}$$

$$= \sqrt{a^2+b^2}\{\cos\alpha\sin\theta + \sin\alpha\cos\theta\}$$

where $\cos\alpha = \frac{a}{\sqrt{a^2+b^2}}$

$\sin\alpha = \frac{b}{\sqrt{a^2+b^2}}$

$$= \sqrt{a^2+b^2}\sin(\alpha+\theta)$$

$\therefore |a\sin\theta + b\cos\theta| \le \sqrt{a^2+b^2}$

$|x| \le 1 \iff -1 \le x \le 1$

$$\Rightarrow \boxed{-\sqrt{a^2+b^2} \le a\sin\theta + b\cos\theta \le \sqrt{a^2+b^2}}$$

Hence Proved.

3.54. Solve : $\sqrt{3}\sin\theta - \cos\theta = \sqrt{2}$

$a = \sqrt{3}$

$b = -1$

$\sqrt{a^2+b^2} = \sqrt{3+1} = 2$

$$\sqrt{3}\sin\theta - \cos\theta = \sqrt{2}$$

$$\therefore 2\left(\frac{\sqrt{3}}{2}\sin\theta - \frac{1}{2}\cos\theta\right) = \sqrt{2}$$

$$\frac{\sqrt{3}}{2}\sin\theta - \frac{1}{2}\cos\theta = \frac{1}{\sqrt{2}}$$

$$\cos\frac{\pi}{6}\sin\theta - \sin\frac{\pi}{6}\cos\theta = \frac{1}{\sqrt{2}}$$

$$\sin\left(\theta - \frac{\pi}{6}\right) = \sin\pi/4$$

$$\theta - \frac{\pi}{6} = n\pi + (-1)^n\frac{\pi}{4} \quad n \in Z$$

$$\boxed{\theta = n\pi + \frac{\pi}{6} + (-1)^n\frac{\pi}{4} \quad n \in Z}$$

3.55 / 133 Solve : $\sqrt{3}\tan^2\theta + (\sqrt{3}-1)\tan\theta - 1 = 0$

$\sqrt{3}\tan^2\theta + \sqrt{3}\tan\theta - \tan\theta - 1 = 0$

$\sqrt{3}\tan\theta(\tan\theta + 1) - 1(\tan\theta + 1) = 0$

$(\sqrt{3}\tan\theta - 1)(\tan\theta + 1) = 0$

$\sqrt{3}\tan\theta - 1 = 0$

$\tan\theta = \frac{1}{\sqrt{3}}$

$\tan\theta = \tan \pi/6$

$\theta = n\pi + \alpha \quad n \in z$

$\boxed{\theta = n\pi + \frac{\pi}{6} \quad n \in z}$

$\tan\theta + 1 = 0$

$\tan\theta = -1$

$\tan\theta = \tan(-\pi/4)$

$\boxed{\theta = n\pi - \frac{\pi}{4} \quad n \in z.}$

Ex : 3.8

1 / 133 Find the Principal solution and general solutions of the following :

(i) $\sin\theta = \frac{-1}{\sqrt{2}}$

$\sin\theta = \sin(-\pi/4)$

$\theta = -\pi/4 \in [-\frac{\pi}{2}, \frac{\pi}{2}]$

$\therefore$ The Principal solution is $-\pi/4 \in [-\frac{\pi}{2}, \frac{\pi}{2}]$.

General Solution : $\theta = n\pi + (-1)^n\alpha \quad n \in z$

$\theta = n\pi + (-1)^n(-\pi/4) \quad n \in z$

$= n\pi + (-1)^{n+1}(\pi/4); n \in z$

(ii) $\cot\theta = \sqrt{3}$

$\tan\theta = \frac{1}{\sqrt{3}} = \tan(\pi/6)$

$\therefore \theta = \pi/6 \in (-\pi/2, \pi/2)$

$\therefore$ Principal solution is $\pi/6 \in (-\pi/2, \pi/2)$

General solution : $\theta = n\pi + \alpha \quad n \in z$

$\theta = n\pi + \pi/6 \quad n \in z.$

(iii) $\tan\theta = -\frac{1}{\sqrt{3}}$

$\tan\theta = \tan(-\frac{\pi}{6})$

$\theta = -\pi/6 \in (-\pi/2, \pi/2)$

$\therefore$ Principal solution is $-\pi/6 \in (-\pi/2, \pi/2)$

General solution : $\theta = n\pi + \alpha \quad n \in z$

$\theta = n\pi + (-\pi/6) \quad n \in z.$

$\frac{2}{133}$ Solve the following equations for which solutions lies in the interval $0^\circ \le \theta \le 360^\circ$

(i) $\sin^4 x = \sin^2 x$

$\sin^4 x - \sin^2 x = 0 \Rightarrow \sin^2 x\,(1-\sin^2 x) = 0$

$\sin^2 x \cos^2 x = 0 \Rightarrow \left(\frac{1}{2}\cdot 2\sin x\cos x\right)^2 = 0$

$\frac{1}{4}\sin^2 2x = 0$

$\sin^2 2x = 0$

$\sin 2x = 0$

$2x = 0, \pi, 2\pi, 3\pi, 4\pi \ldots$

$x = 0, \frac{\pi}{2}, \pi, \frac{3\pi}{2}, 2\pi \ldots$

$x = 0, \frac{\pi}{2}, \pi, \frac{3\pi}{2}, 2\pi \in [0, 2\pi]$.

(ii) $2\cos^2 x + 3\cos x + 1 = 0$

$(2\cos x + 1)(\cos x + 1) = 0$

$\cos x = -\frac{1}{2}$ $\qquad$ $\cos x = -1$

$x = \pi - \frac{\pi}{3} = \frac{2\pi}{3}$ $\qquad$ $x = \pi$

$\therefore\ x = \frac{2\pi}{3}, \pi \in [0, 2\pi]$

(iii) $2\sin^2 x + 1 = 3\sin x$

$2\sin^2 x - 3\sin x + 1 = 0$

$(2\sin x - 1)(\sin x - 1) = 0$

$\sin x = \frac{1}{2}$ $\qquad$ $\sin x = 1$

$\sin x = \sin\frac{\pi}{6}$ $\qquad$ $\sin x = \sin \pi/2$

$x = \frac{\pi}{6}, \frac{5\pi}{6}$

$\therefore\ x = \frac{\pi}{2}, \pi/6, \frac{5\pi}{6}$

(iv) $\cos 2x = 1 - 3\sin x$

$\cancel{1} - 2\sin^2 x = \cancel{1} - 3\sin x$

$2\sin^2 x - 3\sin x = 0$

$\sin x\,(2\sin x - 3) = 0$

$\sin x = 0$ $\qquad$ $\sin x \ne \frac{3}{2}$

$x = 0, \pi$

$\therefore\ x = 0, \pi \in [0, 2\pi]$

$\frac{3}{133}$ Solve the following equations :-

(i) $\sin 5x - \sin x = \cos 3x$

$$\sin 5x - \sin x = \cos 3x$$

$$2\cos 3x \cdot \sin 2x - \cos 3x = 0$$

$$\cos 3x\,(2\sin 2x - 1) = 0$$

$\cos 3x = 0$

$3x = (2n+1)\frac{\pi}{2}$ $n \in Z$

$$\boxed{x = (2n+1)\frac{\pi}{6} \quad n \in Z}$$

$2\sin 2x = 1$

$\sin 2x = \frac{1}{2} = \sin\frac{\pi}{6}$

$2x = n\pi + (-1)^n \alpha$ $n \in Z$

$2x = n\pi + (-1)^n \frac{\pi}{6}$ $n \in Z$

$$\boxed{x = \frac{n\pi}{2} + (-1)^n \cdot \frac{\pi}{12} \quad n \in Z}$$

(ii) $2\cos^2\theta + 3\sin\theta - 3 = 0$

$$2(1 - \sin^2\theta) + 3\sin\theta - 3 = 0$$

$$2 - 2\sin^2\theta + 3\sin\theta - 3 = 0$$

$$2\sin^2\theta - 3\sin\theta + 1 = 0$$

$$(2\sin\theta - 1)(\sin\theta - 1) = 0$$

2

-1 -2

$\frac{-1}{2\sin\theta}$ $\frac{-2}{2\sin\theta}$

$\sin\theta = \frac{1}{2}$

$\sin\theta = \sin\frac{\pi}{6}$

$$\boxed{\theta = n\pi + (-1)^n \frac{\pi}{6} \quad n \in Z}$$

$\sin\theta = 1$

$\sin\theta = \sin\frac{\pi}{2}$

$$\boxed{\theta = n\pi + (-1)^n \frac{\pi}{2},\ n \in Z}$$

(iii) $\cos\theta + \cos 3\theta = 2\cos 2\theta$

$$2\cos 2\theta \cdot \cos\theta = 2\cos 2\theta$$

$$2\cos 2\theta\,(\cos\theta - 1) = 0$$

$2\cos 2\theta = 0$

$\cos 2\theta = 0$

$2\theta = (2n+1)\frac{\pi}{2}$ $n \in Z$

$$\boxed{\theta = (2n+1)\frac{\pi}{4} \quad n \in Z}$$

$\cos\theta = 1$

$\cos\theta = \cos 0$

$\theta = 2n\pi \pm \alpha$ $n \in Z$

$$\boxed{\theta = 2n\pi \quad n \in Z.}$$

(iv) $\sin\theta + \sin 3\theta + \sin 5\theta = 0$

$\sin 5\theta + \sin\theta + \sin 3\theta = 0$

$2\sin 3\theta . \cos 2\theta + \sin 3\theta = 0$

$\sin 3\theta (2\cos 2\theta + 1) = 0$

$\sin 3\theta = 0$

$3\theta = n\pi ; n \in z$

$\boxed{\theta = \frac{n\pi}{3} ; n \in z}$

$2\cos 2\theta + 1 = 0$

$\cos 2\theta = \frac{-1}{2}$

$\cos 2\theta = \cos(\pi - \pi/3)$

$\cos 2\theta = \cos 2\pi/3$

$2\theta = 2n\pi \pm \alpha \quad n \in z$

$2\theta = 2n\pi \pm \frac{2\pi}{3} \quad n \in z$

$\boxed{\theta = n\pi \pm \frac{\pi}{3} ; n \in z}$

(v) $\sin 2\theta - \cos 2\theta - \sin\theta + \cos\theta = 0$

$(\sin 2\theta - \sin\theta) - (\cos 2\theta - \cos\theta) = 0$

$2\cos\frac{3\theta}{2} . \sin\frac{\theta}{2} - (-2\sin\frac{3\theta}{2} . \sin\frac{\theta}{2}) = 0$

$2\sin\theta/2 \left(\cos 3\theta/2 + \sin\frac{3\theta}{2}\right) = 0$

$2\sin\theta/2 = 0$

$\sin\frac{\theta}{2} = 0$

$\frac{\theta}{2} = n\pi ; n \in z$

$\boxed{\theta = 2n\pi \quad n \in z}$

$\cos\frac{3\theta}{2} + \sin\frac{3\theta}{2} = 0$

$\sin\frac{3\theta}{2} = -\cos\frac{3\theta}{2}$

$\tan\frac{3\theta}{2} = -1$

$\tan\frac{3\theta}{2} = \tan(-\pi/4)$

$\frac{3\theta}{2} = n\pi + \alpha \quad n \in z$

$\frac{3\theta}{2} = n\pi - \frac{\pi}{4} ; n \in z$

$3\theta = 2n\pi - \frac{\pi}{2} \quad n \in z$

$\boxed{\theta = \frac{2n\pi}{3} - \frac{\pi}{6} \quad n \in z}$

(vi) $\sin\theta + \cos\theta = \sqrt{2}$

$a = 1 \quad b = 1$

$\sqrt{a^2+b^2} = \sqrt{1+1} = \sqrt{2}$

$\sqrt{2}\left(\frac{1}{\sqrt{2}}\sin\theta + \frac{1}{\sqrt{2}}\cos\theta\right) = \sqrt{2}$

$\sin\frac{\pi}{4} . \sin\theta + \cos\frac{\pi}{4}\cos\theta = 1$

$\cos(\theta - \frac{\pi}{4}) = 1 = \cos 0$

$\theta - \frac{\pi}{4} = 2n\pi + \alpha \quad n \in z$

$\theta - \pi/4 = 2n\pi$

$\boxed{\theta = 2n\pi + \frac{\pi}{4} \quad n \in z}$

(vii) $\sin\theta + \sqrt{3}\cos\theta = 1$

$a = 1$
$b = \sqrt{3}$
$\sqrt{a^2+b^2} = \sqrt{3+1}$
$= 2$

$\frac{1}{2}\sin\theta + \frac{\sqrt{3}}{2}\cos\theta = \frac{1}{2}$

$\sin\frac{\pi}{6}\sin\theta + \cos\frac{\pi}{6}\cos\theta = \frac{1}{2}$

$\cos(\theta - \pi/6) = \cos\frac{\pi}{3}$

$\theta - \pi/6 = 2n\pi \pm \frac{\pi}{3}$

$$\boxed{\theta = 2n\pi \pm \frac{\pi}{3} + \frac{\pi}{6} \qquad n \in Z.}$$

(viii) $\cot\theta + \operatorname{cosec}\theta = \sqrt{3}$

$\frac{\cos\theta}{\sin\theta} + \frac{1}{\sin\theta} = \sqrt{3}$

$1 + \cos\theta = \sqrt{3}\sin\theta$

$a = \sqrt{3}$
$b = 1$
$\sqrt{a^2+b^2} = 2$

$\sqrt{3}\sin\theta - \cos\theta = 1$

$\frac{\sqrt{3}}{2}\sin\theta - \frac{1}{2}\cos\theta = \frac{1}{2}$

$\sin\frac{\pi}{3}\sin\theta - \cos\frac{\pi}{3}\cos\theta = \frac{1}{2}$

$\cos\frac{\pi}{3}\cos\theta - \sin\frac{\pi}{3}\sin\theta = -\frac{1}{2}$

$\cos\left(\theta + \frac{\pi}{3}\right) = \cos(2\pi/3)$

$\theta + \frac{\pi}{3} = 2n\pi \pm \alpha \qquad n \in Z$

$\theta + \frac{\pi}{3} = 2n\pi \pm 2\pi/3$

$$\boxed{\theta = 2n\pi \pm 2\frac{\pi}{3} - \pi/3 \qquad n \in Z.}$$

(ix) $\tan\theta + \tan\left(\theta + \frac{\pi}{3}\right) + \tan\left(\theta + \frac{2\pi}{3}\right) = \sqrt{3}$

$\tan\theta + \frac{\tan\theta + \tan\pi/3}{1 - \tan\theta\tan\pi/3} + \frac{\tan\theta + \tan 2\pi/3}{1 - \tan\theta\tan 2\pi/3} = \sqrt{3}$

$\tan\theta + \frac{\tan\theta + \sqrt{3}}{1 - \sqrt{3}\tan\theta} + \frac{\tan\theta - \sqrt{3}}{1 + \sqrt{3}\tan\theta} = \sqrt{3}$

$\tan\theta + \frac{(\tan\theta + \sqrt{3})(1 + \sqrt{3}\tan\theta) + (\tan\theta - \sqrt{3})(1 - \sqrt{3}\tan\theta)}{1 - 3\tan^2\theta} = \sqrt{3}$

$\tan\theta + \frac{\tan\theta + \sqrt{3}\cancel{\tan^2\theta} + \cancel{\sqrt{3}} + 3\tan\theta + \tan\theta - \sqrt{3}\cancel{\tan^2\theta} - \cancel{\sqrt{3}} + 3\tan\theta}{1 - 3\tan^2\theta} = \sqrt{3}$

$\tan\theta + \frac{8\tan\theta}{1 - 3\tan^2\theta} = \sqrt{3}$

$\frac{\tan\theta - 3\tan^3\theta + 8\tan\theta}{1 - 3\tan^2\theta} = \sqrt{3}$

$\frac{9\tan\theta - 3\tan^3\theta}{1 - 3\tan^2\theta} = \sqrt{3}$

$$\frac{3(3\tan\theta - \tan^3\theta)}{1-3\tan^2\theta} = \sqrt{3}$$

$$3\tan 3\theta = \sqrt{3}$$

$$\tan 3\theta = \frac{1}{\sqrt{3}}$$

$$\tan 3\theta = \tan\frac{\pi}{6}$$

$$3\theta = n\pi + \alpha \quad n\in Z$$

$$3\theta = n\pi + \frac{\pi}{6} \quad n\in Z$$

$$\boxed{\theta = \frac{n\pi}{3} + \frac{\pi}{18} \quad n\in Z}$$

(x) $\cos 2\theta = \frac{\sqrt{5}+1}{4}$

$36^\circ = \frac{2\pi}{10}$

$$\cos 2\theta = \cos 36^\circ$$

$$\cos 2\theta = \cos\frac{2\pi}{10}$$

$$2\theta = 2n\pi \pm \alpha \quad n\in Z$$

$$2\theta = 2n\pi \pm \frac{2\pi}{10}$$

$$\boxed{\theta = n\pi \pm \frac{\pi}{10} \quad n\in Z}$$

(xi) $2\cos^2 x - 7\cos x + 3 = 0$

$(2\cos x - 1)(\cos x - 3) = 0$

6

-1, $-\cancel{6}\ -3$

$\frac{-1}{2\cos x}$, $\frac{-\cancel{6}}{\cancel{2}\cos x}$

$2\cos x - 1 = 0 \qquad \cos x \neq 3$

$$\cos x = \frac{1}{2}$$

$$\cos x = \cos\frac{\pi}{3}$$

$$x = 2n\pi \pm \alpha \quad n\in Z$$

$$\boxed{x = 2n\pi \pm \frac{\pi}{3} \quad n\in Z.}$$