பள்ளிக்கல்வித்துறை

சிவகங்கை மாவட்டம்

சிறப்பு வழிகாட்டி

வகுப்பு - 12

கணிதம்

திரு.ப.மதுசூதன் ரெட்டிஇ இ.ஆ.ப
மாவட்ட ஆட்சியர்
சிவகங்கை மாவட்டம்

திரு.இரா.சுவாமிநாதன்
முதன்மைக்கல்வி அலுவலர்
சிவகங்கை மாவட்டம்

திரு.செ.சண்முகநாதன் மாவட்டக்கல்வி அலுவலர் தேவகோட்டை	திரு.ஜி.முத்துசாமி மாவட்டக்கல்வி அலுவலர் சிவகங்கை	திருமதி.சி.பாலதிரிபுரசுந்தரி மாவட்டக்கல்விஅலுவலர்(பொ) திருப்பத்தூர்

ஒருங்கிணைப்பாளர்
திருமதி.ஜா.பிரபா
தலைமை ஆசிரியர்
அரசு மேல்நிலைப்பள்ளி
தமராக்கி

ஆசிரியர் குழு

திரு.அரு.முத்துராமன் எஸ்.வி.கே. மேல்நிலைப்பள்ளி ஆ.தெக்கூர்	திரு.சு.சந்தானகிருஷ்ணன் முருகப்பா மேல்நிலைப்பள்ளி கல்லல்
திரு.செ.ராஜீஸ் கண்ணன் அரசு மேல்நிலைப்பள்ளி பாகனேரி	திரு.மெ.சுந்தரபாண்டியன் அரசு மாதிரி மேல்நிலைப்பள்ளி கீழப்பூங்குடி
திரு.ச.வினோத்குமார் அரசு மேல்நிலைப்பள்ளி மாங்குடி.	திரு.வீ.ஞானமுருகன் அரசு மேல்நிலைப்பள்ளி ஓ.சிறுவயல்.

1.APPLICATIONS OF MATRICES AND DETERMINANTS

2 MARK QUESTIONS

1. *Find the adjoint of* $\frac{1}{3}\begin{bmatrix}2 & 2 & 1\\ -2 & 1 & 2\\ 1 & -2 & 2\end{bmatrix}$

$\boxed{adj(\lambda A) = \lambda^{n-1} adj(A)}$

$$adj\ A = \left(\frac{1}{3}\right)^{3-1}\begin{bmatrix}2+4 & -2-4 & 4-1\\ 2+4 & 4-1 & -2-4\\ 4-1 & 2+4 & 2+4\end{bmatrix} = \frac{1}{9}\begin{bmatrix}6 & -6 & 3\\ 6 & 3 & -6\\ 3 & 6 & 6\end{bmatrix} = \frac{1}{3}\begin{bmatrix}2 & -2 & 1\\ 2 & 1 & -2\\ 1 & 2 & 2\end{bmatrix}$$

2. *If* $A = \begin{bmatrix}a & b\\ c & d\end{bmatrix}$ *is non − singular, find* A^{-1}.

$|A| = ad - bc$

$adj\ A = \begin{bmatrix}d & -b\\ -c & a\end{bmatrix}$

$$A^{-1} = \frac{1}{|A|}(adj\ A) = \frac{1}{ad-bc}\begin{bmatrix}d & -b\\ -c & a\end{bmatrix}$$

3. *Find the inverse of* $\begin{bmatrix}2 & 3 & 1\\ 3 & 4 & 1\\ 3 & 7 & 2\end{bmatrix}$

$$|A| = \begin{vmatrix}2 & 3 & 1\\ 3 & 4 & 1\\ 3 & 7 & 2\end{vmatrix} = 2(8-7) - 3(6-3) + 1(21-12) = 2 - 9 + 9 = 2 \neq 0$$

$$\begin{matrix}4 & 7 & 3 & 4\\ 1 & 2 & 1 & 1\\ 3 & 3 & 2 & 3\\ 4 & 7 & 3 & 4\end{matrix} \Rightarrow adj\ A = \begin{bmatrix}8-7 & 7-6 & 3-4\\ 3-6 & 4-3 & 3-2\\ 21-12 & 9-14 & 8-9\end{bmatrix} = \begin{bmatrix}1 & 1 & -1\\ -3 & 1 & 1\\ 9 & -5 & -1\end{bmatrix}$$

$$A^{-1} = \frac{1}{|A|} adj\ A \Rightarrow A^{-1} = \frac{1}{2}\begin{bmatrix}1 & 1 & -1\\ -3 & 1 & 1\\ 9 & -5 & -1\end{bmatrix}$$

4. *If* $A = \frac{1}{9}\begin{bmatrix}-8 & 1 & 4\\ 4 & 4 & 7\\ 1 & -8 & 4\end{bmatrix}$, *prove that* $A^{-1} = A^T$.

$A^{-1} = A^T \Longrightarrow AA^T = I$ *is enough to prove..*

$$A^T = \frac{1}{9}\begin{bmatrix}-8 & 4 & 1\\ 1 & 4 & -8\\ 4 & 7 & 4\end{bmatrix}$$

$$AA^T = \frac{1}{9}\begin{bmatrix}-8 & 1 & 4\\ 4 & 4 & 7\\ 1 & -8 & 4\end{bmatrix}\frac{1}{9}\begin{bmatrix}-8 & 4 & 1\\ 1 & 4 & -8\\ 4 & 7 & 4\end{bmatrix} = \begin{bmatrix}1 & 0 & 0\\ 0 & 1 & 0\\ 0 & 0 & 1\end{bmatrix}$$

$\therefore A^{-1} = A^T$

5. *If* $adj\ A = \begin{bmatrix}0 & -2 & 0\\ 6 & 2 & -6\\ -3 & 0 & 6\end{bmatrix}$, *find* A^{-1}.

$\boxed{A^{-1} = \pm\frac{1}{\sqrt{|adjA|}} adjA}$

$|adjA| = 0(12-0) - (-2)(36-18) + 0(0+6) = 0 + 36 + 0 = 36$

$\sqrt{|adjA|} = \sqrt{36} = 6$

$$A^{-1} = \pm\frac{1}{\sqrt{|adjA|}} adjA = \pm\frac{1}{6}\begin{bmatrix}0 & -2 & 0\\ 6 & 2 & -6\\ -3 & 0 & 6\end{bmatrix}$$

6. *Find the rank of the matrix* $\begin{bmatrix}4 & 3 & 1 & -2\\ 3 & -1 & -2 & 4\\ 6 & 7 & -1 & 2\end{bmatrix}$ *using minor method*

$A = \begin{bmatrix}4 & 3 & 1 & -2\\ 3 & -1 & -2 & 4\\ 6 & 7 & -1 & 2\end{bmatrix}$. *The order of A is* 3×4. *So,* $\rho(A) \leq min\{3,4\} = 3$

3×3 *minor:*

$$\begin{vmatrix}4 & 3 & 1\\ -3 & -1 & -2\\ 6 & 7 & -1\end{vmatrix} = 0;\ \begin{vmatrix}4 & 3 & -2\\ -3 & -1 & 4\\ 6 & 7 & 2\end{vmatrix} = 0;\ \begin{vmatrix}4 & 1 & -2\\ -3 & -2 & 4\\ 6 & -1 & 2\end{vmatrix} = 0;\ \begin{vmatrix}3 & 1 & -2\\ -1 & -2 & 4\\ 6 & -1 & 2\end{vmatrix} = 0$$

So, $\rho(A) < 3$

2×2 *minor:*

$\begin{vmatrix}4 & 3\\ -3 & -1\end{vmatrix} = -4 + 9 = 5 \neq 0$.

$\therefore \rho(A) = 2$

7. *Find the rank of* $\begin{bmatrix}1 & 1 & 1 & 3\\ 2 & -1 & 3 & 4\\ 5 & -1 & 7 & 11\end{bmatrix}$ *by Row − Echelon form.*

$$A = \begin{bmatrix}1 & 1 & 1 & 3\\ 2 & -1 & 3 & 4\\ 5 & -1 & 7 & 11\end{bmatrix} \sim \begin{bmatrix}1 & 1 & 1 & 3\\ 0 & -3 & 1 & -2\\ 0 & -6 & 2 & -4\end{bmatrix}\begin{matrix}R_2 \to R_2 - 2R_1\\ R_3 \to R_3 - 5R_1\end{matrix} \sim \begin{bmatrix}1 & 1 & 1 & 3\\ 0 & -3 & 1 & -2\\ 0 & 0 & 0 & 0\end{bmatrix} R_3 \to R_3 - 2R_2$$

The last equivalent matrix is in echelon form and number of non − zero rows is 2.

$\therefore \rho(A) = 2$

3 MARK QUESTIONS

1. *If* $A = \begin{bmatrix}8 & -4\\ -5 & 3\end{bmatrix}$, *then verify that* $A(adjA) = (adjA)A = |A|I_2$.

$|A| = 24 - 20 = 4$

$|A|I_2 = 4\begin{bmatrix}1 & 0\\0 & 1\end{bmatrix} = \begin{bmatrix}4 & 0\\0 & 4\end{bmatrix} \to (1)$

$adjA = \begin{bmatrix}3 & 4\\5 & 8\end{bmatrix}$

$A(adjA) = \begin{bmatrix}8 & -4\\-5 & 3\end{bmatrix}\begin{bmatrix}3 & 4\\5 & 8\end{bmatrix} = \begin{bmatrix}4 & 0\\0 & 4\end{bmatrix} \to (2)$

$(adjA)A = \begin{bmatrix}3 & 4\\5 & 8\end{bmatrix}\begin{bmatrix}8 & -4\\-5 & 3\end{bmatrix} = \begin{bmatrix}4 & 0\\0 & 4\end{bmatrix} \to (3)$

$From\ (1),(2)\&\ (3), A(adjA) = (adjA)A = |A|I_2$

2. $If\ adj\ A = \begin{bmatrix}2 & -4 & 2\\-3 & 12 & -7\\-2 & 0 & 2\end{bmatrix}, find\ A.$

$\boxed{A = \pm\frac{1}{\sqrt{|\text{adjA}|}}\text{adj(adjA)}}$

$|adjA| = 2(24-0) - (-4)(-6-14) + 2(0+24) = 48 - 80 + 48 = 16$

$\sqrt{|\text{adjA}|} = \sqrt{16} = 4$

$$\begin{matrix}12 & 0 & -4 & 12\\-7 & 2 & 2 & -7\\-3 & -2 & 2 & -3\\12 & 0 & -4 & 12\end{matrix} \Rightarrow \text{adj(adjA)} = \begin{bmatrix}24-0 & 0+8 & 28-24\\14+6 & 4+4 & -6+14\\0+24 & 8-0 & 24-12\end{bmatrix} = \begin{bmatrix}24 & 8 & 4\\20 & 8 & 8\\24 & 8 & 12\end{bmatrix}$$

$$A = \pm\frac{1}{\sqrt{|\text{adjA}|}}\text{adj(adjA)} = \pm\frac{1}{4}\begin{bmatrix}24 & 8 & 4\\20 & 8 & 8\\24 & 8 & 12\end{bmatrix} = \pm\begin{bmatrix}6 & 2 & 1\\5 & 2 & 2\\6 & 2 & 3\end{bmatrix}$$

3.Prove that $\begin{bmatrix}cos\theta & -sin\theta\\sin\theta & cos\theta\end{bmatrix}$ is orthogonal

$\boxed{AA^T = A^TA = I_n}$

$A = \begin{bmatrix}\cos\theta & -\sin\theta\\\sin\theta & \cos\theta\end{bmatrix} \Rightarrow A^T = \begin{bmatrix}\cos\theta & \sin\theta\\-\sin\theta & \cos\theta\end{bmatrix}$

$AA^T = \begin{bmatrix}\cos\theta & -\sin\theta\\\sin\theta & \cos\theta\end{bmatrix}\begin{bmatrix}\cos\theta & \sin\theta\\-\sin\theta & \cos\theta\end{bmatrix} = \begin{bmatrix}1 & 0\\0 & 1\end{bmatrix} = I_2 \to (1)$

$A^TA = \begin{bmatrix}\cos\theta & \sin\theta\\-\sin\theta & \cos\theta\end{bmatrix}\begin{bmatrix}\cos\theta & -\sin\theta\\\sin\theta & \cos\theta\end{bmatrix} = \begin{bmatrix}1 & 0\\0 & 1\end{bmatrix} = I_2 \to (2)$

$(From\ (1)\ \&\ (2),\ AA^T = A^TA = I_2.$

4. Verify the property $(A^T)^{-1} = (A^{-1})^T$ with $A = \begin{bmatrix}2 & 9\\1 & 7\end{bmatrix}$.

$A^T = \begin{bmatrix}2 & 1\\9 & 7\end{bmatrix}$; $|A^T| = 14 - 9 = 5 ; adj(A^T) = \begin{bmatrix}7 & -1\\-9 & 2\end{bmatrix}$

$(A^T)^{-1} = \frac{1}{|A^T|}adj(A^T) = \frac{1}{5}\begin{bmatrix}7 & -1\\-9 & 2\end{bmatrix} \to (1)$

$|A| = 14 - 9 = 5 ; adj\ A = \begin{bmatrix}7 & -9\\-1 & 2\end{bmatrix}; A^{-1} = \frac{1}{|A|}adj\ A = \frac{1}{5}\begin{bmatrix}7 & -9\\-1 & 2\end{bmatrix}$

$(A^{-1})^T = \frac{1}{5}\begin{bmatrix}7 & -1\\-9 & 2\end{bmatrix} \to (2)$

$From\ (1)\&\ (2), (A^T)^{-1} = (A^{-1})^T$

5. $.Find\ the\ rank\ of\ \begin{bmatrix}2 & -2 & 4 & 3\\-3 & 4 & -2 & -1\\6 & 2 & -1 & 7\end{bmatrix}\ by\ row-\ echelon\ form.$

$$A = \begin{bmatrix}2 & -2 & 4 & 3\\-3 & 4 & -2 & -1\\6 & 2 & -1 & 7\end{bmatrix} \sim \begin{bmatrix}2 & -2 & 4 & 3\\-6 & 8 & -4 & -2\\6 & 2 & -1 & 7\end{bmatrix} R_2 \to 2R_2$$

$$\sim \begin{bmatrix}2 & -2 & 4 & 3\\0 & 2 & 8 & 7\\0 & 8 & -13 & -2\end{bmatrix} \begin{matrix}R_2 \to R_2 + 3R_1\\R_3 \to R_3 - 3R_1\end{matrix}$$

$$\sim \begin{bmatrix}2 & -2 & 4 & 3\\0 & 2 & 8 & 7\\0 & 0 & -45 & -30\end{bmatrix} R_3 \to R_3 - 4R_2$$

$The\ last\ equivalent\ matrix\ is\ in\ echelon\ form\ and\ number\ of\ non-zero\ rows\ is\ 3.$

$\therefore \rho(A) = 3$

6. $Solve\ the\ equations\ 2x + 5y = -2, x + 2y = -3\ by\ matrix\ inversion\ method.$

$\begin{bmatrix}2 & 5\\1 & 2\end{bmatrix}\begin{bmatrix}x\\y\end{bmatrix} = \begin{bmatrix}-2\\-3\end{bmatrix} \Rightarrow AX = B \Rightarrow X = A^{-1}B$

$|A| = 4 - 5 = -1\ ; adj\ A = \begin{bmatrix}2 & -5\\-1 & 2\end{bmatrix}$

$A^{-1} = \frac{1}{|A|}adjA = \frac{1}{-1}\begin{bmatrix}2 & -5\\-1 & 2\end{bmatrix} = \begin{bmatrix}-2 & 5\\1 & -2\end{bmatrix}$

$X = A^{-1}B \Rightarrow \begin{bmatrix}x\\y\end{bmatrix} = \begin{bmatrix}-2 & 5\\1 & -2\end{bmatrix}\begin{bmatrix}-2\\-3\end{bmatrix} = \begin{bmatrix}4-15\\-2+6\end{bmatrix} = \begin{bmatrix}-11\\4\end{bmatrix}$

$solution:\ (x,y) = (-11,4)$

Do it yourself:

A man is appointed in a job with a monthly salary of certain amount and a fixed amount of annual increment. If his salary was Rs.19,800 per month at the end of the first month after 3 years of service and Rs.23,400 per month at the end of the first month after 9 years of service, find his starting salary and his annual increment. (Use matrix inversion method to solve the problem.) **Hint:** $x + 3y = 19800 ; x + 9y = 23400$

7. $Solve\ the\ following\ equations\ \frac{3}{x} + 2y = 12, \frac{2}{x} + 3y = 13\ by\ Cramer's\ rule\ method.$

$\frac{1}{x} = a, y = b \Rightarrow 3a + 2b = 12 ; 2a + 3b = 13$

$\Delta = \begin{vmatrix} 3 & 2 \\ 2 & 3 \end{vmatrix} = 9 - 4 = 5$

$\Delta_a = \begin{vmatrix} 12 & 2 \\ 13 & 3 \end{vmatrix} = 36 - 26 = 10$

$\Delta_b = \begin{vmatrix} 3 & 12 \\ 2 & 13 \end{vmatrix} = 39 - 24 = 15$

$a = \frac{\Delta_a}{\Delta} = \frac{10}{5} = 2 \Rightarrow x = \frac{1}{a} \Rightarrow x = \frac{1}{2}$

$b = \frac{\Delta_b}{\Delta} = \frac{15}{5} = 3 \Rightarrow y = 3$

$solution : (x, y) = \left(\frac{1}{2}, 3\right)$

Do it yourself:

In a competitive examination, one mark is awarded for every correct answer while $^1/_4$ mark is deducted for every wrong answer. A student answered 100 questions and got 80 marks. How many questions did he answer correctly? (Use Cramer's rule to solve the problem).

Hint: $x + y = 100$; $x - \frac{1}{4}y = 80 \Rightarrow 4x - y = 320 \rightarrow (2)$

5 MARK QUESTIONS

1. $If\ F(\alpha) = \begin{bmatrix} \cos\alpha & 0 & \sin\alpha \\ 0 & 1 & 0 \\ -\sin\alpha & 0 & \cos\alpha \end{bmatrix}, then\ prove\ that\ [F(\alpha)]^{-1} = F(-\alpha)$

$F(-\alpha) = \begin{bmatrix} \cos(-\alpha) & 0 & \sin(-\alpha) \\ 0 & 1 & 0 \\ -\sin(-\alpha) & 0 & \cos(-\alpha) \end{bmatrix} = \begin{bmatrix} \cos\alpha & 0 & -\sin\alpha \\ 0 & 1 & 0 \\ \sin\alpha & 0 & \cos\alpha \end{bmatrix} \rightarrow (1) \left(\because \begin{matrix} \cos(-\alpha) = \cos\alpha \\ \sin(-\alpha) = -\sin\alpha \end{matrix}\right)$

$|F(\alpha)| = \begin{vmatrix} \cos\alpha & 0 & \sin\alpha \\ 0 & 1 & 0 \\ -\sin\alpha & 0 & \cos\alpha \end{vmatrix} = \cos\alpha\,(\cos\alpha) - 0 + \sin\alpha\,(\sin\alpha)$

$= cos^2\alpha + sin^2\alpha = 1$

$adjF(\alpha) = \begin{bmatrix} \cos\alpha - 0 & 0 - 0 & 0 - \sin\alpha \\ 0 - 0 & cos^2\alpha + sin^2\alpha & 0 - 0 \\ 0 + \sin\alpha & 0 - 0 & \cos\alpha - 0 \end{bmatrix} = \begin{bmatrix} \cos\alpha & 0 & -\sin\alpha \\ 0 & 1 & 0 \\ \sin\alpha & 0 & \cos\alpha \end{bmatrix}$

$[F(\alpha)]^{-1} = \frac{1}{|F(\alpha)|} adj\ F(\alpha) \Rightarrow [F(\alpha)]^{-1} = \begin{bmatrix} \cos\alpha & 0 & -\sin\alpha \\ 0 & 1 & 0 \\ \sin\alpha & 0 & \cos\alpha \end{bmatrix} \rightarrow (2)$

$From\ (1) \&\ (2), [F(\alpha)]^{-1} = F(-\alpha)$

2. $If\ \ A = \begin{bmatrix} 8 & -6 & 2 \\ -6 & 7 & -4 \\ 2 & -4 & 3 \end{bmatrix}, verify\ that A(adjA) = (adjA)A = |A|I_3.$

$|A| = 8(21 - 16) - (-6)(-18 + 8) + 2(24 - 14) = 40 - 60 + 20 = 0$

$|A|I_3 = 0 \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \rightarrow (1)$

$\begin{matrix} 7 & -4 & -6 & 7 \\ -4 & 3 & 2 & -4 \\ -6 & 2 & 8 & -6 \\ 7 & -4 & -6 & 7 \end{matrix} \Rightarrow adj\ A = \begin{bmatrix} 21 - 16 & -8 + 18 & 24 - 14 \\ -8 + 18 & 24 - 4 & -12 + 32 \\ 24 - 14 & -12 + 32 & 56 - 36 \end{bmatrix} = \begin{bmatrix} 5 & 10 & 10 \\ 10 & 20 & 20 \\ 10 & 20 & 20 \end{bmatrix}$

$A(adjA) = \begin{bmatrix} 8 & -6 & 2 \\ -6 & 7 & -4 \\ 2 & -4 & 3 \end{bmatrix} \begin{bmatrix} 5 & 10 & 10 \\ 10 & 20 & 20 \\ 10 & 20 & 20 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \rightarrow (2)$

$(adjA)A = \begin{bmatrix} 5 & 10 & 10 \\ 10 & 20 & 20 \\ 10 & 20 & 20 \end{bmatrix} \begin{bmatrix} 8 & -6 & 2 \\ -6 & 7 & -4 \\ 2 & -4 & 3 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \rightarrow (3)$

$From\ (1), (2) \&\ (3), A(adjA) = (adjA)A = |A|I_3$

3. If $A = \begin{bmatrix} 3 & 2 \\ 7 & 5 \end{bmatrix}, B = \begin{bmatrix} -1 & -3 \\ 5 & 2 \end{bmatrix}$, Verify $(AB)^{-1} = B^{-1}A^{-1}$

$AB = \begin{bmatrix} 3 & 2 \\ 7 & 5 \end{bmatrix} \begin{bmatrix} -1 & -3 \\ 5 & 2 \end{bmatrix} = \begin{bmatrix} -3 + 10 & -9 + 4 \\ -7 + 25 & -21 + 10 \end{bmatrix} = \begin{bmatrix} 7 & -5 \\ 18 & -11 \end{bmatrix}$

$|AB| = -77 + 90 = 13$; $adj(AB) = \begin{bmatrix} -11 & 5 \\ -18 & 7 \end{bmatrix}$

$(AB)^{-1} = \frac{1}{|AB|} adj(AB) = \frac{1}{13} \begin{bmatrix} -11 & 5 \\ -18 & 7 \end{bmatrix} \rightarrow (1)$

$|B| = -2 + 15 = 13$; $adj\ B = \begin{bmatrix} 2 & 3 \\ 5 & -1 \end{bmatrix}$; $B^{-1} = \frac{1}{|B|} adj\ B = \frac{1}{13} \begin{bmatrix} 2 & 3 \\ -5 & -1 \end{bmatrix}$

$|A| = 15 - 14 = 1$; $adj\ A = \begin{bmatrix} 5 & -2 \\ -7 & 3 \end{bmatrix}$

$A^{-1} = \frac{1}{|A|} adj\ A = \frac{1}{1} \begin{bmatrix} 5 & -2 \\ -7 & 3 \end{bmatrix} = \begin{bmatrix} 5 & -2 \\ -7 & 3 \end{bmatrix}$

$B^{-1}A^{-1} = \frac{1}{13} \begin{bmatrix} 2 & 3 \\ -5 & -1 \end{bmatrix} \begin{bmatrix} 5 & -2 \\ -7 & 3 \end{bmatrix} = \frac{1}{13} \begin{bmatrix} 10 - 21 & -4 + 9 \\ -25 + 7 & 10 - 3 \end{bmatrix} = \frac{1}{13} \begin{bmatrix} -11 & 5 \\ -18 & 7 \end{bmatrix} \rightarrow (2)$

$From\ (1) \&\ (2), (AB)^{-1} = B^{-1}A^{-1}$

4. Solve the following system of linear equations by matrix inversion method: $2x + 3y - z = 9, x + y + z = 9, 3x - y - z = -1$

$\begin{bmatrix} 2 & 3 & -1 \\ 1 & 1 & 1 \\ 3 & -1 & -1 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 9 \\ 9 \\ -1 \end{bmatrix} \Rightarrow AX = B \Rightarrow X = A^{-1}B$

$|A| = 2(-1 + 1) - 3(-1 - 3) - 1(-1 - 3) = 12 + 4 = 16$

$$\begin{matrix} 1 & -1 & 1 & 1 \\ 1 & -1 & -1 & 1 \\ 1 & 3 & 2 & 1 \\ 1 & -1 & 3 & 1 \end{matrix} \Rightarrow \text{adjA} = \begin{bmatrix} -1+1 & 1+3 & 3+1 \\ 3+1 & -2+3 & -1-2 \\ -1-3 & 9+2 & 2-3 \end{bmatrix} = \begin{bmatrix} 0 & 4 & 4 \\ 4 & 1 & -3 \\ -4 & 11 & -1 \end{bmatrix}$$

$$A^{-1} = \frac{1}{|A|} adjA = \frac{1}{16}\begin{bmatrix} 0 & 4 & 4 \\ 4 & 1 & -3 \\ -4 & 11 & -1 \end{bmatrix}$$

$$X = A^{-1}B \Rightarrow \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \frac{1}{16}\begin{bmatrix} 0 & 4 & 4 \\ 4 & 1 & -3 \\ -4 & 11 & -1 \end{bmatrix}\begin{bmatrix} 9 \\ 9 \\ -1 \end{bmatrix} = \frac{1}{16}\begin{bmatrix} 0+36-4 \\ 36+9+3 \\ -36+99+1 \end{bmatrix} = \frac{1}{16}\begin{bmatrix} 32 \\ 48 \\ 64 \end{bmatrix} = \begin{bmatrix} 2 \\ 3 \\ 4 \end{bmatrix}$$

$solution\ (x, y, z) = (2,3,4)$

5. If $A = \begin{bmatrix} -5 & 1 & 3 \\ 7 & 1 & -5 \\ 1 & -1 & 1 \end{bmatrix}$ and $B = \begin{bmatrix} 1 & 1 & 2 \\ 3 & 2 & 1 \\ 2 & 1 & 3 \end{bmatrix}$, find the products AB and BA and hence solve the system of equations $x + y + 2z = 1, \quad 3x + 2y + z = 7, \quad 2x + y + 3z = 2.$

$$AB = \begin{bmatrix} -5 & 1 & 3 \\ 7 & 1 & -5 \\ 1 & -1 & 1 \end{bmatrix}\begin{bmatrix} 1 & 1 & 2 \\ 3 & 2 & 1 \\ 2 & 1 & 3 \end{bmatrix}$$

$$= \begin{bmatrix} -5+3+6 & -5+2+3 & -10+1+9 \\ 7+3-10 & 7+2-5 & 14+1-15 \\ 1-3+2 & 1-2+1 & 2-1+3 \end{bmatrix}$$

$$= \begin{bmatrix} 4 & 0 & 0 \\ 0 & 4 & 0 \\ 0 & 0 & 4 \end{bmatrix} = 4I_3$$

$$BA = \begin{bmatrix} 1 & 1 & 2 \\ 3 & 2 & 1 \\ 2 & 1 & 3 \end{bmatrix}\begin{bmatrix} -5 & 1 & 3 \\ 7 & 1 & -5 \\ 1 & -1 & 1 \end{bmatrix} = \begin{bmatrix} 4 & 0 & 0 \\ 0 & 4 & 0 \\ 0 & 0 & 4 \end{bmatrix} = 4I_3$$

$$AB = BA = 4I_3 \Rightarrow B^{-1} = \frac{1}{4}A \to (1)$$

$$\begin{bmatrix} 1 & 1 & 2 \\ 3 & 2 & 1 \\ 2 & 1 & 3 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 1 \\ 7 \\ 2 \end{bmatrix} \Rightarrow BX = C \Rightarrow X = B^{-1}C \to (2)$$

$$(1) \Rightarrow B^{-1} = \frac{1}{4}A = \frac{1}{4}\begin{bmatrix} -5 & 1 & 3 \\ 7 & 1 & -5 \\ 1 & -1 & 1 \end{bmatrix}$$

$$(2) \Rightarrow X = B^{-1}C \Rightarrow \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \frac{1}{4}\begin{bmatrix} -5 & 1 & 3 \\ 7 & 1 & -5 \\ 1 & -1 & 1 \end{bmatrix}\begin{bmatrix} 1 \\ 7 \\ 2 \end{bmatrix} = \frac{1}{4}\begin{bmatrix} -5+7+6 \\ 7+7-10 \\ 1-7+2 \end{bmatrix}$$

$$= \frac{1}{4}\begin{bmatrix} 8 \\ 4 \\ -4 \end{bmatrix} = \begin{bmatrix} 2 \\ 1 \\ -1 \end{bmatrix} \qquad solution: (x, y, z) = (2,1,-1)$$

6. *Solve the system equations using Cramer's rule method*: $3x + 3y - z = 11, 2x - y + 2z = 9, 4x + 3y + 2z = 25$

$$\Delta = \begin{vmatrix} 3 & 3 & -1 \\ 2 & -1 & 2 \\ 4 & 3 & 2 \end{vmatrix} = 3(-2-6) - 3(4-8) - 1(6+4) = -24 + 12 - 10 = -22$$

$$\Delta_x = \begin{vmatrix} 11 & 3 & -1 \\ 9 & -1 & 2 \\ 25 & 3 & 2 \end{vmatrix} = 11(-2-6) - 3(18-50) - 1(27+25) = -88 + 96 - 52 = -44$$

$$\Delta_y = \begin{vmatrix} 3 & 11 & -1 \\ 2 & 9 & 2 \\ 4 & 25 & 2 \end{vmatrix} = 3(18-50) - 11(4-8) - 1(50-36) = -96 + 44 - 14 = -66$$

$$\Delta_z = \begin{vmatrix} 3 & 3 & 11 \\ 2 & -1 & 9 \\ 4 & 3 & 25 \end{vmatrix} = 3(-25-27) - 3(50-36) + 11(6+4) = -156 - 42 + 110 = -88$$

$$x = \frac{\Delta_x}{\Delta} = \frac{-44}{-22} = 2\ ;\ y = \frac{\Delta_y}{\Delta} = \frac{-66}{-22} = 3\ ;\ z = \frac{\Delta_z}{\Delta} = \frac{-88}{-22} = 4$$

$solution : (x, y, z) = (2,3,4)$

Do it yourself:

1. A family of 3 people went out for dinner in a restaurant. The cost of two dosai, three idlies and two vadais is $Rs. 150$. The cost of the two dosai, two idlies and four vadais is $Rs. 200$. The cost of five dosai, four idlies and two vadais is $Rs. 250$. The family has $Rs. 350$ in hand and they ate 3 dosai and six idlies and six vadais. Will they be able to manage to pay the bill within the amount they had? **Hint:** $2x + 3y + 2z = 150$, $2x + 2y + 4z = 200$, $5x + 4y + 2z = 250$
2. In a T20 match, Chennai Super Kings needed just 6 runs to win with 1 ball left to go in the last over. The last ball was bowled and the batsman at the crease hit it high up. The ball traversed along a path in a vertical plane and the equation of the path is $y = ax^2 + bx + c$ with respect to a xy-coordinate system in the vertical plane and the ball traversed through the points $(10,8), (20,16), (40,22)$, can you conclude that Chennai Super Kings won the match? Justify your answer. (All distances are measured in meters and the meeting point of the plane of the path with the farthest boundary line is $(70,0)$).
   Hint: $(10,8) \Rightarrow 100a + 10b + c = 8$, $(20,16) \Rightarrow 400a + 20b + c = 16$
   $(40,22) \Rightarrow 1600a + 40b + c = 22$

7. *Solve the system equations using Cramer's rule method*:

$$\frac{3}{x} - \frac{4}{y} - \frac{2}{z} - 1 = 0, \frac{1}{x} + \frac{2}{y} + \frac{1}{z} - 2 = 0, \frac{2}{x} - \frac{5}{y} - \frac{4}{z} + 1 = 0$$

$Let\ \frac{1}{x} = a, \frac{1}{y} = b, \frac{1}{z} = c$

$\Rightarrow 3a-4b-2c=1 \to (1), a+2b+c=2 \to (2), 2a-5b-4c=-1 \to (3)$

$$\Delta = \begin{vmatrix} 3 & -4 & -2 \\ 1 & 2 & 1 \\ 2 & -5 & -4 \end{vmatrix} = 3(-8+5)-(-4)(-4-2)-2(-5-4) = -9-24+18 = -15$$

$$\Delta_a = \begin{vmatrix} 1 & -4 & -2 \\ 2 & 2 & 1 \\ -1 & -5 & -4 \end{vmatrix} = 1(-8+5)-(-4)(-8+1)-2(-10+2) = -3-28+16 = -15$$

$$\Delta_b = \begin{vmatrix} 3 & 1 & -2 \\ 1 & 2 & 1 \\ 2 & -1 & -4 \end{vmatrix} = 3(-8+1)-1(-4-2)-2(-1-4) = -21+6+10 = -5$$

$$\Delta_c = \begin{vmatrix} 3 & -4 & 1 \\ 1 & 2 & 2 \\ 2 & -5 & -1 \end{vmatrix} = 3(-2+10)-(-4)(-1-4)+1(-5-4) = 24-20-9 = -5$$

$$a = \frac{\Delta_a}{\Delta} = \frac{-15}{-15} = 1 \Rightarrow x = \frac{1}{a} = 1$$

$$b = \frac{\Delta_b}{\Delta} = \frac{-5}{-15} = \frac{1}{3} \Rightarrow y = \frac{1}{b} = 3$$

$$c = \frac{\Delta_c}{\Delta} = \frac{-5}{-15} = \frac{1}{3} \Rightarrow z = \frac{1}{c} = 3$$

$solution: (x,y,z) = (1,3,3)$

8. *Solve the system of linear equations by Gaussian elimination method* :

$2x-2y+3z=2, x+2y-z=3, 3x-y+2z=1$

$$[A|B] = \left(\begin{array}{ccc|c} 2 & -2 & 3 & 2 \\ 1 & 2 & -1 & 3 \\ 3 & -1 & 2 & 1 \end{array}\right) \sim \left(\begin{array}{ccc|c} 1 & 2 & -1 & 3 \\ 2 & -2 & 3 & 2 \\ 3 & -1 & 2 & 1 \end{array}\right) R_1 \leftrightarrow R_2$$

$$\sim \left(\begin{array}{ccc|c} 1 & 2 & -1 & 3 \\ 0 & -6 & 5 & -4 \\ 0 & -7 & 5 & -8 \end{array}\right) \begin{array}{l} R_2 \to R_2 - 2R_1 \\ R_3 \to R_3 - 3R_1 \end{array}$$

$$\sim \left(\begin{array}{ccc|c} 1 & 2 & -1 & 3 \\ 0 & -6 & 5 & -4 \\ 0 & 0 & -5 & -20 \end{array}\right) R_3 \to 6R_3 - 7R_2$$

$\Rightarrow x+2y-z=3 \to (1)$; $-6y+5z=-4 \to (2)$; $-5z=-20 \Rightarrow z=4$

$When\ z=4 \Rightarrow -6y+5(4)=-4 \Rightarrow -6y+20=-4$

$\Rightarrow -6y=-24 \Rightarrow y=4$

$When\ \ y=16, z=4$,;$(1) \Rightarrow x+2(4)-4=3 \Rightarrow x+8-4=3 \Rightarrow x=-1$

$solution: (x,y,z) = (-1,4,4)$

9. If ax^2+bx+c is divided by $x+3, x-5$ and $x-1$, the remainders are 21, 61 and 9 respectively. Find a, b and c (Use Gaussian elimination method)

$let\ p(x) = ax^2+bx+c$

$p(-3)=21 \Rightarrow= a(-3)^2+b(-3)+c=21 \Rightarrow 9a-3b+c=21 \to (1)$

$p(5)=61 \Rightarrow a(5)^2+b(5)+c=61 \Rightarrow 25a+5b+c=61 \to (2)$

$p(1)=9 \Rightarrow= a(1)^2+b(1)+c=9 \Rightarrow a+b+c=9 \to (3)$

$$[A|B] = \left(\begin{array}{ccc|c} 9 & -3 & 1 & 21 \\ 25 & 5 & 1 & 61 \\ 1 & 1 & 1 & 9 \end{array}\right) \sim \left(\begin{array}{ccc|c} 1 & 1 & 1 & 9 \\ 25 & 5 & 1 & 61 \\ 9 & -3 & 1 & 21 \end{array}\right) R_1 \leftrightarrow R_3$$

$$\sim \left(\begin{array}{ccc|c} 1 & 1 & 1 & 9 \\ 0 & -20 & -24 & -164 \\ 0 & -12 & -8 & -60 \end{array}\right) \begin{array}{l} R_2 \to R_2 - 25R_1 \\ R_3 \to R_3 - 9R_1 \end{array}$$

$$\sim \left(\begin{array}{ccc|c} 1 & 1 & 1 & 9 \\ 0 & 5 & 6 & 41 \\ 0 & 3 & 2 & -15 \end{array}\right) \begin{array}{l} R_2 \to \left(-\frac{1}{4}\right) R_2 \\ R_3 \to \left(-\frac{1}{4}\right) R_3 \end{array}$$

$$\sim \left(\begin{array}{ccc|c} 1 & 1 & 1 & 9 \\ 0 & 5 & 6 & 41 \\ 0 & 0 & -8 & -48 \end{array}\right) R_3 \to 5R_3 - 3R_2$$

$\Rightarrow a+b+c=9 \to (1)$; $5b+6c=41 \to (2)$; $-8c=-48 \Rightarrow c=6$

$When\ c=6$,; $(2) \Rightarrow 5b+6(6)=41 \Rightarrow 5b=41-36 \Rightarrow 5b=5 \Rightarrow b=1$

$When\ \ b=1, c=6\ \ (1) \Rightarrow a+1+6=9 \Rightarrow a+7=9 \Rightarrow a=2$

$solution: (a,b,c) = (2,1,6)$

Do it yourself:

A boy is walking along the path $y=ax^2+bx+c$ through the points $(-6,8), (-2,-12)$ and (3,8). He wants to meet his friend at $P(7,60)$. Will he meet his friend? (Use Gaussian elimination method)

Hint:$(-6,8) \Rightarrow 36a-6b+c=8, (-2,-12) \Rightarrow 4a-2b+c=-12$

$(3,8) \Rightarrow 9a+3b+c=8$

10. The upward speed $v(t)$of a rocket at time t is approximated by $v(t)=at^2+bt+c, 0 \leq$ t ≤ 100 where $a, b,$ and c are constants. It has been found that the speed at times $t=3, t=6$ and $t=9$ seconds are respectively, 64, 133 and 208 miles per second respectively. Find the speed at time $t\ =15$ seconds. (Use Gaussian elimination method.)

$v(t) = at^2+bt+c$

$v(3)=64 \Rightarrow 9a+3b+c=64 \to (1)$

$v(6)=133 \Rightarrow 36a+6b+c=133 \to (2)$

$v(9)=208 \Rightarrow 81a+9b+c=208 \to (3)$

$$[A|B] = \left(\begin{array}{ccc|c} 9 & 3 & 1 & 64 \\ 36 & 6 & 1 & 133 \\ 81 & 9 & 1 & 208 \end{array}\right) \sim \left(\begin{array}{ccc|c} 9 & 3 & 1 & 64 \\ 0 & -6 & -3 & -123 \\ 0 & -18 & -8 & -368 \end{array}\right) \begin{array}{l} R_2 \to R_2 - 4R_1 \\ R_3 \to R_3 - 9R_1 \end{array}$$

$$\sim\left(\begin{array}{ccc|c}9&3&1&64\\0&2&1&41\\0&18&8&368\end{array}\right)\begin{array}{l}R_2\to\left(-\frac{1}{3}\right)R_2\\R_3\to(-)R_3\end{array}$$

$$\sim\left(\begin{array}{ccc|c}9&3&1&64\\0&2&1&41\\0&0&-1&-1\end{array}\right)R_3\to R_3-9R_2$$

$\Rightarrow 9a+3b+c=64\to(1)\,;2b+c=41\to(2)\,;-c=-1\Rightarrow c=1$

$When\ \ c=1\ \ (2)\Rightarrow 2b+1=41\Rightarrow 2b=40\Rightarrow b=20$

$When\ \ \ b=20,c=1\ (1)\Rightarrow 9a+60+1=64\Rightarrow 9a=3\Rightarrow a=\frac{1}{3}$

$\therefore v(t)=\frac{1}{3}t^2+20t+1$

$v(15)=\frac{1}{3}(15)^2+20(15)+1=75+300+1=376\ mile/sec$

11.Test for consistency and if possible, solve the following systems of equations by Rank method : $x-y+2z=2,\quad 2x+y+4z=7,\quad 4x-y+z=4$

$$AX=B\Rightarrow\begin{bmatrix}1&-1&2\\2&1&4\\4&-1&1\end{bmatrix}\begin{bmatrix}x\\y\\z\end{bmatrix}=\begin{bmatrix}2\\7\\4\end{bmatrix}$$

$$Augmented\ matrix\ [A|B]=\left(\begin{array}{ccc|c}1&-1&2&2\\2&1&4&7\\4&-1&1&4\end{array}\right)$$

$$\sim\left(\begin{array}{ccc|c}1&-1&2&2\\0&3&0&3\\0&3&-7&-4\end{array}\right)\begin{array}{l}R_2\to R_2-2R_1\\R_3\to R_3-4R_1\end{array}$$

$$\sim\left(\begin{array}{ccc|c}1&-1&2&2\\0&3&0&3\\0&0&-7&-7\end{array}\right)R_3\to R_3-R_2$$

$The\ last\ equivalent\ matrix\ is\ in\ echelon\ form.\ \rho([A|B])=3=\rho(A).$

$The\ system\ of\ equations\ is\ consistent\ and\ has\ unique\ solution.$

$$AX=B\Rightarrow\begin{pmatrix}1&-1&2\\0&3&0\\0&0&-7\end{pmatrix}\begin{pmatrix}x\\y\\z\end{pmatrix}=\begin{pmatrix}2\\3\\-7\end{pmatrix}$$

$$\Rightarrow\begin{array}{l}x-y+2z=2\to(1)\\3y=3\Rightarrow y=1\\-7z=-7\Rightarrow z=1\end{array}$$

$When\ \ y=1,z=1\ \ (1)\Rightarrow x-1+2=2\Rightarrow x=1$

$solution:(x,y,z)=(1,1,1)$

12.Test for consistency and if possible, solve the following systems of equations by Rank method : $3x+y+z=2,x-3y+2z=1,7x-y+4z=5$

$$AX=B\Rightarrow\begin{bmatrix}3&1&1\\1&-3&2\\7&-1&4\end{bmatrix}\begin{bmatrix}x\\y\\z\end{bmatrix}=\begin{bmatrix}2\\1\\5\end{bmatrix}$$

$$Augmented\ matrix\ [A|B]=\left(\begin{array}{ccc|c}3&1&1&2\\1&-3&2&1\\7&-1&4&5\end{array}\right)\sim\left(\begin{array}{ccc|c}1&-3&2&1\\3&1&1&2\\7&-1&4&5\end{array}\right)R_1\leftrightarrow R_2$$

$$\sim\left(\begin{array}{ccc|c}1&-3&2&1\\0&10&-5&-1\\0&20&-10&-2\end{array}\right)\begin{array}{l}R_2\to R_2-3R_1\\R_3\to R_3-7R_1\end{array}$$

$$\sim\left(\begin{array}{ccc|c}1&-3&2&1\\0&10&-5&-1\\0&0&0&0\end{array}\right)R_3\to R_3-2R_2$$

$The\ last\ equivalent\ matrix\ is\ in\ echelon\ form.\rho([A|B])=2=\rho(A).$

$The\ system\ of\ equations\ is\ consistent\ and\ has\ infinitely\ many\ solution.$

$$AX=B\Rightarrow\begin{pmatrix}1&-3&2\\0&10&-5\\0&0&0\end{pmatrix}\begin{pmatrix}x\\y\\z\end{pmatrix}=\begin{pmatrix}1\\-1\\0\end{pmatrix}$$

$$\Rightarrow\begin{array}{l}x-3y+2z=1\to(1)\\10y-5z=-1\to(2)\end{array}$$

$When\ z=t\ (2)\Rightarrow 10y-5t=-1\Rightarrow 10y=5t-1\Rightarrow y=\frac{1}{10}(5t-1)$

$when\ \ y=\frac{1}{10}(5t-1),z=t\ (1)\Rightarrow x-3\frac{1}{10}(5t-1)+2t=1$

$\Rightarrow x=1-2t+3\frac{1}{10}(5t-1)=\frac{10-20t+15t-3}{10}=\frac{1}{10}(7-5t)$

$solution:(x,y,z)=\left(\frac{1}{10}(7-5t),\frac{1}{10}(5t-1),t\right);t\in R$

13.Test for consistency and if possible, solve the following systems of equations by Rank method : $2x-y+z=2,6x-3y+3z=6,4x-2y+2z=4$

$$AX=B\Rightarrow\begin{bmatrix}2&-1&1\\6&-3&3\\4&-2&2\end{bmatrix}\begin{bmatrix}x\\y\\z\end{bmatrix}=\begin{bmatrix}2\\6\\4\end{bmatrix}$$

$$Augmented\ matrix\ [A|B]=\left(\begin{array}{ccc|c}2&-1&1&2\\6&-3&3&6\\4&-2&2&4\end{array}\right)$$

$$\sim\left(\begin{array}{ccc|c}2&-1&1&2\\0&0&0&0\\0&0&0&0\end{array}\right)\begin{array}{l}R_2\to R_2-3R_1\\R_3\to R_3-2R_1\end{array}$$

$The\ last\ equivalent\ matrix\ is\ in\ echelon\ form.\ \rho([A|B])=1,\rho(A)=1.$

$The\ system\ of\ equations\ is\ consistent\ and\ has\ infinitely\ many\ solution.$

$$AX=B\Rightarrow\begin{pmatrix}2&-1&1\\0&0&0\\0&0&0\end{pmatrix}\begin{pmatrix}x\\y\\z\end{pmatrix}=\begin{pmatrix}2\\0\\0\end{pmatrix}$$

$$\Rightarrow 2x-y+z=2$$

Put $y = s, z = t$ *in the above equation,*

$2x - s + t = 2 \Longrightarrow 2x = s - t + 2 \Longrightarrow x = \frac{1}{2}(s - t + 2)$

solution: $(x, y, z) = \left(\frac{1}{2}(s - t + 2), s, t\right); s, t \in R$

14.Test for consistency and if possible, solve the following systems of equations by Rank method : $2x + 2y + z = 5, x - y + z = 1, 3x + y + 2z = 4$

$$AX = B \Longrightarrow \begin{bmatrix} 2 & 2 & 1 \\ 1 & -1 & 1 \\ 3 & 1 & 2 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 5 \\ 1 \\ 4 \end{bmatrix}$$

$$Augmented\ matrix\ [A|B] = \left(\begin{array}{ccc|c} 2 & 2 & 1 & 5 \\ 1 & -1 & 1 & 1 \\ 3 & 1 & 2 & 4 \end{array}\right)$$

$$\sim \left(\begin{array}{ccc|c} 1 & -1 & 1 & 1 \\ 2 & 2 & 1 & 5 \\ 3 & 1 & 2 & 4 \end{array}\right) R_1 \leftrightarrow R_2$$

$$\sim \left(\begin{array}{ccc|c} 1 & -1 & 1 & 1 \\ 0 & 4 & -1 & 3 \\ 0 & 4 & -1 & 1 \end{array}\right) \begin{array}{l} R_2 \to R_2 - 2R_1 \\ R_3 \to R_3 - 3R_1 \end{array}$$

$$\sim \left(\begin{array}{ccc|c} 1 & -1 & 1 & 1 \\ 0 & 4 & -1 & 3 \\ 0 & 0 & 0 & -2 \end{array}\right) R_3 \to R_3 - R_2$$

The last equivalent matrix is in echelon form. $\rho([A|B]) = 3, \rho(A) = 2$.

$\rho([A|B]) \neq \rho(A)$, *The system of equations is inconsistent and has no solution.*

15. Find the value of k for which the equations $kx - 2y + z = 1, \ x - 2ky + z = -2$, $x - 2y + kz = 1$ have (i) no solution (ii) unique solution (iii) infinitely many solution

$$AX = B \Longrightarrow \begin{bmatrix} k & -2 & 1 \\ 1 & -2k & 1 \\ 1 & -2 & k \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 1 \\ -2 \\ 1 \end{bmatrix}$$

$$Augmented\ matrix\ [A|B] = \left(\begin{array}{ccc|c} k & -2 & 1 & 1 \\ 1 & -2k & 1 & -2 \\ 1 & -2 & k & 1 \end{array}\right)$$

$$\sim \left(\begin{array}{ccc|c} 1 & -2 & k & 1 \\ 1 & -2k & 1 & -2 \\ k & -2 & 1 & 1 \end{array}\right) R_1 \leftrightarrow R_3$$

$$\sim \left(\begin{array}{ccc|c} 1 & -2 & k & 1 \\ 0 & 2 - 2k & 1 - k & -3 \\ 0 & 2k - 2 & 1 - k^2 & 1 - k \end{array}\right) \begin{array}{l} R_2 \to R_2 - R_1 \\ R_3 \to R_3 - kR_1 \end{array}$$

$$\sim \left(\begin{array}{ccc|c} 1 & -2 & k & 1 \\ 0 & 2(1 - k) & 1 - k & -3 \\ 0 & 0 & 2 - k - k^2 & -k - 2 \end{array}\right) R_3 \to R_3 + R_2$$

$$\sim \left(\begin{array}{ccc|c} 1 & -2 & k & 1 \\ 0 & 2(1 - k) & 1 - k & -3 \\ 0 & 0 & (k + 2)(1 - k) & -(k + 2) \end{array}\right)$$

(i) *When* $k = 1, k \neq -2 \Longrightarrow \rho([A|B]) = 3, \rho(A) = 1 \Longrightarrow \rho([A|B]) \neq \rho(A)$,
The system of equations has no solution .

(ii) *When* $k \neq 1, \ k \neq -2 \Longrightarrow \rho([A|B]) = 3 = \rho(A)$,
The system of equations has unique solution

(iii)*When* $k = -2, k \neq 1 \Longrightarrow; \rho([A|B]) = 2 = \rho(A)$,
The system of equations has infinitely many solution..

16. Investigate the values of λ and μ the system of linear equations $2x + 3y + 5z = 9, \ 7x + 3y - 5z = 8, \ 2x + 3y + \lambda z = \mu$ have
(i) no solution (ii) a unique solution (iii) an infinite number of solutions.

$$AX = B \Longrightarrow \begin{bmatrix} 2 & 3 & 5 \\ 7 & 3 & -5 \\ 2 & 3 & \lambda \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 9 \\ 8 \\ \mu \end{bmatrix}$$

$$Augmented\ matrix\ [A|B] = \left(\begin{array}{ccc|c} 2 & 3 & 5 & 9 \\ 7 & 3 & -5 & 8 \\ 2 & 3 & \lambda & \mu \end{array}\right)$$

$$\sim \left(\begin{array}{ccc|c} 2 & 3 & 5 & 9 \\ 0 & -15 & -25 & --9 \\ 0 & 0 & \lambda - 5 & \mu - 9 \end{array}\right) \begin{array}{l} R_2 \to 2R_2 - 7R_1 \\ R_3 \to R_3 - R_1 \end{array}$$

(i) *When* $\lambda = 5, \mu \neq 9 \Longrightarrow \ \rho([A|B]) = 3, \rho(A) = 2 \Longrightarrow \rho([A|B]) \neq \rho(A)$,
The system of equations has no solution

(ii) *When* $\lambda \neq 5, \mu \neq 9 \Longrightarrow \rho([A|B]) = 3 = \rho(A)$,
The system of equations has unique solution

(iii)*When* $\lambda = 5, \mu = 9 \Longrightarrow \ \rho([A|B]) = 2 = \rho(A)$,
The system of equations has infinitely many solution..

17. Investigate for what values of λ and μ the system of linear equations $x + 2y + z = 7, \ x + y + \lambda z = \mu, \ x + 3y - 5z = 5$ has (i) no solution (ii) a unique solution (iii) an infinite number of solutions.

$$AX = B \Longrightarrow \begin{bmatrix} 1 & 2 & 1 \\ 1 & 1 & \lambda \\ 1 & 3 & -5 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 7 \\ \mu \\ 5 \end{bmatrix}$$

$$Augmented\ matrix\ [A|B] = \left(\begin{array}{ccc|c} 1 & 2 & 1 & 7 \\ 1 & 1 & \lambda & \mu \\ 1 & 3 & -5 & 5 \end{array}\right)$$

$$\sim \left(\begin{array}{ccc|c} 1 & 2 & 1 & 7 \\ 1 & 3 & -5 & 5 \\ 1 & 1 & \lambda & \mu \end{array}\right) R_2 \leftrightarrow R_3$$

$$\sim\left(\begin{array}{ccc|c}1&2&1&7\\0&1&-6&-2\\0&-1&\lambda-1&\mu-7\end{array}\right)\begin{array}{l}R_2\to R_2-R_1\\R_3\to R_3-R_1\end{array}$$

$$\sim\left(\begin{array}{ccc|c}1&2&1&7\\0&1&-6&-2\\0&0&\lambda-7&\mu-9\end{array}\right)R_3\to R_3+R_2$$

(i) $when\ \lambda=7,\mu\neq 9\Longrightarrow\rho([A|B])=3,\rho(A)=2\Longrightarrow\rho([A|B])\neq\rho(A),$
$The\ system\ of\ equations\ \ has\ no\ solution$

(ii) $When\ \lambda\neq 7,\mu\neq 9\Longrightarrow\rho([A|B])=3=\rho(A),$
$The\ system\ of\ equations\ \ has\ unique\ solution$

(iii)$when\ \lambda=7,\mu=9\Longrightarrow\rho([A|B])=2=\rho(A),$
$The\ system\ of\ equations\ has\ infinitely\ many\ solution..$

18. Solve the following system of homogenous equations.
$3x+2y+7z=0,\quad 4x-3y-2z=0,\quad 5x+9y+23z=0$

$$Augmented\ matrix\ [A|O]=\left(\begin{array}{ccc|c}3&2&7&0\\4&-3&-2&0\\5&9&23&0\end{array}\right)$$

$$\sim\left(\begin{array}{ccc|c}3&2&7&0\\0&-17&-34&0\\0&17&34&0\end{array}\right)\begin{array}{l}R_2\to 3R_2-4R_1\\R_3\to 3R_3-5R_1\end{array}$$

$$\sim\left(\begin{array}{ccc|c}3&2&7&0\\0&-17&-34&0\\0&0&0&0\end{array}\right)R_3\to R_3+R_2$$

$\Longrightarrow\rho([A|B])=2=\rho(A)$, $The\ system\ of\ equations\ is\ consistent\ and\ has\ non-trivial\ solution.$

$$AX=O\Longrightarrow\begin{pmatrix}3&2&7\\0&-17&-34\\0&0&0\end{pmatrix}\begin{pmatrix}x\\y\\z\end{pmatrix}=\begin{pmatrix}0\\0\\0\end{pmatrix}\Longrightarrow\begin{array}{l}3x+2y+7z=0\to(1)\\-17y-34z=0\to(2)\end{array}$$

$when\ z=t\quad(2)\Longrightarrow -17y-34t=0\Longrightarrow y=\ -2t$
$when\ \ y=\ -2t,z=t\ \ (1)\Longrightarrow 3x+2(-2t)+7t=0\Longrightarrow 3x=-3t\Longrightarrow x=-t$
$solution: (x,y,z)=(-t,-2t,t); t\in R$

19. Solve the following system of homogenous equations.
$12x+3y-z=0, x-y-2z=0, 3x+y+3z=0$

$$AX=O\Longrightarrow\begin{bmatrix}2&3&-1\\1&-1&-2\\3&1&3\end{bmatrix}\begin{bmatrix}x\\y\\z\end{bmatrix}=\begin{bmatrix}0\\0\\0\end{bmatrix}$$

$$Augmented\ matrix\ [A|O]=\left(\begin{array}{ccc|c}2&3&-1&0\\1&-1&-2&0\\3&1&3&0\end{array}\right)$$

$$\sim\left(\begin{array}{ccc|c}1&-1&-2&0\\2&3&-1&0\\3&1&3&0\end{array}\right)R_1\leftrightarrow R_2$$

$$\sim\left(\begin{array}{ccc|c}1&-1&-2&0\\0&5&3&0\\0&4&9&0\end{array}\right)\begin{array}{l}R_2\to R_2-2R_1\\R_3\to R_3-3R_1\end{array}$$

$$\sim\left(\begin{array}{ccc|c}1&-1&-2&0\\0&5&3&0\\0&0&33&0\end{array}\right)R_3\to 5R_3-4R_2$$

$\Longrightarrow\rho([A|B])=3=\rho(A)$,
$The\ system\ of\ equations\ is\ consistent\ and\ has\ trivial\ solution.$
$(x,y,z)=(0,0,0).$

20. Determine the values of λ for which the following system of equations $x+y+3z=0$, $4x+3y+\lambda z=0$, $2x+y+2z=0$ has (i) a unique solution (ii) a non-trivial solution.

$$AX=O\Longrightarrow\begin{bmatrix}1&1&3\\4&3&\lambda\\2&1&2\end{bmatrix}\begin{bmatrix}x\\y\\z\end{bmatrix}=\begin{bmatrix}0\\0\\0\end{bmatrix}$$

$$Augmented\ matrix\ [A|O]=\left(\begin{array}{ccc|c}1&1&3&0\\4&3&\lambda&0\\2&1&2&0\end{array}\right)$$

$$\sim\left(\begin{array}{ccc|c}1&1&3&0\\2&1&2&0\\4&3&\lambda&0\end{array}\right)R_2\leftrightarrow R_3$$

$$\sim\left(\begin{array}{ccc|c}1&1&3&0\\0&-1&-4&0\\0&-1&\lambda-12&0\end{array}\right)\begin{array}{l}R_2\to R_2-2R_1\\R_3\to R_3-4R_1\end{array}$$

$$\sim\left(\begin{array}{ccc|c}1&1&3&0\\0&-1&-4&0\\0&0&\lambda-8&0\end{array}\right)R_3\to R_3-R_2$$

(i) $when\ \lambda\neq 8\Longrightarrow\rho(A)=\rho([A|O])])=3,$
$The\ system\ of\ equations\ is\ consistent\ and\ has\ trivial\ solution.\ (x,y,z)=(0,0,0)$
(ii)$when\ \ \lambda=8\Longrightarrow\rho(A)=\rho([A|O])])=2,$
$The\ system\ of\ equations\ is\ consistent\ and\ has\ non-trivial\ solution.$

21. By using Gaussian elimination method, balance the chemical reaction equation:
$C_2H_6+\ O_2\to H_2O+CO_2.$

$x_1C_2H_6+x_2O_2\to x_3H_2O+x_4CO_2\to(1)$
$Compare\ C\ on\ both\ sides,\ \ 2x_1-1x_4=0\to(2)$
$Compare\ H\ on\ both\ sides,\ \ 6x_1-2x_3=0\to(2)$
$\div 2\Longrightarrow 3x_1+0x_2-x_3-0x_4=0\to(3)$
$Compare\ O\ on\ both\ sides,\ \ 2x_2-1x_3-2x_4=0\to(4)$

$From\ (2),(3),(4) \Longrightarrow [A|O] = \left(\begin{array}{cccc|c} 2 & 0 & 0 & -1 & 0 \\ 3 & 0 & -1 & 0 & 0 \\ 0 & 2 & -1 & -2 & 0 \end{array}\right)$

$\sim \left(\begin{array}{cccc|c} 2 & 0 & 0 & -1 & 0 \\ 0 & 0 & -2 & 3 & 0 \\ 0 & 2 & -1 & -2 & 0 \end{array}\right) \sim R_2 \to 2R_2 - 3R_1$

$\sim \left(\begin{array}{cccc|c} 2 & 0 & 0 & -1 & 0 \\ 0 & 2 & -1 & -2 & 0 \\ 0 & 0 & -2 & 3 & 0 \end{array}\right) R_2 \leftrightarrow R_3$

$\rho(A) = \rho([A|O]) = 3 < 4$, *The system of equations is consistent and has non − trivial solution.*

$2x_1 - 1x_4 = 0 \to (5)$

$2x_2 - x_3 - 2x_4 = 0 \to (6)$

$-2x_3 + 3x_4 = 0 \to (7)$

$Let\ x_4 = t\ \ (5) \Longrightarrow 2x_1 - t = 0 \Longrightarrow x_1 = \frac{t}{2}$

$x_4 = t, (7) \Longrightarrow -2x_3 + 3x_4 = 0 \Longrightarrow -2x_3 + 3t = 0 \Longrightarrow x_3 = \frac{3t}{2}$

$x_3 = \frac{3t}{2}, x_4 = t, (6) \Longrightarrow 2x_2 - \frac{3t}{2} - 2t = 0 \Longrightarrow x_2 = \frac{7t}{4}$

$(x_1, x_2, x_3, x_4) = \left(\frac{t}{2}, \frac{7t}{4}, \frac{3t}{2}, t\right)$

$t = 4 \Longrightarrow (x_1, x_2, x_3, x_4) = (2,7,6,4)$

The Balanced chemical reaction equation *is* $(1) \Longrightarrow 2C_2H_6 + 7O_2 \to 6H_2O + 4CO_2$

2.COMPLEX NUMBERS

2 MARK QUESTIONS

1. Simplify $:i^{1948} - i^{-1869}$

$i^{1948} - i^{-1869} = i^{1948} - \frac{1}{i^{1869}}$

$= (i^4)^{487}(i)^0 - \frac{1}{(i^4)^{467} i^1}$

$= 1 - \frac{1}{i}$

$= 1 - \left(\frac{1}{i} \times \frac{-i}{-i}\right)$

$=1 + i$

2. Simplify: $\sum_{n=1}^{12} i^n$

$\sum_{n=1}^{12} i^n = i^1 + i^2 + i^3 + i^4 + i^5 + i^6 + i^7 + +i^8 + i^9 + i^{10} + i^{11} + i^{12}$

$= [i^1 + i^2 + i^3 + i^4] + [i^5 + i^6 + i^7 + +i^8] + [i^9 + i^{10} + i^{11} + i^{12}]$

$= 0$

3. Simplify: $i.i^2.i^3.i^4 \ldots\ldots i^{2000}$

$i.i^2.i^3.i^4 \ldots\ldots i^{2000} = i^{(1+2+3\ldots\ldots+2000)}$

$= i^{\frac{2000\times2001}{2}}$

$= i^{1000\times2001}$

$= (i^4)^{250}(i^4)^{500} i^1$

$= i$

4.Find the additive and multiplicative inverse of $Z = 2 + 5i$

Additive inverse of $Z = 2 + 5i$ *is* $-Z = -2 - 5i$

Multiplicative inverse of $z = x + iy$ is $z^{-1} = \left(\frac{x}{x^2+y^2}\right) + i\left(-\frac{y}{x^2+y^2}\right)$

Multiplicative inverse of $Z = 2 + 5i$ *is* $Z^{-1} = \left(\frac{2}{2^2+5^2}\right) + i\left(-\frac{5}{2^2+5^2}\right) = \frac{2}{29} - \frac{5}{29}i$

5. If $z_1 = 3 - 2i$ and $z_2 = 6 + 4i$ find $\frac{z_1}{z_2}$ in the rectangular form

$\frac{z_1}{z_2} = \frac{3-2i}{6+4i} = \frac{3-2i}{6+4i} \times \frac{6-4i}{6-4i}$

$= \frac{18 - 12i - 12i - 8}{6^2 + 4^2}$

$= \frac{10 - 24i}{52}$

$= \frac{5 - 12i}{26}$

6. *write* $\overline{(5+9\imath) + (2-4\imath)}$ *in the rectangular form..*

$\overline{(5+9\imath) + (2-4\imath)} = \overline{5+9\imath} + \overline{2-4\imath}$

$= 5 - 9i + 2 + 4i$

$= 7 - 5i$

7. *Write* $\overline{3\imath} + \frac{1}{2-i}$ *in the rectangular form*

$\overline{3\imath} + \frac{1}{2-i} = -3i + \frac{1}{2-i} \times \frac{2+i}{2+i} = -3i + \frac{2+i}{2^2+1^2} = \frac{2}{5} - 3i + \frac{1}{5}i = \frac{2}{5} - \frac{14}{5}i$

8 If $z = x + iy$ then write $Re\left(\frac{1}{z}\right)$ *in the rectangular form.*

$\therefore Re\left(\frac{1}{z}\right) = Re(z^{-1})$

$= Re\left(\frac{x}{x^2+y^2} - \frac{y}{x^2+y^2}i\right)$

$= \frac{x}{x^2+y^2}$

9 *If* $z = x + iy$ then write $Re(i\bar{z})$ *in the rectangular form.*

$i\bar{z} = i(\overline{x+iy}) = i(x - iy)$

$= ix - i^2y$

$= y + ix$

$\therefore Re(i\bar{z}) = y$

10. *If* $z = x + iy$ then write $Im(3z + 4\bar{z} - 4i)$ *in the rectangular .*

$3z + 4\bar{z} - 4i = 3(x + iy) + 4(\overline{x+iy}) - 4i$

$= 3x + 3yi + 4(x - iy) - 4i$

$= 3x + 3yi + 4x - 4yi - 4i$

$= 7x - yi - 4i$

$= 7x + i(-y - 4)$

$\therefore Im(3z + 4\bar{z} - 4i) = -y - 4$

11. *If* $z_1 = 2 - i$ *and* $z_2 = -4 + 3i$ *then find the inverse of* z_1z_2.

$z_1z_2 = (2 - i)(-4 + 3i) = (-8 + 3) + i(6 + 4) = -5 + 10i$

$(\because z_1z_2 = (ac - bd) + i(ad + bc))$

Inverse of $z_1z_2 = \left(\frac{x}{x^2+y^2}\right) + i\left(-\frac{y}{x^2+y^2}\right)$

$= \left(\frac{-5}{(-5)^2+10^2}\right) + i\left(-\frac{10}{(-5)^2+10^2}\right)$

$= \frac{-5}{125} - \frac{10}{125}i$

$= -\frac{1}{25} - \frac{2}{25}i$

12. *If* $z_1 = 2 - i$ *and* $z_2 = -4 + 3i$ *then find the inverse of* $\frac{z_1}{z_2}$

$\left(\frac{z_1}{z_2}\right)^{-1} = \frac{z_2}{z_1} = \frac{-4+3i}{2-i} = \frac{-4+3i}{2-i} \times \frac{2-i}{2-i}$

$= \frac{(-8-3)+i(-4+6)}{2^2+1^2}$

$= -\frac{11}{5} + \frac{2}{5}i$

13. *Find* $\left|\frac{2+i}{-1+2i}\right|$

$\left|\frac{2+i}{-1+2i}\right| = \frac{|2+i|}{|-1+2i|}$

$= \frac{\sqrt{2^2+1^2}}{\sqrt{(-1)^2+2^2}}$

$= 1$

14. *Find the square root of* $4 + 3i$.

$$\sqrt{a+ib} = \pm\left(\sqrt{\frac{|z|+a}{2}} + i\frac{b}{|b|}\sqrt{\frac{|z|-a}{2}}\right)$$

$a = 4, b = 3, |z| = \sqrt{4^2 + 3^2} = \sqrt{25} = 5$

$\sqrt{4+3i} = \pm\left(\sqrt{\frac{5+4}{2}} + i\frac{3}{3}\sqrt{\frac{5-4}{2}}\right) = \pm\left(\frac{3}{\sqrt{2}} + i\frac{1}{\sqrt{2}}\right)$

15. *Find the modulus of* $2i(3 - 4i)(4 - 3i)$

$|2i(3 - 4i)(4 - 3i)| = |2i||3 - 4i||4 - 3i|$

$= \sqrt{0^2 + 2^2}\sqrt{3^2 + 4^2}\sqrt{4^2 + 3^2}$

$= \sqrt{4}\sqrt{25}\sqrt{25}$

$= 50$

16. *Obtain the cartesian form of the locus of* $z = x + iy$ *for* $[Re(iz)]^2 = 3$.

$z = x + iy$

$iz = i(x + iy) = y - ix \implies Re(iz) = y$

$[Re(iz)]^2 = 3 \implies y^2 = 3$

17 *Obtain the cartesian form of the locus of* $z = x + iy$ *for* $Im[(1 - i)z + 1] = 0$

$z = x + iy$

$(1 - i)z + 1 = (1 - i)(x + iy) + 1 = x + iy - ix + y + 1 = (x + y + 1) + i(y - x)$

$\implies Im[(1 - i)z + 1] = y - x$

$\therefore Im[(x + y + 1) + i(y - x)] = 0 \implies y - x = 0\ (or) x - y = 0$

18 *Obtain the cartesian form of the locus of* $z = x + iy$ *for* $|z| = |z - i|$

$|z| = |z - i| \implies |x + iy| = |x + iy - i|$

$\implies \sqrt{x^2 + y^2} = \sqrt{x^2 + (y - 1)^2}$

Squaring on both sides,

$\implies x^2 + y^2 = x^2 + (y - 1)^2$

$\implies, x^2 + y^2 = x^2 + y^2 - 2y + 1$

$\implies -2y + 1 = 0\ (or) 2y - 1 = 0$

19. *Obtain the cartesian form of the locus of* $z = x + iy$ *for* $|2z - 3 - i| = 3$.

$|2z - 3 - i| = 3 \implies |2(x + iy) - 3 - i| = 3$

$$\Rightarrow |(2x-3)+(2y-1)i| = 3$$
$$\Rightarrow \sqrt{(2x-3)^2+(2y-1)^2} = 3$$

Squaring on both sides,

$$\Rightarrow (2x-3)^2+(2y-1)^2 = 9$$
$$\Rightarrow 4x^2-12x+9+4y^2-4y+1 = 9$$
$$\Rightarrow 4x^2+4y^2-12x-4y+1 = 0$$

20. *Obtain the cartesian form of the locus of* $z = x+iy$ *in* $|z-4| = 16$.

$|z-4| = 16 \Rightarrow |x+iy-4| = 16$

$$\Rightarrow |(x-4)+iy| = 16$$
$$\Rightarrow \sqrt{(x-4)^2+y^2} = 16$$

Squaring on both sides,

$$\Rightarrow (x-4)^2+y^2 = 16^2$$
$$\Rightarrow x^2-8x+16+y^2 = 256$$
$$\Rightarrow x^2+y^2-8x-240 = 0$$

21. Show that the equation $|3z-6+12i| = 8$ represent a circle, and, find its centre and radius

Equation of the circle is: $\boxed{|z-z_0| = r}$

$|3z-6+12i| = 8 \Rightarrow 3|z-2+4i| = 8$

$$\Rightarrow |z-(2-4i)| = \frac{8}{3}$$

centre $= 2-4i = (2,-4)$

Radius $= \frac{8}{3}$ *units.*

22. Show that $|z+2-i| < 2$ represents interior points of a circle. Find its centre and radius

Let $|z+2-i| = 2$

$\Rightarrow |z-(-2+i)| = 2.$

centre $= -2+i = (-2,1)$; *Radius* $= 2$ *units.*

$\therefore |z+2-i| < 2$ represents interior points of a circle

3 MARK QUESTIONS

1. Find the values of the real numbers x and y, if the complex numbers $(3-i)x-(2-i)y+2i+5$ and $2x+(-1+2i)y+3+2i$ are equal.

$z_1 = (3-i)x-(2-i)y+2i+5 = 3x-xi-2y+yi+2i+5$

$$= (3x-2y+5)+i(-x+y+2)$$

$z_2 = 2x+(-1+2i)y+3+2i = 2x-y+2yi+3+2i$

$$= (2x-y+3)+i(2y+2)$$

Given : $z_1 = z_2 \Rightarrow (3x-2y+5)+i(-x+y+2) = (2x-y+3)+i(2y+2)$

Equating the real and imaginary parts on both sides,

$3x-2y+5 = 2x-y+3$	$-x+y+2 = 2y+2$
$3x-2y+5-2x+y-3 = 0$	$-x+y+2-2y-2 = 0$
$x-y = -2 \to (1)$	$-x-y = 0 \to (2)$

$(1)+(2) \Rightarrow -2y = -2 \Rightarrow \boxed{y = 1}$

put $y = -1$ *in* $\Rightarrow x-1 = -2 \Rightarrow \boxed{x = -1}$

2. The complex numbers u, v and w are related by $\frac{1}{u} = \frac{1}{v}+\frac{1}{w}$. If $v = 3-4i$ and $w = 4+3i$, find u in rectangular form

$$\frac{1}{u} = \frac{1}{v}+\frac{1}{w} \Rightarrow \frac{1}{u} = \frac{1}{3-4i}+\frac{1}{4+3i} = \left(\frac{1}{3-4i}\times\frac{3+4i}{3+4i}\right)+\left(\frac{1}{4+3i}\times\frac{4-3i}{4-3i}\right)$$
$$= \frac{3+4i}{3^2+4^2}+\frac{4-3i}{4^2+3^2}$$
$$= \frac{1}{25}(3+4i+4-3i)$$
$$\Rightarrow \frac{1}{u} = \frac{7+i}{25}$$
$$\Rightarrow u = \frac{25}{7+i} = \frac{25}{7+i}\times\frac{7-i}{7-i}$$
$$= \frac{25(7-i)}{7^2+1^2}$$
$$= \frac{1}{2}(7-i)$$

3. Find the least value of the positive integer n for which $\left(\sqrt{3}+i\right)^n$ (i) real (ii) purely imaginary

$\left(\sqrt{3}+i\right)^3 = 3\sqrt{3}+9i-3\sqrt{3}-i = 8i$

$\left(\sqrt{3}+i\right)^6 = \left(\left(\sqrt{3}+i\right)^3\right)^2 = (8i)^2 = -64$

(i) when $n = 6$, $\left(\sqrt{3}+i\right)^n$ *is real.*

(ii) when $n = 3$, $\left(\sqrt{3}+i\right)^n$ *is purely imaginary.*

4. *Show that* $\left(2+i\sqrt{3}\right)^{10}-\left(2-i\sqrt{3}\right)^{10}$ *is purely imaginary.*

$z = \left(2+i\sqrt{3}\right)^{10}-\left(2-i\sqrt{3}\right)^{10}$

$\bar{z} = \overline{\left(2+\imath\sqrt{3}\right)^{10} - \left(2-\imath\sqrt{3}\right)^{10}} = \overline{\left(2+\imath\sqrt{3}\right)^{10}} - \overline{\left(2-\imath\sqrt{3}\right)^{10}}$

$= \left(\overline{2+\imath\sqrt{3}}\right)^{10} - \left(\overline{2-\imath\sqrt{3}}\right)^{10} = \left(2-i\sqrt{3}\right)^{10} - \left(2+i\sqrt{3}\right)^{10}$

$= -\left\{\left(2+i\sqrt{3}\right)^{10} - \left(2-i\sqrt{3}\right)^{10}\right\}$

$= -z$

$\therefore \left(2+i\sqrt{3}\right)^{10} - \left(2-i\sqrt{3}\right)^{10}$ *is purely imaginary.*

5. *Show that* $\left(2+i\sqrt{3}\right)^{10} + \left(2-i\sqrt{3}\right)^{10}$ *is real.*

$z = \left(2+i\sqrt{3}\right)^{10} + \left(2-i\sqrt{3}\right)^{10}$

$\bar{z} = \overline{\left(2+\imath\sqrt{3}\right)^{10} + \left(2-\imath\sqrt{3}\right)^{10}} = \overline{\left(2+\imath\sqrt{3}\right)^{10}} + \overline{\left(2-\imath\sqrt{3}\right)^{10}}$

$= \left(\overline{2+\imath\sqrt{3}}\right)^{10} + \left(\overline{2-\imath\sqrt{3}}\right)^{10} = \left(2-i\sqrt{3}\right)^{10} + \left(2+i\sqrt{3}\right)^{10} = z$

$\therefore \left(2+i\sqrt{3}\right)^{10} - \left(2-i\sqrt{3}\right)^{10}$ *is real*

6 Write $\frac{3+4i}{5-12i}$ in the $x+iy$ form, hence find its real and imaginary parts

$\boxed{\frac{z_1}{z_2} = \frac{a+ib}{c+id} = \left[\frac{ac+bd}{c^2+d^2}\right] + i\left[\frac{bc-ad}{c^2+d^2}\right]}$

$\frac{3+4i}{5-12i} = \left[\frac{15-48}{5^2+(-12)^2}\right] + i\left[\frac{20+36}{5^2+(-12)^2}\right] = \frac{-33+56i}{169} = -\frac{33}{169} + \frac{56}{169}i$

Real part $= -\frac{33}{169}$

Imaginary part $= \frac{56}{169}$

7.Simplify $\left(\frac{1+i}{1-i}\right)^3 - \left(\frac{1-i}{1+i}\right)^3$ into rectangular form..

$\frac{1+i}{1-i} = \left[\frac{1-1}{1^2+(-1)^2}\right] + i\left[\frac{1+1}{1^2+(-1)^2}\right] = \frac{2i}{2} = i$

$\frac{1-i}{1+i} = \left(\frac{1+i}{1-i}\right)^{-1} = -i$

$\left(\frac{1+i}{1-i}\right)^3 - \left(\frac{1-i}{1+i}\right)^3 = (i)^3 - (-i)^3 = -i - i = -2i.$

8. If $\frac{z+3}{z-5i} = \frac{1+4i}{2}$, find the complex number z in the rectangular form

$\frac{z+3}{z-5i} = \frac{1+4i}{2} \Rightarrow 2(z+3) = (1+4i)(z-5i)$

$\Rightarrow 2z + 6 = z - 5i + 4zi - 20i^2$

$\Rightarrow 2z - z - 4zi = -5i + 20 - 6$

$\Rightarrow z - 4zi = 14 - 5i$

$\Rightarrow z(1-4i) = 14 - 5i$

$\Rightarrow z = \frac{14-5i}{1-4i}$ $\qquad \boxed{\frac{z_1}{z_2} = \frac{a+ib}{c+id} = \left[\frac{ac+bd}{c^2+d^2}\right] + i\left[\frac{bc-ad}{c^2+d^2}\right]}$

$= \left[\frac{14+20}{1^2+(-4)^2}\right] + i\left[\frac{-5+56}{1^2+(-4)^2}\right] = \frac{34}{17} + i\frac{51}{17} = 2 + 3i$

9.Which one of the points $10-8i, 11+6i$ is closest to $1+i$.

Distance between the two complex numbers z_1 *and* $z_2 = |z_1 - z_2|$.

$z = 1+i, z_1 = 10-8i, z_2 = 11+6i$

$zz_1 = |1+i-10+8i| = |-9i+9i| = \sqrt{(-9)^2+(9)^2} = \sqrt{162}$

$zz_2 = |1+i-11-6i| = |-10-5i| = \sqrt{(-10)^2+(-5)^2} = \sqrt{125}$

$\sqrt{125} < \sqrt{162}$.

$\therefore$ $11+6i$ *is closest to* $1+i$.

10. If $|z| = 3$, Show that $7 \le |z+6-8i| \le 13$

Let $z_1 = z$ *and* $z_2 = 6-8i$

$\left||z_1| - |z_2|\right| \le |z_1+z_2| \le |z_1| + |z_2|$

$\Rightarrow \left|3 - \sqrt{6^2+8^2}\right| \le |z+6-8i| \le 3 + \sqrt{6^2+8^2}$

$\Rightarrow \left|3 - \sqrt{100}\right| \le |z+6-8i| \le 3 + \sqrt{100}$

$\Rightarrow |3-10| \le |z+6-8i| \le 3+10$

$\Rightarrow |-7| \le |z+6-8i| \le 13$

$\Rightarrow 7 \le |z+6-8i| \le 13$

11. Obtain the Cartesian form of the locus of $z = x+iy$ in $|z+i| = |z-1|$

$|z+i| = |z-1| \Rightarrow |x+iy+i| = |x+iy-1|$

$\Rightarrow |x+i(y+1)| = |(x-1)+iy|$

$\Rightarrow \sqrt{(x)^2+(y+1)^2} = \sqrt{(x-1)^2+(y)^2}$

Squaring on both sides,

$\Rightarrow (x)^2 + (y+1)^2 = (x-1)^2 + (y)^2$

$\Rightarrow x^2 + y^2 + 2y + 1 = x^2 - 2x + 1 + y^2$

$\Rightarrow 2x + 2y = 0$

$\Rightarrow x + y = 0$

12. Obtain the Cartesian form of the locus of $z = x+iy$ in $|z-4|^2 - |z-1|^2 = 16$

$|z-4|^2 - |z-1|^2 = 16 \Rightarrow |x+iy-4|^2 - |x+iy-1|^2 = 16$

$\Rightarrow |(x-4)+iy|^2 - |(x-1)+iy|^2 = 16$

$$\Rightarrow \left(\sqrt{(x-4)^2+y^2}\right)^2 - \sqrt{(x-1)^2+y^2}^{\,2} = 16$$
$$\Rightarrow (x-4)^2+y^2-[(x-1)^2+y^2] = 16$$
$$\Rightarrow x^2-8x+16+y^2-x^2+2x-1-y^2 = 16$$
$$\Rightarrow -6x-1=0$$
$$\Rightarrow 6x+1=0$$

13. If z_1, z_2 and z_3 are complex numbers such that $|z_1| = |z_2| = |z_3| = |z_1+z_2+z_3| = 1$, find the value of $\left|\frac{1}{z_1}+\frac{1}{z_2}+\frac{1}{z_3}\right|$.

$|z_1| = 1 \Rightarrow |z_1|^2 = 1 \Rightarrow z_1\overline{z_1} = 1 \Rightarrow \overline{z_1} = \frac{1}{z_1}$

,JNghyNt $\overline{z_2} = \frac{1}{z_2}$; $\overline{z_3} = \frac{1}{z_3}$

$\left|\frac{1}{z_1}+\frac{1}{z_2}+\frac{1}{z_3}\right| = |\overline{z_1}+\overline{z_2}+\overline{z_3}| = \overline{|z_1+z_2+z_3|} = 1$

14 Find the principal argument $Arg\ z$, when $z = -\frac{2}{1+i\sqrt{3}}$.

$.argz = arg\left(\frac{-2}{1+i\sqrt{3}}\right) = arg(-2) - arg(1+i\sqrt{3}) \to (1)$

$arg(-2) = tan^{-1}\left|\frac{y}{x}\right| = tan^{-1}\left|\frac{0}{-2}\right| = 0$

$-2 \Rightarrow (-,+), \theta$ *lies in the second quadrant*

$\theta = \pi - \alpha \Rightarrow \theta = \pi - 0 = \pi$

$arg(1+i\sqrt{3}) = tan^{-1}\left|\frac{y}{x}\right| = tan^{-1}\left|\frac{\sqrt{3}}{1}\right| = \frac{\pi}{3}$

$1+i\sqrt{3} \Rightarrow (+,+)$, θ *lies in the first quadrant.*

$\theta = \alpha \Rightarrow \theta = \frac{\pi}{3}$

$(1) \Rightarrow arg\left(\frac{-2}{1+i\sqrt{3}}\right) = \pi - \frac{\pi}{3} = \frac{2\pi}{3}$

15. Write in polar form of the following complex number $\frac{i-1}{\cos\frac{\pi}{3}+i\sin\frac{\pi}{3}}$

$i-1 = -1+i = rcis(\theta) \to (1)$

$r = \sqrt{x^2+y^2} = \sqrt{(-1)^2+(1)^2} = \sqrt{2}$

$\alpha = tan^{-1}\left|\frac{y}{x}\right| = tan^{-1}\left|\frac{1}{-1}\right| = \frac{\pi}{4}$

$(-,+)$, θ *lies in the second quadrant*

$\Rightarrow \theta = \pi - \alpha \Rightarrow \theta = \pi - \frac{\pi}{4} = \frac{3\pi}{4}$

$(1) \Rightarrow i-1 = \sqrt{2}cis\left(\frac{3\pi}{4}\right)$

$$\frac{i-1}{\cos\frac{\pi}{3}+i\sin\frac{\pi}{3}} = \frac{\sqrt{2}cis\left(\frac{3\pi}{4}\right)}{cis\left(\frac{\pi}{3}\right)}$$
$$= \sqrt{2}cis\left(\frac{3\pi}{4}-\frac{\pi}{3}\right)$$
$$= \sqrt{2}cis\left(\frac{5\pi}{12}\right)$$
$$= \sqrt{2}cis\left(2k\pi+\frac{5\pi}{12}\right), k \in z$$

16.Find the product of $\left(\cos\frac{\pi}{6}+i\sin\frac{\pi}{6}\right)\left(\cos\frac{\pi}{12}+i\sin\frac{\pi}{12}\right)$ in rectangular form.

$$\left(\cos\frac{\pi}{6}+i\sin\frac{\pi}{6}\right)\left(\cos\frac{\pi}{12}+i\sin\frac{\pi}{12}\right) = \text{cis}\left(\frac{\pi}{6}\right)+\text{cis}\left(\frac{\pi}{12}\right)$$
$$= \text{cis}\left(\frac{\pi}{6}+\frac{\pi}{12}\right)$$
$$= \text{cis}\left(\frac{3\pi}{12}\right)$$
$$= \cos\frac{\pi}{4}+i\sin\frac{\pi}{4}$$
$$= \frac{1}{\sqrt{2}}+i\frac{1}{\sqrt{2}}$$

17. Find the quotient $\frac{\cos\frac{\pi}{6}-i\sin\frac{\pi}{6}}{2\left(\cos\frac{\pi}{3}+i\sin\frac{\pi}{3}\right)}$ in rectangular form.

$$\frac{\cos\frac{\pi}{6}-i\sin\frac{\pi}{6}}{2\left(\cos\frac{\pi}{3}+i\sin\frac{\pi}{3}\right)} = \frac{1}{2}cis\left(-\frac{\pi}{6}-\frac{\pi}{3}\right)$$
$$= \frac{1}{2}cis\left(-\frac{3\pi}{6}\right)$$
$$= \frac{1}{2}cis\left(-\frac{\pi}{2}\right)$$
$$= \frac{1}{2}\left(\cos\frac{\pi}{2}-i\sin\frac{\pi}{2}\right)$$
$$= -\frac{1}{2}i$$

18. Find the value of $\left(\frac{1+\sin\frac{\pi}{10}+i\cos\frac{\pi}{10}}{1+\sin\frac{\pi}{10}-i\cos\frac{\pi}{10}}\right)^{10}$.

$z = \sin\frac{\pi}{10}+i\cos\frac{\pi}{10}$ vd;f. $\Rightarrow \frac{1}{z} = \sin\frac{\pi}{10}-i\cos\frac{\pi}{10}$

$$\left(\frac{1+\sin\frac{\pi}{10}+i\cos\frac{\pi}{10}}{1+\sin\frac{\pi}{10}-i\cos\frac{\pi}{10}}\right)^{10} = \left(\frac{1+z}{1+\frac{1}{z}}\right)^{10} = \left(\frac{1+z}{1+z}\times z\right)^{10} = z^{10}$$

$$z^{10} = \left(\sin\frac{\pi}{10}+i\cos\frac{\pi}{10}\right)^{10}$$
$$= \left[i\left(\cos\frac{\pi}{10}-i\sin\frac{\pi}{10}\right)\right]^{10}$$
$$= i^{10}\left[\cos\left(10\times\frac{\pi}{10}\right)-i\sin\left(10\times\frac{\pi}{10}\right)\right]$$
$$= -1[\cos\pi - i\sin\pi]$$
$$= 1$$

19.Solve the equation $z^3+27=0$

$z^3+27=0 \Rightarrow z^3 = -27$

$\Rightarrow z = (27)^{1/3}(-1)^{1/3}$

$= 3[cis(\pi)]^{1/3}$

$= 3[cis(2k\pi + \pi)]^{1/3}, k = 0,1,2$

$= 3cis(2k+1)\frac{\pi}{3}, k = 0,1,2$

$= 3cis\left(\frac{\pi}{3}\right), 3cis(\pi), 3cis\left(\frac{5\pi}{3}\right)$

20. State and prove the triangle law of inequality. (or)

For any two complex number z_1 and z_2, prove that $|z_1 + z_2| \leq |z_1| + |z_2|$.

PROOF:

$|z_1 + z_2|^2 = (z_1 + z_2)\overline{(z_1 + z_2)}$ $\quad (\because z\bar{z} = |z|^2)$

$= (z_1 + z_2)(\overline{z_1} + \overline{z_2})$ $\quad (\because \overline{z_1 + z_2} = \overline{z_1} + \overline{z_2})$

$= z_1\overline{z_1} + (z_1\overline{z_2} + \overline{z_1}z_2) + z_2\overline{z_2}$

$= z_1\overline{z_1} + (z_1\overline{z_2} + \overline{\overline{z_1}\overline{z_2}}) + z_2\overline{z_2}$ $\quad (\because \bar{\bar{z}} = z)$

$= |z_1|^2 + 2Re(z_1\overline{z_2}) + |z_2|^2$ $\quad (\because z + \bar{z} = 2Re(z))$

$\leq |z_1|^2 + 2|z_1\overline{z_2}| + |z_2|^2$ $\quad (\because Re(z) \leq |z|)$

$= |z_1|^2 + 2|z_1||z_2| + |z_2|^2$ $\quad (\because \overline{z_1 . z_2} = \overline{z_1}.\overline{z_2}\ \&|z| = |\bar{z}|)$

$= (|z_1| + |z_2|)^2$

Squaring on both sides, $|z_1 + z_2| \leq |z_1| + |z_2|$

5 MARK QUESTIONS

1. *Show that* $\left(\frac{19-7i}{9+i}\right)^{12} + \left(\frac{20-5i}{7-6i}\right)^{12}$ *is real.*

$$\frac{19-7i}{9+i} = \frac{19-7i}{9+i} \times \frac{9-i}{9-i}$$

$$= \frac{171 - 19i - 63i + 7i^2}{9^2 + 1^2}$$

$$= \frac{164-82i}{82}$$

$$= 2 - i$$

$$\frac{20-5i}{7-6i} = \frac{20-5i}{7-6i} \times \frac{7+6i}{7+6i}$$

$$= \frac{140 + 120i - 35i - 30i^2}{7^2 + 6^2}$$

$$= \frac{170 + 85i}{85}$$

$$= 2 + i$$

$$z = \left(\frac{19-7i}{9+i}\right)^{12} + \left(\frac{20-5i}{7-6i}\right)^{12} = (2-i)^{12} - (2+i)^{12}$$

$$\bar{z} = \overline{(2-\imath)^{12} - (2+\imath)^{12}}$$

$$= \overline{(2-\imath)^{12}} - \overline{(2+\imath)^{12}}$$

$$= (2+i)^{12} - (2-i)^{12}$$

$$= z$$

$\therefore \left(\frac{19-7i}{9+i}\right)^{12} + \left(\frac{20-5i}{7-6i}\right)^{12}$ *is real.*

2. *Show that* $\left(\frac{19+9i}{5-3i}\right)^{15} - \left(\frac{8+i}{1+2i}\right)^{15}$ *is purely imaginary.*

$$\frac{19+9i}{5-3i} = \frac{19+9i}{5-3i} \times \frac{5+3i}{5+3i}$$

$$= \frac{95 + 57i + 45i - 27}{5^2 + 3^2}$$

$$= \frac{68 + 102i}{34}$$

$$= 2 + 3i$$

$$\frac{8+i}{1+2i} = \frac{8+i}{1+2i} \times \frac{1-2i}{1-2i}$$

$$= \frac{8 - 16i + i + 2}{1^2 + 2^2}$$

$$= \frac{10 - 15i}{5}$$

$$= 2 - 3i$$

$$z = \left(\frac{19+9i}{5-3i}\right)^{15} - \left(\frac{8+i}{1+2i}\right)^{15} = (2+3i)^{15} - (2-3i)^{15}$$

$$\bar{z} = \overline{(2+3\imath)^{15} - (2-3\imath)^{15}} = \overline{(2+3\imath)^{15}} - \overline{(2-3\imath)^{15}}$$

$$= (2-3i)^{15} - (2+3i)^{15}$$

$$= -((2+3i)^{15} - (2-3i)^{15})$$

$$= -z$$

$\therefore \left(\frac{19+9i}{5-3i}\right)^{15} - \left(\frac{8+i}{1+2i}\right)^{15}$ *is purely imaginary*

3. *Show that the points* $1, -\frac{1}{2} + \frac{i\sqrt{3}}{2}$ and $-\frac{1}{2} - \frac{i\sqrt{3}}{2}$ *are the vertices of the equilateral triangle.*

$z_1 = 1, z_2 = -\frac{1}{2} + i\frac{\sqrt{3}}{2}$ *and* $z_3 = -\frac{1}{2} - i\frac{\sqrt{3}}{2}$

$$|z_1 - z_2| = \left|1 - \left(-\frac{1}{2} + i\frac{\sqrt{3}}{2}\right)\right| = \left|\frac{3}{2} - i\frac{\sqrt{3}}{2}\right| = \sqrt{\left(\frac{3}{2}\right)^2 + \left(\frac{\sqrt{3}}{2}\right)^2} = \sqrt{\frac{9}{4} + \frac{3}{4}} = \sqrt{\frac{12}{4}} = \sqrt{3}$$

$|z_2 - z_3| = \left|\left(-\frac{1}{2} + i\frac{\sqrt{3}}{2}\right) - \left(-\frac{1}{2} - i\frac{\sqrt{3}}{2}\right)\right| = \left|-\frac{1}{2} + i\frac{\sqrt{3}}{2} + \frac{1}{2} + i\frac{\sqrt{3}}{2}\right| = \sqrt{\left(\frac{2\sqrt{3}}{2}\right)^2} = \sqrt{3}$

$|z_3 - z_1| = \left|\left(-\frac{1}{2} - i\frac{\sqrt{3}}{2}\right) - 1\right| = \left|-\frac{3}{2} - i\frac{\sqrt{3}}{2}\right| = \sqrt{\left(\frac{3}{2}\right)^2 + \left(\frac{\sqrt{3}}{2}\right)^2} = \sqrt{\frac{9}{4} + \frac{3}{4}} = \sqrt{3}$

$\therefore$ *The given points are the vertices of the equilateral triangle.*

4 Let z_1, z_2 and z_3 be complex numbers such that $|z_1| = |z_2| = |z_3| = r > 0$ and $z_1 + z_2 + z_3 \neq 0$. Prove that $\left|\frac{z_1z_2+z_2z_3+z_3z_1}{z_1+z_2+z_3}\right| = r$.

$|z_1| = r \Rightarrow |z_1|^2 = r^2 \Rightarrow z_1\overline{z_1} = r^2 \Rightarrow z_1 = \frac{r^2}{\overline{z_1}}$ $\boxed{\because z\bar{z} = |z|^2}$

similarly, $z_2 = \frac{r^2}{\overline{z_2}}$; $z_3 = \frac{r^2}{\overline{z_3}}$

$z_1 + z_2 + z_3 = \frac{r^2}{\overline{z_1}} + \frac{r^2}{\overline{z_2}} + \frac{r^2}{\overline{z_3}}$

$\Rightarrow z_1 + z_2 + z_3 = r^2\left(\frac{\overline{z_2}\overline{z_3} + \overline{z_1}\overline{z_3} + \overline{z_1}\overline{z_2}}{\overline{z_1}\overline{z_2}\overline{z_3}}\right)$

$\Rightarrow |z_1 + z_2 + z_3| = |r^2|\left|\frac{\overline{z_2}\overline{z_3} + \overline{z_1}\overline{z_3} + \overline{z_1}\overline{z_2}}{\overline{z_1}\overline{z_2}\overline{z_3}}\right|$

$\Rightarrow |z_1 + z_2 + z_3| = r^2\frac{|\overline{\overline{z_2}\overline{z_3} + \overline{z_1}\overline{z_3} + \overline{z_1}\overline{z_2}}|}{|\overline{\overline{z_1}\overline{z_2}\overline{z_3}}|}$

$\Rightarrow |z_1 + z_2 + z_3| = r^2\frac{|z_1z_2 + z_2z_3 + z_3z_1|}{|z_1||z_2||z_3|}$

$\Rightarrow |z_1 + z_2 + z_3| = r^2\frac{|z_1z_2 + z_2z_3 + z_3z_1|}{r.r.r}$

$\Rightarrow |z_1 + z_2 + z_3| = \frac{|z_1z_2 + z_2z_3 + z_3z_1|}{r}$

$\Rightarrow \left|\frac{z_1z_2 + z_2z_3 + z_3z_1}{z_1 + z_2 + z_3}\right| = r$

5 Let z_1, z_2 and z_3 are three complex numbers such that $|z_1| = 1, |z_2| = 2, |z_3| = 3$ and $|z_1 + z_2 + z_3| = 1$. show that $|9z_1z_2 + 4z_1z_3 + z_2z_3| = 6$.

$|z_1| = 1 \Rightarrow |z_1|^2 = 1 \Rightarrow z_1\overline{z_1} = 1$ $\boxed{\because z\bar{z} = |z|^2}$

$|z_2| = 2 \Rightarrow |z_2|^2 = 4 \Rightarrow z_2\overline{z_2} = 4$

$|z_3| = 3 \Rightarrow |z_3|^2 = 9 \Rightarrow z_3\overline{z_3} = 9$

$|9z_1z_2 + 4z_1z_3 + z_2z_3| = |z_3\overline{z_3}z_1z_2 + z_2\overline{z_2}z_1z_3 + z_1\overline{z_1}z_2z_3|$

$= |(z_1z_2z_3)(\overline{z_1} + \overline{z_2} + \overline{z_3})|$

$= |z_1z_2z_3||\overline{z_1} + \overline{z_2} + \overline{z_3}|$

$= |z_1z_2z_3||\overline{z_1 + z_2 + z_3}|$

$= |z_1||z_2||z_3||z_1 + z_2 + z_3|$

$= 1.2.3.1$

$= 6$

6 Show that the equation $z^3 + 2\bar{z} = 0$ has five solutions.

$z^3 + 2\bar{z} = 0$

$\Rightarrow z^3 = -2\bar{z}$

$\Rightarrow |z^3| = |-2||\bar{z}|$

$\Rightarrow |z^3| = 2|z|$

$\Rightarrow |z|^3 - 2|z| = 0$

$\Rightarrow |z|(|z|^2 - 2) = 0$

$\Rightarrow |z| = 0 \Rightarrow z = 0$ *is one of the solution (or)* $|z|^2 - 2 = 0 \Rightarrow z\bar{z} - 2 = 0 \Rightarrow$ $\boxed{\bar{z} = \frac{2}{z}}$

Put $\bar{z} = \frac{2}{z}$ *in* $z^3 + 2\bar{z} = 0,$

$\Rightarrow z^3 + 2\left(\frac{2}{z}\right) = 0 \Rightarrow z^4 + 4 = 0$ *has four solution.*

$\therefore z^3 + 2\bar{z} = 0$ *has five solutions.*

7 For any two complex numbers z_1 and z_2 such that $|z_1| = |z_2| = 1$ and $z_1z_2 \neq -1$ then show that $\frac{z_1+z_2}{1+z_1z_2}$ is a real number.

$|z_1| = 1 \Rightarrow |z_1|^2 = 1 \Rightarrow z_1\overline{z_1} = 1 \Rightarrow \overline{z_1} = \frac{1}{z_1}$

similarly $\overline{z_2} = \frac{1}{z_2}$.

Let $w = \frac{z_1+z_2}{1+z_1z_2}$

$\bar{w} = \overline{\left(\frac{z_1 + z_2}{1 + z_1z_2}\right)}$

$= \frac{\overline{z_1} + \overline{z_2}}{\overline{1 + z_1z_2}}$

$= \frac{\overline{z_1} + \overline{z_2}}{1 + \overline{z_1}\overline{z_2}}$

$= \frac{\frac{1}{z_1}+\frac{1}{z_2}}{1+\frac{1\ 1}{z_1z_2}}$

$= \frac{z_1+z_2}{1+z_1z_2}$

$= w$

$\bar{w} = w.$

$\therefore W$ is a real

8. If $z = x+iy$ is a complex number such that $\left|\frac{z-4i}{z+4i}\right| = 1$, Show that the locus of z is real axis.

$z = x+iy$

$\left|\frac{z-4i}{z+4i}\right| = 1 \Longrightarrow |z-4i| = |z+4i|$

$\Longrightarrow |x+iy-4i| = |x+iy+4i|$

$\Longrightarrow |x+i(y-4)| = |x+i(y+4)|$

$\Longrightarrow \sqrt{x^2+(y-4)^2} = \sqrt{x^2+(y-4)^2}$

Squaring on both sides,

$\Longrightarrow x^2+(y-4)^2 = x^2+(y-4)^2$

$\Longrightarrow x^2+y^2-8y+16 = x^2+y^2+8y+16$

$\Longrightarrow 8y = 0$

$\Longrightarrow y = 0$

$\therefore$ *The locus of z is real axis.*

9. If $z = x+iy$ is a complex number such that $Im\left(\frac{2z+1}{iz+1}\right) = 0$. Show that the locus of z is $2x^2+2y^2+x-2y = 0$.

$z = x+iy$

$\frac{2z+1}{iz+1} = \frac{2(x+iy)+1}{i(x+iy)+1} = \frac{(2x+1)+i2y}{(1-y)+ix}$

$\boxed{\mathrm{Im}(Z) = \frac{bc-ad}{c^2+d^2}}$

Here $a = 2x+1$, $b = 2y, c = 1-y, d = x$

$Im\left(\frac{2z+1}{iz+1}\right) = 0 \Rightarrow \frac{2y(1-y)-x(2x+1)}{(1-y)^2+x^2} = 0$

$\Longrightarrow 2y-2y^2-2x^2-x = 0$

$\Longrightarrow 2x^2+2y^2+x-2y = 0$

10.If $z = x+iy$ and $arg\left(\frac{z-i}{z+2}\right) = \frac{\pi}{4}$ then show that $x^2+y^2+3x-3y+2 = 0$..

$Z = x+iy$

$\frac{z-i}{z+2} = \frac{x+iy-i}{x+iy+2} = \frac{x+i(y-1)}{(x+2)+iy}$ *Here* $a = x, b = y-1, c = x+2, d = y$

$= \left[\frac{ac+bd}{c^2+d^2}\right] + i\left[\frac{bc-ad}{c^2+d^2}\right]$

$= \left[\frac{x(x+2)+(y-1)y}{(x+2)^2+y^2}\right] + i\left[\frac{(y-1)(x+2)-xy}{(x+2)^2+y^2}\right]$

$= \left[\frac{x^2+y^2+2x-y}{(x+2)^2+y^2}\right] + i\left[\frac{2y-x-2}{(x+2)^2+y^2}\right]$

Here $X = \frac{x^2+y^2+2x-y}{(x+2)^2+y^2}$, $Y = \frac{2y-x-2}{(x+2)^2+y^2}$

$arg\left(\frac{z-i}{z+2}\right) = \frac{\pi}{4} \Rightarrow tan^{-1}\left(\frac{Y}{X}\right) = \frac{\pi}{4}$

$\Rightarrow \frac{Y}{X} = tan\frac{\pi}{4}$

$\Rightarrow \frac{x^2+y^2+2x-y}{(x+2)^2+y^2} = \frac{2y-x-2}{(x+2)^2+y^2}$

$\Longrightarrow \mathrm{x}^2+\mathrm{y}^2+2\mathrm{x}-\mathrm{y}+\mathrm{x}-2\mathrm{y}+2 = 0$

$\Longrightarrow x^2+y^2+3x-3y+2 = 0$

11. If $z = x+iy$ and $arg\left(\frac{z-1}{z+1}\right) = \frac{\pi}{2}$, then show that $x^2+y^2 = 1$.

$z = x+iy$

$\frac{z-1}{z+1} = \frac{x+iy-1}{x+iy+1} = \frac{(x-1)+iy}{(x+1)+iy}$ *Here* $a = x-1, b = y, c = x+1, d = y$

$= \left[\frac{ac+bd}{c^2+d^2}\right] + i\left[\frac{bc-ad}{c^2+d^2}\right]$

$= \left[\frac{(x-1)(x+1)+y^2}{(x+1)^2+y^2}\right] + i\left[\frac{y(x+1)-(x-1)y}{(x+1)^2+y^2}\right]$

$= \left[\frac{x^2-1+y^2}{(x+1)^2+y^2}\right] + i\left[\frac{2y}{(x+1)^2+y^2}\right]$

Here $X = \frac{x^2-1+y^2}{(x+1)^2+y^2}$, $Y = \frac{2y}{(x+1)^2+y^2}$

$arg\left(\frac{z-1}{z+1}\right) = \frac{\pi}{2} \Rightarrow tan^{-1}\left(\frac{Y}{X}\right) = \frac{\pi}{2}$

$\Rightarrow \frac{Y}{X} = tan\frac{\pi}{2}$

$\Rightarrow \frac{Y}{X} = \infty$

$\Longrightarrow \mathrm{X} = 0$

$\Longrightarrow x^2+y^2 = 1$

12. If $2cos\alpha = x+\frac{1}{x}$ and $2cos\beta = y+\frac{1}{y}$, show that $x^my^n + \frac{1}{x^my^n} = 2\cos(m\alpha+n\beta)$.

$x+\frac{1}{x} = 2\cos\alpha \Longrightarrow x = x = \cos\alpha + i\sin\alpha$

$y+\frac{1}{y} = 2\cos\beta \Longrightarrow y = \cos\beta + i\sin\beta$

$x^m = (\cos\alpha + i\sin\alpha)^m = \cos m\alpha + i\sin m\alpha$

$y^n = (\cos\beta + i\sin\beta)^n = \cos n\beta + i\sin n\beta$

$\frac{x^m}{y^n} = \frac{\cos m\alpha + i\sin m\alpha}{\cos n\beta + i\sin n\beta} = \cos(m\alpha - n\beta) + i\sin(m\alpha - n\beta)$

$\frac{y}{x^m}^n = \left(\frac{x^m}{y^n}\right)^{-1} = \cos(m\alpha - n\beta) - i\sin(m\alpha - n\beta)$

$\frac{x^m}{y^n} - \frac{y}{x^m}^n = 2i\sin(m\alpha - n\beta)$

13. . If $2cos\alpha = x + \frac{1}{x}$ and $2cos\beta = y + \frac{1}{y}$, show that $\frac{x^m}{y^n} - \frac{y}{x^m}^n = 2i\sin(m\alpha - n\beta)$.

$x + \frac{1}{x} = 2\cos\alpha \Longrightarrow x = x = \cos\alpha + i\sin\alpha$

$y + \frac{1}{y} = 2\cos\beta \Longrightarrow y = \cos\beta + i\sin\beta$

$x^m y^n = (\cos m\alpha + i\sin m\alpha)(\cos n\beta + i\sin n\beta)$

$\quad = \cos(m\alpha + n\beta) + i\sin(m\alpha + n\beta)$

$\frac{1}{x^m y^n} = \cos(m\alpha + n\beta) - i\sin(m\alpha + n\beta)$

$x^m y^n + \frac{1}{x^m y^n} = 2\cos(m\alpha + n\beta)$

14. Find the cube roots of unity.

$z^3 = 1$

$\Longrightarrow z = (1)^{1/3}$

$\Longrightarrow z = [cis(0)]^{1/3}$

$\Longrightarrow z = [cis(2k\pi + 0)]^{1/3}, k = 0,1,2$

$\Longrightarrow z = cis\left(\frac{2k\pi}{3}\right), k = 0,1,2$

$\Longrightarrow z = cis(0), cis\left(\frac{2\pi}{3}\right), cis\left(\frac{4\pi}{3}\right)$

$\Longrightarrow z = cis(0), cis\left(\pi - \frac{\pi}{3}\right), cis\left(\pi + \frac{\pi}{3}\right)$

$\Longrightarrow z = \cos 0 + i\sin 0, \cos\left(\pi - \frac{\pi}{3}\right) + i\sin\left(\pi - \frac{\pi}{3}\right), \cos\left(\pi + \frac{\pi}{3}\right) + i\sin\left(\pi + \frac{\pi}{3}\right)$

$\Longrightarrow z = 1, -\cos\frac{\pi}{3} + i\sin\frac{\pi}{3}, -\cos\frac{\pi}{3} - i\sin\frac{\pi}{3}$

$\Longrightarrow z = 1, \frac{-1+i\sqrt{3}}{2}, \frac{-1-i\sqrt{3}}{2}$

$\therefore$ *The cube roots of unity are* $1, \omega, \omega^2 = 1, \frac{-1+i\sqrt{3}}{2}, \frac{-1-i\sqrt{3}}{2}$

15. Find the fourth roots of unity.

$z^4 = 1$

$\Longrightarrow z = (1)^{1/4}$

$\Longrightarrow z = [cis(0)]^{1/4}$

$\Longrightarrow z = [cis(2k\pi + 0)]^{1/4}, k = 0,1,2,3$

$\Longrightarrow z = cis\left(\frac{2k\pi}{4}\right), k = 0,1,2,3$

$\Longrightarrow z = cis(0), cis\left(\frac{2\pi}{4}\right), cis\left(\frac{4\pi}{4}\right), cis\left(\frac{6\pi}{4}\right)$

$\Longrightarrow z = 1, i, -1, -i$

$\therefore$ *The fourth roots of unity are* $1, \omega, \omega^2, \omega^3 = 1, i, -1, -i$.

16. Suppose z_1, z_2 and z_3 are the vertices of an equilateral triangle inscribed in the circle $|z| = 2$. If $z_1 = 1 + i\sqrt{3}$, then find z_2 and z_3

z_2 makes an angle $\frac{2\pi}{3}$ with z_1 and z_3 makes an angle $\frac{4\pi}{3}$ with z_1.

$z_1 = 1 + i\sqrt{3}$

$z_2 = z_1 cis\left(\frac{2\pi}{3}\right)$

$\quad = (1 + i\sqrt{3})\left[\cos\left(\frac{2\pi}{3}\right) + i\sin\left(\frac{2\pi}{3}\right)\right]$

$\quad = (1 + i\sqrt{3})[\cos(180 - 60) + i\sin(180 - 60)]$

$\quad = (1 + i\sqrt{3})(-\cos 60 + i\sin 60)$

$\quad = (1 + i\sqrt{3})\left(-\frac{1}{2} + i\frac{\sqrt{3}}{2}\right)$

$\quad = -2$

$z_3 = z_2 cis\left(\frac{2\pi}{3}\right)$

$\quad = -2\left[\cos\left(\frac{2\pi}{3}\right) + i\sin\left(\frac{2\pi}{3}\right)\right]$

$\quad = -2[\cos(180 - 60) + i\sin(180 - 60)]$

$\quad = -2(-\cos 60 + i\sin 60)$

$\quad = -2\left(-\frac{1}{2} + i\frac{\sqrt{3}}{2}\right)$

$\quad = 1 - i\sqrt{3}$

3.THEORY OF EQUATIONS

2 MARK QUESTIONS

1. If α, β and γ are the roots of the quadratic equation $x^3 + px^2 + qx + r = 0$, find the value of $\sum\frac{1}{\beta\gamma}$ in terms of the coefficients.

Equation is $x^3 + px^2 + qx + r = 0$

$\sum_1 = \alpha + \beta + \gamma = -\frac{p}{1} = -p$

$\sum_3 = \alpha\beta\gamma = -\frac{r}{1} = -r$

$\sum\frac{1}{\beta\gamma} = \frac{1}{\beta\gamma} + \frac{1}{\alpha\gamma} + \frac{1}{\alpha\beta} = \frac{\alpha+\beta+\gamma}{\alpha\beta\gamma} = \frac{-p}{-r} = \frac{p}{r}$

2 Find the sum of the squares of the roots of $ax^4 + bx^3 + cx^2 + dx + e = 0, a \neq 0$

$\sum_1 = \alpha + \beta + \gamma + \delta = -\frac{b}{a}$

$\sum_2 = \alpha\beta + \alpha\gamma + \alpha\delta + \beta\gamma + \beta\delta + \gamma\alpha = \frac{c}{a}$

$(\alpha + \beta + \gamma + \delta)^2 = \alpha^2 + \beta^2 + \gamma^2 + \delta^2 - 2\sum\alpha\beta$

$\alpha^2 + \beta^2 + \gamma^2 + \delta^2 = (\alpha + \beta + \gamma + \delta)^2 - 2\sum\alpha\beta$

$$= \left(-\frac{b}{a}\right)^2 - 2\left(\frac{c}{a}\right)$$
$$= \frac{b^2}{a^2} - \frac{2c}{a}$$
$$= \frac{b^2-2ac}{a^2}$$

3. Find the sum of squares of roots of the equation $2x^4 - 8x^3 + 6x^2 - 3 = 0$.

$a = 2, b = -8, c = 6, d = 0, e = -3$

Sum of the squares of the fourth degree equation $= \frac{b^2-2ac}{a^2}$
$$= \frac{(-8)^2-2(2)(6)}{2^2}$$
$$= \frac{40}{4}$$
$$= 10$$

4. If p is real, discuss the nature of the roots of the equation $4x^2 + 4px + p + 2 = 0$, in terms of p.

$Discriminant\ \Delta = b^2 - 4ac$
$$= (4p)^2 - 4(4)(p+2)$$
$$= 16p^2 - 16p - 32$$
$$= 16(p^2 - p - 2)$$
$$= 16(p+1)(p-2)$$

$When\ -1 < p < 2\ then\ \ \Delta < 0 \Longrightarrow Roots\ are\ imaginary.$

$When\ p = -1\ or\ p = 2\ \ then\ \Delta = 0 \Longrightarrow Roots\ are\ real\ and\ equal.$

$When\ -\infty < p < -1\ \ or\ \ 2 < p < \infty\ then\ \Delta > 0 \Longrightarrow Roots\ are\ real\ and\ distinct.$

5.Construct a cubic equation with roots 1,2 *and* 3.

$Let\ \alpha = 1, \beta = 2, \gamma = 3$

$\sum_1 = \alpha + \beta + \gamma = 1 + 2 + 3 = 6$

$\sum_2 = \alpha\beta + \beta\gamma + \gamma\alpha = 2 + 6 + 3 = 11$

$\sum_3 = \alpha\beta\gamma\ = 1 \times 2 \times 3 = 6$

Required cubic equation : $x^3 - (\sum\alpha)x^2 + (\sum\alpha\beta)x - (\sum\alpha\beta\gamma) = 0$
$$\Longrightarrow x^3 - 6x^2 + 11x - 6 = 0$$

DO IT YOUSELF:

Construct a cubic equation with roots $i)\ 1,1\ and -2\ (ii) 2, -2\ and\ \ 4$

6 If p and q are the roots of the equation $lx^2 + nx + n = 0$, show that $\sqrt{\frac{p}{q}} + \sqrt{\frac{q}{p}} + \sqrt{\frac{n}{l}} = 0$.

$p + q = -\frac{n}{l}\ \ ;\ \ pq = \frac{n}{l}$

$$\sqrt{\frac{p}{q}} + \sqrt{\frac{q}{p}} + \sqrt{\frac{n}{l}} = \frac{\sqrt{p}}{\sqrt{q}} + \frac{\sqrt{q}}{\sqrt{p}} + \frac{\sqrt{n}}{\sqrt{l}} = \frac{p+q}{\sqrt{pq}} + \frac{\sqrt{n}}{\sqrt{l}}$$
$$= -\frac{\sqrt{n}}{\sqrt{l}} + \frac{\sqrt{n}}{\sqrt{l}}$$
$$= 0$$

7. Find a polynomial equation of minimum degree with rational coefficients, having $2 - \sqrt{3}i$ as a root.

$x = 2 - \sqrt{3}i \Longrightarrow x - 2 = -\sqrt{3}i$

$squaring\ on\ both\ sides$
$$\Longrightarrow (x-2)^2 = \left(-\sqrt{3}i\right)^2$$
$$\Longrightarrow x^2 - 4x + 4 = 3$$

$\therefore The\ required\ equation\ is\ x^2 - 4x + 1 = 0$

8. Find a polynomial equation of minimum degree with rational coefficients, having $2 - \sqrt{3}$ as a root.

$.x = 2 - \sqrt{3} \Longrightarrow x - 2 = -\sqrt{3}$

$squaring\ on\ both\ sides,$
$$\Longrightarrow (x-2)^2 = \left(-\sqrt{3}\right)^2$$
$$\Longrightarrow x^2 - 4x + 4 = 3$$

$\therefore The\ required\ equation\ is\ x^2 - 4x + 1 = 0$

9 Show that the equation $2x^2 - 6x + 7 = 0$ cannot be satisfied by any real values of x.

$a = 2, b = -6, c = 7$

$\Delta = b^2 - 4ac = (-6)^2 - 4(2)(7)$
$$= 36 - 56$$
$$= -20 < 0$$

$\therefore The\ given\ equation\ has\ only\ imaginary\ roots.$

10.f $x^2 + 2(k+2)x + 9k = 0$ has equal roots. Find k.

$a = 1, b = 2k + 4, c = 9k$

$Since\ has\ equal\ roots, \Delta = 0$
$$\Longrightarrow b^2 - 4ac = 0$$
$$\Longrightarrow (2k+4)^2 - 4(1)(9k) = 0$$
$$\Longrightarrow 4k^2 + 16k + 16 - 36k = 0$$
$$\Longrightarrow 4k^2 - 20k + 16 = 0$$
$$\Longrightarrow k^2 - 5k + 4 = 0$$
$$\Longrightarrow (k-1)(k-4) = 0$$

$\Rightarrow k = 1 \text{ or } 4$

11. Obtain the condition that the roots of $x^3 + px^2 + qx + r = 0$ are in A. P.

Let the roots are $(a-d), a, (a+d)$.

$\sum 1 = a - d + a + a + d = -\frac{b}{a} = -p$

$\Rightarrow 3a = -p$

$\Rightarrow a = -\frac{p}{3}$

since $a = -\frac{p}{3}$ *is the root of* $x^3 + px^2 + qx + r = 0$,

$$\Rightarrow \left(-\frac{p}{3}\right)^3 + p\left(-\frac{p}{3}\right)^2 + q\left(-\frac{p}{3}\right) + r = 0$$

$$\Rightarrow -\frac{p^3}{27} + p\left(\frac{p^2}{9}\right) - \frac{pq}{3} + r = 0$$

$$\Rightarrow -\frac{p^3}{27} + \frac{p^3}{9} + r = \frac{pq}{3}$$

$$(\times)\, 27 \Rightarrow -p^3 + 3p^3 + 27r = 9pq$$

$$\Rightarrow 9pq = 2p^3 + 27r$$

12 If k is real, discuss the nature of the roots of the polynomial equation $2x^2 + kx + k = 0$, in terms of k.

$a = 2, b = k, c = k$

$\Delta = b^2 - 4ac = k^2 - 8k = k(k-8)$

When $k < 0$ *then* $\Delta > 0 \Rightarrow$ *Roots are real.*

When $k = 0$, $k = 8$ *then* $\Delta = 0 \Rightarrow$ *Roots are real and equal.*

When $0 < k < 8$ *then* $\Delta < 0 \Rightarrow$ *Roots are imaginary.*

When $k > 8$ *then* $\Delta > 0 \Rightarrow$ *Roots are real and distinct.*

13.Find a polynomial equation of minimum degree with rational coefficients, having $2 + \sqrt{3}i$ as a root.

$x = 2 + \sqrt{3}i \Rightarrow x - 2 = \sqrt{3}i$

squaring on both sides,

$$\Rightarrow (x-2)^2 = \left(\sqrt{3}i\right)^2$$

$$\Rightarrow x^2 - 4x + 4 = -3$$

$\therefore$ *The required equation is* $x^2 - 4x + 7 = 0$

14. A 12 metre tall tree was broken into twoparts. It was found that the height of the part which was left standing was the cube root of the length of the part that was cut away. Formulate this into a mathematical problem to find the height of the part which was cut away.

Let x be the length of the broken part.

From the given data, $x^3 + x = 12$

$$\Rightarrow x^3 + x - 12 = 0$$

15. Show that the polynomial $9x^9 + 2x^5 - x^4 - 7x^2 + 2$ has at least six imaginary roots.

$P(x) = 9x^9 + 2x^5 - x^4 - 7x^2 + 2$

$P(x)$ *has 2 sign changes, so it has atmost 2 positive real roots.*

$P(-x) = 9(-x)^9 + 2(-x)^5 - (-x)^4 - 7(-x)^2 + 2 = -9x^9 - 2x^5 - x^4 - x^2 + 2$

$P(-x)$ *has 1 sign changes, so it has atmost 1 negative real root.*

No. of imaginary roots $= 9 - (2+1) = 9 - 3 = 6$.

$\therefore P(x)$ *has* at least six imaginary roots.

16. Discuss the nature of the roots of the following polynomial $x^{2018} + 1947x^{1950} + 15x^8 + 26x^6 + 2019$.

$P(x) = x^{2018} + 1947x^{1950} + 15x^8 + 26x^6 + 2019$

$P(x)$ *has no sign changes, so it has no positive real roots.*

$P(-x) = x^{2018} + 1947x^{1950} + 15x^8 + 26x^6 + 2019$

$P(-x)$ *has no sign changes, so it has no negative real roots.*

$\therefore P(x)$ *has* only imaginary roots.

3 MARK QUESTIONS

1.If α and β are the roots of the quadratic equation $17x^2 + 43x - 73 = 0$, construct a quadratic equation whose roots are $\alpha + 2$ and $\beta + 2$.

$\alpha + \beta = -\frac{43}{17}$; $\alpha\beta = -\frac{73}{17}$

Sum of the roots $= \alpha + 2 + \beta + 2 = \alpha + \beta + 4 = -\frac{43}{17} + 4 = \frac{25}{17}$

product of the roots $= (\alpha+2)(\beta+2) = \alpha\beta + 2\alpha + 2\beta + 4$

$$= \alpha\beta + 2(\alpha+\beta) + 4$$

$$= -\frac{73}{17} + 2\left(-\frac{43}{17}\right) + 4$$

$$= \frac{-73-86}{17} + 4$$

$$= \frac{-159+68}{17}$$

$$= -\frac{91}{17}$$

Required quadratic equation is $x^2 - (\text{sum of roots})x + (\text{product of roots}) = 0$

$$\Rightarrow x^2 - \left(\frac{25}{17}\right)x + \left(-\frac{91}{17}\right) = 0$$

$$\Rightarrow 17x^2 - 25x - 91 = 0$$

2.If α and β are the roots of the quadratic equation $2x^2 - 7x + 13 = 0$, construct a quadratic equation whose roots are α^2 and β^2.

$\alpha + \beta = \dfrac{7}{2}$; $\alpha\beta = \dfrac{13}{2}$

$$Sum\ of\ the\ roots = \alpha^2 + \beta^2 = (\alpha+\beta)^2 - 2\alpha\beta = \left(\frac{7}{2}\right)^2 - 2\left(\frac{13}{2}\right)$$
$$= \frac{49}{4} - 13$$
$$= -\frac{3}{4}$$

$$product\ of\ the\ roots = \alpha^2\beta^2 = (\alpha\beta)^2 = \left(\frac{13}{2}\right)^2 = \frac{169}{4}$$

$$Required\ quadratic\ equation\ is\ x^2 - (sum\ of\ roots)x + (product\ of\ roots) = 0$$
$$\Rightarrow x^2 - \left(-\frac{3}{4}\right)x + \left(\frac{169}{4}\right) = 0$$
$$\Rightarrow 4x^2 + 3x + 169 = 0$$

3.If α , β and γ are the roots of the cubic equation $x^3 + 2x^2 + 3x + 4 = 0$, form a cubic equation whose roots are $2\alpha, 2\beta, 2\gamma$

$Equation: x^3 + 2x^2 + 3x + 4 = 0$ $Roots: \alpha, \beta\ and\ \gamma$

$\sum_1 = \alpha + \beta + \gamma = -\frac{b}{a} = -\frac{2}{1} = -2$

$\sum_2 = \alpha\beta + \beta\gamma + \gamma\alpha = \frac{c}{a} = \frac{3}{1} = 3$

$\sum_3 = \alpha\beta\gamma = -\frac{d}{a} = -\frac{4}{1} = -4$

$Roots: 2\alpha, 2\beta, 2\gamma$

$\sum_1 = 2\alpha + 2\beta + 2\gamma = 2(\alpha+\beta+\gamma) = 2(-2) = -4$

$\sum_2 = (2\alpha)(2\beta) + (2\beta)(2\gamma) + (2\gamma)(2\alpha) = 4(\alpha\beta + \beta\gamma + \gamma\alpha) = 4(3) = 12$

$\sum_3 = (2\alpha)(2\beta)(2\gamma) = 8(\alpha\beta\gamma) = 8 \times (-4) = -32$

$$Required\ cubic\ equation\ is\ x^3 - (\textstyle\sum 1)x^2 + (\sum 2)x - (\sum 3) = 0$$
$$\Rightarrow x^3 - (-4)x^2 + 12x - (-32) = 0$$
$$\Rightarrow x^3 + 4x^2 + 12x + 32 = 0$$

DO IT YOURSELF :

If α , β and γ are the roots of the cubic equation $x^3 + 2x^2 + 3x + 4 = 0$, form a cubic equation whose roots are $(i)\ \frac{1}{\alpha}, \frac{1}{\beta}, \frac{1}{\gamma}$ $(ii) -\alpha, -\beta, -\gamma$

4.If α, β and γ are the roots of the polynomial equation $ax^3 + bx^2 + cx + d = 0$, find the value of $\sum \frac{\alpha}{\beta\gamma}$ in terms of the coefficients.

$Equation: ax^3 + bx^2 + cx + d = 0;$ $Roots: \alpha, \beta\ and\ \gamma$

$\sum_1 = \alpha + \beta + \gamma = -\frac{b}{a}$

$\sum_2 = \alpha\beta + \beta\gamma + \gamma\alpha = \frac{c}{a}$

$\sum_3 = \alpha\beta\gamma = -\frac{d}{a}$

$$\sum 1 = \sum\frac{\alpha}{\beta\gamma} = \frac{\alpha}{\beta\gamma} + \frac{\beta}{\alpha\gamma} + \frac{\gamma}{\alpha\beta} = \frac{\alpha^2+\beta^2+\gamma^2}{\alpha\beta\gamma} = \frac{(\alpha+\beta+\gamma)^2 - 2\sum\alpha\beta}{\alpha\beta\gamma} = \frac{\left(-\frac{b}{a}\right)^2 - 2\left(\frac{c}{a}\right)}{-\frac{d}{a}}$$
$$= \left(\frac{b^2}{a^2} - \frac{2c}{a}\right)\left(-\frac{a}{d}\right)$$
$$= \left(\frac{b^2 - 2ac}{a^2}\right)\left(-\frac{a}{d}\right)$$
$$= \frac{2ac - b^2}{ad}$$

5.Form a polynomial equation with integer coefficients with $\sqrt{\frac{\sqrt{2}}{\sqrt{3}}}$ as a root.

$$x = \sqrt{\frac{\sqrt{2}}{\sqrt{3}}}. \Rightarrow x^2 = \frac{\sqrt{2}}{\sqrt{3}}$$
$$\Rightarrow x^4 = \frac{2}{3}$$
$$\Rightarrow 3x^4 = 2$$
$$\Rightarrow 3x^4 - 2 = 0$$

6.Find a polynomial equation of minimum degree with rational coefficients, having $\sqrt{5} - \sqrt{3}$ as a root.

$$\sqrt{5} - \sqrt{3}\ is\ one\ of\ the\ root \quad \Rightarrow x = \sqrt{5} - \sqrt{3}$$
$$\Rightarrow x + \sqrt{3} = \sqrt{5}$$
$$Squaring\ on\ both\ sides, \left(x + \sqrt{3}\right)^2 = \sqrt{5}^2$$
$$\Rightarrow x^2 + 2\sqrt{3}x + 3 = 5$$
$$\Rightarrow x^2 - 2 = 2\sqrt{3}x$$
$$Again\ squaring\ on\ both\ sides, (x^2 - 2)^2 = \left(2\sqrt{3}x\right)^2$$
$$\Rightarrow x^4 - 4x^2 + 4 = 12x^2$$
$$\Rightarrow x^4 - 16x^2 + 4 = 0$$

7. Prove that the straight line and parabola cannot intersect at more than two points.

$Equation\ of\ the\ straight\ line\ is\ y = mx + c \rightarrow (1)$

$Equation\ of\ the\ parabola\ is\ y^2 = 4ax \rightarrow (2)$

$From\ (1)\ \&\ (2) \Rightarrow (mx + c)^2 = 4ax$

$$\Rightarrow m^2x^2 + 2mcx + c^2 = 4ax$$
$$\Rightarrow m^2x^2 + 2mcx + c^2 - 4ax = 0$$
$$\Rightarrow m^2x^2 + (2mc - 4a)x + c^2 = 0$$

Since it is a quadratic equation, it has only two solutions, So the straight line and parabola cannot intersect at more than two points.

8. Prove that a line cannot intersect a circle at more than two points.

$Equation\ of\ the\ circle\ is\ x^2 + y^2 = r^2 \rightarrow (1)$

$Equation\ of\ the\ straight\ line\ is\ \ y = mx + c \rightarrow (2)$

$From\ (1)\ \&\ (2) \Longrightarrow x^2 + (mx + c)^2 = r^2$

$\Longrightarrow x^2 + m^2x^2 + 2mcx + c^2 = r^2$

$\Longrightarrow (1 + m^2)x^2 + 2mcx + (c^2 - r^2) = 0$

Since it is a quadratic equation, it has only two solutions, So a line cannot intersect a circle at more than two points.

9.Solve the equation $x^4 - 9x^2 + 20 = 0$.

$Let\ x^2 = y \rightarrow (1).$

$x^4 - 9x^2 + 20 = 0 \Longrightarrow y^2 - 9y + 20 = 0$

$\Longrightarrow (y - 4)(y - 5) = 0$

$-\frac{4}{y}$	$\mathbf{20}$ / $\mathbf{-9}$	$-\frac{5}{y}$

$\Longrightarrow y = 4,5$

$Sub.\ y = 4,5\ in\ (1),$

$\Longrightarrow x^2 = 4\ \ ;\ \ x^2 = 5$

$\Longrightarrow x = \pm 2 \quad \Longrightarrow x = \pm\sqrt{5}$

$Thus\ the\ roots\ are 2, -2, -\sqrt{5}\ and\ \sqrt{5}.$

10.$Solve\ the\ equation\ x^3 - 3x^2 - 33x + 35 = 0$.

$sum\ of\ all\ term\ co.eff = 1 - 3 - 33 + 35 = 0$

$\therefore (x - 1)\ is\ a\ factor.$

	1	−3	−33	35
1	0	1	−2	−35
	1	−2	−35	0

$Quotient = x^2 - 2x - 35$

$\frac{5}{x}$	$\mathbf{-35}$ / $\mathbf{-2}$	$-\frac{7}{x}$

$= (x + 5)(x - 7)$

$\Longrightarrow x = -5,7$

$Thus\ the\ roots\ are\ 1, -5\ and\ 7$

11. $Solve\ the\ equation\ 2x^3 + 11x^2 - 9x - 18 = 0.$

$sum\ of\ odd\ term\ co.eff = 2 - 9 = -7$

$sum\ of\ even\ term\ co.eff = 11 - 18 = -7$

$\therefore (x + 1)\ is\ a\ factor.$

	2	11	−9	−18
−1	0	−2	−9	18
	2	9	−18	0

$Quotient = 2x^2 + 9x - 18$

$\frac{-3}{2x}$	$\mathbf{-36}$ / $\mathbf{9}$	$\frac{12}{2x}$

$= (2x - 3)(x + 6)$

$x = \frac{3}{2}, -6$

$Thus\ the\ roots\ are\ -1, \frac{3}{2}\ and\ -6.$

12.$Solve\ the\ equation\ x^3 - 5x^2 - 4x + 20 = 0.$

$x^3 - 5x^2 - 4x + 20 = 0 \Longrightarrow x^2(x - 5) - 4(x - 5) = 0$

$\Longrightarrow (x^2 - 4)(x - 5) = 0$

$\Longrightarrow x^2 - 4 = 0\ ; x - 5 = 0$

$\Longrightarrow x^2 = 4\ ; x = 5$

$\Longrightarrow x = \pm 2\ , x = 5$

13.$Solve\ the\ equation\ 7x^3 - 43x^2 - 43x + 7 = 0.$

$sum\ of\ odd\ term\ co.eff\ = 7 - 43 = -36$

$sum\ of\ even\ term\ co.eff\ = -43 + 7 = -36$

$\therefore (x + 1)\ is\ a\ factor.$

	7	−43	−43	7
−1	0	−7	50	−7
	7	−50	7	0

$Quotient\ = 7x^2 - 50x + 7$

$-\frac{49}{7x}$	$\mathbf{49}$ / $\mathbf{-50}$	$-\frac{1}{7x}$

$= (x - 7)(7x - 1)$

$\Longrightarrow x = 7, \frac{1}{7}$

$Thus\ the\ roots\ are\ -1,7\ and\ \frac{1}{7}.$

14. Solve: $8x^{\frac{3}{2n}} - 8x^{\frac{-3}{2n}} = 63$

$Let\ y = x^{\frac{3}{2n}}$

$8x^{\frac{3}{2n}} - 8x^{\frac{-3}{2n}} = 63$

$\Rightarrow 8x^{\frac{3}{2n}} - \frac{8}{x^{\frac{3}{2n}}} = 63$

$\Rightarrow 8y - \frac{8}{y} = 63$

$\Rightarrow \frac{8y^2-8}{y} = 63$

$\Rightarrow 8y^2 - 8 = 63y$

$\Rightarrow 8y^2 - 63y - 8 = 0$

$\Rightarrow 8y^2 - 64y + y - 8 = 0$

$\Rightarrow 8y(y-8) + 1(y-8) = 0$

$\Rightarrow (y-8)(8y+1) = 0$

$\Rightarrow \boxed{y = 8}\ (or)\ \ y = -\frac{1}{8}$ (Not a solution)

$y = x^{\frac{3}{2n}} \Rightarrow x^{\frac{3}{2n}} = 8$

$\Rightarrow x^{\frac{3}{2n}} = 2^3$

$\Rightarrow x^{\frac{1}{2n}} = 2$

$\Rightarrow x = 2^{2n}$

$\Rightarrow \boxed{x = 4^n}$

15. Find all real numbers satisfying $4^x - 3(2^{x+2}) + 2^5 = 04^x - 3(2^{x+2}) + 2^5 = 0$

$\Rightarrow (2^x)^2 - 3(2^x 2^2) + 32 = 0$

$\Rightarrow (2^x)^2 - 12(2^x) + 32 = 0$

$Let\ 2^x = y.$

$\Rightarrow (y)^2 - 12(y) + 32 = 0$

$\Rightarrow y^2 - 4y - 8y + 32 = 0$

$\Rightarrow y(y-4) - 8(y-4) = 0$

$\Rightarrow (y-4)(y-8) = 0$

$\Rightarrow y = 4{,}8$

$Sub. the\ value\ of\ y\ in 2^x = y,$

$y = 4 \Rightarrow 2^x = 4 \Rightarrow 2^x = 2^2 \Rightarrow \boxed{x = 2}$

$y = 8 \Rightarrow 2^x = 8 \Rightarrow 2^x = 2^3 \Rightarrow \boxed{x = 3}$

$Thus\ the\ roots\ are\ 2\ and\ 3.$

16. Find solution if any of the equation $2\cos^2 x - 9\cos x + 4 = 0$.

$Let\ \cos x = y \rightarrow (1)$

$2cos^2 x - 9\cos x + 4 = 0$

$\Rightarrow 2y^2 - 9y + 4 = 0$

$\Rightarrow 2y^2 - y - 8y + 4 = 0$

$\Rightarrow y(2y-1) - 4(2y-1) = 0$

$\Rightarrow (y-4)(2y-1) = 0$

$\Rightarrow y = 4, \frac{1}{2}$

$y = \frac{1}{2} \Rightarrow \cos x = \frac{1}{2}$ $\Rightarrow \cos x = \cos\frac{\pi}{3}$ $\Rightarrow x = 2n\pi \pm \frac{\pi}{3}, n \in z$	$y = 4$ $\Rightarrow \cos x = 4$ Not a solution

17. $Solve : sin^2 x - 5\sin x + 4 = 0$

$sin^2 x - 5\sin x + 4 = 0$

$Let\ \sin x = y \rightarrow (1)$

$sin^2 x - 5\sin x + 4 = 0$

$\Rightarrow y^2 - 5y + 4 = 0$

$\Rightarrow (y-1)(y-4) = 0$

$\Rightarrow y = 1{,}4$

$Sub. y = 1{,}4\ in\ (1),$

$\sin x = 1$ $\Rightarrow \sin x = \sin\frac{\pi}{2}$ $\Rightarrow x = 2n\pi + \frac{\pi}{2}, n \in z$	$\sin x = 4$ Not a solution

5 MARK QUESTIONS

1. If $2 + i$ and $3 - \sqrt{2}$ are roots of the equation $x^6 - 13x^5 + 62x^4 - 126x^3 + 65x^2 + 127x - 140 = 0.$ Find all roots.

$Since\ 2 + i\ and\ 3 - \sqrt{2}\ are\ the\ two\ roots\ then\ 2 - i\ and\ \sqrt{3} + \sqrt{2}\ are\ also\ the$
$other\ two\ roots. Let\ the\ remaining\ two\ rooys\ be\ \alpha\ and\ \beta.$

$x^6 - 13x^5 + 62x^4 - 126x^3 + 65x^2 + 127x - 140 = 0$

$\Rightarrow a = 1, b = -13, c = 62, d = -126, e = 65, f = 127, g = -140$

$\sum_1 = -\frac{b}{a} \Rightarrow 2 + i + 2 - i + 3 - \sqrt{2} + 3 + \sqrt{2} + \alpha + \beta = 13$

$\Rightarrow \alpha + \beta = 3$

$\sum_6 = \frac{g}{a} \Rightarrow (2+i)(2-i)(3-\sqrt{2})(3+\sqrt{2})\alpha\beta = -140$

$\Rightarrow (5)(7)\alpha\beta = -140$

$\Rightarrow \alpha\beta = -4$

Quadratic equation : $x^2 - (\alpha+\beta)x + \alpha\beta = 0$

$\Rightarrow x^2 - 3x - 4 = 0$

$\Rightarrow (x+1)(x-4) = 0$

$\Rightarrow x = -1{,}4$

Thus the roots are $2 + i, 2 - i, 3 - \sqrt{2}, 3 + \sqrt{2}, -1\ and\ 4.$

2. Find all zeros of the polynomial $x^6 - 3x^5 - 5x^4 + 22x^3 - 39x^2 - 39x + 135$, if it is known that $1 + 2i$ and $\sqrt{3}$ are two of its zeros.

Since $1+2i$ and $\sqrt{3}$ *are the two roots then* $1-2i$ and $-\sqrt{3}$ *are also the other two roots. Let the remaining two rooys be* α *and* β.

$x^6-3x^5-5x^4+22x^3-39x^2-39x+135$

$\Rightarrow a=1, b=-3, c=-5, d=22, e=-39, f=-39, g=135$

$\sum_1=-\frac{b}{a} \Rightarrow 1+2i+1-2i+\sqrt{3}-\sqrt{3}+\alpha+\beta=3$

$\Rightarrow \alpha+\beta=1$

$\sum_6=\frac{g}{a} \Rightarrow (1+2i)(1-2i)(\sqrt{3})(-\sqrt{3})\alpha\beta=135$

$\Rightarrow (5)(3)\alpha\beta=135$

$\Rightarrow \alpha\beta=9$

Quadratic equation: $x^2-(\alpha+\beta)x+\alpha\beta=0$

$\Rightarrow x^2-x-9=0$

$\Rightarrow x=\frac{-b\pm\sqrt{b^2-4ac}}{2a}$

$=\frac{-(-1)\pm\sqrt{(-1)^2-4(1)(-9)}}{2(1)}$

$=\frac{1\pm\sqrt{1+36}}{2}$

$=\frac{1\pm\sqrt{37}}{2}$

$=\frac{1+\sqrt{37}}{2}, \frac{1-\sqrt{37}}{2}$

Thus the roots are $1+2i, 1-2i, \sqrt{3}, -\sqrt{3}, \frac{1+\sqrt{37}}{2}$ *and* $\frac{1-\sqrt{37}}{2}$.

3.Solve the equation $6x^4-5x^3-38x^2-5x+6=0$ if it is known that $\frac{1}{3}$ is a solution.

Since it is type I reciprocal equation, $\frac{1}{3}$ *is one of the root then its reciprocal* 3 *is another root.*

Let α *be another root then its reciprocal* $\frac{1}{\alpha}$ *is also root.*

$\sum 1=\frac{5}{6} \Rightarrow 3+\frac{1}{3}+\alpha+\frac{1}{\alpha}=\frac{5}{6} \Rightarrow \frac{10}{3}+\frac{\alpha^2+1}{\alpha}=\frac{5}{6}$

$\Rightarrow \frac{\alpha^2+1}{\alpha}=\frac{5}{6}-\frac{10}{3}$

$\Rightarrow \frac{\alpha^2+1}{\alpha}=\frac{5-20}{6}$

$\Rightarrow \frac{\alpha^2+1}{\alpha}=-\frac{5}{2}$

$\Rightarrow 2\alpha^2+2=-5\alpha$

$\Rightarrow 2\alpha^2+5\alpha+2=0$

4

$\frac{4}{2\alpha}$ | $\frac{1}{2\alpha}$

5

$\Rightarrow (\alpha+2)(2\alpha+1)=0$

$\Rightarrow \alpha=-2, -\frac{1}{2}$

Thus the roots are $-2, -\frac{1}{2}, 3$ *and* $\frac{1}{3}$.

4. *Solve* : $(x-5)(x-7)(x+6)(x+4)=504$

$(x-5)(x-7)(x+6)(x+4)=504$

$\Rightarrow [(x-5)(x+4)][(x-7)(x+6)]=504$

$\Rightarrow [x^2-x-20][x^2-x-42]=504$

Let $x^2-x=y \rightarrow (1)$.

$\Rightarrow [y-20][y-42]=504$

$\Rightarrow y^2-62y+840-504=0$

$\Rightarrow y^2-62y+336=0$

$\Rightarrow y^2-6y-56y+336=0$

$\Rightarrow y(y-6)-56(y-6)=0$

$\Rightarrow (y-6)(y-56)=0$

$\Rightarrow y=6,56$

sub. $y=6,56$ *in* (1),

$\Rightarrow x^2-x=6$	$\Rightarrow x^2-x=56$
$\Rightarrow x^2-x-6=0$	$\Rightarrow x^2-x-56=0$
$\Rightarrow (x+2)(x-3)=0$	$\Rightarrow (x+7)(x-8)=0$
$\Rightarrow x=-2,3$	$\Rightarrow x=-7,8$

Thus the roots are $-7, -2, 3$ *and* 8.

5. *Solve* : $(2x-1)(x+3)(x-2)(2x+3)+20=0$

$\Rightarrow [(2x-1)(2x+3)][(x+3)(x-2)]+20=0$

$\Rightarrow [4x^2+4x-3][x^2+x-6]+20=0$

$\Rightarrow [4(x^2+x)-3][x^2+x-6]+20=0$

Let $x^2+x=y \rightarrow (1)$.

$\Rightarrow [4y-3][y-6]+20=0$

$\Rightarrow 4y^2-24y-3y+18+20=0$

$\Rightarrow 4y^2-27y+38=0$

$\Rightarrow 4y^2-8y-19y+38=0$

$\Rightarrow 4y(y-2)-19(y-2)=0$

$\Rightarrow (y-2)(4y-19)=0$

$\Rightarrow y=2, \frac{19}{4}$

sub. $y=2, \frac{19}{4}$ *in* (1),

$\Rightarrow x^2+x=2$ $\Rightarrow x^2+x-2=0$ $\Rightarrow (x+2)(x-1)=0$ $\Rightarrow x=-2,1$	$\Rightarrow x^2+x=\frac{19}{4}$ $\Rightarrow 4x^2+4x-19=0$ $\Rightarrow x=\frac{-b\pm\sqrt{b^2-4ac}}{2a}$ $\Rightarrow x=\frac{-(4)\pm\sqrt{(4)^2-4(4)(-19)}}{2(4)}$ $\Rightarrow x=\frac{-4\pm\sqrt{320}}{8}$ $\Rightarrow x=\frac{-4\pm4\sqrt{20}}{8}$ $\Rightarrow x=\frac{-1\pm\sqrt{4\times5}}{2}$ $\Rightarrow x=\frac{-1\pm2\sqrt{5}}{2}$

Thus the roots are $-2,1,\frac{-1+2\sqrt{5}}{2}$ *and* $\frac{-1-2\sqrt{5}}{2}$.

6. *Solve*: $6x^4-35x^3+62x^2-35x+6=0$

$6x^4-35x^3+62x^2-35x+6=0$

It is type I reciprocal equation, divide it by x^2,

$6x^2-35x+62-\frac{35}{x}+\frac{6}{x^2}=0 \Rightarrow 6\left(x^2+\frac{1}{x^2}\right)-35\left(x+\frac{1}{x}\right)+62=0 \to (1)$

$\boxed{y=x+\frac{1}{x}} \Rightarrow y^2=\left(x+\frac{1}{x}\right)^2 \Rightarrow y^2=x^2+\frac{1}{x^2}+2 \Rightarrow x^2+\frac{1}{x^2}=y^2-2$

$(1) \Rightarrow 6(y^2-2)-35y+62=0$

$\Rightarrow 6y^2-12-35y+62=0$

$\Rightarrow 6y^2-35y+50=0$

300 ; $\frac{-15}{6y}$ | $\frac{-20}{6y}$; **−35**

$\Rightarrow (2y-5)(3y-10)=0$

$\Rightarrow \boxed{y=\frac{5}{2}\ ;y=\frac{10}{3}}$

Sub. the values of y *in* $y=x+\frac{1}{x}$,

$y=\frac{5}{2}\Rightarrow x+\frac{1}{x}=\frac{5}{2}$ $\Rightarrow \frac{x^2+1}{x}=\frac{5}{2}$ $\Rightarrow 2x^2+2=5x$ $\Rightarrow 2x^2-5x+2=0$ (**4** ; $\frac{-4}{2x}$ | $\frac{-1}{2x}$; **−5**) $\Rightarrow (x-2)(2x-1)=0$	$y=\frac{10}{3}\Rightarrow x+\frac{1}{x}=\frac{10}{3}$ $\Rightarrow \frac{x^2+1}{x}=\frac{10}{3}$ $\Rightarrow 3x^2+3=10x$ $\Rightarrow 3x^2-10x+3=0$ (**9** ; $\frac{-9}{3x}$ | $\frac{-1}{3x}$; **−10**) $\Rightarrow (x-3)(3x-1)=0$
$\Rightarrow x=2,\frac{1}{2}$	$\Rightarrow x=3,\frac{1}{3}$

Thus the roots are $2,\frac{1}{2},3$ *and* $\frac{1}{3}$.

7. *Solve* : $x^4+3x^3-3x-1=0$

$x^4+3x^3-3x-1=0$

It is type II reciprocal equation. So 1 *and* −1 *are the two roots* .

Let the other two roots be α *and* $\frac{1}{\alpha}$.

$\sum 1=-1+1+\alpha+\frac{1}{\alpha}=-3 \Rightarrow \frac{\alpha^2+1}{\alpha}=-3$

$\Rightarrow \alpha^2+1=-3\alpha$

$\Rightarrow \alpha^2+3\alpha+1=0$

$\Rightarrow \alpha=\frac{-b\pm\sqrt{b^2-4ac}}{2a}$

$=\frac{-(3)\pm\sqrt{(3)^2-4(1)(1)}}{2(1)}$

$=\frac{-3\pm\sqrt{9-4}}{2}$

$=\frac{-3\pm\sqrt{5}}{2}$

Thus the roots are $-1,1,\frac{-3+\sqrt{5}}{2}$ *and* $\frac{-3-\sqrt{5}}{2}$

8. *Solve the eqaution* $x^4-10x^3+26x^2-10x+1=0$.

It is type I reciprocal equation, divide it by x^2,

$\Rightarrow \left(x^2+\frac{1}{x^2}\right)-10\left(x+\frac{1}{x}\right)+26=0 \to (1)$

$\boxed{y=x+\frac{1}{x}} \Rightarrow y^2=\left(x+\frac{1}{x}\right)^2$

$\Rightarrow y^2=x^2+\frac{1}{x^2}+2$

$\Rightarrow x^2+\frac{1}{x^2}=y^2-2$

$(1) \Rightarrow y^2-2-10y+26=0$

$\Rightarrow y^2-10y+24=0$

24 ; $\frac{-6}{y}$ | $\frac{-4}{y}$; **−10**

$\Rightarrow (y-6)(y-4)=0$

$\Rightarrow y=6,4$

Sub. the values of y *in* $y=x+\frac{1}{x}$,

$x+\frac{1}{x}=6$	$x+\frac{1}{x}=4$
$\Rightarrow x^2-6x+1=0$	$\Rightarrow x^2-4x+1=0$
$\Rightarrow x=\frac{-b\pm\sqrt{b^2-4ac}}{2a}$	$\Rightarrow x=\frac{-b\pm\sqrt{b^2-4ac}}{2a}$
$\Rightarrow x=\frac{-(-6)\pm\sqrt{(6)^2-4(1)(1)}}{2(1)}$	$\Rightarrow x=\frac{-(-4)\pm\sqrt{(4)^2-4(1)(1)}}{2(1)}$
$\Rightarrow x=\frac{6\pm\sqrt{36-4}}{2}$	$\Rightarrow x=\frac{4\pm\sqrt{12}}{2}$
$\Rightarrow x=\frac{6\pm\sqrt{32}}{2}$	$\Rightarrow x=\frac{4\pm 2\sqrt{3}}{2}$
$\Rightarrow x=\frac{6\pm 4\sqrt{2}}{2}$	$\Rightarrow x=2\pm\sqrt{3}$
$\Rightarrow x=3\pm 2\sqrt{2}$	

Thus the roots are $3\pm 2\sqrt{2}$ *and* $2\pm\sqrt{3}$

4.INVERSE TRIGONOMETRIC FUNCTIONS

2 MARK QUESTIONS

1. Find the domain of the following $f(x)=sin^{-1}\left(\frac{x^2+1}{2x}\right)$

Domain of sin^{-1} *is* $[-1,1]$

$So, -1\le\frac{x^2+1}{2x}\le 1$

$\Rightarrow -2x\le x^2+1\le 2x$

$\Rightarrow -2x\le x^2+1$	$\Rightarrow x^2+1\le 2x$
$\Rightarrow 0\le x^2+1+2x$	$\Rightarrow x^2+1-2x\le 0$
$\Rightarrow 0\le (x+1)^2$	$\Rightarrow (x-1)^2\le 0$
$\Rightarrow (x+1)^2\ge 0$	$\Rightarrow x-1\le 0$
$\Rightarrow x+1\ge 0$	$\Rightarrow x\le 1\rightarrow(2)$
$\Rightarrow x\ge -1\rightarrow(1)$	

From $(1),(2)\Rightarrow\ -1\le x\le 1$

Domain of $f(x)=sin^{-1}\left(\frac{x^2+1}{2x}\right)$ *is* $[-1,1]$.

2. Find the value of $sin^{-1}\left(\sin\frac{5\pi}{9}\cos\frac{\pi}{9}+\cos\frac{5\pi}{9}\sin\frac{\pi}{9}\right)$

$sin^{-1}\left(\sin\frac{5\pi}{9}\cos\frac{\pi}{9}+\cos\frac{5\pi}{9}\sin\frac{\pi}{9}\right)$

$=sin^{-1}\left(\sin\left(\frac{5\pi}{9}+\frac{\pi}{9}\right)\right)$ $\qquad(\because \sin(A+B)=sinAcosB+cosAsinB)$

$=sin^{-1}\left(\sin\left(\frac{6\pi}{9}\right)\right)$

$=sin^{-1}\left(\sin\left(\frac{2\pi}{3}\right)\right)$

$=sin^{-1}\left(\sin\left(\pi-\frac{\pi}{3}\right)\right)$

$=sin^{-1}\left(\sin\left(\frac{\pi}{3}\right)\right)$

$=\frac{\pi}{3}\in\left[-\frac{\pi}{2},\frac{\pi}{2}\right]$

3.Find the domain of $sin^{-1}(2-3x^2)$.

Domain of sin^{-1} *is* $[-1,1]$

$So, -1\le 2-3x^2\le 1.$

$-1\le 2-3x^2$	$2-3x^2\le 1$
$\Rightarrow -3\le -3x^2$	$\Rightarrow -3x^2\le -1$
$\Rightarrow 3x^2\le 3$	$\Rightarrow 3x^2\ge 1$
$\Rightarrow x^2\le 1\rightarrow(1)$	$\Rightarrow x^2\ge\frac{1}{3}\rightarrow(2)$

From $(1),(2)\Rightarrow\ \frac{1}{3}\le x^2\le 1\Rightarrow\frac{1}{\sqrt{3}}\le|x|\le 1\Rightarrow x\in\left[-1,-\frac{1}{\sqrt{3}}\right]\cup\left[\frac{1}{\sqrt{3}},1\right].$

$(\because$ *If* $a\le|x|\le b$ *then* $x\in[-b,-a]\cup[a,b])$

4. Find the value of $cos^{-1}\left(\cos\frac{\pi}{7}\cos\frac{\pi}{17}-\sin\frac{\pi}{7}\sin\frac{\pi}{17}\right)$

$cos^{-1}\left(\cos\frac{\pi}{7}\cos\frac{\pi}{17}-\sin\frac{\pi}{7}\sin\frac{\pi}{17}\right)$

$=cos^{-1}\left(\cos\left(\frac{\pi}{7}+\frac{\pi}{17}\right)\right)$

$=cos^{-1}\left(\cos\left(\frac{17\pi+7\pi}{119}\right)\right)$

$=cos^{-1}\left(\cos\left(\frac{24\pi}{119}\right)\right)$

$=\frac{24\pi}{119}$

5. Find the domain of $f(x)=sin^{-1}\left(\frac{|x|-2}{3}\right)+cos^{-1}\left(\frac{1-|x|}{4}\right)$.

Domain of sin^{-1} *is* $[-1,1]$ *and domain of* $cos^{-1}x$ *is* $[-1,1]$

$So, -1\le\frac{|x|-2}{3}\le 1\ ;\ -1\le\frac{1-|x|}{4}\le 1$

$-1\le\frac{|x|-2}{3}\le 1$	$-1\le\frac{1-|x|}{4}\le 1$
$\Rightarrow\ -3\le|x|-2\le 3$	$\Rightarrow -4\le 1-|x|\le 4$
$\Rightarrow -1\le|x|\le\ 5$	$\Rightarrow -5\le -|x|\le 3$

$\Longrightarrow \lvert x\rvert \le 5$	$\Longrightarrow -3 \le \lvert x\rvert \le 5$ $\Longrightarrow \lvert x\rvert \le 5$

$\therefore x \in [-5,5]$

6. Find the value of $cos^{-1}\left(\cos\left(\frac{4\pi}{3}\right)+cos^{-1}\left(\cos\left(\frac{5\pi}{4}\right)\right)\right)$

$cos^{-1}\left(\cos\left(\frac{4\pi}{3}\right)+cos^{-1}\left(\cos\left(\frac{5\pi}{4}\right)\right)\right)$

$= cos^{-1}\left(\cos\left(\pi+\frac{\pi}{3}\right)+cos^{-1}\left(\cos\left(\pi+\frac{\pi}{4}\right)\right)\right)$

$= cos^{-1}\left(-\cos\left(\frac{\pi}{3}\right)+cos^{-1}\left(-\cos\left(\frac{\pi}{4}\right)\right)\right)$

$= cos^{-1}\left(\cos\left(\pi-\frac{\pi}{3}\right)+cos^{-1}\left(\cos\left(\pi-\frac{\pi}{4}\right)\right)\right)$

$= cos^{-1}\left(\cos\left(\frac{2\pi}{3}\right)+cos^{-1}\left(\cos\left(\frac{3\pi}{4}\right)\right)\right)$

$= \frac{2\pi}{3}+\frac{3\pi}{4}$

$= \frac{17\pi}{12}$

7.Find the domain of $cos^{-1}\left(\frac{2+\sin x}{3}\right)$

$Domain\ of\ cos^{-1}x\ is\ [-1,1].$

$So, -1 \le \frac{2+\sin x}{3} \le 1$

$\Longrightarrow -3 \le 2+\sin x \le 3$

$\Longrightarrow -5 \le sinx \le 1$

$\Longrightarrow -1 \le sinx \le 1$

$\Longrightarrow -sin^{-1}(1) \le x \le sin^{-1}(1)$

$\Longrightarrow -\frac{\pi}{2} \le x \le \frac{\pi}{2}$

$\therefore\ Domain\ of\ cos^{-1}\left(\frac{2+\sin x}{3}\right)\ is\ \left[-\frac{\pi}{2},\frac{\pi}{2}\right]$

8. Prove that $\tan\left(sin^{-1}(x)\right)=\frac{x}{\sqrt{1-x^2}}, -1<x<1.$

$When\ x=0, then\ both\ sides\ become\ 0\ \rightarrow (1)$

$When\ 0<x<1.\ Let\ \theta = sin^{-1}x.\ So, 0<\theta<\frac{\pi}{2}$

$\theta = sin^{-1}x \Longrightarrow \sin\theta = \frac{x}{1} \Longrightarrow\ Oppsite\ side = x, Hypotenuse\ = 1$

$Adjacent\ side\ = \sqrt{1-x^2}.$

$\tan\theta = \frac{x}{\sqrt{1-x^2}} \Longrightarrow \tan\left(sin^{-1}(x)\right) = \frac{x}{\sqrt{1-x^2}} \rightarrow (2)$

$When\ -1<x<0.\ Let\ \theta = sin^{-1}x.\ so, -\frac{\pi}{2}<\theta<0$

$\theta = sin^{-1}x \Longrightarrow \sin\theta = \frac{x}{1} \Longrightarrow\ \tan\theta = \frac{x}{\sqrt{1-x^2}} \Longrightarrow \tan\left(sin^{-1}(x)\right) = \frac{x}{\sqrt{1-x^2}} \rightarrow (3)$

$From\ (1),(2),(3) \Longrightarrow\ \tan\ \left(sin^{-1}(x)\right) = \frac{x}{\sqrt{1-x^2}}, -1<x<1$

9. Find the value of $\sin\left(tan^{-1}\left(\frac{1}{2}\right)-cos^{-1}\left(\frac{4}{5}\right)\right)$

$sin(x-y) = sinxcosy - cosxsiny.$

$Let\ tan^{-1}\left(\frac{1}{2}\right) = x\ ; cos^{-1}\left(\frac{4}{5}\right) = y$

$tan^{-1}\left(\frac{1}{2}\right) = x \Longrightarrow \tan x = \frac{1}{2}$

$sec^2x = 1+tan^2x = 1+\frac{1}{4} = \frac{5}{4} \Longrightarrow secx = \frac{\sqrt{5}}{2} \Longrightarrow cosx = \frac{2}{\sqrt{5}}$

$sinx = \sqrt{1-cos^2x} = \sqrt{1-\frac{4}{5}} = \sqrt{\frac{1}{5}} = \frac{1}{\sqrt{5}}$

$cos^{-1}\left(\frac{4}{5}\right) = y\ \Longrightarrow cosy = \frac{4}{5}$

$siny = \sqrt{1-cos^2y} = \sqrt{1-\frac{16}{25}} = \sqrt{\frac{9}{25}} = \frac{3}{5}$

$\sin\left(tan^{-1}\left(\frac{1}{2}\right)-cos^{-1}\left(\frac{4}{5}\right)\right) = sin(x-y)\ = sinxcosy - cosxsiny$

$= \frac{1}{\sqrt{5}}.\frac{4}{5}-\frac{2}{\sqrt{5}}.\frac{3}{5}$

$= \frac{4-6}{5\sqrt{5}}$

$= -\frac{2}{5\sqrt{5}}$

$= -\frac{2\sqrt{5}}{25}$

10. Find the value of:$\cos\left(sin^{-1}\left(\frac{4}{5}\right)-tan^{-1}\left(\frac{3}{4}\right)\right)$

$cos(x-y) = cosxcosy + sinxsiny.$

$Let\ sin^{-1}\left(\frac{4}{5}\right) = x\ ; tan^{-1}\left(\frac{3}{4}\right) = y$

$sin^{-1}\left(\frac{4}{5}\right) = x \Longrightarrow sinx = \frac{4}{5}$

$cosx = \sqrt{1-sin^2x} = \sqrt{1-\frac{16}{25}} = \sqrt{\frac{9}{25}} = \frac{3}{5}$

$tan^{-1}\left(\frac{3}{4}\right) = y \Longrightarrow tany = \frac{3}{4}$

$sec^2y = 1+tan^2y = 1+\frac{9}{16} = \frac{25}{16} \Longrightarrow secy = \frac{5}{4} \Longrightarrow cosy = \frac{4}{5}$

$siny = \sqrt{1 - cos^2y} = \sqrt{1 - \frac{16}{25}} = \sqrt{\frac{9}{25}} = \frac{3}{5}$

$\cos\left(sin^{-1}\left(\frac{4}{5}\right) - tan^{-1}\left(\frac{3}{4}\right)\right) = cos(x - y)$

$= cosxcosy + sinxsiny$

$= \frac{3}{5}.\frac{4}{5} + \frac{4}{5}.\frac{3}{5}$

$= \frac{24}{25}$

11. Find the value of $tan^{-1}(-1) + cos^{-1}\left(\frac{1}{2}\right) + sin^{-1}\left(-\frac{1}{2}\right)$

$tan^{-1}(-1) = tan^{-1}\left(-tan\frac{\pi}{4}\right) = tan^{-1}\left(tan\left(-\frac{\pi}{4}\right)\right) = -\frac{\pi}{4} \in \left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$

$cos^{-1}\left(\frac{1}{2}\right) = cos^{-1}\left(cos\left(\frac{\pi}{3}\right)\right) = \frac{\pi}{3} \in [0, \pi]$

$sin^{-1}\left(-\frac{1}{2}\right) = sin^{-1}\left(-sin\frac{\pi}{6}\right) = sin^{-1}\left(sin\left(-\frac{\pi}{6}\right)\right) = -\frac{\pi}{6} \in \left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$

$tan^{-1}(-1) + cos^{-1}\left(\frac{1}{2}\right) + sin^{-1}\left(-\frac{1}{2}\right) = -\frac{\pi}{4} + \frac{\pi}{3} - \frac{\pi}{6} = -\frac{\pi}{12}$

12. Find the value of $sin^{-1}(-1) + cos^{-1}\left(\frac{1}{2}\right) + cot^{-1}(2)$

$sin^{-1}(-1) + cos^{-1}\left(\frac{1}{2}\right) + cot^{-1}(2)$

$= sin^{-1}\left(sin\left(-\frac{\pi}{2}\right)\right) + cos^{-1}\left(\cos\left(\frac{\pi}{3}\right)\right) + cot^{-1}(2)$

$= -\frac{\pi}{2} + \frac{\pi}{3} + cot^{-1}(2)$

$= -\frac{\pi}{6} + cot^{-1}(2)$

13. Find the value of $cot^{-1}(1) + sin^{-1}\left(-\frac{\sqrt{3}}{2}\right) - sec^{-1}(-\sqrt{2})$

$cot^{-1}(1) + sin^{-1}\left(-\frac{\sqrt{3}}{2}\right) - sec^{-1}(-\sqrt{2})$

$= \tan(1) + sin^{-1}\left(-\sin\left(\frac{\sqrt{3}}{2}\right)\right) - \cos\left(-\frac{1}{\sqrt{2}}\right)$

$= \tan\left(\frac{\pi}{4}\right) + sin^{-1}\left(\sin\left(-\frac{\pi}{3}\right)\right) - \cos\left(\pi - \frac{\pi}{4}\right)$

$= \frac{\pi}{4} - \frac{\pi}{3} - \frac{3\pi}{4}$

$= -\frac{2\pi}{4} - \frac{\pi}{3}$

$= -\frac{\pi}{2} - \frac{\pi}{3}$

$= -\frac{5\pi}{6}$

14. Show that $\cot^{-1}\left(\frac{1}{\sqrt{x^2-1}}\right) = \sec^{-1}x, |x| > 1$

Let $cot^{-1}\left(\frac{1}{\sqrt{x^2-1}}\right) = y \to (1)$.

$\Rightarrow coty = \frac{1}{\sqrt{x^2-1}}$

$\Rightarrow tany = \frac{\sqrt{x^2-1}}{1}$; $opposite\ side = \sqrt{x^2-1}$; $Adjacent\ side = 1$

$Hypotonuse = \sqrt{\left(\sqrt{x^2-1}\right)^2 + (1)^2} = \sqrt{x^2-1+1} = \sqrt{x^2} = x$

$secy = \frac{x}{1} \Rightarrow sec^{-1}x = y \to (2)$

$(1), (2) \Rightarrow cot^{-1}\left(\frac{1}{\sqrt{x^2-1}}\right) = sec^{-1}x$

15. *Find the value of* $\cot\left(sin^{-1}\frac{3}{5} + sin^{-1}\frac{4}{5}\right)$

$sin^{-1}x + sin^{-1}y = sin^{-1}\left[x\sqrt{1-y^2} + y\sqrt{1-x^2}\right]$

$sin^{-1}\frac{3}{5} + sin^{-1}\frac{4}{5} = sin^{-1}\left[\frac{3}{5}\sqrt{1-\left(\frac{4}{5}\right)^2} + \frac{4}{5}\sqrt{1-\left(\frac{3}{5}\right)^2}\right]$

$= sin^{-1}\left[\frac{3}{5}\sqrt{\frac{9}{25}} + \frac{4}{5}\sqrt{\frac{16}{25}}\right]$

$= sin^{-1}\left[\frac{3}{5}.\frac{3}{5} + \frac{4}{5}.\frac{4}{5}\right]$

$= sin^{-1}\left[\frac{9}{25} + \frac{16}{25}\right]$

$= sin^{-1}\left[\frac{25}{25}\right]$

$= sin^{-1}[1]$

$= sin^{-1}\left(sin\frac{\pi}{2}\right)$

$= \frac{\pi}{2}$

$\therefore \cot\left(sin^{-1}\frac{3}{5} + sin^{-1}\frac{4}{5}\right) = \cot\left(\frac{\pi}{2}\right) = 0$

16. *Find the value of* $\tan\left(sin^{-1}\frac{3}{5} + cot^{-1}\frac{3}{2}\right)$

$sin^{-1}\frac{3}{5} = x \Rightarrow \sin x = \frac{3}{5}$

$\Rightarrow cosx = \sqrt{1 - sin^2x}$

$= \sqrt{1 - \frac{9}{25}}$

$= \sqrt{\frac{16}{25}}$

$= \frac{4}{5}$

$\tan x = \frac{\sin x}{\cos x} = \frac{3/5}{4/5} = \frac{3}{4}$

$cot^{-1}\frac{3}{2} = y \Rightarrow \cot y = \frac{3}{2}$

$\Rightarrow \tan y = \frac{2}{3}$

$\tan(x+y) = \frac{\tan x + \tan y}{1 - \tan x \tan y}$

$= \frac{\frac{3}{4}+\frac{2}{3}}{1-\left(\frac{3}{4}\right)\left(\frac{2}{3}\right)}$

$= \frac{17}{6}$

17. *Prove that* $sin^{-1}\frac{3}{5} - cos^{-1}\frac{12}{13} = sin^{-1}\frac{16}{65}$

$sin^{-1}x - sin^{-1}y = sin^{-1}\left(x\sqrt{1-y^2} - y\sqrt{1-x^2}\right)$

$y = cos^{-1}\frac{12}{13} \Rightarrow cosy = \frac{12}{13}$

$siny = \sqrt{1-cos^2y} = \sqrt{1-\left(\frac{12}{13}\right)^2} = \sqrt{\frac{169-144}{169}} = \sqrt{\frac{25}{169}} = \frac{5}{13} \Rightarrow y = sin^{-1}\left(\frac{5}{13}\right)$

$\therefore cos^{-1}\frac{12}{13} = sin^{-1}\left(\frac{5}{13}\right)$

$sin^{-1}\frac{3}{5} - cos^{-1}\frac{12}{13} = sin^{-1}\frac{3}{5} - sin^{-1}\left(\frac{5}{13}\right)$

$= sin^{-1}\left(\frac{3}{5}\sqrt{1-\left(\frac{5}{13}\right)^2} - \frac{5}{13}\sqrt{1-\left(\frac{3}{5}\right)^2}\right)$

$= sin^{-1}\left(\frac{3}{5}\sqrt{\frac{169-25}{169}} - \frac{5}{13}\sqrt{\frac{25-9}{25}}\right)$

$= sin^{-1}\left(\frac{3}{5}\sqrt{\frac{144}{169}} - \frac{5}{13}\sqrt{\frac{16}{25}}\right)$

$= sin^{-1}\left(\frac{3}{5}.\frac{12}{13} - \frac{5}{13}.\frac{4}{5}\right)$

$= sin^{-1}\left(\frac{36-20}{65}\right)$

$= sin^{-1}\frac{16}{65}$

18. If $\tan^{-1}x + \tan^{-1}y + \tan^{-1}z = \pi$, show that $x+y+z = xyz$.

$tan^{-1}x + tan^{-1}y = tan^{-1}\left(\frac{x+y}{1-xy}\right)$

$tan^{-1}x + tan^{-1}y + tan^{-1}z = tan^{-1}\left(\frac{x+y}{1-xy}\right) + tan^{-1}z$

$= tan^{-1}\left(\frac{\frac{x+y}{1-xy}+z}{1-\left(\frac{x+y}{1-xy}\right)z}\right)$

$= tan^{-1}\left(\frac{x+y+z-xy/_{1-xy}}{1-xy-xz-yz/_{1-xy}}\right)$

$= tan^{-1}\left[\frac{x+y+z-xyz}{1-xy-yz-zx}\right]$

$tan^{-1}x + tan^{-1}y + tan^{-1}z = \pi \Rightarrow tan^{-1}\left[\frac{x+y+z-xyz}{1-xy-yz-zx}\right] = \pi$

$\Rightarrow \frac{x+y+z-xyz}{1-xy-yz-zx} = tan\pi = 0$

$\Rightarrow x+y+z-xyz = 0$

19. *Solve* : $cot^{-1}x - cot^{-1}(x+2) = \frac{\pi}{12}, x > 0$

$cot^{-1}x - cot^{-1}(x+2) = \frac{\pi}{12}$

$\Rightarrow tan^{-1}\left(\frac{1}{x}\right) - tan^{-1}\left(\frac{1}{x+2}\right) = \frac{\pi}{12}$

$\Rightarrow tan^{-1}\left(\frac{\frac{1}{x}-\frac{1}{x+2}}{1+\left(\frac{1}{x}\right)\left(\frac{1}{x+2}\right)}\right) = \frac{\pi}{12}$

$\Rightarrow tan^{-1}\left(\frac{x+2-x}{x^2+2x+1}\right) = \frac{\pi}{12}$

$\Rightarrow \frac{2}{(x+1)^2} = \tan\frac{\pi}{12}$

$\Rightarrow \frac{2}{(x+1)^2} = tan15 \rightarrow (1)$

$tan15 = \tan(45-30) = \frac{\tan 45 - \tan 30}{1+\tan 45 \tan 30}$

$= \frac{1-\frac{1}{\sqrt{3}}}{1+\frac{1}{\sqrt{3}}} = \frac{\sqrt{3}-1}{\sqrt{3}+1}$

$= \frac{\sqrt{3}-1}{\sqrt{3}+1} \times \frac{\sqrt{3}-1}{\sqrt{3}-1}$

$= \frac{3-2\sqrt{3}+1}{3-1}$

$= \frac{4-2\sqrt{3}}{2}$

$= 2-\sqrt{3}$

$(1) \Rightarrow \frac{2}{(x+1)^2} = 2-\sqrt{3} \Rightarrow (x+1)^2 = \frac{2}{2-\sqrt{3}}$

$\Rightarrow (x+1)^2 = \frac{2}{2-\sqrt{3}} \times \frac{2+\sqrt{3}}{2+\sqrt{3}}$

$\Rightarrow (x+1)^2 = \frac{4+2\sqrt{3}}{4-3}$

$\Rightarrow (x+1)^2 = 4+2\sqrt{3}$

$\Rightarrow (x+1)^2 = 3+2\sqrt{3}+1$

$\Rightarrow (x+1)^2 = \sqrt{3}^2 + 2\sqrt{3} + 1^2$

$\Rightarrow (x+1)^2 = \left(\sqrt{3}+1\right)^2$

$\Rightarrow x+1 = \sqrt{3}+1$

$\Rightarrow x = \sqrt{3}$

5.TWO DIMENSIONAL ANALYTICAL GEOMETRY -11

2 MARK QUESTIONS

1. Find the equation of the circle with centre $(2,-1)$ and passing through the point $(3,6)$ in standard form.

$point = P\ (3,6)\quad centre\ C = (2,-1)$

$Radius\ CP = r = \sqrt{(3-2)^2 + (6+1)^2} = \sqrt{1+49} = \sqrt{50}$

$Equation\ of\ the\ circle\ is\ (x-h)^2 + (y-k)^2 = r^2$

$$\Longrightarrow (x-2)^2 + (y+1)^2 = \sqrt{50}^2$$
$$\Longrightarrow x^2 - 4x + 4 + y^2 + 2y + 1 = 50$$
$$\Longrightarrow x^2 + y^2 - 4x + 2y - 45 = 0$$

2.Obtain the equation of the circle for which $(3,4)$and $(2,-7)$ are the ends of a diameter.

$Equation\ of\ the\ circle\ passing\ through\ the\ end\ points(x_1,y_1)\ \&\ (x_2,y_2)\ of\ the\ diameter\ is$

$(x-x_1)(x-x_2) + (y-y_1)(y-y_2) = 0$

$\Longrightarrow (x-3)(x-2) + (y-4)(y+7) = 0$

$\Longrightarrow x^2 - 5x + 6 + y^2 + 3y - 28 = 0$

$\Longrightarrow x^2 + y^2 - 5x + 3y - 22 = 0$

3.*Find the centre and radius of the circle* $x^2 + y^2 + 6x - 4y + 4 = 0$.

$x^2 + y^2 + 6x - 4y + 4 = 0$

$2g = 6\ ; 2f = -4\ ; c = 4 \Longrightarrow g = 3\ ; f = -2, c = 4$

$centre = (-g,-f) = (-3,2)$

$radius = \sqrt{\sqrt{g^2 + f^2 - c}} = \sqrt{9+4-4} = \sqrt{9} = 3\ units.$

4.*Find the centre and radius of the circle* $2x^2 + 2y^2 - 6x + 4y + 2 = 0$

$\div 2 \Rightarrow x^2 + y^2 - 3x + 2y + 1 = 0$

$2g = -3\ ; 2f = 2\ ; c = 1 \Longrightarrow g = -\frac{3}{2}\ ; f = 1, c = 1$

$centre = (-g,-f) = \left(\frac{3}{2}, -1\right)$

$radius = \sqrt{\sqrt{g^2 + f^2 - c}} = \sqrt{\frac{9}{4} + 1 - 1} = \frac{3}{2}\ units$

5. Find the equation of the parabola with focus $(-\sqrt{2}, 0)$, and directrix $x = \sqrt{2}$

From the given data the parabola is open leftwards, $V(h,k) = (0,0)\ ; VF = a = \sqrt{2}$

$Equation\ of\ the\ Parabola\ is\ (y-k)^2 = -4a(x-h)$

$$\Rightarrow (y-0)^2 = -4\sqrt{2}(x-0)$$
$$\Rightarrow y^2 = -4\sqrt{2}x$$

6. Find the equation of the parabola whose vertex is $(5,-2)$ and focus $(2,-2)$.

From the given data the parabola is open leftwards, $V(h,k) = (5,-2),\ \ focus\ F = (2,-2)$

$VF = a = |x_2 - x_1| = |2-5| = 3$

$Equation\ of\ the\ Parabola\ is\ (y-k)^2 = -4a(x-h)$

$$\Rightarrow (y+2)^2 = -4(3)(x-5)$$
$$\Rightarrow (y+2)^2 = -12(x-5)$$

7. The orbit of Halley's Comet is an ellipse 36.18 astronomical units long and by 9.12 astronomical units wide. Find its eccentricity.

$2a = 36.18 \Longrightarrow a = 18.09$

$2b = 9.12 \Longrightarrow b = 4.56$

$$e = \sqrt{\frac{a^2-b^2}{a^2}} = \sqrt{\frac{18.09^2 - 4.56^2}{18.09^2}}$$
$$= \sqrt{\frac{(18.09+4.56)(18.09-4.56)}{18.09^2}}$$
$$= \frac{\sqrt{22.65\times13.53}}{18.09}$$
$$= \frac{\sqrt{306.4545}}{18.09}$$
$$= \frac{17.51}{18.09}$$
$$e \approx 0.97$$

8.Find the equation of the tangent at $t = 2$ to the parabola $y^2 = 8x$

$y^2 = 8x \Longrightarrow y^2 = 4(2)x \Longrightarrow a = 2$

Equation of the tangent to the parabola at $t\ is\ yt = x + at^2$

$$\Longrightarrow 2y = x + 2(2)^2$$
$$\Longrightarrow x - 2y + 8 = 0$$

9. The equation $y = \frac{1}{32}x^2$ models cross sections of parabolic mirrors that are used for solar energy. There is a heating tube located at the focus of each parabola; how high is this tube located above the vertex of the parabola?

$Equation\ of\ the\ Parabola\ is\ y = \frac{1}{32}x^2$

$$\Longrightarrow x^2 = 32y$$
$$\Longrightarrow x^2 = 4(8)y$$
$$\Longrightarrow \boxed{a = 8}$$

$Focus\ F = (a,o) = (8,0)$

Thus the tube is located 8 unit above the vertex of the parabola.

10. If the equation of the ellipse is $\frac{(x-11)^2}{484} + \frac{y^2}{64} = 1$ (x and y are measured in centimeters) where to the nearest centimeter, should the patient's kidney stone be placed so that the reflected sound hits

the kidney stone?

$\frac{(x-11)^2}{484}+\frac{y^2}{64}=1 \Rightarrow a^2=484, b^2=64$

$e=\sqrt{\frac{a^2-b^2}{a^2}}=\sqrt{\frac{484-64}{484}}=\sqrt{\frac{420}{484}}=\frac{20.5}{22}$

$ae=22\times\frac{20.5}{22}=20.5$

$Focus\ \ F=(ae,0)=(15,0)$

$\therefore$ *The kidney stone must be placed at a distance of* $20.5cm$ *from the centre.*

22. The maximum and minimum distances of the Earth from the Sun respectively are $152\times 10^6\ km$ and $94.5\times 10^6\ km$. The Sun is at one focus of the elliptical orbit. Find the distance from the Sun to the other focus.

$Maximum\ distance\ =152\times10^6 \quad \Rightarrow a+ae=152\times10^6 \rightarrow (1)$

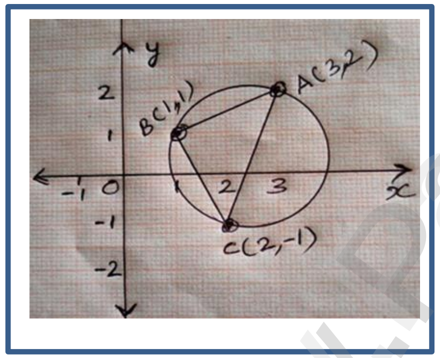
$Minimum\ distance=94.5\times10^6 \ \Rightarrow\ a-ae=94.5\times10^6\rightarrow(2)$

$(1)-(2)\Rightarrow 2ae=57.5\times10^6=575\times10^5$

$\therefore$ The distance from the Sun to the other focus. $=575\times10^5\ km$.

5 MARK QUESTIONS

1.Find the equation of the circle passing through the points $(1,1),(2,-1)$ and $(3,2)$,



$Let\ A(3,2)\ B(1,1),\ C(2,-1)$

$slope=\frac{y_2-y_1}{x_2-x_1}$

$Slope\ of\ AB, m_1=\frac{1-2}{1-3}=\frac{-1}{-2}=\frac{1}{2}$

$Slope\ of\ BC, m_2=\frac{-1-1}{2-1}=\frac{-2}{1}=-2$

$m_1\times m_2=\frac{1}{2}\times-2=-1$

$\Rightarrow AB\ \perp BC\ .\ So\ \angle ABC=\frac{\pi}{2}$

W.K.T the angle in a semi circle is right angle.

$A(3,2)$ and $C(2,-1)$ are the end points of the diameter.

Equation of the circle passing through the end points (x_1,y_1) & (x_2,y_2) *of the diameter is*

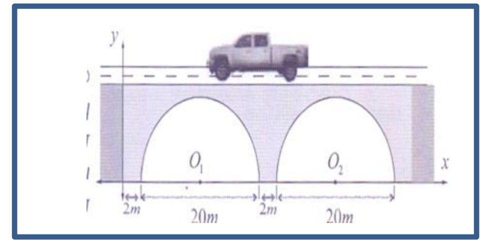
$(x-x_1)(x-x_2)+(y-y_1)(y-y_2)=0$

$\Rightarrow (x-3)(x-2)+(y-2)(y+1)=0$

$\Rightarrow x^2-5x+6+y^2-y-2=0$

$\Rightarrow x^2+y^2-5x-y+4=0$

2. A road bridge over an irrigation canal has two semi circular vents each with a span of $20m$ and the supporting pillars of width $2m$. Use Fig to write the equations that represent the semi-vertical vents.



FIRST VENT:

$centre\ c=(12,0) \qquad radius=10$மீ

$Equation\ of\ the\ first\ vent\ (x-h)^2+(y-k)^2=r^2$

$\Rightarrow (x-12)^2+(y-0)^2=10^2$

$\Rightarrow x^2-24x+144+y^2=100$

$\Rightarrow x^2+y^2-24x+144=0$

SECOND VENT:

$centre\ c=(34,0)\ \ radius=10$மீ

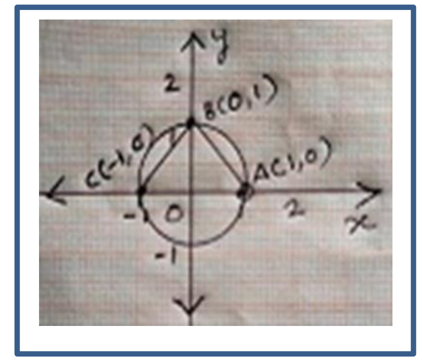
$Equation\ of\ the\ second\ vent\ (x-h)^2+(y-k)^2=r^2$

$\Rightarrow (x-34)^2+(y-0)^2=10^2$

$\Rightarrow x^2-68x+1156+y^2=100$

$\Rightarrow x^2+y^2-68x+1056=0$

3. Find the equation of the circle passing through the points $(1,0),(-1,0)$ *and* $(0,1)$



$Let\ A(1,0),\ B(0,1),\ C(-1,0).$

$Slope\ of\ AB,\ m_1=\frac{1-0}{0-1}=-1$

$Slope\ of\ BC, m_2=\frac{0-1}{-1-0}=1$

$m_1\times m_2=(-1)(1)=-1$

$\Rightarrow AB\ \perp BC. So, \angle ABC=\frac{\pi}{2}$

W.K.T the angle in a semi circle is right angle.

$A(1,0)$ and $C(-1,0)$ are the end points of the diameter .

tpl;lj;jpd; Kidg;Gs;spfshFk;.

Equation of the circle passing through the end points (x_1,y_1) & (x_2,y_2) *of the diameter is*

$(x-x_1)(x-x_2)+(y-y_1)(y-y_2)=0$

$\Rightarrow (x-1)(x+1)+(y-0)(y-0)=0$

$\Rightarrow x^2-1+y^2=0$

$\Rightarrow x^2+y^2=1$

7. Find the vertex, focus, directrix, and length of the latusrectum of the parabola x^2-4x-

$5y-1=0$

$x^2-4x-5y-1=0 \Rightarrow x^2-4x=5y+1$

$\Rightarrow x^2-4x+4=5y+1+4$

$\Rightarrow (x-2)^2=5(y+1)$

$\Rightarrow 4a=5$

$\Rightarrow a=\frac{5}{4}$

The parabola is open upwards.

	Referred to X, Y	*Referred to x, y* $X=x-2, Y=y+1$ $\Rightarrow x=X+2\,; y=Y-1$
vertex	$V(0,0)$	$x=0+2\,; y=0-1$ $\Rightarrow V=(2,-1)$
Focus	$F(0,a)=\left(0,\frac{5}{4}\right)$	$x=0+2\,; y=\frac{5}{4}-1=\frac{1}{4}$ $\Rightarrow F=\left(2,\frac{1}{4}\right)$
Equation of Latus rectum	$Y=-a \Rightarrow Y=-\frac{5}{4}$	$y=-\frac{5}{4}-1 \Rightarrow y=-\frac{9}{4}$ $\Rightarrow 4y+9=0$
Length of Latus rectum	$4a=4\left(\frac{5}{4}\right)=5$	5

9. Find the vertex, focus, directrix, and length of the latusrectum of the parabola $y^2-4y-8x+12=0$

$y^2-4y-8x+12=0 \Rightarrow y^2-4y=8x-12$

$\Rightarrow y^2-4y+4=8x-12+4$

$\Rightarrow (y-2)^2=8x-8$

$\Rightarrow (y-2)^2=8(x-1)$

$\Rightarrow (y-2)^2=4(2)(x-1)$

$a=2\,, V(h,k)=(1,2)$

The parabola is rightwards..

	Referred to X, Y	*Referred to x, y* $X=x-2, Y=y+1$ $\Rightarrow x=X+2\,; y=Y-1$
vertex	$V(0,0)$	$x=0+1\,; y=0+2$ $\Rightarrow V=(1,2)$
Focus	$F(a,0)=(2,0)$	$x=2+1\,; y=0+2$

		$\Rightarrow F=(3.2)$
Equation of Latus rectum	$X=-a \Rightarrow X=-2$	$x=-2+1 \Rightarrow x=-1$ $(or)\ x+1=0$
Length of Latus rectum	$4a=4(2)=8$	8

10. Identify the type of conic and find centre, foci, vertices, and directrices of $\frac{(x-3)^2}{225}+\frac{(y-4)^2}{289}=1$

$\frac{(x-3)^2}{225}+\frac{(y-4)^2}{289}=1 \Rightarrow \frac{X^2}{225}+\frac{Y^2}{289}=1, X=x-3, Y=y-4$

It is an ellipse, the major axis is parallel to $y-axis$.

$a^2=289 \Rightarrow a=17, b^2=225$

$e=\sqrt{\frac{a^2-b^2}{a^2}}=\sqrt{\frac{289-225}{289}}=\sqrt{\frac{64}{289}}=\frac{8}{17}$

$ae=17\times\frac{8}{17}=8\ ;\ \frac{a}{e}=\frac{17}{8/17}=\frac{289}{8}$

	Referred to X, Y	*Referred to x, y* $X=x-3, Y=y-4$ $\Rightarrow x=X+3\,; y=Y+4$
Centre	$C(0,0)$	$x=0+3\,; y=0+4 \Rightarrow C(3,4)$
Vertices	$A(0,a)=(0,17)$	$x=0+3\,; y=17+4 \Rightarrow A(3,21)$
	$A'(0,-a)=(0,-17)$	$x=0+3\,; y=-17+4 \Rightarrow A'(3,-13)$
Foci	$F_1(0,ae)=(0,8)$	$x=0+3\,; y=8+4 \Rightarrow F_1(3,12)$
	$F_2(0,-ae)=(0,-8)$	$x=0+3\,; y=-8+4 \Rightarrow F_2(3,-4)$
Equation of directrices	$Y=\frac{a}{e} \Rightarrow Y=\frac{289}{8}$	$y=\frac{289}{8}+4 \Rightarrow y=\frac{321}{8}$
	$Y=-\frac{a}{e} \Rightarrow Y=-\frac{289}{8}$	$y=-\frac{289}{8}+4 \Rightarrow y=-\frac{257}{8}$

11. Identify the type of conic and find centre, foci, vertices, and directrices of $\frac{(x+1)^2}{100}+\frac{(y-2)^2}{64}=1$

$\frac{(x+1)^2}{100}+\frac{(y-2)^2}{64}=1 \Rightarrow \frac{X^2}{100}+\frac{Y^2}{64}=1, X=x+1, Y=y-2$

It is an ellipse, the major axis is parallel to $x-axis$.

$a^2=100 \Rightarrow a=10, b^2=64$

$e=\sqrt{\frac{a^2-b^2}{a^2}}=\sqrt{\frac{100-64}{100}}=\sqrt{\frac{36}{100}}=\frac{6}{10}$

$ae=10\times\frac{6}{10}=6\ ;\ \frac{a}{e}=\frac{10}{6/10}=\frac{100}{6}$

	$Referred\ to\ X,Y$	$Referred\ to\ x,y$ $X=x+1, Y=y-2$ $\Rightarrow x=X-1\,;y=Y+2$
Centre	$C(0,0)$	$x=0-1\,;y=0+2 \Rightarrow C(-1,2)$
Vertices	$A(a,0)=(10,0)$	$x=10-1\,;y=0+2 \Rightarrow A(9,2)$
	$A'(-a,0)=(-10,0)$	$x=-10-1\,;y=0+2 \Rightarrow A'(-11,2)$
Foci	$F_1(ae,0)=(6,0)$	$x=6-1\,;y=0+2 \Rightarrow F_1(5,2)$
	$F_2(-ae,0)=(-6,0)$	$x=-6-1\,;y=0+2 \Rightarrow F_2(-7,2)$
Equation of directrices	$X=\frac{a}{e} \Rightarrow X=\frac{100}{6}$	$x=\frac{100}{6}-1 \Rightarrow x=\frac{94}{6}=\frac{47}{3}$
	$X=-\frac{a}{e} \Rightarrow X=-\frac{100}{6}$	$x=-\frac{100}{6}-1 \Rightarrow x=-\frac{106}{6}=-\frac{53}{3}$

13. Find the equations of tangent and normal to the ellipse $x^2+4y^2=32$ when $\theta=\frac{\pi}{4}$

$x^2+4y^2=32 \Rightarrow \frac{x^2}{32}+\frac{y^2}{8}=1 \Rightarrow a^2=32, b^2=8 \Rightarrow a=4\sqrt{2}, b=2\sqrt{2}$

$Equation\ of\ the\ tangent\ to\ the\ ellipse\ at\ \theta\ is\ \frac{x\cos\theta}{a}+\frac{y\sin\theta}{b}=1$

$$\Rightarrow \frac{x\cos\frac{\pi}{4}}{4\sqrt{2}}+\frac{y\sin\frac{\pi}{4}}{2\sqrt{2}}=1$$
$$\Rightarrow \frac{x/\sqrt{2}}{4\sqrt{2}}+\frac{y/\sqrt{2}}{2\sqrt{2}}=1$$
$$\Rightarrow \frac{x}{8}+\frac{y}{4}=1$$
$$\Rightarrow \frac{x+2y}{8}=1$$
$$\Rightarrow x+2y=8$$
$$\Rightarrow x+2y-8=0$$

$Equation\ of\ the\ normal\ to\ the\ ellipse\ at\ \theta\ is\ \frac{ax}{\cos\theta}-\frac{by}{\sin\theta}=a^2-b^2$

$$\Rightarrow \frac{4\sqrt{2}x}{\cos\frac{\pi}{4}}-\frac{2\sqrt{2}y}{\sin\frac{\pi}{4}}=32-8$$
$$\Rightarrow \frac{4\sqrt{2}x}{1/\sqrt{2}}-\frac{2\sqrt{2}y}{1/\sqrt{2}}=24$$
$$\Rightarrow 8x-4y-24=0$$
$$\div 4 \Rightarrow 2x-y-6=0$$

14. Show that the line $x-y+4=0$ is a tangent to the ellipse $x^2+3y^2=12$. Also find the coordinates of the point of contact.

$x-y+4=0 \Rightarrow y=x+4$

$Compare\ with\ y=mx+c \Rightarrow \boxed{m=1, c=4}$

$x^2+3y^2=12 \Rightarrow \frac{x^2}{12}+\frac{y^2}{4}=1 \Rightarrow \boxed{a^2=12, b^2=4}$

$Condition: c^2=a^2m^2+b^2$

$c^2=4^2=16 \rightarrow (1)$ $\qquad a^2m^2+b^2=12(1)^2+4=16 \rightarrow (2)$

$From\ (1),(2) \Rightarrow\ c^2=a^2m^2+b^2.$

$\therefore$ The line $x-y+4=0$ is a tangent to the ellipse $x^2+3y^2=12$

$Point\ of\ contact\ =\left(\frac{-a^2m}{c},\frac{b^2}{c}\right)=\left(\frac{-12(1)}{4},\frac{4}{4}\right)=(-3,1)$

15. Find the equations of the tangent and normal to hyperbola $12x^2-9y^2=108$ at $\theta=\frac{\pi}{3}$.(Use parametric form).

$12x^2-9y^2=108$

$\div 108 \Rightarrow \frac{x^2}{9}-\frac{y^2}{12}=1$

$\Rightarrow a^2=9, b^2=12$

$\Rightarrow a=3, b=2\sqrt{3}$

$Equation\ of\ the\ tangent\ to\ the$ hyperbola $at\ \theta\ is\ \frac{x\sec\theta}{a}-\frac{y\tan\theta}{b}=1$

$$\Rightarrow \frac{x\sec\frac{\pi}{3}}{3}-\frac{y\tan\frac{\pi}{3}}{2\sqrt{3}}=1$$
$$\Rightarrow \frac{2x}{3}-\frac{\sqrt{3}y}{2\sqrt{3}}=1$$
$$\Rightarrow \frac{2x}{3}-\frac{y}{2}=1$$
$$\Rightarrow \frac{4x-3y}{6}=1$$
$$\Rightarrow 4x-3y=6$$
$$\Rightarrow 4x-3y-6=0$$

$Equation\ of\ the\ normal\ to\ the$ hyperbola $at\ \theta\ is\ \frac{ax}{\sec\theta}+\frac{by}{\tan\theta}=a^2+b^2$

$$\Rightarrow \frac{3x}{\sec\frac{\pi}{3}}+\frac{2\sqrt{3}y}{\tan\frac{\pi}{3}}=(3)^2+\left(2\sqrt{3}\right)^2$$
$$\Rightarrow \frac{3x}{2}+\frac{2\sqrt{3}y}{\sqrt{3}}=9+12$$
$$\Rightarrow \frac{3x}{2}+y=21$$
$$\Rightarrow \frac{3x+2y}{2}=21$$
$$\Rightarrow 3x+2y=42$$
$$\Rightarrow 3x+2y-42=0$$

16. A bridge has a parabolic arch that is 10m high in the centre and $30m$ wide at the bottom. Find

the height of the arch $6m$ from the centre, on either sides.

$Equation\ of\ the\ parabola\ is\ x^2 = -4ay \rightarrow (1)$

$A(15,-10) \Rightarrow (15)^2 = -4a(-10)$

$\Rightarrow \boxed{4a = \frac{225}{10}}$

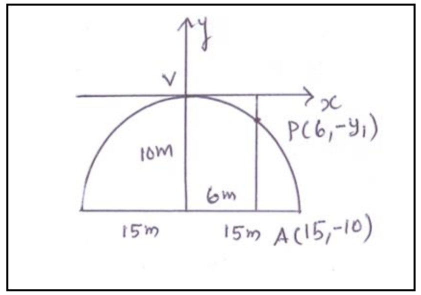
$(1) \Longrightarrow x^2 = -\left(\frac{225}{10}\right)y$

$P(6,-y_1) \Rightarrow (6)^2 = -\left(\frac{225}{10}\right)(-y_1)$

$\Longrightarrow y_1 = \frac{36\times10}{225}$

$= 1.6m$

$Height = 10 - 1.6 = 8.4m$



17. At a water fountain, water attains a maximum height of $4m$ at horizontal distance of $0.5\ m$ from its origin. If the path of water is a parabola,find the height of water at a horizontal distance of $0.75m$ from the point of origin.

$Equation\ of\ the\ parabola\ is\ (x-h)^2 = -4a(y-k)$

$V(0.5,4) \Rightarrow (x-0.5)^2 = -4a(y-4) \rightarrow (1)$

$Since\ O(0,0)\ lies\ on\ the\ parabola$

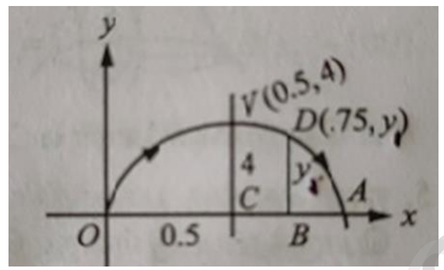
$\Rightarrow (0-0.5)^2 = -4a(0-4) \Rightarrow 4a = \frac{(0.25)^2}{4}$

$(1) \Rightarrow (x-0.5)^2 = -\frac{(0.25)^2}{4}(y-4)$

$D(0.75,y_1) \Rightarrow (0.75-0.5)^2 = -\frac{(0.25)^2}{4}(y_1-4)$

$\Longrightarrow y_1 = \frac{0.25\times0.25\times4}{0.25} = 1m$

$Height = 4 - 1 = 3m$



19. Parabolic cable of a $60m$ portion of the roadbed of a suspension bridge are positioned as shown below. Vertical Cables are to be spaced every 6m along this portion of the roadbed. Calculate the lengths of first two of these vertical cables from the vertex.

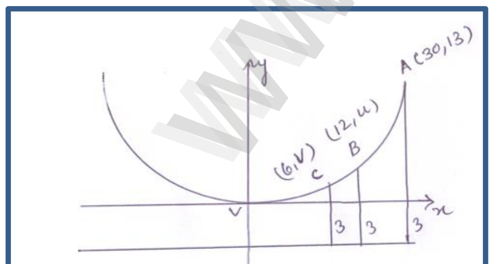
$Equation\ of\ the\ parabola\ is\ x^2 = 4ay \rightarrow (1)$

$A(30,13) \Rightarrow (30)^2 = 4a(13) \Longrightarrow \boxed{4a = \frac{900}{13}}$

$\therefore (1) \Longrightarrow x^2 = \left(\frac{900}{13}\right)y \rightarrow (2)$

$Sub.B(12,u)\ in\ (2),$

$(12)^2 = \left(\frac{900}{13}\right)u \Longrightarrow u = \frac{12\times12\times13}{900} = 2.08m$



$Length\ of\ the\ cable\ at\ a\ distance\ of 12m = 3 + 2.08 = 5.08m$

$Sub.C(6,v)\ in\ (2) \Rightarrow (6)^2 = \left(\frac{900}{13}\right)v \Longrightarrow v = \frac{6\times6\times13}{900} = 0.52m$

$Length\ of\ the\ cable\ at\ a\ distance\ of\ 6m = 3 + 0.52 = 3.52m$

20. Assume that water issuing from the end of a horizontal pipe, $7.5m$ above the ground, describes a parabolic path. The vertex of the parabolic path is at the end of the pipe. At a position $2.5m$ below the line of the pipe, the flow of water has curved outward $3m$ beyond the vertical line through the end of the pipe. How far beyond this vertical line will the water strike the ground?

$x^2 = -4ay$

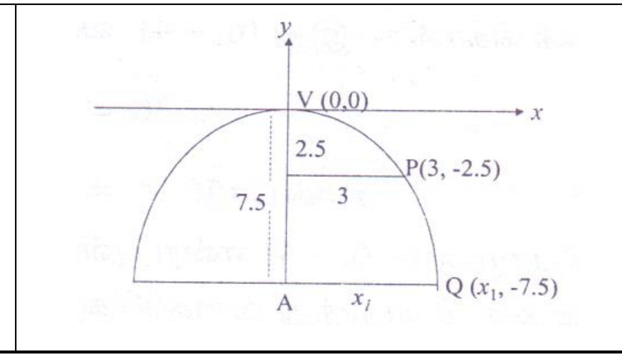
$P(3,-2.5) \Rightarrow (3)^2 = -4a(-2.5)$

$\Rightarrow a = \frac{9}{10}$

$x^2 = -4\times\frac{9}{10}\times y$

$Q(x_1,-7.5) \Rightarrow$

$(x_1)^2 = -4\times\frac{9}{10}(-7.5) \Rightarrow x_1 = 3\sqrt{3}m$



21. On lighting a rocket cracker it gets projected in a parabolic path and reaches a maximum height of $4m$ when it is $6m$ away from the point of projection. Finally it reaches the ground $12m$ away from the starting point. Find the angle of projection.

$x^2 = -4ay$

$(6,-4) \Rightarrow (6)^2 = -4a(-4) \Rightarrow 4a = 9$

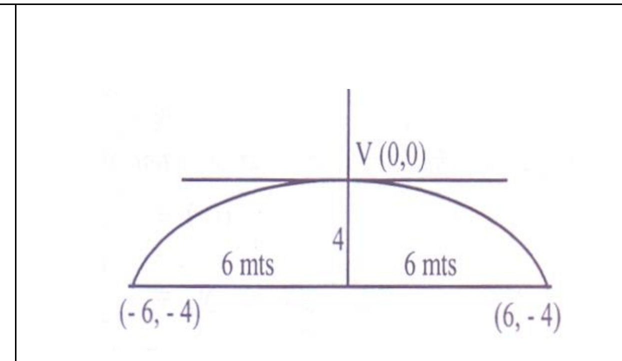
$x^2 = -9y$

$Diff.w.r.to\ x,$

$\frac{dy}{dx} = -\frac{2x}{9}$

$Slope\ at\ (-6,-4)\ is\ \Longrightarrow \tan\theta = -\frac{2x}{9}$

$\theta = tan^{-1}\frac{4}{3}$



24. A tunnel through a mountain for a four lane highway is to have a elliptical opening. The total width of the highway (not the opening) is to be $16m$, and the height at the edge of the road must be sufficient for a truck $4m$ high to clear if the highest point of the opening is to be $5m$ approximately How wide must the opening be?

$Given\ b = 5 \Longrightarrow b^2 = 25$

$Equation\ of\ the\ ellipse\ is\ \frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$

$\Rightarrow \frac{x^2}{a^2}+\frac{y^2}{25}=1 \to (1)$

$P=(8,4) \Rightarrow \frac{8^2}{a^2}+\frac{4^2}{25}=1$

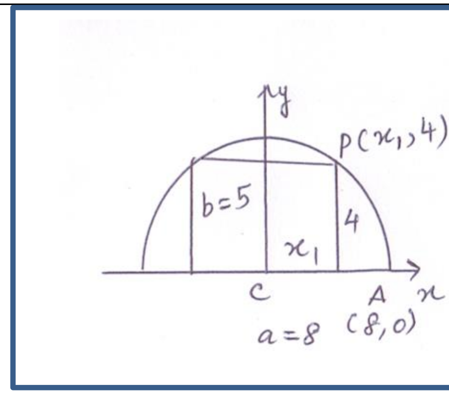
$\Rightarrow a^2=\frac{25}{9}\times 64$

$\Rightarrow a=\frac{40}{3}$

$\Rightarrow 2a=\frac{80}{3}$

$\Rightarrow 2a=26.66$

$\therefore$ *Width of the opening* $=26.66m$



25 A rod of length $1.2m$ moves with its ends always touching the coordinate axes. The locus of a point Pon the rod, which is $0.3m$ from the end in contact with x −axis is an ellipse. Find the eccentricity.

$\angle PAO=\angle BPQ=\theta$

ΔPQB,y; $\cos\theta=\frac{x_1}{0.9}=\frac{x_1}{9/10}=\frac{10x_1}{9}$

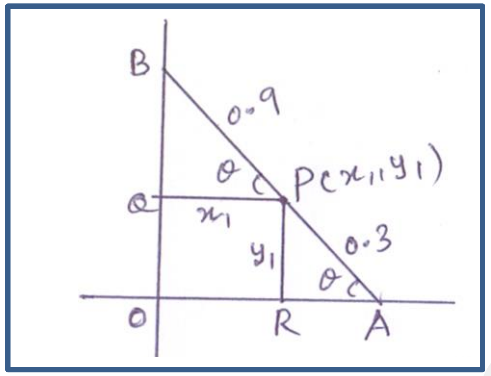
ΔARP,y; $\sin\theta=\frac{y_1}{0.3}=\frac{y_1}{3/10}=\frac{10y_1}{3}$

$W.K.T.\ \sin^2\theta+\cos^2\theta=1$

$\left(\frac{10y_1}{3}\right)^2+\left(\frac{10x_1}{9}\right)^2=1$

Locus of P is $\frac{100x^2}{81}+\frac{100y^2}{9}=1$

$\Rightarrow \frac{x^2}{81/100}+\frac{y^2}{9/100}=1$



$e=\sqrt{\frac{a^2-b^2}{a^2}}=\sqrt{\frac{81/100-9/100}{81/100}}$

$=\frac{\sqrt{72/100}}{9/10}$

$=\frac{\sqrt{36\times2}/10}{9/10}$

$=\frac{6\sqrt{2}}{9}$

$=\frac{2\sqrt{2}}{3}$

26. A semielliptical archway over a one-way road has a height of 3m and a width of $12m$. The truck has a width of $3m$ and a height of $2.7m$. Will the truck clear the opening of the archway?

$b=3 \Rightarrow b^2=9$

Equation of the ellipse is $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1 \Rightarrow \frac{x^2}{36}+\frac{y^2}{9}=1 \to (1)$

$P=\left(\frac{3}{2},y_1\right) \Rightarrow \frac{\left(\frac{3}{2}\right)^2}{36}+\frac{{y_1}^2}{9}=1$

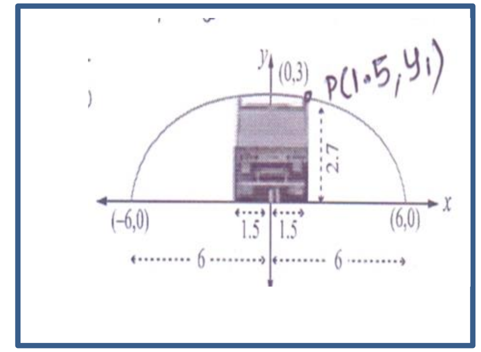
$\Rightarrow \frac{9/4}{36}+\frac{{y_1}^2}{9}=1$

$\Rightarrow \frac{{y_1}^2}{9}=1-\frac{9}{144}$

$\Rightarrow \frac{{y_1}^2}{9}=\frac{135}{144}$

$\Rightarrow {y_1}^2=\frac{9\times135}{144}$

$\Rightarrow y_1=\frac{3\sqrt{135}}{12}\approx 2.9$



Height of the archway $=2.9m>2.7m=$ *Height of the truck.*

$\therefore$ The truck clear the opening of the archway.

27. Cross section of a Nuclear cooling tower is in the shape of a hyperbola with equation $\frac{x^2}{30^2}-\frac{y^2}{44^2}=1$. The tower is $150m$ tall and the distance from the top of the tower to the centre of the hyperbola is half the distance from the base of the tower to the centre of the hyperbola. Find the diameter of the top and base of the tower

Let the length of the base from the centre $=\ l$

The length of the top from the centre $=\frac{l}{2}$

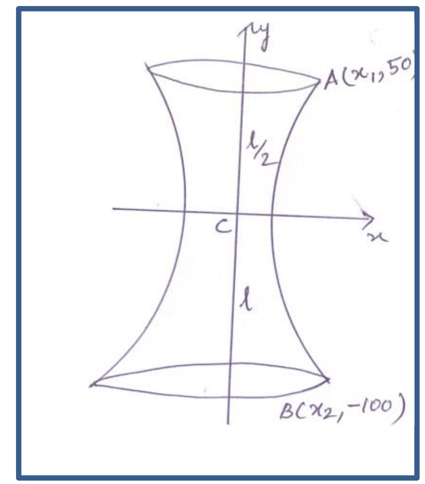
Height of the tower $=150m$

$\Rightarrow l+\frac{l}{2}=150$

$\Rightarrow \frac{3l}{2}=150$

$\Rightarrow l=150\times\frac{2}{3}$

$\Rightarrow l=100m\quad So,\ \frac{l}{2}=50m$



Points on the hyperbola are $A=(x_1,50)$ *and*

$B=(x_2,-100)$

Equation of the hyperbola is $\frac{x^2}{30^2}-\frac{y^2}{44^2}=1$

$A=(x_1,50) \Rightarrow \frac{{x_1}^2}{30^2}-\frac{50^2}{44^2}=1$

$\Rightarrow \frac{{x_1}^2}{900}=1+\frac{2500}{1936}$

$\Rightarrow \frac{{x_1}^2}{900}=\frac{4436}{1936}$

$\Rightarrow {x_1}^2 = \frac{4436}{1936} \times 900$

$\Rightarrow x_1 = \frac{\sqrt{4436}}{44} \times 30 = \frac{66.6 \times 30}{44} = 45.41$

$\therefore$ *Diameter of the top* $= 2x_1 = 2(45.41) = 90.82m$

$B = (x_2, -100) \Rightarrow \frac{{x_2}^2}{30^2} - \frac{(-100)^2}{44^2} = 1 \Rightarrow \frac{{x_2}^2}{900} = 1 + \frac{10000}{1936}$

$\Rightarrow \frac{{x_2}^2}{900} = \frac{11936}{1936}$

$\Rightarrow {x_2}^2 = \frac{11936}{1936} \times 900$

$\Rightarrow x_2 = \frac{\sqrt{11936}}{44} \times 30 = 74.49$

$\therefore$ *Diameter of the base* $= 2x_2 = 2(74.49) = 148.98m \approx 149m$

6.VECTOR ALGEBRA

2 & 3 MARK QUESTIONS

1.A particle acted on by constant forces $8\hat{\imath} + 2\hat{\jmath} - 6\hat{k}$ and $6\hat{\imath} + 2\hat{\jmath} - 2\hat{k}$ is displaced from the point (1,2,3) to the point (5,4,1). Find the total work done by the forces.

$\overrightarrow{F_1} = 8\vec{i} + 2\vec{j} - 6\vec{k}, \overrightarrow{F_2} = 6\vec{i} + 2\vec{j} - 2\vec{k}$; $\overrightarrow{OA} = \vec{i} + 2\vec{j} + 3\vec{k}, \overrightarrow{OB} = 5\vec{i} + 4\vec{j} + \vec{k}$

$\vec{F} = \overrightarrow{F_1} + \overrightarrow{F_2} = 8\vec{i} + 2\vec{j} - 6\vec{k} + 6\vec{i} + 2\vec{j} - 2\vec{k} = 14\vec{i} + 4\vec{j} - 8\vec{k}$

$\vec{d} = \overrightarrow{OB} - \overrightarrow{OA} = 5\vec{i} + 4\vec{j} + \vec{k} - (\vec{i} + 2\vec{j} + 3\vec{k}) = 4\vec{i} + 2\vec{j} - 2\vec{k}$

work done by the force $w = \vec{F}.\vec{d}$

$= (14\vec{i} + 4\vec{j} - 8\vec{k}).(4\vec{i} + 2\vec{j} - 2\vec{k})$

$= 56 + 8 + 16$

$= 80$

2. Find the volume of the parallelepiped whose coterminous edges are represented by the vectors $-6\hat{\imath} + 14\hat{\jmath} + 10\hat{k}, 14\hat{\imath} - 10\hat{\jmath} - 6\hat{k}$ and $2\hat{\imath} + 4\hat{\jmath} - 2\hat{k}$.

$\vec{a} = -6\vec{i} + 14\vec{j} + 10\vec{k},\ \vec{b} = 14\vec{i} - 10\vec{j} - 6\vec{k},\ \vec{c} = 2\vec{i} + 4\vec{j} - 2\vec{k}$

volume of the parallelepiped $= [\vec{a}\ \ \vec{b}\ \ \vec{c}] = \begin{vmatrix} -6 & 14 & 10 \\ 14 & -10 & -6 \\ 2 & 4 & -2 \end{vmatrix}$

$= -6(20 + 24) - 14(-28 + 12) + 10(56 + 20)$

$= -264 + 224 + 760$

$= 720$

3.Determine whether the three vectors $2\hat{\imath} + 3\hat{\jmath} + \hat{k}, \hat{\imath} - 2\hat{\jmath} + 2\hat{k}$ and $3\hat{\imath} + \hat{\jmath} + 3\hat{k}$ are coplanar.

$\vec{a} = 2\vec{i} + 3\vec{j} + \vec{k},\ \vec{b} = \vec{i} - 2\vec{j} + 2\vec{k},\ \vec{c} = 3\vec{i} + \vec{j} + 3\vec{k}$

If $\vec{a}, \vec{b}, \vec{c}$ *are coplanar then* $[\vec{a}, \vec{b}, \vec{c}] = 0$

$[\vec{a}, \vec{b}, \vec{c}] = \begin{vmatrix} 2 & 3 & 1 \\ 1 & -2 & 2 \\ 3 & 1 & 3 \end{vmatrix} = 2(-8) - 3(-3) + 1(7) = 0$

4. If $2\hat{\imath} - \hat{\jmath} + 3\hat{k}, 3\hat{\imath} + 2\hat{\jmath} + \hat{k}, \hat{\imath} + m\hat{\jmath} + 4\hat{k}$ are coplanar, find the value of m.

Condition for coplanar vectors is $[\vec{a}, \vec{b}, \vec{c}] = 0$

$\Rightarrow \begin{vmatrix} 2 & -1 & 3 \\ 3 & 2 & 1 \\ 1 & m & 4 \end{vmatrix} = 0$

$\Rightarrow 2(8 - m) - (-1)(12 - 1) + 3(3m - 2) = 0$

$\Rightarrow 16 - 2m + 11 + 9m - 6 = 0$

$\Rightarrow 7m + 21 = 0$

$\Rightarrow 7m = -21$

$\Rightarrow m = -3$

5. Show that the points $(2, 3, 4), (-1, 4, 5)$ and $(8, 1, 2)$ are collinear

$[\vec{a}\ \ \vec{b}\ \ \vec{c}] = \begin{vmatrix} 2 & 3 & 4 \\ -1 & 4 & 5 \\ 8 & 1 & 2 \end{vmatrix} = 6 + 126 - 132 = 0$

$\therefore$ *The given points are collinear.*

6. If the Cartesian equation of a plane is $3x - 4y + 3z = -8$, find the vector equation of the plane in the standard form.

$3x - 4y + 3z = -8$

$\Rightarrow (x\vec{i} + y\vec{j} + z\vec{k}).(3\vec{i} - 4\vec{j} + 3\vec{k}) = -8$

$\Rightarrow \vec{r}.(3\vec{i} - 4\vec{j} + 3\vec{k}) = -8$

7. Find the intercepts cut off by the plane $\vec{r} \cdot (6\hat{\imath} + 4\hat{\jmath} - 3\hat{k}) = 12$ on the coordinate axes

Let $\vec{r} = x\vec{i} + y\vec{j} + z\vec{k}$

$\vec{r}.(6\vec{i} + 4\vec{j} - 3\vec{k}) = 12 \Rightarrow (x\vec{i} + y\vec{j} + z\vec{k}).(6\vec{i} + 4\vec{j} - 3\vec{k}) = 12$

$\Rightarrow 6x + 4y - 3z = 12$

$\div 12 \Rightarrow \frac{x}{2} + \frac{y}{3} - \frac{z}{4} = 1$

$\Rightarrow \frac{x}{2} + \frac{y}{3} + \frac{z}{-4} = 1$

x − *intercept* $= 2$

y − *intercept* $= 3$

z − *intercept* $= -4$

8. Find the angle between the planes $\vec{r} \cdot (\hat{\imath} + \hat{\jmath} - 2\hat{k}) = 3$ and $2x - 2y + z = 2$

$\vec{r}.(\vec{i} + \vec{j} - 2\vec{k}) = 3 \Rightarrow \overrightarrow{n_1} = \vec{i} + \vec{j} - 2\vec{k}$

$2x - 2y + z = 2 \Rightarrow \overrightarrow{n_2} = 2\vec{i} - 2\vec{j} + \vec{k}$

Angle between the two planes is $\cos\theta = \frac{|\overrightarrow{n_1}.\overrightarrow{n_2}|}{|\overrightarrow{n_1}||\overrightarrow{n_2}|}$

$\Rightarrow \cos\theta = \frac{|2-2-2|}{\sqrt{1^2+1^2+(-2)^2}\sqrt{2^2+(-2)^2+1^2}}$

$= \frac{2}{\sqrt{6}\sqrt{9}}$

$= \frac{2}{3\sqrt{6}}$

$\Rightarrow \theta = \cos^{-1}\left(\frac{2}{3\sqrt{6}}\right)$

9. Find the angle between the line $\vec{r} = (2\hat{\imath} - \hat{\jmath} + \hat{k}) + t(\hat{\imath} + 2\hat{\jmath} - 2\hat{k})$ and the plane $\vec{r}\cdot(6\hat{\imath} + 3\hat{\jmath} + 2\hat{k}) = 8$..

$\vec{r} = (2\vec{i} - \vec{j} + \vec{k}) + t(\vec{i} + 2\vec{j} - 2\vec{k}) \Rightarrow \vec{v} = \vec{i} + 2\vec{j} - 2\vec{k}$

$\vec{r}.(6\vec{i} + 3\vec{j} + 2\vec{k}) = 8 \Rightarrow \vec{n} = 6\vec{i} + 3\vec{j} + 2\vec{k}$

Angle between the line and a plane is $\sin\theta = \frac{|\vec{v}.\vec{n}|}{|\vec{v}||\vec{n}|}$

$\Rightarrow \sin\theta = \frac{|6+6-4|}{\sqrt{1^2+2^2+(-2)^2}\sqrt{6^2+3^2+2^2}}$

$= \frac{8}{\sqrt{9}\sqrt{49}}$

$= \frac{8}{21}$

$\Rightarrow \theta = \sin^{-1}\left(\frac{8}{21}\right)$

10. Find the length of the perpendicular from the point $(1, -2, 3)$ to the plane $x - y + z = 5$.

Length of the perpendicular $= \left|\frac{ax_1+by_1+cz_1+d}{\sqrt{a^2+b^2+c^2}}\right|$

$(x_1, y_1, z_1) = (1, -2, 3)$

$x - y + z - 5 = 0 \Rightarrow a = 1,\ b = -1,\ c = 1,\ d = -5$

Length of the perpendicular $= \frac{|1(1)-1(-2)+1(3)-5|}{\sqrt{1^2+(-1)^2+1^2}} = \frac{1}{\sqrt{3}}$

11. Find the distance of a point $(2, 5, -3)$ from the plane $\vec{r}\cdot(6\hat{\imath} - 3\hat{\jmath} + 2\hat{k}) = 5$.

Distance from a point to the plane $= \left|\frac{ax_1+by_1+cz_1+d}{\sqrt{a^2+b^2+c^2}}\right|$

$(x_1, y_1, z_1) = (2, 5, -3)$

$\vec{r}.(6\vec{i} - 3\vec{j} + 2\vec{k}) = 5 \Rightarrow a = 6,\ b = -3,\ c = 2,\ d = -5$

Distance $= \frac{|6(2)-3(5)+2(-3)-5|}{\sqrt{6^2+(-3)^2+2^2}} = 2$

5 MARK QUESTIONS

1. By vector method, prove that $\cos(\alpha + \beta) = \cos\alpha\cos\beta - \sin\alpha\sin\beta$.

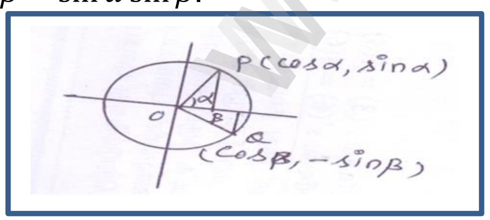
$\overrightarrow{OP} = cos\alpha\vec{i} + sin\alpha\,\vec{j}$

$\overrightarrow{OQ} = cos\beta\vec{i} - sin\beta\,\vec{j}$

$\overrightarrow{OP}.\overrightarrow{OQ} = cos\alpha cos\beta - sin\alpha sin\beta$

$\overrightarrow{OP}.\overrightarrow{OQ} = \cos(\alpha + \beta)$

$\cos(\alpha + \beta) = cos\alpha cos\beta - sin\alpha sin\beta$



2. Prove by vector method that $\sin(\alpha - \beta) = \sin\alpha\cos\beta - \cos\alpha\sin\beta$.

$\overrightarrow{OP} = cos\alpha\vec{i} + sin\alpha\,\vec{j}$;

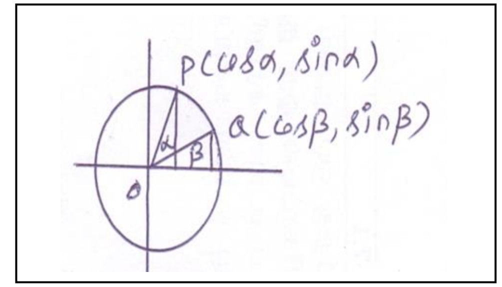
$\overrightarrow{OQ} = cos\beta\vec{i} + sin\beta\,\vec{j}$

$\overrightarrow{OQ} \times \overrightarrow{OP} = \sin(\alpha - \beta)\,\vec{k}$

$\overrightarrow{OQ} \times \overrightarrow{OP} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ cos\beta & sin\beta & 0 \\ cos\alpha & sin\alpha & 0 \end{vmatrix}$

$= (sin\alpha cos\beta - cos\alpha sin\beta)\vec{k}$

$\sin(\alpha - \beta) = sin\alpha cos\beta - cos\alpha sin\beta$



3. Using vector method, prove that $\cos(\alpha - \beta) = \cos\alpha\cos\beta + \sin\alpha\sin\beta$.

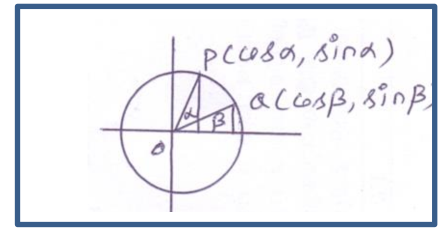
$.\overrightarrow{OP} = cos\alpha\vec{i} + sin\alpha\,\vec{j}$

$\overrightarrow{OQ} = cos\beta\vec{i} + sin\beta\,\vec{j}$

$\overrightarrow{OP}.\overrightarrow{OQ} = cos\alpha cos\beta + sin\alpha sin\beta$

$\overrightarrow{OP}.\overrightarrow{OQ} = \cos(\alpha - \beta)$

$\cos(\alpha - \beta) = cos\alpha cos\beta + sin\alpha sin\beta$



4.Prove by vector method that $\sin(\alpha + \beta) = \sin\alpha\cos\beta + \sin\alpha\cos\beta$.

$\overrightarrow{OP} = cos\alpha\vec{i} + sin\alpha\,\vec{j}$

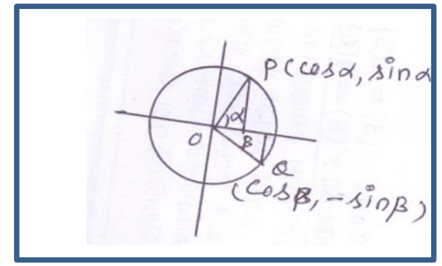
$\overrightarrow{OQ} = cos\beta\vec{i} - sin\beta\,\vec{j}$

$\overrightarrow{OQ} \times \overrightarrow{OP} = \sin(\alpha + \beta)\,\vec{k}$

$\overrightarrow{OQ} \times \overrightarrow{OP} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ cos\beta & -sin\beta & 0 \\ cos\alpha & sin\alpha & 0 \end{vmatrix}$

$= (sin\alpha cos\beta + cos\alpha sin\beta)\vec{k}$

$\sin(\alpha + \beta) = sin\alpha cos\beta + cos\alpha sin\beta$



5. Prove by vector method that the perpendiculars (alttitudes) from the vertices to the opposite sides of a triangle are concurrent.

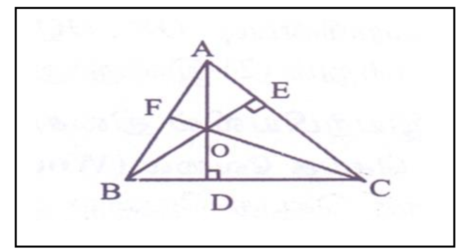
$\overrightarrow{OA} = \vec{a}, \overrightarrow{OB} = \vec{b}, \overrightarrow{OC} = \vec{c}$

$\overrightarrow{OA} \perp \overrightarrow{BC} \Rightarrow \vec{a}.\vec{c} - \vec{a}.\vec{b} = 0 \rightarrow (1)$

$\overrightarrow{OB} \perp \overrightarrow{CA} \Rightarrow \vec{b}.\vec{a} - \vec{b}.\vec{c} = 0 \rightarrow (2)$

$(1) + (2) \Rightarrow (\vec{a} - \vec{b}).\vec{c} = 0$

$\Rightarrow \overrightarrow{BA}.\overrightarrow{OC} = 0 \Rightarrow BA \perp CF$



6. If $\vec{a}=\hat{\imath}-\hat{\jmath},\ \vec{b}=\hat{\imath}-\hat{\jmath}-4\hat{k}, \vec{c}=3\hat{\jmath}-\hat{k}$ and $\vec{d}=2\hat{\imath}+5\hat{\jmath}+\hat{k}$, verify that $(\vec{a}\times\vec{b})\times(\vec{c}\times\vec{d})=[\vec{a},\vec{b},\vec{d}]\vec{c}-[\vec{a},\vec{b},\vec{c}]\vec{d}$

$$(\vec{a}\times\vec{b})=\begin{vmatrix}\vec{\imath} & \vec{\jmath} & \vec{k}\\ 1 & -1 & 0\\ 1 & -1 & -4\end{vmatrix}=\vec{\imath}(4-0)-\vec{\jmath}(-4-0)+\vec{k}(-1+1)=4\vec{\imath}+4\vec{\jmath}+0\vec{k}$$

$$(\vec{c}\times\vec{d})=\begin{vmatrix}\vec{\imath} & \vec{\jmath} & \vec{k}\\ 0 & 3 & -1\\ 2 & 5 & 1\end{vmatrix}=\vec{\imath}(3+5)-\vec{\jmath}(0+2)+\vec{k}(0-6)=8\vec{\imath}-2\vec{\jmath}-6\vec{k}$$

$$(\vec{a}\times\vec{b})\times(\vec{c}\times\vec{d})=\begin{vmatrix}\vec{\imath} & \vec{\jmath} & \vec{k}\\ 4 & 4 & 0\\ 8 & -2 & -6\end{vmatrix}= =-24\vec{\imath}+24\vec{\jmath}-40\vec{k}\to(1)$$

$$[\vec{a}\ \vec{b}\ \vec{d}]=\begin{vmatrix}1 & -1 & 0\\ 1 & -1 & -4\\ 2 & 5 & 1\end{vmatrix}=19+9+0=28$$

$$[\vec{a}\ \vec{b}\ \vec{d}]\vec{c}=28(3\vec{\jmath}-\vec{k})=84\vec{\jmath}-28\vec{k}$$

$$[\vec{a}\ \vec{b}\ \vec{c}]=\begin{vmatrix}1 & -1 & 0\\ 1 & -1 & -4\\ 0 & 3 & -1\end{vmatrix}=13-1+0=12$$

$$[\vec{a}\ \vec{b}\ \vec{c}]\vec{d}=12(2\vec{\imath}+5\vec{\jmath}+\vec{k})=24\vec{\imath}+60\vec{\jmath}+12\vec{k}$$

$$[\vec{a}\ \vec{b}\ \vec{d}]\vec{c}-[\vec{a}\ \vec{b}\ \vec{c}]\vec{d}=84\vec{\jmath}-28\vec{k}-24\vec{\imath}-60\vec{\jmath}-12\vec{k}=-24\vec{\imath}+24\vec{\jmath}-40\vec{k}\to(2)$$

From (1), (2): $(\vec{a}\times\vec{b})\times(\vec{c}\times\vec{d})=[\vec{a}\ \vec{b}\ \vec{d}]\vec{c}-[\vec{a}\ \vec{b}\ \vec{c}]\vec{d}$

7. If $\vec{a}=2\hat{\imath}+3\hat{\jmath}-\hat{k},\ \vec{b}=3\hat{\imath}+5\hat{\jmath}+2\hat{k},\ \vec{c}=-\hat{\imath}-2\hat{\jmath}+3\hat{k}$, verify that

$$(\vec{a}\times\vec{b})\times\vec{c}=(\vec{a}\cdot\vec{c})\vec{b}-(\vec{b}\cdot\vec{c})\vec{a}$$

$$\vec{a}\times\vec{b}=\begin{vmatrix}\vec{\imath} & \vec{\jmath} & \vec{k}\\ 2 & 3 & -1\\ 3 & 5 & 2\end{vmatrix}=11\vec{\imath}-7\vec{\jmath}+\vec{k}$$

$$(\vec{a}\times\vec{b})\times\vec{c}=\begin{vmatrix}\vec{\imath} & \vec{\jmath} & \vec{k}\\ 11 & -7 & 1\\ -1 & -2 & 3\end{vmatrix}=-19\vec{\imath}-34\vec{\jmath}-29\vec{k}\to(1)$$

$$\vec{a}.\vec{c}=(2\vec{\imath}+3\vec{\jmath}-\vec{k}).(-\vec{\imath}-2\vec{\jmath}+3\vec{k})=-2-6-3=-11$$

$$(\vec{a}.\vec{c})\vec{b}=-11(3\vec{\imath}+5\vec{\jmath}+2\vec{k})=-33\vec{\imath}-55\vec{\jmath}-22\vec{k}$$

$$\vec{b}.\vec{c}=(3\vec{\imath}+5\vec{\jmath}+2\vec{k}).(-\vec{\imath}-2\vec{\jmath}+3\vec{k})=-3-10+6=-7$$

$$(\vec{b}.\vec{c})\vec{a}=-7(2\vec{\imath}+3\vec{\jmath}-\vec{k})=-14\vec{\imath}-21\vec{\jmath}+7\vec{k}$$

$$(\vec{a}.\vec{c})\vec{b}-(\vec{b}.\vec{c})\vec{a}=-33\vec{\imath}-55\vec{\jmath}-22\vec{k}+14\vec{\imath}+21\vec{\jmath}-7\vec{k}=-19\vec{\imath}-34\vec{\jmath}-29\vec{k}\to(2)$$

From (1), (2) $\Rightarrow(\vec{a}\times\vec{b})\times\vec{c}=(\vec{a}.\vec{c})\vec{b}-(\vec{b}.\vec{c})\vec{a}$

8. Find the vector equation in parametric form and Cartesian equations of a straight passing through the points $(-5,7,-4)$ and $(13,-5,2)$. Find the point where the straight line crosses the xy −plane.

$(x_1,y_1,z_1)=(-5,7,-4)$, $(x_2,y_2,z_2)=(13,-5,2)$

Equation of the straight line passing through two points is $\frac{x-x_1}{x_2-x_1}=\frac{y-y_1}{y_2-y_1}=\frac{z-z_1}{z_2-z_1}$

$$\Rightarrow\frac{x+5}{13+5}=\frac{y-7}{-5-7}=\frac{z+4}{2+4}$$

$$\Rightarrow\frac{x+5}{18}=\frac{y-7}{-12}=\frac{z+4}{6}\to(1)$$

Equation of xy plane is $z=0$

$$\therefore(1)\Rightarrow\frac{x+5}{18}=\frac{y-7}{-12}=\frac{0+4}{6}$$

$$\Rightarrow\frac{x+5}{18}=\frac{y-7}{-12}=\frac{2}{3}$$

$$\Rightarrow\frac{x+5}{18}=\frac{2}{3};\frac{y-7}{-12}=\frac{2}{3}$$

$$\Rightarrow x+5=12\,;y-7=-8$$

$$\Rightarrow x=7;y=-1$$

$\therefore$ *The required point* $=(7,-1,0)$

9. Find the points where the straight line passes through (6,7,4) and (8,4,9) cuts the xz and yz planes.

$(x_1,y_1,z_1)=(6,7,4)$, $(x_2,y_2,z_2)=(8,4,9)$

Equation of the straight line passing through two points is: $\frac{x-x_1}{x_2-x_1}=\frac{y-y_1}{y_2-y_1}=\frac{z-z_1}{z_2-z_1}$

$$\Rightarrow\frac{x-6}{8-6}=\frac{y-7}{4-7}=\frac{z-4}{9-4}$$

$$\Rightarrow\frac{x-6}{2}=\frac{y-7}{-3}=\frac{z-4}{5}\to(1)$$

Equation of xz plane is $y=0$

$$\therefore(1)\Rightarrow\frac{x-6}{2}=\frac{0-7}{-3}=\frac{z-4}{5}$$

$$\Rightarrow\frac{x-6}{2}=\frac{-7}{-3};\ \frac{z-4}{5}=\frac{-7}{-3}$$

$$\Rightarrow\frac{x-6}{2}=\frac{7}{3};\frac{z-4}{5}=\frac{7}{3}$$

$$\Rightarrow 3x-18=14\,;3z-12=35$$

$$\Rightarrow 3x=32;3z=47$$

$$\Rightarrow x=\frac{32}{3};z=\frac{47}{3}$$

$\therefore$ *The point at xz plane* $=\left(\frac{32}{3},0,\frac{47}{3}\right)$

Equation of yz plane is $x=0$

$$\therefore(1)\Rightarrow\frac{x-6}{2}=\frac{y-7}{-3}=\frac{z-4}{5}$$

$$\Rightarrow\frac{0-6}{2}=\frac{y-7}{-3}=\frac{z-4}{5}$$

$$\Rightarrow\frac{y-7}{-3}=-3;\ \frac{z-4}{5}=-3$$

$\Longrightarrow y = 16, z = -11$

$\therefore$ *The point at yz plane* $= (0,16,-11)$

10. Find the non parametric form of vector equation, and Cartesian equation of the plane passing through the point $(2,3,6)$ and parallel to the straight lines $\frac{x-1}{2} = \frac{y+1}{3} = \frac{z-3}{1}$ and $\frac{x+3}{2} = \frac{y-3}{-5} = \frac{z+1}{-3}$.

$Point\ \vec{a} = 2\vec{i} + 3\vec{j} + 6\vec{k}$

$Parallel\ vectors\ \vec{b} = 2\vec{i} + 3\vec{j} + \vec{k}, \vec{c} = 2\vec{i} - 5\vec{j} - 3\vec{k}$

$$\vec{b} \times \vec{c} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 2 & 3 & 1 \\ 2 & -5 & -3 \end{vmatrix} = -4\vec{i} + 8\vec{j} - 16\vec{k}$$

Non parametric form of vector equation $is\ (\vec{r} - \vec{a}).(\vec{b} \times \vec{c}) = 0$

$$\Longrightarrow [\vec{r} - (2\vec{i} + 3\vec{j} + 6\vec{k})](-4\vec{i} + 8\vec{j} - 16\vec{k}) = 0$$
$$\Longrightarrow \vec{r}.(-4\vec{i} + 8\vec{j} - 16\vec{k}) - (-8 + 24 - 96) = 0$$
$$\Longrightarrow \vec{r}.(-4\vec{i} + 8\vec{j} - 16\vec{k}) + 80 = 0$$
$$\div 4 \Longrightarrow \vec{r}.(-\vec{i} + 2\vec{j} - 4\vec{k}) + 20 = 0$$

Cartesian equation : $Let\ \vec{r} = x\vec{i} + y\vec{j} + z\vec{k}$

$$\vec{r}.(-\vec{i} + 2\vec{j} - 4\vec{k}) + 20 \Longrightarrow (x\vec{i} + y\vec{j} + z\vec{k}).(-\vec{i} + 2\vec{j} - 4\vec{k}) + 20 = 0$$
$$\Longrightarrow -x + 2y - 4z + 20 = 0$$
$$\Longrightarrow x - 2y + 4z - 20 = 0$$

11. Find the non-parametric form of vector equation, and Cartesian equation of the plane passing through the point $(0, 1, -5)$ and parallel to the straight lines $\vec{r} = (\hat{\imath} + 2\hat{\jmath} - 4\hat{k}) + s(2\hat{\imath} + 3\hat{\jmath} + 6\hat{k})$ and $\vec{r} = (\hat{\imath} - 3\hat{\jmath} + 5\hat{k}) + t(\hat{\imath} + \hat{\jmath} - \hat{k})$.

$Point\ \vec{a} = 0\vec{i} + \vec{j} - 5\vec{k}$

$Parallel\ vectors\ \vec{b} = 2\vec{i} + 3\vec{j} + 6\vec{k}, \vec{c} = \vec{i} + \vec{j} - \vec{k}$

$$\vec{b} \times \vec{c} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 2 & 3 & 6 \\ 1 & 1 & -1 \end{vmatrix} = -9\vec{i} + 8\vec{j} - \vec{k}$$

Non parametric form of vector equation is $(\vec{r} - \vec{a}).(\vec{b} \times \vec{c}) = 0$

$$\Longrightarrow [\vec{r} - (0\vec{i} + \vec{j} - 5\vec{k})](-9\vec{i} + 8\vec{j} - \vec{k}) = 0$$
$$\Longrightarrow \vec{r}.(-9\vec{i} + 8\vec{j} - \vec{k}) - (0 + 8 + 5) = 0$$
$$\Longrightarrow \vec{r}.(-9\vec{i} + 8\vec{j} - \vec{k}) - 13 = 0$$

Cartesian equation : $Let\ \vec{r} = x\vec{i} + y\vec{j} + z\vec{k}$

$$\vec{r}.(-9\vec{i} + 8\vec{j} - \vec{k}) - 13 = 0 \Longrightarrow (x\vec{i} + y\vec{j} + z\vec{k}).(-9\vec{i} + 8\vec{j} - \vec{k}) - 13 = 0$$
$$\Longrightarrow -9x + 8y - z - 13 = 0$$
$$\Longrightarrow 9x - 8y + z + 13 = 0$$

12. Find the non-parametric form of vector equation and Cartesian equation of the plane passing through the point $(1, -2, 4)$ and perpendicular to the plane $x + 2y - 3z = 11$ and parallel to the line $\frac{x+7}{3} = \frac{y+3}{-1} = \frac{z}{1}$.

$Point\ \vec{a} = \vec{i} - 2\vec{j} + 4\vec{k}$

$Parallel\ vectors\ \vec{b} = \vec{i} + 2\vec{j} - 3\vec{k}, \vec{c} = 3\vec{i} - \vec{j} + \vec{k}$

$$\vec{b} \times \vec{c} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 1 & 2 & -3 \\ 3 & -1 & 1 \end{vmatrix} = -\vec{i} - 10\vec{j} - 7\vec{k}$$

Non parametric form of vector equation $is\ (\vec{r} - \vec{a}).(\vec{b} \times \vec{c}) = 0$

$$\Longrightarrow [\vec{r} - (\vec{i} - 2\vec{j} + 4\vec{k})](-\vec{i} - 10\vec{j} - 7\vec{k}) = 0$$
$$\Longrightarrow \vec{r}.(-\vec{i} - 10\vec{j} - 7\vec{k}) - (-1 + 20 - 28) = 0$$
$$\Longrightarrow \vec{r}.(-\vec{i} - 10\vec{j} - 7\vec{k}) + 9 = 0$$

Cartesian equation : $Let\ \vec{r} = x\vec{i} + y\vec{j} + z\vec{k}$

$$\vec{r}.(-\vec{i} - 10\vec{j} - 7\vec{k}) + 9 = 0 \Longrightarrow (x\vec{i} + y\vec{j} + z\vec{k}).(-\vec{i} - 10\vec{j} - 7\vec{k}) + 9 = 0$$
$$\Longrightarrow -x - 10y - 7z + 9 = 0$$
$$\Longrightarrow x + 10y + 7z - 9 = 0$$

13.Find the parametric form of vector equation, and Cartesian equations of the plane containing the line $\vec{r} = (\hat{\imath} - \hat{\jmath} + 3\hat{k}) + t(2\hat{\imath} - \hat{\jmath} + 4\hat{k})$ and perpendicular to plane $\vec{r} \cdot (\hat{\imath} + 2\hat{\jmath} + \hat{k}) = 8$.

$Point\ \vec{a} = \vec{i} - \vec{j} + 3\vec{k}$

$Parallel\ vectors\ \vec{b} = 2\vec{i} - \vec{j} + 4\vec{k}, \vec{c} = \vec{i} + 2\vec{j} + \vec{k}$

Parametric form of vector equation $is\ \vec{r} = \vec{a} + s\vec{b} + t\vec{c}$

$$\Longrightarrow \vec{r} = (\vec{i} - \vec{j} + 3\vec{k}) + s(2\vec{i} - \vec{j} + 4\vec{k}) + t(\vec{i} + 2\vec{j} + \vec{k}); s, t \in R$$

Cartesian equation is $\begin{vmatrix} x - x_1 & y - y_1 & z - z_1 \\ b_1 & b_1 & b_1 \\ c_2 & c_2 & c_2 \end{vmatrix} = 0$

$$\Longrightarrow \begin{vmatrix} x - 1 & y + 1 & z - 3 \\ 2 & -1 & 4 \\ 1 & 2 & 1 \end{vmatrix} = 0$$
$$\Longrightarrow (x - 1)(-1 - 8) - (y + 1)(2 - 4) + (z - 3)(4 + 1) = 0$$
$$\Longrightarrow -9(x - 1) + 2(y + 1) + 5(z - 3) = 0$$
$$\Longrightarrow -9x + 9 + 2y + 2 + 5z - 15 = 0$$
$$\Longrightarrow -9x + 2y + 5z - 4 = 0$$
$$\Longrightarrow 9x - 2y - 5z + 4 = 0$$

14. Find the non-parametric form of vector equation, and Cartesian equations of the plane $\vec{r} = (6\hat{\imath} - \hat{\jmath} + \hat{k}) + s(-\hat{\imath} + 2\hat{\jmath} + \hat{k}) + t(-5\hat{\imath} - 4\hat{\jmath} - 5\hat{k})$.

$Point\ \vec{a} = 6\vec{i} - \vec{j} + \vec{k}$

$Parallel\ vectors\ \vec{b} = -\vec{i} + 2\vec{j} + \vec{k}\ , \vec{c} = \overrightarrow{-5i} - 4\vec{j} - 5\vec{k}$

$$\vec{b} \times \vec{c} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ -1 & 2 & 1 \\ -5 & -4 & -5 \end{vmatrix} = -6\vec{i} - 10\vec{j} + 14\vec{k}$$

Non parametric form of vector equation is $(\vec{r} - \vec{a}) . (\vec{b} \times \vec{c}) = 0$

$$\Rightarrow [\vec{r} - (6\vec{i} - \vec{j} + \vec{k})](-6\vec{i} - 10\vec{j} + 14\vec{k}) = 0$$

$$\Rightarrow \vec{r}.(-6\vec{i} - 10\vec{j} + 14\vec{k}) - (-36 + 10 + 14) = 0$$

$$\Rightarrow \vec{r}.(-6\vec{i} - 10\vec{j} + 14\vec{k}) + 12 = 0$$

$$\div -2 \Rightarrow \vec{r}.(3\vec{i} + 5\vec{j} - 7\vec{k}) - 6 = 0$$

Cartesian equation : $Let\ \ \vec{r} = x\vec{i} + y\vec{j} + z\vec{k}$

$\vec{r}.(3\vec{i} + 5\vec{j} - 7\vec{k}) - 6 = 0 \Rightarrow (x\vec{i} + y\vec{j} + z\vec{k}).(3\vec{i} + 5\vec{j} - 7\vec{k}) - 6 = 0$

$$\Rightarrow 3x + 5y - 7z - 6 = 0$$

15 Find parametric form of vector equation and Cartesian equations of the plane passing through the points $(2, 2, 1), (9,3,6)$ and and perpendicular to the plane $2x + 6y + 6z = 9$.

$Points\ \vec{a} = 2\vec{i} + 2\vec{j} + \vec{k}, \vec{b} = 9\vec{i} + 3\vec{j} + 6\vec{k}$

$Parallel\ vector\ \vec{c} = 2\vec{i} + 6\vec{j} + 6\vec{k}$

$\vec{b} - \vec{a} = 9\vec{i} + 3\vec{j} + 6\vec{k} - 2\vec{i} - 2\vec{j} - \vec{k} = 7\vec{i} + \vec{j} + 5\vec{k}$

Parametric form of vector equation is $\vec{r} = \vec{a} + s(\vec{b} - \vec{a}) + t\vec{c}$

$$\Rightarrow \vec{r} = (2\vec{i} + 2\vec{j} + \vec{k}) + s(7\vec{i} + \vec{j} + 5\vec{k}) + t(2\vec{i} + 6\vec{j} + 6\vec{k})$$

Cartesian equation is $\begin{vmatrix} x - x_1 & y - y_1 & z - z_1 \\ x_2 - x_1 & y_2 - y_1 & z_2 - z_1 \\ c_1 & c_1 & c_1 \end{vmatrix} = 0$

$$\Rightarrow \begin{vmatrix} x - 2 & y - 2 & z - 1 \\ 7 & 1 & 5 \\ 2 & 6 & 6 \end{vmatrix} = 0$$

$$\Rightarrow (x - 2)(6 - 30) - (y - 2)(42 - 10) + (z - 1)(42 - 2) = 0$$

$$\Rightarrow -24(x - 2) - 32(y - 2) + 40(z - 1) = 0$$

$$\Rightarrow -24x + 48 - 32y + 64 + 40z - 40 = 0$$

$$\Rightarrow -24x - 32y + 40z + 72 = 0$$

$$\div (-8) \Rightarrow 3x + 4y - 5z - 9 = 0$$

16 Find the vector parametric, vector non-parametric and Cartesian form of the equation of the plane passing through the points $(-1, 2, 0), (2, 2, -1)$ and parallel to the straight line $\frac{x-1}{1} = \frac{2y+1}{2} = \frac{z+1}{-1}$.

$Points\ \ \vec{a} = -\vec{i} + 2\vec{j} + 0\vec{k},\ \ \vec{b} = 2\vec{i} + 2\vec{j} - \vec{k}$

$\frac{x-1}{1} = \frac{2y+1}{2} = \frac{z+1}{-1} \Rightarrow \frac{x-1}{1} = \frac{y+1/2}{1} = \frac{z+1}{-1}$

$Parallel\ vector\ \vec{c} = \vec{i} + \vec{j} - \vec{k}$

$\vec{b} - \vec{a} = 2\vec{i} + 2\vec{j} - \vec{k} + \vec{i} - 2\vec{j} - 0\vec{k} = 3\vec{i} + 0\vec{j} - \vec{k}$

Parametric form of vector equation is $\vec{r} = \vec{a} + s(\vec{b} - \vec{a}) + t\vec{c}$

$$\Rightarrow \vec{r} = (-\vec{i} + 2\vec{j} + 0\vec{k}) + s(3\vec{i} + 0\vec{j} - \vec{k}) + t(\vec{i} + \vec{j} - \vec{k})$$

$$(\vec{b} - \vec{a}) \times \vec{c} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 3 & 0 & -1 \\ 1 & 1 & -1 \end{vmatrix} = \vec{i}(0 + 1) - \vec{j}(-3 + 1) + \vec{k}(3 - 0) = \vec{i} + 2\vec{j} + 3\vec{k}$$

Non parametric form of vector equation is $(\vec{r} - \vec{a}) . ((\vec{b} - \vec{a}) \times \vec{c}) = 0$

$$\Rightarrow (\vec{r} - (-\vec{i} + 2\vec{j} + 0\vec{k})) . (\vec{i} + 2\vec{j} + 3\vec{k}) = 0$$

$$\Rightarrow \vec{r}.(\vec{i} + 2\vec{j} + 3\vec{k}) - (-1 + 4 + 0) = 0$$

$$\Rightarrow \vec{r}.(\vec{i} + 2\vec{j} + 3\vec{k}) - 3 = 0$$

Cartesian equation: $Let\ \ \vec{r} = x\vec{i} + y\vec{j} + z\vec{k}$

$\vec{r}.(\vec{i} + 2\vec{j} + 3\vec{k}) - 3 \Rightarrow (x\vec{i} + y\vec{j} + z\vec{k}).(\vec{i} + 2\vec{j} + 3\vec{k}) - 3 = 0$

$$\Rightarrow x + 2y + 3z - 3 = 0$$

17.Find parametric form of vector equation and Cartesian equations of the plane passing through the points $(2, 2, 1), (1, -2, 3)$ and parallel to the straight line passing through the points $(2, 1, -3)$ and $(-1, 5, -8)$.

$Points\ \ \vec{a} = 2\vec{i} + 2\vec{j} + \vec{k}, \vec{b} = \vec{i} - 2\vec{j} + 3\vec{k}$

$Parallel\ vector\ \vec{c} = (2,1,-3) - (-1,5,-8) = 3\vec{i} - 4\vec{j} + 5\vec{k}$

$\vec{b} - \vec{a} = \vec{i} - 2\vec{j} + 3\vec{k} - 2\vec{i} - 2\vec{j} - \vec{k} = -\vec{i} - 4\vec{j} + 2\vec{k}$

Parametric form of vector equation is $\vec{r} = \vec{a} + s(\vec{b} - \vec{a}) + t\vec{c}$

$$\Rightarrow \vec{r} = (2\vec{i} + 2\vec{j} + \vec{k}) + s(-\vec{i} - 4\vec{j} + 2\vec{k}) + t(3\vec{i} - 4\vec{j} + 5\vec{k})$$

Cartesian equation is $\begin{vmatrix} x - x_1 & y - y_1 & z - z_1 \\ x_2 - x_1 & y_2 - y_1 & z_2 - z_1 \\ c_1 & c_1 & c_1 \end{vmatrix} = 0$

$$\Rightarrow \begin{vmatrix} x - 2 & y - 2 & z - 1 \\ -1 & -4 & 2 \\ 3 & -4 & 5 \end{vmatrix} = 0$$

$$\Rightarrow (x - 2)(-20 + 8) - (y - 2)(-5 - 6) + (z - 1)(4 + 12) = 0$$

$$\Rightarrow -12(x - 2) + 11(y - 2) + 16(z - 1) = 0$$

$$\Rightarrow -12x + 24 + 11y - 22 + 16z - 16 = 0$$

$$\Rightarrow -12x + 11y + 16z - 14 = 0$$

$$\Rightarrow 12x - 11y - 16z + 14 = 0$$

18.Find the parametric vector, non-parametric vector and Cartesian form of the equations of the plane passing through the non-collinear points $(3, 6, -2), (-1, -2, 6)$, and $(6, -4, -2)$.

$Points\ \ \vec{a} = 3\vec{i} + 6\vec{j} - 2\vec{k}, \vec{b} = -\vec{i} - 2\vec{j} + 6\vec{k};\quad \vec{c} = 6\vec{i} - 4\vec{j} - 2\vec{k}$

$\vec{b}-\vec{a}=-\vec{\imath}-2\vec{\jmath}+6\vec{k}-3\vec{\imath}-6\vec{\jmath}+2\vec{k}==-4\vec{\imath}-8\vec{\jmath}+8\vec{k}$

$\vec{c}-\vec{a}=6\vec{\imath}-4\vec{\jmath}-2\vec{k}-3\vec{\imath}-6\vec{\jmath}+2\vec{k}=3\vec{\imath}-10\vec{\jmath}+0\vec{k}$

Parametric form of vector equation is $\vec{r}=\vec{a}+s(\vec{b}-\vec{a})+t(\vec{c}-\vec{a})$

$$\Rightarrow \vec{r}=(3\vec{\imath}+6\vec{\jmath}-2\vec{k})+s(-4\vec{\imath}-8\vec{\jmath}+8\vec{k})+t(3\vec{\imath}-10\vec{\jmath}+0\vec{k})$$

$$(\vec{b}-\vec{a})\times(\vec{c}-\vec{a})=\begin{vmatrix}\vec{\imath} & \vec{\jmath} & \vec{k}\\ -4 & -8 & 8\\ 3 & -10 & 0\end{vmatrix}=80\vec{\imath}+24\vec{\jmath}+64\vec{k}$$

$$\div 8, =10\vec{\imath}+3\vec{\jmath}+8\vec{k}$$

Non parametric form of vector equation is $(\vec{r}-\vec{a}).\left((\vec{b}-\vec{a})\times(\vec{c}-\vec{a})\right)=0$

$$\Rightarrow \left(\vec{r}-(3\vec{\imath}+6\vec{\jmath}-2\vec{k})\right).(10\vec{\imath}+3\vec{\jmath}+8\vec{k})=0$$

$$\Rightarrow \vec{r}.(10\vec{\imath}+3\vec{\jmath}+8\vec{k})-(30+18-16)=0$$

$$\Rightarrow \vec{r}.(10\vec{\imath}+3\vec{\jmath}+8\vec{k})-32=0$$

Cartesian equation: Let $\vec{r}=x\vec{\imath}+y\vec{\jmath}+z\vec{k}$

$\vec{r}.(10\vec{\imath}+3\vec{\jmath}+8\vec{k})-32=0 \Rightarrow (x\vec{\imath}+y\vec{\jmath}+z\vec{k}).(10\vec{\imath}+3\vec{\jmath}+8\vec{k})-32=0$

$$\Rightarrow 10x+3y+8z-32=0$$

19. If the straight lines $\frac{x-1}{1}=\frac{y-2}{2}=\frac{z-3}{m^2}$ and $\frac{x-3}{1}=\frac{y-2}{m^2}=\frac{z-1}{2}$ are coplanar, find the distinct real values of m.

$\frac{x-1}{1}=\frac{y-2}{2}=\frac{z-3}{m^2} \Rightarrow \begin{matrix}\vec{a}=\vec{\imath}+2\vec{\jmath}+3\vec{k}\\ \vec{b}=\vec{\imath}+2\vec{\jmath}+m^2\vec{k}\end{matrix}$ $\frac{x-3}{1}=\frac{y-2}{m^2}=\frac{z-1}{2} \Rightarrow \begin{matrix}\vec{c}=3\vec{\imath}+2\vec{\jmath}+\vec{k}\\ \vec{d}=\vec{\imath}+m^2\vec{\jmath}+2\vec{k}\end{matrix}$

$\vec{c}-\vec{a}=3\vec{\imath}+2\vec{\jmath}+\vec{k}-\vec{\imath}-2\vec{\jmath}-3\vec{k}=2\vec{\imath}+0\vec{\jmath}-2\vec{k}$

$$(\vec{c}-\vec{a}).(\vec{b}\times\vec{d})=0 \Rightarrow \begin{vmatrix}2 & 0 & -2\\ 1 & 2 & m^2\\ 1 & m^2 & 2\end{vmatrix}=0$$

$$\Rightarrow 2(4-m^4)-0-2(m^2-2)=0$$

$$\Rightarrow 8-2m^4-2m^2+4=0$$

$$\Rightarrow 2m^4+2m^2-12=0$$

$$\div 2 \Rightarrow m^4+m^2-6=0$$

$$\Rightarrow (m^2-2)(m^2+3)=0$$

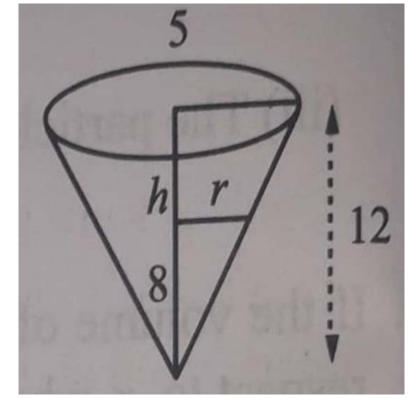
$$\Rightarrow m^2=2\ ;\ m^2=-3$$

$$\Rightarrow m=\pm\sqrt{2}\in R\ ;\ m=\pm\sqrt{3}i\notin R$$

7. APPLICATIONS OF DIFFERENTIAL CALCULUS

5 MARK QUESTIONS

1. A conical water tank with vertex down of 12 meters height has a radius of 5 meters at the top. If water flows into the tank at a rate 10 cubic m/min, how fast is the depth of the water increases when the water is 8 meters deep?



$clearly, \frac{r}{h}=\frac{5}{12} \Rightarrow r=\frac{5h}{12}$

$From\ the\ given\ data, \frac{dv}{dt}=10cu.m/min$

$To\ find: when\ h=8m, \frac{dh}{dt}=?$

$Volume\ of\ the\ cone\ v=\frac{1}{3}\pi r^2h=\frac{1}{3}\pi\left(\frac{5h}{12}\right)^2h$

$$\Rightarrow v=\frac{25\pi}{3\times144}h^3$$

$D.w.r.to\ t\ on\ both\ sides, \Rightarrow \frac{dv}{dt}=\frac{25\pi}{3\times144}(3h^2)\frac{dh}{dt}$

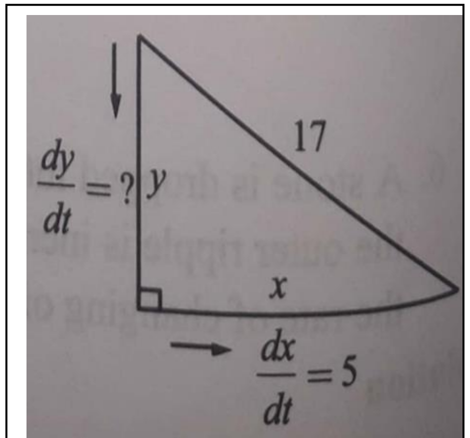
$$\Rightarrow 10=\frac{25\pi}{3\times144}[3(8)^2]\frac{dh}{dt}$$

$$\Rightarrow \frac{dh}{dt}=\frac{10\times3\times144}{25\pi\times3\times64}$$

$$\Rightarrow \frac{dh}{dt}=\frac{9}{10\pi}m/min$$

2. A ladder 17 meter long is against the wall. The base of the ladder is pulled away from the wall at a rate of 5m/s. When the base of the ladder is 8 meters from the wall, (i) how fast is the top of the ladder moving down the wall? (ii) at what rate, the area of the triangle formed by the ladder, wall, and the floor, is changing?

$(i) From\ the\ given\ data, x=8, \frac{dx}{dt}=5$ $To\ find: \frac{dy}{dt}=?$



$From\ the\ fig., x^2+y^2=17^2 \Rightarrow x^2+y^2=289$

$$\Rightarrow 8^2+y^2=289$$

$$\Rightarrow y^2=289-64$$

$$\Rightarrow y^2=225$$

$$\Rightarrow y=15$$

$x^2+y^2=289$

$D.w.r.to\ t\ on\ both\ sides,$

$\Rightarrow 2x\frac{dx}{dt}+2y\frac{dy}{dt}=0$

$\Rightarrow x\frac{dx}{dt}+y\frac{dy}{dt}=0$

$\Rightarrow 8(5)+15\frac{dy}{dt}=0$

$\Rightarrow 15\frac{dy}{dt}=-40$

$\Rightarrow \frac{dy}{dt}=-\frac{40}{15}=-\frac{8}{3}m/sec$

$\therefore$ *Height is decreasing at the rate of* $\frac{8}{3}\ m/sec$

(ii) *Area of the triangle*, $A = \frac{1}{2}xy$

$D.w.r.to\ t\ on\ both\ sides, \Longrightarrow \frac{dA}{dt} = \frac{1}{2}\left[x\frac{dy}{dt} + y\frac{dx}{dt}\right]$

$= \frac{1}{2}\left[8\left(-\frac{8}{3}\right) + 15(5)\right]$

$= \frac{1}{2}\left[-\frac{64}{3} + 75\right]$

$= \frac{1}{2}\left[\frac{-64+225}{3}\right]$

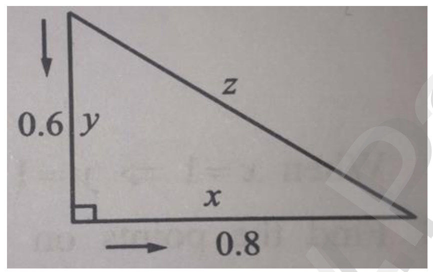
$= \frac{1}{2}\left[\frac{161}{3}\right]$

$= \frac{161}{6}$

$= 26.83\ m^2/sec.$

3. A police jeep, approaching an orthogonal intersection from the northern direction, is chasing a speeding car that has turned and moving straight east. When the jeep is 0.6 km north of the intersection and the car is 0.8 km to the east. The police determine with a radar that the distance between them and the car is increasing at 20 km/hr. If the jeep is moving at 60km/hr at the instant of measurement, what is the speed of the car?

$From\ the\ given\ data, x = 0.8, y = 0.6, \frac{dy}{dt} = -60, \frac{dz}{dt} = 20, \frac{dx}{dt} = ?$



$From\ the\ fig., x^2 + y^2 = z^2 \Longrightarrow 0.8^2 + 0.6^2 = z^2$

$\Longrightarrow z = 1$

$x^2 + y^2 = z^2$

$D.w.r.to\ t\ on\ both\ sides \Longrightarrow 2x\frac{dx}{dt} + 2y\frac{dy}{dt} = 2z\frac{dz}{dt}$

$\Longrightarrow x\frac{dx}{dt} + y\frac{dy}{dt} = z\frac{dz}{dt}$

$\Longrightarrow 0.8\frac{dx}{dt} + 0.6\,(-60) = 1(20)$

$\Longrightarrow 0.8\frac{dx}{dt} - 36 = 20$

$\Longrightarrow 0.8\frac{dx}{dt} = 56$

$\Longrightarrow \frac{dx}{dt} = \frac{56}{0.8}$

$\Longrightarrow \frac{dx}{dt} = 70km/hr$

4. If we blow air into a balloon of spherical shape at a rate of $1000cm^3$ per second, at what rate the radius of the balloon changes when the radius is 7cm? Also compute the rate at which the surface area changes.

$From\ the\ given\ data, \frac{dv}{dt} = 1000, r = 7, \frac{dr}{dt} = ?$

$Volume\ of\ the\ sphere, v = \frac{4}{3}\pi r^3$

$D.w.r.to\ t\ on\ both\ sides, \Longrightarrow \frac{dv}{dt} = \frac{4}{3}\pi\left(3r^2.\frac{dr}{dt}\right)$

$\Longrightarrow 1000 = \frac{4}{3}\pi.3.(7)^7.\frac{dr}{dt}$

$\Longrightarrow \frac{dr}{dt} = \frac{1000\times3}{4.\pi.3.7.7}$

$\Longrightarrow \frac{dr}{dt} = \frac{250}{49\pi}\ cm/sec$

$Surface\ area\ of\ the\ sphere,\ s = 4\pi r^2$

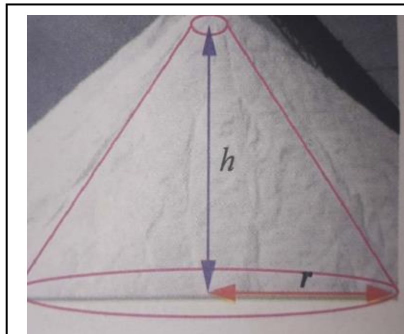
$D.w.r.to\ t\ on\ both\ sides, \Longrightarrow \frac{ds}{dt} = 4\pi\left(2r.\frac{dr}{dt}\right)$

$\Longrightarrow \frac{ds}{dt} = 4\pi.2.7.\frac{250}{49\pi}$

$\Longrightarrow \frac{ds}{dt} = \frac{2000}{7}\ cm^2/sec$

5. Salt is poured from a conveyer belt at a rate of 30 cubic meter per minute forming a conical pile with a circular base whose height and diameter of base are always equal. How fast is the height of the pile increasing when the pile is 10 metre high?

$Given, d = h \Longrightarrow 2r = h \Longrightarrow r = \frac{h}{2}$



$From\ the\ given\ data, \frac{dv}{dt} = 30\ m^3/min$

$To\ find: when\ h = 10m,\ \frac{dh}{dt} = ?$

$Volume\ of\ the\ cone, v = \frac{1}{3}\pi r^2 h$

$= \frac{1}{3}\pi\left(\frac{h}{2}\right)^2 h$

$\Longrightarrow v = \frac{\pi}{12}h^3$

$D.w.r.to\ t\ on\ both\ sides, \Longrightarrow \frac{dv}{dt} = \frac{\pi}{12}(3h^2)\frac{dh}{dt}$

$\Longrightarrow 30 = \frac{\pi}{4}[(10)^2]\frac{dh}{dt}$

$\Longrightarrow \frac{dh}{dt} = \frac{30\times4}{\pi\times100}$

$\Longrightarrow \frac{dh}{dt} = \frac{6}{5\pi}\ m/min$

6. A road running north to south crosses a road going east to west at the point P. Car A is driving north along the first road,and car B is driving east along the second road. At a particular time car A is 10 kilometers to the north of P and travelling at 80km/hr, while car B is 15 kilometers to the east of P and travelling at 100km/hr. How fast is the distance between the two cars changing?

From the given data, $x = 15, y = 10, \frac{dx}{dt} = 100, \frac{dy}{dt} = 80, \frac{dz}{dt} = ?$

From the fig., $x^2 + y^2 = z^2$

$$\Rightarrow 15^2 + 10^2 = z^2$$
$$\Rightarrow z^2 = 225 + 100$$
$$\Rightarrow z = \sqrt{325} = \sqrt{25 \times 13} = 5\sqrt{13}$$
$$x^2 + y^2 = z^2$$

D.w.r.to t on both sides $\Rightarrow 2x\frac{dx}{dt} + 2y\frac{dy}{dt} = 2z\frac{dz}{dt}$

$$\Rightarrow x\frac{dx}{dt} + y\frac{dy}{dt} = z\frac{dz}{dt}$$
$$\Rightarrow 15(100) + 10\,(80) = 5\sqrt{13}\frac{dz}{dt}$$
$$\Rightarrow \frac{dz}{dt} = \frac{2300}{5\sqrt{13}}$$
$$\Rightarrow \frac{dz}{dt} = \frac{460}{\sqrt{13}}$$
$$\Rightarrow \frac{dz}{dt} \approx 127.6\ km/hr$$

7.Find the equations of the tangents to the curve $y = 1 + x^3$ for which the tangent is orthogonal with the line $x + 12y = 12$.

Let the point of contact be (x_1, y_1)

$y = 1 + x^3 \Rightarrow y_1 = 1 + {x_1}^3 \rightarrow (1)$

slope of $x + 12y = 12$ *is* $= -\frac{1}{12}$

$\therefore$*Slope of the tangent,* $m = 12 \rightarrow (2) \quad (\because \perp)$

Diff. $y = 1 + x^3$ *w.r.to x on both sides,* $\frac{dy}{dx} = 3x^2$

$$\Rightarrow \frac{dy}{dx}_{(x_1,y_1)} = 3{x_1}^2 \rightarrow (3)$$

From (2) & (3), $3{x_1}^2 = 12 \Rightarrow {x_1}^2 = 4 \Rightarrow \boxed{x_1 = \pm 2}$

Sub. $x_1 = \pm 2$ *in eqn* (1),

$x_1 = 2 \Rightarrow y_1 = 1 + 2^3 = 9$

$x_1 = -2 \Rightarrow y_1 = 1 + (-2)^3 = 1 - 8 = -7$

$\therefore$ *The point of contacts are* (2,9) *and* (−2,−7).

Eqn of the tangent : $y - y_1 = m(x - x_1)$

Case(i): $(x_1, y_1) = (2,9), m = 12 \Rightarrow y - 9 = 12(x - 2) \Rightarrow 12x - y - 15 = 0$

case(ii): $(x_1, y_1) = (-2,-7), m = 12 \Rightarrow y + 7 = 12(x + 2) \Rightarrow 12x - y + 17 = 0$

8.Find the equations of the tangents to the curve $y = \frac{x+1}{x-1}$ which are parallel to the line $x + 2y = 6$.

Let the point of contact be (x_1, y_1)

$y = \frac{x+1}{x-1} \Rightarrow y_1 = \frac{x_1+1}{x_1-1} \rightarrow (1)$

Slope of $x + 2y = 6$ *is* $= -\frac{1}{2}$

Slope of the tangent, $m - \frac{1}{2} \rightarrow (2) \quad (\because \|)$

Diff. $y = \frac{x+1}{x-1}$ *w.r.to x on both sides,* $\frac{dy}{dx} = \frac{x-1-x-1}{(x-1)^2}$

$$\Rightarrow \frac{dy}{dx}_{(x_1,y_1)} = -\frac{2}{(x_1-1)^2} \rightarrow (3)$$

From (2) & (3), $-\frac{2}{(x_1-1)^2} = -\frac{1}{2} \Rightarrow (x_1 - 1)^2 = 4 \Rightarrow x_1 - 1 = \pm 2$

$\Rightarrow x_1 = -1, 3$

Sub. $x_1 = -1, 3$ *in eqn* (1),

$x_1 = -1 \Rightarrow y_1 = \frac{-1+1}{-1-1} = 0$

$x_1 = 3 \Rightarrow y_1 = \frac{3+1}{3-1} = 2$

$\therefore$ *The point of contacts are* (−1,0) *and* (3,2).

Eqn of the tangent: $y - y_1 = m(x - x_1)$

case (i): $(x_1, y_1) = (-1,0), m = -\frac{1}{2} \Rightarrow y - 0 = -\frac{1}{2}(x + 1) \Rightarrow x + 2y + 1 = 0$

case (ii): $(x_1, y_1) = (3,2), m = -\frac{1}{2} \Rightarrow y - 2 = -\frac{1}{2}(x - 3) \Rightarrow x + 2y - 7 = 0$

9..Find the equation of the tangent and normal to the curve given by $x = 7cost,\ y = 2sint, t \in \mathbb{R}$ at any point on the curve.

$x = 7cost, y = 2sint$

D.w.r.to t on both sides, $\frac{dx}{dt} = -7sint$; $\frac{dy}{dt} = 2cost$

Slope $m = \frac{dy}{dx} = \frac{\frac{dy}{dt}}{\frac{dx}{dt}} = -\frac{2cost}{7sint}$

$\therefore$ *The point of contact is* $(x_1, y_1) = (7cost, 2sint)$

Eqn of the tangent: $y - y_1 = m(x - x_1)$

$\Rightarrow y - 2sint = -\frac{2cost}{7sint}(x - 7cost)$

$\Rightarrow 7sinty - 14sin^2t = -2costx + 14cos^2t$

$\Rightarrow 2costx + 7sinty = 14(sin^2t + cos^2t)$

$\Rightarrow 2costx + 7sinty = 14$

Eqn of the normal : $7sintx - 2costy = k$

$(7cost, 2sint) \Rightarrow 7sint(7cost) - 2cost(2sint) = k$

$\Rightarrow 49sintcost - 4sintcost = k$

$\Rightarrow k = 45sintcost$

$\therefore$ *Eqn of the normal is* $7sintx - 2costy = 45sintcost$

10.Find the equation of the tangent and normal at any point to the Lissajous curve given by $x = 2cos3t$ and $y = 3sin2t$, $t \in \mathbb{R}$.

$x = 2cos3t \Rightarrow \frac{dx}{dt} = -6\sin 3t$

$y = 3sin2t \Rightarrow \frac{dy}{dt} = 6\cos 2t$

$slope\ m, \frac{dy}{dx} = \frac{\frac{dy}{dt}}{\frac{dx}{dt}} = -\frac{6\cos 2t}{6\sin 3t} = -\frac{\cos 2t}{\sin 3t}$

$Eqn\ of\ the\ tangent: y - y_1 = m(x - x_1)$

$\Rightarrow y - 3sin2t = -\frac{\cos 2t}{\sin 3t}(x - 2cos3t)$

$\Rightarrow x\cos 2t + y\sin 3t = 3sin2t\sin 3t + 2\cos 2t\, cos3t$

$Eqn\ of\ the\ normal: y - y_1 = -\frac{1}{m}(x - x_1)$

$\Rightarrow y - 3sin2t = \frac{\sin 3t}{\cos 2t}(x - 2cos3t)$

$\Rightarrow x\sin 3t - y\cos 2t = 2\sin 3t\, cos3t - 3sin2t\cos 2t$

11. Find the angle between the rectangular hyperbola $xy = 2$ and the parabola $x^2 + 4y = 0$.

$xy = 2 \Rightarrow y = \frac{2}{x} \to (1);\ \ x^2 + 4y = 0 \to (2)$

$Solvingthe\ \ eqns$ (1) & (2),

$(2) \Rightarrow x^2 + 4\left(\frac{2}{x}\right) = 0 \Rightarrow x^3 + 8 = 0 \Rightarrow x = -2$

$sub. x = -2\ in\ (1),\ y = \frac{2}{-2} = -1$

$\therefore The\ point\ of\ intersection = (-2, -1)$

$Diff.\ xy = 2\ w.r.to\ x\ on\ both\ sides,\ \ x\frac{dy}{dx} + y = 0 \Rightarrow \frac{dy}{dx} = -\frac{y}{x}$

$\Rightarrow m_1 = \frac{dy}{dx}_{((-2,-1))} = -\frac{-1}{-2}$

$\Rightarrow m_1 = -\frac{1}{2}$

$Diff. x^2 + 4y = 0\ w.r.to\ x\ on\ both\ sides, 2x + 4\frac{dy}{dx} = 0 \Rightarrow \frac{dy}{dx} = -\frac{4}{2x}$

$\Rightarrow m_2 = \frac{dy}{dx}_{((-2,-1))} = -\frac{4}{-4}$

$\Rightarrow m_2 = 1$

$Angle\ between\ the\ curves, \tan\theta = \left|\frac{m_1 - m_2}{1 + m_1 m_2}\right| = \left|\frac{-\frac{1}{2} - 1}{1 + \left(-\frac{1}{2}\right)(1)}\right| = \left|\frac{-\frac{3}{2}}{1 - \frac{1}{2}}\right| = 3$

$\Rightarrow \theta = tan^{-1}(3)$

12.Find the angle between $y = x^2$ and $y = (x-3)^2$.

$solving\ the\ given\ eqns \Rightarrow x^2 = (x-3)^2 \Rightarrow x = \frac{3}{2}$

$sub.\ x = \frac{3}{2}\ in\ y = x^2 \Rightarrow y = \frac{9}{4}$

$\therefore The\ point\ of\ intersection = \left(\frac{3}{2}, \frac{9}{4}\right)$

$y = x^2 \Rightarrow \frac{dy}{dx} = 2x \Rightarrow m_1 = \frac{dy}{dx}_{\left(\frac{3}{2},\frac{9}{4}\right)} = 2\left(\frac{3}{2}\right) = 3$

$y = (x-3)^2 \Rightarrow \frac{dy}{dx} = 2(x-3) \Rightarrow m_2 = \frac{dy}{dx}_{\left(\frac{3}{2},\frac{9}{4}\right)} = 2\left(\frac{3}{2} - 3\right) = -3$

$Angle\ between\ the\ curves,\ \tan\theta = \left|\frac{m_1 - m_2}{1 + m_1 m_2}\right| = \left|\frac{3+3}{1+(-3)(3)}\right| = \left|\frac{6}{-8}\right| = \frac{3}{4}$

$\Rightarrow \theta = tan^{-1}(\frac{3}{4})$

13.Find the angle between the curves $y = x^2$ and $x = y^2$ at their points of intersection (0,0) and (1,1).

$y = x^2 \Rightarrow \frac{dy}{dx} = 2x \Rightarrow m_1 = 2x$

$x = y^2 \Rightarrow 2y\frac{dy}{dx} = 1 \Rightarrow m_2 = \frac{1}{2y}$

$Angle\ between\ the\ curves,\ \tan\theta = \left|\frac{m_1 - m_2}{1 + m_1 m_2}\right| = \left|\frac{2x - \frac{1}{2y}}{1 + (2x)\left(\frac{1}{2y}\right)}\right|$

$(0,0) \Rightarrow\ \ \tan\theta = \left|\frac{0}{1}\right| = \infty \Rightarrow \theta = \frac{\pi}{2}$

$(1,1) \Rightarrow\ \ \tan\theta = \left|\frac{2 - \frac{1}{2}}{1 + 2\left(\frac{1}{2}\right)}\right| = \frac{3}{4} \Rightarrow \theta = tan^{-1}(\frac{3}{4})$

14.Show that the two curves $x^2 - y^2 = r^2$ and $xy = c^2$ where c, r are constants, cut orthogonally.

$Let\ the\ point\ of\ intersection\ be\ (x_1, y_1)$

$x^2 - y^2 = r^2 \Rightarrow {x_1}^2 - {y_1}^2 = r^2 \to (1)$

$xy = c^2 \Rightarrow x_1 y_1 = c^2 \to (2)$

$Diff.\ x^2 - y^2 = r^2\ w.r.to\ x\ on\ both\ sides, 2x - 2y\frac{dy}{dx} = 0$

$\Rightarrow \frac{dy}{dx} = \frac{x}{y}$

$\Rightarrow m_1 = \frac{dy}{dx}_{(x_1, y_1)} = \frac{x_1}{y_1}$

$Diff.\ xy = c^2 w.r.to\ x\ on\ both\ sides, x\frac{dy}{dx} + y = 0$

$\Rightarrow \frac{dy}{dx} = -\frac{y}{x}$

$\Rightarrow m_2 = \frac{dy}{dx}_{(x_1,y_1)} = -\frac{y_1}{x_1}$

$Product\ of\ the\ slopes, m_1.m_2 = \frac{x_1}{y_1} \times -\frac{y_1}{x_1} = -1$

∴The given curves cut orthogonally.

15.If the curves $ax^2 + by^2 = 1$ and $cx^2 + dy^2 = 1$ intersect each other orthogonally then, show that $\frac{1}{a} - \frac{1}{b} = \frac{1}{c} - \frac{1}{d}$.

$Let\ the\ point\ of\ intersection\ be\ (x_1, y_1)$

$ax^2 + by^2 = 1 \Rightarrow 2ax + 2by\frac{dy}{dx} = 0 \Rightarrow \frac{dy}{dx} = -\frac{ax}{by} \Rightarrow m_1 = -\frac{ax_1}{by_1}$

$cx^2 + dy^2 = 1 \Rightarrow 2cx + 2dy\frac{dy}{dx} = 0 \Rightarrow \frac{dy}{dx} = -\frac{cx}{dy} \Rightarrow m_2 = -\frac{cx_1}{dy_1}$

$m_1.m_2 = -1 \Rightarrow -\frac{ax_1}{by_1} \times -\frac{cx_1}{dy_1} = -1 \Rightarrow acx_1{}^2 + bdy_1{}^2 = 0 \rightarrow (1)$

$Also, (a-c)x_1{}^2 + (b-d)y_1{}^2 = 0 \rightarrow (2)$

$From\ (1)\ \&\ (2),\ \frac{a-c}{ac} = \frac{b-d}{bd} \Rightarrow \frac{1}{a} - \frac{1}{b} = \frac{1}{c} - \frac{1}{d}$

16.Prove that the ellipse $x^2 + 4y^2 = 8$ and the hyperbola $x^2 - 2y^2 = 4$ intersect orthogonally.

$Let\ the\ point\ of\ intersection\ be\ (a, b)$

$\left.\begin{array}{l} x^2 + 4y^2 = 8 \Rightarrow a^2 + 4b^2 = 8 \\ x^2 - 2y^2 = 4 \Rightarrow a^2 - 2b^2 = 4 \end{array}\right\} \rightarrow (1)$

$x^2 + 4y^2 = 8 \Rightarrow 2x + 8y\frac{dy}{dx} = 0 \Rightarrow \frac{dy}{dx} = -\frac{x}{4y} \Rightarrow m_1 = -\frac{a}{4b}$

$x^2 - 2y^2 = 4 \Rightarrow 2x - 4y\frac{dy}{dx} = 0 \Rightarrow \frac{dy}{dx} = \frac{x}{2y} \Rightarrow m_1 = \frac{a}{2b}$

$m_1.m_2 = -\frac{a}{4b}\left(\frac{a}{2b}\right) = -\frac{a^2}{8b^2} \rightarrow (2)$

$By\ proportionality\ (1) \Rightarrow \frac{a^2}{-16-16} = \frac{b^2}{-8+4} \Rightarrow \frac{a^2}{b^2} = 8$

$(2) \Rightarrow m_1.m_2 = -\frac{1}{8}(8) = -1$

∴The given curves cut orthogonally.

17. A garden is to be laid out in a rectangular area and protected by wire fence. What is the largest possible area of the fenced garden with 40 meters of wire.

$Let\ the\ length\ and\ breadth\ of\ the\ rectangular\ garden\ be\ x\ and\ y,\ (x, y > 0)$

$Length\ of\ the\ wire = 40\ m \Rightarrow x + x + y + y = 40$

$\Rightarrow 2(x + y) = 40$

$\Rightarrow x + y = 20$

$\Rightarrow y = 20 - x \rightarrow (1)$

$Area\ of\ the\ rectangular\ garden,\ A = xy$

$\Rightarrow A(x) = x(20 - x)$

$\Rightarrow A(x) = 20x - x^2$

$\Rightarrow A'(x) = 20 - 2x$

$\Rightarrow A''(x) = -2$

$A'(x) = 0 \Rightarrow 20 - 2x = 0 \Rightarrow 2x = 20 \Rightarrow x = 10.$

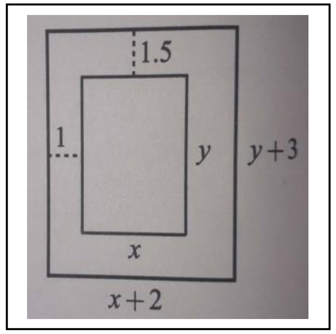
$When\ x = 10 \Rightarrow A''(x) = -2 < 0,\ hence\ area\ is\ max.$

$Sub.\ x = 6\ in\ (1),\ y = 20 - x = 20 - 10 = 10$

$Area\ of\ the\ garden,\ A = xy = 10 \times 10 = 100m^2.$

18. A rectangular page is to contain 24 cm^2 of print. The margins at the top and bottom of the page are 1.5 cm and the margins at other sides of the page is 1 cm. What should be the dimensions of the page so that the area of the paper used is minimum.

$Let\ the\ length\ and\ breadth\ of\ the\ printed\ area\ be\ x\ and\ y,\ (x, y > 0)$



$Printed\ area, xy = 24 \Rightarrow y = \frac{24}{x} \rightarrow (1)$

$Length\ of\ the\ page = x + 1 + 1 = x + 2$

$Breadth\ of\ the\ page = y + 1.5 + 1.5 = y + 3$

$Area\ of\ the\ paper, A = (x + 2)(y + 3) \Rightarrow A(x) = (x + 2)\left(\frac{24}{x} + 3\right)$

$\Rightarrow A(x) = 24 + 3x + \frac{48}{x} + 6$

$\Rightarrow A(x) = 3x + \frac{48}{x} + 30$

$\Rightarrow A'(x) = 3 - \frac{48}{x^2}$

$\Rightarrow A''(x) = \frac{96}{x^3}$

$A'(x) = 0 \Rightarrow 3 - \frac{48}{x^2} = 0 \Rightarrow 3x^2 - 48 = 0 \Rightarrow x^2 = 16 \Rightarrow x = 4$

$When\ x = 4 \Rightarrow A''(x) > 0,\ hence\ A(x)\ is\ mimimum.$

$sub.\ x = 4\ in\ (1),\ y = \frac{24}{x} \Rightarrow y = \frac{24}{4} = 6.$

$\therefore Length\ of\ the\ paper = x + 2 = 4 + 2 = 6cm.$

$Breadth\ of\ the\ paper = y + 3 = 6 + 3 = 9cm.$

19. A farmer plans to fence a rectangular pasture adjacent to a river. The pasture must contain 1,80,000 sq.mtrs in order to provide enough grass for herds. No fencing is needed along the river. What is the length of the minimum needed fencing material?

$Let\ the\ length\ and\ breadth\ of\ the\ rectangular$ pasture $be\ x\ and\ y,\ (x, y > 0)$

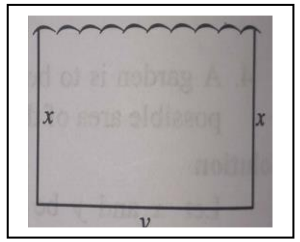
$Area, xy = 180000 \Rightarrow y = \frac{180000}{x} \rightarrow (1)$

$Perimeter, P = x + y + y = x + 2y \Rightarrow P(x) = x + 2\left(\frac{180000}{x}\right)$

$\Rightarrow P(x) = x + \frac{360000}{x}$

$\Rightarrow P'(x) = 1 - \frac{360000}{x^2}$

$\Rightarrow P''(x) = \frac{720000}{x^3}$



$P'(x) = 0 \Rightarrow 1 - \frac{360000}{x^2} = 0 \Rightarrow x^2 - 360000 = 0 \Rightarrow x^2 = 360000 \Rightarrow x = 600$

$When\ x = 600 \Rightarrow P''(x) = \frac{720000}{600^3} > 0, hence\ P(x)\ is\ minimum.$

$Sub.\ x = 600\ in\ (1) \Rightarrow y = \frac{180000}{x} \Rightarrow y = \frac{180000}{600} = 300.$

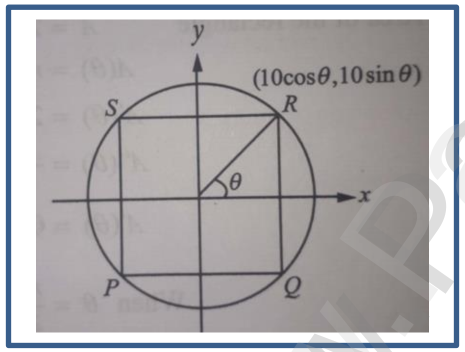
$\therefore Length\ of\ the\ fence = x + 2y = 600 + 600 = 1200m.$

20.Find the dimensions of the rectangle with maximum area that can be inscribed in a circle of radius 10 cm.

$Let\ the\ length\ and\ breadth\ of\ the\ rectangle\ be\ 2x\ and\ 2y,$

$A\ point\ on\ the\ circle\ with\ radius\ r\ is\ P(x,y) = (r\cos\theta, r\sin\theta)$

$r = 10\ cm.$



$Length\ of\ the\ rectangle\ ,2x = 2r\cos\theta = 20\cos\theta.$

$Breadth\ of\ the\ rectangle\ ,2y = 2r\sin\theta = 20\sin\theta.$

$Area\ of\ the\ rectangle, A = (2x)(2y)$

$A(\theta) = (20\cos\theta)(20\sin\theta)$

$= 200(2\sin\theta\cos\theta)$

$= 200\sin 2\theta$

$\Rightarrow A'(\theta) = 400\cos 2\theta$

$\Rightarrow A''(\theta) = -800\sin 2\theta$

$A'(\theta) = 0 \Rightarrow 400\cos 2\theta = 0 \Rightarrow \cos 2\theta = 0 \Rightarrow 2\theta = \frac{\pi}{2} \Rightarrow \theta = \frac{\pi}{4}$

$When\ \theta = \frac{\pi}{4} \Rightarrow A''(\theta) = -800\sin 2\left(\frac{\pi}{4}\right) = -800 < 0$

$\therefore Area\ is\ maximum\ at\ \theta = \frac{\pi}{4}$

$\therefore Length\ of\ the\ rectangle\ 2x = 20\cos\frac{\pi}{4} = 20\left(\frac{1}{\sqrt{2}}\right) = 10\sqrt{2}\ cm.$

$Breadth\ of\ the\ rectangle\ 2y = 20\sin\frac{\pi}{4} = 20\left(\frac{1}{\sqrt{2}}\right) = 10\sqrt{2}\ cm$

21. Prove that among all the rectangles of the given perimeter, the square has the maximum area.

$Let\ the\ length\ and\ breadth\ of\ the\ rectangle\ be\ x\ and\ y$

$Perimeter\ of\ the\ rectangle, P = 2x + 2y \Rightarrow y = \frac{P}{2} - x \to (1)$

$Area\ of\ the\ rectangle, A = xy \Rightarrow A(x) = x\left(\frac{P}{2} - x\right)$

$\Rightarrow A(x) = \frac{px}{2} - x^2$

$\Rightarrow A'(x) = \frac{P}{2} - 2x$

$\Rightarrow A''(x) = -2 < 0.$

$\therefore Area\ is\ maximum$

$A'(x) = 0 \Rightarrow \frac{P}{2} - 2x = 0 \Rightarrow 2x = \frac{P}{2} \Rightarrow x = \frac{P}{4}$

$sub.\ x = \frac{P}{4}\ in\ (1),\ y = \frac{P}{2} - x = \frac{P}{2} - \frac{P}{4} = \frac{P}{4}$

$\because x = y = \frac{P}{4}, It\ is\ a\ square$

Hence all the rectangles of the given perimeter is the square with the maximum area.

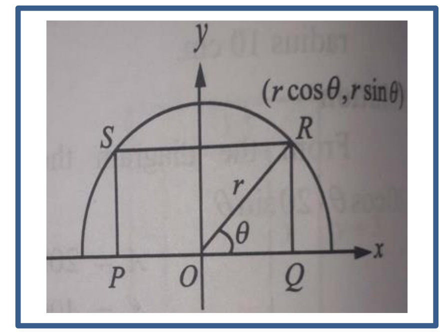
22.Find the dimensions of the largest rectangle that can be inscribed in a semi -circle of radius r cm.

$Let\ the\ length\ and\ breadth\ of\ the\ rectangle\ be\ 2x\ and\ 2y,$

$A\ point\ on\ the\ circle\ with\ radius\ r\ is\ P(x,y) = (r\cos\theta, r\sin\theta)$

$Length\ of\ the\ rectangle\ ,2x = 2r\cos\theta.$

$Breadth\ of\ the\ rectangle\ , y = r\sin\theta.$



$Area\ of\ the\ rectangle, A = (2x)(y)$

$A(\theta) = (2r\cos\theta)(r\sin\theta)$

$A(\theta) = r^2(2\sin\theta\cos\theta)$

$= r^2\sin 2\theta$

$\Rightarrow A'(\theta) = 2r^2\cos 2\theta$

$\Rightarrow A''(\theta) = -4r^2\sin 2\theta$

$A'(\theta) = 0 \Rightarrow 2r^2\cos 2\theta = 0 \Rightarrow \cos 2\theta = 0 \Rightarrow 2\theta = \frac{\pi}{2} \Rightarrow \theta = \frac{\pi}{4}$

$When\ \theta = \frac{\pi}{4} \Rightarrow A''(\theta) = -4r^2\sin 2\left(\frac{\pi}{4}\right) = -4r^2 < 0$

$\therefore Area\ is\ maximum\ at\ \theta = \frac{\pi}{4}$

$\therefore Length\ of\ the\ rectangle\ 2x = 2r\cos\frac{\pi}{4} = 2r\left(\frac{1}{\sqrt{2}}\right) = \sqrt{2}\ rcm.$

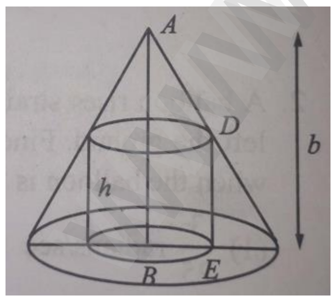
$Breadth\ of\ the\ rectangle\ y = r\sin\frac{\pi}{4} = r\left(\frac{1}{\sqrt{2}}\right) = \frac{r}{\sqrt{2}}\ cm$

23. A manufacturer wants to design an open box having a square base and a surface area of 108 sq.cm. Determine the dimensions of the box for the maximum volume.

Let the length, breadth and height of the box be x, x and y. $(x, y > 0)$
Surface area of the box $= 108$
$\Rightarrow$ *Area of the square base + Area of 4 rectangular sides* $= 108$
$\Rightarrow x^2 + 4xy = 108$
$\Rightarrow 4xy = 108 - x^2$
$\Rightarrow y = \frac{108-x^2}{4x} \to (1)$
Volume of the box, $V =$ *Base area* $\times$ *Height*
$\Rightarrow V = x^2 y$
$\Rightarrow V(x) = x^2\left(\frac{108-x^2}{4x}\right)$
$\Rightarrow V(x) = 27x - \frac{x^3}{4}$
$\Rightarrow V'(x) = 27 - \frac{3x^2}{4}$
$\Rightarrow V''(x) = -\frac{6x}{4} = -\frac{3x}{2}$
$V'(x) = 0 \Rightarrow 27 - \frac{3x^2}{4} = 0 \Rightarrow 108 - 3x^2 = 0$
$\Rightarrow 3x^2 = 108$
$\Rightarrow x^2 = 36$
$\Rightarrow x = 6$
When $x = 6 \Rightarrow V''(x) = -\frac{3(6)}{2} = -9 < 0$, *Hence V is maximum.*
Sub. $x = 6$ *in (1)* $\Rightarrow y = \frac{108-x^2}{4x} = \frac{108-6^2}{4(6)} = \frac{108-36}{24} = \frac{72}{24} = 3$
$\therefore$ *The dimensions of the box is* $6cm \times 6cm \times 3cm$.

24. A hollow cone with base radius a cm and height b cm is placed on a table. Show that the volume of the largest cylinder that can be hidden underneath is $\frac{4}{9}$ times volume of the cone.

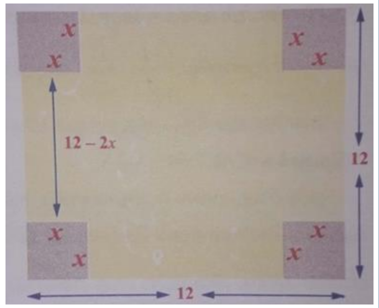
$AB = b, BC = a, BE = r$
$\Delta ABC \sim \Delta DEF \Rightarrow \frac{b}{a} = \frac{h}{a-r} \Rightarrow h = (a-r)\frac{b}{a}$
Volume of the cylinder, $V = \pi r^2 h \Rightarrow V = \pi r^2 (a-r)\frac{b}{a}$
$\Rightarrow V = \frac{\pi b}{a}(ar^2 - r^3)$
$V' = \frac{\pi b}{a}(2ar - 3r^2)$
$V'' = \frac{\pi b}{a}(2a - 6r)$
$V' = 0 \Rightarrow \frac{\pi b}{a}(2ar - 3r^2) \Rightarrow r = \frac{2a}{3}$



When $r = \frac{2a}{3} \Rightarrow V'' < 0$, *hence volume is maximum..*
Volume of the cylinder, $V = \pi r^2 (a-r)\frac{b}{a} = \pi\left(\frac{2a}{3}\right)^2\left(a - \frac{2a}{3}\right)\frac{b}{a} = \frac{4}{9}\left(\frac{1}{3}\pi a^2 b\right)$
$= \frac{4}{9} \times$ *Volume of the cone,*

25. We have a 12 square unit piece of thin material and want to make an open box by cutting small squares from the corners of our material and folding the sides up. The question is, which cut produces the box of maximum value?

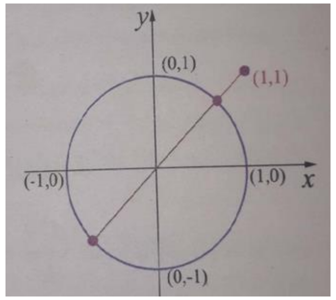
Let x be the length of the side of the cut out square piece.
Length of the box , $l = 12 - x - x = 12 - 2x$
Breadth of the box , $b = 12 - x - x = 12 - 2x$
Height of the box , $h = x, \ x \in (0,6)$.
Volume of the box, $V = lbh \Rightarrow V(x) = (12-2x)(12-2x)x$
$\Rightarrow V(x) = (12-2x)^2 x$
$\Rightarrow V(x) = (144 - 48x + 4x^2)x$
$\Rightarrow V(x) = 4x^3 - 48x^2 + 144x$
$\Rightarrow V'(x) = 12x^2 - 96x + 144$
$\Rightarrow V''(x) = 24x - 96$



$V'(x) = 0 \Rightarrow 12x^2 - 96x + 144 = 0$
$\div 12 \Rightarrow x^2 - 8x + 12 = 0$
$\Rightarrow (x-2)(x-6) = 0 \Rightarrow x = 2, x = 6$ *is not admissible*
when $x = 2 \Rightarrow V''(x) = 24(2) - 96 < 0$, *Hence V is maximum.*
$\therefore$ *The length of the side of the cutted out square piece is 2 units.*

26.Find the points on the unit circle $x^2 + y^2 = 1$ nearest and farthest from (1,1).

Let the required point be $A(x, y)$ *and* $B(1,1)$
$AB = d = \sqrt{(x-1)^2 + (y-1)^2} \Rightarrow d^2 = (x-1)^2 + (y-1)^2$
$\Rightarrow D = (x-1)^2 + (y-1)^2$
$\Rightarrow \frac{dD}{dx} = 2(x-1) + 2(y-1)\frac{dy}{dx} \to (1)$
$x^2 + y^2 = 1 \Rightarrow 2x + 2y\frac{dy}{dx} = 0 \Rightarrow \frac{dy}{dx} = -\frac{x}{y} \to (2)$
$\Rightarrow \frac{d^2y}{dx^2} = -\frac{y - x\frac{dy}{dx}}{y^2} = -\frac{y - x\left(-\frac{x}{y}\right)}{y^2} = -\frac{x^2+y^2}{y^3} \to (3)$
Sub (2) in (1) $\Rightarrow \frac{dD}{dx} = 2(x-1) + 2(y-1)\left(-\frac{x}{y}\right)$
$\frac{dD}{dx} = 0 \Rightarrow 2(x-1) + 2(y-1)\left(-\frac{x}{y}\right) = 0$



$\Rightarrow x-1-x+\frac{x}{y}=0 \Rightarrow \frac{x}{y}=1 \Rightarrow x=y$

$x^2+y^2=1 \Rightarrow x^2+x^2=1 \Rightarrow 2x^2=1 \Rightarrow x^2=\frac{1}{2} \Rightarrow x=\pm\frac{1}{\sqrt{2}}$.

Hence the nearest and farthest distances would get at the points $\left(\frac{1}{\sqrt{2}},\frac{1}{\sqrt{2}}\right)$ *and* $\left(-\frac{1}{\sqrt{2}},-\frac{1}{\sqrt{2}}\right)$.

$\frac{dD}{dx}=2(x-1)+2(y-1)\frac{dy}{dx} \Rightarrow \frac{d^2D}{dx^2}=2\left(\frac{x^2+y^2}{y^3}\right)$

$\frac{d^2D}{dx^2}_{\left(\frac{1}{\sqrt{2}},\frac{1}{\sqrt{2}}\right)}>0\ ;\ \frac{d^2D}{dx^2}_{\left(-\frac{1}{\sqrt{2}},-\frac{1}{\sqrt{2}}\right)}<0$

$Nearest\ distance=\sqrt{\left(\frac{1}{\sqrt{2}}-1\right)^2+\left(\frac{1}{\sqrt{2}}-1\right)^2}=\sqrt{2}-1$

$Farthest\ distance=\sqrt{\left(-\frac{1}{\sqrt{2}}-1\right)^2+\left(-\frac{1}{\sqrt{2}}-1\right)^2}=\sqrt{2}+1$

27. Prove that among all the rectangles of the given area square has the least perimeter.

Let the length and breadth of the rectangle be x and y,

$Perimeter\ of\ the\ rectangle, P=2x+2y \Rightarrow y=\frac{P}{2}-x \rightarrow (1)$

$Area\ of\ the\ rectangle, A=xy \Rightarrow y=\frac{A}{x} \rightarrow (1)$

$Perimeter\ of\ the\ rectangle, P=2x+2y \Rightarrow P(x)=2x+2\left(\frac{A}{x}\right) \Rightarrow P(x)=2x+\frac{2A}{x}$

$\Rightarrow P'(x)=2-\frac{2A}{x^2}$

$\Rightarrow P''(x)=\frac{4A}{x^3}$

$P'(x)=0 \Rightarrow 2-\frac{2A}{x^2}=0 \Rightarrow 2x^2-2A=0 \Rightarrow x^2=A \Rightarrow x=\sqrt{A}$

$when\ x=\sqrt{A} \Rightarrow\ P''(x)=\frac{4A}{\sqrt{A}^3}>0.\ \therefore P(x)\ is\ minimum$

$Sub.\ x=\sqrt{A}\ in\ (1) \Rightarrow\ y=\frac{A}{\sqrt{A}}=\sqrt{A}$

$\because x=y=\sqrt{A}$, all the rectangles of the given area are square with least perimeter.

28. For the function $f(x)=4x^3+3x^2-6x+1$ find the intervals of monotonicity, local extrema, intervals of concavity and points of inflexion.

$f(x)=4x^3+3x^2-6x+1$

$f'(x)=12x^2+6x-6$

$f'(x)=0 \Rightarrow 12x^2+6x-6=0$

$\Rightarrow 6(x+1)(2x-1)=0$

$\Rightarrow x=-1,\frac{1}{2}$ $\quad -\infty \quad -1 \quad \frac{1}{2} \quad \infty$

Intervals	$sign\ of\ f'(x)$	Monotonicity
$(-\infty,-1)$	+	Strictly increasing
$\left(-1,\frac{1}{2}\right)$	−	Strictly decreasing
$\left(\frac{1}{2},\infty\right)$	+	Strictly increasing

$Local\ maximum\ value=f(-1)=6$

$Local\ minimum\ value=f\left(\frac{1}{2}\right)=-\frac{3}{4}$

$f(x)=4x^3+3x^2-6x+1$

$f''(x)=24x+6$

$f''(x)=0 \Rightarrow x=-\frac{1}{4}$ $\quad -\infty \quad -\frac{1}{4} \quad \infty$

Intervals	$sign\ of\ f''(x)$	concavity
$\left(-\infty,-\frac{1}{4}\right)$	−	Concave downwards
$\left(-\frac{1}{4},\infty\right)$	+	Concave upwards

$f\left(-\frac{1}{4}\right)=4\left(-\frac{1}{4}\right)^3+3\left(-\frac{1}{4}\right)^2-6\left(-\frac{1}{4}\right)+1=\frac{21}{8}$

Point of inflexion $=\left(-\frac{1}{4},\frac{21}{8}\right)$

29. Expand $\log(1+x)$ as a Maclaurin's series upto 4 non-zero terms for $-1<x\leq 1$.

Maclaurin's series $: f(x)=f(0)+\frac{f'(0)}{1!}x+\frac{f''(0)}{2!}x^2+\frac{f'''(0)}{3!}x^3+\cdots$

Derivative of $f(x)$	Derivative of $\log(1+x)$	Value for $x=0$
$f(x)$	$\log(1+x)$	0
$f'(x)$	$\frac{1}{1+x}$	1
$f''(x)$	$-\frac{1}{(1+x)^2}$	−1
$f'''(x)$	$\frac{2}{(1+x)^3}$	1
$f^{iv}(x)$	$-\frac{6}{(1+x)^4}$	−6

$\log(1+x)=x-\frac{x^2}{2}+\frac{x^3}{3}-\frac{x^4}{4}+\cdots;\ -1<x\leq 1$

30. Expand $tanx$ in ascending powers of x upto 5^{th} power for $\frac{-\pi}{2} < x < \frac{\pi}{2}$.

Maclaurin's series : $f(x) = f(0) + \frac{f'(0)}{1!}x + \frac{f''(0)}{2!}x^2 + \frac{f'''(0)}{3!}x^3 + \cdots$

Derivative of $f(x)$	Derivative of $tanx$	Value for $x = 0$
$f(x)$	$\tan x$	0
$f'(x)$	$sec^2x = 1 + tan^2x = 1 + (f(x))^2$	1
$f''(x)$	$2f(x)f'(x)$	0
$f'''(x)$	$2\left[f(x)f''(x) + (f'(x))^2\right]$	2
$f^{1v}(x)$	$2[f(x)f'''(x) + f'(x)f''(x) + 2f'(x)f''(x)]$	0
$f^{v}(x)$	$2\left[f(x)f'^{v}(x) + f'''(x)f'(x) + f'(x)f'''(x) + (f''(x))^2 + 2(f'(x)f'^{v}(x) + f''(x)f'''(x))\right]$	16

$\tan x = x + \frac{1}{3}x^3 + \frac{2}{15}x^5 + \cdots; -\frac{\pi}{2} < x < \frac{\pi}{2}$

31. Write down the Taylor series expansion, of the function $logx$ about $x = 1$ upto three non − zero terms for $x > 0$.

$f(x) = \log x \Rightarrow f(1) = \log 1 = 0$

$f'(x) = \frac{1}{x} \Rightarrow f'(1) = \frac{1}{1} = 1$

$f''(x) = -\frac{1}{x^2} \Rightarrow f''(x) = -\frac{1}{(1)^2} = -1$

$f'''(x) = \frac{2}{x^3} \Rightarrow f'''(1) = \frac{2}{(1)^3} = 2$

Taylor series: $f(x) = f(a) + f'(a)\frac{(x-a)}{1!} + f''(a)\frac{(x-a)^2}{2!} + f'''(a)\frac{(x-a)^3}{3!} + \cdots$

$\log x = 0 + \frac{(x-1)}{1!} - 1\frac{(x-1)^2}{2!} + 2\frac{(x-1)^3}{3!} + \cdots$

$\Rightarrow \log x = \frac{(x-1)}{1} - \frac{(x-1)^2}{2} + \frac{(x-1)^3}{3} - \cdots$

32. Expand $sinx$ in ascending powers $x - \frac{\pi}{4}$ upto three non − zero terms.

$f(x) = sinx \Rightarrow f\left(\frac{\pi}{4}\right) = sin\frac{\pi}{4} = \frac{1}{\sqrt{2}}$

$f'(x) = cosx \Rightarrow f'\left(\frac{\pi}{4}\right) = cos\frac{\pi}{4} = \frac{1}{\sqrt{2}}$

$f''(x) = -sinx \Rightarrow f''\left(\frac{\pi}{4}\right) = -sin\frac{\pi}{4} = -\frac{1}{\sqrt{2}}$

Taylor series: $f(x) = f(a) + f'(a)\frac{(x-a)}{1!} + f''(a)\frac{(x-a)^2}{2!} + f'''(a)\frac{(x-a)^3}{3!} + \cdots$

$\therefore sinx = \frac{1}{\sqrt{2}} + \frac{1}{\sqrt{2}}\frac{\left(x-\frac{\pi}{4}\right)}{1!} - \frac{1}{\sqrt{2}}\frac{\left(x-\frac{\pi}{4}\right)^2}{2!} + \cdots$

$= \frac{1}{\sqrt{2}}\left[1 + \frac{\left(x-\frac{\pi}{4}\right)}{1} - \frac{\left(x-\frac{\pi}{4}\right)^2}{2} + \cdots\right]$

33. Evaluate : $\lim_{x\to\frac{\pi}{2}}(sinx)^{tanx}$

$y = (sinx)^{tanx}$ vd;f.

$logy = log(sinx)^{tanx} \Rightarrow logy = tanxlog(sinx)$

$\Rightarrow \lim_{x\to\frac{\pi}{2}} logy = \lim_{x\to\frac{\pi}{2}} tanxlog(sinx)$

$\Rightarrow \lim_{x\to\frac{\pi}{2}} logy = \lim_{x\to\frac{\pi}{2}} \frac{log(sinx)}{\cot x}$ ($\frac{0}{0}$ tbtk;)

$\Rightarrow \lim_{x\to\frac{\pi}{2}} logy = \lim_{x\to\frac{\pi}{2}} \frac{\frac{1}{sinx}.cosx}{-cosec^2x}$

$\Rightarrow \lim_{x\to\frac{\pi}{2}} logy = \lim_{x\to\frac{\pi}{2}} \frac{\frac{1}{sinx}.cosx}{-1/_{sin^2x}}$

$\Rightarrow \lim_{x\to\frac{\pi}{2}} logy = \lim_{x\to\frac{\pi}{2}}(-sinxcosx)$

$\Rightarrow \lim_{x\to\frac{\pi}{2}} logy = -sin\frac{\pi}{2}cos\frac{\pi}{2}$

$\Rightarrow \lim_{x\to\frac{\pi}{2}} logy = 0$

$\Rightarrow log\lim_{x\to\infty} y = 0$

$\Rightarrow \lim_{x\to\infty} y = e^0$

$\Rightarrow \lim_{x\to\frac{\pi}{2}}(sinx)^{tanx} = 1$

34.Evaluate : $\lim_{x\to0^+} x^x$

$y = x^x$ vd;f.

$logy = logx^x \Rightarrow logy = xlogx$

$\Rightarrow \lim_{x\to0^+} logy = \lim_{x\to0^+} xlogx$

$\Rightarrow \lim_{x\to0^+} logy = \lim_{x\to0^+} \frac{logx}{1/_x}$ ($\frac{\infty}{\infty}$ tbtk;)

$\Rightarrow \lim_{x\to0^+} logy = \lim_{x\to0^+} \frac{1/_x}{-1/_{x^2}}$

$\Rightarrow \lim_{x\to0^+} logy = \lim_{x\to0^+} -x$

$\Rightarrow \lim_{x\to0^+} logy = 0$

$\Rightarrow log\lim_{x\to0^+} y = 0$

$\Rightarrow \lim_{x\to0^+} y = e^0$

$\Rightarrow \lim_{x\to0^+} x^x = 1$

8.DIFFERENTIALS AND PARTIAL DERIVATIVES

5 MARK QUESTIONS

1.Assuming $\log_{10} e = 0.4343, find\ an\ approximate\ value\ of\ \log_{10} 1003$.

$f(x) = \log_{10} x \quad , x_o = 1000$ vd;f.

$f(x_o) = \log_{10} 1000 = 3$

$f'(x) = \frac{1}{x}.\log_{10} e = \frac{0.4343}{x}$

$f'(x_o) = \frac{0.4343}{1000}$

$L(x) = f(x_o) + f'(x_o)(x - x_o)$

$L(x) = 3 + \frac{0.4343}{1000}(x - 1000)$

$\therefore \log_{10} 1003 \approx 3 + \frac{0.4343}{1000}(1003 - 1000) \approx 3 + \frac{0.4343}{1000}(3) \approx 3 + 0.0013 \approx 3.0013$

2.The trunk of a tree has diameter 30 cm. During the following year, the circumference grew 6 cm. (i) Approximately, how much did the tree's diameter grow? (Ii) What is the percentage increase in area of the tree's cross section?

Diameter $d = 30$, Radius $r = 15$

(i) Increase in circumference $= 6$

$\Longrightarrow 2\pi r_2 - 2\pi r_1 = 6$

$\Longrightarrow 2r_2 - 2r_1 = \frac{6}{\pi}$

Approximate change in diameter$= \frac{6}{\pi}\ cm$

$(ii) 2r_2 - 2r_1 = \frac{6}{\pi} \Longrightarrow r_2 - r_1 = \frac{3}{\pi} \Longrightarrow dr = \frac{3}{\pi}$

$A = \pi r^2 \Longrightarrow dA = 2\pi r dr = 2\pi(15)\left(\frac{3}{\pi}\right) = 90cm^2$

Percentage increase $= \frac{dA}{A} \times 100 = \frac{90}{\pi(15)^2} \times 100 = \frac{40}{\pi}\%$

3. The relation between number of words y a person learns in x hours is given by $y = 52\sqrt{x}, 0 \le x \le 9$. What is the approximate number of words learned when x changes from (i) 1 to 1.1 hour? (ii) 4 to 4.1 hour?

$y = 52\sqrt{x} \Longrightarrow dy = 52\left(\frac{1}{2\sqrt{x}}\right)dx = \frac{26}{\sqrt{x}}dx$

$(i) x = 1, dx = 1.1 - 1 = 0.1;\ dy = \frac{26}{\sqrt{1}}(0.1) = 2.6 \approx 3$ words.

$(ii) x = 4, dx = 4.1 - 4 = 0.1;\ dy = \frac{26}{\sqrt{4}}(0.1) = 1.3 \approx 1$ word.

4.The time T, taken for a complete oscillation of a single pendulum with length l, is given by the equation $T = 2\pi\sqrt{\frac{l}{g}}$, where g is a constant. Find the approximate percentage error in the calculated value of T corresponding to an error of 2 percent in the value of l.

$dl = (2\%)l = \frac{2l}{100}$

$T = 2\pi\sqrt{\frac{l}{g}} \Longrightarrow \log T = \log\left(2\pi\sqrt{\frac{l}{g}}\right)$

$\Longrightarrow \log T = \log 2\pi + \frac{1}{2}\log l - \frac{1}{2}\log g$

$\Longrightarrow \frac{dT}{T} = 0 + \frac{dl}{2l} - 0$

$\Longrightarrow \frac{dT}{T} \times 100 = \frac{dl}{2l} \times 100$

$\Longrightarrow \frac{dT}{T} \times 100 = \frac{2l \times 100}{2l \times 100} = 1\%$

5.The radius of a circular plate is measured as 12.65 cm instead of the actual length 12.5 cm, find the following in calculating the area of the circular plate: (i) Absolute error (ii) Relative error (iii) Percentage error.

$r = 12.65, dr = 12.5 - 12.65 = -0.15$

(i)Area of the circle $A = \pi r^2$

Approximate error $dA = 2\pi r dr = 2\pi(12.65)(-0.15) = -3.795\pi$

Actual error $= A(12.5) - A(12.65)$

$= \pi(12.5)^2 - \pi(12.65)^2$

$= 156.25\pi - 160.0225\pi$

$= -3.7725\pi$

(i) Absolute error $=$ Actual error $-$ Approximate error

$= -3.7725\pi + 3.795\pi$

$= 0.0225\pi cm^2$

(ii) Relative error $= \frac{\text{Absolute error}}{\text{Actual error}} = -\frac{0.0225\pi}{3.7725\pi} = 0.00596 \approx 0.0060cm^2$

(*iii*) Percentage error = Relative error × 100 = 0.0060 × 100 ≈ 0.6%

6. A right circular cylinder has radius $r = 10\ cm$ and height $h = 20\ cm$. Suppose that the radius of the cylinder is increased from 10 cm to 10.1 cm and the height does not change. Estimate the change in the volume of the cylinder. Also, calculate the relative error and percentage error.

$r = 10\ cm\ , dr = 10.1 - 10 = 0.1\ cm, h = 20\ cm.$

Volume of the cylinder $V = \pi r^2 h = 20\pi r^2$

$$\Rightarrow dV = 40\pi r dr = 40\pi(10)(0.1) = 40\pi cm^3.$$

Approximate error $= 40\pi cm^3$

Actual error= $V(10.1) - V(10) = 2040.2\pi - 2000\pi = 40.2\pi cm^3$

Absolute error = Actual error − Approximate error= $40.2\pi - 40\pi = 0.2\pi$

Relative error= $\frac{\text{Absolute error}}{\text{Actual error}} = \frac{0.2\pi}{40.2\pi} = 0.00497$

Percentage error = Relative error × 100 = 0.00497 × 100 = 0.497%

7.If $u = sin^{-1}\left(\frac{x+y}{\sqrt{x}+\sqrt{y}}\right)$, Show that $x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} = \frac{1}{2}tanu$

$f(x,y) = \frac{x+y}{\sqrt{x}+\sqrt{y}} = \sin u$

$f(tx,ty) = \frac{tx+ty}{\sqrt{tx}+\sqrt{ty}} = t^{1/2}f(x,y)$

$Order\ n = \frac{1}{2}$

Euler's Theorem : $x\frac{\partial f}{\partial x} + y\frac{\partial f}{\partial y} = nf$

$$\Rightarrow x\frac{\partial(\sin u)}{\partial x} + y\frac{\partial(\sin u)}{\partial y} = \frac{1}{2}\sin u$$

$$\Rightarrow x\cos u\frac{\partial u}{\partial x} + y\cos u\frac{\partial u}{\partial y} = \frac{1}{2}\sin u$$

$$\Rightarrow x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} = \frac{1}{2}\tan u$$

8. If $u(x,y) = \frac{x^2+y^2}{\sqrt{x+y}}$, prove that $x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} = \frac{3}{2}u$.

$u(x,y) = \frac{x^2+y^2}{\sqrt{x+y}}$,

$u(tx,ty) = \frac{(tx)^2+(ty)^2}{\sqrt{tx+ty}} = t^{3/2}.u(x,y)$

$Order\ n = \frac{3}{2}$

Euler's Theorem : $x\frac{\partial f}{\partial x} + y\frac{\partial f}{\partial y} = nf$

$$\Rightarrow x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} = \frac{3}{2}u$$

9. If $v(x,y) = log\left(\frac{x^2+y^2}{x+y}\right)$, prove that $x\frac{\partial v}{\partial x} + y\frac{\partial v}{\partial y} = 1$

$v(x,y) = \log\left(\frac{x^2+y^2}{x+y}\right) \Rightarrow e^v = \frac{x^2+y^2}{x+y} = f(x,y)$

$\Rightarrow f(x,y) = \frac{x^2+y^2}{x+y}$

$f(tx,ty) = \frac{(tx)^2+(ty)^2}{tx+ty} = t^1 f(x,y)$

$Order\ n = 1$

Euler's Theorem : $x\frac{\partial f}{\partial x} + y\frac{\partial f}{\partial y} = nf$

$$\Rightarrow x\frac{\partial(e^v)}{\partial x} + y\frac{\partial(e^v)}{\partial y} = 1.e^v$$

$$\Rightarrow xe^v\frac{\partial v}{\partial x} + ye^v\frac{\partial v}{\partial y} = e^v$$

$$\Rightarrow x\frac{\partial v}{\partial x} + y\frac{\partial v}{\partial y} = 1$$

10. If $w(x,y,z) = log\left(\frac{5x^3y^4+7y^2xz^4-75y^3z^4}{x^2+y^2}\right)$, find $x\frac{\partial w}{\partial x} + y\frac{\partial w}{\partial y} + z\frac{\partial w}{\partial z}$.

$w(x,y,z) = \log\left(\frac{5x^3y^4+7y^2xz^4-75y^3z^4}{x^2+y^2}\right)$

$\Rightarrow e^w = \frac{5x^3y^4+7y^2xz^4-75y^3z^4}{x^2+y^2} = f(x,y,z)$

$\Rightarrow f(x,y,z) = \frac{5x^3y^4+7y^2xz^4-75y^3z^4}{x^2+y^2}$

$\Rightarrow f(tx,ty,tz) = \frac{t^7(5x^3y^4+7y^2xz^4-75y^3z^4)}{t^2(x^2+y^2)} = t^5.f(x,y,z)$

$Order\ n = 5$

Euler's Theorem : $x\frac{\partial f}{\partial x} + y\frac{\partial f}{\partial y} = nf$

$$\Rightarrow x\frac{\partial(e^w)}{\partial x} + y\frac{\partial(e^w)}{\partial y} + z\frac{\partial(e^w)}{\partial z} = 5e^v$$

$$\Rightarrow xe^w\frac{\partial v}{\partial x} + ye^w\frac{\partial v}{\partial y} + ze^w\frac{\partial w}{\partial z} = 5e^v$$

$$\Rightarrow x\frac{\partial w}{\partial x} + y\frac{\partial w}{\partial y} + z\frac{\partial w}{\partial z} = 5$$

10.ORDINARY DIFFERENTIAL EQUATIONS

5 MARK QUESTIONS

1.The growth of a population is proportional to the number present. If the population of a colony double in 50 years, in how many years will the population become triple?

t	A
0	A_0
50	$2A_0$
?	$3A_0$

$\frac{dA}{dt} \propto A \Rightarrow \frac{dA}{dt} = kA \Rightarrow \frac{dA}{A} = kdt,$

Integrating on both sides, $\boxed{A = ce^{kt}} \rightarrow (1)$

$\underline{t = 0, A = A_0 :-}$

$(1) \Rightarrow A_0 = ce^{k(0)} \Rightarrow \boxed{c = A_0}$

$\therefore (1) \Rightarrow \boxed{A = A_0 e^{kt}} \rightarrow (2)$

$\underline{t = 50, A = 2A_0 :-}$

$(2) \Rightarrow 2A_0 = A_0 e^{50k} \Rightarrow 2 = e^{50k} \Rightarrow 50k = \log 2 \Rightarrow k = \frac{1}{50}\log 2$

$\therefore (2) \Rightarrow A = A_0 (e)^{\left(\frac{1}{50}\log 2\right)t} \rightarrow (3)$

$\underline{t = ?, A = 3A_0 :-}$

$(3) \Rightarrow 3A_0 = A_0 (e)^{\left(\frac{1}{50}\log 2\right)t} \Rightarrow 3 = (e)^{\left(\frac{1}{50}\log 2\right)t}$

$\Rightarrow \log 3 = \frac{1}{50}\log 2 . t \Rightarrow \boxed{t = 50\left(\frac{\log 3}{\log 2}\right)}$

$\therefore$ The population is triples in $50\left(\frac{\log 3}{\log 2}\right)$years.

2.The rate of increase in the number of bacteria in a certain bacteria culture is proportional to the number present. Given that the number triples in 5 hours, find how many bacteria will be present after 10 hours?

t	A
0	A_0
5	$3A_0$
10	?

$\frac{dA}{dt} \propto A \Rightarrow \frac{dA}{dt} = kA \Rightarrow \frac{dA}{A} = kdt,$

Integrating on both sides, $\boxed{A = ce^{kt}} \rightarrow (1)$

$\underline{t = 0, A = A_0 :-}$

$(1) \Rightarrow A_0 = ce^{k(0)} \Rightarrow \boxed{c = A_0}$

$\therefore (1) \Rightarrow \boxed{A = A_0 e^{kt}} \rightarrow (2)$

$\underline{t = 5, A = 3A_0 :-}$

$(2) \Rightarrow 3A_0 = A_0 e^{5k} \Rightarrow \boxed{e^{5k} = 3}$

$\underline{t = 10, A = ? :-}$

$(2) \Rightarrow A = A_0 (e)^{10t} \Rightarrow A = A_0 (e^{5k})^2 = A_0 (3)^2 \Rightarrow \boxed{A = 9A_0}$

Hence after 10 hours the number of bacteria is 9 times the original number of bacteria.

3.Find the population of a city at any time t, given that the rate of increase of population is proportional to the population at that instant and that in a period of 40 years the population increased from 3,00,000 to 4,00,000.

t	A
0	300000
40	400000
t	?

$\frac{dA}{dt} \propto A \Rightarrow \frac{dA}{dt} = kA \Rightarrow \frac{dA}{A} = kdt,$

Integrating on both sides, $\boxed{A = ce^{kt}} \rightarrow (1)$

$\underline{t = 0, A = 300000 :-}$

$(1) \Rightarrow 300000 = ce^{k(0)} \Rightarrow \boxed{c = 300000}$

$\therefore (1) \Rightarrow \boxed{A = 300000e^{kt}} \rightarrow (2)$

$\underline{t = 40, A = 400000 :-}$

$(2) \Rightarrow 400000 = 300000e^{40k} \Rightarrow \frac{4}{3} = e^{40k} \Rightarrow 40k = \log\left(\frac{4}{3}\right)$

$\Rightarrow k = \frac{1}{40}\log\left(\frac{4}{3}\right) \Rightarrow \boxed{k = \left(\log\left(\frac{4}{3}\right)\right)^{\left(\frac{1}{40}\right)}}$

$\underline{t = t, A = ? :-}$

$(2) \Rightarrow A = 300000(e)^{\left(\log\left(\frac{4}{3}\right)\right)^{\left(\frac{1}{40}\right)t}} \Rightarrow \boxed{A = 300000\left(\log\left(\frac{4}{3}\right)\right)^{\left(\frac{t}{40}\right)}}$

4.The engine of a motor boat moving at 10 m/ is shut off. Given that the retardation at any subsequent time (after shutting off the engine) equal to the velocity at that time. Find the velocity after 2 seconds of switching off the engine.

t	A
0	10
2	?

$A = -\frac{dA}{dt} \Rightarrow \frac{dA}{A} = -dt$

Integrating on both sides, $\boxed{A = ce^{-t}} \rightarrow (1)$

$\underline{t = 0, A = 10 :-}$

$(1) \Rightarrow 10 = ce^{-(0)} \Rightarrow \boxed{c = 10}$

$\therefore (1) \Rightarrow \boxed{A = 10e^{-t}} \rightarrow (2)$

$\underline{t = 2, A = ? :-}$

$(2) \Rightarrow A = 10e^{-2} \Rightarrow \boxed{A = \frac{10}{e^2}}$

$\therefore$ velocity $= \frac{10}{e^2}\frac{m}{s}$.

5.Suppose a person deposits 10,000 Indian rupees in a bank account at the rate of 5% per annum compounded continuously. How much money will be in his bank account 18 months later?

t	A
0	1000
$18 = \frac{3}{2}$	?

$\frac{dA}{dt} \propto A \Rightarrow \frac{dA}{dt} = kA \Rightarrow \frac{dA}{A} = kdt,$

Integrating on both sides, $\boxed{A = ce^{kt}} \rightarrow (1)$

$k = 5\% = \frac{5}{100} = 0.05$

$\therefore (1) \Rightarrow \boxed{A = ce^{0.05t}} \rightarrow (2)$

$\underline{t = 0, A = 10000 :-}$

$(2) \Rightarrow 10000 = ce^{0.05(0)} \Rightarrow \boxed{c = 10000}$

$\therefore(1) \Rightarrow \boxed{A = 10000e^{0.05t}} \rightarrow (3)$

$t = \frac{3}{2}, A = ? :-$

$(3) \Rightarrow A = 10000e^{\left(0.05\times\frac{3}{2}\right)} = 10000e^{\left(\frac{0.15}{2}\right)} \Rightarrow \boxed{A = 10000e^{0.075}}.$

6.Assume that the rate at which radioactive nuclei decay is proportional to the number of such nuclei that are present in a given sample. In a certain sample 10% of the original numberof radioactive nuclei have undergone disintegration in a period of 100 year. What percentage of the original radioactive nuclei will remain after 1000 years?

$\frac{dA}{dt} \propto A \Rightarrow \frac{dA}{dt} = kA \Rightarrow \frac{dA}{A} = kdt,$

Integrating on both sides, $\boxed{A = ce^{kt}} \rightarrow (1)$

t	A
0	100
100	90
1000	?

$t = 0, A = 100 :-$

$100 = ce^{k(0)} \Rightarrow \boxed{c = 100}$

$\therefore(1) \Rightarrow \boxed{A = 100e^{kt}} \rightarrow (2)$

$t = 100, A = 90 :-$

$(2) \Rightarrow 90 = 100e^{100k} \Rightarrow \boxed{e^{100k} = \frac{9}{10}}$

$t = 1000, A = ? :-$

$(2) \Rightarrow A = 100(e)^{(1000)k} = 100(e^{100k})^{(10)} = 100\left(\frac{9}{10}\right)^{10}$

Hence the radioactive nuclei remain after 1000years is$100\left(\frac{9}{10}\right)^{10}$

7.A radioactive isotope has an initial mass 200mg. Which 2 years later is 150mg. Find the expression for the amount of the amount of the isotope remaining at any time. What is its half life? (half life means the time taken for the radioactivity of a specified isotope to fall to half its original value)

$\frac{dA}{dt} \propto A \Rightarrow \frac{dA}{dt} = -kA \Rightarrow \frac{dA}{A} = -kdt,$

Integrating on both sides, $\boxed{A = ce^{-kt}} \rightarrow (1)$

t	A
0	200
2	150
?	100

$t = 0, A = 200 :-$

$200 = ce^{k(0)} \Rightarrow \boxed{c = 200}$

$\therefore(1) \Rightarrow \boxed{A = 200e^{-kt}} \rightarrow (2)$

$t = 2, A = 150 :-$

$(2) \Rightarrow 150 = 200e^{-2k} \Rightarrow \frac{150}{200} = e^{-2k} \Rightarrow \frac{3}{4} = e^{-2k} \Rightarrow e^{2k} = \frac{4}{3}$

$\Rightarrow 2k = \log\left(\frac{4}{3}\right) \Rightarrow \boxed{k = \frac{1}{2}\log\left(\frac{4}{3}\right)}$

$t = ?, A = 100 :-$

$(2) \Rightarrow 100 = 200e^{-\frac{1}{2}\log\left(\frac{4}{3}\right)t} \Rightarrow \frac{100}{200} = e^{-\frac{1}{2}\log\left(\frac{4}{3}\right)t} \Rightarrow \log\left(\frac{1}{2}\right) = -\frac{t}{2}\log\left(\frac{4}{3}\right)$

$\Rightarrow \log\left(\frac{1}{2}\right)\frac{t}{2}\log\left(\frac{3}{4}\right) \Rightarrow \boxed{t = \frac{2\log\left(\frac{1}{2}\right)}{\log\left(\frac{3}{4}\right)}}$

8.Water at100^0C cools in 10 minutes to 80^0C in a room temperature of 25^0C. Find (i) The temperature of water after 20 minutes (ii) The time when the temperature is 40^0C.$\left[\log_e\frac{11}{15} = -0.3101;\ \log_e 5 = 1.6094\right]$

$\frac{dT}{dt} \propto (T-S) \Rightarrow \frac{dT}{dt} = k(T-S) \Rightarrow \frac{dT}{T-S} = kdt$

Integrating on both sides, $\boxed{T - S = ce^{kt}}$

$\Rightarrow T - 25 = ce^{kt} \rightarrow (1)$

t	T	S
0	100	25
10	80	
20	?	
?	40	

$t = 0, T = 100 :-$

$(1) \Rightarrow 100 - 25 = ce^{k(0)} \Rightarrow \boxed{c = 75}$

$\therefore(1) \Rightarrow T - 25 = 75e^{kt} \rightarrow (2)$

$t = 10, T = 80 :-$

$(2) \Rightarrow 80 - 25 = 75e^{10k} \Rightarrow 55 = 75e^{10k}$

$\Rightarrow \boxed{e^{10k} = \frac{55}{75}} \Rightarrow 10k = \log\left(\frac{55}{75}\right) \Rightarrow \boxed{k = \frac{1}{10}\log\left(\frac{55}{75}\right)}$

$(i) t = 20, T = ? :-$

$(2) \Rightarrow T - 25 = 75e^{20k} \Rightarrow T = 25 + 75(e^{10k})^{(2)}$

$= 25 + 75\left(\frac{55}{75}\right)^2$

$\Rightarrow \boxed{T = 65.33^0C}$

$(ii) t = ?, T = 40 :-$

$(2) \Rightarrow 40 - 25 = 75e^{tk} \Rightarrow 15 = 75e^{tk} \Rightarrow \frac{15}{75} = e^{tk} \Rightarrow \frac{1}{5} = e^{tk}$

$\Rightarrow tk = -\log 5 \Rightarrow t = \frac{1}{k}(-\log 5) = -\frac{1}{10}\log\left(\frac{55}{75}\right)\log 5$

$= -(-0.3101)(1.6094) \Rightarrow t = 51.81$minutes.

9.At 10.00 A.M. a woman took a cup of hot instant coffee from her microwave oven and placed it on a nearby kitchen counter to cool. At this instant the temperature of the coffee was 180^0F, and 10 minutes later it was 160^0F.Assume that constant temperature of the kitchen was 70^0F. (i) what was the temperature of the coffee at 10.15A.M.? (ii) The woman likes to drink coffee when its temperature is between 130^0F and 140^0Fbetween what times should she have drunk the coffee?

$\frac{dT}{dt} \propto (T-S) \Rightarrow \frac{dT}{dt} = k(T-S) \Rightarrow \frac{dT}{T-S} = kdt$

Integrating on both sides, $\boxed{T - S = ce^{kt}}$

$\Rightarrow T-70=ce^{kt} \rightarrow (1)$

t	T	S
0	180	70
10	160	
15	?	
?	130	
?	140	

$t=0, T=180:-$

$(1) \Rightarrow 180-70=ce^{k(0)} \Rightarrow \boxed{c=110}$

$\therefore (1) \Rightarrow T-70=110e^{kt} \rightarrow (2)$

$t=10, T=160:-$

$(2) \Rightarrow 160-70=110e^{10k} \Rightarrow 90=110e^{10k}$

$\Rightarrow \boxed{e^{10k}=\frac{9}{11}}$

$\Rightarrow 10k=\log\left(\frac{9}{11}\right) \Rightarrow \boxed{k=\frac{1}{10}(-0.2006)=-0.02006}$

$(i) t=15, T=?:-$

$(2) \Rightarrow T-70=110e^{15k} \Rightarrow T=70+110(e^{10k})^{(3/2)}$

$=70+110\left(\frac{9}{11}\right)^{(3/2)} \Rightarrow \boxed{T=151.4^0F}$

$(ii) t=?, T=130:-$

$(2) \Rightarrow 130-70=110e^{tk} \Rightarrow 60=110e^{tk} \Rightarrow \frac{6}{11}=e^{tk} \Rightarrow tk=\log\left(\frac{6}{11}\right)$

$=\frac{1}{k}\log\left(\frac{6}{11}\right)=-\frac{1}{0.02006}\log\left(\frac{6}{11}\right)=30.216$

$t=?, T=140:-$

$(2) \Rightarrow 140-70=110e^{tk} \Rightarrow 70=110e^{tk} \Rightarrow \frac{7}{11}=e^{tk} \Rightarrow tk=\log\left(\frac{7}{11}\right)$

$=\frac{1}{k}\log\left(\frac{7}{11}\right)=-\frac{1}{0.02006}\log\left(\frac{7}{11}\right)=22.53$

$\therefore$ she drunk coffee between 10.22A.M and 10.30A.M.

10.A pot of boiling water at $100^0C$is removed from a stove at time $t=0$and left to gool in the kitchen. After 5 minutes, the water temperature has decreased to 80^0C, and another 5 minutes later it has dropped to 65^0C . Determine the temperature of the kitchen.

t	T	S
0	100	?
5	80	
10	65	

$\frac{dT}{dt} \propto (T-S) \Rightarrow \frac{dT}{dt}=k(T-S) \Rightarrow \frac{dT}{T-S}=kdt$

Integrating on both sides, $\boxed{T-S=ce^{kt}} \rightarrow (1)$

$t=0, T=100:-$

$(1) \Rightarrow 100-S=ce^{k(0)} \Rightarrow \boxed{c=100-S}$

$\therefore (1) \Rightarrow T-S=(100-S)e^{kt} \rightarrow (2)$

$t=5, T=80:-$

$(2) \Rightarrow 80-S=(100-S)e^{5k} \Rightarrow \boxed{e^{5k}=\frac{80-S}{100-S}}$

$t=10, T=65:-$

$(2) \Rightarrow 65-S=(100-S)e^{10k}$

$\Rightarrow 65-S=(100-S)e^{10k}$

$\Rightarrow 65-S=(100-S)(e^{5k})^2 \Rightarrow 65-S=(100-S)\left(\frac{80-S}{100-S}\right)^2$

$\Rightarrow 65-S=\frac{(80-S)^2}{100-S} \Rightarrow (65-S)(100-S)=(80-S)^2$

$\Rightarrow 6500-65s-100S+S^2=6400-160S+S^2 \Rightarrow 5S=100$

$\Rightarrow \boxed{S=20^0C}$

11.In a murder investigation a corpse was found by a detective at exactly 8 p.m. Being alert, the detective also measured the body temperature and found it to be 70^0F. Two hours later, the detective measured the body temperature again and found it to be 60^0F.If the room temperature is 50^0F, and assuming that the body temperature of the person before death was 98.6^0F,at what time did the murder occur ? [$\log 2.43=0.88789;\ \log 0.5=-0.69315$]

$$\frac{dT}{dt} \propto (T-S) \Rightarrow \frac{dT}{dt}=k(T-S) \Rightarrow \frac{dT}{T-S}=kdt$$

Integrating on both sides,T-S= ce^{kt}

$\Rightarrow \boxed{T-50=ce^{kt}} \rightarrow (1)$

t	T	S
0	70	50
2	60	
t_1	98.6	

$t=0, T=70:-$

$(1) \Rightarrow 70-50=ce^{k(0)} \Rightarrow \boxed{c=20}$

$\therefore (1) \Rightarrow T-50=20e^{kt} \rightarrow (2)$

$t=2, T=60:-$

$(2) \Rightarrow 60-50=20e^{2k} \Rightarrow e^{2k}=\frac{1}{2} \Rightarrow 2k=\log\left(\frac{1}{2}\right)$

$\Rightarrow k=\frac{1}{2}\log\left(\frac{1}{2}\right) \Rightarrow \boxed{k=\frac{1}{2}(-0.69315)}$

$t=t_1, T=98.6:-$

$(2) \Rightarrow 98.6-50=20e^{kt_1}$

$\Rightarrow e^{kt_1}=\frac{48.6}{20}=2.43 \Rightarrow kt_1=\log 2.43 \Rightarrow t_1=\frac{1}{k}(\log 2.43)$

$\Rightarrow t_1=\frac{2}{-0.69315}(0.88789) \Rightarrow t_1 \approx -2.56$

$\therefore$Time of murder$\approx 8.00-2.56 \approx 5.30$p.m..

12. A tank contains 1000 litres of water in which 100 grams of salt is dissolved. Brine runs in a rate of 10 litres per minute, and each litre contains 5 grams of dissolved salt. The mixture of the tank is kept uniform by stirring. Brine runs out at 10 litres per minute. Find the amount of salt at any time t.

t	A
0	100

In flow rate= $10 \times \frac{5}{1}=50$

Out flow rate= $10 \times \frac{A}{1000}=\frac{A}{100}$

$\frac{dA}{dt}$ =In flow rate- Out flow rate

$\frac{dA}{dt}=50-\frac{A}{100} \Rightarrow \frac{dA}{dt}=\frac{5000-A}{100} \Rightarrow \frac{dA}{5000-A}=\frac{dt}{100} \Rightarrow \frac{dA}{A-5000}=-\frac{dt}{100}$

Integrating on both sides, $\boxed{A - 5000 = ce^{-0.01t}} \to (1)$

$\underline{t = 0, A = 100}$:−

$100 - 5000 = ce^{-0.01(0)} \Longrightarrow \boxed{c = -4900}$

$\therefore (1) \Longrightarrow A - 5000 = -4900e^{-0.01t}$

$\Longrightarrow \boxed{A = 5000 - 4900e^{-0.01t}}$

13. A tank initially contains 50 litres of pure water. Starting at time t=0 a brine containing with 2 grams of dissolved salt per litre flows into the tank at the rate of 3 litres per minute. The mixture is kept uniform by stirring and the well- stirred mixture simultaneously flows out of the tank at the same rate. Find the amount of salt present in the tank at any time $t > 0$

In flow rate= $3 \times \frac{2}{1} = 6$

Out flow rate = $3 \times \frac{A}{50} = \frac{3A}{50}$

$\frac{dA}{dt} =$ In flow rate − Out flow rate

t	A
0	0

$\frac{dA}{dt} = 6 - \frac{3A}{50} \Longrightarrow \frac{dA}{dt} = \frac{300-3A}{50} \Longrightarrow \frac{dA}{100-A} = \frac{3}{50}dt \Longrightarrow \frac{dA}{A-100} = -\frac{3}{50}dt$

Integrating on both sides, $\boxed{A - 100 = ce^{-\frac{3}{50}t}} \to (1)$

$\underline{t = 0, A = 0}$:−

$0 - 100 = ce^{-\frac{3}{50}(0)} \Longrightarrow \boxed{c = -100}$

$\therefore (1) \Longrightarrow A - 100 = -100e^{-\frac{3}{50}t}$

$\Longrightarrow A = 100 - 100e^{-\frac{3}{50}t} \Longrightarrow \boxed{A = 100\left(1 - e^{-\frac{3}{50}t}\right)}$

14.The equation of electromotive force for an electric circuit containing resistance and self-inductance is $E = Ri + L\frac{di}{dt}$, where E is the electromotive force is given to the circuit, R the resistance and L, the coefficient of induction. Find the current I at time t when E=0.

$E = Ri + L\frac{di}{dt} \Longrightarrow \div L \Longrightarrow \frac{E}{L} = \frac{R}{L}i + \frac{di}{dt} \Longrightarrow \boxed{\frac{di}{dt} + \frac{R}{L}i = \frac{E}{L}}$ It is linear in i.

$P = \frac{R}{L}$; $Q = \frac{E}{L}$

$I.F = e^{\int pdt} = e^{\int \frac{R}{L}dt} = e^{\left(\frac{Rt}{L}\right)}$

Solution: $i(I.F) = \int Q(I.F)dt + c$

$\Longrightarrow i\left(e^{\left(\frac{Rt}{L}\right)}\right) = \int \frac{E}{L}\left(e^{\left(\frac{Rt}{L}\right)}\right)dt + c$

$\Longrightarrow i\left(e^{\left(\frac{Rt}{L}\right)}\right) = \frac{E}{L} \times \frac{L}{R} \times e^{\left(\frac{Rt}{L}\right)} + c$

$\Longrightarrow i\left(e^{\left(\frac{Rt}{L}\right)}\right) = \frac{E}{R}e^{\left(\frac{Rt}{L}\right)} + c \quad \div e^{\left(\frac{Rt}{L}\right)} \Longrightarrow \boxed{i = \frac{E}{R} + Ce^{\left(-\frac{Rt}{L}\right)}}$

11.PROBABILITY DISTRIBUTION

2 MARK QUESTIONS

1. An urn contains 2 white balls and 3 red balls. A sample of 3 balls are chosen at random from the urn. If X denotes the number of red balls chosen, find the values taken by the random variable X and its number of inverse images. Number of white balls= 2.Number of red balls= 3. The random variable X takes the values 1,2,3.

$n(s) = 5C_3 = \frac{5.4.3}{1.2.3} = 10$

$X(1) = 3C_1 \times 2C_2 = 3 \times 1 = 3$

$X(2) = 3C_2 \times 2C_1 = 3 \times 2 = 6$

$X(3) = 3C_3 \times 2C_0 = 1 \times 1 = 1$

X	1	2	3	Total
Number of points in inverse images	3	6	1	10

2. Suppose X is the number of tails occurred when three fair coins are tossed once simultaneously. Find the values of the random variable X and number of points in its inverse images.

$S = \{HHH, HHT, HTH, THH, TTT, TTH, THT, HTT\} n(s) = 8$.

Let X be the random variable of getting tails. The values of X are 0,1,2,3

X	0	1	2	3	Total
Number of points in inverse images	1	3	3	1	8

3. In a pack of 52cards, two cards are drawn at random simultaneously. If the number of black cards drawn is a random variable, find the values of the random variable and the number of points in its inverse images.

$n(s) = 52C_2 = \frac{52.51}{1.2} = 1326.$

Number of black cards= 26.Number of red cards = 26.

Let X be the random variable of getting black cards. The values of X are 0,1,2.

$X(No\ black\ card) \Longrightarrow 26C_0 \times 26C_2 = 1.\frac{26.25}{1.2} = 325$

$X(One\ black\ card) \Longrightarrow 26C_1 \times 26C_1 = 26.26 = 676$

$X(Two\ black\ card) \Longrightarrow 26C_2 \times 26C_0 = \frac{26.25}{1.2}.1 = 325$

X	0	1	2	Total
Number of points in inverse images	325	676	325	1326

4. A six sided die is marked '2' on one face, '3' on two of its faces, and '4' on remaining three faces. The die is thrown twice. If X denotes the total score in two throws, find the values of the random variable and number of points in its inverse images.

Numbers on the face 2,3,3,4,4,4. $n(s) = 36.$

Let X be the random variable of getting total of face values. The values of X are 4,5,6,7,8.

I/II	2	3	3	4	4	4
2	4	5	5	6	6	6
3	5	6	6	7	7	7
3	5	6	6	7	7	7
4	6	7	7	8	8	8
4	6	7	7	8	8	8
4	6	7	7	8	8	8

From the table,

X	4	5	6	7	8	Total
Number of points in inverse images	1	4	10	12	9	36

5. Three fair coins are tossed simultaneously. Find the probability mass function for number of heads occurred.

$S=\{HHH,HHT,HTH,THH,TTT,TTH,THT,HTT\} \Rightarrow n(S)=8$

Let X be the random variable of getting heads. The values of X are 0,1,2,3

$f(0)=p(X=0)=\frac{1}{8}$ $\quad f(1)=p(X=1)=\frac{3}{8}$

$f(2)=p(X=2)=\frac{3}{8}$ $\quad f(3)=p(X=3)=\frac{1}{8}$

∴ The Probability mass function is,

x	0	1	2	3
$f(x)$	$\frac{1}{8}$	$\frac{3}{8}$	$\frac{3}{8}$	$\frac{1}{8}$

6. The probability density function of X is given by $f(x)=\begin{cases}kxe^{-2x}, & x>0\\ 0, & x\le 0\end{cases}$. Find the value of k.

$\int_{-\infty}^{\infty} f(x)dx = 1 \Rightarrow \int_0^{\infty} kxe^{-2x} = 1$

$\Rightarrow k\left[\frac{1!}{2^{1+1}}\right]=1$

$\Rightarrow \boxed{k=4}$

7. If X is the random variable with distribution function F(x) given by, $F(x)=\begin{cases}0, & x<0\\ x, & 0\le x<1\\ 1, & 1\le x\end{cases}$ then find (i) the probability density function $f(x)$ (ii) $P(0.2 \le X \le 0.7)$.

(i) The probability density function $f(x)=\begin{cases}1, 0\le x<1\\ 0, otherwise\end{cases}$

$(ii) P(0.2\le x\le 0.7)=F(0.7)-F(0.2)=0.7-0.2=0.5$

8. If X is a random variable with distribution function F(x) given by, $F(x)=\begin{cases}0, & -\infty<x<0\\ \frac{1}{2}(x^2+x), & 0\le x<1\\ 1, & 1\le x<\infty\end{cases}$ then find (i) the probability density function $f(x)$ (ii) $P(0.3\le X\le 0.6)$

(i) The probability density function $f(x)=\begin{cases}\frac{1}{2}(2x+1), & 0\le x<1\\ 0, & otherwise\end{cases}$

$(ii)\ P(0.3\le x\le 0.6)=F(0.6)-F(-0.3)$

$=\frac{1}{2}\{(0.6^2+0.6)-(0.3^2+0.3)\}$

$=0.285$

9. Find the binomial distribution if five fair coins are tossed once and X denotes the number of heads.

$p=\frac{1}{2} \Rightarrow q=1-p=\frac{1}{2}, n=5$

$f(x)=nc_x p^x q^{n-x}$

$f(x)=5c_x\left(\frac{1}{2}\right)^x\left(\frac{1}{2}\right)^{5-x}, x=0,1,2,\dots,n$

10.Using binomial distribution find the mean and variance of X if a fair coin is tossed 100 times, and X denote the number of heads .

$n=100, p=\frac{1}{2}, q=\frac{1}{2}$

$Mean=np=(100)\left(\frac{1}{2}\right)=50$

$Variance=npq=(100)\left(\frac{1}{2}\right)\left(\frac{1}{2}\right)=25$

11. The probability that a certain kind of component will survive a electrical test is $\frac{3}{4}$. Find the probability that exactly 3 of the 5 components tested survive.

$n=5, p=\frac{3}{4}, q=\frac{1}{4}$

$f(x)=nc_x p^x q^{n-x} \Rightarrow P(3)=5c_3\left(\frac{3}{4}\right)^3\left(\frac{1}{4}\right)^2$

3MARK QUESTIONS

1.Suppose two coins are tossed once. If X denotes the number of tails, (i) write down the sample space (ii) find the inverse image of 1 (iii) the values of the random variable and number of elements in its inverse images.

$(i) S=\{HH,HT,TH,TT\} \Rightarrow n(S)=4$

$(ii) Inverse\ image\ of\ 1=\{HT,TH\}$

(iii) $X = 0,1,2$

X	0	1	2	Total
Number of points in inverse images	1	2	1	4

2 Two fair coins are tossed simultaneously (equivalent to a fair coin is tossed twice). Find the probability mass function for number of heads occurred. $S = \{HH, HT, TH, TT\} \Rightarrow n(S) = 4$

Let X be the random variable of getting heads and the values of X are 0,1,2.

$f(0) = p(X = 0) = \frac{1}{4}$ $\quad f(1) = p(X = 1) = \frac{2}{4}$

$f(2) = p(X = 2) = \frac{1}{4}$

∴ The Probability mass function is,

x	0	1	2
$f(x)$	$\frac{1}{4}$	$\frac{2}{4}$	$\frac{1}{4}$

3. Two balls are chosen randomly from an urn containing 6 red and 8 black balls. Suppose that we win Rs.15 for each red ball selected and we lose Rs.10 for each black ball selected. If X denotes the winning amount, find the values of X and number of points in its inverse images.

Red(R)	Black (B)	Total
6	8	14

Let X be the winning amount. $n(S) = 14c_2 = \frac{14.13}{1.2} = 91$

$X(2\ red\ balls) = 15 + 15 = 30.$ $\quad X(1\ red\ balls) = 15 - 10 = 5.$

$X(0\ red\ balls) = -10 - 10 = -20.$

The values of X are $30, 5$ and -20.

$X(30) = 6c_2 \times 8c_0 = \frac{6.5}{1.2} \times 1 = 15.$

$X(5) = 6c_1 \times 8c_1 = 6 \times 8 = 48.$

$X(-20) = 6c_0 \times 8c_2 = 1 \times \frac{8.7}{1.2} = 28.$

X	30	5	−20	Total
Number of points in inverse images	15	48	28	91

4. Two balls are chosen randomly from an urn containing 6 white and 4 black balls. Suppose that we win Rs.30 for each red ball selected and we lose Rs.20 for each black ball selected. If X denotes the winning amount, find the values of X and number of points in its inverse images.

White(W)	Black (B)	Total
6	4	10

$n(s) = 10C_2 = \frac{10.9}{1.2} = 45$

Let X be the winning amount.

$X(2\ black\ balls) = 30 + 30 = 60.$

$X(1\ black\ ball) = 30 - 20 = 10.$

$X(0\ black\ ball) = -20 - 20 = -40.$

The values of X are $60, 10$ *and* -40

$X = 60 \Rightarrow 4C_2 \times 6C_0 = 6.1 = 6$

$X = 10 \Rightarrow 4C_1 \times 6C_1 = 4.6 = 24$

$X = -40 \Rightarrow 4C_0 \times 6C_2 = 1.15 = 15$

X	60	10	−40	Total
Number of points in inverse images	6	24	15	45

5.An urn contains 5 mangoes and 4 apples. Three fruits are taken at random. If the number of black cards drawn is a random variable, find the values of the random variable and the number of points in its inverse images

Mango(M)	Apple(A)	Total
5	4	9

Let X be the random variable of getting apple and the values of X are 0,1,2 *and* 3.

$n(S) = 9c_3 = \frac{9.8.7}{1.2.3} = 84$

$X(MMM) = 5c_3 \times 4c_0 = \frac{5.4.3}{1.2.3} \times 1 = 10$

$X(MMA) = 5c_2 \times 4c_1 = \frac{5.4}{1.2} \times 4 = 40$

$X(MAA) = 5c_1 \times 4c_2 = 5 \times \frac{4.3}{1.2} = 30$

$X(AAA) = 5c_0 \times 4c_3 = 1 \times 4 = 4$

X	0	1	2	3	Total
Number of points in inverse images	10	40	30	4	84

6. Find the probability mass function and cumulative distribution function of number of girl child in families with 4 children, assuming equal probabilities for boys and girls.

$n(S) = 2^4 = 16.$

Let X be the random variable of selecting girl and the values of X are0,1,2,3,4.

$f(0) = P(X = 0) = \frac{4C_0}{16} = \frac{1}{16}$

$f(1) = P(X = 1) = \frac{4C_1}{16} = \frac{4}{16}$

$f(2) = P(X = 2) = \frac{4C_2}{16} = \frac{6}{16}$

$f(3) = P(X = 3) = \frac{4C_3}{16} = \frac{4}{16} p(X = 4) = \frac{4C_4}{16} = \frac{1}{16}$

∴ The Probability mass function:

x	0	1	2	3	4
$f(x)$	$\frac{1}{16}$	$\frac{4}{16}$	$\frac{6}{16}$	$\frac{4}{16}$	$\frac{1}{16}$

∴ The cumulative distribution function:

x	0	1	2	3	4
$F(x)$	$\frac{1}{16}$	$\frac{5}{16}$	$\frac{11}{16}$	$\frac{15}{16}$	1

7. A pair of fair dice is rolled once. Find the probability mass function to get the number of fours.

$$S = \begin{Bmatrix} (1,1) & (1,2) & (1,3) & (1,4) & (1,5) & (1,6) \\ (2,1) & (2,2) & (2,3) & (2,4) & (2,5) & (2,6) \\ (3,1) & (3,2) & (3,3) & (3,4) & (3,5) & (3,6) \\ (4,1) & (4,2) & (4,3) & (4,4) & (4,5) & (4,6) \\ (5,1) & (5,2) & (5,3) & (5,4) & (5,5) & (5,6) \\ (6,1) & (6,2) & (6,3) & (6,4) & (6,5) & (6,6) \end{Bmatrix} \Longrightarrow n(S) = 36$$

Let X be the random variable of getting 4 *and the values of X are*0,1,2.

$f(0) = p(X = 0) = \frac{25}{36}$

$f(1) = p(X = 1) = \frac{10}{36}$

$f(2) = p(X = 2) = \frac{1}{36}$

∴ The probability mass function is

x	0	1	2
$f(x)$	$\frac{25}{36}$	$\frac{10}{36}$	$\frac{1}{36}$

8. If the probability mas function $f(x)$ of a random variable X is

x	1	2	3	4
$f(x)$	$\frac{1}{12}$	$\frac{5}{12}$	$\frac{5}{12}$	$\frac{1}{12}$

Find (i)its cumulative distribution function, hence find (ii) $P(X \leq 3)$ and$(iii)P(X \geq 2)$

(i) The cumulative distribution function:

x	1	2	3	4
$F(x)$	$\frac{1}{12}$	$\frac{6}{12}$	$\frac{11}{12}$	1

$(ii)P(X \leq 3) = F(3) = \frac{11}{12}$

$(iii)P(X \geq 2) = 1 - P(X < 2) = 1 - F(1) = 1 - \frac{1}{12} = \frac{11}{12}$

9. Find the probability mass function f(x) of the discrete random variable X whose cumulative distribution function F(x) is given by

$$F(x) = \begin{cases} 0 & -\infty < x < -2 \\ 0.25 & -2 \leq x < -1 \\ 0.60 & -1 \leq x < 0 \\ 0.90 & 0 \leq x < 1 \\ 1 & 1 \leq x < \infty \end{cases}$$

. *Also find* $(i)\ P(X < 0)$ and $(ii)P(X \geq -1)$.

The values of the discrete random variable X are $-2, -1, 0, 1$.

∴ The Probability mass function $f(x)$:

x	-2	-1	0	1
$F(x)$	0.25	0.60	0.90	1
$f(x)$	0.25	0.35	0.30	0.10

$(i)P(X < 0) = P(X = -2) + P(X = -1) = 0.25 + 0.35 = 0.60$

$(ii)P(X \geq -1) = P(X = -1) + P(X = 0) + P(X = 1) = 0.35 + 0.30 + 0.10 = 0.75$

10. The cumulative distribution function of a discrete random variable is given by

$$F(x) = \begin{cases} 0 & ; -\infty < x < 0 \\ \frac{1}{2} & ; 0 \leq x < 1 \\ \frac{3}{5} & ; 1 \leq x < 2 \\ \frac{4}{5} & ; 2 \leq x < 3 \\ \frac{9}{10} & ; 3 \leq x < 4 \\ 1 & ; 4 \leq x < \infty \end{cases}$$

. Find (i)*the* probability mass function $(ii)\ p(X < 3)$ and $(iii)P(X \geq 2)$.

The values of the discrete random variable X are 0,1,2,3,4.

(i)The Probability mass function $f(x)$:

x	0	1	2	3	4
$F(x)$	$\frac{1}{2}$	$\frac{3}{5}$	$\frac{4}{5}$	$\frac{9}{10}$	1
$f(x)$	$\frac{1}{2}$ *or* $\frac{5}{10}$	$\frac{1}{5}$ *or* $\frac{2}{10}$	$\frac{1}{10}$	$\frac{1}{10}$	$\frac{1}{10}$

$(i) P(X<3) = P(X=0)+P(X=1)+P(X=2) = \frac{5}{10}+\frac{1}{10}+\frac{2}{10} = \frac{8}{10} = \frac{4}{5}$

$(ii) P(X \geq 2) = P(X=2)+P(X=3)+P(X=4) = \frac{2}{10}+\frac{1}{10}+\frac{1}{10} = \frac{4}{10} = \frac{2}{5}$

11. Two balls are drawn in succession without replacement from an urn containing 4 red balls and 3 black balls. Let X be the possible outcomes drawing red balls. Find the probability mass function and mean for X.

X *denote the number of red balls.* $X = 0,1,2.$

$P(X=0) = \frac{3}{7}\times\frac{2}{6} = \frac{1}{7}$

$P(X=1) = \left(\frac{4}{7}\times\frac{3}{6}\right)\times\left(\frac{3}{7}\times\frac{4}{6}\right) = \frac{4}{7}$

$P(X=2) = \frac{4}{7}\times\frac{3}{6} = \frac{2}{7}$

Probability mass function

x	0	1	2
$F(x)$	$\frac{1}{7}$	$\frac{4}{7}$	$\frac{2}{7}$

$Mean = \sum x f(x) = \left(0\times\frac{1}{7}\right)+\left(1\times\frac{4}{7}\right)+\left(2\times\frac{2}{7}\right) = \frac{8}{7}$

12. The time to failure in thousands of hours of an electronic equipment used in a manufactured computer has the density function $f(x) = \begin{cases} 3e^{-3x}, for x > 0 \\ 0, otherwise \end{cases}$

Find the expected life of this electronic equipment

$Mean = \int_a^b x f(x) dx$

$= \int_0^\infty x(3e^{-3x}) dx$

$= \left[-xe^{-3x} - \frac{1}{3}e^{-3x}\right]_0^\infty$

$= \frac{1}{3}$

5 MARK QUESTIONS

1. Suppose a pair of unbiased dice is rolled once. If X denotes the total score of two dice, write down (i)the sample space (ii)the values taken by the random variable X, (iii)the inverse image of 10, and (iv) the number of elements in inverse image of X.

$$(i) S = \begin{Bmatrix} (1,1) & (1,2) & (1,3) & (1,4) & (1,5) & (1,6) \\ (2,1) & (2,2) & (2,3) & (2,4) & (2,5) & (2,6) \\ (3,1) & (3,2) & (3,3) & (3,4) & (3,5) & (3,6) \\ (4,1) & (4,2) & (4,3) & (4,4) & (4,5) & (4,6) \\ (5,1) & (5,2) & (5,3) & (5,4) & (5,5) & (5,6) \\ (6,1) & (6,2) & (6,3) & (6,4) & (6,5) & (6,6) \end{Bmatrix} \Rightarrow n(S) = 36$$

(ii) *The values of* X *are* $2,3,4,5,6,7,8,9,10,11$ *and* $12.$

(iii) The inverse image of $10 = \{(4,6),(5,5),(6,4)\}.$

(iv)

X	0	1	2	Total
Number of elements in inverse images	1	2	1	4

2. A six sided die is marked '1' on one face, '2' on two of its faces, and '3' on remaining three faces. The die is thrown twice. If X denotes the total score in two throws, find (i) the probability mass function (ii) the cumulative distribution function $(iii) P(3 \leq X \leq 6)$ $(iv)\ P(X \geq 4)$.

The numbers on the faces are 1,2,2,3,3,3. $n(s) = 36$.

I/II	1	2	2	3	3	3
1	2	3	3	4	4	4
2	3	4	4	5	5	5
2	3	4	4	5	5	5
3	4	5	5	6	6	6
3	4	5	5	6	6	6
3	4	5	5	6	6	6

The values of X are 2,3,4,5,6.

From the table,

(i) The probability mass function is

x	2	3	4	5	6
$f(x)$	$\frac{1}{36}$	$\frac{4}{36}$	$\frac{10}{36}$	$\frac{12}{36}$	$\frac{9}{36}$

(ii) The cumulative distribution function:

x	2	3	4	5	6
$F(x)$	$\frac{1}{36}$	$\frac{5}{36}$	$\frac{15}{36}$	$\frac{27}{36}$	1

$(iii) P(3 \leq X \leq 6) = P(X=3)+P(X=4)+P(X=5) = \frac{4}{36}+\frac{10}{36}+\frac{12}{36} = \frac{26}{36} = \frac{13}{18}$

$(iv) P(X \geq 4) = P(X=4)+P(X=5)+P(X=6) = \frac{10}{36}+\frac{12}{36}+\frac{9}{36} = \frac{31}{36}$

3. A six sided die is marked '1' on one face, '3' on two of its faces, and '5' on remaining three faces. The die is thrown twice. If X denotes the total score in two throws, find (i)the probability mass function (ii)the cumulative distribution function $(iii) P(4 \leq X \leq 10)$ $(iv) P(X \geq 6)$.

The numbers on the faces are 1,3,3,5,5,5. $n(s) = 36$

I/II	1	3	3	5	5	5
1	2	4	4	6	6	6
3	4	6	6	8	8	8
3	4	6	6	8	8	8
5	6	8	8	10	10	10
5	6	8	8	10	10	10
5	6	8	8	10	10	10

The values of X are 2,4,6,8,10.

From the table,

(i)The probability mass function is

x	2	4	6	8	10
$f(x)$	$\frac{1}{36}$	$\frac{4}{36}$	$\frac{10}{36}$	$\frac{12}{36}$	$\frac{9}{36}$

(ii)The cumulative distribution function:

x	2	4	6	8	10
$F(x)$	$\frac{1}{36}$	$\frac{5}{36}$	$\frac{15}{36}$	$\frac{27}{36}$	1

$(iii)P(4 \le X < 10) = P(X=4) + P(X=6) + P(X=8) = \frac{4}{36} + \frac{10}{36} + \frac{12}{36} = \frac{26}{36} = \frac{13}{18}$

$(iv)P(X \ge 6) = P(X=6) + P(X=8) + P(X=10) = \frac{10}{36} + \frac{12}{36} + \frac{9}{36} = \frac{31}{36}$

4. A random variable X has the following probability mass function

x	1	2	3	4	5	6
$f(x)$	k	$2k$	$6k$	$5k$	$6k$	$10k$

Find $(i)\ p(2 < X < 6)$ $(ii)\ p(2 \le X < 5)$ $(iii)\ p(X \le 4)$ $(iv) p(3 < X)$

$\sum \mathrm{f(x)} = 1 \Rightarrow \mathrm{k + 2k + 6k + 5k + 6k + 10k = 1} \Rightarrow 30\mathrm{k} = 1 \Rightarrow \boxed{\mathrm{k} = \frac{1}{30}}$

$(i)\ p(2 < X < 6) = f(3) + f(4) + f(5) = 6k + 5k + 6k = 17k = \frac{17}{30}$

$(ii)\ p(2 \le X < 5) = f(2) + f(3) + f(4) = 2k + 6k + 5k = 13k = \frac{13}{30}$

$(iii)\ p(X \le 4) = f(1) + f(2) + f(3) + f(4) = k + 2k + 6k + 5k = 14k = \frac{14}{30}$

$(iv)p(3 < X) = f(4) + f(5) + f(6) = 5k + 6k + 10k = 21k = \frac{21}{30}$

5. Suppose a discrete random variable can only take the values 0,1and 2. The probability mass function is defined by $f(x) = \begin{cases} \frac{x^2+1}{k}, x = 0,1,2 \\ 0, x - otherwise \end{cases}$ .Find (i) the value of k (ii)cumulative distribution function$(iii)P(X \ge 1)$.

$f(x) = \frac{x^2+1}{k}$

$f(0) = \frac{0^2+1}{k} = \frac{1}{k};$ $f(1) = \frac{1^2+1}{k} = \frac{2}{k};$ $f(2) = \frac{2^2+1}{k} = \frac{5}{k}.$

$(i) \sum \mathrm{f(x)} = 1 \Rightarrow \frac{1}{k} + \frac{2}{k} + \frac{5}{k} = 1 \Rightarrow \frac{8}{k} = 1 \Rightarrow \boxed{k = 8}$

The probability mass function is:

x	0	1	2
$f(x)$	$\frac{1}{8}$	$\frac{2}{8}$	$\frac{5}{8}$

(ii)The cumulative distribution function is

x	0	1	2
$F(x)$	$\frac{1}{8}$	$\frac{3}{8}$	1

$(iii)P(X \ge 1) = 1 - P(X < 1) = 1 - P(X \le 0) = 1 - \frac{1}{8} = \frac{7}{8}$

6. A random variable X has the following probability mass function

x	1	2	3	4	5
$f(x)$	k^2	$2k^2$	$3k^2$	$2k$	$3k$

Find $(i)$the value of k $(ii)\ p(2 \le X < 5)$ $(iii)p(3 < X)$.

$(i) \sum \mathrm{f(x)} = 1 \Rightarrow k^2 + 2k^2 + 3k^2 + 2k + 3k = 1$

$\Rightarrow 6k^2 + 5k - 1 = 0$

$\Rightarrow (6k - 1)(k + 1) = 0$

$\Rightarrow k = \frac{1}{6}, k = -1(Not\ a\ soln.)$

$(ii)\ p(2 \le X < 5) = P(X=2) + P(X=3) + P(X=4) = 2k^2 + 3k^2 + 2k$

$= 5k^2 + 2k$

$= 5\left(\frac{1}{6}\right)^2 + 2\left(\frac{1}{6}\right)$

$= \frac{5}{36} + \frac{2}{6} = \frac{17}{36}$

$(iii)p(3 < X) = P(X=4) + P(X=5) = 2k + 3k = 5k = \frac{5}{6}$

7.Find the constant C such that the function $f(x) = \begin{cases} Cx^2, & 1 < x < 4 \\ 0, & x - \text{otherwise} \end{cases}$ is a density function,and compute$(i)P(1.5 < X < 3.5)(ii)P(X \leq 2)(iii)P(3 < X)$.

Since f(x) is a p.d.f $\int_{-\infty}^{\infty} f(x)dx = 1 \Rightarrow \int_1^4 Cx^2 dx = 1 \Rightarrow C\left[\frac{x^3}{3}\right]_1^4 = 1$

$$\Rightarrow C\left[\frac{4^3}{3} - \frac{1^3}{3}\right] = 1$$

$$\Rightarrow C\left[\frac{64-1}{3}\right] = 1$$

$$\Rightarrow C[21] = 1$$

$$\Rightarrow \boxed{C = \frac{1}{21}}$$

$$(i)P(1.5 < X < 3.5) = \int_{1.5}^{3.5} \frac{1}{21}x^2 dx = \frac{1}{21}\left[\frac{x^3}{3}\right]_{1.5}^{3.5} = \frac{1}{21}\left[\frac{3.5^3 - 1.5^3}{3}\right] = \frac{79}{126}$$

$$(ii)P(X \leq 2) = \int_1^2 \frac{1}{21}x^2 dx = \frac{1}{21}\left[\frac{x^3}{3}\right]_1^2 = \frac{1}{21}\left[\frac{2^3 - 1^3}{3}\right] = \frac{7}{63}$$

$$(iii)P(3 < X) = \int_3^4 \frac{1}{21}x^2 dx = \frac{1}{21}\left[\frac{x^3}{3}\right]_3^4 = \frac{1}{21}\left[\frac{4^3 - 3^3}{3}\right] = \frac{37}{63}$$

8.If X is the random variable with probability density function f(x) given by,

$f(x) = \begin{cases} x-1, & 1 \leq x < 2 \\ -x+3, & 2 \leq x < 3 \\ 0, & x - otherwise \end{cases}$. Find the (i) the distribution function F(X)

$(ii)P(1.5 \leq X \leq 2.5)$.

$(i)F(x) = P(X = x) = \int_{-\infty}^{x} f(u)du$

$\boxed{When\ x < 1}$,

$$F(x) = \int_{-\infty}^{x} 0du = 0$$

$\boxed{When\ 1 \leq x < 2}$,

$$F(x) = \int_{-\infty}^{1} f(u)du + \int_1^x (u-1)du = 0 + \left[\frac{(u-1)^2}{2}\right]_1^x = \frac{(x-1)^2}{2} - 0 = \frac{(x-1)^2}{2}$$

$\boxed{When\ 2 \leq x < 3}$,

$$F(x) = \int_{-\infty}^{1} f(u)du + \int_1^2 (u-1)du + \int_2^x (-u+3)du$$

$$= 0 + \left[\frac{(u-1)^2}{2}\right]_1^2 + \left[-\frac{(-u+3)^2}{2}\right]_2^x$$

$$= \left(\frac{1}{2} - 0\right) + \left[-\frac{(-x+3)^2}{2} + \frac{1}{2}\right]$$

$$= 1 - \frac{(3-x)^2}{2}$$

$\boxed{When\ X \geq 3}$,

$$F(x) = \int_{-\infty}^{1} f(u)du + \int_1^2 (u-1)du + \int_2^3 (-u+3)du + \int_3^x 0du$$

$$= 0 + \left[\frac{(u-1)^2}{2}\right]_1^2 + \left[-\frac{(-u+3)^2}{2}\right]_2^3 + 0$$

$$= \left(\frac{1}{2} - 0\right) + \left(-0 + \frac{1}{2}\right)$$

$$= 1$$

$$\therefore F(x) = \begin{cases} 0, & -\infty < x < 1 \\ \frac{(x-1)^2}{2}, & 1 \leq x < 2 \\ 1 - \frac{(3-x)^2}{2}, & 2 \leq x < 3 \\ 1, & 3 \leq x < \infty \end{cases}$$

$(ii)P(1.5 \leq X \leq 2.5) = F(2.5) - F(1.5) = \left[1 - \frac{(3-2.5)^2}{2}\right] - \left[\frac{(1.5-1)^2}{2}\right] = \frac{1.75 - 0.25}{2} = 0.75$

9. If X is the random variable with probability density function f(x) given by,

$f(x) = \begin{cases} x, 0 < x < 1 \\ 2-x, 1 \leq x < 2 \\ 0, otherwise \end{cases}$. $Find\ \ (i)P(0.2 \leq x < 0.6)\ \ (ii)P(1.2 \leq x < 1.8)$

$(iii)P(0.5 \leq x < 1.5)$.

$(i)P(0.2 \leq x < 0.6) = \int_{0.2}^{0.6} xdx = \left[\frac{x^2}{2}\right]_{0.2}^{0.6} = \frac{1}{2}[0.36 - 0.04] = 0.16$

$(ii)P(1.2 \leq x < 1.8) = \int_{0.2}^{0.6}(2-x)dx = \left[2x - \frac{x^2}{2}\right]_{1.2}^{1.8} = (3.60 - 1.62) - (2.4 - 0.72) = 0.3$

$(iii)P(0.5 \leq x < 1.5) = \int_{0.5}^{1} xdx + \int_1^{1.5}(2-x)dx$

$$= \left[\frac{x^2}{2}\right]_{0.5}^{1} + \left[2x - \frac{x^2}{2}\right]_1^{1.5}$$

$$= (0.5 - 0.125) + (3 - 0.125) - (2 - 0.5) = 0.75$$

10. Suppose the amount of milk sold daily at a milk booth is distributed with a minimum of 200 litres and a maximum of 600 litres with probability density function

$f(x) = \begin{cases} k & 200 \leq x \leq 600 \\ 0 & otherwise \end{cases}$ $Find\ (i)\ the\ value\ of\ k\ \ (ii)\ the\ distribution\ function$

$(iii)\ the\ probability\ that\ daily\ sales\ will\ fall\ between\ 300\ litres\ and\ 500\ litres?$

(i) Since f(x) is a p.d.f $\int_{-\infty}^{\infty} f(x)dx = 1 \Rightarrow \int_{200}^{600} kdx = 1 \Rightarrow k[x]_{200}^{600} = 1$

$\Rightarrow k(600-200)=1$

$\Rightarrow k=\frac{1}{400}$

$(ii) F(x)=P(X=x)=\int_{-\infty}^{x} f(u)du$

When $x<200$,

$F(x)=\int_{-\infty}^{x} 0du=0$

When $200\le x\le 600$,

$F(x)=\int_{-\infty}^{200} 0du+\int_{200}^{x}\frac{1}{400}du=0+\frac{1}{400}[u]_{200}^{x}=\frac{x}{400}-\frac{1}{2}$

When $x>600$,

$F(x)=\int_{-\infty}^{200} 0du+\int_{200}^{600}\frac{1}{400}du+\int_{600}^{x}\frac{1}{400}du$

$=0+\frac{1}{400}[x]_{200}^{600}+0$

$=1$

$$\therefore F(x)=\begin{cases}0 &,x<200\\ \frac{x}{400}-\frac{1}{2} &,200\le x\le 600\\ 1 &,x>600\end{cases}$$

$(iii) P(300\le x\le 500)=F(500)-F(300)=\left[\frac{500}{400}-\frac{1}{2}\right]-\left[\frac{300}{400}-\frac{1}{2}\right]=\frac{5}{4}-\frac{1}{2}-\frac{3}{4}+\frac{1}{2}=\frac{1}{2}$

11 *The probability density function X is given by* $f(x)=\begin{cases}ke^{-x/3}, x>0\\ 0, x\le 0\end{cases}$

Find (i) the value of k (ii) the distribution function $(iii) P(X<3)$ $(iv) P(5\le X)$ $(v) P(X\le 4)$.

(i) Since f(x) is a p.d.f $\int_{-\infty}^{\infty} f(x)dx=1\Rightarrow\int_{0}^{\infty} ke^{-x/3}dx=1\Rightarrow k\left[-3e^{-x/3}\right]_{0}^{\infty}=1$

$\Rightarrow -3k(0-1)=1$

$\Rightarrow k=\frac{1}{3}$

$(ii) F(x)=P(X=x)=\int_{-\infty}^{x} f(u)du$

When $x\le 0$,

$F(x)=\int_{-\infty}^{x} 0du=0$

When $x>0$,

$F(x)=\int_{-\infty}^{0} 0du+\int_{0}^{x}\frac{1}{3}e^{-u/3}du$

$=0+\frac{1}{3}\left[-3e^{-u/3}\right]_{0}^{x}$

$=1-e^{-x/3}$

$$\therefore F(x)=\begin{cases}0 & x\le 0\\ 1-e^{-x/3} & x>0\end{cases}$$

$(iii) P(X<3)=F(3)=1-e^{-3/3}=1-\frac{1}{e}$

$(iv) P(5\le X)=1-P(x<5)=1-\left(1-e^{-5/3}\right)=e^{-5/3}$

$(v) P(X\le 4)=F(4)=1-e^{-4/3}$

12. Let X be a random variable denoting the life time of an electrical equipment having probability density function $f(x)=\begin{cases}ke^{-2x}, & x>0\\ 0, & x\le 0\end{cases}$ Find (i) the value of k (ii) Distribution function (iii) $P(X<2)$ (iv) calculate the probability that X is at least for four unit of time (v) $P(X=3)$

(i) Since f(x) is a p.d.f $\int_{-\infty}^{\infty} f(x)dx=1\Rightarrow\int_{0}^{\infty} ke^{-2x}dx=1\Rightarrow k\left[\frac{e^{-2x}}{2}\right]_{0}^{\infty}=1$

$\Rightarrow k\left(0-\frac{1}{2}\right)=1$

$\Rightarrow k=2$

$(ii) F(x)=P(X=x)=\int_{-\infty}^{x} f(u)du$

When $x\le 0$,

$F(x)=\int_{-\infty}^{x} 0du=0$

When $x>0$,

$F(x)=\int_{-\infty}^{0} 0du+\int_{0}^{x} 2e^{-2u}du$

$=0+2\left[\frac{e^{-2u}}{-2}\right]_{0}^{x}$

$=1-e^{-2x}$

$$\therefore F(x)=\begin{cases}0 & x\le 0\\ 1-e^{-2x} & x>0\end{cases}$$

$(iii) P(X<2)=F(2)=1-e^{-2(2)}=1-e^{-4}$

$(iv) P(X\ge 4)=1-P(X<4)$

$=1-F(4)$

$=1-(1-e^{-2\times 4})$

$=e^{-8}$

$(v) P(X=3)=0$

13. Suppose that f(x) given below represents a probability mass function,

x	1	2	3	4	5	6
$f(x)$	c^2	$2c^2$	$3c^2$	$4c^2$	c	$2c$

Find (i) the value of c (ii) Mean and Variance..

$(i)\ \sum f(x) = 1 \Rightarrow 9c^2 + 3c = 1 \quad \Rightarrow c = \frac{1}{5}\ (or) -\frac{1}{2}\ (not\ a\ solution)$

x	1	2	3	4	5	6
$f(x)$	$\frac{1}{25}$	$\frac{2}{25}$	$\frac{3}{25}$	$\frac{4}{25}$	$\frac{1}{5}$	$\frac{2}{5}$
$xf(x)$	$\frac{1}{25}$	$\frac{4}{25}$	$\frac{9}{25}$	$\frac{16}{25}$	$\frac{5}{5}$	$\frac{12}{5}$
$x^2f(x)$	$\frac{1}{25}$	$\frac{8}{25}$	$\frac{27}{25}$	$\frac{64}{25}$	$\frac{25}{5}$	$\frac{72}{5}$

$Mean\ E(X) = \sum xf(x) = \frac{115}{25} = 4.6$

$Variance\ V(X) = E(X^2) - \left(E(X)\right)^2$

$= \sum x^2 f(x) - [\sum xf(x)]^2$

$= \frac{585}{25} - \left[\frac{115}{25}\right]^2$

$= 2.24$

14. Two balls are chosen randomly from an urn containing 8 white and 4 black balls. Suppose that we win Rs.20 for each black ball selected and we lose Rs.10 for each white ball selected. Find the expected winning amount and variance.

$No.of\ white\ balls = 8.$

$No.of\ white\ balls = 4.$

$n(s) = 12C_2 = \frac{12.11}{1.2} = 66$

X be the winning amount.$X = 0,1,2$

$2\ black\ balls\ means\ winning\ amount = 20 + 20 = 40.$

$1\ black\ ball\ means\ winning\ amount = 20 - 10 = 10.$

$0\ black\ ball\ means\ winning\ amount = -10 - 10 = -20.$

$Values\ of\ X\ are\ 60,10,-40.$

$X = 40 \Rightarrow 4C_2 \times 8C_0 = 6.1 = 6$

$X = 10 \Rightarrow 4C_1 \times 8C_1 = 4.8 = 32$

$X = -20 \Rightarrow 4C_0 \times 8C_2 = 1.15 = 28$

Probability mass function is :

X	40	10	-20	Total
$f(x)$	$\frac{6}{66}$	$\frac{32}{66}$	$\frac{28}{66}$	1

$Mean\ E(X) = \sum xf(x) = 40\left(\frac{6}{66}\right) + 10\left(\frac{32}{66}\right) + (-20)\left(\frac{28}{66}\right) = 0$

$Expected\ winning\ amount = 0$

$E(X^2) = \sum x^2 f(x) = 40^2\left(\frac{6}{66}\right) + 10^2\left(\frac{32}{66}\right) + (-20)^2\left(\frac{28}{66}\right) = \frac{4000}{11}$

$[E(X)]^2 = 0$

$Variance\ V(X) = E(X^2) - \left(E(X)\right)^2$

$= \frac{4000}{11} - 0 = \frac{4000}{11}$

15. A multiple choice examination has ten questions, each question has four distracters with exactly one correct answer. Suppose a student answers by guessing and if X denotes the number of correct answers, find (i) binomial distribution (ii) probability that the student will get seven correct answers (iii) the probability of getting at least one correct answer.

$Let\ X\ be\ the\ no.of\ questions\ .X = 0,1,2,\ldots,10$

$p = \frac{1}{4}$

$\Rightarrow q = \frac{3}{4}\ , n = 10$

$(i)\ \ Binomial\ distribution: f(x) = nc_x p^x q^{n-x}$

$\Rightarrow f(x) = 10c_x\left(\frac{1}{4}\right)^x\left(\frac{3}{4}\right)^{10-x}, x = 0,1,2,\ldots.,10$

$(ii) p(X = 7) = 10c_7\left(\frac{1}{4}\right)^7\left(\frac{3}{4}\right)^3 = 120\left(\frac{3^3}{4^{10}}\right)$

$(iii) p(X \geq 1) = 1 - p(X < 1)$

$= 1 - 10c_0\left(\frac{1}{4}\right)^0\left(\frac{3}{4}\right)^{10}$

$= 1 - \left(\frac{3}{4}\right)^{10}$

16.On the average,20% of the products manufactured by ABC company are found to be defective. If we select 6 of these products at random and X denotes the number of defective products, find the probability that (i) two products are defective (ii) at most one product is defective (iii) at least two products are defective.

$n = 6$

$Probability\ for\ defective\ item, p = 20\% = \frac{20}{100} = \frac{1}{5}$

$\Rightarrow q = \frac{4}{5}$

$Binomial\ distribution: f(x) = nc_x p^x q^{n-x}$

$$\Rightarrow f(x) = 6c_x\left(\frac{1}{5}\right)^x\left(\frac{4}{5}\right)^{6-x}, x = 0,1,2,\dots,6$$

$(i)p(X = 2) = 6c_2\left(\frac{1}{5}\right)^2\left(\frac{4}{5}\right)^4 = 15\left(\frac{4^4}{5^6}\right)$

$(ii)p(X \le 1) = p(X = 0) + p(X = 1)$

$$= 6c_0\left(\frac{1}{5}\right)^0\left(\frac{4}{5}\right)^6 + 6c_1\left(\frac{1}{5}\right)^1\left(\frac{4}{5}\right)^5$$

$$= \left(\frac{4}{5}\right)^6 + 6\left(\frac{4^5}{5^6}\right)$$

$$= 2\left(\frac{4}{5}\right)^5$$

$(iii)p(X \ge 2) = 1 - p(X < 2)$

$$= 1 - p(X \le 1)$$

$$= 1 - 2\left(\frac{4}{5}\right)^5$$

DO IT YOURSELF :

A retailer purchases a certain kind of electronic device from a manufacturer. The manufacturer indicates that the defective rate of the device is 5%. The inspector of the retailer randomly picks 10 items from a shipment. What is the probability that there will be (i) at least one defective item (ii) exactly two defective items?.

17. If the probability that a fluorescent light has a useful life of at least 600 hours is 0.9, find the probabilities that among 12 such lights (i) exactly 10 will have a useful life of at least 600 hours (ii) at least 11 will have a useful life of at least 600 hours (iii) at least 2 will not have a useful life of at least 600 hours

$n = 12, p = 0.9, q = 0.1$

$Binomial\ distribution: f(x) = nc_x p^x q^{n-x}$

$$\Rightarrow f(x) = 12c_x(0.9)^x(0.1)^{12-x}, x = 0,1,2,\dots,12$$

$(i)p(X = 10) = 12c_{10}(0.9)^{10}(0.1)^2$

$(ii)p(X \ge 11) = p(X = 11) + p(X = 12)$

$$= 12c_{11}(0.9)^{11}(0.1)^1 + 12c_{12}(0.9)^{12}(0.1)^0$$

$$= 12(0.9)^{11}(0.1) + (0.9)^{12}$$

$$= 2.1(0.9)^{11}$$

$(iii)p(X < 11) = 1 - p(X \ge 11)$

$$= 1 - 2.1(0.9)^{11}$$

12 DISCRETE MATHEMATICS

2 MARK QUESTIONS

1.Define Boolean Matrix.

A Boolean Matrixis a real matrix whose entries are either 0 or 1.

2.Define Join of A and B.

$A \vee B = [a_{ij}] \vee [b_{ij}] = [a_{ij} \vee b_{ij}] = [c_{ij}]$ where $c_{ij} = \begin{cases} 1, a_{ij} = 1 \text{ or } b_{ij} = 1 \\ 0, a_{ij} = 0 \text{ and } b_{ij} = 0 \end{cases}$

3.Define Meet of A and B.

$A \wedge B = [a_{ij}] \wedge [b_{ij}] = [a_{ij} \wedge b_{ij}] = [c_{ij}]$ where $c_{ij} = \begin{cases} 1, a_{ij} = 1 \text{ and } b_{ij} = 1 \\ 0, a_{ij} = 0 \text{ or } b_{ij} = 0 \end{cases}$

4. Write each of the following sentences in symbolic form using statement variables $p$and q.

(i) 19 is not a prime number and all the angles of a triangle are equal.

(ii) 19 is a prime number or all the angles of a triangle are not equal.

(iii) 19 is a prime number and all the angles of a triangle are equal.

(iv) 19 is not a prime number

$(i)\neg p \wedge q$, $(ii) p \vee \neg q$, $(iii) p \wedge q$, $(iv)\neg p$

5.How many rows are needed for following statement formulae?$(i) p \vee \neg t \wedge (p \vee \neg s)$

$(ii)\big((p \wedge q) \vee (\neg r \vee \neg s)\big) \wedge (\neg t \wedge v)$

(i)Number of rows$= 2^n = 2^3 = 8$

(ii)Number of rows $= 2^n = 2^6 = 64$

6 Consider p → q: If today is Monday, then $4 + 4 = 8$

p: Today is Monday. q: $4 + 4 = 8$

p	q	$p \to q$
$T\ (or) F$	T	T

3 MARK QUESTIONS

1. Verify (i)closure property (ii)commutative property and (iii)associative property of the following operations on the given set $a * b = a^b, \forall a, b \in N$.

$(i)\forall a, b \in N \Rightarrow a * b = a^b \in N$.Hence closure is true.

$(ii)\forall a, b \in N \Rightarrow a * b = a^b \ne b * a = b^a \Rightarrow a * b \ne b * a$.

Hence commutative is true.

$(iii) Let\ a = 2, b = 3, c = 4.$

$a * (b * c) = a * (b^c) = a^{(b^c)} = 2^{(3^4)} = 2^{81}$

$(a * b) * c = (a^b) * c = a^{bc} = 2^{(3\times 4)} = 2^{12}$

$\Rightarrow a * (b * c) \ne (a * b) * c.$

Hence associative is not true..

2. Let $A = \begin{bmatrix} 0 & 1 \\ 1 & 1 \end{bmatrix}, B = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$ be any two Boolean matrices of the same type. Find $A \vee B$ and

$A \wedge B$.

$A \vee B = \begin{bmatrix} 0 & 1 \\ 1 & 1 \end{bmatrix} \vee \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}$

$A \wedge B = \begin{bmatrix} 0 & 1 \\ 1 & 1 \end{bmatrix} \wedge \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 0 & 1 \end{bmatrix}$

3. On $\mathbb{Z}$, $define \otimes by (m \otimes n) = m^n + n^m;\ \forall m, n \in \mathbb{Z}.\ \ Is \otimes$ binary on $\mathbb{Z}$?

$(m \otimes n) = m^n + n^m;\ \forall m, n \in \mathbb{Z}$.
Consider, $m = 2, n = -2 \in \mathbb{Z}$.
$(2 \otimes -2) = 2^{-2} + (-2)^2 = \frac{1}{4} + 4 = \frac{17}{4} \notin \mathbb{Z}$
Hence $\otimes$ is not binary on $\mathbb{Z}$.

4. Let$*$ be defined on $\mathbb{R}$ by$(a * b) = a + b + ab - 7$.Is $*$ binary on $\mathbb{R}$? If so, find$3 * \left(-\frac{7}{15}\right)$.

Let $a, b \epsilon R. \Rightarrow a, b, ab \in R \Rightarrow a + b + ab - 7 \in R$
$\therefore (a * b) = a + b + ab - 7 \in R$
Hence $*$ is binary on$R$.
$(a * b) = a + b + ab - 7$
$\Rightarrow 3 * \left(-\frac{7}{15}\right) = 3 - \frac{7}{15} + 3\left(-\frac{7}{15}\right) - 7 = 3 - \frac{7}{15} - \frac{21}{15} - 7 = -\frac{88}{15}$

5. Fill in the following table so that the binary operation $*$ on$A = \{a, b, c\}$is commutative.

$*$	a	b	c
a	b		
b	c	b	a
c	a		c

$Since\ \ *$ satisfies commutative, $b * a = c \ \Rightarrow a * b = c$
$b * c = a \ \Rightarrow c * b = a$; $c * a = a \Rightarrow a * c = a$

$*$	a	b	c
a	b	c	a
b	c	b	a
c	a	a	c

6. Write the statements in words corresponding to $\neg p$, p ∧ q, p ∨ q and $q \vee \neg p$, where p is “ It is cold” and q is “It is raining”.

$\neg p$:. It is not cold.
p ∧ q: It is cold and raining.
p ∨ q: It is cold or raining.
$q \vee \neg p$: It is raining or it is not cold.

7. Let p:Jupiter is a planet and$q$:India is an island be any two simple statement. Give verbal sentence describing each of the following statements.
$(i) \neg p$ $(ii)\ p \wedge \neg q$ $(iii)\ \neg p \vee q$ (iv) $p \rightarrow \neg q$ $(v)\ p \leftrightarrow q$

$(i) \neg p$:Jupiter is not a planet.
$(ii)\ p \wedge \neg q$:Jupiter is a planet and India is not an island.
$(iii)\ \neg p \vee$ q:Jupiter is not a planet or India is an island .
(iv) $p \rightarrow \neg q$:If Jupiter is a planet then India is not an island .
$(v)\ p \leftrightarrow q$:Jupiter is a planet if and only if India is an island.

8. Determine the truth value of each of the following statements.
(i)6 + 2 = 5 then the milk is white.
(ii)China is in Europe or$\sqrt{3}$is an integer.
$(iii) It\ i\ not\ true\ that\ 5 + 5 = 9\ or\ Earth\ is\ a\ planet.$
(iv)11is a prime number and all the sides of a rectangle are equal.

(i)

p	q	p → q
F	T	T

(ii)

p	q	p ∨ q
F	F	F

(iii)

$\neg p$	q	p ∨ q
T	T	T

(iv)

p	q	p ∧ q
T	F	F

9. Verify whether the following compound proposition is tautology or contradiction or contingency $(p \wedge q) \wedge \neg(p \vee q)$

p	q	$p \wedge q$	p ∨ q	¬(p ∨ q)	$(p \wedge q) \wedge \neg(p \vee q)$
T	T	T	T	F	F
T	F	F	T	F	F
F	T	F	T	F	F
F	F	F	F	T	F

Since the last column contains only F, it is a contradiction.

10. Verify whether the following compound proposition is tautology or contradiction or contingency $((p \vee q) \wedge \neg p) \neg q$

p	q	p ∨ q	$\neg p$	$(p \vee q) \wedge \neg p$	$((p \vee q) \wedge \neg p) \rightarrow q$
T	T	T	F	F	T
T	F	T	F	F	T
F	T	T	T	T	T
F	F	F	T	F	T

Since the last column contains only T, it is a tautology.

11. Write down the (i)conditional statement (ii)converse statement (iii)inverse statement and (iv)contrapositive statement for the two statements p and q.

p: *The number of primes is infinite* q: *Ooty is in kerala*

(i)conditional statement($p \to q$):If the number of primes is infinite then Ooty is in kerala.

(ii)converse statement($q \to p$):If Ooty is in kerala then the number of primes is infinite.

(iii)Inverse statement ($\neg p \to \neg q$):If the number of primes is not infinite then Ooty is not in kerala.

(iv)contrapositive statement($\neg q \to \neg p$):If Ooty is not in kerala then the number of primes is not infinite.

5 MARK QUESTIONS

1. Verify the (i) closure property (ii) commutative property (iii) associative property (iv) existence of identity and (v) existence of inverse for the arithmetic operation + on $\mathbb{Z}$.

$(i)\forall\, m,n \in \mathbb{Z} \Longrightarrow m+n \in \mathbb{Z}$. So the closure property is satisfied .

$(ii)\forall\, m,n \in \mathbb{Z} .\Longrightarrow m+n = n+m \in \mathbb{Z}$. So the commutative property is satisfied.

$(iii)\forall\, m,n,p \in \mathbb{Z} \Longrightarrow m+(n+p) = (m+n)+p \in \mathbb{Z}$. So the associative property is satisfied.

$(iv)\mathbf{0} \in \mathbb{Z}$ is an identity element.

$(v)\forall\, m \in \mathbb{Z} \Longrightarrow -\boldsymbol{m} \in \mathbb{Z}$ is an inverse element.

2. Verify (i)closure property, (ii)commutative property,(iii)associative property,(iv)existence of identity and (v) existence of inverse for the operation $m * n = m + n - mn\,; m,n \in \mathbb{Z}$

$(i)\forall\, m,n \in \mathbb{Z} \Longrightarrow m*n = m+n-mn\,; m,n \in \mathbb{Z}$. So the closure property is satisfied.

$(\boldsymbol{ii}\forall\, m,n \in \mathbb{Z} \Longrightarrow m*n = m+n-mn = n*m \ \in \mathbb{Z}$. So the commutative property is satisfied.

$(iii\forall\, m,n,p \in \mathbb{Z} .\Longrightarrow m*(n*p) = (m*n)*p \in \mathbb{Z}$. So the associative property is satisfied.

$(iv) e = 0$ is an identity element.

(v)*For* $1 \in z$, *Inverse element does not exists.*

3. Verify (i)closure property, (ii)commutative property,(iii)associative property,(iv)existence of identity and (v) existence of inverse for the operation $+_5$ on $\mathbb{Z}_5$. Using the corresponding to addition modulo 5.

$\mathbb{Z}_5 = \{[0],[1],[2],[3],[4]\}$

$+_5$	0	1	2	3	4
0	0	1	2	3	4
1	1	2	3	4	0
2	2	3	4	0	1
3	3	4	0	1	2
4	4	0	1	2	3

(i) All the entries in the table are the elements of $\mathbb{Z}_5$. So the closure property is satisfied.

(ii) The entries are symmetrically placed with respect to the main diagonal, So the commutative property is satisfied.

(iii) All the entries in the table are the elements of $\mathbb{Z}_5$. So the associative property is satisfied.

$(iv)\mathbf{0} \in \mathbb{Z}_5$ is an identity element

(v)The inverse elements of 0,1,2,3 and 4 are 0,4,3,2 and 1 respectively

4. Verify (i)closure property, (ii)commutative property,(iii)associative property,(iv)existence of identity and (v) existence of inverse for the operation $\times_{11}$ on a subset $A = \{1,3,4,5,9\}$ of the subset of remainders $\{0,1,2,3,4,5,6,7,8,9,10\}$

$A = \{1,3,4,5,9\}$

$\times_{11}$	1	3	4	5	9
1	1	3	4	5	9
3	3	9	1	4	5
4	4	1	5	9	3
5	5	4	9	3	1
9	9	5	3	1	4

(i) All the entries in the table are the elements of A. So the closure property is satisfied.

(ii) The entries are symmetrically placed with respect to the main diagonal, So the commutative property is satisfied.

(iii) All the entries in the table are the elements of A. So the associative property is satisfied.

$(iv)\mathbf{1} \in \boldsymbol{A}$ is an identity element

(v)The inverse elements of 1,3,4,5 and 9 are 1,4,3,9 and 5 respectively.

5.(i)Define an operation $*$ on $\mathbb{Q}$ as follows: $a*b = \left(\frac{a+b}{2}\right)$; $\forall a,b \in \mathbb{Q}$.Examine the closure, commutative and associative properties satisfied by $*$ on $\mathbb{Q}$.

(ii) Define an operation $*$ on $\mathbb{Q}$ as follows: $a*b = \left(\frac{a+b}{2}\right)$; $\forall a,b \in \mathbb{Q}$.Examine the existence of identity and inverse for the operation $*$ on $\mathbb{Q}$

(i)**closure** : $\forall\, a,b \in \mathbb{Q} \Longrightarrow a+b \in \mathbb{Q} \ *b = \left(\frac{a+b}{2}\right) \in \mathbb{Q}$. So the closure property is satisfied.

commutative: $\forall\, a,b \in \mathbb{Q}.\ \ a*b = \left(\frac{a+b}{2}\right) = \left(\frac{b+a}{2}\right) = b*a \in \mathbb{Q}$.

So the commutative property is satisfied.

Associative : $\forall\, a,b,c \in \mathbb{Q}.\ \ a*(b*c) = \frac{2a+b+c}{4}$.

$$(a*b)*c = \frac{a+b+2c}{4}$$

$\Longrightarrow a*(b*c) \neq (a*b)*c$

So the associative property is not satisfied ..

(ii)**Identity :** Identity element does not exists

Inverse: Inverse element does not exists.

6. $Let\ A = \begin{bmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ 1 & 0 & 0 & 1 \end{bmatrix}, B = \begin{bmatrix} 0 & 1 & 0 & 1 \\ 1 & 0 & 1 & 0 \\ 1 & 0 & 0 & 1 \end{bmatrix}, C = \begin{bmatrix} 1 & 1 & 0 & 1 \\ 0 & 1 & 1 & 0 \\ 1 & 1 & 1 & 1 \end{bmatrix}$ be any three Boolean matrices of the same type. $Find\ (i) A \vee B \quad (ii)\ A \wedge B \quad (iii)(A \vee B) \wedge C \quad (iv)(A \wedge B) \vee C$

$$(i) A \vee B = \begin{bmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ 1 & 0 & 0 & 1 \end{bmatrix} \vee \begin{bmatrix} 0 & 1 & 0 & 1 \\ 1 & 0 & 1 & 0 \\ 1 & 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \\ 1 & 0 & 0 & 1 \end{bmatrix}$$

$$(ii) A \wedge B = \begin{bmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ 1 & 0 & 0 & 1 \end{bmatrix} \wedge \begin{bmatrix} 0 & 1 & 0 & 1 \\ 1 & 0 & 1 & 0 \\ 1 & 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 1 \end{bmatrix}$$

$$(iii)(A \vee B) \wedge C = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \\ 1 & 0 & 0 & 1 \end{bmatrix} \wedge \begin{bmatrix} 1 & 1 & 0 & 1 \\ 0 & 1 & 1 & 0 \\ 1 & 1 & 1 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 1 & 0 & 1 \\ 0 & 1 & 1 & 0 \\ 1 & 0 & 0 & 1 \end{bmatrix}$$

$$(iv)(A \wedge B) \vee C = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 1 \end{bmatrix} \vee \begin{bmatrix} 1 & 1 & 0 & 1 \\ 0 & 1 & 1 & 0 \\ 1 & 1 & 1 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 1 & 0 & 1 \\ 0 & 1 & 1 & 0 \\ 1 & 1 & 1 & 1 \end{bmatrix}$$

7.$(i) Let\ M = \left\{ \begin{pmatrix} x & x \\ x & x \end{pmatrix} : x \in R - \{0\} \right\}$ and let $*$ be the matrix multiplication. Determine whether M is closed under $*$. If so, examine the commutative and associative properties satisfied by $*$ on M.

(ii) $Let\ M = \left\{ \begin{pmatrix} x & x \\ x & x \end{pmatrix} : x \in R - \{0\} \right\}$ and let $*$ be the matrix multiplication. Determine whether M is closed under $*$. If so, examine the existence of identity and inverse properties satisfied by $*$ on M.

(i) ***Closure*** :

$A = \begin{pmatrix} x & x \\ x & x \end{pmatrix}, B = \begin{pmatrix} y & y \\ y & y \end{pmatrix} \in M,\ x, y \in R - \{0\}$

$AB = \begin{pmatrix} 2xy & 2xy \\ 2xy & 2xy \end{pmatrix} \in M$

So the closure property is satisfied.

Commutative:

$A = \begin{pmatrix} x & x \\ x & x \end{pmatrix}, B = \begin{pmatrix} y & y \\ y & y \end{pmatrix} \in M, x, y \in R - \{0\}.$

$AB = \begin{pmatrix} xy & xy \\ xy & xy \end{pmatrix} = \begin{pmatrix} yx & yx \\ yx & yx \end{pmatrix} = \begin{pmatrix} y & y \\ y & y \end{pmatrix} \begin{pmatrix} x & x \\ x & x \end{pmatrix} = BA.$

So the commutative property is satisfied.

Asociative: Matrix multiplication is always associative

(ii) ***Closure*** :

$A = \begin{pmatrix} x & x \\ x & x \end{pmatrix}, B = \begin{pmatrix} y & y \\ y & y \end{pmatrix} \in M,\ x, y \in R - \{0\}$

$AB = \begin{pmatrix} 2xy & 2xy \\ 2xy & 2xy \end{pmatrix} \in M$

So the closure property is satisfied.

Identity:

$\boldsymbol{E} = \begin{pmatrix} \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & \frac{1}{2} \end{pmatrix} \in \boldsymbol{M}$ is an identity element

Inverse:

$\boldsymbol{A^{-1}} = \begin{pmatrix} \frac{1}{4x} & \frac{1}{4x} \\ \frac{1}{4x} & \frac{1}{4x} \end{pmatrix} \in \boldsymbol{M}$ is an inverse element

8.$(i) Let\ A\ be\ \mathbb{Q}\backslash\{1\}$. Define $*$ on A by $x * y = x + y - xy$. Is $*$ binary on A?. If so, examine the commutative and associative properties satisfied by $*$ on A.

$(ii) Let\ A\ be\ \mathbb{Q}\backslash\{1\}$. Define $*$ on A by $x * y = x + y - xy$. Is $*$ binary on A?. If so, examine the existence of identity and inverse properties satisfied by $*$ on A.

$A = \mathbb{Q}\backslash\{1\}.\ \ x * y = x + y - xy.$

(i) ***Closure*** $\forall\ x, y \in A; x \neq 1,\ y \neq 1.$

We have to prove that $x + y - xy \neq 1$

on contrary consider, $x + y - xy = 1$.

$x + y - xy = 1$

$\Rightarrow x + y - xy - 1 = 0$

$\Rightarrow x = 1, y = 1$ Contradiction.

So $x + y - xy \neq 1$.

So the closure property is satisfied.

Commutative $\forall x, y \in A; x \neq 1,\ y \neq 1$

$x * y = x + y - xy = y + x - yx = y * x$

So the commutative property is satisfied.

Asociative:: $\forall x, y, z \in A; x \neq 1, y \neq 1, z \neq 1$

$x * (y * z) = x + y + z - xy - yz - zx + xyz = (x * y) * z$

So the associative property is satisfied.

(ii) ***Closure*** $\forall\ x, y \in A; x \neq 1,\ y \neq 1.$

We have to prove that $x + y - xy \neq 1$

on contrary consider, $x + y - xy = 1$.

$x + y - xy = 1$
$\Rightarrow x + y - xy - 1 = 0$
$\Rightarrow x = 1, y = 1$ Contradiction.
So $x + y - xy \neq 1$.
So the closure property is satisfied.

Identity: $e = 0 \in A$ is an identity element

Inverse: $x^{-1} = -\frac{x}{1-x}$ is an inverse element.

9.Construct a truth table for $(p \veebar q) \wedge (p \veebar \neg q)$..

p	q	$p \veebar q$	$\neg q$	$p \veebar \neg q$	$(p \veebar q) \wedge (p \veebar \neg q)$
T	T	F	F	T	F
T	F	T	T	F	F
F	T	T	F	F	F
F	F	F	T	T	F

10. Prove that $p \to q \equiv \neg p \vee q$. .

p	q	$p \to q$	$\neg q$	$\neg p \vee q$
T	T	T	F	T
T	F	F	T	F
F	T	T	F	T
F	F	T	T	T

From the table, $p \to q \equiv \neg p \vee q$.

11.. Prove that $p \leftrightarrow q \equiv (p \to q) \wedge (q \to p)$.

p	q	$p \leftrightarrow q$	$p \to q$	$q \to p$	$(p \to q) \wedge (q \to p)$
T	T	T	T	T	T
T	F	F	F	T	F
F	T	F	T	F	F
F	F	T	T	T	T

From the table, $p \leftrightarrow q \equiv (p \to q) \wedge (q \to p)$

12. Using the equivalence property, show that $p \leftrightarrow q \equiv (p \wedge q) \vee (\neg p \wedge \neg q)$

$p \leftrightarrow q \equiv (p \to q) \wedge (q \to p)$ $(\because p \leftrightarrow q \equiv (p \to q) \wedge (q \to p))$

$\equiv (\neg p \vee q) \wedge (\neg q \vee p)$ $(\because p \to q \equiv \neg p \vee q)$

$\equiv (\neg p \vee q) \wedge (p \vee \neg q)$ (*Commutative law*)

$\equiv [\neg p \wedge (p \vee \neg q)] \vee [q \wedge (p \vee \neg q)]$ (*Distributive law*)

$\equiv [(\neg p \wedge p) \vee (\neg p \wedge \neg q)] \vee [(q \wedge p) \vee (q \wedge \neg q)]$ (*Distributive law*)

$\equiv [\mathbb{F} \vee (\neg p \wedge \neg q)] \vee [(q \wedge p) \vee \mathbb{F}]$ (*Compliment law*)

$\equiv (\neg p \wedge \neg q) \vee (q \wedge p)$ (*Identity law*)

$\equiv (p \wedge q) \vee (\neg p \wedge \neg q)$ (*Commutative law*)

14..Verify whether the following compound proposition is tautology or contradiction or contingency. $(p \to q) \leftrightarrow (\neg p \to q)$.

p	q	$p \to q$	$\neg p$	$\neg p \to q$	$(p \to q) \leftrightarrow (\neg p \to q)$
T	T	T	F	T	T
T	F	F	F	T	F
F	T	T	T	T	T
F	F	T	T	F	F

Since the last column contains both T and F, it is a contingency

15. Verify whether the following compound proposition is tautology or contradiction or contingency. $((p \to q) \wedge (q \to r)) \to (p \to r)$

p	q	r	$p \to q$	$q \to r$	$(p \to q) \wedge (q \to r)$	$p \to r$	$((p \to q) \wedge (q \to r)) \to (p \to r)$
T	T	T	T	T	T	T	T
T	T	F	T	F	F	F	T
T	F	T	F	T	F	T	T
T	F	F	F	T	F	F	T
F	T	T	T	T	T	T	T
F	T	F	T	F	F	T	T
F	F	T	T	T	T	T	T
F	F	F	T	T	T	T	T

Since the last column contains only T, it is a tautology.

16. Show that $\neg(p \wedge q) \equiv \neg p \vee \neg q$

p	q	$p \wedge q$	$\neg(p \wedge q)$	$\neg p$	$\neg q$	$\neg p \vee \neg q$
T	T	T	F	F	F	F
T	F	F	T	F	T	T
F	T	F	T	T	F	T
F	F	F	T	T	T	T

From the table, $\neg(p \wedge q) \equiv \neg p \vee \neg q$.

17. Show that $\neg(p \to q) \equiv p \wedge \neg q$.

p	q	$p \to q$	$\neg(p \to q)$	$\neg q$	$p \wedge \neg q$
T	T	T	F	F	F
T	F	F	T	T	T
F	T	T	F	F	F
F	F	T	F	T	F

From the table, $\neg(p \to q) \equiv p \wedge \neg q$.

18. Prove that $q \to p \equiv \neg p \to \neg q$

p	q	$q \to p$	$\neg p$	$\neg q$	$\neg p \to \neg q$
T	T	T	F	F	T
T	F	T	F	T	T
F	T	F	T	F	F
F	F	T	T	T	T

From the table, $q \to p \equiv \neg p \to \neg q$

19 Show that $p \to q$ and $q \to p$ are not equivalent.

p	q	$p \to q$	$q \to p$
T	T	T	T
T	F	F	T
F	T	T	F
F	F	T	T

From the table, $p \to q \not\equiv q \to p$

20. Show that $\neg(p \leftrightarrow q) \equiv p \leftrightarrow \neg q$..

p	q	$p \leftrightarrow q$	$\neg(p \leftrightarrow q)$	$\neg q$	$p \leftrightarrow \neg q$
T	T	T	F	F	F
T	F	F	T	T	T
F	T	F	T	F	T
F	F	T	F	T	F

From the table, $\neg(p \leftrightarrow q) \equiv p \leftrightarrow \neg q$

21. Check whether the statement $p \to (q \to p)$ is a tautology or a contradiction without using the truth table.

$$p \to (q \to p) \equiv p \to (\neg q \vee p)$$
$$\equiv \neg p \vee (\neg q \vee p)$$
$$\equiv \neg p \vee (p \vee \neg q) \quad (Commutative\ law)$$
$$\equiv (\neg p \vee p) \vee \neg q \quad (Associative\ law)$$
$$\equiv \mathbb{T} \vee \neg q \quad (Compliment\ law)$$
$$\equiv \mathbb{T}$$

$\therefore p \to (q \to p)$ is a tautology.

22. Using the truth table check whether the statement $\neg(p \vee q) \vee (\neg p \wedge q)$ and $\neg p$ are logically equivalent..

p	q	$p \vee q$	$\neg(p \vee q)$	$\neg p$	$\neg p \wedge q$	$\neg(p \vee q) \vee (\neg p \wedge q)$
T	T	T	F	F	F	F
T	F	T	F	F	F	F
F	T	T	F	T	T	T
F	F	F	T	T	F	T

From the table, $\neg(p \vee q) \vee (\neg p \wedge q) \equiv \neg p$

23. Prove $p \to (q \to r) \equiv (p \wedge q) \to r$ without using truth table.

$$p \to (q \to r) \equiv p \to (\neg q \vee r)$$
$$\equiv \neg p \vee (\neg q \vee r)$$
$$\equiv (\neg p \vee \neg q) \vee r \quad (Associative\ law)$$

$\equiv \neg(p \wedge q) \vee r \qquad (DeMorgan\ law)$

$\equiv (p \wedge q) \rightarrow r$

24 Prove that $\rightarrow (\neg q \vee r) \equiv \neg p \vee (\neg q \vee r)$ using truth table

p	q	r	$\neg q$	$\neg q \vee r$	$p \rightarrow (\neg q \vee r)$	$\neg p$	$\neg p \vee (\neg q \vee r)$
T	T	T	F	T	T	F	T
T	T	F	F	F	F	F	F
T	F	T	T	T	T	F	T
T	F	F	T	T	T	F	T
F	T	T	F	T	T	T	T
F	T	F	F	F	T	T	T
F	F	T	T	T	T	T	T
F	F	F	T	T	T	T	T

$From\ the\ table, p \rightarrow (\neg q \vee r) \equiv \neg p \vee (\neg q \vee r) p \rightarrow (\neg q \vee r) \equiv \neg p \vee (\neg q \vee r)$

9.APPLICATION OF INTEGRATION

5 MARK QUESTIONS

1.Find the area of the region bounded by the line $6x + 5y = 30, x - axis$ and the lines $x = -1\ and\ x = 3.$

$y = \frac{1}{5}(30 - 6x), x = -1, x = 3.$

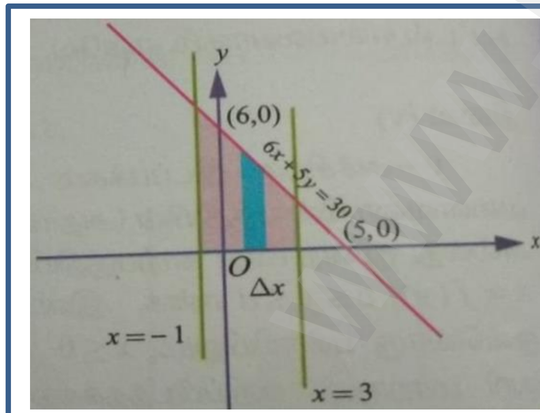
Area $A = \int_a^b y dx$

$= \int_{-1}^{3} \frac{1}{5}(30 - 6x)\, dx$

$= \frac{1}{5}\left[30x - \frac{6x^2}{2}\right]_{-1}^{3}$

$= \frac{1}{5}\{[90 - 27] - [-30 - 3]\}$

$= \frac{96}{5}$ sq.units.



2.Find the area of the region bounded by the line $7x - 5y = 35, x - axis$ and the lines $x = -2\ and\ x = 3.$

$y = \frac{1}{5}[7x - 35], x = -2, x = 3.$

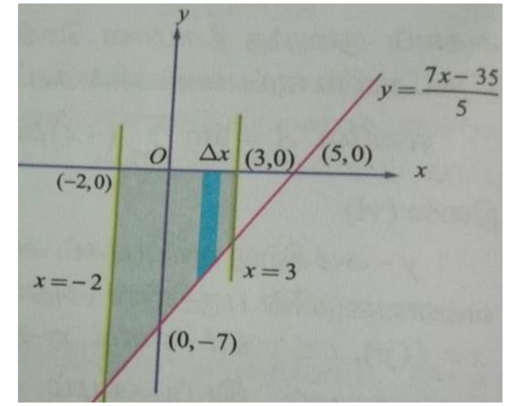
Area $A = \int_a^b y dx$

$= \int_{-2}^{3} \frac{1}{5}[7x - 35]\, dx$

$= \frac{1}{5}\left[\frac{7x^2}{2} - 35x\right]_{-2}^{3}$

$= \frac{1}{5}\left\{\left[\frac{63}{2} - 105\right] - [14 + 70]\right\}$

$= \frac{63}{2}$ sq.units.



3.Find the area of the region bounded by the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$

$\boldsymbol{y} = \frac{b}{a}\sqrt{a^2 - x^2}\ ; x = 0\ , x = a$

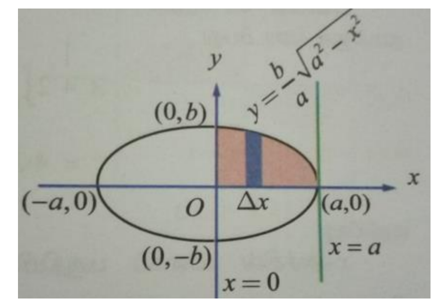
Area $A = 4\ \int_0^a y dx$

$= 4\ \int_0^a \frac{b}{a}\sqrt{a^2 - x^2}\, dx$

$= \frac{4b}{a}\left[\frac{x}{2}\sqrt{a^2 - x^2} + \frac{a^2}{2} sin^{-1}\left(\frac{x}{a}\right)\right]_0^a$

$= \frac{4b}{a}\left\{\left(\frac{a}{2}\sqrt{a^2 - a^2} + \frac{a^2}{2} sin^{-1}\left(\frac{a}{a}\right)\right) - \left(\frac{0}{2}\sqrt{a^2 - 0^2} + \frac{a^2}{2} sin^{-1}\left(\frac{0}{a}\right)\right)\right\}$

$= \pi ab$ sq.units.



4.Find the area of the region bounded between the parabola $y^2 = 4ax$ and its latus rectum

$y = 2\sqrt{a}\sqrt{x}, x = 0, x = a.$

Area $A = 2\ \int_0^a y dx$

$= 2\ \int_0^a 2\sqrt{a}\sqrt{x} dx$

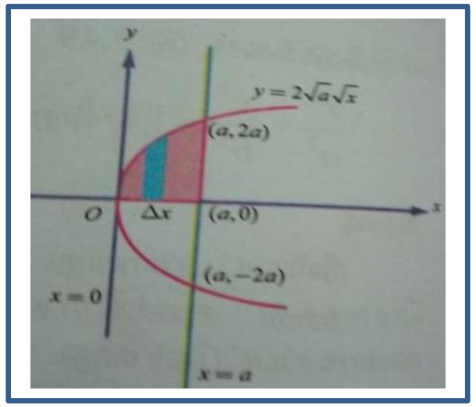
$= 4\sqrt{a}\left[\frac{x^{3/2}}{\frac{3}{2}}\right]_0^a$

$= 4\sqrt{a} \times \frac{2}{3}\left\{\left[a^{3/2}\right] - \left[0^{3/2}\right]\right\}$

$= \frac{8}{3} \times a^{1/2} \times a^{3/2}$

$= \frac{8a^{4/2}}{3}$

$= \frac{8a^2}{3}$ sq.units.



5. Find the area of the region bounded by the $y-axis$ and the parabola $x = 5 - 4y - y^2$

$Equation\ of\ y - axis\ is\ \ x = 0$

$0 = 5 - 4y - y^2 \Rightarrow y^2 + 4y - 5 = 0$

$\Rightarrow (y+5)(y-1) = 0$

$\Rightarrow y = -5, 1$

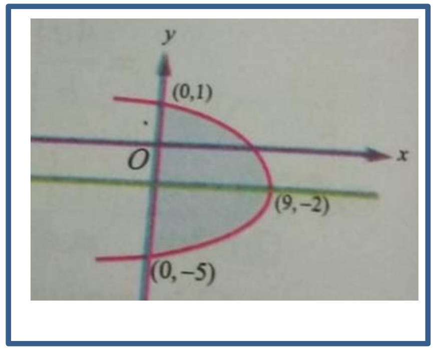
Area $A = \int_a^b x dy$

$= \int_{-5}^{1}(5 - 4y - y^2)dy$

$= \left[5y - \frac{4y^2}{2} - \frac{y^3}{3}\right]_{-5}^{1}$

$= \left[5(1) - \frac{4(1)^2}{2} - \frac{(1)^3}{3}\right] - \left[5(-5) - \frac{4(-5)^2}{2} - \frac{(-5)^3}{3}\right]$

$= 36$ sq.units.



6.Find the area of the region bounded by $x - axis$, the sine curve $y = \sin x$, $the\ lines\ x = 0\ and\ x = 2\pi$

$Above\ x - axis\text{:}\ \ y = \sin x, x = 0, x = \pi.$

$Below\ x - axis\text{:}\ y = \sin x, x = \pi, x = 2\pi.$

Area $A = \int_a^b ydx + \int_c^d -ydx$

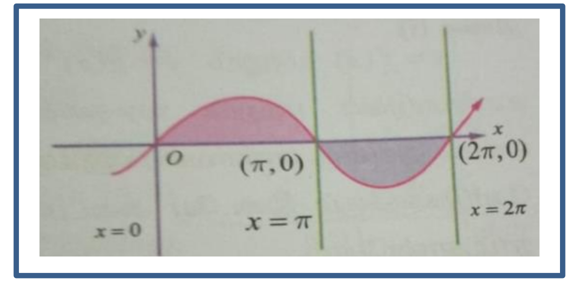
$= \int_0^{\pi} \sin x\, dx - \int_{\pi}^{2\pi} \sin x\, dx$

$= [-\cos x]_0^{\pi} - [-\cos x]_{\pi}^{2\pi}$

$= [-\cos\pi + \cos 0] - \ [-\cos 2\pi + \cos\pi]$

$= [1 + 1] - [-1 - 1]$

$= 4$ sq.units.



7. Find the area of the region bounded by $x - axis$, the curve $y = |\cos x|$, $the\ lines\ x = 0\ and\ x = \pi.$

$y = |\cos x| = \begin{cases} \cos x, 0 \le x \le \frac{\pi}{2} \\ -\cos x, \frac{\pi}{2} \le x \le \pi \end{cases}$

Area $A = \int_a^b ydx$

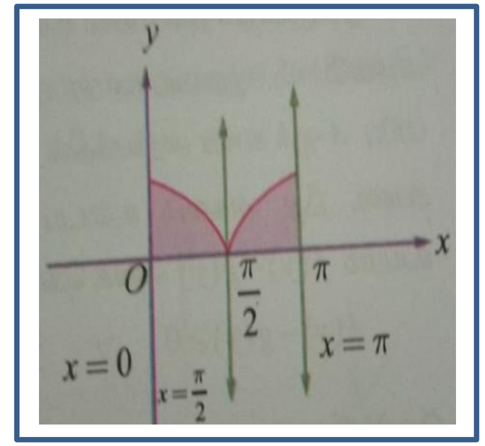
$= \int_0^{\frac{\pi}{2}} \cos x\, dx + \int_{\frac{\pi}{2}}^{\pi}(-\cos x)dx$

$= [\sin x]_0^{\frac{\pi}{2}} - [\sin x]_{\frac{\pi}{2}}^{\pi}$

$= \left[\sin\frac{\pi}{2} - \sin 0\right] - \left[\sin\pi - \sin\frac{\pi}{2}\right]$

$= (1 - 0) - (0 - 1)$

$= 2$ sq.units.



8. Find the area of the region bounded between the parabolas $y^2 = 4x\ and\ x^2 = 4y$

$y^2 = 4x \rightarrow (1)\ \Rightarrow y = 2x^{1/2}$

$x^2 = 4y \Rightarrow y = \frac{x^2}{4} \rightarrow (2)$

$(1) \Rightarrow \left(\frac{x^2}{4}\right)^2 = 4x \Rightarrow x^4 = 64x$

$\Rightarrow x(x^3 - 64) = 0$

$\Rightarrow x = 0, x = 4$

Area $A = \int_a^b (y_U - y_L)dx = \int_0^4 \left(2x^{1/2} - \frac{x^2}{4}\right) dx$

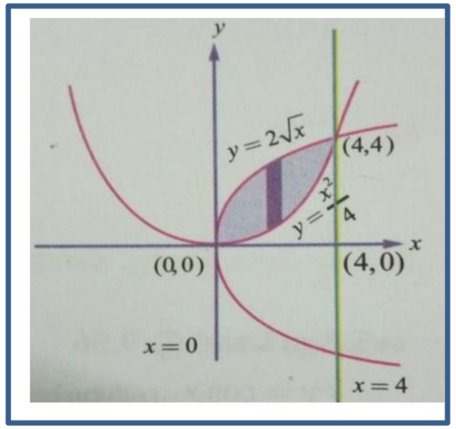
$= \left[\frac{2x^{3/2}}{\frac{3}{2}} - \frac{x^3}{4\times 3}\right]_0^4$

$= \left[\frac{4}{3}(4)^{3/2} - \frac{4^3}{12}\right] - [0]$

$= \frac{4}{3}(8) - \frac{64}{12}$

$= \frac{32}{3} - \frac{64}{12}$

$= \frac{16}{3}$ sq.units.



9. Find the area of the region bounded between the parabola$x^2 = y$ and $y = |x|$

$y = x^2 \rightarrow (1)$

$y = |x| = \begin{cases} x, x \geq 0 \\ -x, x < 0 \end{cases} \rightarrow (2)$

Solving equations (1) & (2),

Case (i):

$x^2 = x \Rightarrow x(x-1) = 0 \Rightarrow x = 0, x = 1$

$x = 1 \Rightarrow \quad y = 1$

Case (ii):

$x^2 = -x \Rightarrow x(x+1) = 0 \Rightarrow x = 0, x = -1$

$x = -1 \Rightarrow \ y = 1$

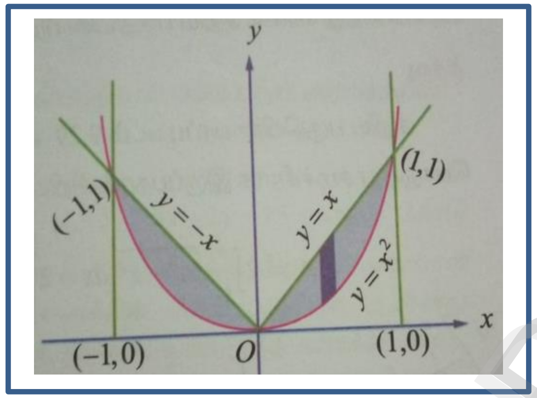
Area $A = \int_a^b (y_U - y_L)dx$

$= \int_0^1 (x - x^2)dx$

$= \left[\frac{x^2}{2} - \frac{x^3}{3}\right]_0^1$

$= \left[\frac{1^2}{2} - \frac{1^3}{3}\right] - [0]$

$= \frac{1}{3}$ sq.units.



10. Find the area of the region bounded by $y = \cos x$ and $y = \sin x$ $, the$ $lines$ $x = \frac{\pi}{4}$ and $x = \frac{5\pi}{4}$.

Required area is covered by:

$y = \sin x, \frac{\pi}{4} \leq x \leq \frac{5\pi}{4}$

$y = \cos x, \frac{\pi}{4} \leq x \leq \frac{5\pi}{4}$

Area $A = \int_a^b (y_U - y_L)dx$

$= \int_{\frac{\pi}{4}}^{\frac{5\pi}{4}} (\sin x - \cos x)dx$

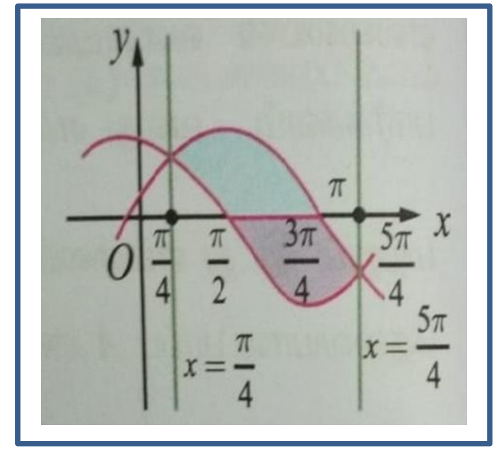
$= [-\cos x - \sin x]_{\frac{\pi}{4}}^{\frac{5\pi}{4}}$

$= \left[-\cos\frac{5\pi}{4} - \sin\frac{5\pi}{4}\right] - \left[-\cos\frac{\pi}{4} - \sin\frac{\pi}{4}\right]$

$= \left[-\left(-\frac{1}{\sqrt{2}}\right) - \left(-\frac{1}{\sqrt{2}}\right)\right] - \left[-\left(-\frac{1}{\sqrt{2}}\right) - \frac{1}{\sqrt{2}}\right]$

$= \frac{2}{\sqrt{2}} + \frac{2}{\sqrt{2}}$

$= 2\sqrt{2}$ sq.units.



11.The region enclosed by the circle $x^2 + y^2 = a^2$ is divided into two segments by the line $x = h$. Find the area of the smaller region.

$x^2 + y^2 = a^2 \Rightarrow y = \sqrt{a^2 - x^2}$,

$x = h, x = a.$

Area $A = 2\int_a^b y dx$

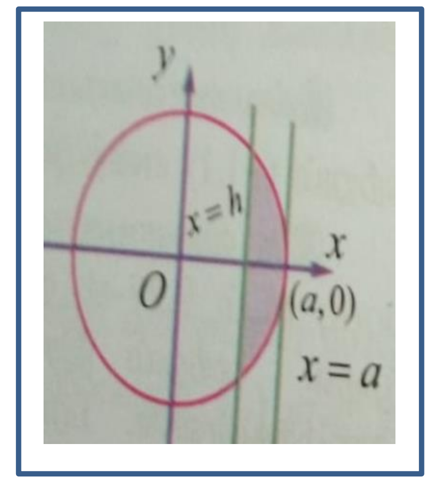
$= 2\int_h^a \sqrt{a^2 - x^2} dx$

$= 2\left[\frac{x\sqrt{a^2-x^2}}{2} + \frac{a^2}{2} sin^{-1}\left(\frac{x}{a}\right)\right]_h^a$

$= 2\left[\frac{a\sqrt{a^2-a^2}}{2} + \frac{a^2}{2} sin^{-1}\left(\frac{a}{a}\right)\right] - 2\left[\frac{h\sqrt{a^2-h^2}}{2} + \frac{a^2}{2} sin^{-1}\left(\frac{h}{a}\right)\right]$

$= a^2 sin^{-1}\left(\frac{a}{a}\right) - h\sqrt{a^2 - h^2} - \ a^2 sin^{-1}\left(\frac{h}{a}\right)$

$= a^2\left(\frac{\pi}{2}\right) - h\sqrt{a^2 - h^2} - \ a^2 sin^{-1}\left(\frac{h}{a}\right)$



$= a^2\left(\frac{\pi}{2} - sin^{-1}\left(\frac{h}{a}\right)\right) - h\sqrt{a^2 - h^2}$

$= a^2 cos^{-1}\left(\frac{h}{a}\right) - h\sqrt{a^2 - h^2}$ sq.units.

12. Find the area of the region in the first quadrant bounded by the parabola $y^2 = 4x, the\ line$ $x + y = 3$ and $y - axis$.

$x + y = 3 \Rightarrow x = 3 - y$

$y^2 = 4x \Rightarrow y^2 = 4(3 - y)$

$\Rightarrow y^2 + 4y - 12 = 0$

$\Rightarrow (y - 2)(y + 6) = 0$

$\Rightarrow y = 2, y = -6$

$y - intercept\ of\ x + y = 3\ is\ 3.$

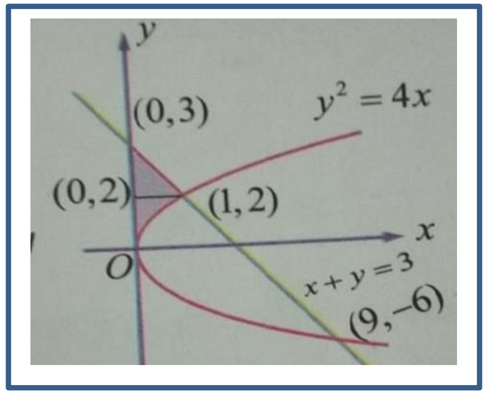
Area $A = \int_a^b xdy + \int_c^d xdy$

$= \int_0^2 \frac{y^2}{4}dy + \int_2^3 (3 - y)dy$

$= \left[\frac{y^3}{12}\right]_0^2 + \left[3y - \frac{y^2}{2}\right]_2^3$

$= \left(\frac{8}{12} - 0\right) + \left(9 - \frac{9}{2}\right) - \left(6 - \frac{4}{2}\right)$

$= \frac{7}{6}$ sq.units.



13.Find, by integration, the area of the region bounded by the lines $5x - 2y = 15, x + y + 4 = 0\ and\ the\ \ x - axis.$

$5x - 2y = 15 \Rightarrow x = \frac{2y+15}{5} \rightarrow (1)$

$x + y + 4 = 0 \Rightarrow x = -y - 4 \rightarrow (2)$

Solving equations (1) & (2),

$\Rightarrow \frac{2y+15}{5} = -y - 4$

$\Rightarrow 2y + 15 = -5y - 20$

$\Rightarrow 7y = -35$

$\Rightarrow y = -5$

$y = -5 \Rightarrow\ \ x = 5 - 4 = 1.$

Point of intersection $= (1, -5)$

$x - intercept\ of\ 5x - 2y = 15\ is\ 3.$

$x - intercept\ of\ x + y + 4 = 0\ is - 4.$

Area $A = \int_a^b (X_R - X_L)dy$

$= \int_{-5}^{0}\left(\frac{2y+15}{5} - (-y - 4)\right)dy$

$= \int_{-5}^{0}\left(\frac{2y+15+5y+20}{5}\right)dy$

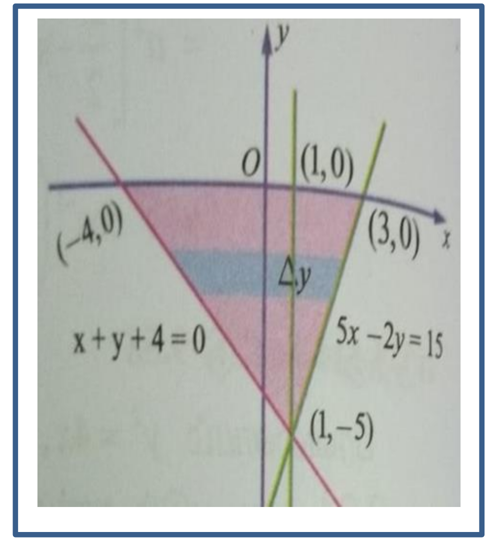
$= \int_{-5}^{0}\left(\frac{7y+35}{5}\right)dy$

$= \frac{1}{5}\left[\frac{7y^2}{2} + 35y\right]_{-5}^{0}$

$= \frac{1}{5}(0) - \frac{1}{5}\left(\frac{175}{2} - 175\right)$

$= -\frac{1}{5}\left(-\frac{175}{2}\right)$

$= \frac{35}{2}$ sq.units.



14.Using integration, find the area of the region bounded by triangle ABC, whose vertices A,B and C are $(-1,1), (3,2)\ and\ (0,5)$ respectively

$A(-1,1), B(3,2), C(0,5)$

$Equation\ of\ AB: \frac{y-y_1}{y_2-y_1} = \frac{x-x_1}{x_2-x_1} \Rightarrow \frac{y-1}{2-1} = \frac{x+1}{3+1}$

$\Rightarrow y = \frac{1}{4}(x + 5)$

$Equation\ of BC\ \ : \frac{y-5}{2-5} = \frac{x-0}{3-0} \Rightarrow y = -x + 5$

$Equation\ of AC: \frac{y-1}{5-1} = \frac{x+1}{0+1} \Rightarrow y = 4x + 5$

$Area\ of\ \Delta ABC = Area\ of\ DACO +$

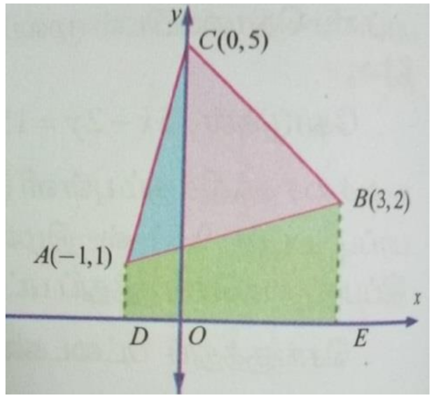
$Area\ of\ OCBE - Area\ of\ DABE$

$= \int_{-1}^{0}(4x+5)dx + \int_{0}^{3}(-x+5)dx - \frac{1}{4}\int_{-1}^{3}(x+5)dx$

$= \left[\frac{4x^2}{2}+5x\right]_{-1}^{0} + \left[-\frac{x^2}{2}+5x\right]_{0}^{3} - \frac{1}{4}\left[\frac{x^2}{2}+5x\right]_{-1}^{3}$

$= [(0)-(2-5)] + \left[\left(-\frac{9}{2}+15\right)-(0)\right] - \frac{1}{4}\left[\left(\frac{9}{2}+15\right)-\left(\frac{1}{2}-5\right)\right]$

$= \frac{15}{2}$ sq.units.



15. Using integration, find the area of the region bounded by $x-axis$, the tangent and normal to the circle $x^2+y^2=4$ drawn at $(1,\sqrt{3})$.

Equation of tangent $xx_1+yy_1=4 \Rightarrow x+\sqrt{3}y=4$

$\Rightarrow x = 4-\sqrt{3}y$

Equation of normal : $\sqrt{3}x - y + k = 0 \Rightarrow \sqrt{3}-\sqrt{3}+k=0 \Rightarrow k=0$

Equation of normal $is\ \sqrt{3}x - y = 0$

$\Rightarrow y = \sqrt{3}x$

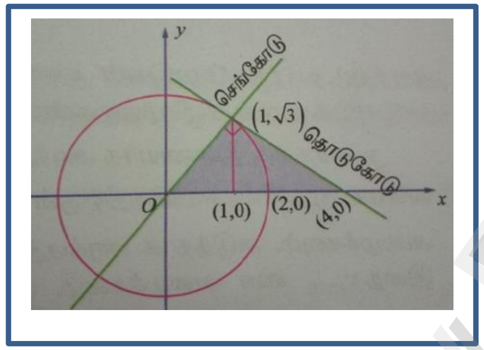
$\Rightarrow x = \frac{1}{\sqrt{3}}y$

Required area is bounded by

$y=\sqrt{3}x, x+\sqrt{3}y=4, y=0, y=\sqrt{3}$

$Area\ \ A = \int_a^b (X_R - X_L)dy$

$= \int_0^{\sqrt{3}}\left((4-\sqrt{3}y)-\frac{1}{\sqrt{3}}y\right)dy$



$= \left[4y - \sqrt{3}\frac{y^2}{2} - \frac{1}{\sqrt{3}}\frac{y^2}{2}\right]_0^{\sqrt{3}}$

$= \left(4\sqrt{3} - \frac{3\sqrt{3}}{2} - \frac{3}{2\sqrt{3}}\right) - (0)$

$= \frac{24-9-3}{2\sqrt{3}}$

$= \frac{12}{2\sqrt{3}}$

$= 2\sqrt{3}$ sq.units.

16. Find the area of the region bounded by the line $3x-2y+6=0, x=-3. x=1\ and\ x-axis$.

$3x-2y+6=0 \Rightarrow y=\frac{1}{2}(3x+6)$

$x\ and\ y\ intercepts\ are\ -2,3$ respectively.

$Area\ \ A = \int_a^b -ydx + \int_b^c ydx$

$= -\int_{-3}^{-2}\frac{1}{2}(3x+6)dx + \int_{-2}^{1}\frac{1}{2}(3x+6)dx$

$= -\frac{1}{2}\left[\frac{3x^2}{2}+6x\right]_{-3}^{-2} + \frac{1}{2}\left[\frac{3x^2}{2}+6x\right]_{-2}^{1}$

$= -\frac{1}{2}\left[\left(\frac{12}{2}-12\right)-\left(\frac{27}{2}-18\right)\right] + \frac{1}{2}\left[\left(\frac{3}{2}+6\right)-\left(\frac{12}{2}-12\right)\right]$

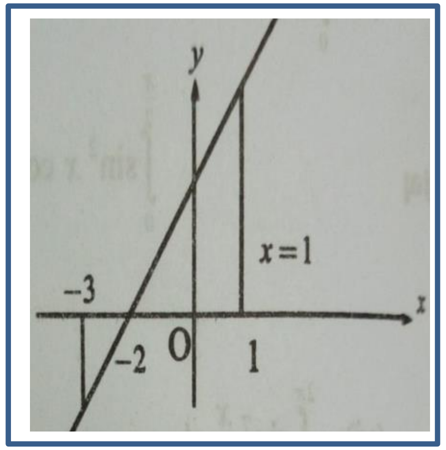
$= -\frac{1}{2}\left(-6+\frac{9}{2}\right) + \frac{1}{2}\left(\frac{15}{2}+\frac{12}{2}\right)$

$= -\frac{1}{2}\left(-\frac{3}{2}\right) + \frac{1}{2}\left(\frac{27}{2}\right)$

$= \frac{3}{4} + \frac{27}{4}$

$= \frac{30}{4}$

$= \frac{15}{2}$ sq.units.



17. Find the area of the region bounded by the line $2x-y+1=0, y=-1. y=3\ and\ y-axis$.

$2x-y+1=0 \Rightarrow x=\frac{1}{2}(y-1)$

x *and* y *intercepts are* $-\frac{1}{2},1$ respectively.

Area $A=\int_a^b -x dy+\int_b^c x dy$

$=-\int_{-1}^{1}\frac{1}{2}(y-1)dy+\int_1^3\frac{1}{2}(y-1)dy$

$=-\frac{1}{2}\left[\frac{y^2}{2}-y\right]_{-1}^{1}+\frac{1}{2}\left[\frac{y^2}{2}-y\right]_1^3$

$=-\frac{1}{2}\left[\left(\frac{1}{2}-1\right)-\left(\frac{1}{2}+1\right)\right]+$

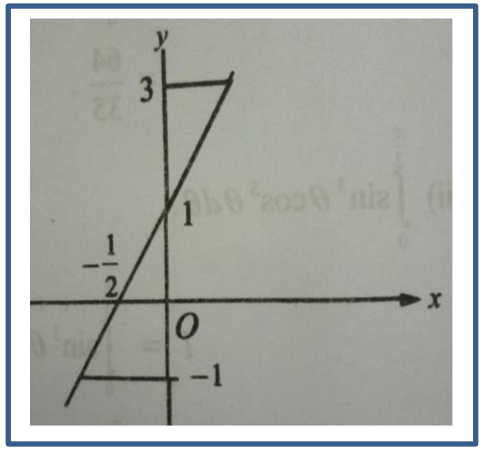
$\frac{1}{2}\left[\left(\frac{9}{2}-3\right)-\left(\frac{1}{2}-1\right)\right]$

$=-\frac{1}{2}\left(-\frac{1}{2}-\frac{3}{2}\right)+\frac{1}{2}\left(\frac{3}{2}+\frac{1}{2}\right)$

$=-\frac{1}{2}(-2)+\frac{1}{2}(2)$

$=1+1$

$=2$ sq.units.



18. Find the area of the region bounded by the curve $2+x-x^2+y=0, x-axis, x=-3$ *and* $x=3$.

$2+x-x^2+y=0 \Rightarrow y=x^2-x-2$

$y=x^2-x-2 \Rightarrow y=(x+1)(x-2)$

x *intercepts are* $-1,2$.

y *intercept is* -2 .

Area $A=\int_a^b y dx+\int_b^c -y dx+\int_c^d y dx$

$=\int_{-3}^{-1}(x^2-x-2)dx-$

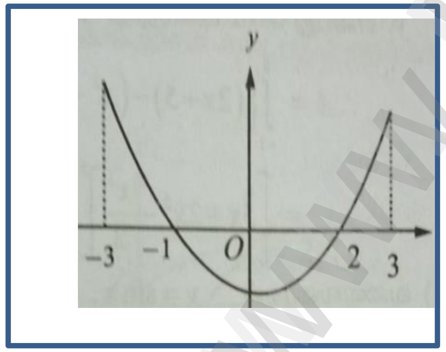
$\int_{-1}^{2}(x^2-x-2)dx+\int_2^3(x^2-x-2)dx$

$=\left[\frac{x^3}{3}-\frac{x^2}{2}-2x\right]_{-3}^{-1}-\left[\frac{x^3}{3}-\frac{x^2}{2}-2x\right]_{-1}^{2}+$

$\left[\frac{x^3}{3}-\frac{x^2}{2}-2x\right]_2^3$

$=\frac{26}{3}+\frac{9}{2}+\frac{11}{6}$

$=15$ sq.units.



19. Find the area of the region bounded by the line $y=2x+5$ and the parabola $y=x^2-2x$.

$y=2x+5 \rightarrow (1)$

$y=x^2-2x \rightarrow (2)$

Solving equations (1) & (2),

$x^2-2x=2x+5 \Rightarrow x^2-4x-5=0$

$\Rightarrow (x+1)(x-5)=0$

$\Rightarrow x=-1,5$

Area $A=\int_a^b (y_U-y_L)dx$

$=\int_{-1}^{5}[(2x+5)-(x^2-2x)]dx$

$=\int_{-1}^{5}(5+4x-x^2)dx$

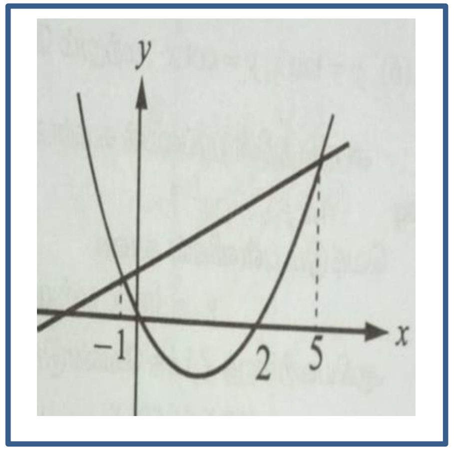
$=\left[5x+\frac{4x^2}{2}-\frac{x^3}{3}\right]_{-1}^{5}$

$=\left(25+50-\frac{125}{3}\right)-\left(-5+2+\frac{1}{3}\right)$

$=\frac{100}{3}+\frac{8}{3}$

$=\frac{108}{3}$

$=36$ sq.units.



20.Find the area of the region bounded between the curves $y=\sin x$ *and* $y=\cos x$ *and the lines* $x=0$ and $x=\pi$.

$y=\sin x \rightarrow (1)$

$y=\cos x \rightarrow (2)$

Solving equations (1) & (2),

$\sin x = \cos x \Rightarrow x = \frac{\pi}{4}$

Area $A = \int_a^b (y_U - y_L)dx + \int_b^c (y_U - y_L)dx$

$= \int_0^{\frac{\pi}{4}} (\cos x - \sin x)dx + \int_{\frac{\pi}{4}}^{\pi} (\sin x - \cos x)dx$

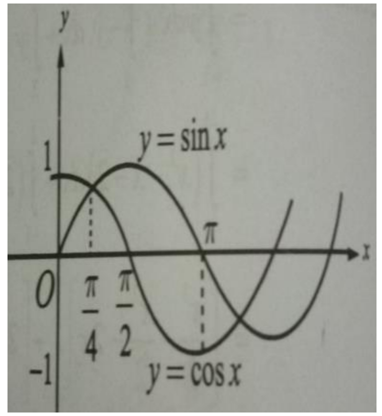
$= [\sin x + \cos x]_0^{\frac{\pi}{4}} + [-\cos x - \sin x]_{\frac{\pi}{4}}^{\pi}$

$= \left[\left(\frac{1}{\sqrt{2}} + \frac{1}{\sqrt{2}}\right) - (0+1)\right] + \left[(1-0) - \left(-\frac{1}{\sqrt{2}} - \frac{1}{\sqrt{2}}\right)\right]$

$= \frac{2}{\sqrt{2}} - 1 + 1 + \frac{2}{\sqrt{2}}$

$= \sqrt{2} + \sqrt{2}$

$= 2\sqrt{2}$ sq.units.



21. Find the area of the region bounded by $y = tanx$, $y = \cot x$ and between the lines $x = 0$ and $x = \frac{\pi}{2}$.

$y = tanx \rightarrow (1)$

$y = \cot x \rightarrow (2)$

Solving equations (1) & (2),

$tanx = \cot x \Rightarrow x = \frac{\pi}{4}$

Area $A = \int_a^b ydx + \int_b^c ydx$

$= \int_0^{\frac{\pi}{4}} tanxdx + \int_{\frac{\pi}{4}}^{\frac{\pi}{2}} \cot x\, dx$

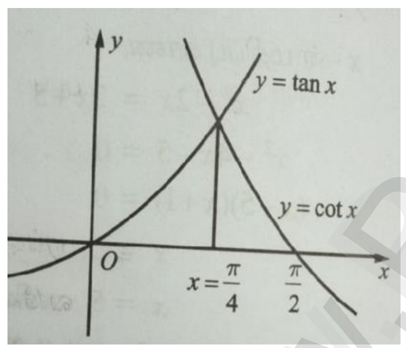
$= [\log \sec x]_0^{\frac{\pi}{4}} + [\log \sin x]_{\frac{\pi}{4}}^{\frac{\pi}{2}}$

$= \log \sec\frac{\pi}{4} - \log \sec 0 + \log \sin\frac{\pi}{2} - \log \sin\frac{\pi}{4}$

$= \log\sqrt{2} - 0 + 0 - \log\frac{1}{\sqrt{2}}$

$= \log(\sqrt{2} \times \sqrt{2})$

$= \log 2$ sq.units.



22. Find the area of the region bounded by the parabola $y^2 = x$ and the line $y = x - 2$.

$x = y^2 \rightarrow (1)$

$x = y + 2 \rightarrow (2)$

Solving equations (1) & (2),

$y^2 = y + 2 \Rightarrow y^2 - y - 2 = 0 \Rightarrow (y+1)(y-2) = 0$

$\Rightarrow y = -1, 2$

Area $A = \int_a^b (X_R - X_L)dy$

$= \int_{-1}^{2} (y + 2 - y^2)dy$

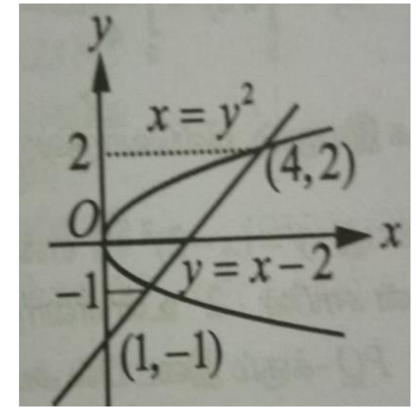
$= \left[\frac{y^2}{2} + 2y - \frac{y^3}{3}\right]_{-1}^{2}$

$= \left(2 + 4 - \frac{8}{3}\right) - \left(\frac{1}{2} - 2 + \frac{1}{3}\right)$

$= \frac{10}{3} + \frac{7}{6}$

$= \frac{27}{6}$

$= \frac{9}{2}$ sq.units.



23.Father of a family wishes to divide his square field bounded by $x = 0, x = 4, y = 4$ and $y = 0$ along the curves $y^2 = 4x$ and $x^2 = 4y$ into three equal parts for his wife, daughter and son. Is it possible to divide? If so, find the area to be divided among them?

Area $A_1 = \int_a^b ydx = \int_0^4 \frac{x^2}{4} dx$

$= \frac{1}{4}\left[\frac{x^3}{3}\right]_0^4$

$= \frac{1}{12}(64 - 0)$

$= \frac{16}{3}$ sq.units.

Area $A_2 = \int_a^b (y_U - y_L)dx$

$= \int_0^4 \left(2x^{1/2} - \frac{x^2}{4}\right) dx$

$= \left[2.\frac{x^{3/2}}{3/2} - \frac{1}{4}\frac{x^3}{3}\right]_0^4$

$= \left(\frac{32}{3} - \frac{16}{3}\right) - (0)$

$= \frac{16}{3}$ sq.units.

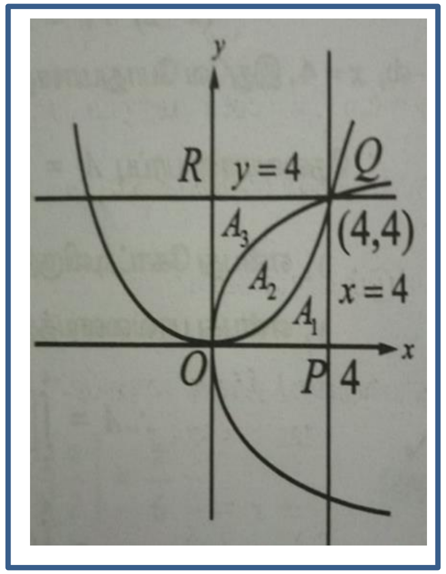
Area $A_3 = \int_a^b x dy = \int_0^4 \frac{y^2}{4} dy$

$= \frac{1}{4}\left[\frac{y^3}{3}\right]_0^4$

$= \frac{1}{12}(64 - 0)$

$= \frac{16}{3}$ sq.units.

Yes, it is possible to divide among them and the area is $\frac{16}{3}$ sq.units.



24.The curve $y = (x-2)^2 + 1$ has a minimum point at P. The point Q on the curve is such that the slope of PQ is 2. Find the area bounded by the curve and the chord PQ.

$y = (x-2)^2 + 1 \Rightarrow (x-2)^2 = (y-1)$, is a parabola.

$\Rightarrow\ vertex = (2,1)$

$slope\ of\ PQ = 2$

$\Rightarrow y = mx + c$

$\Rightarrow y = 2x + c$

$Since\ it\ passing\ through\ (2,1)$

$1 = 2(2) + c \Rightarrow c = -3$

So $y = 2x - 3 \rightarrow (1)$

$y = (x-2)^2 + 1 \rightarrow (2)$

Solving equations (1) & (2),

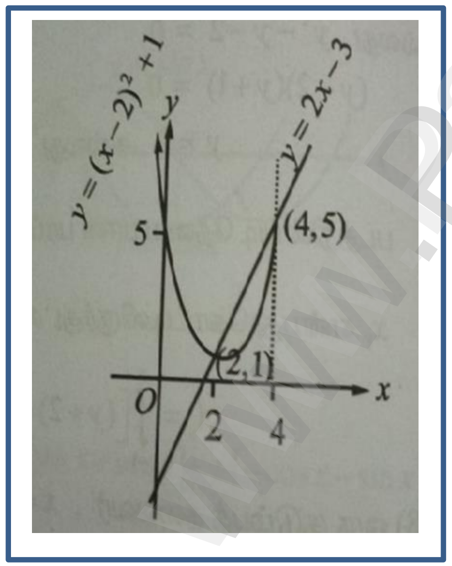
$(x-2)^2 + 1 = 2x - 3$

$\Rightarrow x^2 - 6x + 8 = 0$

$\Rightarrow (x-2)(x-4) = 0$

$\Rightarrow x = 2,4.$

Area $A = \int_a^b (y_U - y_L) dx$



$= \int_2^4 [(2x-3) - ((x-2)^2 + 1)] dx$

$= \int_2^4 (2x - 3 - x^2 + 4x - 4 - 1) dx$

$= \int_2^4 (6x - x^2 - 8) dx$

$= \left[\frac{6x^2}{2} - \frac{x^3}{3} - 8x\right]_2^4$

$= \left(48 - \frac{64}{3} - 32\right) - \left(12 - \frac{8}{3} - 16\right)$

$= -\frac{16}{3} + \frac{20}{3}$

$= \frac{4}{3}$ Sq.units.

25.Find the area common to the circle $x^2 + y^2 = 16$ and the parabola $y^2 = 6x$.

$x^2 + y^2 = 16 \rightarrow (1)$

$y^2 = 6x \rightarrow (2)$

Solving equations (1) and (2),

$x^2 + 6x = 16 \Rightarrow x^2 + 6x - 16 = 0$

$\Rightarrow (x-2)(x+8) = 0$

$\Rightarrow x = 2$ is a solution

$x^2 + y^2 = 16 \Rightarrow y = \sqrt{4^2 - x^2}$

$y^2 = 6x \Rightarrow y = \sqrt{6}x^{1/2}$

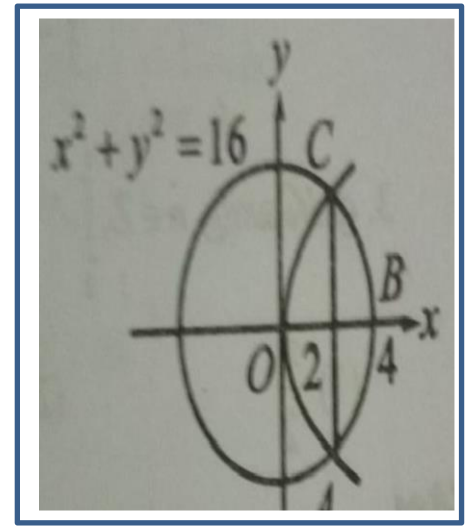
Area $A = 2\left[\int_a^b y dx + \int_b^c y dx\right]$

$= 2\left[\int_0^2 \sqrt{6}x^{1/2} dx + \int_2^4 \sqrt{4^2 - x^2} dx\right]$

$= 2\left[\left[\sqrt{6}.\frac{x^{3/2}}{3/2}\right]_0^2 + \left[\frac{x}{2}\sqrt{4^2 - x^2} + \frac{4^2}{2} sin^{-1}\left(\frac{x}{4}\right)\right]_2^4\right]$

$= \frac{4\sqrt{6}}{3}\left[(2\sqrt{2}) - (0)\right] + \frac{2}{2}\left[\begin{array}{c}(0 + 16sin^{-1}(1)) - \\ \left(2\sqrt{12} + 16sin^{-1}\left(\frac{1}{2}\right)\right)\end{array}\right]$

$= \frac{8\sqrt{12}}{3} + \frac{16\pi}{2} - 2\sqrt{12} - \frac{16\pi}{6}$



$= \frac{2\sqrt{12}}{3} + 8\pi - \frac{8\pi}{3}$

$= \frac{2\sqrt{4\times3}}{3} + \frac{16\pi}{3}$

$= \frac{4\sqrt{3}}{3} + \frac{16\pi}{3}$

$= \frac{4}{3}\left(4\pi + \sqrt{3}\right)$ Sq.units.

26.Find the volume of a sphere of radius a.

$x^2 + y^2 = a^2 \Rightarrow y^2 = a^2 - x^2$

$x = -a, x = a$

$volume\ V = \int_a^b \pi y^2 dx$

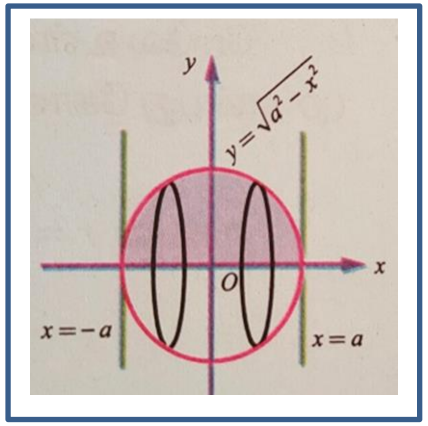
$= \pi \int_{-a}^{a} (a^2 - x^2) dx$

$= 2\pi \int_0^a (a^2 - x^2) dx$

$= 2\pi \left[a^2 x - \frac{x^3}{3}\right]_0^a$

$= 2\pi \left(a^3 - \frac{a^3}{3}\right)$

$= \frac{4}{3}\pi a^3$



27. Find the volume of a right-circular cone of base radius r and height h

$y = \frac{r}{h}x, x = 0, x = h$

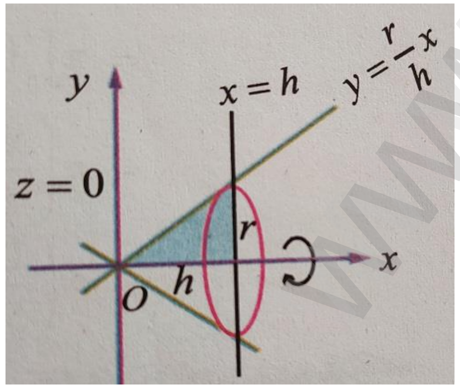
$volumeV = \int_a^b \pi y^2 dx$

$= \pi \int_0^h \left(\frac{r}{h}x\right)^2 dx$

$= \frac{\pi r^2}{h^2}\left[\frac{x^3}{3}\right]_0^h$

$= \frac{\pi r^2}{h^2}\left[\frac{h^3}{3} - 0\right]$

$= \frac{1}{3}\pi r^2 h$



28. Find the volume of the solid formed by revolving the region bounded by the parabola $y = x^2$, x-axis, ordinates $x = 0$ and $x = 1$ about x-axis

$y = x^2, x = 0,\ x = 1$

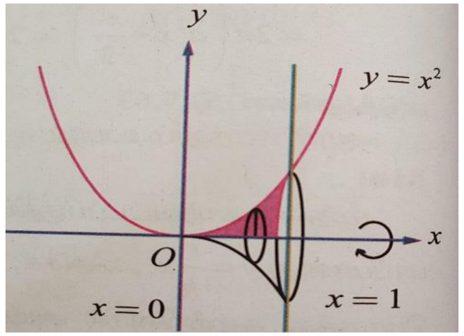
$volumeV = \int_a^b \pi y^2 dx$

$= \pi \int_0^1 (x^2)^2 dx$

$= \pi \int_0^1 x^4 dx$

$= \pi \left[\frac{x^5}{5}\right]_0^1$

$= \frac{\pi}{5}$



29.Find the volume of the solid formed by revolving the region bounded by the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1,\ a > b$ about the major axis

$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1 \Rightarrow y^2 = \frac{b^2}{a^2}(a^2 - x^2)$

$x = -a, x = a$

$volumeV = \int_a^b \pi y^2 dx$

$= \pi \int_{-a}^{a} \frac{b^2}{a^2}(a^2 - x^2) dx$

$= 2\pi \frac{b^2}{a^2} \int_0^a (a^2 - x^2) dx$

$= 2\pi \frac{b^2}{a^2}\left[a^2 x - \frac{x^3}{3}\right]_0^a$

$= 2\pi \frac{b^2}{a^2}\left(a^3 - \frac{a^3}{3}\right)$

$= \frac{4}{3}\pi ab^3$