3. TRIGONOMETRY

Basic Trigonometric ratios using right triangle.

$$\sin\theta = \frac{\text{opp. side}}{\text{hypotenuse}}$$

$$\cos\theta = \frac{\text{adjacent side}}{\text{hypotenuse.}}$$

$$\tan\theta = \frac{\text{opp. side}}{\text{adjacent side.}}$$

$\operatorname{cosec}\theta = \dfrac{1}{\sin\theta}$ $\quad$ $\sec\theta = \dfrac{1}{\cos\theta}$ $\quad$ $\cot\theta = \dfrac{1}{\tan\theta}$

$\sin\theta \cdot \operatorname{cosec}\theta = 1$ $\quad$ $\operatorname{cosec}\theta \cdot \sec\theta = 1$ $\quad$ $\tan\theta \cot\theta = 1$

$\tan\theta = \dfrac{\sin\theta}{\cos\theta}$ $\quad$ $\cot\theta = \dfrac{\cos\theta}{\sin\theta}.$

Basic Trigonometric Identities:

$\sin^2\theta + \cos^2\theta = 1$

$\cos^2\theta = 1 - \sin^2\theta$

$\sin^2\theta = 1 - \cos^2\theta$

$\operatorname{cosec}^2\theta - \cot^2\theta = 1$

$\operatorname{cosec}^2\theta = 1 + \cot^2\theta$

$\cot^2\theta = \operatorname{cosec}^2\theta - 1$

$\sec^2\theta - \tan^2\theta = 1$

$\sec^2\theta = 1 + \tan^2\theta$

$\tan^2\theta = \sec^2\theta - 1.$

Exact values of trigonometric functions of widely used angles:

θ	0°	30°	45°	60°	90°
$\sin\theta$	0	$\frac{1}{2}$	$\frac{1}{\sqrt{2}}$	$\frac{\sqrt{3}}{2}$	1
$\cos\theta$	1	$\frac{\sqrt{3}}{2}$	$\frac{1}{\sqrt{2}}$	$\frac{1}{2}$	0
$\tan\theta$	0	$\frac{1}{\sqrt{3}}$	1	$\sqrt{3}$	∞

3.1 Prove that $\dfrac{\tan\theta+\sec\theta-1}{\tan\theta-\sec\theta+1} = \dfrac{1+\sin\theta}{\cos\theta}$

$$L.H.S = \frac{\tan\theta+\sec\theta-1}{\tan\theta-\sec\theta+1} = \frac{\tan\theta+\sec\theta-(\sec^2\theta-\tan^2\theta)}{\tan\theta-\sec\theta+1}$$

$$= \frac{(\tan\theta+\sec\theta)-(\sec\theta+\tan\theta)(\sec\theta-\tan\theta)}{(\tan\theta-\sec\theta+1)}$$

$$= \frac{(\tan\theta+\sec\theta)\cancel{\{1-\sec\theta+\tan\theta\}}}{\cancel{(\tan\theta-\sec\theta+1)}}$$

$$= \tan\theta+\sec\theta = \frac{\sin\theta}{\cos\theta}+\frac{1}{\cos\theta} = \frac{1+\sin\theta}{\cos\theta} = R.H.S$$

3.2 Prove that $(\sec A-\operatorname{cosec}A)(1+\tan A+\cot A)$
$= \tan A\sec A - \cot A.\operatorname{cosec}A$

$$L.H.S = (\sec A-\operatorname{cosec}A)(1+\tan A+\cot A)$$

$$= \left(\frac{1}{\cos A}-\frac{1}{\sin A}\right)\left(1+\frac{\sin A}{\cos A}+\frac{\cos A}{\sin A}\right)$$

$$= \frac{\sin A-\cos A}{\sin A\cos A}\left[\frac{\sin A\cos A+\sin^2A+\cos^2A}{\sin A\cos A}\right]$$

$$= \frac{(\sin A-\cos A)(\sin^2A+\sin A\cos A+\cos^2A)}{\sin^2A\cos^2A}$$

$$= \frac{\sin^3A-\cos^3A}{\sin^2A\cos^2A} \quad \text{------------ (1)}$$

$$R.H.S = \tan A\sec A - \cot A.\operatorname{cosec}A$$

$$= \frac{\sin A}{\cos A}.\frac{1}{\cos A} - \frac{\cos A}{\sin A}.\frac{1}{\sin A}$$

$$= \frac{\sin A}{\cos^2A} - \frac{\cos A}{\sin^2A}$$

$$= \frac{\sin^3A-\cos^3A}{\sin^2A\cos^2A} \quad \text{------------ (2)}$$

From (1) & (2)

L.H.S = R.H.S

Hence Proved.

Ex : 3.1

3\. If $a\cos\theta - b\sin\theta = c$, show that $a\sin\theta + b\cos\theta = \pm\sqrt{a^2+b^2-c^2}$

Proof :-

$$a\cos\theta - b\sin\theta = c$$

Squaring on both sides,

$$(a\cos\theta - b\sin\theta)^2 = c^2$$

$$a^2\cos^2\theta + b^2\sin^2\theta - 2ab\sin\theta\cos\theta = c^2$$

$$a^2(1-\sin^2\theta) + b^2(1-\cos^2\theta) - 2ab\sin\theta\cos\theta = c^2$$

$$a^2 - a^2\sin^2\theta + b^2 - b^2\cos^2\theta - 2ab\sin\theta\cos\theta = c^2$$

$$-a^2\sin^2\theta - b^2\cos^2\theta - 2ab\sin\theta\cos\theta = c^2 - a^2 - b^2$$

$$a^2\sin^2\theta + b^2\cos^2\theta + 2ab\sin\theta\cos\theta = a^2+b^2-c^2$$

$$(a\sin\theta + b\cos\theta)^2 = a^2+b^2-c^2$$

$$a\sin\theta + b\cos\theta = \pm\sqrt{a^2+b^2-c^2}$$

Hence Proved.

4\. If $\sin\theta + \cos\theta = m$ show that $\cos^6\theta + \sin^6\theta = \dfrac{4-3(m^2-1)^2}{4}$ where $m^2 \le 2$

$$\sin\theta + \cos\theta = m$$

squaring on both sides,

$$\sin^2\theta + \cos^2\theta + 2\sin\theta\cos\theta = m^2$$

$$1 + 2\sin\theta\cos\theta = m^2$$

$$2\sin\theta\cos\theta = m^2 - 1$$

$$\sin\theta\cos\theta = \frac{m^2-1}{2}$$

$$\cos^6\theta + \sin^6\theta = (\cos^2\theta)^3 + (\sin^2\theta)^3$$

$$= (\cos^2\theta + \sin^2\theta)^3 - 3\sin^2\theta\cos^2\theta(\sin^2\theta + \cos^2\theta)$$

$$= 1 - 3\sin^2\theta\cos^2\theta$$

$$\left\{ a^3+b^3 = (a+b)^3 - 3ab(a+b) \right\}$$

$$= 1 - 3\left[\frac{m^2-1}{2}\right]^2$$

$$= 1 - \frac{3(m^2-1)}{4}$$

$$= \frac{4-3(m^2-1)}{4}$$

5. If $\dfrac{\cos^4\alpha}{\cos^2\beta} + \dfrac{\sin^4\alpha}{\sin^2\beta} = 1$ then Prove that

(i) $\sin^4\alpha + \sin^4\beta = 2\sin^2\alpha\sin^2\beta$

(ii) $\dfrac{\cos^4\beta}{\cos^2\alpha} + \dfrac{\sin^4\beta}{\sin^2\alpha} = 1.$

Solution :-

$$\frac{\cos^4\alpha}{\cos^2\beta} + \frac{\sin^4\alpha}{\sin^2\beta} = 1$$

$$\cos^4\alpha\sin^2\beta + \sin^4\alpha\cos^2\beta = \cos^2\beta\sin^2\beta$$

$$\cos^4\alpha(1-\cos^2\beta) + (1-\cos^2\alpha)^2\cos^2\beta = \cos^2\beta(1-\cos^2\beta)$$

$$\cos^4\alpha - \cos^4\alpha\cos^2\beta + (1+\cos^4\alpha - 2\cos^2\alpha)\cos^2\beta = \cos^2\beta - \cos^4\beta$$

$$\cos^4\alpha - \cancel{\cos^4\alpha\cos^2\beta} + \cos^2\beta + \cancel{\cos^4\alpha\cos^2\beta} - 2\cos^2\alpha\cos^2\beta = \cos^2\beta - \cos^4\beta$$

$$\cos^4\alpha + \cancel{\cos^2\beta} - 2\cos^2\alpha\cos^2\beta = \cancel{\cos^2\beta} - \cos^4\beta$$

$$\cos^4\alpha + \cos^4\beta - 2\cos^2\alpha\cos^2\beta = 0$$

$$(\cos^2\alpha - \cos^2\beta)^2 = 0$$

$$\cos^2\alpha - \cos^2\beta = 0$$

$$\boxed{\cos^2\alpha = \cos^2\beta}$$

$$1 - \sin^2\alpha = 1 - \sin^2\beta$$

$$\boxed{\sin^2\alpha = \sin^2\beta}$$

(i) $\sin^4\alpha + \sin^4\beta = 2\sin^2\alpha\sin^2\beta$

L.H.S $= \sin^4\alpha + \sin^4\beta$ $\quad \left\{a^2+b^2 = (a-b)^2 + 2ab\right\}$

$= (\sin^2\alpha - \sin^2\beta)^2 + 2\sin^2\alpha\sin^2\beta$

$= 0 + 2\sin^2\alpha\sin^2\beta$

$= 2\sin^2\alpha\sin^2\beta$

$=$ R.H.S

(ii) $\dfrac{\cos^4\beta}{\cos^2\alpha} + \dfrac{\sin^4\beta}{\sin^2\alpha} = 1$

L.H.S $= \dfrac{\cos^4\beta}{\cos^2\alpha} + \dfrac{\sin^4\beta}{\sin^2\alpha} = \dfrac{(\cos^2\beta)^2}{\cos^2\alpha} + \dfrac{(\sin^2\beta)^2}{\sin^2\alpha}$

$= \dfrac{(\cos^2\alpha)^2}{\cos^2\alpha} + \dfrac{(\sin^2\alpha)^2}{\sin^2\alpha} = \cos^2\alpha + \sin^2\alpha$

$= 1$

$=$ R.H.S

$\left\{\because \begin{array}{l}\sin^2\alpha = \sin^2\beta \\ \cos^2\alpha = \cos^2\beta\end{array}\right\}$

6. If $y = \frac{2\sin\alpha}{1+\cos\alpha+\sin\alpha}$ then Prove that $\frac{1-\cos\alpha+\sin\alpha}{1+\sin\alpha} = y$

$$y = \frac{2\sin\alpha}{1+\cos\alpha+\sin\alpha} \times \frac{1+\sin\alpha-\cos\alpha}{1+\sin\alpha-\cos\alpha}$$

$$= \frac{2\sin\alpha(1+\sin\alpha-\cos\alpha)}{(1+\sin\alpha)^2-\cos^2\alpha} = \frac{2\sin\alpha(1+\sin\alpha-\cos\alpha)}{1+\sin^2\alpha+2\sin\alpha-\cos^2\alpha}$$

$$= \frac{2\sin\alpha(1+\sin\alpha-\cos\alpha)}{\sin^2\alpha+2\sin\alpha+\sin^2\alpha} = \frac{2\cancel{\sin\alpha}(1+\sin\alpha-\cos\alpha)}{2\cancel{\sin\alpha}(1+\sin\alpha)}$$

$$y = \frac{1-\cos\alpha+\sin\alpha}{1+\sin\alpha}$$

Hence Proved.

7. If $x=\sum_{n=0}^{\infty}\cos^{2n}\theta$ $y=\sum_{n=0}^{\infty}\sin^{2n}\theta$ and $z=\sum_{n=0}^{\infty}\cos^{2n}\theta.\sin^{2n}\theta$

$0<\theta<\pi/2$, then show that $xyz = x+y+z$.

Solution :-

$$x = \sum_{n=0}^{\infty}\cos^{2n}\theta = 1+\cos^2\theta+\cos^4\theta+\cos^6\theta+\cdots\infty$$

$$x = \frac{1}{1-\cos^2\theta} = \frac{1}{\sin^2\theta}.$$

$$y = \sum_{n=0}^{\infty}\sin^{2n}\theta = 1+\sin^2\theta+\sin^4\theta+\sin^6\theta+\cdots\infty.$$

$$y = \frac{1}{1-\sin^2\theta} = \frac{1}{\cos^2\theta}$$

$$z = \sum_{n=0}^{\infty}\cos^{2n}\theta.\sin^{2n}\theta = 1+\sin^2\theta\cos^2\theta+\sin^4\theta\cos^4\theta+\cdots\infty$$

$$z = \frac{1}{1-\sin^2\theta.\cos^2\theta}.$$

$$\text{L.H.S} = xyz = \frac{1}{\sin^2\theta}\cdot\frac{1}{\cos^2\theta}\cdot\frac{1}{1-\sin^2\theta\cos^2\theta}$$

$$= \frac{1}{\sin^2\theta.\cos^2\theta(1-\sin^2\theta\cos^2\theta)} \quad \text{———— (1)}$$

$$\text{R.H.S} = x+y+z = \frac{1}{\sin^2\theta}+\frac{1}{\cos^2\theta}+\frac{1}{1-\sin^2\theta\cos^2\theta}$$

$$= \frac{\sin^2\theta+\cos^2\theta}{\sin^2\theta.\cos^2\theta}+\frac{1}{1-\sin^2\theta\cos^2\theta} = \frac{1}{\sin^2\theta\cos^2\theta}+\frac{1}{1-\sin^2\theta\cos^2\theta}$$

$$= \frac{1-\cancel{\sin^2\theta\cos^2\theta}+\cancel{\sin^2\theta\cos^2\theta}}{\sin^2\theta.\cos^2\theta(1-\sin^2\theta\cos^2\theta)}$$

$$= \frac{1}{\sin^2\theta\cos^2\theta(1-\sin^2\theta\cos^2\theta)} \quad \text{———— (2)}$$

from (1) & (2) L.H.S = R.H.S.

8. If $\tan^2\theta = 1-k^2$, Show that $\sec\theta + \tan^3\theta \csc\theta = (2-k^2)^{3/2}$. Also find the values of k for which this result holds.

$$\tan^2\theta = 1-k^2$$

$$\begin{aligned} L.H.S &= \sec\theta + \tan^3\theta \cdot \csc\theta \\ &= \sec\theta\left[1+\tan^3\theta \cdot \frac{\csc\theta}{\sec\theta}\right] \\ &= \sec\theta\left[1+\tan^2\theta\right] \\ &= \sqrt{1+\tan^2\theta}\,(1+\tan^2\theta) \\ &= (1+\tan^2\theta)^{3/2} \\ &= \left[1+(1-k^2)\right]^{3/2} = (2-k^2)^{3/2} = R.H.S. \end{aligned}$$

$\tan^2\theta \geq 0 \;\; \forall\; \theta$

$$\Rightarrow 1-a^2 \geq 0$$

$$a^2-1 \leq 0 \Rightarrow a^2 \leq 1$$

$$|a| \leq 1$$

$$\Rightarrow \boxed{-1 \leq a \leq 1} \quad \text{———} \; (1)$$

Since L.H.S of $\sec\theta + \tan^3\theta \csc\theta = (2-a^2)^{3/2}$ is real for all $\theta \in R$. So RHS must also be real.

$$2-a^2 \geq 0$$

$$a^2-2 \leq 0$$

$$a^2 \leq 2$$

$$|a| \leq \sqrt{2}$$

$$\boxed{-\sqrt{2} \leq a \leq a} \quad \text{———} \; (2)$$

from (1) & (2)

The given relation holds true for all $\theta \in [-1, 1]$.

9. If $\sec\theta + \tan\theta = P$, obtain the values of $\sec\theta$, $\tan\theta$ and $\sin\theta$ in terms of P.

$$P = \sec\theta + \tan\theta$$

$$\frac{1}{P} = \frac{1}{\sec\theta+\tan\theta} \times \frac{\sec\theta - \tan\theta}{\sec\theta - \tan\theta}$$

$$= \frac{\sec\theta - \tan\theta}{\sec^2\theta - \tan^2\theta} = \sec\theta - \tan\theta.$$

$P = \sec\theta + \tan\theta$ $\quad \frac{1}{P} = \sec\theta - \tan\theta$

$$P + \frac{1}{P} = \sec\theta + \tan\theta + \sec\theta - \tan\theta$$

$$2\sec\theta = P + \frac{1}{P}$$

$$\sec\theta = \frac{1}{2}\left(P + \frac{1}{P}\right)$$

$$P - \frac{1}{P} = \cancel{\sec\theta} + \tan\theta - \cancel{\sec\theta} + \tan\theta$$

$$2\tan\theta = P - \frac{1}{P}$$

$$\tan\theta = \frac{1}{2}\left(P - \frac{1}{P}\right)$$

$$\frac{P - \frac{1}{P}}{P + \frac{1}{P}} = \frac{\cancel{2}\tan\theta}{\cancel{2}\sec\theta}$$

$$\frac{P^2 - 1}{P^2 + 1} = \sin\theta.$$

$$\therefore \sin\theta = \frac{P^2 - 1}{P^2 + 1}.$$

---

10. If $\cot\theta(1+\sin\theta) = 4m$ and $\cot\theta(1-\sin\theta) = 4n$ then Prove that $(m^2 - n^2)^2 = mn$.

$\cot\theta(1+\sin\theta) = 4m$ $\quad \cot\theta(1-\sin\theta) = 4n$

$\cot\theta + \cos\theta = 4m$ $\quad \cot\theta - \cos\theta = 4n$

$$(\cot\theta + \cos\theta)^2 - (\cot\theta - \cos\theta)^2 = (4m)^2 - (4n)^2$$

$$\{(a+b)^2 - (a-b)^2 = 4ab\}$$

$$4\cot\theta\cos\theta = 16(m^2 - n^2)$$

$$4\frac{\cos^2\theta}{\sin\theta} = 16(m^2 - n^2)$$

$$\frac{\cos^2\theta}{\sin\theta} = 4(m^2 - n^2)$$

Squaring on both sides

$$\frac{\cos^4\theta}{\sin^2\theta} = 16(m^2 - n^2)^2 \quad \text{——— (1)}$$

$$(\cot\theta + \cos\theta)(\cot\theta - \cos\theta) = 4m \cdot 4n$$

$$\cot^2\theta - \cos^2\theta = 16mn$$

$$\cos^2\theta\left(\frac{1}{\sin^2\theta} - 1\right) = 16mn$$

$$\frac{\cos^4\theta}{\sin^2\theta} = 16mn \quad \text{——— (2)}$$

from (1) & (2)

$$16(m^2 - n^2)^2 = 16mn$$

$$(m^2 - n^2)^2 = mn$$

Hence Proved

11. If $\operatorname{cosec}\theta - \sin\theta = a^3$ and $\sec\theta - \cos\theta = b^3$ then Prove that $a^2b^2(a^2+b^2) = 1$.

$\operatorname{cosec}\theta - \sin\theta = a^3$

$\frac{1}{\sin\theta} - \sin\theta = a^3$

$\frac{1-\sin^2\theta}{\sin\theta} = a^3$

$\frac{\cos^2\theta}{\sin\theta} = a^3$

$a = \frac{\cos^{2/3}\theta}{\sin^{1/3}\theta}$

$\sec\theta - \cos\theta = b^3$

$\frac{1}{\cos\theta} - \cos\theta = b^3$

$\frac{1-\cos^2\theta}{\cos\theta} = b^3$

$\frac{\sin^2\theta}{\cos\theta} = b^3$

$b = \frac{\sin^{2/3}\theta}{\cos^{1/3}\theta}$.

To Prove : $a^2b^2(a^2+b^2) = 1$.

$$\text{L.H.S} = a^2b^2(a^2+b^2) = \frac{(\cos^{4/3}\theta)}{\sin^{2/3}\theta} \cdot \frac{\sin^{4/3}\theta}{\cos^{2/3}\theta}\left[\frac{\cos^{4/3}\theta}{\sin^{2/3}\theta} + \frac{\sin^{4/3}\theta}{\cos^{2/3}\theta}\right]$$

$$= \cancel{\cos^{2/3}\theta} \cdot \cancel{\sin^{2/3}\theta}\left[\frac{\cos^2\theta + \sin^2\theta}{\cancel{\sin^{2/3}\theta} \cdot \cancel{\cos^{2/3}\theta}}\right]$$

$= 1$

$=$ R.H.S.

12. Eliminate 'θ' from the equations $a\sec\theta - c\tan\theta = b$ and $b\sec\theta + d\tan\theta = c$

$$a\sec\theta - c\tan\theta - b = 0$$
$$b\sec\theta + d\tan\theta - c = 0$$

$\sec\theta$	$\tan\theta$	1	
$-c$	$-b$	a	$-c$
d	$-c$	b	d

$$\frac{\sec\theta}{c^2+bd} = \frac{\tan\theta}{-b^2+ac} = \frac{1}{ad+bc}$$

$\sec\theta = \frac{c^2+bd}{ad+bc}$ $\qquad$ $\tan\theta = \frac{ac-b^2}{ad+bc}$

W.K.T

$$\sec^2\theta - \tan^2\theta = 1$$

$$\left(\frac{c^2+bd}{ad+bc}\right)^2 - \left(\frac{ac-b^2}{ad+bc}\right)^2 = 1$$

$$(c^2+bd)^2 - (ac-b^2)^2 = (ad+bc)^2$$

$$(c^2+bd)^2 = (ad+bc)^2 + (ac-b^2)^2$$

3.3 Eliminate θ from $a\cos\theta = b$ and $c\sin\theta = d$ where a, b, c, d are constants.

$a\cos\theta = b$ $\qquad$ $c\sin\theta = d$

$\cos\theta = \frac{b}{a}$ $\qquad$ $\sin\theta = \frac{d}{c}$

W.K.T $\sin^2\theta + \cos^2\theta = 1$

$$\frac{d^2}{c^2} + \frac{b^2}{a^2} = 1$$

$$a^2d^2 + b^2c^2 = a^2c^2$$

3.3. Radian Measure :

The radian measure of an angle is the ratio of the arc length it subtends, to the radius of the circle in which it is the central angle.

Consider a circle of radius r. Let S be the arc length subtending an angle θ at the centre.

$$\theta = \frac{\text{arc length}}{\text{radius}} = \frac{S}{r} \text{ radians.}$$

$$\theta = \frac{S}{r}$$

$$S = r\theta$$

Relationship between Degree and Radian Measures :

In a unit circle, a full rotation corresponds to 360° whereas, a full rotation is related to 2π radians., the circumference of the unit circle.

$$2\pi = 360^\circ \qquad \pi = 180^\circ$$

$$1 \text{ radian} = \left(\frac{180}{\pi}\right)^\circ \text{ (or) } 1^\circ = \frac{\pi}{180} \text{ radians.}$$

$$x \text{ radians} = \left(\frac{180x}{\pi}\right)^\circ \text{ (or) } x^\circ = \frac{\pi x}{180} \text{ radians.}$$

Remarks :

* Arc length $= \begin{cases} \frac{\theta}{360} \times 2\pi r & \theta \text{ in deg} \\ r\theta & \theta \text{ in rad} \end{cases}$

* Area of the sector $= \begin{cases} \frac{\theta}{360} \times \pi r^2 \text{ (or) } \frac{Sr}{2} & \theta \text{ in deg} \\ \frac{\theta}{2} r^2 & \theta \text{ in rad} \end{cases}$

* Area of the segment $= \frac{\theta}{360}\pi r^2 - \frac{1}{2}r^2\sin\theta.$

3.4 Convert (i) 18° to radians

(ii) -108° to radians.

(i) $18^\circ = 18^\circ \times \frac{\pi}{180} = \frac{\pi}{10}$ radians.

(ii) $-108^\circ = -108 \times \frac{\pi}{180} = -\frac{3\pi}{5}$ radians.

3.5. Convert : (i) $\frac{\pi}{5}$ radians to degree.

(ii) 6 radians to degree.

(i) $\frac{\pi}{5} = \frac{\pi}{5} \times \frac{180}{\pi} = 36^\circ$

(ii) $6 = 6 \times \frac{180}{\pi}$

$= 6 \times 180 \times \frac{7}{22}$

$= 3 \times 180 \times \frac{7}{11} = \frac{3780}{11}$

$= \left(343\frac{7}{11}\right)^\circ$

$$\begin{array}{r} 343 \\ 11 \overline{)3780} \\ 33 \\ \hline 48 \\ 44 \\ \hline 40 \\ 33 \\ \hline 7 \end{array}$$

Ex : 3.2

1 Express each of the following angles in radian measure.

(i) $30^\circ = 30 \times \frac{\pi}{180} = \frac{\pi}{6}$ radians.

(ii) $135^\circ = 135 \times \frac{\pi}{180} = \frac{3\pi}{4}$ radians.

(iii) $-205^\circ = -205 \times \frac{\pi}{180} = -\frac{41\pi}{36}$ radians.

(iv) $150^\circ = 150 \times \frac{\pi}{180} = \frac{5\pi}{6}$ radians.

(v) $330^\circ = 330 \times \frac{\pi}{180} = \frac{11\pi}{6}$ radians.

2. Find the degree measure corresponding to the following radian measures.

(i) $\frac{\pi}{3} = \frac{\pi}{3} \times \frac{180}{\pi} = 60^\circ$

(ii) $\frac{\pi}{9} = \frac{\pi}{9} \times \frac{180}{\pi} = 20^\circ$

(iii) $\frac{2\pi}{5} = \frac{2\pi}{5} \times \frac{180}{\pi} = 72^\circ$

(iv) $\frac{7\pi}{3} = \frac{7\pi}{3} \times \frac{180}{\pi} = 420^\circ$

(v) $\frac{10\pi}{9} = \frac{10\pi}{9} \times \frac{180}{\pi} = 200^\circ$

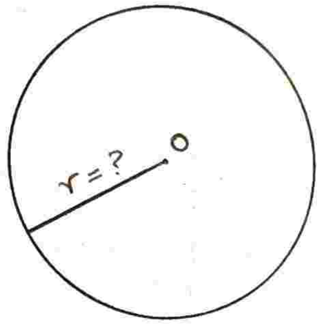
3. What must be the radius of a circular running path, around which an athlete must run 5 times in order to describe 1 km?

Distance travelled in 5 rounds

$= 1 \text{ km} = 1000 \text{m}$

Distance travelled in 1 round

$= \frac{1000}{5} = 200 \text{m}.$



Perimeter $= 200 \text{m}$

$2\pi r = 200$

$2 \times \frac{22}{7} \times r = 200$

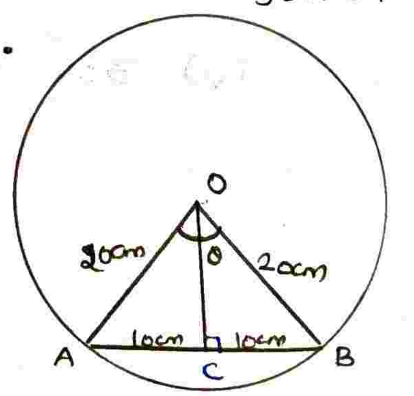
$r = \frac{200 \times 7}{2 \times 22} = \frac{350}{11}$

$= 31.82 \text{ m}.$

$\therefore$ The radius of the circular Path $= 31.82 \text{m}.$

4. In a circle of diameter 40 cm, a chord is of length 20 cm. Find the length of the minor arc of the chord.

O be the centre of the circle.



$OA = OB = 20 \text{cm}$

$AB = 20 \text{ cm}$

$AC = BC = 10 \text{cm}.$

$\angle AOB = \theta \qquad \angle AOC = \theta/2$

Consider the ΔAOC

$$\sin \theta/2 = \frac{AC}{OA} = \frac{10}{20} = \frac{1}{2}$$

$$\theta/2 = 30^\circ$$

$$\theta = 60^\circ = \frac{\pi}{3} \text{ radians.}$$

$$\text{arc length} = r\theta$$

$$= 20 \times \frac{\pi}{3} = \frac{20\pi}{3} \text{ cm}$$

$$= \frac{20 \times 22}{3 \times 7} = 20.95 \text{ cm.}$$

$\therefore$ The length of the minor arc is 20.95 cm.

5. Find the degree measure of the angle subtended at the centre of circle of radius 100 cm by an arc of length 22 cm.

radius $r = 100$ cm

arc length $= 22$ cm.

$$s = r\theta = 22$$

$$100 \times \theta = 22$$

$$\theta = \frac{22}{100} \text{ rad}$$

$$= \frac{22}{100} \times \frac{180}{\pi} \text{ deg}$$

$$= \frac{\cancel{22}}{\cancel{100}_{5}} \times \cancel{180}^{9} \times \frac{7}{\cancel{22}} = \frac{63}{5}$$

$$\theta = 12^\circ 36'$$

6. what is the length of the arc intercepted by a central angle of measure 41° in a circle of radius 10 ft?

$\theta = 41^\circ = 41 \times \frac{\pi}{180}$ rad $\qquad$ radius $= 10$ ft

$$\text{arc length } s = r\theta$$

$$= 1\cancel{0} \times 41 \times \frac{\cancel{22}^{11}}{7} \times \frac{1}{\cancel{180}_{9}} = \frac{451}{63}$$

$$= 7.1587$$

$$= 7.16 \text{ ft.}$$

$\therefore$ The length of the arc is 7.16 ft.

7. If in two circles, arcs of the same length subtend angles 60° and 75° at the centre, find the ratio of their radii

Let r_1 and r_2 be the two radii of the circles.

$\theta_1 = 60^\circ = \frac{\pi}{3}$ rad

$\theta_2 = 75^\circ = 75 \times \frac{\pi}{180} = \frac{5\pi}{12}$

$S_1 = S_2 \Rightarrow r_1\theta_1 = r_2\theta_2$

$\Rightarrow r_1 \frac{\cancel{\pi}}{3} = r_2 \cdot \frac{5\cancel{\pi}}{12}$

$\frac{r_1}{\cancel{3}} = \frac{5r_2}{\cancel{12}_{4}}$

$\frac{r_1}{r_2} = \frac{5}{4}$

$r_1 : r_2 = 5 : 4.$

8. The Perimeter of a certain sector of a circle is equal to the length of the arc of a semicircle having the same radius. Express the angle of the sector in degrees, min, and sec.

Let r be the radius of the Sector.

Given that

Perimeter of the Sector $= \frac{1}{2}(2\pi r)$

$2r + r\theta = \pi r$

$\cancel{r}(2+\theta) = \pi \cancel{r}$

$\theta = (\pi - 2)$ rad

$\theta = (\pi - 2) \times \frac{180}{\pi}$ deg

$= 180 - \frac{360}{\pi}$

$= 180 - 360 \times \frac{7}{22} = 180^\circ - \frac{1260}{11}$

$= 180^\circ - 114^\circ 32' 43''$

$\theta = 65^\circ 27' 17''$

$179^\circ 59' 60''$
$114^\circ 32' 43''$
$65\ 27\ 17$

9. An airplane Propeller rotates 1000 times per minute. Find the number of degrees that a Point on the edge of the Propeller will rotate in 1 Second.

No of rotations in 1 min = 1000

No of rotations in 1 sec = $\frac{1000}{60} = \frac{50}{3}$

The angle rotated in 1 rotation = $360°$

∴ The angle rotated in $\frac{50}{3}$ rotations

$= \frac{50}{3} \times 360 = 6000°$

10. A train is moving on a circular track of 1500m radius at the rate of 66 km/h what angle will it turn in 20 seconds.?

radius = 1500 m.

Speed of the train = 66 km/hr

$= \frac{66000}{60 \times 60}$ m/sec

$= \frac{110}{6}$ m/sec.

Train will travel in 20 sec $= 20 \times \frac{110}{6}$

$= \frac{1100}{3}$ m.

arc length $= \frac{1100}{3}$ m

$r\theta = \frac{1100}{3}$

$15\cancel{0}\cancel{0} \times \theta = \frac{11\cancel{0}\cancel{0}}{3}$

$\theta = \frac{11}{3} \times \frac{1}{15}$ rad

$= \frac{11}{3} \times \frac{11}{15} \times \frac{180}{\pi}$ deg

$= \frac{\cancel{11}}{\cancel{3}} \times \frac{1}{\cancel{15}} \times \cancel{180} \times \frac{7}{\cancel{22}}$

$= 14°$

$\boxed{\theta = 14°}$

11. A circular metallic plate of radius 8cm and thickness 6mm is melted and molded into a Pie of radius 16cm and thickness 4mm. Find the angle of the sector.

Circular metallic Plate: (cylinder)

radius = 8cm

height $= 6\text{mm} = \frac{6}{10}$ cm.

Pie

radius = 16cm

thickness (height) $= 4\text{mm} = \frac{4}{10}$ cm.

Given that

Vol. of Pie = vol. of circular Plate

$$\frac{\theta}{360} \times \cancel{\pi} R^2 H = \cancel{\pi} r^2 h$$

$$\frac{\theta}{\underset{\cancel{96}\ 45}{\cancel{360}}} \times \frac{\cancel{4}}{\cancel{10}} \times \overset{2}{\cancel{16}} \times \overset{2}{\cancel{16}} = \cancel{8} \times \cancel{8} \times \frac{\overset{3}{\cancel{6}}}{\cancel{10}}$$

$$\frac{\theta}{45} = 3$$

$$\theta = 3(45)$$

$\boxed{\theta = 135^\circ}$ (or) $\theta = \frac{3\pi}{4}$.

3.6 Find the length of an arc of a circle of radius 5cm subtending a central angle measuring 15°.

radius = 5 cm

$\theta = 15^\circ = 15 \times \frac{\pi}{180} = \frac{\pi}{12}$ rad

arc length $= r\theta$

$= 5 \times \frac{\pi}{12} = \frac{5\pi}{12}$ cm,

3.7 If the arcs of same length in two circles subtend central angles 30° and 80°. find the ratio of their radii

Let r_1, r_2 be the radii of two given Circles.

$\theta_1 = 30^\circ = \pi/6$ $\quad \theta_2 = 80^\circ = \frac{4\pi}{9}$ rad.

Given that $S_1 = S_2$

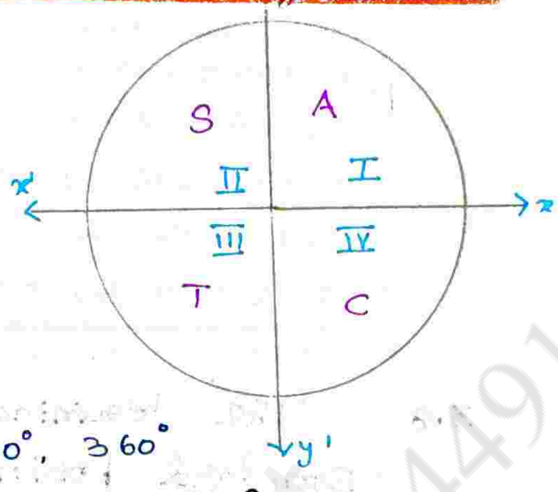
$$r_1\theta_1 = r_2\theta_2$$

$$r_1 \frac{\cancel{\pi}}{\cancel{6}_2} = r_2 \cdot \frac{4\cancel{\pi}}{\cancel{9}_3} \Rightarrow \frac{r_1}{r_2} = \frac{8}{3}$$

$\therefore \boxed{r_1 : r_2 = 8 : 3}$

Trigonometric functions and their Properties:

$\sin(-\theta) = -\sin\theta$
$\cos(-\theta) = \cos\theta$
$\tan(-\theta) = -\tan\theta$
$\operatorname{cosec}(-\theta) = -\operatorname{cosec}\theta$
$\sec(-\theta) = \sec\theta$
$\cot(-\theta) = -\cot\theta$



$90^\circ, 270^\circ$

$\sin\theta \rightarrow \cos\theta$
$\cos\theta \rightarrow \sin\theta$
$\tan\theta \rightarrow \cot\theta$
$\operatorname{cosec}\theta \rightarrow \sec\theta$
$\sec\theta \rightarrow \operatorname{cosec}\theta$
$\cot\theta \rightarrow \tan\theta$

$180^\circ, 360^\circ$

$\sin\theta \rightarrow \sin\theta$
$\cos\theta \rightarrow \cos\theta$
$\tan\theta \rightarrow \tan\theta$
$\operatorname{cosec}\theta \rightarrow \operatorname{cosec}\theta$
$\sec\theta \rightarrow \sec\theta$
$\cot\theta \rightarrow \cot\theta$.

$90-\theta$	$90+\theta$
$\sin(90-\theta) = \cos\theta$	$\sin(90+\theta) = \cos\theta$
$\cos(90-\theta) = \sin\theta$	$\cos(90+\theta) = -\sin\theta$
$\tan(90-\theta) = \cot\theta$	$\tan(90+\theta) = -\cot\theta$
$\operatorname{cosec}(90-\theta) = \sec\theta$	$\operatorname{cosec}(90+\theta) = \sec\theta$
$\sec(90-\theta) = \operatorname{cosec}\theta$	$\sec(90+\theta) = -\operatorname{cosec}\theta$
$\cot(90-\theta) = \tan\theta$	$\cot(90+\theta) = -\tan\theta$
$180-\theta$	**$180+\theta$**
$\sin(180-\theta) = \sin\theta$	$\sin(180+\theta) = -\sin\theta$
$\cos(180-\theta) = -\cos\theta$	$\cos(180+\theta) = -\cos\theta$
$\tan(180-\theta) = -\tan\theta$	$\tan(180+\theta) = \tan\theta$
$\operatorname{cosec}(180-\theta) = \operatorname{cosec}\theta$	$\operatorname{cosec}(180+\theta) = -\operatorname{cosec}\theta$
$\sec(180-\theta) = -\sec\theta$	$\sec(180+\theta) = -\sec\theta$
$\cot(180-\theta) = -\cot\theta$	$\cot(180+\theta) = \cot\theta$.
$270-\theta$	**$270+\theta$**
$\sin(270-\theta) = -\cos\theta$	$\sin(270+\theta) = -\cos\theta$
$\cos(270-\theta) = -\sin\theta$	$\cos(270+\theta) = \sin\theta$
$\tan(270-\theta) = \cot\theta$	$\tan(270+\theta) = -\cot\theta$
$\operatorname{cosec}(270-\theta) = -\sec\theta$	$\operatorname{cosec}(270+\theta) = -\sec\theta$
$\sec(270-\theta) = -\operatorname{cosec}\theta$	$\sec(270+\theta) = \operatorname{cosec}\theta$
$\cot(270-\theta) = \tan\theta$	$\cot(270+\theta) = -\tan\theta$

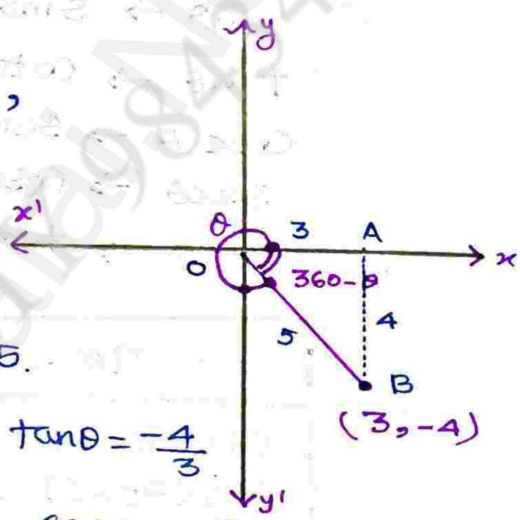
$360 - \theta$	$360 + \theta$
$\sin(360-\theta) = -\sin\theta$	$\sin(360+\theta) = \sin\theta$
$\cos(360-\theta) = \cos\theta$	$\cos(360+\theta) = \cos\theta$
$\tan(360-\theta) = -\tan\theta$	$\tan(360+\theta) = \tan\theta$
$\operatorname{cosec}(360-\theta) = -\operatorname{cosec}\theta$	$\operatorname{cosec}(360+\theta) = \operatorname{cosec}\theta$
$\sec(360-\theta) = \sec\theta$	$\sec(360+\theta) = \sec\theta$
$\cot(360-\theta) = -\cot\theta$	$\cot(360+\theta) = \cot\theta$

3.8 The terminal side of an angle θ in standard position passes through the point $(3,-4)$. Find the six trigonometric function values at an angle θ.

By the given data,

θ lies in IV quadrant

In IV^{th} quad $\cos\theta$ and $\sec\theta$ only positive.



$$OB = \sqrt{3^2+(-4)^2} = \sqrt{9+16} = 5.$$

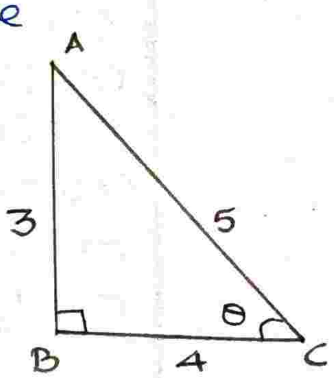
$\sin\theta = \frac{-4}{5}$ $\quad\cos\theta = \frac{3}{5}$ $\quad\tan\theta = \frac{-4}{3}$

$\operatorname{cosec}\theta = \frac{-5}{4}$ $\quad\sec\theta = \frac{5}{3}$ $\quad\cot\theta = -\frac{3}{4}$.

3.9 If $\sin\theta = \frac{3}{5}$ and the angle θ is in the second quadrant, then find the values of other five trigonometric functions:

θ lies in II quadrant, In II quadrant $\sin\theta$ and $\operatorname{cosec}\theta$ only positive

$$\sin\theta = \frac{3}{5}$$



$$AC^2 = AB^2 + BC^2$$
$$25 = 9 + BC^2$$
$$BC^2 = 16 \qquad \boxed{BC = 4}$$

$\cos\theta = \frac{-4}{5}$ $\quad\tan\theta = \frac{-3}{4}$

$\operatorname{cosec}\theta = \frac{5}{3}$ $\quad\sec\theta = \frac{-5}{4}$ $\quad\cot\theta = -\frac{4}{3}$

3.10 Find the values of

(i) $\sin(-45^\circ)$ (ii) $\cos(-45^\circ)$ (iii) $\cot(-45^\circ)$

(i) $\sin(-\theta) = -\sin\theta$

$\sin(-45^\circ) = -\sin 45^\circ = \frac{-1}{\sqrt{2}}$

(ii) $\cos(-\theta) = \cos\theta$

$\cos(-45^\circ) = \cos 45^\circ = \frac{1}{\sqrt{2}}$

(iii) $\cot(-\theta) = -\cot\theta$

$\cot(-45^\circ) = -\cot 45^\circ = -1.$

3.11 Find the value of (i) $\sin 150^\circ$ (ii) $\cos 135^\circ$ (iii) $\tan 120^\circ$

(i) $\sin 150^\circ = \sin(180 - 30^\circ) = \sin 30^\circ = \frac{1}{2}.$

(ii) $\cos 135^\circ = \cos(180 - 45^\circ) = -\cos 45^\circ = \frac{-1}{\sqrt{2}}$

(iii) $\tan 120^\circ = \tan(180 - 60^\circ) = -\tan 60^\circ = -\sqrt{3}.$

3.12 Find the value of

(i) $\sin 765^\circ$ (ii) $\operatorname{cosec}(-1410^\circ)$ (iii) $\cot\left(-\frac{15\pi}{4}\right)$

(i) $\sin 765^\circ = \sin(2\times 360^\circ + 45^\circ) = \sin 45^\circ = \frac{1}{\sqrt{2}}$

(ii) $\operatorname{cosec}(-1410^\circ) = -\operatorname{cosec} 1410$

$= -\operatorname{cosec}[4\times 360 - 30^\circ]$

$= \operatorname{cosec} 30^\circ = 2.$

(iii) $\cot\left(-\frac{15\pi}{4}\right) = -\cot\frac{15\pi}{4} = -\cot\left(4\pi - \frac{\pi}{4}\right)$

$= \cot\frac{\pi}{4} = 1..$

3.13 Prove that $\tan 315^\circ \cdot \cot(-405^\circ) + \cot 495^\circ \tan(-585^\circ) = 2.$

$\tan 315^\circ = \tan(360 - 45^\circ) = -\tan 45^\circ = -1$

$\cot(-405^\circ) = -\cot 405^\circ = -\cot(360 + 45^\circ) = -\cot 45^\circ = -1.$

$\cot(495^\circ) = \cot(360 + 135^\circ) = \cot 135^\circ$

$= \cot(180 - 45^\circ) = -\cot 45^\circ = -1$

$\tan(-585^\circ) = -\tan(585^\circ) = -\tan(360 + 225^\circ)$

$= -\tan 225^\circ = -\tan(180 + 45^\circ) = -\tan 45^\circ = -1$

$L.H.S = (-1)(-1) + (-1)(-1) = 1 + 1 = 2$

$= R.H.S.$

Ex: 3.3.

1. Find the values of

(i) $\sin 480^\circ = \sin(360+120^\circ) = \sin 120^\circ$
$= \sin(180-60^\circ) = \sin 60^\circ = \sqrt{3}/2$

(ii) $\sin(-1110^\circ) = -\sin(1110^\circ) = -\sin(3\times 360+30^\circ)$
$= -\sin 30^\circ = -\frac{1}{2}$

(iii) $\cos 300^\circ = \cos(360-60^\circ)$
$= \cos 60^\circ = \frac{1}{2}$

(iv) $\tan 1050^\circ = \tan(3\times 360-30^\circ)$
$= -\tan 30^\circ = \frac{-1}{\sqrt{3}}$

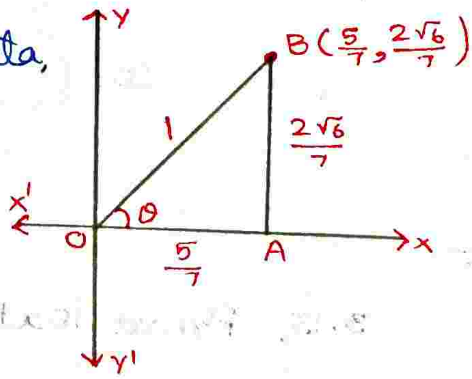
v) $\cot 660^\circ = \cot(2\times 360-60^\circ)$
$= -\cot 60^\circ = -1/\sqrt{3}$

vi) $\tan \frac{19\pi}{3} = \tan\left(6\pi+\frac{\pi}{3}\right) = \tan\frac{\pi}{3} = \sqrt{3}$

(vii) $\sin\left(-\frac{11\pi}{3}\right) = -\sin\frac{11\pi}{3} = -\sin\left(4\pi-\frac{\pi}{3}\right)$
$= +\sin\frac{\pi}{3} = \frac{\sqrt{3}}{2}$

2. $\left(\frac{5}{7}, \frac{2\sqrt{6}}{7}\right)$ is a Point on the terminal side of an angle θ in standard Position. Determine the trigonometric function values of angle 'θ'

From the given data,
θ lies in I quadrant.



$OB^2 = OA^2 + AB^2$
$= \frac{25}{49} + \frac{24}{49} = \frac{49}{49} = 1$

$\boxed{OB = 1}$

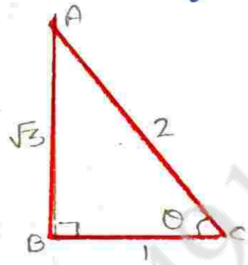
$\sin\theta = \frac{AB}{OB} = \frac{2\sqrt{6}}{7}$ $\cos\theta = \frac{OA}{OB} = \frac{5}{7}$ $\tan\theta = \frac{2\sqrt{6}}{5}$

$\operatorname{cosec}\theta = \frac{7}{2\sqrt{6}}$ $\sec\theta = \frac{7}{5}$ $\cot\theta = \frac{5}{2\sqrt{6}}$

3. Find the values of other five trigonometric functions for the following:

(i) $\cos\theta = -\frac{1}{2}$, θ lies in the III quadrant.

In III quadrant $\tan\theta$ and $\cot\theta$ are only Positive.



$AC^2 = AB^2 + BC^2$

$4 = AB^2 + 1$

$AB^2 = 3$

$AB = \sqrt{3}$

$\cos\theta = -\frac{1}{2}$

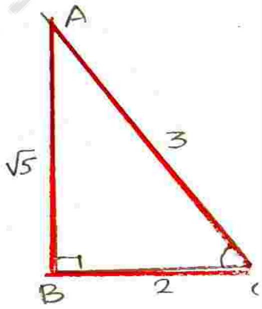
$\sin\theta = -\frac{\sqrt{3}}{2}$

$\tan\theta = \sqrt{3}$

$\operatorname{cosec}\theta = -\frac{2}{\sqrt{3}}$ $\sec\theta = -2$ $\cot\theta = \frac{1}{\sqrt{3}}$.

(ii) $\cos\theta = \frac{2}{3}$, θ lies in I quadrant.

In I quadrant all functions are Positive.



$AC^2 = AB^2 + BC^2$

$9 = AB^2 + 4 \Rightarrow AB^2 = 5$

$AB = \sqrt{5}$

$\sin\theta = \frac{\sqrt{5}}{3}$ $\cos\theta = \frac{2}{3}$ $\tan\theta = \frac{\sqrt{5}}{2}$

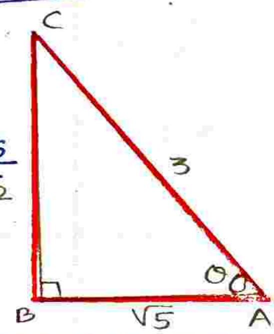
$\operatorname{cosec}\theta = \frac{3}{\sqrt{5}}$ $\sec\theta = \frac{3}{2}$ $\cot\theta = \frac{2}{\sqrt{5}}$.

(iii) $\sin\theta = -\frac{2}{3}$ θ lies in IV quadrant.

In IV quadrant $\cos\theta$, $\sec\theta$ are Positive.

$AC^2 = AB^2 + BC^2 \Rightarrow 9 = AB^2 + 4$

$AB^2 = 5$ $\boxed{AB = \sqrt{5}}$



(i) $\sin\theta = -\frac{2}{3}$ $\cos\theta = \frac{\sqrt{5}}{3}$ $\tan\theta = \frac{-2}{\sqrt{5}}$

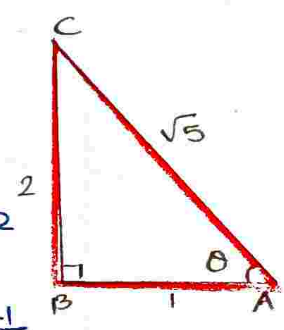
$\operatorname{cosec}\theta = -\frac{3}{2}$ $\sec\theta = \frac{3}{\sqrt{5}}$ $\cot\theta = -\frac{\sqrt{5}}{2}$

(iv) $\tan\theta = -2$; θ lies in the II quadrant.

In II quadrant, $\sin\theta$, $\operatorname{cosec}\theta$ only Positive.

$AC^2 = AB^2 + BC^2 = 4 + 1 = 5$

$AC = \sqrt{5}$

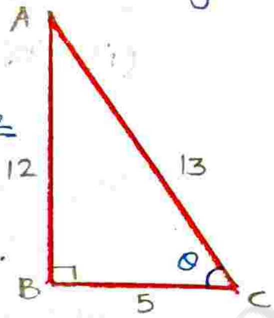


(i) $\sin\theta = \frac{2}{\sqrt{5}}$ $\cos\theta = -\frac{1}{\sqrt{5}}$ $\tan\theta = -2$

$\operatorname{cosec}\theta = \frac{\sqrt{5}}{2}$ $\sec\theta = -\sqrt{5}$ $\cot\theta = -\frac{1}{2}$

(v) $\sec\theta = \frac{13}{5}$, θ lies in the IV quadrant.

In IV quadrant, $\cos\theta$ and $\sec\theta$ only Positive.



i) $\sin\theta = \frac{-12}{13}$ $\quad \cos\theta = \frac{5}{3}$ $\quad \tan\theta = \frac{-12}{5}$

$\operatorname{cosec}\theta = \frac{-13}{12}$ $\quad \sec\theta = \frac{13}{5}$ $\quad \cot\theta = \frac{-5}{12}$.

4. Prove that $\frac{\cot(180^\circ+\theta)\sin(90-\theta)\cos(-\theta)}{\sin(270+\theta)\tan(-\theta)\operatorname{cosec}(360+\theta)} = \cos^2\theta\cot\theta$

$$L.H.S = \frac{\cot(180+\theta)\sin(90-\theta)\cos(-\theta)}{\sin(270+\theta)\tan(-\theta)\operatorname{cosec}(360+\theta)}$$

$$= \frac{\cot\theta . \cos\theta . \cos\theta}{(-\cos\theta)(-\tan\theta) . \operatorname{cosec}\theta}$$

$$= \frac{\cot\theta . \cos^2\theta}{\tan\theta . \cot\theta} = \cot\theta . \cos^2\theta = R.H.S$$

5. Find all the angles between 0° and 360° which satisfy the equation $\sin^2\theta = \frac{3}{4}$

$$\sin^2\theta = \frac{3}{4}$$

$$\sin\theta = \pm\frac{\sqrt{3}}{2}$$

$\sin\theta = \frac{\sqrt{3}}{2}$ $\qquad\qquad$ $\sin\theta = -\frac{\sqrt{3}}{2} = 240^\circ$

$\theta = 60^\circ, 120^\circ$ $\qquad\qquad$ $\sin\theta = \sin(-60^\circ)$

$= \sin(360-300^\circ)$

$\theta = 300^\circ$

$\therefore \theta = 60^\circ, 120^\circ, 240^\circ, 300^\circ$

6. Show that $\sin^2\frac{\pi}{18} + \sin^2\frac{\pi}{9} + \sin^2\frac{7\pi}{18} + \sin^2\frac{4\pi}{9} = 2$

$$L.H.S = \sin^2\frac{\pi}{18} + \sin^2\frac{\pi}{9} + \sin^2\frac{7\pi}{18} + \sin^2\frac{4\pi}{9}$$

$$= \sin^2 10^\circ + \sin^2 20^\circ + \sin^2 70^\circ + \sin^2 80^\circ$$

$$= \sin^2 10^\circ + \sin^2 20^\circ + \sin^2(90-20) + \sin^2(90-10)$$

$$= \sin^2 10^\circ + \sin^2 20^\circ + \cos^2 20^\circ + \cos^2 10^\circ$$

$$= (\sin^2 10^\circ + \cos^2 10^\circ) + (\sin^2 20^\circ + \cos^2 20^\circ)$$

$$= 1+1 = 2 = R.H.S.$$

Periodicity of Trigonometric Functions :

A function f is said to be a Periodic function with Period P. if there exists a Smallest Positive number p such that $f(x+p) = f(x)$ for all x in the domain.

$\sin x$, $\cos x$, $\operatorname{cosec} x$ and $\sec x$ are Periodic functions with Period 2π.

$\tan x$ and $\cot x$ are Periodic functions with Period π.

odd and even trigonometric functions :

A real valued function $f(x)$ is an even function if it Satisfies $f(-x) = f(x)$ for all real number x. and an odd function if it satisfies $f(-x) = -f(x)$ for all real values of x.

$\cos x$ and $\sec x$ are even functions

$\cos(-x) = \cos x$ $\qquad$ $\sec(-x) = \sec x$.

$\sin x$, $\tan x$, $\operatorname{cosec} x$ and $\cot x$ are odd functions.

$\sin(-x) = -\sin x$ $\qquad$ $\tan(-x) = -\tan x$

$\operatorname{cosec}(-x) = -\operatorname{cosec} x$ $\qquad$ $\cot(-x) = -\cot x$.

3.14 Determine whether the following functions are even, odd or neither.

(i) $\sin^2 x - 2\cos^2 x - \cos x$

$$f(x) = \sin^2 x - 2\cos^2 x - \cos x$$

$$f(-x) = [\sin(-x)]^2 - 2[\cos(-x)]^2 - \cos(-x)$$

$$= (-\sin x)^2 - 2\cos^2 x - \cos x$$

$$= \sin^2 x - 2\cos^2 x - \cos x$$

$$f(-x) = f(x)$$

$\therefore f(x)$ is an even function.

(ii) $f(x) = \sin\cos x$

$f(-x) = \sin(\cos(-x))$

$= \sin\cos x = f(x)$

$f(-x) = f(x)$

$\therefore f(x)$ is an even function.

(iii) $\cos(\sin x)$

$f(x) = \cos\sin x$

$f(-x) = \cos\sin(-x)$

$= \cos[-\sin x]$

$= \cos(\sin x)$

$f(-x) = f(x)$

$\therefore f(x)$ is an even function.

(iv) $\sin x + \cos x$

$f(x) = \sin x + \cos x$

$f(-x) = \sin(-x) + \cos(-x)$

$= -\sin x + \cos x$

$f(-x) \neq f(x)$, $f(-x) \neq -f(x)$

$\therefore f(x)$ is neither an odd nor an even function.

Trigonometric Identities :

Compound angle identities:

$$\sin(\alpha+\beta) = \sin\alpha\cos\beta + \cos\alpha\sin\beta$$

$$\sin(\alpha-\beta) = \sin\alpha\cos\beta - \cos\alpha\sin\beta$$

$$\cos(\alpha+\beta) = \cos\alpha\cos\beta - \sin\alpha\sin\beta$$

$$\cos(\alpha-\beta) = \cos\alpha\cos\beta + \sin\alpha\sin\beta$$

$$\tan(\alpha+\beta) = \frac{\tan\alpha + \tan\beta}{1 - \tan\alpha\tan\beta}$$

$$\tan(\alpha-\beta) = \frac{\tan\alpha - \tan\beta}{1 + \tan\alpha\tan\beta}$$

Ex: 3.4

1. If $\sin x = \frac{15}{17}$ and $\cos y = \frac{12}{13}$ $\quad x \in (0, \frac{\pi}{2})$, $y \in (0, \frac{\pi}{2})$

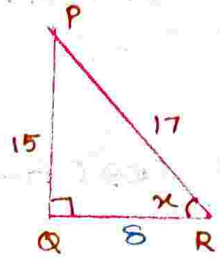
From ΔPQR

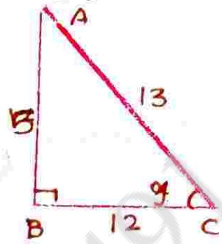
$PR^2 = PQ^2 + QR^2$

$17^2 = 15^2 + QR^2$

$QR^2 = 289 - 225 = 64$

$\boxed{QR = 8}$





From ΔABC, $\quad AC^2 = AB^2 + BC^2$

$13^2 = AB^2 + 12^2$

$AB^2 = 169 - 144 = 25$

$\boxed{AB = 5}$

$\sin x = \frac{15}{17}$ $\qquad \sin y = \frac{5}{13}$

$\cos x = \frac{8}{17}$ $\qquad \cos y = \frac{12}{13}$

$\tan x = \frac{15}{8}$ $\qquad \tan y = \frac{5}{12}$.

(i) $\sin(x+y) = \sin x \cos y + \cos x \sin y$

$= \frac{15}{17} \cdot \frac{12}{13} + \frac{8}{17} \cdot \frac{5}{13} = \frac{180}{221} + \frac{40}{221} = \frac{220}{221}$

(ii) $\cos(x-y) = \cos x \cos y + \sin x \sin y$

$= \frac{8}{17} \cdot \frac{12}{13} + \frac{15}{17} \cdot \frac{5}{13} = \frac{96}{221} + \frac{75}{221} = \frac{171}{221}$

(iii) $\tan(x+y) = \frac{\tan x + \tan y}{1 - \tan x \tan y} = \frac{\frac{15}{8} + \frac{5}{12}}{1 - \frac{15}{8} \cdot \frac{5}{12}} = \frac{180+40}{96-75}$

$= \frac{220}{21}$

---

2. If $\sin A = \frac{3}{5}$ and $\cos B = \frac{9}{41}$, $A, B \in (0, \pi/2)$

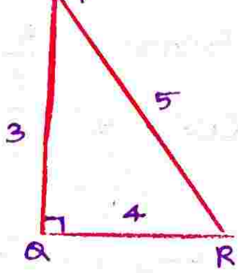
find the value of (i) $\sin(A+B)$ (ii) $\cos(A-B)$

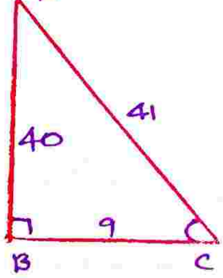
From the ΔPQR

$PR^2 = PQ^2 + QR^2$

$25 = 9 + QR^2 \Rightarrow QR^2 = 16$

$\boxed{QR = 4}$





From the ΔABC

$AC^2 = AB^2 + BC^2$

$41^2 = AB^2 + 9^2 \Rightarrow 1681 = AB^2 + 81$

$AB^2 = 1600$

$\boxed{AB = 40}$

$\sin A = \frac{3}{5}$ $\quad\quad$ $\sin B = \frac{40}{41}$

$\cos A = \frac{4}{5}$ $\quad\quad$ $\cos B = \frac{9}{41}$

$\left.\right\}$ A, B lies in I quad

(i) $\sin(A+B) = \sin A \cos B + \cos A \sin B$

$= \frac{3}{5}\cdot\frac{9}{41} + \frac{4}{5}\cdot\frac{40}{41} = \frac{27}{205} + \frac{160}{205} = \frac{187}{205}$

(ii) $\cos(A-B) = \cos A \cos B + \sin A \sin B$

$= \frac{4}{5}\cdot\frac{9}{41} + \frac{3}{5}\cdot\frac{40}{41}$

$= \frac{36}{205} + \frac{120}{205} = \frac{156}{205}$.

---

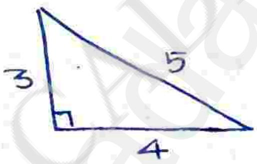
3. Find $\cos(x-y)$ given that $\cos x = -\frac{4}{5}$ with $x \in (\pi, \frac{3\pi}{2})$ and $\sin y = -\frac{24}{25}$ with $y \in (\pi, \frac{3\pi}{2})$

Since x and y both are lies in the third quadrant.

In third quadrant $\tan x$ only Positive.

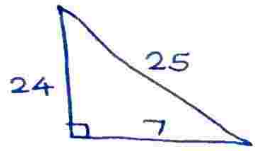
$\cos x = -\frac{4}{5}$

$\sin x = -\frac{3}{5}$



$\sin y = \frac{-24}{25}$

$\cos y = \frac{-7}{25}$



$\cos(x-y) = \cos x \cos y + \sin x \sin y$

$= \left(\frac{-4}{5}\right)\left(\frac{-7}{25}\right) + \left(\frac{-3}{5}\right)\left(\frac{-24}{25}\right)$

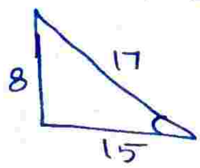
$= \frac{28}{125} + \frac{72}{125} = \frac{100}{125} = \frac{4}{5}$

---

4. Find $\sin(x-y)$ given that $\sin x = \frac{8}{17}$ $x \in (0, \frac{\pi}{2})$ and $\cos y = \frac{-24}{25}$ with $y \in (\pi, \frac{3\pi}{2})$

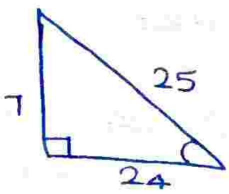
$\sin x = \frac{8}{17}$ $\quad$ x lies in I quadrant

$\cos x = \frac{15}{17}$



$\cos y = \frac{-24}{25}$

$\sin y = \frac{-7}{25}$



y lies in III quad

$$\sin(x-y) = \sin x\cos y - \cos x \sin y$$
$$= \frac{8}{17}\cdot\left(\frac{-24}{25}\right) - \frac{15}{17}\cdot\left(\frac{-7}{25}\right)$$
$$= \frac{-192}{425} + \frac{105}{425} = \frac{-87}{425}$$

5. Find the value of (i) $\cos 105^\circ$ (ii) $\sin 105^\circ$ (iii) $\tan\frac{7\pi}{12}$

(i) $\cos 105^\circ = \cos(60+45^\circ)$

$\cos(A+B) = \cos A\cos B - \sin A \sin B$

$\cos 105^\circ = \cos 60^\circ \cos 45^\circ - \sin 60^\circ \sin 45^\circ$

$$= \frac{1}{2}\cdot\frac{1}{\sqrt{2}} - \frac{\sqrt{3}}{2}\cdot\frac{1}{\sqrt{2}} = \frac{1-\sqrt{3}}{2\sqrt{2}}.$$

(ii) $\sin 105^\circ = \sin(60+45^\circ)$

$\sin(A+B) = \sin A\cos B + \cos A \sin B$

$\sin 105^\circ = \sin 60^\circ \cos 45^\circ + \cos 60^\circ \sin 45^\circ$

$$= \frac{\sqrt{3}}{2}\cdot\frac{1}{\sqrt{2}} + \frac{1}{2}\cdot\frac{1}{\sqrt{2}} = \frac{\sqrt{3}+1}{2\sqrt{2}}.$$

(iii) $\tan\frac{7\pi}{12} = \tan 105^\circ = \tan(60+45^\circ)$

$$\tan(A+B) = \frac{\tan A + \tan B}{1-\tan A\tan B}$$

$$\tan 105^\circ = \frac{\tan 60^\circ + \tan 45^\circ}{1-\tan 60^\circ \tan 45^\circ} = \frac{\sqrt{3}+1}{1-\sqrt{3}}$$

$$= \frac{1+\sqrt{3}}{1-\sqrt{3}} \times \frac{1+\sqrt{3}}{1+\sqrt{3}} = \frac{4+2\sqrt{3}}{-2}$$

$$= -(2+\sqrt{3}).$$

$\sin 15^\circ = \frac{\sqrt{3}-1}{2\sqrt{2}}$	$\cos 15^\circ = \frac{\sqrt{3}+1}{2\sqrt{2}}$	$\tan 15^\circ = \frac{\sqrt{3}-1}{\sqrt{3}+1} = 2-\sqrt{3}$
$\sin 75^\circ = \frac{\sqrt{3}+1}{2\sqrt{2}}$	$\cos 75^\circ = \frac{\sqrt{3}-1}{2\sqrt{2}}$	$\tan 75^\circ = 2+\sqrt{3}$
$\sin 105^\circ = \frac{\sqrt{3}+1}{2\sqrt{2}}$	$\cos 105^\circ = -\frac{(\sqrt{3}-1)}{2\sqrt{2}}$	$\tan 105^\circ = -(2+\sqrt{3})$

6. Prove that (i) $\cos(30+x) = \dfrac{\sqrt{3}\cos x - \sin x}{2}$

(ii) $\cos(\pi+\theta) = -\cos\theta$ (iii) $\sin(\pi+\theta) = -\sin\theta$

(i) $\cos(30+x) = \cos 30^\circ \cos x - \sin 30^\circ \sin x$

$= \frac{\sqrt{3}}{2}\cos x - \frac{1}{2}\sin x$

$= \frac{1}{2}(\sqrt{3}\cos x - \sin x)$

$=$ R.H.S.

(ii) $\cos(\pi+\theta) = \cos\pi\cos\theta - \sin\pi\sin\theta$

$= (-1)\cos\theta - 0\cdot\sin\theta$

$= -\cos\theta$

$=$ R.H.S

(iii) $\sin(\pi+\theta) = \sin\pi\cos\theta + \cos\pi\sin\theta$

$= 0\cdot\cos\theta + (-1)\sin\theta)$

$= -\sin\theta$

$=$ R.H.S

$\sin n\pi = 0$ $\cos n\pi = (-1)^n$

$\left.\begin{matrix}\sin\pi\\2\pi\\3\pi\\\vdots\end{matrix}\right\} = 0$ $\left.\begin{matrix}\cos\pi\\3\pi\\5\pi\\\vdots\end{matrix}\right\} = -1$ $\left.\begin{matrix}\cos 2\pi\\4\pi\\6\pi\\\vdots\end{matrix}\right\} = 1$

---

7. Find a quadratic equation whose roots are $\sin 15^\circ$ and $\cos 15^\circ$

We know that

$\sin 15^\circ = \dfrac{\sqrt{3}-1}{2\sqrt{2}}$ $\cos 15^\circ = \dfrac{\sqrt{3}+1}{2\sqrt{2}}$

Sum of the roots $= \alpha+\beta = \dfrac{\sqrt{3}-1}{2\sqrt{2}} + \dfrac{\sqrt{3}+1}{2\sqrt{2}}$

$= \dfrac{\sqrt{3}-1+\sqrt{3}+1}{2\sqrt{2}} = \dfrac{2\sqrt{3}}{2\sqrt{2}} = \dfrac{\sqrt{3}}{\sqrt{2}}$

Product of the roots $= \alpha\beta = \left(\dfrac{\sqrt{3}-1}{2\sqrt{2}}\right)\left(\dfrac{\sqrt{3}+1}{2\sqrt{2}}\right)$

$= \dfrac{3-1}{8} = \dfrac{2}{8} = \dfrac{1}{4}.$

$\therefore$ The Required equation is

$x^2 - (\alpha+\beta)x + \alpha\beta = 0$

$$x^2 - \frac{\sqrt{3}}{\sqrt{2}}x + \frac{1}{4} = 0$$

$$\boxed{4\sqrt{2}x^2 - 4\sqrt{3}x + \sqrt{2} = 0}$$

(or)

$$4x^2 - 2\sqrt{6}x + 1 = 0$$

8. Expand $\cos(A+B+C)$. Hence Prove that $\cos A\cos B\cos C = \sin A\sin B\cos C + \sin B\sin C\cos A + \sin C\sin A\cos B$. if $A+B+C = \pi/2$.

$$\cos(A+B+C) = \cos(A+\overline{B+C})$$

$$= \cos A\cos(B+C) - \sin A\sin(B+C)$$

$$= \cos A\{\cos B\cos C - \sin B\sin C\} - \sin A\{\sin B\cos C + \cos B\sin C\}$$

$$= \cos A\cos B\cos C - \cos A\sin B\sin C - \sin A\sin B\cos C - \sin A\cos B\sin C.$$

If $A+B+C = \frac{\pi}{2}$

$$\cos(A+B+C) = \cos A\cos B\cos C - \cos A\sin B\sin C - \sin A\sin B\cos C - \sin A\cos B\sin C$$

$$\cos(\pi/2) = \cos A\cos B\cos C - \cos A\sin B\sin C - \sin A\sin B\cos C - \sin A\cos B\sin C$$

$$0 = \cos A\cos B\cos C - \cos A\sin B\sin C - \sin A\sin B\cos C - \sin A\cos B\sin C$$

$$\therefore \cos A\cos B\cos C = \cos A\sin B\sin C + \sin A\sin B\cos C + \sin A\cos B\sin C$$

Hence Proved.

9. Prove that

(i) $\sin(45+\theta) - \sin(45-\theta) = \sqrt{2}\sin\theta$

(ii) $\sin(30+\theta) + \cos(60+\theta) = \cos\theta$

(i) $\sin(A+B) + \sin(A-B) = 2\sin A\cos B$

$\sin(A+B) - \sin(A-B) = 2\cos A\sin B$

$$\sin(45+\theta) - \sin(45-\theta) = 2\cos 45^\circ \cdot \sin\theta$$

$$= 2 \cdot \frac{1}{\sqrt{2}} \cdot \sin\theta$$

$$= \sqrt{2}\sin\theta$$

$$= \text{R.H.S}$$

(ii) $\sin(30+\theta) + \cos(60+\theta)$

$$= \{\sin 30^\circ \cos\theta + \cos 30^\circ \sin\theta\} + \{\cos 60^\circ \cos\theta - \sin 60^\circ \sin\theta\}$$

$$= \frac{1}{2}\cos\theta + \cancel{\frac{\sqrt{3}}{2}\sin\theta} + \frac{1}{2}\cos\theta - \cancel{\frac{\sqrt{3}}{2}\sin\theta}$$

$$= \frac{1}{2}\cos\theta + \frac{1}{2}\cos\theta = \cos\theta = R.H.S$$

10. If $a\cos(x+y) = b\cos(x-y)$, show that $(a+b)\tan x = (a-b)\cot y$.

$$\boxed{\begin{aligned}\cos(A-B)+\cos(A+B) &= 2\cos A\cos B\\ \cos(A-B)-\cos(A+B) &= 2\sin A\sin B\end{aligned}}$$

$$a\cos(x+y) = b\cos(x-y)$$

$$\frac{a}{b} = \frac{\cos(x-y)}{\cos(x+y)}$$

$$\frac{a+b}{a-b} = \frac{\cos(x-y)+\cos(x+y)}{\cos(x-y)-\cos(x+y)}$$

$$\frac{a+b}{a-b} = \frac{\cancel{2}\cos x\cos y}{\cancel{2}\sin x\sin y}$$

$$\frac{a+b}{a-b} = \cot x \cdot \cot y$$

$$(a+b)\tan x = (a-b)\cot y$$

Hence Proved.

11. Prove that $\sin 105^\circ + \cos 105^\circ = \cos 45^\circ$

we know that

$$\sin 105^\circ = \frac{\sqrt{3}+1}{2\sqrt{2}} \qquad \cos 105^\circ = -\frac{(\sqrt{3}-1)}{2\sqrt{2}}$$

$$\sin 105^\circ + \cos 105^\circ = \frac{\sqrt{3}+1}{2\sqrt{2}} + \frac{1-\sqrt{3}}{2\sqrt{2}}$$

$$= \frac{\sqrt{3}+1+1-\sqrt{3}}{2\sqrt{2}} = \frac{2}{2\sqrt{2}} = \frac{1}{\sqrt{2}}$$

$$= \frac{1}{\sqrt{2}}$$

$$= \cos 45^\circ$$

$$= R.H.S.$$

12. Prove that $\sin 75^\circ - \sin 15^\circ = \cos 105^\circ + \cos 15^\circ$

$$R.H.S = \cos 105^\circ + \cos 15^\circ$$
$$= \cos(90+15^\circ) + \cos(90-75^\circ)$$
$$= -\sin 15^\circ + \sin 75^\circ$$
$$= \sin 75^\circ - \sin 15^\circ$$
$$= L.H.S.$$

---

13. Show that $\tan 75^\circ + \cot 75^\circ = 4$

we know that

$$\tan 75^\circ = 2+\sqrt{3} \text{ (or) } \frac{\sqrt{3}+1}{\sqrt{3}-1}$$
$$\cot 75^\circ = \tan 15^\circ = 2-\sqrt{3} \text{ (or) } \frac{\sqrt{3}-1}{\sqrt{3}+1}$$
$$\tan 75^\circ + \cot 75^\circ = 2+\sqrt{3}+2-\sqrt{3} = 4 = R.H.S.$$

---

14. Prove that $\cos(A+B)\cdot\cos C - \cos(B+C)\cos A = \sin B \sin(C-A)$

$$L.H.S = \cos(A+B)\cdot\cos C - \cos(B+C)\cdot\cos A$$
$$= (\cos A\cos B - \sin A\sin B)\cdot\cos C - [\cos B\cos C - \sin B\sin C]\cos A$$
$$= \cos A\cancel{\cos B}\cos C - \sin A\sin B\cos C - \cos A\cancel{\cos B}\cos C + \cos A\sin B\sin C$$
$$= \cos A\sin B\sin C - \sin A\sin B\cos C$$
$$= \sin B\{\sin C\cos A - \cos C\cdot\sin A\}$$
$$= \sin B\sin(C-A) = R.H.S$$

---

15. Prove that $\sin(n+1)\theta\sin(n-1)\theta + \cos(n+1)\theta\cos(n-1)\theta = \cos 2\theta,\ n\in Z.$

$$\boxed{\sin A\sin B + \cos A\cos B = \cos(A-B)}$$

$$L.H.S = \sin(n+1)\theta\sin(n-1)\theta + \cos(n+1)\theta\cdot\cos(n-1)\theta$$
$$= \cos\{(n+1)\theta - (n-1)\theta\}$$
$$= \cos\{(\cancel{n}+1-\cancel{n}+1)\theta\} = \cos 2\theta \quad n\in Z.$$
$$= R.H.S$$

16. If $x\cos\theta = y\cos\left(\theta+\frac{2\pi}{3}\right) = z\cos\left(\theta+\frac{4\pi}{3}\right)$ find the value of $xy+yz+zx$

$$x\cos\theta = y\cos\left(\theta+\frac{2\pi}{3}\right) = z\cos\left(\theta+\frac{4\pi}{3}\right) = k$$

$x\cos\theta = k \qquad y\cos\left(\theta+\frac{2\pi}{3}\right) = k \qquad z\cos\left(\theta+\frac{4\pi}{3}\right) = k$

$\frac{k}{x} = \cos\theta \qquad \frac{k}{y} = \cos\left(\theta+\frac{2\pi}{3}\right) \qquad \frac{k}{z} = \cos\left(\theta+\frac{4\pi}{3}\right)$

$$\frac{k}{x}+\frac{k}{y}+\frac{k}{z} = \cos\theta + \cos\left(\theta+\frac{2\pi}{3}\right) + \cos\left(\theta+\frac{4\pi}{3}\right)$$

$$= \cos\theta + \cos\theta\cos\frac{2\pi}{3} - \sin\theta\sin\frac{2\pi}{3} + \cos\theta\cos\frac{4\pi}{3} - \sin\theta\sin\frac{4\pi}{3} \quad \text{---} (1)$$

$\sin 2\pi/3 = \sin 120^\circ = \sin(90+30^\circ) = \cos 30^\circ = \frac{\sqrt{3}}{2}$

$\cos\frac{2\pi}{3} = \cos 120^\circ = \cos(90+30^\circ) = -\sin 30^\circ = -\frac{1}{2}$

$\cos\frac{4\pi}{3} = \cos 240^\circ = \cos(270-30^\circ) = -\sin 30^\circ = -\frac{1}{2}$

$\sin\frac{4\pi}{3} = \sin 240^\circ = \sin(270-30^\circ) = -\cos 30^\circ = -\frac{\sqrt{3}}{2}$

sub the values in (1)

$$\frac{k}{x}+\frac{k}{y}+\frac{k}{z} = \cancel{\cos\theta} - \cancel{\frac{1}{2}\cos\theta} - \cancel{\frac{\sqrt{3}}{2}\sin\theta} - \cancel{\frac{1}{2}\cos\theta} + \cancel{\frac{\sqrt{3}}{2}\sin\theta}$$

$$= 0$$

$$\frac{1}{x}+\frac{1}{y}+\frac{1}{z} = 0$$

$$\frac{xy+yz+zx}{xyz} = 0$$

$$\boxed{xy+yz+zx = 0}$$

17. Prove that :

(i) $\sin(A+B)\sin(A-B) = \sin^2 A - \sin^2 B$

L.H.S $= (\sin A\cos B + \cos A\sin B)(\sin A\cos B - \cos A\sin B)$

$= \sin^2 A\cos^2 B - \cos^2 A\sin^2 B$

$= \sin^2 A(1-\sin^2 B) - (1-\sin^2 A)\sin^2 B$

$= \sin^2 A - \cancel{\sin^2 A\sin^2 B} - \sin^2 B + \cancel{\sin^2 A\sin^2 B}$

$= \sin^2 A - \sin^2 B = R.H.S$

$\sin(A+B)\cdot\sin(A-B) = \sin^2 A - \sin^2 B.$

(ii) $\cos(A+B)\cdot\cos(A-B) = \cos^2A - \sin^2B$ (or)

$= \cos^2B - \sin^2A$

$$\begin{aligned}
L.H.S &= \cos(A+B)\cdot\cos(A-B)\\
&= (\cos A\cos B - \sin A\sin B)(\cos A\cos B + \sin A\sin B)\\
&= \cos^2A\cos^2B - \sin^2A\sin^2B \quad \text{--- (1)}\\
&= \cos^2A(1-\sin^2B) - (1-\cos^2A)\sin^2B\\
&= \cos^2A - \cancel{\cos^2A\sin^2B} - \sin^2B + \cancel{\cos^2A\sin^2B}\\
&= \cos^2A - \sin^2B = R.H.S
\end{aligned}$$

from (1)

$$\begin{aligned}
L.H.S &= \cos^2A\cos^2B - \sin^2A\sin^2B\\
&= (1-\sin^2A)\cos^2B - \sin^2A(1-\cos^2B)\\
&= \cos^2B - \cancel{\sin^2A\cos^2B} - \sin^2A + \cancel{\sin^2A\cos^2B}\\
&= \cos^2B - \sin^2A = R.H.S
\end{aligned}$$

Hence Proved.

$$\boxed{\begin{aligned}
\sin(A+B)\cdot\sin(A-B) &= \sin^2A - \sin^2B\\
&= \cos^2B - \cos^2A\\
\cos(A+B)\cdot\cos(A-B) &= \cos^2A - \sin^2B\\
&= \cos^2B - \sin^2A.
\end{aligned}}$$

(iii) $\sin^2(A+B) - \sin^2(A-B) = \sin 2A\cdot\sin 2B$

[W.K.T $\sin^2A - \sin^2B = \sin(A+B)\cdot\sin(A-B)$]

$$\begin{aligned}
L.H.S &= \sin^2(A+B) - \sin^2(A-B)\\
&= \sin(A+B+A-B)\cdot\sin(A+B-\overline{A-B})\\
&= \sin(2A)\cdot\sin(\cancel{A}+B-\cancel{A}+B)\\
&= \sin 2A\cdot\sin 2B\\
&= R.H.S
\end{aligned}$$

(iv) $\cos 8\theta \cdot \cos 2\theta = \cos^2 5\theta - \sin^2 3\theta$

$$L.H.S = \cos 8\theta \cdot \cos 2\theta$$
$$= \cos(5\theta + 3\theta) \cdot \cos(5\theta - 3\theta)$$
$$= \cos^2 5\theta - \sin^2 3\theta$$
$$= R.H.S$$

---

18. Show that $\cos^2 A + \cos^2 B - 2\cos A \cos B \cos(A+B) = \sin^2(A+B)$.

$$L.H.S = \cos^2 A + \cos^2 B - 2\cos A \cos B \cdot \cos(A+B)$$
$$= \cos^2 A + (1 - \sin^2 B) - 2\cos A \cos B \cdot \cos(A+B)$$
$$= \cos^2 A - \sin^2 B - 2\cos A \cos B \cos(A+B) + 1$$
$$= \cos(A+B) \cdot \cos(A-B) - 2\cos A \cos B \cdot \cos(A+B) + 1$$
$$= \cos(A+B)\{\cos(A-B) - 2\cos A \cos B\} + 1$$
$$= \cos(A+B)\{\cos A \cos B + \sin A \sin B - 2\cos A \cos B\} + 1$$
$$= \cos(A+B)\{-\cos A \cos B + \sin A \sin B\} + 1$$
$$= -\cos(A+B)\{\cos A \cos B - \sin A \sin B\} + 1$$
$$= -\cos(A+B)\cos(A+B) + 1$$
$$= 1 - \cos^2(A+B)$$
$$= \sin^2(A+B) = R.H.S$$

---

19. If $\cos(\alpha - \beta) + \cos(\beta - \gamma) + \cos(\gamma - \alpha) = -\frac{3}{2}$

Prove that $\cos\alpha + \cos\beta + \cos\gamma = \sin\alpha + \sin\beta + \sin\gamma = 0$.

$$\cos(\alpha - \beta) + \cos(\beta - \gamma) + \cos(\gamma - \alpha) = -\frac{3}{2}$$

$$\cos\alpha\cos\beta + \sin\alpha\sin\beta + \cos\beta\cos\gamma + \sin\beta\sin\gamma + \cos\gamma\cos\alpha + \sin\gamma\sin\alpha = -\frac{3}{2}$$

$$2\cos\alpha\cos\beta + 2\sin\alpha\sin\beta + 2\cos\beta\cos\gamma + 2\sin\beta\sin\gamma + 2\cos\gamma\cos\alpha + 2\sin\gamma\sin\alpha = -3$$

$$2\cos\alpha\cos\beta + 2\sin\alpha\sin\beta + 2\cos\beta\cos\gamma + 2\sin\beta\sin\gamma + 2\cos\gamma\cos\alpha + 2\sin\gamma\sin\alpha + 3 = 0$$

$$2\cos\alpha\cos\beta + 2\sin\alpha\sin\beta + 2\cos\beta\cos\gamma + 2\sin\beta\sin\gamma + 2\cos\gamma\cos\alpha + 2\sin\gamma\sin\alpha + 1 + 1 + 1 = 0$$

$$2\cos\alpha\cos\beta + 2\sin\alpha\sin\beta + 2\cos\beta\cos\gamma + 2\sin\beta\sin\gamma + 2\cos\gamma\cos\alpha + 2\sin\gamma\sin\alpha + (\sin^2\alpha+\cos^2\alpha) + (\sin^2\beta+\cos^2\beta) + (\sin^2\gamma+\cos^2\gamma) = 0$$

$$[\sin^2\alpha+\sin^2\beta+\sin^2\gamma + 2\sin\alpha\sin\beta + 2\sin\beta\sin\gamma + 2\sin\gamma\sin\alpha] + [\cos^2\alpha+\cos^2\beta+\cos^2\gamma + 2\cos\alpha\cos\beta + 2\cos\beta\cos\gamma + 2\cos\gamma\cos\alpha] = 0$$

$$[\sin\alpha+\sin\beta+\sin\gamma]^2 + [\cos\alpha+\cos\beta+\cos\gamma]^2 = 0$$

$$\sin\alpha+\sin\beta+\sin\gamma = 0 \qquad \cos\alpha+\cos\beta+\cos\gamma = 0$$

---

20. S.T

$$\boxed{\begin{array}{ll} \text{(i)} & \tan(45+A) = \dfrac{1+\tan A}{1-\tan A} \\ \text{(ii)} & \tan(45-A) = \dfrac{1-\tan A}{1+\tan A} \end{array}}$$

(i) $L.H.S = \tan(45+A) = \dfrac{\tan 45 + \tan A}{1 - \tan 45 \tan A}$

$$= \frac{1+\tan A}{1-\tan A} = R.H.S$$

(ii) $L.H.S = \tan(45-A) = \dfrac{\tan 45 - \tan A}{1 + \tan 45 \tan A}$

$$= \frac{1-\tan A}{1+\tan A} = R.H.S$$

---

21. Prove that $\cot(A+B) = \dfrac{\cot A \cot B - 1}{\cot A + \cot B}$

$R.H.S = \dfrac{\cot A \cot B - 1}{\cot A + \cot B} = \dfrac{\frac{1}{\tan A}\cdot\frac{1}{\tan B} - 1}{\frac{1}{\tan A} + \frac{1}{\tan B}}$

$$= \frac{1 - \tan A \tan B}{\tan A + \tan B} = \frac{1}{\tan(A+B)}$$

$$= \cot(A+B)$$

$$= L.H.S.$$

---

22. If $\tan x = \frac{n}{n+1}$ and $\tan y = \frac{1}{2n+1}$ find $\tan(x+y)$

$$\tan(x+y) = \frac{\tan x + \tan y}{1 - \tan x \tan y}$$

$$= \frac{\frac{n}{n+1} + \frac{1}{2n+1}}{1 - \frac{n}{n+1} \cdot \frac{1}{2n+1}} = \frac{n(2n+1) + (n+1)}{(n+1)(2n+1) - n}$$

$$= \frac{2n^2+n+n+1}{2n^2+n+2n+1-n} = \frac{2n^2+2n+1}{2n^2+2n+1} = 1.$$

$$\tan(x+y) = 1.$$

---

23. Prove that $\tan\left(\frac{\pi}{4}+\theta\right) \cdot \tan\left(\frac{3\pi}{4}+\theta\right) = -1$

$$\tan\left(\frac{\pi}{4}+\theta\right) = \frac{1+\tan\theta}{1-\tan\theta}$$

$$\tan\left(\frac{3\pi}{4}+\theta\right) = \tan\left(\pi - \frac{\pi}{4} - \theta\right)$$

$$= -\tan\left(\frac{\pi}{4}-\theta\right) = -\left(\frac{1-\tan\theta}{1+\tan\theta}\right)$$

$$\tan\left(\frac{\pi}{4}+\theta\right) \cdot \tan\left(\frac{3\pi}{4}+\theta\right) = -\left(\frac{1+\tan\theta}{1-\tan\theta}\right)\left(\frac{1-\tan\theta}{1+\tan\theta}\right)$$

$$= -1$$

Hence Proved.

---

24. Find the values of $\tan(\alpha+\beta)$, given that $\cot\alpha = \frac{1}{2}$ $\alpha \in \left(\pi, \frac{3\pi}{2}\right)$ and $\sec\beta = -\frac{5}{3}$ $\beta \in \left(\frac{\pi}{2}, \pi\right)$

$$\cot\alpha = \frac{1}{2}$$

$$\tan\alpha = 2$$

$$\sec\beta = -\frac{5}{3}$$

$$\sec^2\beta = \frac{25}{9}$$

$$\sec^2\beta - 1 = \frac{16}{9}$$

$$\tan^2\beta = \frac{16}{9}$$

$$\tan\beta = \frac{-4}{3}$$

β lies in II quad

$$\tan(\alpha+\beta) = \frac{\tan\alpha + \tan\beta}{1 - \tan\alpha \tan\beta}$$

$$= \frac{2-\frac{4}{3}}{1+2(4/3)} = \frac{6-4}{3+8} = \frac{2}{11}$$

$$\tan(\alpha+\beta) = \frac{2}{11}.$$

25. If $\theta+\varphi=\alpha$ and $\tan\theta = k\tan\phi$ then Prove that $\sin(\theta-\varphi) = \frac{k-1}{k+1}.\sin\alpha$

$\theta+\phi=\alpha$

$$\tan\theta = k\tan\varphi$$

$$\frac{\tan\theta}{\tan\varphi} = \frac{k}{1}$$

$$\frac{\tan\theta+\tan\varphi}{\tan\theta-\tan\varphi} = \frac{k+1}{k-1}$$

$$\frac{\frac{\sin\theta}{\cos\theta}+\frac{\sin\varphi}{\cos\varphi}}{\frac{\sin\theta}{\cos\theta}-\frac{\sin\varphi}{\cos\varphi}} = \frac{k+1}{k-1}$$

$$\frac{\sin\theta\cos\varphi+\cos\theta.\sin\varphi}{\sin\theta\cos\varphi-\cos\theta.\sin\varphi} = \frac{k+1}{k-1}$$

$$\frac{\sin(\theta+\varphi)}{\sin(\theta-\varphi)} = \frac{k+1}{k-1} \qquad [\theta+\varphi=\alpha]$$

$$\frac{\sin\alpha}{\sin(\theta-\varphi)} = \frac{k+1}{k-1}$$

$$\therefore \sin(\theta-\varphi) = \frac{k-1}{k+1}.\sin\alpha$$

Hence Proved

3.15 Find the values of (i) $\cos 15^\circ$ (ii) $\tan 165^\circ$

(i) $\cos 15^\circ = \cos(45-30^\circ)$

$$= \cos 45^\circ\cos 30^\circ + \sin 45.\sin 30$$

$$= \frac{1}{\sqrt2}.\frac{\sqrt3}{2}+\frac{1}{\sqrt2}.\frac{1}{2} = \frac{\sqrt3+1}{2\sqrt2}$$

(ii) $\tan 165^\circ = \tan(180-15) = -\tan 15^\circ$

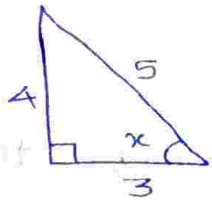
$$= -\tan(45-30^\circ) = -\left\{\frac{\tan 45-\tan 30^\circ}{1+\tan 45\tan 30^\circ}\right\}$$

$$= -\left\{\frac{1-\frac{1}{\sqrt3}}{1+\frac{1}{\sqrt3}}\right\} = -\left\{\frac{\sqrt3-1}{\sqrt3+1}\right\} = \frac{1-\sqrt3}{1+\sqrt3}$$

3.16. If $\sin x = \frac{4}{5}$ $x \in (0, \frac{\pi}{2})$ and $\cos y = \frac{-12}{13}$ $y \in (\frac{\pi}{2}, \pi)$ then find (i) $\sin(x-y)$ (ii) $\cos(x-y)$

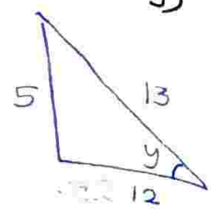
$\sin x = \frac{4}{5}$

$\cos x = \frac{3}{5}$



$\cos y = -\frac{12}{13}$

$\sin y = \frac{5}{13}$



(i) $\sin(x-y) = \sin x \cos y + \cos x \sin y$

$= \frac{4}{5} \cdot \left(\frac{-12}{13}\right) - \frac{3}{5}\left(\frac{5}{13}\right)$

$= \frac{-48}{65} - \frac{15}{65} = \frac{-63}{65}.$

(ii) $\cos(x-y) = \cos x \cos y + \sin x \sin y$

$= \frac{3}{5}\left(\frac{-12}{13}\right) + \frac{4}{5}\left(\frac{5}{13}\right)$

$= \frac{-36}{65} + \frac{20}{65} = \frac{-16}{65}.$

3.17. Prove that $\cos\left(\frac{3\pi}{4} + x\right) - \cos\left(\frac{3\pi}{4} - x\right) = -\sqrt{2}\sin x$

$$\boxed{\cos(A+B) - \cos(A-B) = -2\sin A \sin B}$$

$\cos\left(\frac{3\pi}{4} + x\right) - \cos\left(\frac{3\pi}{4} - x\right) = -2\sin\frac{3\pi}{4} \cdot \sin x$

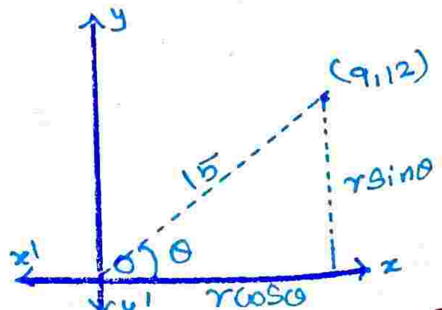
$= -2 \cdot \sin\left(\pi - \frac{\pi}{4}\right) \cdot \sin x$

$= -2\sin\frac{\pi}{4}\sin x$

$= -2 \cdot \frac{1}{\sqrt{2}}\sin x = -\sqrt{2}\sin x$

$= R.H.S.$

3.18 Point $(9, 12)$ rotates around the origin in a plane through $60°$ in the anticlockwise direction to a new position B.
Find the Co-ordinates of the Point B.



Let $A(9,12) = A(r\cos\theta, r\sin\theta)$

$r = \sqrt{9^2+12^2} = \sqrt{81+144} = \sqrt{225} = 15$ units

$\therefore A(15\cos\theta, 15\sin\theta)$

Now the Point B is given by

$B(15\cos(\theta+60^\circ), 15\sin(\theta+60^\circ))$

$15\cos(\theta+60^\circ) = 15\{\cos\theta\cos 60^\circ - \sin\theta\sin 60^\circ\}$

$= 15\cos\theta\cdot\cos 60^\circ - 15\sin\theta\cdot\sin 60^\circ$

$= 9\cdot\frac{1}{2} - 12\cdot\frac{\sqrt{3}}{2} = \frac{9}{2} - 6\sqrt{3}$

$= \frac{3}{2}(3-4\sqrt{3})$

$15\sin(\theta+60^\circ) = 15(\sin\theta\cos 60^\circ + \cos\theta\sin 60^\circ)$

$= 15\sin\theta\cdot\cos 60^\circ + 15\cos\theta\cdot\sin 60^\circ$

$= 12\cdot\frac{1}{2} + 9\cdot\frac{\sqrt{3}}{2} = 6 + \frac{9\sqrt{3}}{2}$

$= \frac{3}{2}(4+3\sqrt{3})$

$\therefore$ The new Position B is $\left(\frac{3}{2}(3-4\sqrt{3}), \frac{3}{2}(4+\sqrt{3})\right)$

3.19 A ripple tank demonstrates the effect of two water waves being added together. The two waves are described by $h = 8\cos t$ and $h = 6\sin t$ where $t \in [0, 2\pi)$ is in seconds and h is the height in millimeters above still water. Find the maximum height of the resultant wave and the value of 't' at which it occurs.

Let H be the height of the resultant wave at any time 't'

$\therefore H = 8\cos t + 6\sin t = k\cos(t-\alpha)$

$= k[\cos t\cos\alpha + \sin t\sin\alpha]$

$k\cos\alpha = 8$ $\qquad$ $k\sin\alpha = 6$

$k^2\cos^2\alpha = 64$ $\qquad$ $k^2\sin^2\alpha = 36$

$k^2 = 64+36 = 100$ $\qquad$ $\boxed{k=10}$

$\cos\alpha = \frac{8}{10} = \frac{4}{5}$ $\qquad$ $\sin\alpha = \frac{6}{10} = \frac{3}{4}$

$\tan\alpha = \frac{3}{4}$

$\therefore H = 10\cos(t-\alpha)$

Thus, the maximum of $H = 10$ mm.

The maximum occurs when $t = \alpha$

where $\tan\alpha = \frac{3}{4}$.

3.20 Expand: (i) $\sin(A+B+C)$ (ii) $\tan(A+B+C)$

$$\sin(A+B+C) = \sin(A+\overline{B+C})$$

$$= \sin A\cos(B+C) + \cos A\sin(B+C)$$

$$= \sin A[\cos B\cos C - \sin B\sin C] + \cos A[\sin B\cos C + \cos B\sin C]$$

$$= \sin A\cos B\cos C - \sin A\sin B\sin C + \cos A\sin B\cos C + \cos A\cos B\sin C$$

(ii) $\tan(A+B+C) = \tan(A+\overline{B+C})$

$$= \frac{\tan A + \tan(B+C)}{1-\tan A\tan(B+C)}$$

$$= \frac{\tan A + \left\{\dfrac{\tan B+\tan C}{1-\tan B\tan C}\right\}}{1-\tan A\left\{\dfrac{\tan B+\tan C}{1-\tan B\tan C}\right\}}$$

$$= \frac{\tan A - \tan A\tan B\tan C + \tan B + \tan C}{1-\tan B\tan C - \tan A(\tan B+\tan C)}$$

$$= \frac{\tan A + \tan B + \tan C - \tan A\tan B\tan C}{1-\tan A\tan B - \tan B\tan C - \tan C\tan A}.$$

12. Prove that $\sin 75^\circ - \sin 15^\circ = \cos 105^\circ + \cos 15^\circ$

$$R.H.S = \cos 105^\circ + \cos 15^\circ$$
$$= \cos(90+15^\circ) + \cos(90-75^\circ)$$
$$= -\sin 15^\circ + \sin 75^\circ$$
$$= \sin 75^\circ - \sin 15^\circ$$
$$= L.H.S.$$

13. Show that $\tan 75^\circ + \cot 75^\circ = 4$

we know that

$$\tan 75^\circ = 2+\sqrt{3} \text{ (or) } \frac{\sqrt{3}+1}{\sqrt{3}-1}$$
$$\cot 75^\circ = \tan 15^\circ = 2-\sqrt{3} \text{ (or) } \frac{\sqrt{3}-1}{\sqrt{3}+1}$$
$$\tan 75^\circ + \cot 75^\circ = 2+\sqrt{3}+2-\sqrt{3} = 4 = R.H.S.$$

14. Prove that $\cos(A+B)\cdot\cos C - \cos(B+C)\cos A = \sin B \sin(C-A)$

$$L.H.S = \cos(A+B)\cdot\cos C - \cos(B+C)\cdot\cos A$$
$$= (\cos A\cos B - \sin A\sin B)\cdot\cos C - [\cos B\cos C - \sin B \sin C]\cos A$$
$$= \cos A\cancel{\cos B}\cos C - \sin A\sin B\cos C - \cos A\cos\cancel{B}\cos C + \cos A \sin B \sin C$$
$$= \cos A \sin B \sin C - \sin A \sin B \cos C$$
$$= \sin B\{\sin C\cos A - \cos C.\sin A\}$$
$$= \sin B \sin(C-A) = R.H.S$$

15. Prove that $\sin(n+1)\theta \sin(n-1)\theta + \cos(n+1)\theta\cos(n-1)\theta = \cos 2\theta,\ n\in Z.$

$$\boxed{\sin A\sin B + \cos A\cos B = \cos(A-B)}$$

$$L.H.S = \sin(n+1)\theta\sin(n-1)\theta + \cos(n+1)\theta.\cos(n-1)\theta$$
$$= \cos\{(n+1)\theta - (n-1)\theta\}$$
$$= \cos\{(\cancel{n}+1-\cancel{n}+1)\theta\} = \cos 2\theta \quad n\in Z.$$
$$= R.H.S$$

16. If $x\cos\theta = y\cos\left(\theta+\frac{2\pi}{3}\right) = z\cos\left(\theta+\frac{4\pi}{3}\right)$ find the value of $xy+yz+zx$

$$x\cos\theta = y\cos\left(\theta+\frac{2\pi}{3}\right) = z\cos\left(\theta+\frac{4\pi}{3}\right) = k$$

$$x\cos\theta = k \qquad y\cos\left(\theta+\frac{2\pi}{3}\right) = k \qquad z\cos\left(\theta+\frac{4\pi}{3}\right) = k$$

$$\frac{k}{x} = \cos\theta \qquad \frac{k}{y} = \cos\left(\theta+\frac{2\pi}{3}\right) \qquad \frac{k}{z} = \cos\left(\theta+\frac{4\pi}{3}\right)$$

$$\frac{k}{x}+\frac{k}{y}+\frac{k}{z} = \cos\theta+\cos\left(\theta+\frac{2\pi}{3}\right)+\cos\left(\theta+\frac{4\pi}{3}\right)$$

$$= \cos\theta+\cos\theta\cos\frac{2\pi}{3}-\sin\theta\sin\frac{2\pi}{3}+\cos\theta\cos\frac{4\pi}{3}-\sin\theta\sin\frac{4\pi}{3} \quad \text{--- (1)}$$

$$\sin 2\pi/3 = \sin 120^\circ = \sin(90+30^\circ) = \cos 30^\circ = \frac{\sqrt{3}}{2}$$

$$\cos\frac{2\pi}{3} = \cos 120^\circ = \cos(90+30^\circ) = -\sin 30^\circ = -\frac{1}{2}$$

$$\cos\frac{4\pi}{3} = \cos 240^\circ = \cos(270-30^\circ) = -\sin 30^\circ = -\frac{1}{2}$$

$$\sin\frac{4\pi}{3} = \sin 240^\circ = \sin(270-30^\circ) = -\cos 30^\circ = -\frac{\sqrt{3}}{2}$$

sub the values in (1)

$$\frac{k}{x}+\frac{k}{y}+\frac{k}{z} = \cancel{\cos\theta}-\cancel{\frac{1}{2}\cos\theta}-\cancel{\frac{\sqrt{3}}{2}\sin\theta}-\cancel{\frac{1}{2}\cos\theta}+\cancel{\frac{\sqrt{3}}{2}\sin\theta}$$

$$= 0$$

$$\frac{1}{x}+\frac{1}{y}+\frac{1}{z} = 0$$

$$\frac{xy+yz+zx}{xyz} = 0$$

$$\boxed{xy+yz+zx = 0}$$

17. Prove that:

(i) $\sin(A+B)\sin(A-B) = \sin^2 A - \sin^2 B$

$$\text{L.H.S} = (\sin A\cos B+\cos A\sin B)(\sin A\cos B-\cos A\sin B)$$

$$= \sin^2 A\cos^2 B - \cos^2 A\sin^2 B$$

$$= \sin^2 A(1-\sin^2 B) - (1-\sin^2 A)\sin^2 B$$

$$= \sin^2 A - \cancel{\sin^2 A\sin^2 B} - \sin^2 B + \cancel{\sin^2 A\sin^2 B}$$

$$= \sin^2 A - \sin^2 B = \text{R.H.S}$$

$$\sin(A+B)\cdot\sin(A-B) = \sin^2 A - \sin^2 B.$$

(ii) $\cos(A+B)\cdot\cos(A-B) = \cos^2 A - \sin^2 B$ (or)

$= \cos^2 B - \sin^2 A$

L.H.S $= \cos(A+B)\cdot\cos(A-B)$

$= (\cos A\cos B - \sin A\sin B)(\cos A\cos B + \sin A\sin B)$

$= \cos^2 A\cos^2 B - \sin^2 A\sin^2 B$ ——— (1)

$= \cos^2 A(1-\sin^2 B) - (1-\cos^2 A)\sin^2 B$

$= \cos^2 A - \cancel{\cos^2 A\sin^2 B} - \sin^2 B + \cancel{\cos^2 A\sin^2 B}$

$= \cos^2 A - \sin^2 B$ = R.H.S

from (1)

L.H.S $= \cos^2 A\cos^2 B - \sin^2 A\sin^2 B$

$= (1-\sin^2 A)\cos^2 B - \sin^2 A(1-\cos^2 B)$

$= \cos^2 B - \cancel{\sin^2 A\cos^2 B} - \sin^2 A + \cancel{\sin^2 A\cos^2 B}$

$= \cos^2 B - \sin^2 A$ = R.H.S

Hence Proved.

$$\boxed{\begin{aligned}\sin(A+B)\cdot\sin(A-B) &= \sin^2 A - \sin^2 B\\ &= \cos^2 B - \cos^2 A\\ \cos(A+B)\cdot\cos(A-B) &= \cos^2 A - \sin^2 B\\ &= \cos^2 B - \sin^2 A.\end{aligned}}$$

(iii) $\sin^2(A+B) - \sin^2(A-B) = \sin 2A\cdot\sin 2B$

[W.K.T $\sin^2 A - \sin^2 B = \sin(A+B)\cdot\sin(A-B)$]

L.H.S $= \sin^2(A+B) - \sin^2(A-B)$

$= \sin(A+B+A-B)\cdot\sin(A+B-\overline{A-B})$

$= \sin(2A)\cdot\sin(\cancel{A}+B-\cancel{A}+B)$

$= \sin 2A\cdot\sin 2B$

= R.H.S

(iv) $\cos 8\theta \cdot \cos 2\theta = \cos^2 5\theta - \sin^2 3\theta$

$L.H.S = \cos 8\theta \cdot \cos 2\theta$

$= \cos(5\theta + 3\theta) \cdot \cos(5\theta - 3\theta)$

$= \cos^2 5\theta - \sin^2 3\theta$

$= R.H.S$

18. Show that $\cos^2 A + \cos^2 B - 2\cos A \cos B \cos(A+B) = \sin^2(A+B)$.

$L.H.S = \cos^2 A + \cos^2 B - 2\cos A \cos B \cdot \cos(A+B)$

$= \cos^2 A + (1 - \sin^2 B) - 2\cos A \cos B \cdot \cos(A+B)$

$= \cos^2 A - \sin^2 B - 2\cos A \cos B \cos(A+B) + 1$

$= \cos(A+B) \cdot \cos(A-B) - 2\cos A \cos B \cdot \cos(A+B) + 1$

$= \cos(A+B)\{\cos(A-B) - 2\cos A \cos B\} + 1$

$= \cos(A+B)\{\cos A \cos B + \sin A \sin B - 2\cos A \cos B\} + 1$

$= \cos(A+B)\{-\cos A \cos B + \sin A \sin B\} + 1$

$= -\cos(A+B)\{\cos A \cos B - \sin A \sin B\} + 1$

$= -\cos(A+B)\cos(A+B) + 1$

$= 1 - \cos^2(A+B)$

$= \sin^2(A+B) = R.H.S$

19. If $\cos(\alpha - \beta) + \cos(\beta - \gamma) + \cos(\gamma - \alpha) = -\frac{3}{2}$

Prove that $\cos\alpha + \cos\beta + \cos\gamma = \sin\alpha + \sin\beta + \sin\gamma = 0$

$\cos(\alpha - \beta) + \cos(\beta - \gamma) + \cos(\gamma - \alpha) = -\frac{3}{2}$

$\cos\alpha\cos\beta + \sin\alpha\sin\beta + \cos\beta\cos\gamma + \sin\beta\sin\gamma + \cos\gamma\cos\alpha + \sin\gamma\sin\alpha = -\frac{3}{2}$

$2\cos\alpha\cos\beta + 2\sin\alpha\sin\beta + 2\cos\beta\cos\gamma + 2\sin\beta\sin\gamma + 2\cos\gamma\cos\alpha + 2\sin\gamma\sin\alpha = -3$

$2\cos\alpha\cos\beta + 2\sin\alpha\sin\beta + 2\cos\beta\cos\gamma + 2\sin\beta\sin\gamma + 2\cos\gamma\cos\alpha + 2\sin\gamma\sin\alpha + 3 = 0$

$$2\cos\alpha\cos\beta+2\sin\alpha\sin\beta+2\cos\beta\cos\gamma+2\sin\beta\sin\gamma+2\cos\gamma\cos\alpha+2\sin\gamma\sin\alpha+1+1+1=0$$

$$2\cos\alpha\cos\beta+2\sin\alpha\sin\beta+2\cos\beta\cos\gamma+2\sin\beta\sin\gamma+2\cos\gamma\cos\alpha+2\sin\gamma\sin\alpha+(\sin^2\alpha+\cos^2\alpha)+(\sin^2\beta+\cos^2\beta)+(\sin^2\gamma+\cos^2\gamma)=0$$

$$[\sin^2\alpha+\sin^2\beta+\sin^2\gamma+2\sin\alpha\sin\beta+2\sin\beta\sin\gamma+2\sin\gamma\sin\alpha]+[\cos^2\alpha+\cos^2\beta+\cos^2\gamma+2\cos\alpha\cos\beta+2\cos\beta\cos\gamma+2\cos\gamma\cos\alpha]=0$$

$$[\sin\alpha+\sin\beta+\sin\gamma]^2+[\cos\alpha+\cos\beta+\cos\gamma]^2=0$$

$$\sin\alpha+\sin\beta+\sin\gamma=0 \qquad \cos\alpha+\cos\beta+\cos\gamma=0$$

20. S.T

$$\boxed{\begin{array}{ll}\text{(i)} & \tan(45+A)=\dfrac{1+\tan A}{1-\tan A}\\ \text{(ii)} & \tan(45-A)=\dfrac{1-\tan A}{1+\tan A}\end{array}}$$

(i) $L.H.S=\tan(45+A)=\dfrac{\tan 45+\tan A}{1-\tan 45\tan A}$

$$=\frac{1+\tan A}{1-\tan A}=R.H.S$$

(ii) $L.H.S=\tan(45-A)=\dfrac{\tan 45-\tan A}{1+\tan 45\tan A}$

$$=\frac{1-\tan A}{1+\tan A}=R.H.S$$

21. Prove that $\cot(A+B)=\dfrac{\cot A\cot B-1}{\cot A+\cot B}$

$R.H.S=\dfrac{\cot A\cot B-1}{\cot A+\cot B}=\dfrac{\frac{1}{\tan A}\cdot\frac{1}{\tan B}-1}{\frac{1}{\tan A}+\frac{1}{\tan B}}$

$$=\frac{1-\tan A\tan B}{\tan A+\tan B}=\frac{1}{\tan(A+B)}$$

$$=\cot(A+B)$$

$$=L.H.S.$$

22. If $\tan x = \frac{n}{n+1}$ and $\tan y = \frac{1}{2n+1}$ find $\tan(x+y)$

$$\tan(x+y) = \frac{\tan x + \tan y}{1 - \tan x \tan y}$$

$$= \frac{\frac{n}{n+1} + \frac{1}{2n+1}}{1 - \frac{n}{n+1}\cdot\frac{1}{2n+1}} = \frac{n(2n+1) + (n+1)}{(n+1)(2n+1) - n}$$

$$= \frac{2n^2+n+n+1}{2n^2+n+2n+1-n} = \frac{2n^2+2n+1}{2n^2+2n+1} = 1.$$

$$\tan(x+y) = 1.$$

---

23. Prove that $\tan\left(\frac{\pi}{4}+\theta\right)\cdot\tan\left(\frac{3\pi}{4}+\theta\right) = -1$

$$\tan\left(\frac{\pi}{4}+\theta\right) = \frac{1+\tan\theta}{1-\tan\theta}$$

$$\tan\left(\frac{3\pi}{4}+\theta\right) = \tan\left(\pi - \frac{\pi}{4} - \theta\right)$$

$$= -\tan\left(\frac{\pi}{4}-\theta\right) = -\left(\frac{1-\tan\theta}{1+\tan\theta}\right)$$

$$\tan\left(\frac{\pi}{4}+\theta\right)\cdot\tan\left(\frac{3\pi}{4}+\theta\right) = -\left(\frac{1+\tan\theta}{1-\tan\theta}\right)\left(\frac{1-\tan\theta}{1+\tan\theta}\right)$$

$$= -1$$

Hence Proved.

---

24. Find the values of $\tan(\alpha+\beta)$, given that $\cot\alpha = \frac{1}{2}$ $\alpha \in \left(\pi, \frac{3\pi}{2}\right)$ and $\sec\beta = -\frac{5}{3}$ $\beta \in \left(\frac{\pi}{2}, \pi\right)$

$\cot\alpha = \frac{1}{2}$

$\tan\alpha = 2$

$\sec\beta = -\frac{5}{3}$

$\sec^2\beta = \frac{25}{9}$

$\sec^2\beta - 1 = \frac{16}{9}$

$\tan^2\beta = \frac{16}{9}$

$\tan\beta = -\frac{4}{3}$ β lies in II quad

$$\tan(\alpha+\beta) = \frac{\tan\alpha + \tan\beta}{1 - \tan\alpha\tan\beta}$$

$$= \frac{2 - \frac{4}{3}}{1+2(4/3)} = \frac{6-4}{3+8} = \frac{2}{11}$$

$$\tan(\alpha+\beta) = \frac{2}{11}.$$

25. If $\theta+\varphi = \alpha$ and $\tan\theta = k\tan\varphi$ then prove that $\sin(\theta-\varphi) = \frac{k-1}{k+1}\cdot\sin\alpha$

$\theta+\varphi=\alpha$

$$\tan\theta = k\tan\varphi$$

$$\frac{\tan\theta}{\tan\varphi} = \frac{k}{1}$$

$$\frac{\tan\theta+\tan\varphi}{\tan\theta-\tan\varphi} = \frac{k+1}{k-1}$$

$$\frac{\frac{\sin\theta}{\cos\theta}+\frac{\sin\varphi}{\cos\varphi}}{\frac{\sin\theta}{\cos\theta}-\frac{\sin\varphi}{\cos\varphi}} = \frac{k+1}{k-1}$$

$$\frac{\sin\theta\cos\varphi+\cos\theta\cdot\sin\varphi}{\sin\theta\cos\varphi-\cos\theta\cdot\sin\varphi} = \frac{k+1}{k-1}$$

$$\frac{\sin(\theta+\varphi)}{\sin(\theta-\varphi)} = \frac{k+1}{k-1} \quad [\theta+\varphi=\alpha]$$

$$\frac{\sin\alpha}{\sin(\theta-\varphi)} = \frac{k+1}{k-1}$$

$$\therefore \sin(\theta-\varphi) = \frac{k-1}{k+1}\cdot\sin\alpha$$

Hence Proved

3.15 Find the values of (i) $\cos 15^\circ$ (ii) $\tan 165^\circ$

(i) $\cos 15^\circ = \cos(45-30^\circ)$

$= \cos 45^\circ\cos 30^\circ + \sin 45\cdot\sin 30$

$= \frac{1}{\sqrt2}\cdot\frac{\sqrt3}{2} + \frac{1}{\sqrt2}\cdot\frac{1}{2} = \frac{\sqrt3+1}{2\sqrt2}$

(ii) $\tan 165^\circ = \tan(180-15^\circ) = -\tan 15^\circ$

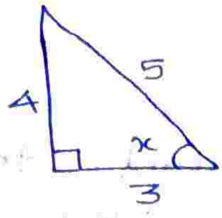
$= -\tan(45-30^\circ) = -\left\{\frac{\tan 45-\tan 30^\circ}{1+\tan 45\tan 30^\circ}\right\}$

$= -\left\{\frac{1-\frac{1}{\sqrt3}}{1+\frac{1}{\sqrt3}}\right\} = -\left\{\frac{\sqrt3-1}{\sqrt3+1}\right\} = \frac{1-\sqrt3}{1+\sqrt3}$

3.16. If $\sin x = \frac{4}{5}$ $x \in (0, \frac{\pi}{2})$ and $\cos y = \frac{-12}{13}$ $y \in (\frac{\pi}{2}, \pi)$ then find (i) $\sin(x-y)$ (ii) $\cos(x-y)$

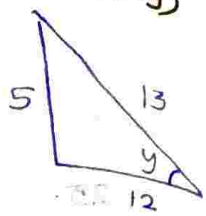
$\sin x = \frac{4}{5}$

$\cos x = \frac{3}{5}$



$\cos y = -\frac{12}{13}$

$\sin y = \frac{5}{13}$



(i) $\sin(x-y) = \sin x \cos y - \cos x \sin y$

$= \frac{4}{5}.\left(\frac{-12}{13}\right) - \frac{3}{5}\left(\frac{5}{13}\right)$

$= \frac{-48}{65} - \frac{15}{65} = \frac{-63}{65}.$

(ii) $\cos(x-y) = \cos x \cos y + \sin x \sin y$

$= \frac{3}{5}\left(\frac{-12}{13}\right) + \frac{4}{5}\left(\frac{5}{13}\right)$

$= \frac{-36}{65} + \frac{20}{65} = \frac{-16}{65}.$

3.17. Prove that $\cos\left(\frac{3\pi}{4}+x\right) - \cos\left(\frac{3\pi}{4}-x\right) = -\sqrt{2}\sin x$

$$\boxed{\cos(A+B) - \cos(A-B) = -2\sin A \sin B}$$

$\cos\left(\frac{3\pi}{4}+x\right) - \cos\left(\frac{3\pi}{4}-x\right) = -2\sin\frac{3\pi}{4}.\sin x$

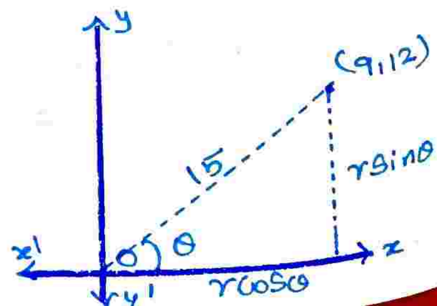
$= -2.\sin\left(\pi - \frac{\pi}{4}\right).\sin x$

$= -2\sin\frac{\pi}{4}\sin x$

$= -2.\frac{1}{\sqrt{2}}\sin x = -\sqrt{2}\sin x$

$= R.H.S.$

3.18 Point (9, 12) rotates around the origin in a Plane through 60° in the anticlockwise direction to a new Position B. Find the Co-ordinates of the Point B.



Let $A(9,12) = A(r\cos\theta, r\sin\theta)$

$$r = \sqrt{9^2+12^2} = \sqrt{81+144} = \sqrt{225} = 15 \text{ units}$$

$\therefore A(15\cos\theta, 15\sin\theta)$

Now the Point B is given by

$$B(15\cos(\theta+60^\circ), 15\sin(\theta+60^\circ))$$

$$15\cos(\theta+60^\circ) = 15\{\cos\theta\cos 60^\circ - \sin\theta\sin 60^\circ\}$$
$$= 15\cos\theta \cdot \cos 60^\circ - 15\sin\theta \cdot \sin 60^\circ$$
$$= 9 \cdot \tfrac{1}{2} - 12 \cdot \tfrac{\sqrt{3}}{2} = \tfrac{9}{2} - 6\sqrt{3}$$
$$= \tfrac{3}{2}(3-4\sqrt{3})$$

$$15\sin(\theta+60^\circ) = 15(\sin\theta\cos 60^\circ + \cos\theta\sin 60^\circ)$$
$$= 15\sin\theta \cdot \cos 60^\circ + 15\cos\theta \cdot \sin 60^\circ$$
$$= 12 \cdot \tfrac{1}{2} + 9 \cdot \tfrac{\sqrt{3}}{2} = 6 + \tfrac{9\sqrt{3}}{2}$$
$$= \tfrac{3}{2}(4+3\sqrt{3})$$

$\therefore$ The new Position B is $\left(\tfrac{3}{2}(3-4\sqrt{3}), \tfrac{3}{2}(4+\sqrt{\ }\right.$

3.19 A ripple tank demonstates the effect of two water waves being added together. The two waves are described by $h = 8\cos t$ and $h = 6\sin t$ where $t \in [0, 2\pi)$ is in seconds and h is the height in millimeters above still water. Find the maximum height of the resultant wave and the value of 't' at which it occurs.

Let H be the height of the resultant wave at any time 't'

$$\therefore H = 8\cos t + 6\sin t = K\cos(t-\alpha)$$
$$= K[\cos t\cos\alpha + \sin t\sin\alpha]$$

$K\cos\alpha = 8$ $\quad$ $K\sin\alpha = 6$

$K^2\cos^2\alpha = 64$ $\quad$ $K^2\sin^2\alpha = 36$

$K^2 = 64+36 = 100$ $\quad$ $\boxed{K=10}$.

$\cos\alpha = \frac{8}{10} = \frac{4}{5}$ $\quad$ $\sin\alpha = \frac{6}{10} = \frac{3}{4}$

$\tan\alpha = \frac{3}{4}$

$\therefore H = 10\cos(t-\alpha)$

Thus, the maximum of $H = 10$ mm.

The maximum occurs when $t = \alpha$ where $\tan\alpha = \frac{3}{4}$.

3.20 Expand: (i) $\sin(A+B+C)$ (ii) $\tan(A+B+C)$

$$\sin(A+B+C) = \sin(A+\overline{B+C})$$
$$= \sin A\cos(B+C) + \cos A\sin(B+C)$$
$$= \sin A[\cos B\cos C - \sin B\sin C] + \cos A[\sin B\cos C + \cos B\sin C]$$
$$= \sin A\cos B\cos C - \sin A\sin B\sin C + \cos A\sin B\cos C + \cos A\cos B\sin C$$

(ii) $\tan(A+B+C) = \tan(A+\overline{B+C})$

$$= \frac{\tan A + \tan(B+C)}{1-\tan A\tan(B+C)}$$
$$= \frac{\tan A + \left\{\frac{\tan B+\tan C}{1-\tan B\tan C}\right\}}{1-\tan A\left\{\frac{\tan B+\tan C}{1-\tan B\tan C}\right\}}$$
$$= \frac{\tan A - \tan A\tan B\tan C + \tan B + \tan C}{1-\tan B\tan C - \tan A(\tan B+\tan C)}$$
$$= \frac{\tan A + \tan B + \tan C - \tan A\tan B\tan C}{1-\tan A\tan B - \tan B\tan C - \tan C\tan A}.$$

Multiple Angle Identities :-

$\sin 2A = 2\sin A \cos A$

$\cos 2A = 2\cos^2 A - 1$

$= 1 - 2\sin^2 A$

$= \cos^2 A - \sin^2 A$

$$\tan 2A = \frac{2\tan A}{1 + \tan^2 A}$$

$1 + \cos 2A = 2\cos^2 A$

$$\cos^2 A = \frac{1 + \cos 2A}{2}$$

$1 - \cos 2A = 2\sin^2 A$

$$\sin^2 A = \frac{1 - \cos 2A}{2}$$

$$\sin 2A = \frac{2\tan A}{1 + \tan^2 A}$$

$$\cos 2A = \frac{1 - \tan^2 A}{1 + \tan^2 A}.$$

$$\tan^2 A = \frac{1 - \cos 2A}{1 + \cos 2A}$$

Half Angle Identities :-

$\sin A = 2\sin \frac{A}{2} . \cos \frac{A}{2}$

$\cos A = \cos^2 \frac{A}{2} - \sin^2 \frac{A}{2}$

$= 2\cos^2 \frac{A}{2} - 1$

$= 1 - 2\sin^2 A/2$

$$\tan A = \frac{2\tan A/2}{1 - \tan^2 A/2}.$$

$1 + \cos A = 2\cos^2 A/2$

$$\cos^2 A/2 = \frac{1 + \cos A}{2}$$

$1 - \cos A = 2\sin^2 A/2$

$$\sin^2 A/2 = \frac{1 - \cos A}{2}$$

$$\sin A = \frac{2\tan A/2}{1 + \tan^2 A/2}$$

$$\cos A = \frac{1 - \tan^2 A/2}{1 + \tan^2 A/2}$$

$$\tan^2 A/2 = \frac{1 - \cos A}{1 + \cos A}.$$

Triple Angle Identities:

$$\sin 3A = 3\sin A - 4\sin^3 A$$

$$\cos 3A = 4\cos^3 A - 3\cos A$$

$$\tan 3A = \frac{3\tan A - \tan^3 A}{1 - 3\tan^2 A}.$$

$$\sin^3 A = \frac{1}{4}\left[3\sin A - \sin 3A\right]$$

$$\cos^3 A = \frac{1}{4}\left[3\cos A + \cos 3A\right]$$

3.22 Find the value of $\sin 22\frac{1}{2}^\circ$

Let $\theta = 45^\circ$

We know that

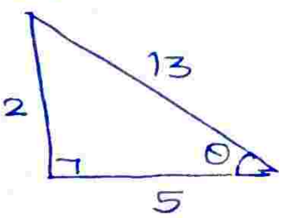
$$\sin^2 \theta/2 = \frac{1-\cos\theta}{2}$$

$$\sin^2 22\frac{1}{2}^\circ = \frac{1-\cos 45^\circ}{2} = \frac{1-\frac{1}{\sqrt{2}}}{2}$$

$$= \frac{\sqrt{2}-1}{2\sqrt{2}} \times \frac{\sqrt{2}}{\sqrt{2}} = \frac{2-\sqrt{2}}{4}$$

$$\sin 22\frac{1}{2}^\circ = \sqrt{\frac{2-\sqrt{2}}{4}} = \frac{\sqrt{2-\sqrt{2}}}{2}$$

3.23 Find the value of $\sin 2\theta$ when $\sin\theta = \frac{12}{13}$. θ lies in the first quadrant.



$$\sin\theta = \frac{12}{13}$$

$$\cos\theta = \frac{5}{13}$$

$$\sin 2\theta = 2\sin\theta\cos\theta$$

$$= 2 \cdot \frac{12}{13} \cdot \frac{5}{13} = \frac{120}{169}$$

$$\sin 2\theta = \frac{120}{169}.$$

3.24 Prove that $\sin 4A = 4\sin A\cos^3 A - 4\cos A \cdot \sin^3 A$

$$\text{R.H.S} = 4\sin A\cos^3 A - 4\cos A\sin^3 A$$
$$= 4\sin A\cos A(\cos^2 A - \sin^2 A)$$
$$= 2(2\sin A\cos A)(\cos^2 A - \sin^2 A)$$
$$= 2(\sin 2A)(\cos 2A) \quad = \sin 2(2A)$$
$$= \sin 4A \quad = \text{L.H.S}$$

3.25 Prove that $\sin x = 2^{10} \cdot \sin\left(\frac{x}{2^{10}}\right)\cos\left(\frac{x}{2}\right)\left(\cos\frac{x}{2^2}\right)\cdots\cos\left(\frac{x}{2^{10}}\right)$

$$\text{L.H.S} = \sin x$$
$$= 2\sin\frac{x}{2} \cdot \cos\frac{x}{2}$$
$$= 2\cos\frac{x}{2}\left\{\sin\frac{x}{2}\right\}$$
$$= 2\cos\frac{x}{2}\left\{2\sin\frac{x}{2^2} \cdot \cos\frac{x}{2^2}\right\}$$
$$= 4\cos\frac{x}{2} \cdot \cos\frac{x}{2^2}\left\{\sin\frac{x}{2^2}\right\}$$
$$= 2^2 \cdot \cos\frac{x}{2} \cdot \cos\frac{x}{2^2}\left(2\sin\frac{x}{2^3} \cdot \cos\frac{x}{2^3}\right)$$
$$= 2^3 \cdot \cos\frac{x}{2} \cdot \cos\frac{x}{2^2} \cdot \cos\frac{x}{2^3} \cdot \sin\frac{x}{2^3}$$

Applying repeatedly the half angle sine formula,

$$\sin x = 2^{10} \cdot \cos\frac{x}{2} \cdot \cos\frac{x}{2^2} \cdot \cdots \cos\frac{x}{2^{10}} \cdot \sin\frac{x}{2^{10}}$$

Hence Proved.

3.26 Prove that $\dfrac{\sin\theta + \sin 2\theta}{1+\cos\theta+\cos 2\theta} = \tan\theta$

$$\text{L.H.S} = \frac{\sin\theta + \sin 2\theta}{\cos\theta + 1 + \cos 2\theta}$$
$$= \frac{\sin\theta + 2\sin\theta\cos\theta}{\cos\theta + 2\cos^2\theta}$$
$$= \frac{\sin\theta(1+2\cos\theta)}{\cos\theta(1+2\cos\theta)} = \frac{\sin\theta}{\cos\theta}$$
$$= \tan\theta \quad = \text{R.H.S}$$

Hence Proved.

3.27. Prove that $1-\frac{1}{2}\sin 2x = \frac{\sin^3 x+\cos^3 x}{\sin x+\cos x}$

$$\text{R.H.S} = \frac{\sin^3 x+\cos^3 x}{\sin x+\cos x} \qquad \left[\because a^3+b^3=(a+b)(a^2-ab+b^2)\right]$$

$$= \frac{\cancel{(\sin x+\cos x)}(\sin^2 x+\cos^2 x-\sin x\cos x)}{\cancel{(\sin x+\cos x)}}$$

$$= 1-\sin x\cos x$$

$$= 1-\frac{1}{2}(2\sin x\cos x)$$

$$= 1-\frac{1}{2}\sin 2x = \text{L.H.S}$$

Hence Proved.

3.28 Find x such that $-\pi \le x \le \pi$ and $\cos 2x = \sin x$

$$\cos 2x = \sin x$$

$$1-2\sin^2 x = \sin x$$

$$2\sin^2 x+\sin x-1=0$$

-2; -1, 2; $\frac{-1}{2\sin x}$, $\frac{2}{2\sin x}$

$$(2\sin x-1)(\sin x+1)=0$$

$\sin x = \frac{1}{2}$ $\qquad$ $\sin x = -1$

$x = \frac{\pi}{6}, \frac{5\pi}{6} \in [-\pi,\pi]$ $\qquad$ $x = -\frac{\pi}{2} \in [-\pi,\pi]$

$\therefore$ The Solutions are $-\frac{\pi}{2}, \frac{\pi}{6}, \frac{5\pi}{6}$.

3.29 Find the values of (i) $\sin 18^\circ$ (ii) $\cos 18^\circ$ (iii) $\sin 72^\circ$ (iv) $\cos 36^\circ$ (v) $\sin 54^\circ$

Let $\theta = 18^\circ$

$$5\theta = 90^\circ$$

$$2\theta+3\theta = 90^\circ$$

$$2\theta = 90-3\theta$$

$$\sin 2\theta = \sin(90-3\theta)$$

$$\sin 2\theta = \cos 3\theta$$

$$2\sin\theta\cos\theta = 4\cos^3\theta-3\cos\theta$$

$\div \cos\theta$ $\qquad$ $2\sin\theta = 4\cos^2\theta-3$

$$2\sin\theta = 4(1-\sin^2\theta)-3$$
$$2\sin\theta = 4-4\sin^2\theta-3$$
$$4\sin^2\theta+2\sin\theta-1=0$$

$a=4$
$b=2$
$c=-1$

$$\sin\theta = \frac{-b\pm\sqrt{b^2-4ac}}{2a}$$
$$= \frac{-2\pm\sqrt{4+16}}{8} = \frac{-2\pm\sqrt{20}}{8}$$
$$= \frac{-2\pm2\sqrt5}{8} = \frac{-1\pm\sqrt5}{4}$$

θ is acute angle

$$\sin\theta = \frac{\sqrt5-1}{4} \qquad \sin\theta \neq \frac{-1-\sqrt5}{4}$$

$$\therefore \sin 18^\circ = \frac{\sqrt5-1}{4}.$$

(i) $\sin 18^\circ = \frac{\sqrt5-1}{4}$

(ii) $\cos 18^\circ = \sqrt{1-\sin^2 18^\circ} = \sqrt{1-\left(\frac{\sqrt5-1}{4}\right)^2}$
$$= \sqrt{\frac{16-5-1+2\sqrt5}{16}} = \frac{\sqrt{10+2\sqrt5}}{4}$$

(iii) $\sin 72^\circ = \sin(90-18^\circ) = \cos 18^\circ = \frac{\sqrt{10+2\sqrt5}}{4}$

(iv) $\cos 36^\circ = 1-2\sin^2 18^\circ$ (or) $2\cos^2 18^\circ - 1$
$$= 2\left(\frac{10+2\sqrt5}{16}\right)-1 = \frac{20+4\sqrt5-16}{16}$$
$$= \frac{4+4\sqrt5}{16} = \frac{1+\sqrt5}{4} = \frac{\sqrt5+1}{4}$$

$$\cos 36^\circ = \frac{\sqrt5+1}{4}$$

(v) $\sin 54^\circ = \sin(90-36^\circ) = \cos 36^\circ = \frac{\sqrt5+1}{4}.$

$\sin 18^\circ = \frac{\sqrt5-1}{4}$	$\cos 18^\circ = \frac{\sqrt{10+2\sqrt5}}{4}$
$\sin 36^\circ = \frac{\sqrt{10-2\sqrt5}}{4}$	$\cos 36^\circ = \frac{\sqrt5+1}{4}$
$\sin 54^\circ = \frac{\sqrt5+1}{4}$	$\cos 54^\circ = \frac{\sqrt{10-2\sqrt5}}{4}$
$\sin 72^\circ = \frac{\sqrt{10+2\sqrt5}}{4}$	$\cos 72^\circ = \frac{\sqrt5-1}{4}.$

3.30 If $\tan\frac{\theta}{2} = \sqrt{\frac{1-a}{1+a}}\tan\frac{\varphi}{2}$ then Prove that

$$\cos\phi = \frac{\cos\theta - a}{1 - a\cos\theta}$$

$$\tan\frac{\varphi}{2} = \sqrt{\frac{1+a}{1-a}}\tan\frac{\theta}{2}$$

Proof :

$$\cos\varphi = \frac{1-\tan^2\varphi/2}{1+\tan^2\frac{\varphi}{2}}$$

$$= \frac{1 - \frac{1+a}{1-a}\tan^2\frac{\theta}{2}}{1 + \frac{1+a}{1-a}\tan^2\frac{\theta}{2}}$$

$$= \frac{1-a-(1+a)\tan^2\frac{\theta}{2}}{1-a+(1+a)\tan^2\frac{\theta}{2}}$$

$$= \frac{1-a-\tan^2\frac{\theta}{2} - a\tan^2\frac{\theta}{2}}{1-a+\tan^2\frac{\theta}{2}+a\tan^2\frac{\theta}{2}}$$

$$= \frac{1-\tan^2\frac{\theta}{2} - a(1+\tan^2\theta/2)}{1+\tan^2\frac{\theta}{2} - a(1-\tan^2\theta/2)}$$

$\div (1+\tan^2\theta/2)$

$$= \frac{\frac{1-\tan^2\theta/2}{1+\tan^2\theta/2} - a}{1 - a\left(\frac{1-\tan^2\theta/2}{1+\tan^2\theta/2}\right)}$$

$$= \frac{\cos\theta - a}{1 - a\cos\theta} = R.HS$$

Hence Proved.

3.31 Find the value of $\sqrt{3}\operatorname{cosec}20^\circ - \sec 20^\circ$

$$\sqrt{3}\operatorname{cosec}20^\circ - \sec 20^\circ = \frac{\sqrt{3}}{\sin 20^\circ} - \frac{1}{\cos 20^\circ}$$

$$= \frac{\sqrt{3}\cos 20^\circ - \sin 20^\circ}{\sin 20^\circ \cos 20^\circ}$$

$$= \frac{2\left(\frac{\sqrt{3}}{2}\cos 20^\circ - \frac{1}{2}\sin 20^\circ\right)}{\sin 20^\circ \cos 20^\circ}$$

$$= \frac{2\cdot 2\left[\sin 60^\circ \cos 20^\circ - \cos 60^\circ \sin 20^\circ\right]}{2\sin 20^\circ \cos 20^\circ}$$

$$= \frac{4\sin(60-20^\circ)}{\sin 2(20^\circ)} = \frac{4\sin 40^\circ}{\sin 40^\circ} = 4.$$

Ex: 3.5

1. Find the value of cos2A, A lies in the first quadrant, when

(i) $\cos A = \frac{15}{17}$ (ii) $\sin A = \frac{4}{5}$ (iii) $\tan A = \frac{16}{63}$

(i) $\cos A = \frac{15}{17}$

$$\cos 2A = 2\cos^2 A - 1 = 2\left(\frac{225}{289}\right) - 1$$

$$= \frac{450 - 289}{289} = \frac{161}{289}$$

(ii) $\sin A = \frac{4}{5}$

$$\cos 2A = 1 - 2\sin^2 A = 1 - 2\left(\frac{16}{25}\right)$$

$$= \frac{25 - 32}{25} = \frac{-7}{25}$$

(iii) $\tan A = \frac{16}{63}$

$$\cos 2A = \frac{1 - \tan^2 A}{1 + \tan^2 A} = \frac{1 - \left(\frac{16}{63}\right)^2}{1 + \left(\frac{16}{63}\right)^2}$$

$$= \frac{1 - \frac{256}{3969}}{1 + \frac{256}{3969}} = \frac{3969 - 256}{3969 + 256}$$

$$= \frac{3713}{4225}$$

2. If θ is acute angle. Then find,

(i) $\sin\left(\frac{\pi}{4} - \frac{\theta}{2}\right)$ when $\sin\theta = \frac{1}{25}$

(ii) $\cos\left(\frac{\pi}{4} + \frac{\theta}{2}\right)$ when $\sin\theta = \frac{8}{9}$

(i) $$\sin\left(\frac{\pi}{4} - \frac{\theta}{2}\right) = \sin\frac{\pi}{4}\cos\frac{\theta}{2} - \cos\frac{\pi}{4}\sin\theta/2$$

$$= \frac{1}{\sqrt{2}}\cos\frac{\theta}{2} - \frac{1}{\sqrt{2}}\sin\theta/2$$

$$= \frac{1}{\sqrt{2}}(\cos\theta/2 - \sin\theta/2)$$

$$\sin^2\left(\frac{\pi}{4} - \frac{\theta}{2}\right) = \frac{1}{2}\left[1 - 2\sin\theta/2 \cdot \cos\theta/2\right]$$

$$= \frac{1}{2}\left[1 - \sin\theta\right]$$

$$= \frac{1}{2}\left(1 - \frac{1}{25}\right) = \frac{1}{2}\left(\frac{24}{25}\right) = \frac{12}{25}$$

$$\sin\left(\frac{\pi}{4} - \frac{\theta}{2}\right) = \sqrt{\frac{12}{25}} = \frac{2\sqrt{3}}{5}.$$

(ii) $\cos\left(\frac{\pi}{4}+\frac{\theta}{2}\right) = \cos\frac{\pi}{4}\cos\frac{\theta}{2} - \sin\frac{\pi}{4}\sin\frac{\theta}{2}$

$= \frac{1}{\sqrt{2}}\left(\cos\theta/2 - \sin\theta/2\right)$

$\cos^2\left(\frac{\pi}{4}+\frac{\theta}{2}\right) = \frac{1}{2}\left[1 - 2\sin\theta/2\cos\theta/2\right]$

$= \frac{1}{2}\left(1-\sin\theta\right)$

$= \frac{1}{2}\left(1-\frac{8}{9}\right) = \frac{1}{2}\left(\frac{1}{9}\right) = \frac{1}{18}$

$\cos\left(\frac{\pi}{4}+\frac{\theta}{2}\right) = \frac{1}{3\sqrt{2}}$

---

3. If $\cos\theta = \frac{1}{2}\left(a+\frac{1}{a}\right)$ show that $\cos 3\theta = \frac{1}{2}\left(a^3+\frac{1}{a^3}\right)$

$$\boxed{\begin{aligned} a^3+b^3 &= (a+b)^3 - 3ab(a+b) \\ a^3+\frac{1}{a^3} &= \left(a+\frac{1}{a}\right)^3 - 3\left(a+\frac{1}{a}\right)\end{aligned}}$$

$\cos\theta = \frac{1}{2}\left(a+\frac{1}{a}\right) \Rightarrow a+\frac{1}{a} = 2\cos\theta.$

$a^3+\frac{1}{a^3} = \left(a+\frac{1}{a}\right)^3 - 3\left(a+\frac{1}{a}\right)$

$= (2\cos\theta)^3 - 3(2\cos\theta)$

$= 8\cos^3\theta - 6\cos\theta$

$= 2(4\cos^3\theta - 3\cos\theta)$

$= 2\cos 3\theta$

$\frac{1}{2}\left(a^3+\frac{1}{a^3}\right) = \cos 3\theta$

Hence Proved.

---

4. Prove that $\cos 5\theta = 16\cos^5\theta - 20\cos^3\theta + 5\cos\theta$

L.H.S $= \cos 5\theta = \cos(2\theta+3\theta)$

$= \cos 2\theta\cos 3\theta - \sin 2\theta \sin 3\theta$

$= (2\cos^2\theta-1)(4\cos^3\theta - 3\cos\theta) - 2\sin\theta\cos\theta(3\sin\theta - 4\sin^3\theta)$

$= (2\cos^2\theta-1)(4\cos^3\theta-3\cos\theta) - 2\sin^2\theta\cos\theta(3-4\sin^2\theta)$

$= (2\cos^2\theta-1)(4\cos^3\theta-3\cos\theta) - 2\cos\theta(1-\cos^2\theta)(3-4(1-\cos^2\theta))$

$= (2\cos^2\theta-1)(4\cos^3\theta-3\cos\theta) - 2\cos\theta(1-\cos^2\theta)(4\cos^2\theta-1)$

$= 8\cos^5\theta - 6\cos^3\theta - 4\cos^3\theta + 3\cos\theta - 2\cos\theta\{4\cos^2\theta - 1 - 4\cos^4\theta + \cos^2\theta\}$

$= 8\cos^5\theta - 6\cos^3\theta - 4\cos^3\theta + 3\cos\theta - 8\cos^3\theta + 2\cos\theta + 8\cos^5\theta - 2\cos^3\theta$

$= 16\cos^5\theta - 20\cos^3\theta + 5\cos\theta$

$= R.H.S.$

5. Prove that $\sin 4\alpha = 4\tan\alpha\left(\frac{1-\tan^2\alpha}{(1+\tan^2\alpha)^2}\right)$

$L.H.S = \sin 4\alpha = 2\sin 2\alpha \cdot \cos 2\alpha$

$= 2\left(\frac{2\tan\alpha}{1+\tan^2\alpha}\right)\left(\frac{1-\tan^2\alpha}{1+\tan^2\alpha}\right)$

$= 4\tan\alpha\left(\frac{1-\tan^2\alpha}{(1+\tan^2\alpha)^2}\right) = R.H.S.$

6. If $A+B=45^\circ$ Show that $(1+\tan A)(1+\tan B)=2$

$A+B=45^\circ$

$\tan(A+B) = \tan 45^\circ$

$\frac{\tan A+\tan B}{1-\tan A\tan B} = 1$

$\tan A+\tan B = 1-\tan A\tan B$

$\tan A+\tan B+\tan A\tan B = 1$

$1+\tan A+\tan B+\tan A\tan B = 1+1$

$(1+\tan A)+\tan B(1+\tan A) = 2$

$(1+\tan A)(1+\tan B) = 2$

Hence Proved.

7. Prove that $(1+\tan 1^\circ)(1+\tan 2^\circ)(1+\tan 3^\circ)\ldots(1+\tan 44^\circ)$ is a multiple of 4.

We know that

$A+B=45^\circ$

$(1+\tan A)(1+\tan B) = 2$

$(1+\tan 1^\circ)(1+\tan 44^\circ) = 2$

$(1+\tan 2^\circ)(1+\tan 43^\circ) = 2$

$\vdots \qquad \vdots$

$(1+\tan 22^\circ)(1+\tan 23^\circ) = 2$

$\therefore (1+\tan 1^\circ)(1+\tan 2^\circ)(1+\tan 3^\circ)\ldots(1+\tan 44^\circ)$

$= 2.2.2.$ up to 22 times

$= 2^{22} = (2^2)^{11} = 4^{11}$

which is multiple of 4.

8. Prove that $\tan\left(\frac{\pi}{4}+\theta\right) - \tan\left(\frac{\pi}{4}-\theta\right) = 2\tan 2\theta$

$$L.H.S = \tan\left(\frac{\pi}{4}+\theta\right) - \tan\left(\frac{\pi}{4}-\theta\right)$$

$$= \frac{1+\tan\theta}{1-\tan\theta} - \frac{1-\tan\theta}{1+\tan\theta}$$

$$= \frac{(1+\tan\theta)^2 - (1-\tan\theta)^2}{(1-\tan\theta)(1+\tan\theta)}$$

$$= \frac{4\tan\theta}{1-\tan^2\theta} \qquad \left[\because (a+b)^2-(a-b)^2 = 4ab\right]$$

$$= 2\frac{(2\tan\theta)}{1-\tan^2\theta}$$

$$= 2\tan 2\theta = R.H.S$$

---

9. Show that $\cot\left(7\frac{1}{2}\right)^\circ = \sqrt{2}+\sqrt{3}+\sqrt{4}+\sqrt{6}$

Let $\theta = 15^\circ$

$$\cot\theta/2 = \frac{\cos\theta/2}{\sin\theta/2} = \frac{2\cos^2\theta/2}{2\sin\theta/2\cos\theta/2}$$

$$= \frac{1+\cos\theta}{\sin\theta} = \frac{1+\cos 15^\circ}{\sin 15^\circ}$$

$$= \frac{1+\frac{\sqrt{3}+1}{2\sqrt{2}}}{\frac{\sqrt{3}-1}{2\sqrt{2}}} = \frac{2\sqrt{2}+\sqrt{3}+1}{\sqrt{3}-1} \times \frac{\sqrt{3}+1}{\sqrt{3}+1}$$

$$= \frac{2\sqrt{6}+2\sqrt{2}+3+\sqrt{3}+\sqrt{3}+1}{3-1}$$

$$= \frac{2\sqrt{6}+2\sqrt{2}+2\sqrt{3}+4}{2} = \sqrt{6}+\sqrt{2}+\sqrt{3}+2$$

$$= \sqrt{6}+\sqrt{2}+\sqrt{3}+\sqrt{4}$$

$$\therefore \cot 7\frac{1}{2}^\circ = \sqrt{2}+\sqrt{3}+\sqrt{4}+\sqrt{6}$$

Hence Proved.

11. Prove that $32\sqrt{3}\cdot \sin\frac{\pi}{48}\cdot\cos\frac{\pi}{48}\cdot\cos\frac{\pi}{24}\cdot\cos\frac{\pi}{12}\cdot\cos\frac{\pi}{6} = 3$

$$L.H.S = 32\sqrt{3}\,\sin\frac{\pi}{48}\cdot\cos\frac{\pi}{48}\cdot\cos\frac{\pi}{24}\cdot\cos\frac{\pi}{12}\cdot\cos\frac{\pi}{6}$$

$$= 16\sqrt{3}\left(2\sin\frac{\pi}{48}\cdot\cos\frac{\pi}{48}\right)\cdot\cos\frac{\pi}{24}\cdot\cos\frac{\pi}{12}\cdot\cos\frac{\pi}{6}$$

$$= 16\sqrt{3}\,\sin\frac{\pi}{24}\cdot\cos\frac{\pi}{24}\cdot\cos\frac{\pi}{12}\cdot\cos\frac{\pi}{6}$$

$$= 8\sqrt{3}\left[2\sin\frac{\pi}{24}\cdot\cos\frac{\pi}{24}\right]\cdot\cos\frac{\pi}{12}\cdot\cos\frac{\pi}{6}$$

$$= 8\sqrt{3}\,\sin\frac{\pi}{12}\cdot\cos\frac{\pi}{12}\cdot\cos\frac{\pi}{6}$$

$$= 4\sqrt{3}\left[2\sin\frac{\pi}{12}\cdot\cos\frac{\pi}{12}\right]\cdot\cos\frac{\pi}{6}$$

$$= 4\sqrt{3}\,\sin\frac{\pi}{6}\cdot\cos\frac{\pi}{6}$$

$$= 4\sqrt{3}\cdot\frac{1}{2}\cdot\frac{\sqrt{3}}{2} = 3 = R.H.S$$

3.21 A football player can kick a football from the ground level with an initial velocity of 80 ft/sec. Find the maximum horizontal distance the football travells and at what angle? [Take $g = 32$]

$u = 80$ ft/sec $\qquad g = 32$

Horizontal Range

$$\boxed{R = \frac{u^2\sin 2\alpha}{g}} = \frac{80^2\sin 2\alpha}{32}$$

$$= 10\times 20\,\sin 2\alpha$$

$$= 200\,\sin 2\alpha$$

when $\alpha = 45^\circ$

$$R = 200\sin 90 = 200 \text{ ft}.$$

∴ The maximum distance $= 200$ ft.

$$\boxed{\alpha = 45^\circ}$$

3.32 Prove that $\cos A \cos 2A \cos 2^2A \cdot \cos 2^3A \cdots \cos 2^{n-1}A = \dfrac{\sin 2^n A}{2^n \sin A}.$

$$L.H.S = \cos A \cos 2A \cos 2^2A \cdots \cos 2^{n-1}A$$

$$= \frac{1}{2\sin A}(2\sin A\cos A).\cos 2A.\cos 2^2A \cdots \cos 2^{n-1}A$$

$$= \frac{1}{2\sin A}.\sin 2A.\cos 2A.\cos 2^2A \cdots \cos 2^{n-1}A$$

$$= \frac{1}{2^2\sin A}(2\sin 2A\cos 2A).\cos 2^2A \cdots \cos 2^{n-1}A$$

$$= \frac{1}{2^2\sin A}.\sin 2^2A.\cos 2^2A \cdots \cos 2^{n-1}.A$$

Continuing the process, we get.

$$= \frac{\sin 2^n.A}{2^n.\sin A} = R.H.S$$

---

10. Prove that $(1+\sec 2\theta)(1+\sec 4\theta)\cdots(1+\sec 2^n\theta) = \tan 2^n\theta.\cot\theta.$

$$L.H.S = \left(1+\frac{1}{\cos 2\theta}\right)\left(1+\frac{1}{\cos 4\theta}\right)\cdots\frac{1}{1+\cos 2^n\theta}$$

$$= \frac{1+\cos 2\theta}{\cos 2\theta}.\frac{1+\cos 2^2\theta}{\cos 2^2\theta}.\frac{1+\cos 2^3\theta}{\cos 2^3\theta}\cdots\frac{1+\cos 2^n\theta}{\cos 2^n\theta}$$

$$= \frac{(2\cos^2\theta)(2\cos^2 2\theta)(2\cos^2 2^2\theta)\cdots(2\cos^2 2^{n-1}\theta)}{\cos 2\theta.\cos 2^2\theta\,\cos 2^3\theta\cdots\cos 2^n\theta}$$

$$= \frac{2^n\cos\theta(\cos\theta.\cos^2 2\theta.\cos^2 2^2\theta.\cos^2 2^3\theta\cdots\cos^2 2^{n-1}\theta)}{\cos 2\theta.\cos 2^2\theta.\cos 2^3\theta\cdots\cos 2^n\theta}$$

$$= \frac{2^n\cos\theta}{\cos 2^n\theta}\left[\cos\theta.\cos 2\theta.\cos 2^2\theta.\cos 2^3\theta\cdots\cos 2^{n-1}\theta\right]$$

$$= \frac{\cancel{2^n}\cos\theta}{\cos 2^n\theta}\left[\frac{\sin 2^n\theta}{\cancel{2^n}.\sin\theta}\right] \qquad [\text{Eg } 3.32]$$

$$= \tan 2^n\theta.\cot\theta$$

$$= R.H.S.$$

Product to Sum and Sum to Product Identities

$\sin(A+B) = \sin A \cos B + \cos A \sin B$
$\sin(A-B) = \sin A \cos B - \cos A \sin B$
$\cos(A+B) = \cos A \cos B - \sin A \sin B$
$\cos(A-B) = \cos A \cos B + \sin A \sin B$.

$\sin(A+B) + \sin(A-B) = 2\sin A \cos B$
$\sin(A+B) - \sin(A-B) = 2\cos A \sin B$
$\cos(A+B) + \cos(A-B) = 2\cos A \cos B$
$\cos(A+B) - \cos(A-B) = -2\sin A \sin B$

Let $A+B = C$ $\quad$ $A-B = D$

$A = \frac{C+D}{2}$ $\quad$ $B = \frac{C-D}{2}$

$\sin C + \sin D = 2\sin\frac{C+D}{2} \cdot \cos\frac{C-D}{2}$
$\sin C - \sin D = 2\cos\frac{C+D}{2} \cdot \sin\frac{C-D}{2}$
$\cos C + \cos D = 2\cos\frac{C+D}{2} \cdot \cos\frac{C-D}{2}$
$\cos C - \cos D = -2\sin\frac{C+D}{2} \cdot \sin\frac{C-D}{2}$

$\sin(60-A) \cdot \sin A \cdot \sin(60+A) = \frac{1}{4}\sin 3A$
$\cos(60-A) \cos A \cdot \cos(60+A) = \frac{1}{4}\cos 3A$
$\tan(60-A) \tan A \cdot \tan(60+A) = \tan 3A$

3.33 Express each of the following Product as a Sum or difference.

(i) $\sin 40^\circ \cos 30^\circ$

$$\sin A \cos B = \frac{1}{2}[\sin(A+B) + \sin(A-B)]$$

$$\sin 40 \cdot \cos 30^\circ = \frac{1}{2}[\sin(40+30^\circ) + \sin(40-30^\circ)]$$

$$= \frac{1}{2}(\sin 70^\circ + \sin 10^\circ)$$

(ii) $\cos 110^\circ \sin 55^\circ$

$$\cos A \sin B = \frac{1}{2}\left[\sin(A+B) - \sin(A-B)\right]$$

$$\cos 110^\circ \sin 55^\circ = \frac{1}{2}\left[\sin(110^\circ + 55^\circ) - \sin(110 - 55^\circ)\right]$$

$$= \frac{1}{2}\left[\sin 165^\circ - \sin 55^\circ\right]$$

(iii) $\sin \frac{x}{2} \cdot \cos \frac{3x}{2}$

$$\sin A \cos B = \frac{1}{2}\left[\sin(A+B) + \sin(A-B)\right]$$

$$\sin \frac{x}{2} \cdot \cos \frac{3x}{2} = \frac{1}{2}\left[\sin\left(\frac{x}{2} + \frac{3x}{2}\right) + \sin\left(\frac{x}{2} - \frac{3x}{2}\right)\right]$$

$$= \frac{1}{2}\left[\sin 2x + \sin(-x)\right]$$

$$= \frac{1}{2}\left[\sin 2x - \sin x\right] \quad \left[\because \sin(-x) = -\sin x\right]$$

3.34 Express each of the following sum or difference as a Product.

(i) $\sin 50^\circ + \sin 20^\circ$

$$\sin C + \sin D = 2\sin \frac{C+D}{2} \cdot \cos \frac{C-D}{2}$$

$$\sin 50^\circ + \sin 20^\circ = 2\sin \frac{50+20}{2} \cdot \cos \frac{50-20}{2}$$

$$= 2\sin 35^\circ \cdot \cos 15^\circ$$

(ii) $\cos 6\theta + \cos 2\theta$

$$\cos C + \cos D = 2\cos \frac{C+D}{2} \cdot \cos \frac{C-D}{2}$$

$$\cos 6\theta + \cos 2\theta = 2\cos \frac{6\theta + 2\theta}{2} \cdot \cos \frac{6\theta - 2\theta}{2}$$

$$= 2\cos 4\theta \cdot \cos 2\theta.$$

(iii) $\cos \frac{3x}{2} - \cos \frac{9x}{2}$

$$\cos C - \cos D = -2\sin \frac{C+D}{2} \cdot \sin \frac{C-D}{2}$$

$$\cos \frac{3x}{2} - \cos \frac{9x}{2} = -2\sin\left(\frac{\frac{3x}{2} + \frac{9x}{2}}{2}\right)\sin\left(\frac{\frac{3x}{2} - \frac{9x}{2}}{2}\right)$$

$$= -2\sin 3x \sin\left(-\frac{3x}{2}\right)$$

$$= 2\sin 3x \cdot \sin \frac{3x}{2}$$

$\left[\sin(-x) = -\sin x\right]$

3.35 Find the value of $\sin 34^\circ + \cos 64^\circ - \cos 4^\circ$

$$\sin 34^\circ + \cos 64^\circ - \cos 4^\circ = \sin 34^\circ - 2\sin \frac{64+4}{2} \cdot \sin \frac{64-4}{2}$$

$$= \sin 34^\circ - 2\sin 34^\circ \cdot \sin 30^\circ$$

$$= \sin 34^\circ - \cancel{2}\sin 34^\circ \cdot \frac{1}{\cancel{2}}$$

$$= \sin 34^\circ - \sin 34^\circ = 0.$$

3.36 Show that $\cos 36^\circ \cos 72^\circ \cos 108^\circ \cdot \cos 144^\circ = \frac{1}{16}$

$$L.H.S = \cos 36^\circ \cdot \cos 72^\circ \cos 108^\circ \cdot \cos 144^\circ$$

$$= \cos 36^\circ \sin 18^\circ \cos(90+18^\circ) \cos(180-36^\circ)$$

$$= \cos 36^\circ \sin 18^\circ (-\sin 18^\circ)(-\cos 36^\circ)$$

$$= \sin^2 18^\circ \cdot \cos^2 36^\circ \qquad \left[\cos 72^\circ = \sin 18^\circ\right]$$

$$= \left(\frac{\sqrt{5}-1}{4}\right)^2 \left(\frac{\sqrt{5}+1}{4}\right)^2 \qquad \left[\because \sin 18^\circ = \frac{\sqrt{5}-1}{4},\ \cos 36^\circ = \frac{\sqrt{5}+1}{4}\right.$$

$$= \frac{\left[(\sqrt{5}-1)(\sqrt{5}+1)\right]^2}{16 \times 16}$$

$$= \frac{16}{16(16)} = \frac{1}{16} = R.H.S$$

3.37 Simplify : $\dfrac{\sin 75^\circ - \sin 15^\circ}{\cos 75^\circ + \cos 15^\circ}$

$$\frac{\sin 75^\circ - \sin 15^\circ}{\cos 75^\circ + \cos 15^\circ} = \frac{\cancel{2}\cos \frac{75+15}{2} \cdot \sin \frac{75-15}{2}}{\cancel{2}\cos \frac{75+15}{2} \cdot \cos \frac{75-15}{2}}$$

$$= \frac{\cancel{2}\cancel{\cos 45^\circ} \sin 30^\circ}{\cancel{2}\cancel{\cos 45^\circ} \cos 30^\circ} = \tan 30^\circ = \frac{1}{\sqrt{3}}.$$

3.38 Show that $\cos 10^\circ \cos 30^\circ \cos 50^\circ \cos 70^\circ = \frac{3}{16}$

$$\boxed{\cos(60-A) \cdot \cos A \cdot \cos(60+A) = \frac{1}{4} \cdot \cos 3A}$$

$$L.H.S = \cos 30 \left\{ \cos 50^\circ \cos 10^\circ \cos 70^\circ \right\} \qquad \boxed{A = 10^\circ}$$

$$= \cos 30 \left\{ \frac{1}{4} \cos 3(10^\circ) \right\}$$

$$= \cos 30^\circ \left\{ \frac{1}{4} \cos 30^\circ \right\}$$

$$= \frac{\sqrt{3}}{2} \cdot \frac{1}{4} \cdot \frac{\sqrt{3}}{2} = \frac{3}{16} = R.H.S$$

Ex : 3.6

1 **Express each of the following as a sum or difference :**

(i) $\sin 35^\circ \cos 28^\circ$

$$\sin A \cos B = \frac{1}{2}\left[\sin(A+B) + \sin(A-B)\right]$$

$$\sin 35^\circ \cos 28^\circ = \frac{1}{2}\left[\sin(35+28) + \sin(35-28)\right]$$

$$= \frac{1}{2}\left[\sin 63 + \sin 7^\circ\right]$$

(ii) $\sin 4x \cos 2x$

$$\sin A \cos B = \frac{1}{2}\left[\sin(A+B) + \sin(A-B)\right]$$

$$\sin 4x \cos 2x = \frac{1}{2}\left[\sin(4x+2x) + \sin(4x-2x)\right]$$

$$= \frac{1}{2}\left[\sin 6x + \sin 2x\right]$$

(iii) $2\sin 10\theta \cos 2\theta$

$$2\sin A \cos B = \frac{1}{2}\left[\sin(A+B) + \sin(A-B)\right]$$

$$2\sin 10\theta \cos 2\theta = \frac{1}{2}\left[\sin(10\theta+2\theta) + \sin(10\theta-2\theta)\right]$$

$$= \frac{1}{2}\left[\sin 12\theta + \sin 8\theta\right]$$

(iv) $\cos 5\theta \cos 2\theta$

$$\cos A \cos B = \frac{1}{2}\left[\cos(A+B) + \cos(A-B)\right]$$

$$\cos 5\theta \cos 2\theta = \frac{1}{2}\left[\cos(5\theta+2\theta) + \cos(5\theta-2\theta)\right]$$

$$= \frac{1}{2}\left[\cos 7\theta + \cos 3\theta\right]$$

(v) $\sin 5\theta \sin 4\theta$

$$\sin A \sin B = \frac{1}{2}\left[\cos(A-B) - \cos(A+B)\right]$$

$$\sin 5\theta \sin 4\theta = \frac{1}{2}\left[\cos(5\theta-4\theta) - \cos(5\theta+4\theta)\right]$$

$$= \frac{1}{2}\left[\cos\theta + \cos 9\theta\right]$$

2. Express each of the following as a Product:

(i) $\sin 75^\circ - \sin 35^\circ$

$$\sin C - \sin D = 2\cos\frac{C+D}{2}\cdot\sin\frac{C-D}{2}$$

$$\sin 75^\circ - \sin 35^\circ = 2\cos\frac{75+35}{2}\cdot\sin\frac{75-35}{2}$$

$$= 2\cos 55^\circ\cdot\sin 20^\circ$$

(ii) $\cos 65^\circ + \cos 15^\circ$

$$\cos C + \cos D = 2\cos\frac{C+D}{2}\cdot\sin\frac{C-D}{2}$$

$$\cos 65^\circ + \cos 15^\circ = 2\cos\frac{65+15}{2}\cdot\cos\frac{65-15}{2}$$

$$= 2\cos 40^\circ\cdot\cos 25^\circ$$

(iii) $\sin 50^\circ + \sin 40^\circ$

$$\sin C + \sin D = 2\sin\frac{C+D}{2}\cdot\cos\frac{C-D}{2}$$

$$\sin 50^\circ + \sin 40^\circ = 2\sin\frac{50+40}{2}\cdot\cos\frac{50-40}{2}$$

$$= 2\sin 45^\circ\cdot\cos 5^\circ$$

(iv) $\cos 35^\circ - \cos 75^\circ$

$$\cos C - \cos D = -2\sin\frac{C+D}{2}\cdot\sin\frac{C-D}{2}$$

$$\cos 35^\circ - \cos 75^\circ = -2\sin\frac{35+75}{2}\cdot\sin\frac{35-75}{2}$$

$$= -2\sin 55^\circ\cdot\sin(-20)$$

$$= 2\sin 55^\circ\sin 20^\circ.$$

3. Show that $\sin 12^\circ\cdot\sin 48^\circ\cdot\sin 54^\circ = \frac{1}{8}$

$$L.H.S = \sin 12^\circ\cdot\sin 48^\circ\ \sin 54^\circ$$

$$= \frac{1}{2}\left[\cos(48-12) - \cos(48+12)\right]\cdot\sin 54^\circ$$

$$= \frac{1}{2}\left[\cos 36^\circ - \cos 60^\circ\right]\sin 54^\circ$$

$$= \frac{1}{2}\left[\frac{\sqrt{5}+1}{4} - \frac{1}{2}\right]\frac{\sqrt{5}+1}{4}$$

$$= \frac{1}{2}\left[\frac{\sqrt{5}+1-2}{4}\right]\left[\frac{\sqrt{5}+1}{4}\right]$$

$$= \frac{1}{2}\left[\frac{(\sqrt{5}-1)(\sqrt{5}+1)}{16}\right]$$

$$= \frac{1}{2}\left(\frac{4}{16}\right) = \frac{1}{8} = R.H.S$$

$$\boxed{\sin 54^\circ = \cos 36^\circ = \frac{\sqrt{5}+1}{4}}$$

4. Show that $\cos\frac{\pi}{15}\cdot\cos\frac{2\pi}{15}\cdot\cos\frac{3\pi}{15}\cdot\cos\frac{4\pi}{15}\cdot\cos\frac{5\pi}{15}\cos\frac{6\pi}{15}\cos\frac{7\pi}{15}=\frac{1}{128}$

$$\boxed{\cos(60-A)\ \cos A\cdot\cos(60+A)=\frac{1}{4}\cos 3A}$$

$$L.H.S=\cos\frac{\pi}{15}\cdot\cos\frac{2\pi}{15}\cdot\cos\frac{3\pi}{15}\cos\frac{4\pi}{15}\cos\frac{5\pi}{15}\cdot\cos\frac{6\pi}{15}\cdot\cos\frac{7\pi}{15}$$

$$=\cos 12^\circ\cos 24^\circ\cos 36^\circ\cos 48^\circ\cos 60^\circ\cos 72^\circ\cos 84^\circ$$

$$=\cos 60^\circ\{\cos 48^\circ\cdot\cos 12^\circ\cos 72^\circ\}\{\cos 36^\circ\cos 24^\circ\cos 84^\circ\}$$

$$=\cos 60^\circ\left\{\frac{1}{4}\cos 3(12^\circ)\right\}\left\{\frac{1}{4}\cos 3(24^\circ)\right\}$$

$$=\cos 60^\circ\left\{\frac{1}{4}\cos 36^\circ\right\}\left\{\frac{1}{4}\cos 72^\circ\right\}$$

$$=\frac{1}{2}\cdot\frac{1}{4}\cdot\frac{\sqrt{5}+1}{4}\cdot\frac{1}{4}\cdot\frac{\sqrt{5}-1}{4}$$

$$=\frac{1}{2\times16\times16}(\sqrt{5}+1)(\sqrt{5}-1)=\frac{\cancel{4}}{2\times\cancel{16}\times16}=\frac{1}{128}=R.H.S$$

5. Show that $\dfrac{\sin 8x\cdot\cos x-\sin 6x\cdot\cos 3x}{\cos 2x\cdot\cos x-\sin 3x\cdot\sin 4x}=\tan 2x$

$$L.H.S=\frac{\sin 8x\cos x-\sin 6x\cdot\cos 3x}{\cos 2x\cos x-\sin 3x\sin 4x}$$

$$=\frac{\frac{1}{2}(\sin 9x+\sin 7x)-\frac{1}{2}[\sin(9x+\sin 3x]}{\frac{1}{2}(\cos 3x+\cos x)+\frac{1}{2}(\cos 7x-\cos x)}$$

$$=\frac{\cancel{\sin 9x}+\sin 7x-\cancel{\sin 9x}-\sin 3x}{\cos 3x+\cancel{\cos x}+\cos 7x-\cancel{\cos x}}$$

$$=\frac{\sin 7x-\sin 3x}{\cos 7x+\cos 3x}$$

$$=\frac{\cancel{2}\cos\frac{7x+3x}{2}\cdot\sin\left(\frac{7x-3x}{2}\right)}{\cancel{2}\cos\frac{7x+3x}{2}\cdot\cos\left(\frac{7x-3x}{2}\right)}$$

$$=\frac{\cancel{\cos 5x}\sin 2x}{\cancel{\cos 5x}\cos 2x}=\tan 2x$$

$=R.H.S$

6. Show that $\frac{(\cos\theta-\cos3\theta)(\sin8\theta+\sin2\theta)}{(\sin5\theta-\sin\theta)(\cos4\theta-\cos6\theta)}=1$

$$L.H.S = \frac{(\cos\theta-\cos3\theta)(\sin8\theta+\sin2\theta)}{(\sin5\theta-\sin\theta)(\cos4\theta-\cos6\theta)}$$

$$= \frac{-2\sin\left(\frac{\theta+3\theta}{2}\right)\sin\left(\frac{\theta-3\theta}{2}\right).2\sin\left(\frac{8\theta+2\theta}{2}\right).\cos\frac{8\theta-2\theta}{2}}{2\cos\left(\frac{5\theta+\theta}{2}\right)\sin\left(\frac{5\theta-\theta}{2}\right).\left(-2\sin\left(\frac{4\theta+6\theta}{2}\right).\sin\left(\frac{4\theta-6\theta}{2}\right)\right)}$$

$$= \frac{-\cancel{4}\ \cancel{\sin2\theta}.\cancel{\sin(-\theta)}\ \cancel{\sin5\theta}.\cancel{\cos3\theta}}{-\cancel{4}\ \cancel{\cos3\theta}.\cancel{\sin2\theta}.\cancel{\sin5\theta}.\cancel{\sin(-\theta)}} = 1$$

$= R.H.S$

7. $\sin x+\sin2x+\sin3x = \sin2x(1+2\cos x)$

$L.H.S = \sin x+\sin2x+\sin3x$

$= \sin2x + (\sin x+\sin3x)$

$= \sin2x + \left[2\sin\frac{x+3x}{2}.\cos\frac{x-3x}{2}\right]$

$= \sin2x + (2\sin2x.\cos(-x)]$

$= \sin2x + 2\sin2x\cos x$

$= \sin2x(1+2\cos x) = R.H.S$

8. Prove that $\frac{\sin4x+\sin2x}{\cos4x+\cos2x} = \tan3x$

$$L.H.S = \frac{\sin4x+\sin2x}{\cos4x+\cos2x}$$

$$= \frac{2\sin\frac{4x+2x}{2}.\cos\left(\frac{4x-2x}{2}\right)}{2\cos\left(\frac{4x+2x}{2}\right).\cos\left(\frac{4x-2x}{2}\right)}$$

$$= \frac{2\sin3x.\cos x}{2\cos3x.\cos x} = \frac{\sin3x}{\cos3x}$$

$= \tan3x = R.H.S$

9. Prove that $1+\cos 2x+\cos 4x+\cos 6x = 4\cos x\cos 2x\cos 3x$

$$\text{L.H.S} = 1+\cos 2x+\cos 4x+\cos 6x$$
$$= 2\cos^2 x + 2\cos\left(\frac{6x+4x}{2}\right)\cdot\cos\left(\frac{6x-4x}{2}\right)$$
$$= 2\cos^2 x + 2\cos 5x\cdot\cos x$$
$$= 2\cos x\,(\cos x+\cos 5x)$$
$$= 2\cos x\left\{2\cos\frac{5x+x}{2}\cdot\cos\frac{5x-x}{2}\right\}$$
$$= 2\cos x\{2\cos 3x\cdot\cos 2x\}$$
$$= 4\cos x\cos 2x\cos 3x$$
$$= \text{R.H.S}$$

10. Prove that $\sin\frac{\theta}{2}\sin\frac{7\theta}{2}+\sin\frac{3\theta}{2}\sin\frac{11\theta}{2} = \sin 2\theta\cdot\sin 5\theta$

$$\text{L.H.S} = \sin\frac{\theta}{2}\sin\frac{7\theta}{2}+\sin\frac{3\theta}{2}\cdot\sin\frac{11\theta}{2}$$
$$= \frac{1}{2}\left[\cos\left(\frac{\theta}{2}-\frac{7\theta}{2}\right)-\cos\left(\frac{\theta}{2}+\frac{7\theta}{2}\right)\right]+\frac{1}{2}\left[\cos\left(\frac{3\theta}{2}-\frac{11\theta}{2}\right)-\cos\left(\frac{3\theta}{2}+\frac{11\theta}{2}\right)\right]$$
$$= \frac{1}{2}\left[\cos(-3\theta)-\cos 4\theta\right]+\frac{1}{2}\left(\cos(-4\theta)-\cos 7\theta\right)$$
$$= \frac{1}{2}\left[\cos 3\theta-\cancel{\cos 4\theta}+\cancel{\cos 4\theta}-\cos 7\theta\right]$$
$$= \frac{1}{2}\left[\cos 3\theta-\cos 7\theta\right]$$
$$= \frac{1}{2}\left[-2\sin\left(\frac{3\theta+7\theta}{2}\right)\cdot\sin\left(\frac{3\theta-7\theta}{2}\right)\right]$$
$$= -1\sin 5\theta\sin(-2\theta)$$
$$= \sin 5\theta\cdot\sin 2\theta = \sin 2\theta\sin 5\theta = \text{R.H.S.}$$

11. Prove that $\cos(30-A)\cos(30+A)+\cos(45-A)\cos(45+A) = \cos 2A+\frac{1}{4}$

$$\text{L.H.S} = \cos(30-A)\cos(30+A)+\cos(45-A)\cos(45+A)$$
$$= \frac{1}{2}\left[\cos(30-\cancel{A}+30+\cancel{A})+\cos(\cancel{30}-A-\cancel{30}-A)\right]+\frac{1}{2}\left[\cos(45-\cancel{A}+45+\cancel{A})+\cos(\cancel{45}-A-\cancel{45}-A)\right]$$
$$= \frac{1}{2}\left[\cos 60^\circ+\cos(-2A)+\cos 90^\circ+\cos(-2A)\right]$$
$$= \frac{1}{2}\left[\cos 60^\circ+\cos 2A+0+\cos 2A\right]$$
$$= \frac{1}{2}\left[\frac{1}{2}+2\cos 2A\right] = \frac{1}{4}+\cos 2A$$
$$= \text{R.H.S.}$$

Another method :

$\cos(30+A)\cos(30-A) + \cos(45+A)\cos(45-A)$

$= \cos^2 30^\circ - \sin^2 A + \cos^2 45^\circ - \sin^2 A$

$= \frac{3}{4} - \sin^2 A + \frac{1}{2} - \sin^2 A$

$= \frac{5}{4} - 2\sin^2 A$

$= \frac{1}{4} + 1 - 2\sin^2 A = \frac{1}{4} + \cos 2A$

$= R.H.S$

$$\boxed{\cos(A+B)\cdot\cos(A-B) = \cos^2 A - \sin^2 B.}$$

12. Prove that $\dfrac{\sin x + \sin 3x + \sin 5x + \sin 7x}{\cos x + \cos 3x + \cos 5x + \cos 7x} = \tan 4x$

$L.H.S = \dfrac{\sin x + \sin 3x + \sin 5x + \sin 7x}{\cos x + \cos 3x + \cos 5x + \cos 7x}$

$= \dfrac{(\sin x + \sin 5x) + (\sin 3x + \sin 7x)}{(\cos x + \cos 5x) + (\cos 3x + \cos 7x)}$

$= \dfrac{2\sin 3x \cdot \cos 2x + 2\sin 5x \cdot \cos 2x}{2\cos 3x \cdot \cos 2x + 2\cos 5x \cos 2x}$

$= \dfrac{\cancel{2\cos 2x}(\sin 3x + \sin 5x)}{\cancel{2\cos 2x}(\cos 3x + \cos 5x)}$

$= \dfrac{2\sin 4x \cdot \cos x}{2\cos 4x \cdot \cos x} = \dfrac{\sin 4x}{\cos 4x} = \tan 4x$

$= R.H.S$

13. Prove that $\dfrac{\sin(4A-2B) + \sin(4B-2A)}{\cos(4A-2B) + \cos(4B-2A)} = \tan(A+B)$

$L.H.S = \dfrac{\sin(4A-2B) + \sin(4B-2A)}{\cos(4A-2B) + \cos(4B-2A)}$

$= \dfrac{\cancel{2}\sin\left(\frac{4A-2B+4B-2A}{2}\right)\cdot\cos\left(\frac{4A-2B-4B+2A}{2}\right)}{\cancel{2}\cos\left(\frac{4A-2B+4B-2A}{2}\right)\cdot\cos\left(\frac{4A-2B-4B+2A}{2}\right)}$

$= \dfrac{\sin\left(\frac{2A+2B}{2}\right)\cancel{\cos\left(\frac{6A-6B}{2}\right)}}{\cos\left(\frac{2A+2B}{2}\right)\cancel{\cos\left(\frac{6A-6B}{2}\right)}}$

$= \tan(A+B)$

$= R.H.S$

14. Show that $\cot(A+15^\circ) - \tan(A-15^\circ) = \dfrac{4\cos 2A}{1+2\sin 2A}$

$$L.H.S = \cot(A+15^\circ) - \tan(A-15^\circ)$$

$$= \frac{\cos(A+15^\circ)}{\sin(A+15^\circ)} - \frac{\sin(A-15^\circ)}{\cos(A-15^\circ)}$$

$$= \frac{\cos(A+15^\circ)\cos(A-15^\circ) - \sin(A+15^\circ)\sin(A-15^\circ)}{\sin(A+15^\circ)\cos(A-15^\circ)}$$

$$= \frac{(\cos^2 A - \sin^2 15^\circ) - (\sin^2 A - \sin^2 15^\circ)}{\frac{1}{2}[\sin(A+\cancel{15^\circ}+A-\cancel{15^\circ}) + \sin(\cancel{A}+15^\circ-\cancel{A}+15^\circ)]}$$

$$= \frac{\cos^2 A - \cancel{\sin^2 15^\circ} - \sin^2 A + \cancel{\sin^2 15^\circ}}{\frac{1}{2}(\sin 2A + \sin 30^\circ)}$$

$$= \frac{2(\cos^2 A - \sin^2 A)}{\sin 2A + \frac{1}{2}} = \frac{2\cos 2A}{2\sin 2A + 1} \times \frac{2}{1}$$

$$= \frac{4\cos 2A}{1+2\sin 2A} = R.H.S.$$

Conditional Trigonometric Identities:

3.39 If $A+B+C = \pi$. Prove the following:

(i) $\cos A + \cos B + \cos C = 1 + 4\sin\frac{A}{2}.\sin\frac{B}{2}.\sin\frac{C}{2}$

(ii) $\sin\left(\frac{A}{2}\right)\sin\left(\frac{B}{2}\right)\sin\left(\frac{C}{2}\right) \leq \frac{1}{8}$

(iii) $1 < \cos A + \cos B + \cos C \leq \frac{3}{2}$

$$A+B+C = \pi \qquad \frac{A+B+C}{2} = \frac{\pi}{2}$$

(i) $\cos A + \cos B + \cos C$

$$= 2\cos\frac{A+B}{2}.\cos\frac{A-B}{2} + \cos C$$

$$= 2\cos\left(\frac{\pi}{2} - \frac{C}{2}\right).\cos\frac{A-B}{2} + 1 - 2\sin^2\frac{C}{2}$$

$$= 2\sin\frac{C}{2}.\cos\frac{A-B}{2} + 1 - 2\sin^2\frac{C}{2}$$

$$= 1 + 2\sin\frac{C}{2}\left[\cos\frac{A-B}{2} - \sin\frac{C}{2}\right]$$

$$= 1 + 2\sin\frac{C}{2}\left[\cos\frac{A-B}{2} - \sin\left(\frac{\pi}{2} - \frac{A+B}{2}\right)\right]$$

$$= 1 + 2\sin\frac{C}{2}\left[\cos\frac{A-B}{2} - \cos\frac{A+B}{2}\right]$$

$$= 1 + 2\sin\frac{C}{2}\left\{2\sin\frac{B}{2}.\sin\frac{A}{2}\right\}$$

$$= 1 + 4\sin\frac{A}{2}\sin\frac{B}{2}\sin\frac{C}{2}.$$

(ii) Let $u = \sin\frac{A}{2}.\sin\frac{B}{2}.\sin\frac{C}{2}$

$$= -\frac{1}{2}\left[\cos\frac{A+B}{2} - \cos\frac{A-B}{2}\right].\sin\frac{C}{2}$$

$$= -\frac{1}{2}\left[\cos\frac{A+B}{2} - \cos\frac{A-B}{2}\right].\sin\left(\frac{\pi}{2} - \frac{A+B}{2}\right)$$

$$u = -\frac{1}{2}\left[\cos\frac{A+B}{2} - \cos\frac{A-B}{2}\right]\cos\left(\frac{A+B}{2}\right)$$

$$2u = -\cos^2\frac{A+B}{2} + \cos\frac{A-B}{2}.\cos\frac{A+B}{2}$$

$$\cos^2\frac{A+B}{2} + \cos\frac{A-B}{2}.\cos\frac{A+B}{2} + 2u = 0$$

It is a quadratic equation on $\cos\frac{A+B}{2}$

$\cos\frac{A+B}{2}$ is a real number

It has solution.

Discriminant ≥ 0

$b^2 - 4ac \geq 0$

$a = 1$

$b = \cos\left(\frac{A-B}{2}\right)$

$c = 2u$

$$\cos^2\left(\frac{A-B}{2}\right) - 8u \geq 0$$

$$8u \leq \cos^2\frac{A-B}{2}$$

$$u \leq \frac{1}{8}\cos^2\frac{A-B}{2}$$

$$u \leq \frac{1}{8}$$

$$\therefore \sin\frac{A}{2}.\sin\frac{B}{2}.\sin\frac{C}{2} \leq \frac{1}{8}$$

(iii) From (i) and (ii)

$$\cos A + \cos B + \cos C > 1$$

$$\cos A + \cos B + \cos C \leq 1 + 4\left(\frac{1}{8}\right)$$

$$\cos A + \cos B + \cos C \leq \frac{3}{2}$$

$$\therefore 1 < \cos A + \cos B + \cos C \leq \frac{3}{2}$$

Hence Proved.

3.40 Prove that $\sin\frac{A}{2}+\sin\frac{B}{2}+\sin\frac{C}{2}$
$$=1+4\sin\left(\frac{\pi-A}{4}\right)\sin\left(\frac{\pi-B}{4}\right).\sin\left(\frac{\pi-C}{4}\right)$$
If $A+B+C=\pi$

$A+B+C=\pi$

$\frac{A+B+C}{2}=\frac{\pi}{2}$

$\frac{A+B}{2}=\frac{\pi}{2}-\frac{C}{2}$

$\frac{A+B}{4}=\frac{\pi}{4}-\frac{C}{4}$

$$L.H.S=\sin\frac{A}{2}+\sin\frac{B}{2}+\sin\frac{C}{2}$$
$$=\cos\left(\frac{\pi}{2}-\frac{A}{2}\right)+\cos\left(\frac{\pi}{2}-\frac{B}{2}\right)+\cos\left(\frac{\pi}{2}-\frac{C}{2}\right)$$
$$=\left[2\cos\frac{\frac{\pi}{2}-\frac{A}{2}+\frac{\pi}{2}-\frac{B}{2}}{2}.\cos\left(\frac{\frac{\pi}{2}-\frac{A}{2}-\frac{\pi}{2}+\frac{B}{2}}{2}\right)\right]+\left(1-2\sin^2\left(\frac{\frac{\pi}{2}-\frac{C}{2}}{2}\right)\right)$$
$$=2\cos\left(\frac{\pi}{2}-\frac{A+B}{4}\right).\cos\left(\frac{B-A}{4}\right)+1-2\sin^2\left(\frac{\pi}{4}-\frac{C}{4}\right)$$
$$=2\sin\left(\frac{\pi}{4}-\frac{C}{4}\right).\cos\frac{B-A}{4}+1-2\sin^2\left(\frac{\pi}{4}-\frac{C}{4}\right)$$
$$=1+2\sin\left(\frac{\pi}{4}-\frac{C}{4}\right)\left\{\cos\frac{B-A}{4}-\sin\left(\frac{\pi}{4}-\frac{C}{4}\right)\right\}$$
$$=1+2\sin\left(\frac{\pi-C}{4}\right)\left\{\cos\frac{B-A}{4}-\sin\left(\frac{A+B}{4}\right)\right\}$$
$$=1+2\sin\left(\frac{\pi-C}{4}\right)\left\{\cos\frac{B-A}{4}-\cos\left(\frac{\pi}{2}-\frac{A+B}{4}\right)\right\}$$
$$=1+2\sin\left(\frac{\pi-C}{4}\right)\left\{2\sin\left(\frac{\frac{B-A}{4}+\frac{\pi}{2}-\frac{A+B}{4}}{2}\right)\sin\left(\frac{\frac{\pi}{2}-\frac{A+B}{4}-\frac{B-A}{4}}{2}\right)\right.$$
$$=1+2\sin\frac{\pi-C}{4}\left\{2\sin\left(\frac{\pi-A}{4}\right).\sin\left(\frac{\pi-B}{4}\right)\right.$$
$$=1+4\sin\frac{\pi-A}{4}.\sin\frac{\pi-B}{4}.\sin\frac{\pi-C}{4}$$
$= R.H.S.$

3.41 If $A+B+C=\pi$ Prove that $\cos^2A+\cos^2B+\cos^2C$
$=1-2\cos A\cos B\cos C$

$A+B+C=\pi$

$A+B=\pi-C$

$$L.H.S=\cos^2A+\cos^2B+\cos^2C$$
$$=\frac{1+\cos2A}{2}+\frac{1+\cos2B}{2}+\frac{1+\cos2C}{2}$$
$$=\frac{3}{2}+\frac{1}{2}\left[\cos2A+\cos2B+\cos2C\right]$$
$$=\frac{3}{2}+\frac{1}{2}\left\{2\cos(A+B).\cos(A-B)+2\cos^2C-1\right\}$$
$$=\frac{3}{2}+\cos(A+B)\cos(A-B)+\cos^2C-\frac{1}{2}$$
$$=1+\cos(\pi-C)\cos(A-B)+\cos^2C$$
$$=1-\cos C.\cos(A-B)+\cos^2C$$

$= 1 - \cos C \{ \cos(A-B) - \cos C \}$

$= 1 - \cos C \{ \cos(A-B) - \cos(\pi - \overline{A+B}) \}$

$= 1 - \cos C \{ \cos(A-B) + \cos(A+B) \}$

$= 1 - \cos C \{ 2\cos A \cos B \}$

$= 1 - 2\cos A \cos B \cos C = R.H.S.$

Ex: 3.7.

1 If $A+B+C = \pi$ Prove that

(i) $\sin 2A + \sin 2B + \sin 2C = 4 \sin A \sin B \sin C$

$A+B+C=\pi$

$A+B = \pi - C$

$C = \pi - \overline{A+B}$

$L.H.S = \sin 2A + \sin 2B + \sin 2C$

$= 2\sin(A+B)\cos(A-B) + 2\sin C \cos C$

$= 2\sin(\pi - C)\cos(A-B) + 2\sin C \cos C$

$= 2\sin C \cdot \cos(A-B) + 2\sin C \cos C$

$= 2\sin C \{ \cos(A-B) + \cos C \}$

$= 2\sin C \{ \cos(A-B) + \cos(\pi - \overline{A+B}) \}$

$= 2\sin C \{ \cos(A-B) - \cos(A+B) \}$

$= 2\sin C (2\sin A \sin B)$

$= 4\sin A \sin B \sin C = R.H.S$

(ii) $\cos A + \cos B - \cos C = -1 + 4\cos\frac{A}{2} \cdot \cos\frac{B}{2} \cdot \sin\frac{C}{2}$

$A+B+C=\pi$

$\frac{A+B}{2} = \frac{\pi}{2} - \frac{C}{2}$

$\frac{C}{2} = \frac{\pi}{2} - \frac{A+B}{2}$

$L.H.S = \cos A + \cos B - \cos C$

$= 2\cos\frac{A+B}{2} \cdot \cos\left(\frac{A-B}{2}\right) - \cos C$

$= 2\cos\left(\frac{\pi}{2} - \frac{C}{2}\right) \cdot \cos\left(\frac{A-B}{2}\right) - \left(1 - 2\sin^2\frac{C}{2}\right)$

$= -1 + 2\sin\frac{C}{2} \cos\frac{A-B}{2} + 2\sin^2\frac{C}{2}$

$= -1 + 2\sin\frac{C}{2} \left\{ \cos\frac{A-B}{2} + \sin\frac{C}{2} \right\}$

$= -1 + 2\sin\frac{C}{2} \left\{ \cos\frac{A-B}{2} + \sin\left(\frac{\pi}{2} - \frac{A+B}{2}\right) \right\}$

$= -1 + 2\sin\frac{C}{2} \left[\cos\frac{A-B}{2} + \cos\frac{A+B}{2} \right]$

$= -1 + 2\sin\frac{C}{2} \left[2\cos\frac{A}{2} \cdot \cos\frac{B}{2} \right]$

$= -1 + 4\cos\frac{A}{2} \cdot \cos\frac{B}{2} \cdot \sin\frac{C}{2}$

$= R.H.S.$

(iii) $\sin^2 A + \sin^2 B + \sin^2 C = 2 + 2\cos A \cos B \cos C$

$$L.H.S = \sin^2 A + \sin^2 B + \sin^2 C$$
$$= \frac{1-\cos 2A}{2} + \frac{1-\cos 2B}{2} + \frac{1-\cos 2C}{2}$$
$$= \frac{3}{2} - \frac{1}{2}\left[\cos 2A + \cos 2B + \cos 2C\right]$$
$$= \frac{3}{2} - \frac{1}{2}\left\{2\cos(A+B)\cos(A-B) + 2\cos^2 C - 1\right\}$$
$$= \frac{3}{2} - \frac{1}{2}\left[2\cos(A+B)\cos A-B\right] - \cos^2 C + \frac{1}{2}$$
$$= 2 - \frac{1}{2}\left[2\cos(\pi - C)\cos(A-B)\right] - \cos^2 C$$
$$= 2 - \left[-\cos C \cdot \cos(A-B)\right] - \cos^2 C$$
$$= 2 + \cos C \cos(A-B) - \cos^2 C$$
$$= 2 + \cos C\left[\cos(A-B) - \cos C\right]$$
$$= 2 + \cos C\left[\cos(A-B) - \cos(\pi - \overline{A+B})\right]$$
$$= 2 + \cos C\left[\cos(A-B) + \cos(A+B)\right]$$
$$= 2 + \cos C\left[2\cos A \cos B\right]$$
$$= 2 + 2\cos A \cos B \cos C = R.H.S$$

(iv) $\sin^2 A + \sin^2 B - \sin^2 C = 2\sin A \sin B \cos C$

$$L.H.S = \sin^2 A + \sin^2 B - \sin^2 C$$
$$= \frac{1-\cos 2A}{2} + \frac{1-\cos 2B}{2} - \frac{1-\cos 2C}{2}$$
$$= \frac{1}{2} + \cancel{\frac{1}{2}} - \cancel{\frac{1}{2}} - \frac{1}{2}\left[\cos 2A + \cos 2B - \cos 2C\right]$$
$$= \frac{1}{2} - \frac{1}{2}\left[2\cos(A+B)\cdot\cos(A-B) - 2\cos^2 C + 1\right]$$
$$= \frac{1}{2} - \frac{1}{2}\left[2\cos(\pi - C)\cdot\cos(A-B) - 2\cos^2 C + 1\right]$$
$$= \cancel{\frac{1}{2}} - \left[\cos(\pi - C)\cdot\cos(A-B)\right] + \cos^2 C - \cancel{\frac{1}{2}}$$
$$= \cos C \cos(A-B) + \cos^2 C$$
$$= \cos C\left[\cos(A-B) + \cos C\right]$$
$$= \cos C\left[\cos(A-B) + \cos(\pi - \overline{A+B})\right]$$
$$= \cos C\left[\cos(A-B) - \cos(A+B)\right]$$
$$= \cos C\left[2\sin A \sin B\right]$$
$$= 2\sin A \sin B \cos C = R.H.S$$

(v) $\tan\frac{A}{2}\tan\frac{B}{2}+\tan\frac{B}{2}\tan\frac{C}{2}+\tan\frac{C}{2}\tan\frac{A}{2}=1.$

$A+B+C=180^\circ \qquad \frac{A+B}{2}=90-\frac{C}{2}$

$$\frac{A+B}{2}=90-\frac{C}{2}$$

$$\tan\left(\frac{A+B}{2}\right)=\tan\left(90-\frac{C}{2}\right)$$

$$\frac{\tan\frac{A}{2}+\tan\frac{B}{2}}{1-\tan\frac{A}{2}.\tan\frac{B}{2}}=\cot\frac{C}{2}=\frac{1}{\tan\frac{C}{2}}$$

$$\tan\frac{C}{2}\left(\tan\frac{A}{2}+\tan\frac{B}{2}\right)=1-\tan A/2\tan\frac{B}{2}$$

$$\tan\frac{C}{2}\tan\frac{A}{2}+\tan\frac{C}{2}.\tan\frac{B}{2}=1-\tan\frac{A}{2}\tan\frac{B}{2}$$

$$\therefore \tan\frac{A}{2}\tan\frac{B}{2}+\tan\frac{B}{2}\tan\frac{C}{2}+\tan\frac{C}{2}.\tan\frac{A}{2}=1$$

Hence Proved.

(vi) $\sin A+\sin B+\sin C=4\cos\frac{A}{2}\cos\frac{B}{2}.\cos\frac{C}{2}$

$A+B+C=\pi$

$\frac{A+B}{2}=\frac{\pi}{2}-\frac{C}{2}$

$\frac{C}{2}=\frac{\pi}{2}-\frac{A+B}{2}$

$$\begin{aligned}
L.H.S &= \sin A+\sin B+\sin C\\
&= 2\sin\frac{A+B}{2}.\cos\frac{A-B}{2}+\sin C\\
&= 2\sin\frac{A+B}{2}.\cos\frac{A-B}{2}+2\sin\frac{C}{2}.\cos\frac{C}{2}\\
&= 2\sin\left(\frac{\pi}{2}-\frac{C}{2}\right).\cos\frac{A-B}{2}+2\sin\frac{C}{2}.\cos\frac{C}{2}\\
&= 2\cos\frac{C}{2}.\cos\frac{A-B}{2}+2\sin\frac{C}{2}.\cos\frac{C}{2}\\
&= 2\cos\frac{C}{2}\left\{\cos\frac{A-B}{2}+\sin\frac{C}{2}\right\}\\
&= 2\cos\frac{C}{2}\left\{\cos\frac{A-B}{2}+\sin\left(\frac{\pi}{2}-\frac{A+B}{2}\right)\right\}\\
&= 2\cos\frac{C}{2}\left[\cos\frac{A-B}{2}+\cos\frac{A+B}{2}\right\}\\
&= 2\cos\frac{C}{2}\left[2\cos\frac{A}{2}.\cos\frac{B}{2}\right]\\
&= 4\cos\frac{A}{2}.\cos\frac{B}{2}.\cos\frac{C}{2}\\
&= R.H.S.
\end{aligned}$$

(vii) $\sin(B+C-A)+\sin(C+A-B)+\sin(A+B-C) = 4\sin A\sin B\sin C$

$A+B+C=\pi$

$A+B=\pi-C \qquad B+C=\pi-A \qquad C+A=\pi-B$

$$\text{L.H.S} = \sin(\pi-A-A)+\sin(\pi-B-B)+\sin(\pi-C-C)$$
$$= \sin(\pi-2A)+\sin(\pi-2B)+\sin(\pi-2C)$$
$$= \sin 2A+\sin 2B+\sin 2C$$
$$= 2\sin(A+B)\cos(A-B)+2\sin C\cos C$$
$$= 2\sin(\pi-C)\cos(A-B)+2\sin C\cos C$$
$$= 2\sin C\cos(A-B)+2\sin C\cos C$$
$$= 2\sin C(\cos(A-B)+\cos C)$$
$$= 2\sin C[\cos(A-B)+\cos(\pi-\overline{A+B})]$$
$$= 2\sin C[\cos(A-B)-\cos(A+B)]$$
$$= 2\sin C[2\sin A\sin B]$$
$$= 4\sin A\sin B\sin C$$
$$= \text{R.H.S}$$

2. If $A+B+C=2S$ then Prove that

$\sin(S-A)\cdot\sin(S-B)+\sin S\sin(S-C)=\sin A\sin B$

$$\text{L.H.S} = \sin(S-A)\cdot\sin(S-B)+\sin C\cdot\sin(S-C)$$
$$= \frac{1}{2}\left\{\cos[(S-A)-(S-B)]-\cos(S-A+S-B)\right] + \frac{1}{2}[\cos(S-\overline{S-C})-\cos(S+S-C)]$$
$$= \frac{1}{2}[\cos(B-A)-\cos(2S-\overline{A+B})]+\frac{1}{2}[\cos C-\cos(2S-C)]$$
$$= \frac{1}{2}[\cos(A-B)-\cos(C)]+\frac{1}{2}[\cos C-\cos(A+B)]$$
$$= \frac{1}{2}[\cos(A-B)-\cancel{\cos C}+\cancel{\cos C}-\cos(A+B)]$$
$$= \frac{1}{2}[\cos(A-B)-\cos(A+B)]$$
$$= \frac{1}{2}[2\sin A\sin B] = \sin A\sin B = \text{R.HS}$$

$A+B+C=2S$

$C=2S-\overline{A+B} \qquad A+B=2S-C$

$\sin A\sin B=\frac{1}{2}[\cos(A-B)-\cos(A+B)]$

3. If $x+y+z = xyz$ then Prove that

$$\frac{2x}{1-x^2} + \frac{2y}{1-y^2} = \frac{2z}{1-z^2} = \frac{2x}{1-x^2} \cdot \frac{2y}{1-y^2} \cdot \frac{2z}{1-z^2}$$

Let $x = \tan A \quad y = \tan B \quad z = \tan C$

$$\frac{2x}{1-x^2} = \frac{2\tan A}{1-\tan^2 A} = \tan 2A \qquad \text{lly} \quad \frac{2y}{1-y^2} = \tan 2B$$

$$\frac{2z}{1-z^2} = \tan 2C$$

Given that $x+y+z = xyz$

$$\boxed{\begin{gathered}\tan A + \tan B + \tan C = \tan A \tan B \tan C \\ \Rightarrow A+B+C = 180^\circ\end{gathered}}$$

multiply by 2

$$2A+2B+2C = 360^\circ$$

$$2A+2B = 360 - 2C$$

$$\tan(2A+2B) = \tan(360-2C)$$

$$\frac{\tan 2A + \tan 2B}{1 - \tan 2A \tan B} = -\tan 2C$$

$$\tan 2A + \tan 2B = -\tan 2C\,(1 - \tan 2A \tan 2B)$$

$$\tan 2A + \tan 2B + \tan 2C = \tan 2A \tan 2B \tan 2C$$

$$\frac{2x}{1-x^2} + \frac{2y}{1-y^2} + \frac{2z}{1-z^2} = \frac{2x}{1-x^2} \cdot \frac{2y}{1-y^2} \cdot \frac{2z}{1-z^2}$$

Hence Proved.

4. If $A+B+C = \frac{\pi}{2}$ Prove the following:

(i) $\sin 2A + \sin 2B + \sin 2C = 4\cos A \cos B \cos C$

$$\begin{aligned}
L.H.S &= \sin 2A + \sin 2B + \sin 2C \\
&= 2\sin(A+B)\cos(A-B) + 2\sin C\cos C \\
&= 2\sin\left(\tfrac{\pi}{2} - C\right)\cos(A-B) + 2\sin C \cos C \\
&= 2\cos C \cdot \cos(A-B) + 2\sin C \cos C \\
&= 2\cos C\left[\cos(A-B) + \sin\left(\tfrac{\pi}{2} - \overline{A+B}\right)\right] \\
&= 2\cos C\left[\cos(A-B) + \cos(A+B)\right] \\
&= 2\cos C\left[2\cos A \cos B\right] \\
&= 4\cos A \cos B \cos C \\
&= R.H.S.
\end{aligned}$$

(ii) $\cos 2A + \cos 2B + \cos 2C = 1 + 4\sin A \sin B \sin C$

$A+B+C = \frac{\pi}{2}$

$A+B = \frac{\pi}{2} - \frac{C}{2}$

$L.H.S = \cos 2A + \cos 2B + \cos 2C$

$= 2\cos(A+B)\cos(A-B) + \cos 2C$

$= 2\cos(\frac{\pi}{2} - C)\cos(A-B) + 1 - 2\sin^2 C$

$= 2\sin C \cos(A-B) + 1 - 2\sin^2 C$

$= 1 + 2\sin C\,[\cos(A-B) - \sin C]$

$= 1 + 2\sin C\,[\cos(A-B) - \sin(\frac{\pi}{2} - \overline{A+B})]$

$= 1 + 2\sin C\,[\cos(A-B) - \cos(A+B)]$

$= 1 + 2\sin C\,[2\sin A \sin B]$

$= 1 + 4\sin A \sin B \sin C = R.H.S.$

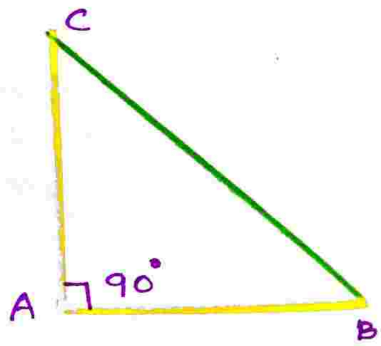
---

5. If ΔABC is a right angle and if $\angle A = \frac{\pi}{2}$ then Prove that

(i) $\cos^2 B + \cos^2 C = 1$ (ii) $\sin^2 B + \sin^2 C = 1$

(iii) $\cos B - \cos C = -1 + 2\sqrt{2}\cos\frac{B}{2}\cdot\sin\frac{C}{2}$

(i) $\cos^2 B + \cos^2 C = 1$



$\angle A = 90^\circ$

$\cos B = \frac{AB}{BC}$ $\cos C = \frac{AC}{BC}$

$\cos^2 B + \cos^2 C = \frac{AB^2}{BC^2} + \frac{AC^2}{BC^2}$

$= \frac{AB^2 + AC^2}{BC^2} = \frac{BC^2}{BC^2} = 1$

$\therefore \cos^2 B + \cos^2 C = 1.$

(ii) $\sin^2 B + \sin^2 C = 1$

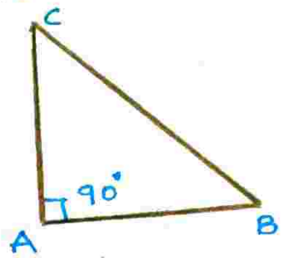
$\sin B = \frac{AC}{BC}$ $\sin C = \frac{AB}{BC}$

$\sin^2 B + \sin^2 C = \frac{AC^2}{BC^2} + \frac{AB^2}{BC^2} = \frac{AC^2 + AB^2}{BC^2}$

$= \frac{BC^2}{BC^2} = 1$

$\therefore \sin^2 B + \sin^2 C = 1.$

(iii) $\cos B - \cos C = -1 + 2\sqrt{2}\cos\frac{B}{2}\cdot\sin\frac{C}{2}$



$B + C = 90$

$B = 90 - C$

$C = 90 - B$

$\frac{B}{2} = 45 - \frac{C}{2}$

$\frac{C}{2} = 45 - \frac{B}{2}$

$$\begin{aligned}
L.H.S &= \cos B - \cos C \\
&= \cos(90 - C) - \cos C \\
&= \sin C - \cos C \\
&= 2\sin\frac{C}{2}\cdot\cos\frac{C}{2} - \left(1 - 2\sin^2\frac{C}{2}\right) \\
&= 2\sin\frac{C}{2}\cdot\cos\frac{C}{2} - 1 + 2\sin^2\frac{C}{2} \\
&= -1 + 2\sin\frac{C}{2}\left[\cos\frac{C}{2} + \sin\frac{C}{2}\right] \\
&= -1 + 2\sin\frac{C}{2}\left[\cos\left(45 - \frac{B}{2}\right) + \sin\left(45 - \frac{B}{2}\right)\right] \\
&= -1 + 2\sin\frac{C}{2}\left\{\cos 45\cos\frac{B}{2} + \sin 45\sin\frac{B}{2} + \sin 45\cos\frac{B}{2} - \cos 45\cdot\sin\frac{B}{2}\right\} \\
&= -1 + 2\sin\frac{C}{2}\left[\frac{1}{\sqrt{2}}\cos\frac{B}{2} + \cancel{\frac{1}{\sqrt{2}}\sin\frac{B}{2}} + \frac{1}{\sqrt{2}}\cos\frac{B}{2} - \cancel{\frac{1}{\sqrt{2}}\sin\frac{B}{2}}\right] \\
&= -1 + 2\sin\frac{C}{2}\left[\frac{2}{\sqrt{2}}\cos\frac{B}{2}\right] \\
&= -1 + 2\sqrt{2}\cos\frac{B}{2}\cdot\sin\frac{C}{2} = R.H.S
\end{aligned}$$

Hence Proved.

Trigonometric Equations:

General Solutions:

$$\boxed{\begin{aligned}
\sin\theta = 0 &\Rightarrow \theta = n\pi, \quad n \in Z \\
\cos\theta = 0 &\Rightarrow \theta = (2n+1)\frac{\pi}{2}, \quad n \in Z \\
\tan\theta = 0 &\Rightarrow \theta = n\pi, \quad n \in Z
\end{aligned}}$$

$$\boxed{\begin{aligned}
\sin\theta = \sin\alpha &\Rightarrow \theta = n\pi + (-1)^n\alpha \quad n \in Z \\
\cos\theta = \cos\alpha &\Rightarrow \theta = 2n\pi \pm \alpha, \quad n \in Z \\
\tan\theta = \tan\alpha &\Rightarrow \theta = n\pi + \alpha \quad n \in Z.
\end{aligned}}$$

$$\boxed{\begin{aligned}
\sin^2\theta = \sin^2\alpha &\Rightarrow x = n\pi \pm \alpha, \quad n \in Z \\
\cos^2\theta = \cos^2\alpha &\Rightarrow x = n\pi \pm \alpha, \quad n \in Z \\
\tan^2\theta = \tan^2\alpha &\Rightarrow x = n\pi \pm \alpha, \quad n \in Z.
\end{aligned}}$$

Principal Solution :

The smallest numerical value of unknown angle satisfying the equation in the interval $[0, 2\pi]$ or $[-\pi, \pi]$ is called a Principal Solution.

* Principal value of sine function lies in the interval $\left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$ and hence lies in I and IV quadrant.

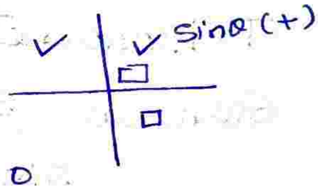
* Principal value of cosine function lies in the interval $[0, \pi]$ and hence lies in I & II quadrant.

* Principal value of tangent function lies in the interval $\left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$ and hence lies in I & IV quadrant,

---

3.42 Find the Principal solution of

(i) $\sin\theta = \frac{1}{2}$

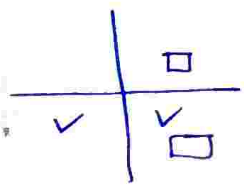


$\sin\theta = \frac{1}{2} = \sin\frac{\pi}{6}$

$\theta = \frac{\pi}{6} \in \left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$

[* $\sin\theta > 0$
* Principal value lies in I & II quad]

(ii) $\sin\theta = -\frac{\sqrt{3}}{2}$

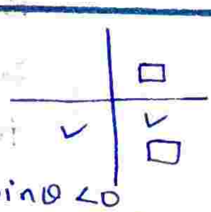


$\sin\theta = -\frac{\sqrt{3}}{2} = \sin(-\pi/3)$

$\theta = -\frac{\pi}{3} \in \left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$

[$\sin\theta < 0$
Principal vale lies in I & IV quad]

(iii) $\operatorname{cosec}\theta = -2$



$\operatorname{cosec}\theta = -2$

$\sin\theta = -\frac{1}{2}$

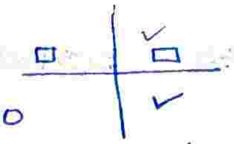
$\sin\theta = \sin\left(-\frac{\pi}{6}\right)$

$\theta = -\frac{\pi}{6} \in \left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$

[* $\sin\theta < 0$
* Principal value lies in I & IV quad]

(iv) $\cos\theta = \frac{1}{2}$

$\cos\theta = \cos\frac{\pi}{3}$



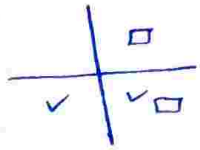
$\theta = \frac{\pi}{3} \in [0, \pi]$

[* $\cos\theta > 0$
* Principal value lies in I and II quad]

3.43 Find the General Solution of $\sin\theta = -\frac{\sqrt{3}}{2}$

$\sin\theta = -\frac{\sqrt{3}}{2}$

$\sin\theta = \sin(-\frac{\pi}{3})$



$\theta = -\frac{\pi}{3} \in [-\frac{\pi}{2}, \frac{\pi}{2}]$

General solution

$\sin\theta = \sin\alpha$

$\theta = n\pi + (-1)^n \alpha, \quad n \in Z$

$\theta = n\pi + (-1)^n (-\frac{\pi}{3}) \quad n \in Z$

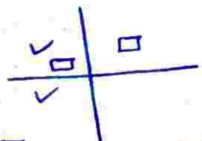
$\boxed{= n\pi + (-1)^{n+1} \cdot \frac{\pi}{3} \quad n \in Z.}$

3.44 / 129 Find the General Solution of

(i) $\sec\theta = -2$ (ii) $\tan\theta = \sqrt{3}$

(i) $\sec\theta = -2$

$\cos\theta = -\frac{1}{2}$



$\cos\theta = \cos(\pi - \frac{\pi}{3})$

$= \frac{2\pi}{3} \in [0, \pi]$

[* $\cos\theta < 0$
* Principal value lies in I & II quad]

General Solution :

$\cos\theta = \cos\alpha$ $\qquad \alpha = \frac{2\pi}{3}$

$\theta = 2n\pi \pm \alpha \quad n \in Z$

$\boxed{\theta = 2n\pi \pm \frac{2\pi}{3} \quad n \in Z.}$

(ii) $\tan\theta = \sqrt{3}$

$\tan\theta = \tan\frac{\pi}{3}$

$\theta = \frac{\pi}{3} \in (-\pi/2, \frac{\pi}{2})$

$\alpha = \frac{\pi}{3}$

General Solution :

$\tan\theta = \tan\alpha$

$\theta = n\pi + \alpha \quad n \in Z$

$\boxed{\theta = n\pi + \frac{\pi}{3} \quad n \in Z.}$

3.45 Solve : $3\cos^2\theta = \sin^2\theta$

$$3\cos^2\theta = \sin^2\theta$$
$$3\cos^2\theta = 1-\cos^2\theta$$
$$4\cos^2\theta = 1 \Rightarrow \cos^2\theta = \frac{1}{4}$$
$$\frac{1+\cos 2\theta}{2} = \frac{1}{4}$$
$$1+\cos 2\theta = \frac{1}{2}$$
$$\cos 2\theta = \frac{1}{2}-1 = \frac{-1}{2}$$
$$\cos 2\theta = \cos(\pi - \pi/3)$$
$$\cos 2\theta = \cos 2\pi/3$$
$$2\theta = 2\pi/3$$

$$\boxed{\begin{aligned} 2\theta &= 2n\pi \pm \alpha \quad n\in Z \\ 2\theta &= 2n\pi \pm 2\pi/3 \quad n\in Z \\ \theta &= n\pi \pm \pi/3 \quad n\in Z \end{aligned}}$$

$\frac{3.46}{130}$ Solve : $\sin x + \sin 5x = \sin 3x$

$$\sin x + \sin 5x = \sin 3x$$
$$2\sin\left(\frac{5x+x}{2}\right)\cdot\cos\left(\frac{5x-x}{2}\right) = \sin 3x$$
$$2\sin 3x\cdot\cos 2x - \sin 3x = 0$$
$$\sin 3x\,(2\cos 2x - 1) = 0$$

$\sin 3x = 0$

$\sin 3x = \sin 0$

$3x = 0$

$3x = n\pi \quad n\in Z$

$\boxed{x = \frac{n\pi}{3} \quad n\in Z}$

$2\cos 2x - 1 = 0$

$\cos 2x = \frac{1}{2}$

$\cos 2x = \cos\frac{\pi}{3}$

$2x = 2n\pi \pm \alpha \quad n\in Z$

$= 2n\pi \pm \frac{2\pi}{6} \quad n\in Z$

$\boxed{x = n\pi \pm \frac{\pi}{6} \quad n\in Z}$

$\frac{3.47}{130}$ Solve: $\cos x + \sin x = \cos 2x + \sin 2x$

$$\cos x - \cos 2x = \sin 2x - \sin x$$
$$-2\sin\left(\frac{x+2x}{2}\right)\cdot\sin\left(\frac{x-2x}{2}\right) = 2\cos\left(\frac{x+2x}{2}\right)\cdot\sin\left(\frac{2x-x}{2}\right)$$
$$\Rightarrow -2\sin\frac{3x}{2}\cdot\sin\left(\frac{-x}{2}\right) = 2\cos\frac{3x}{2}\cdot\sin\frac{x}{2}$$
$$2\sin\frac{3x}{2}\sin\frac{x}{2} - 2\cos\frac{3x}{2}\sin\frac{x}{2} = 0$$

$$2\sin\frac{x}{2}\left(\sin\frac{3x}{2}-\cos\frac{3x}{2}\right)=0$$

$$2\sin\frac{x}{2}=0$$

$$\sin\frac{x}{2}=0$$

$$\sin\frac{x}{2}=\sin 0$$

$$\frac{x}{2}=n\pi\pm\alpha \quad n\in z$$

$$\frac{x}{2}=n\pi \quad n\in z$$

$$\boxed{x=2n\pi \quad n\in z.}$$

$$\sin\frac{3x}{2}-\cos\frac{3x}{2}=0$$

$$\sin\frac{3x}{2}=\cos\frac{3x}{2}$$

$$\tan\frac{3x}{2}=1$$

$$\tan\frac{3x}{2}=\tan\frac{\pi}{4}$$

$$\frac{3x}{2}=n\pi+\alpha \quad n\in z$$

$$\frac{3x}{2}=n\pi+\frac{\pi}{4} \quad n\in z$$

$$3x=2n\pi+\frac{\pi}{2} \quad n\in z$$

$$\boxed{x=\frac{2n\pi}{3}+\frac{\pi}{6} \quad n\in z}$$

$\frac{3.48}{130}$ Solve the equation : $\sin 9\theta=\sin\theta$

$$\sin 9\theta=\sin\theta$$

$$\sin 9\theta-\sin\theta=0$$

$$2\cos\frac{9\theta+\theta}{2}\cdot\sin\left(\frac{9\theta-\theta}{2}\right)=0$$

$$2\cos 5\theta\cdot\sin 4\theta=0$$

$$2\cos 5\theta=0$$

$$\cos 5\theta=0$$

$$5\theta=(2n+1)\frac{\pi}{2} \quad n\in z$$

$$\theta=(2n+1)\frac{\pi}{10} \quad n\in z.$$

$$\sin 4\theta=0$$

$$4\theta=n\pi \quad n\in z$$

$$\theta=\frac{n\pi}{4} \quad n\in z.$$

$\frac{3.49}{131.}$ Solve : $\tan 2x=-\cot\left(x+\frac{\pi}{3}\right)$

$$\tan 2x=\tan\left(\frac{\pi}{2}+\left(x+\frac{\pi}{3}\right)\right)$$

$$\tan 2x=\tan\left(\frac{5\pi}{6}+x\right)$$

$$2x=n\pi+\alpha \quad n\in z$$

$$2x=n\pi+\frac{5\pi}{6}+x \quad n\in z$$

$$2x-x=n\pi+\frac{5\pi}{6}$$

$$x=n\pi+\frac{5\pi}{6} \quad n\in z.$$

3.50 / 131.

Solve : $\sin x - 3\sin 2x + \sin 3x = \cos x - 3\cos 2x + \cos 3x$

$\sin x - 3\sin 2x + \sin 3x = \cos x - 3\cos 2x + \cos 3x$

$\sin 3x + \sin x - 3\sin 2x = \cos 3x + \cos x - 3\cos 2x$

$2\sin 2x . \cos x - 3\sin 2x = 2\cos 2x \cdot \cos x - 3\cos 2x$

$\sin 2x(2\cos x - 3) = \cos 2x(2\cos x - 3)$

$\sin 2x(2\cos x - 3) - \cos 2x(2\cos x - 3)$

$(\sin 2x - \cos 2x)(2\cos x - 3) = 0$

$\sin 2x - \cos 2x = 0$ $\qquad$ $2\cos x - 3 = 0$

$\tan 2x = 1 = \tan \pi/4$ $\qquad$ $2\cos x = 3$

$2x = n\pi + \alpha$ $\qquad$ $\boxed{\cos x \neq \frac{3}{2}}$

$2x = n\pi + \frac{\pi}{4}$

$\boxed{x = \frac{n\pi}{2} + \frac{\pi}{8}, n \in z}$

3.51 / 131

Solve : $\sin x + \cos x = 1 + \sin x \cos x$

$\sin x + \cos x = 1 + \sin x \cos x$ ——— ①

Let $t = \sin x + \cos x$

$t^2 = (\sin x + \cos x)^2 = \sin^2 x + \cos^2 x + 2\sin x \cos x$

$t^2 = 1 + 2\sin x \cos x$

$\sin x \cos x = \frac{t^2 - 1}{2}$

① becomes

$t = 1 + \frac{t^2 - 1}{2} \Rightarrow 2t = 2 + t^2 - 1$

$t^2 - 2t + 1 = 0$

$(t-1)^2 = 0 \Rightarrow t = 1$

$\therefore \sin x + \cos x = 1$

$\sqrt{2}\left(\frac{1}{\sqrt{2}}\sin x + \frac{1}{\sqrt{2}}\cos x\right) = 1$

$\frac{1}{\sqrt{2}}\sin x + \frac{1}{\sqrt{2}}\cos x = \frac{1}{\sqrt{2}}$

$\cos\left(\frac{\pi}{4} - x\right) = \frac{1}{\sqrt{2}} = \cos\frac{\pi}{4}$

$\frac{\pi}{4} - x = 2n\pi \pm \pi/4 \quad n \in z$

$x - \frac{\pi}{4} = 2n\pi \pm \pi/4 \quad n \in z$

$\boxed{x = 2n\pi \pm \frac{\pi}{2} \text{ or } x = 2n\pi \quad n \in z.}$

$\frac{3.52}{132}$ Solve : $2\sin^2 x + \sin^2 2x = 2$

$$2\sin^2 x + (2\sin x \cos x)^2 = 2$$
$$\sin^2 x + 2\sin^2 x \cos^2 x = 1$$
$$\cancel{1} - \cos^2 x + 2\sin^2 x \cos^2 x = \cancel{1}$$
$$\cos^2 x(2\sin^2 x - 1) = 0$$

$\cos^2 x = 0$

$\cos x = 0$

$$\boxed{x = (2n+1)\frac{\pi}{2} \quad n \in z}$$

$2\sin^2 x - 1 = 0$

$2\sin^2 x = 1$

$\sin^2 x = \frac{1}{2}$

$\sin^2 x = \sin^2 \frac{\pi}{4}$

$x = n\pi \pm \alpha \quad n \in z$

$$\boxed{x = n\pi \pm \frac{\pi}{4} \quad n \in z.}$$

3.53 Prove that for any a and b $-\sqrt{a^2+b^2} \leq a\sin\theta + b\cos\theta \leq \sqrt{a^2+b^2}$

Proof :-

$$a\sin\theta + b\cos\theta = \sqrt{a^2+b^2}\left\{\frac{a}{\sqrt{a^2+b^2}}\sin\theta + \frac{b}{\sqrt{a^2+b^2}}\cos\theta\right\}$$
$$= \sqrt{a^2+b^2}\{\cos\alpha \sin\theta + \sin\alpha \cos\theta\}$$

where $\cos\alpha = \frac{a}{\sqrt{a^2+b^2}}$

$\sin\alpha = \frac{b}{\sqrt{a^2+b^2}}$

$$= \sqrt{a^2+b^2}\sin(\alpha+\theta)$$

$\therefore |a\sin\theta + b\cos\theta| \leq \sqrt{a^2+b^2}$

$|x| \leq 1 \Leftrightarrow -1 \leq x \leq 1$

$$\Rightarrow \boxed{-\sqrt{a^2+b^2} \leq a\sin\theta + b\cos\theta \leq \sqrt{a^2+b^2}}$$

Hence Proved.

3.54. Solve : $\sqrt{3}\sin\theta - \cos\theta = \sqrt{2}$

$a = \sqrt{3}$

$b = -1$

$\sqrt{a^2+b^2} = \sqrt{3+1} = 2$

$$\sqrt{3}\sin\theta - \cos\theta = \sqrt{2}$$
$$\therefore 2\left(\frac{\sqrt{3}}{2}\sin\theta - \frac{1}{2}\cos\theta\right) = \sqrt{2}$$
$$\frac{\sqrt{3}}{2}\sin\theta - \frac{1}{2}\cos\theta = \frac{1}{\sqrt{2}}$$
$$\cos\frac{\pi}{6}\sin\theta - \sin\frac{\pi}{6}\cos\theta = \frac{1}{\sqrt{2}}$$
$$\sin\left(\theta - \frac{\pi}{6}\right) = \sin\pi/4$$
$$\theta - \frac{\pi}{6} = n\pi + (-1)^n\frac{\pi}{4} \quad n \in z$$
$$\boxed{\theta = n\pi + \frac{\pi}{6} + (-1)^n\frac{\pi}{4} \quad n \in z}$$

$\frac{3.55}{133}$ Solve : $\sqrt{3}\tan^2\theta + (\sqrt{3}-1)\tan\theta - 1 = 0$

$$\sqrt{3}\tan^2\theta + \sqrt{3}\tan\theta - \tan\theta - 1 = 0$$
$$\sqrt{3}\tan\theta(\tan\theta + 1) - 1(\tan\theta + 1) = 0$$
$$(\sqrt{3}\tan\theta - 1)(\tan\theta + 1) = 0$$

$\sqrt{3}\tan\theta - 1 = 0$

$\tan\theta = \frac{1}{\sqrt{3}}$

$\tan\theta = \tan \pi/6$

$\theta = n\pi + \alpha \quad n \in z$

$\boxed{\theta = n\pi + \frac{\pi}{6} \quad n \in z}$

$\tan\theta + 1 = 0$

$\tan\theta = -1$

$\tan\theta = \tan(-\pi/4)$

$\boxed{\theta = n\pi - \frac{\pi}{4} \quad n \in z.}$

Ex : 3.8

$\frac{1}{133}$ Find the Principal solution and general Solutions of the following :

(i) $\sin\theta = \frac{-1}{\sqrt{2}}$

$\sin\theta = \sin(-\pi/4)$

$\theta = -\pi/4 \in [-\frac{\pi}{2}, \frac{\pi}{2}]$

$\therefore$ The Principal solution is $-\pi/4 \in [-\frac{\pi}{2}, \frac{\pi}{2}]$.

General Solution : $\theta = n\pi + (-1)^n\alpha \quad n \in z$

$$\theta = n\pi + (-1)^n(-\pi/4) \quad n \in z$$
$$= n\pi + (-1)^{n+1}(\pi/4);\ n \in z$$

(ii) $\cot\theta = \sqrt{3}$

$\tan\theta = \frac{1}{\sqrt{3}} = \tan(\pi/6)$

$\therefore \theta = \pi/6 \in (-\pi/2, \pi/2)$

$\therefore$ Principal solution is $\pi/6 \in (-\pi/2, \pi/2)$

General solution : $\theta = n\pi + \alpha \quad n \in z$

$\theta = n\pi + \pi/6 \quad n \in z.$

(iii) $\tan\theta = -\frac{1}{\sqrt{3}}$

$\tan\theta = \tan(-\frac{\pi}{6})$

$\theta = -\pi/6 \in (-\pi/2, \pi/2)$

$\therefore$ Principal solution is $-\pi/6 \in (-\pi/2, \pi/2)$

General solution : $\theta = n\pi + \alpha \quad n \in z$

$\theta = n\pi + (-\pi/6) \quad n \in z.$

2/133 Solve the following equations for which solutions lies in the interval $0^\circ \le \theta \le 360^\circ$

(i) $\sin^4 x = \sin^2 x$

$\sin^4 x - \sin^2 x = 0 \Rightarrow \sin^2 x(1-\sin^2 x) = 0$

$\sin^2 x \cos^2 x = 0 \Rightarrow \left(\frac{1}{2}\cdot 2\sin x \cos x\right)^2 = 0$

$\frac{1}{4}\sin^2 2x = 0$

$\sin^2 2x = 0$

$\sin 2x = 0$

$2x = 0, \pi, 2\pi, 3\pi, 4\pi \ldots$

$x = 0, \frac{\pi}{2}, \pi, \frac{3\pi}{2}, 2\pi \ldots$

$x = 0, \frac{\pi}{2}, \pi, \frac{3\pi}{2}, 2\pi \in [0, 2\pi]$.

(ii) $2\cos^2 x + 3\cos x + 1 = 0$

$(2\cos x+1)(\cos x+1) = 0$

$\cos x = -\frac{1}{2}$ $\quad$ $\cos x = -1$

$\frac{1}{2\cos x}$ $\quad$ $\frac{\cancel{2}\ 1}{2\cos x}$

$x = \pi - \frac{\pi}{3} = \frac{2\pi}{3}$ $\quad$ $x = \pi$

$\therefore x = \frac{2\pi}{3}, \pi \in [0, 2\pi]$

(iii) $2\sin^2 x + 1 = 3\sin x$

$2\sin^2 x - 3\sin x + 1 = 0$

$\frac{-1}{2\sin x}$ $\quad$ $\frac{-\cancel{2}}{\cancel{2}\sin x}$

$(2\sin x - 1)(\sin x - 1) = 0$

$\sin x = \frac{1}{2}$ $\quad$ $\sin x = 1$

$\sin x = \sin\frac{\pi}{6}$ $\quad$ $\sin x = \sin \pi/2$

$x = \frac{\pi}{6}, \frac{5\pi}{6}$

$\therefore x = \frac{\pi}{2}, \pi/6, \frac{5\pi}{6}$

(iv) $\cos 2x = 1 - 3\sin x$

$\cancel{1} - 2\sin^2 x = \cancel{1} - 3\sin x$

$2\sin^2 x - 3\sin x = 0$

$\sin x(2\sin x - 3) = 0$

$\sin x = 0$ $\quad$ $\sin x \ne \frac{3}{2}$

$x = 0, \pi$

$\therefore x = 0, \pi \in [0, 2\pi]$

$\frac{3}{133}$ Solve the following equations :-

(i) $\sin 5x - \sin x = \cos 3x$

$$\sin 5x - \sin x = \cos 3x$$

$$2\cos 3x \cdot \sin 2x - \cos 3x = 0$$

$$\cos 3x\,(2\sin 2x - 1) = 0$$

$\cos 3x = 0$

$3x = (2n+1)\frac{\pi}{2} \quad n \in Z$

$\boxed{x = (2n+1)\frac{\pi}{6} \quad n \in Z}$

$2\sin 2x = 1$

$\sin 2x = \frac{1}{2} = \sin\frac{\pi}{6}$

$2x = n\pi + (-1)^n \alpha \quad n \in Z$

$2x = n\pi + (-1)^n \cdot \frac{\pi}{6} \quad n \in Z$

$\boxed{x = \frac{n\pi}{2} + (-1)^n \cdot \frac{\pi}{12} \quad n \in Z}$

(ii) $2\cos^2\theta + 3\sin\theta - 3 = 0$

$$2(1 - \sin^2\theta) + 3\sin\theta - 3 = 0$$

$$2 - 2\sin^2\theta + 3\sin\theta - 3 = 0$$

$$2\sin^2\theta - 3\sin\theta + 1 = 0$$

$$(2\sin\theta - 1)(\sin\theta - 1) = 0$$

= 2

$-1 \quad -2$

$\overline{2\sin\theta} \quad \overline{2\sin\theta}$

$\sin\theta = \frac{1}{2}$

$\sin\theta = \sin\frac{\pi}{6}$

$\boxed{\theta = n\pi + (-1)^n \frac{\pi}{6} \quad n \in Z}$

$\sin\theta = 1$

$\sin\theta = \sin\frac{\pi}{2}$

$\boxed{\theta = n\pi + (-1)^n \frac{\pi}{2}; n \in Z}$

(iii) $\cos\theta + \cos 3\theta = 2\cos 2\theta$

$$2\cos 2\theta \cdot \cos\theta = 2\cos 2\theta$$

$$2\cos 2\theta(\cos\theta - 1) = 0$$

$2\cos 2\theta = 0$

$\cos 2\theta = 0$

$2\theta = (2n+1)\frac{\pi}{2} \quad n \in Z$

$\boxed{\theta = (2n+1)\frac{\pi}{4} \quad n \in Z}$

$\cos\theta = 1$

$\cos\theta = \cos 0$

$\theta = 2n\pi \pm \alpha \quad n \in Z$

$\boxed{\theta = 2n\pi \quad n \in Z.}$

(iv) $\sin\theta + \sin 3\theta + \sin 5\theta = 0$

$$\sin 5\theta + \sin\theta + \sin 3\theta = 0$$

$$2\sin 3\theta \cdot \cos 2\theta + \sin 3\theta = 0$$

$$\sin 3\theta \,(2\cos 2\theta + 1) = 0$$

$$\sin 3\theta = 0$$

$$3\theta = n\pi \,;\, n\in z$$

$$\boxed{\theta = \frac{n\pi}{3} \;;\; n\in z}$$

$$2\cos 2\theta + 1 = 0$$

$$\cos 2\theta = \frac{-1}{2}$$

$$\cos 2\theta = \cos(\pi - \pi/3)$$

$$\cos 2\theta = \cos 2\pi/3$$

$$2\theta = 2n\pi \pm \alpha \quad n\in z$$

$$2\theta = 2n\pi \pm \frac{2\pi}{3} \quad n\in z$$

$$\boxed{\theta = n\pi \pm \frac{\pi}{3} \;,\; n\in z}$$

(v) $\sin 2\theta - \cos 2\theta - \sin\theta + \cos\theta = 0$

$$(\sin 2\theta - \sin\theta) - (\cos 2\theta - \cos\theta) = 0$$

$$2\cos\frac{3\theta}{2}\cdot\sin\frac{\theta}{2} - \left(-2\sin\frac{3\theta}{2}\cdot\sin\frac{\theta}{2}\right) = 0$$

$$2\sin\theta/2\left(\cos 3\theta/2 + \sin\frac{3\theta}{2}\right) = 0$$

$$2\sin\theta/2 = 0$$

$$\sin\frac{\theta}{2} = 0$$

$$\frac{\theta}{2} = n\pi \,;\, n\in z$$

$$\boxed{\theta = 2n\pi \quad n\in z}$$

$$\cos\frac{3\theta}{2} + \sin\frac{3\theta}{2} = 0$$

$$\sin\frac{3\theta}{2} = -\cos\frac{3\theta}{2}$$

$$\tan\frac{3\theta}{2} = -1$$

$$\tan\frac{3\theta}{2} = \tan(-\pi/4)$$

$$\frac{3\theta}{2} = n\pi + \alpha \quad n\in z$$

$$\frac{3\theta}{2} = n\pi - \frac{\pi}{4} \;,\; n\in z$$

$$3\theta = 2n\pi - \frac{\pi}{2} \quad n\in z$$

$$\boxed{\theta = \frac{2n\pi}{3} - \frac{\pi}{6} \quad n\in z}$$

(vi) $\sin\theta + \cos\theta = \sqrt{2}$

$a = 1 \quad b = 1$

$$\sqrt{a^2+b^2} = \sqrt{1+1} = \sqrt{2}$$

$$\sqrt{2}\left(\frac{1}{\sqrt{2}}\sin\theta + \frac{1}{\sqrt{2}}\cos\theta\right) = \sqrt{2}$$

$$\sin\frac{\pi}{4}\cdot\sin\theta + \cos\frac{\pi}{4}\cos\theta = 1$$

$$\cos\left(\theta - \frac{\pi}{4}\right) = 1 = \cos 0$$

$$\theta - \frac{\pi}{4} = 2n\pi + \alpha \quad n\in z$$

$$\theta - \pi/4 = 2n\pi$$

$$\boxed{\theta = 2n\pi + \frac{\pi}{4} \quad n\in z}$$

(vii) $\sin\theta + \sqrt{3}\cos\theta = 1$

$a = 1$

$b = \sqrt{3}$

$\sqrt{a^2+b^2} = \sqrt{3+1} = 2$

$\frac{1}{2}\sin\theta + \frac{\sqrt{3}}{2}\cos\theta = \frac{1}{2}$

$\sin\frac{\pi}{6}\sin\theta + \cos\frac{\pi}{6}\cos\theta = \frac{1}{2}$

$\cos(\theta - \pi/6) = \cos\frac{\pi}{3}$

$\theta - \pi/6 = 2n\pi \pm \frac{\pi}{3}$

$$\boxed{\theta = 2n\pi \pm \frac{\pi}{3} + \frac{\pi}{6} \qquad n \in z.}$$

(viii) $\cot\theta + \operatorname{cosec}\theta = \sqrt{3}$

$\frac{\cos\theta}{\sin\theta} + \frac{1}{\sin\theta} = \sqrt{3}$

$1 + \cos\theta = \sqrt{3}\sin\theta$

$a = \sqrt{3}$

$b = 1$

$\sqrt{3}\sin\theta - \cos\theta = 1$

$\sqrt{a^2+b^2} = 2$

$\frac{\sqrt{3}}{2}\sin\theta - \frac{1}{2}\cos\theta = \frac{1}{2}$

$\sin\frac{\pi}{3}\sin\theta - \cos\frac{\pi}{3}\cos\theta = \frac{1}{2}$

$\cos\frac{\pi}{3}\cos\theta - \sin\frac{\pi}{3}\sin\theta = -\frac{1}{2}$

$\cos(\theta + \frac{\pi}{3}) = \cos(2\pi/3)$

$\theta + \frac{\pi}{3} = 2n\pi \pm \alpha \quad n \in z$

$\theta + \frac{\pi}{3} = 2n\pi \pm 2\pi/3$

$$\boxed{\theta = 2n\pi \pm 2\frac{\pi}{3} - \pi/3 \qquad n \in z.}$$

(ix) $\tan\theta + \tan(\theta + \frac{\pi}{3}) + \tan(\theta + \frac{2\pi}{3}) = \sqrt{3}$

$\tan\theta + \frac{\tan\theta + \tan\pi/3}{1 - \tan\theta\tan\pi/3} + \frac{\tan\theta + \tan 2\pi/3}{1 - \tan\theta\tan 2\pi/3} = \sqrt{3}$

$\tan\theta + \frac{\tan\theta + \sqrt{3}}{1 - \sqrt{3}\tan\theta} + \frac{\tan\theta - \sqrt{3}}{1 + \sqrt{3}\tan\theta} = \sqrt{3}$

$\tan\theta + \frac{(\tan\theta + \sqrt{3})(1 + \sqrt{3}\tan\theta) + (\tan\theta - \sqrt{3})(1 - \sqrt{3}\tan\theta)}{1 - 3\tan^2\theta} = \sqrt{3}$

$\tan\theta + \frac{\tan\theta + \sqrt{3}\cancel{\tan^2\theta} + \cancel{\sqrt{3}} + 3\tan\theta + \tan\theta - \cancel{\sqrt{3}\tan^2\theta} - \cancel{\sqrt{3}} + 3\tan^2\theta}{1 - 3\tan^2\theta} = \sqrt{3}$

$\tan\theta + \frac{8\tan\theta}{1 - 3\tan^2\theta} = \sqrt{3}$

$\frac{\tan\theta - 3\tan^3\theta + 8\tan\theta}{1 - 3\tan^2\theta} = \sqrt{3}$

$\frac{9\tan\theta - 3\tan^3\theta}{1 - 3\tan^2\theta} = \sqrt{3}$

$$\frac{3(3\tan\theta - \tan^3\theta)}{1-3\tan^2\theta} = \sqrt{3}$$

$$3\tan 3\theta = \sqrt{3}$$

$$\tan 3\theta = \frac{1}{\sqrt{3}}$$

$$\tan 3\theta = \tan\frac{\pi}{6}$$

$$3\theta = n\pi + \alpha \quad n \in Z$$

$$3\theta = n\pi + \frac{\pi}{6} \quad n \in Z$$

$$\boxed{\theta = \frac{n\pi}{3} + \frac{\pi}{18} \quad n \in Z}$$

(x) $\cos 2\theta = \frac{\sqrt{5}+1}{4}$

$36^\circ = \frac{2\pi}{10}$

$$\cos 2\theta = \cos 36^\circ$$

$$\cos 2\theta = \cos\frac{2\pi}{10}$$

$$2\theta = 2n\pi \pm \alpha \quad n \in Z$$

$$2\theta = 2n\pi \pm \frac{2\pi}{10}$$

$$\boxed{\theta = n\pi \pm \frac{\pi}{10} \quad n \in Z}$$

(xi) $2\cos^2 x - 7\cos x + 3 = 0$

$$(2\cos x - 1)(\cos x - 3) = 0$$

6

$-\cancel{1}$ $\quad$ $-\cancel{6}$ -3

$\frac{}{2\cos x}$ $\quad$ $\frac{}{\cancel{2}\cos x}$

$2\cos x - 1 = 0 \qquad \cos x \neq 3$

$$\cos x = \frac{1}{2}$$

$$\cos x = \cos\frac{\pi}{3}$$

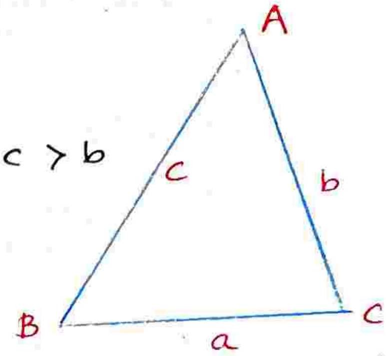
$$x = 2n\pi \pm \alpha \quad n \in Z$$

$$\boxed{x = 2n\pi \pm \frac{\pi}{3} \quad n \in Z.}$$

3.7. Properties of triangle:

In ΔABC, $A+B+C=\pi$

$a+b>c$, $b+c>a$, $a+c>b$



Law of Sines:

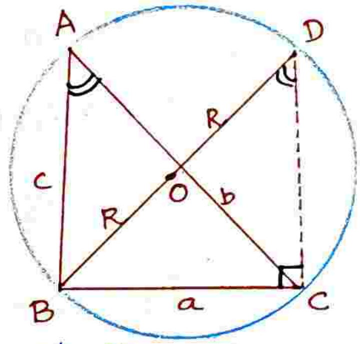
In any triangle, the lengths of the sides are proportional to the sines of the opposite angles.

> In ΔABC, $\dfrac{a}{\sin A} = \dfrac{b}{\sin B} = \dfrac{c}{\sin C} = 2R$
>
> where R is the circumcentre of the triangle.

Proof :-

The angle A of the ΔABC. is either acute, or right or abtuse.

Let O be the centre of the circum circle of ΔABC and R, its radius.



Case I : $\angle A$ is acute

Produce BO to meet the circle at D.

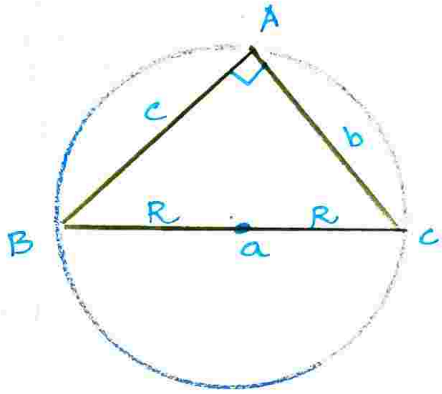
$\angle BDC = \angle BAC = A$.

$\angle BCD = 90^\circ$

$\sin \angle BDC = \dfrac{BC}{BD}$

$\sin A = \dfrac{a}{2R} \Rightarrow$

$\Rightarrow \dfrac{a}{\sin A} = 2R$



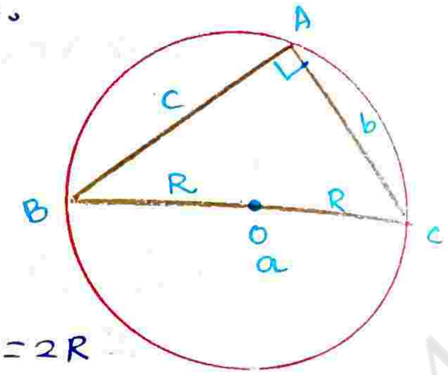
Ily we can prove $\dfrac{b}{\sin B} = 2R$

$\dfrac{c}{\sin C} = 2R$

$\therefore \dfrac{a}{\sin A} = \dfrac{b}{\sin B} = \dfrac{c}{\sin C} = 2R$

Case (ii) $\angle A$ is right angle:

In this case O must be on the side BC of the ΔABC.



$$\frac{a}{\sin A} = \frac{BC}{\sin 90^\circ} = \frac{2R}{1}$$

$$\frac{a}{\sin A} = 2R.$$

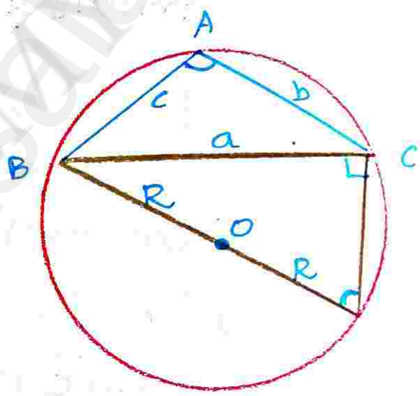
Illy we can Prove

$$\frac{b}{\sin B} = 2R \qquad \frac{c}{\sin C} = 2R$$

$$\therefore \frac{a}{\sin A} = \frac{b}{\sin B} = \frac{c}{\sin C} = 2R.$$

Case (iii) $\angle A$ is abtuse:

Produce BO to meet the circle at D.



$$\angle BDC + \angle BAC = 180^\circ$$

$$\angle BDC = 180 - \angle BAC$$

$$= 180^\circ - A$$

$$\angle BCD = 90^\circ$$

$$\sin \angle BDC = \frac{BC}{BD}$$

$$\sin(180 - A) = \frac{a}{2R}$$

$$\sin A = \frac{a}{2R} \Rightarrow \frac{a}{\sin A} = 2R$$

Illy we can Prove

$$\frac{b}{\sin B} = 2R \qquad \frac{c}{\sin C} = 2R$$

$$\therefore \frac{a}{\sin A} = \frac{b}{\sin B} = \frac{c}{\sin C} = 2R$$

Results:

* The law of Sines can be written as ~~a~~ Collection of three equations

$$\frac{a}{b} = \frac{\sin A}{\sin B} \qquad \frac{b}{c} = \frac{\sin B}{\sin C} \qquad \frac{c}{a} = \frac{\sin C}{\sin a}$$

* $a = 2R\sin A \qquad b = 2R\sin B \qquad c = 2R\sin C.$

Napier's Formula :

In ΔABC, we have

(i) $\tan \frac{A-B}{2} = \frac{a-b}{a+b} \cdot \cot \frac{C}{2}$

(ii) $\tan \frac{B-C}{2} = \frac{b-c}{b+c} \cot \frac{A}{2}$

(iii) $\tan \frac{C-A}{2} = \frac{c-a}{c+a} \cdot \cot \frac{B}{2}$

Proof :

We know that,

$a = 2R\sin A \qquad b = 2R\sin B \qquad c = 2R\sin C$

$$\frac{a-b}{a+b} = \frac{2R\sin A - 2R\sin B}{2R\sin A + 2R\sin B} = \frac{\sin A - \sin B}{\sin A + \sin B}$$

$$= \frac{\cancel{2}\cos\frac{A+B}{2} \cdot \sin\frac{A-B}{2}}{\cancel{2}\sin\frac{A+B}{2} \cdot \cos\frac{A-B}{2}}$$

$A+B+C=\pi$

$\frac{A+B}{2} = \frac{\pi}{2} - \frac{C}{2}$

$$= \cot\frac{A+B}{2} \cdot \tan\frac{A-B}{2}$$

$$= \cot\left(\frac{\pi}{2} - \frac{C}{2}\right) \cdot \tan\frac{A-B}{2}$$

$$= \tan\frac{C}{2} \cdot \tan\frac{A-B}{2}$$

$$\frac{a-b}{a+b} \cdot \cot\frac{C}{2} = \tan\frac{A-B}{2}$$

$$\therefore \tan\frac{A-B}{2} = \frac{a-b}{a+b} \cdot \cot\frac{C}{2}$$

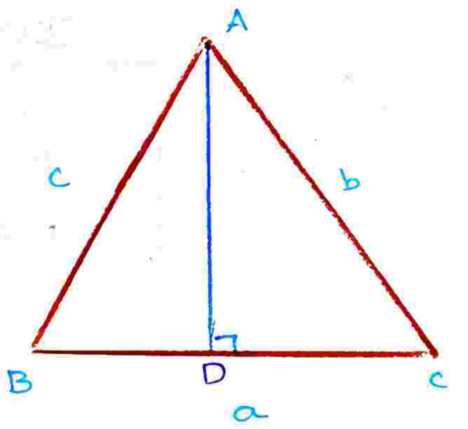
llly we can prove (ii) & (iii) results.

Laws of Cosines :

(i) $\cos A = \frac{b^2+c^2-a^2}{2bc}$

(ii) $\cos B = \frac{c^2+a^2-b^2}{2ac}$

(iii) $\cos C = \frac{a^2+b^2-c^2}{2ab}$



Proof :

In ΔABC, $AD \perp BC$

In ΔABD,

$$AB^2 = AD^2 + BD^2$$

$$c^2 = AD^2 + BD^2 \quad \text{--------- (1)}$$

In ΔADC,

$$\sin C = \frac{AD}{AC} \Rightarrow AD = AC \sin C$$

$$AD = b \sin C$$

$\cos C = \frac{DC}{AC}$

$DC = b\cos C$

$$BD = BC - DC = a$$

$$= a - b\cos C$$

(1) becomes,

$$c^2 = (b\sin C)^2 + (a - b\cos C)^2$$

$$= b^2\sin^2 C + a^2 - 2ab\cos C + b^2\cos^2 C$$

$$= b^2(\sin^2 C + \cos^2 C) + a^2 - 2ab\cos C$$

$$c^2 = a^2 + b^2 - 2ab\cos C$$

$$2ab\cos C = a^2 + b^2 - c^2$$

$$\cos C = \frac{a^2 + b^2 - c^2}{2ab}$$

Ily we can Prove (i) & (ii) results

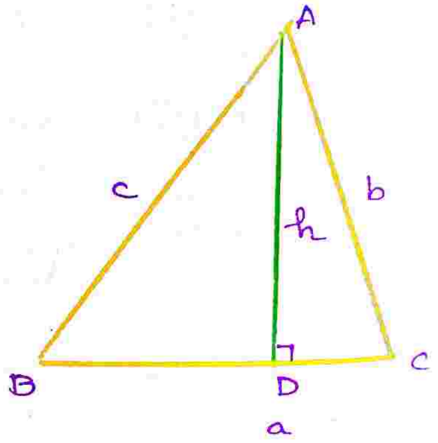
Results :

$$\boxed{\begin{aligned} a^2 &= b^2 + c^2 - 2bc \cdot \cos A \\ b^2 &= a^2 + c^2 - 2ac \cos B \\ c^2 &= a^2 + b^2 - 2ab \cos C \end{aligned}}$$

Projection Formula :

In ΔABC we have

$$\boxed{\begin{aligned} a &= b\cos C + c\cos B \\ b &= c\cos A + a\cos C \\ c &= a\cos B + b\cos A. \end{aligned}}$$



Proof :-

In ΔABC,

$BC = a \quad AB = c \quad AC = b$

Draw $AD \perp BC$

$a = BC = BD + DC$

$= \frac{BD}{AB} \cdot AB + \frac{DC}{AC} \cdot AC$

$= (\cos B) \cdot c + (\cos C) \cdot b$

$\therefore a = c\cos B + b\cos C$

Ily we can Prove (ii) & (iii) results.

Area of the triangle:

In ΔABC, area of the triangle is

$\Delta = \frac{1}{2}ab\sin C = \frac{1}{2}bc\sin A = \frac{1}{2}ca\sin B$

Proof:

In ΔABC, Draw $AD \perp BC$.

In ΔADC,

$\sin C = \frac{AD}{AC}$

$AD = b\sin C$

area of $\Delta ABC = \frac{1}{2}bh$

$= \frac{1}{2} \times BC \times AD$

$= \frac{1}{2} \times a \times b\sin C$

$\Delta = \frac{1}{2}ab\sin C$

Ily we can Prove

$\Delta = \frac{1}{2}bc\sin A = \frac{1}{2}ac\sin B$

Other forms of area of triangle:

$$\boxed{\begin{array}{l} \Delta = \frac{abc}{4R} \\ \Delta = 2R^2\sin A\sin B\sin C \\ \Delta = \sqrt{s(s-a)(s-b)(s-c)} \qquad s = \frac{a+b+c}{2} \end{array}}$$

Half angle formulae:

In any triangle ABC,

(i) $\sin\frac{A}{2} = \sqrt{\frac{(s-b)(s-c)}{bc}}$ (ii) $\sin\frac{B}{2} = \sqrt{\frac{(s-c)(s-a)}{ac}}$

(iii) $\sin\frac{C}{2} = \sqrt{\frac{(s-a)(s-b)}{ab}}$ (iv) $\cos\frac{A}{2} = \sqrt{\frac{s(s-a)}{bc}}$

(v) $\cos\frac{B}{2} = \sqrt{\frac{s(s-b)}{ac}}$ (vi) $\cos\frac{C}{2} = \sqrt{\frac{s(s-c)}{ab}}$

(vii) $\tan\frac{A}{2} = \sqrt{\frac{(s-b)(s-c)}{s(s-a)}}$ (viii) $\tan\frac{B}{2} = \sqrt{\frac{(s-c)(s-a)}{s(s-b)}}$

(ix) $\tan\frac{C}{2} = \sqrt{\frac{(s-a)(s-b)}{s(s-c)}}$

Proof :

$$\sin\frac{A}{2} = \sqrt{\frac{(s-b)(s-c)}{bc}}$$

We know that $\cos 2A = 1-2\sin^2 A$

$$2\sin^2 A = 1-\cos 2A$$

$$2\sin^2\frac{A}{2} = 1-\cos A$$

$$2\sin^2\frac{A}{2} = 1-\frac{b^2+c^2-a^2}{2bc}$$

$$= \frac{2bc-b^2-c^2+a^2}{2bc}$$

$$= \frac{a^2-(b^2+c^2-2bc)}{2bc} = \frac{a^2-(b-c)^2}{2bc}$$

$$= \frac{(a+b-c)(a-b+c)}{2bc}$$

$$= \frac{(a+b+c-2c)(a+b+c-2b)}{2bc}$$

$$= \frac{(2s-2c)(2s-2b)}{2bc}$$

$$2\sin^2\frac{A}{2} = \frac{4(s-c)(s-b)}{2bc}$$

$$\sin^2\frac{A}{2} = \frac{(s-c)(s-b)}{bc}$$

$$\sin\frac{A}{2} = \sqrt{\frac{(s-c)(s-b)}{bc}}$$

Illy we can Prove

$\sin\frac{B}{2} = \sqrt{\frac{(s-c)(s-a)}{ac}}$ $\sin\frac{C}{2} = \sqrt{\frac{(s-a)(s-b)}{ab}}$

(ii) $\cos\frac{A}{2} = \sqrt{\frac{s(s-a)}{bc}}$

We know that, $\cos 2A = 2\cos^2 A - 1$

$$2\cos^2 A = 1 + \cos 2A$$

$$2\cos^2\frac{A}{2} = 1 + \cos A$$

$$= 1 + \frac{b^2+c^2-a^2}{2bc}$$

$$= \frac{2bc + b^2 + c^2 - a^2}{2bc}$$

$$= \frac{(b+c)^2 - a^2}{2bc}$$

$$= \frac{(b+c+a)(b+c-a)}{2bc}$$

$$= \frac{(a+b+c)(a+b+c-2a)}{2bc}$$

$$= \frac{2s(2s-2a)}{2bc}$$

$$2\cos^2\frac{A}{2} = \frac{2s(s-a)}{bc}$$

$$\cos^2\frac{A}{2} = \frac{s(s-a)}{bc}$$

$$\cos\frac{A}{2} = \sqrt{\frac{s(s-a)}{bc}}$$

Ily we can Prove

$\cos\frac{B}{2} = \sqrt{\frac{s(s-b)}{ac}}$ $\cos\frac{C}{2} = \sqrt{\frac{s(s-c)}{ab}}$

(iii) $\tan\frac{A}{2} = \sqrt{\frac{(s-b)(s-c)}{s(s-a)}}$

$$\tan\frac{A}{2} = \frac{\sin A/2}{\cos A/2} = \sqrt{\frac{(s-b)(s-c)}{bc}} \times \sqrt{\frac{bc}{s(s-a)}}$$

$$= \sqrt{\frac{(s-b)(s-c)}{s(s-a)}}$$

Ily we can Prove

$$\tan\frac{B}{2} = \sqrt{\frac{(s-c)(s-a)}{s(s-b)}}$$

$$\tan\frac{C}{2} = \sqrt{\frac{(s-a)(s-b)}{s(s-c)}}$$

3.57/140 In a ΔABC, Prove that $b^2\sin 2C + c^2 \sin 2B = 2bc \sin A$

We know that

$a = 2R\sin A \quad b = 2R\sin B \quad c = 2R\sin C$

$$L.H.S = b^2\sin 2C + c^2\sin 2B$$
$$= (2R\sin B)^2 \sin 2C + (2R\sin C)^2 . \sin 2B$$
$$= 4R^2\sin^2 B \sin 2C + 4R^2\sin^2 C \sin 2B$$
$$= 4R^2\left[2\sin^2 B \sin C \cos C + 2\sin^2 C . \sin B \cos B\right]$$
$$= 8R^2 \sin B \sin C\left[\sin B \cos C + \cos B \sin C\right]$$
$$= 8R^2 \sin B \sin C \sin(B+C) \qquad \left[\begin{array}{l} A+B+C=\pi \\ B+C=\pi-A \end{array}\right.$$
$$= 8R^2\left(\frac{b}{2R}\right)\left(\frac{c}{2R}\right)\sin(\pi - A)$$
$$= 2bc\sin A = R.H.S.$$

3.58/140 In a ΔABC, Prove that $\sin\left(\frac{B-C}{2}\right) = \frac{b-c}{a}.\cos\frac{A}{2}$

We know that

$a = 2R\sin A \quad b = 2R\sin B \quad c = 2R\sin C$

$$R.H.S = \frac{b-c}{a}\cos\frac{A}{2} = \frac{2R\sin B - 2R\sin C}{2R\sin A}\cos\frac{A}{2}$$
$$= \frac{\sin B - \sin C}{\sin A} . \cos\frac{A}{2}$$
$$= \frac{2\cos\frac{B+C}{2} . \sin\frac{B-C}{2}}{2\sin\frac{A}{2} . \cancel{\cos\frac{A}{2}}} . \cancel{\cos\frac{A}{2}}$$
$$= \frac{\cancel{2}\cos\left(\frac{\pi}{2} - \frac{A}{2}\right) . \sin\frac{B-C}{2}}{\cancel{2}\sin A/2}$$
$$= \frac{\sin A/2 . \sin\frac{B-C}{2}}{\sin A/2} = \sin\frac{B-C}{2}$$
$$= L.H.S$$

3.59 / 141 If the three angles in a triangle are in the ratio 1:2:3 then Prove that the Corresponding sides are in the ratio $1:\sqrt{3}:2$

Let the angles be $\theta, 2\theta, 3\theta$

$$\theta+2\theta+3\theta=180^\circ$$
$$6\theta=180^\circ$$
$$\boxed{\theta=30^\circ}$$

W.K.T $\dfrac{a}{\sin A}=\dfrac{b}{\sin B}=\dfrac{c}{\sin C}$

$$\frac{a}{\sin 30^\circ}=\frac{b}{\sin 60^\circ}=\frac{c}{\sin 90^\circ}$$

$$a:b:c=\sin 30^\circ:\sin 60^\circ:\sin 90^\circ$$
$$=\frac{1}{2}:\frac{\sqrt{3}}{2}:1$$
$$a:b:c=1:\sqrt{3}:2$$

3.60 / 141 In a ΔABC, Prove that $(b+c)\cos A+(c+a)\cos B+(a+b)\cos C=a+b+c$

$$L.H.S=(b+c)\cos A+(c+a)\cos B+(a+b)\cos C$$
$$=b\cos A+c\cos A+c\cos B+a\cos B+a\cos C+b\cos C$$
$$=[b\cos A+a\cos B]+[c\cos A+a\cos C]+[c\cos B+b\cos C]$$
$$=c+b+a=a+b+c=R.H.S$$

Hence Proved.

3.61 / 141 In a ΔABC, Prove that $\dfrac{a^2+b^2}{a^2+c^2}=\dfrac{1+\cos(A-B)\cos C}{1+\cos(A-C)\cos B}$

W.K.T $a=2R\sin A$ $b=2R\sin B$ $c=2R\sin C$

$$L.H.S=\frac{a^2+b^2}{a^2+c^2}=\frac{4R^2\sin^2 A+4R^2\sin^2 B}{4R^2\sin^2 A+4R^2\sin^2 C}$$

$$=\frac{\sin^2 A+\sin^2 B}{\sin^2 A+\sin^2 C}=\frac{1-\cos^2 A+\sin^2 B}{1-\cos^2 A+\sin^2 C}$$

$$=\frac{1+\sin^2 B-\cos^2 A}{1+\sin^2 C-\cos^2 A}=\frac{1+\cos(A+B)\cos(A-B)}{1+\cos(A+C)\cdot\cos(A-C)}$$

$$=\frac{1+\cos(\pi-C)\cdot\cos(A-B)}{1+\cos(\pi-B)\cdot\cos(A-C)}=\frac{1+\cos(A-B)\cdot\cos C}{1+\cos(A-C)\cdot\cos B}$$
$$=R.H.S$$

3.62 / 142. Derive cosine formula using the law of sines in a ΔABC.

W.K.T $a = 2R\sin A \quad b = 2R\sin B \quad c = 2R\sin C$

Cosine formula

$$\cos A = \frac{b^2+c^2-a^2}{2bc}$$

$$R.H.S = \frac{b^2+c^2-a^2}{2bc} = \frac{4R^2\sin^2 B + 4R^2\sin^2 C - 4R^2\sin^2 A}{2(2R\sin B)(2R\sin C)}$$

$$= \frac{\sin^2 B + \sin^2 C - \sin^2 A}{2\sin B\sin C}$$

$$= \frac{\sin^2 B + \sin(C+A).\sin(C-A)}{2\sin B\sin C}$$

$$= \frac{\sin^2 B + \sin(\pi - B)\sin(C-A)}{2\sin B\sin C}$$

$$= \frac{\sin^2 B + \sin B.\sin(C-A)}{2\sin B\sin C}$$

$$= \frac{\cancel{\sin B}\left[\sin B + \sin(C-A)\right]}{2\cancel{\sin B}\sin C}$$

$$= \frac{\sin(\pi - \overline{C+A}) + \sin(C-A)}{2\sin C}$$

$$= \frac{\sin(C+A) + \sin(C-A)}{2\sin C}$$

$$= \frac{2\sin C.\cos A}{2\sin C} = \cos A = R.H.S$$

Hence Proved.

Ex : 3.9

1 / 142 In a ΔABC, if $\frac{\sin A}{\sin C} = \frac{\sin(A-B)}{\sin(B-C)}$, Prove that a^2, b^2, c^2 are in A.P

We know that

$a = 2R\sin A \quad b = 2R\sin B \quad c = 2R\sin C$

$$\frac{\sin A}{\sin C} = \frac{\sin(A-B)}{\sin(B-C)}$$

$$\sin A \cdot \sin(B-C) = \sin C \cdot \sin(A-B)$$
$$\sin(\pi - \overline{B+C})\sin(B-C) = \sin(\pi - \overline{A+B})\sin(A-B)$$
$$\sin(B+C)\sin(B-C) = \sin(A+B)\cdot\sin(A-B)$$
$$\sin^2 B - \sin^2 C = \sin^2 A - \sin^2 B$$
$$\left(\frac{b}{2R}\right)^2 - \left(\frac{c}{2R}\right)^2 = \left(\frac{a}{2R}\right)^2 - \left(\frac{b}{2R}\right)^2$$
$$b^2 - c^2 = a^2 - b^2$$
$$2b^2 = a^2 + c^2$$
$\therefore$ a^2, b^2, c^2 are in A.P

2/142 The angles of a triangle ABC, are in A.P and if $b : c = \sqrt{3} : \sqrt{2}$ find $\angle A$.

$$b : c = \sqrt{3} : \sqrt{2}$$
$$\frac{b}{c} = \frac{\sqrt{3}}{\sqrt{2}} = \frac{\sqrt{3}/2}{\sqrt{2}/2}$$

$\frac{b}{c} = \frac{\sin B}{\sin C}$

$$\frac{b}{c} = \frac{\sqrt{3}/2}{1/\sqrt{2}} = \frac{\sin 60^\circ}{\sin 45^\circ}$$
$$\therefore \angle B = 60^\circ \quad \angle C = 45^\circ$$
$$\angle A + \angle B + \angle C = 180^\circ$$
$$\angle A = 180 - 105^\circ = 75^\circ$$
$$\therefore \angle A = 75^\circ$$

3/143 In a ΔABC, if $\cos C = \frac{\sin A}{2\sin B}$ show that the triangle is isosceles.

$$\cos C = \frac{\sin A}{2\sin B}$$
$$\frac{a^2+b^2-c^2}{2ab} = \frac{a/2R}{2b/2R}$$
$$\frac{a^2+b^2-c^2}{\cancel{2}a\cancel{b}} = \frac{a}{2\cancel{b}}$$
$$\cancel{a^2} + b^2 - c^2 = \cancel{a^2}$$
$$b^2 - c^2 = 0$$
$$b^2 = c^2$$
$$\boxed{b = c} \qquad \therefore \angle B = \angle C$$
$\therefore$ ΔABC is an isosceles.

$\frac{4}{143}$. In a ΔABC, Prove that $\frac{\sin B}{\sin C} = \frac{c - a\cos B}{b - a\cos C}$

$$R.H.S = \frac{c - a\cos B}{b - a\cos C} = \frac{\cancel{a\cos B} + b\cos A - \cancel{a\cos B}}{\cancel{a\cos C} + c\cos A - \cancel{a\cos C}}$$

$$= \frac{b\cos A}{c\cos A} = \frac{b}{c} = \frac{2R\sin B}{2R\sin C}$$

$$= \frac{\sin B}{\sin C} = L.H.S$$

Hence Proved.

$\frac{5}{143}$ In a ΔABC, Prove that

$a\cos A + b\cos B + c\cos C = 2a\sin B\sin C$

We know that

$a = 2R\sin A \quad b = 2R\sin B \quad c = 2R\sin C$

$$L.H.S = 2R\sin A\cos A + 2R\sin B\cos B + 2R\sin C\cos C$$

$$= R[\sin 2A + \sin 2B + \sin 2C]$$

$$= R\{2\sin(A+B).\cos(A-B) + 2\sin C\cos C\}$$

$$= 2R\{\sin(\pi - C)\cos(A-B) + \sin C\cos C\}$$

$$= 2R\sin C[\cos(A-B) + \cos C]$$

$$= 2R\sin C[\cos(A-B) + \cos(\pi - \overline{A+B})]$$

$$= 2R\sin C[\cos(A-B) - \cos(A+B)]$$

$$= 2R\sin C[2\sin A\sin B]$$

$$= 2(2R\sin A).\sin B\sin C$$

$$= 2a\sin B\sin C = R.H.S$$

$\frac{6}{143}$ In ΔABC, $\angle A = 60°$. Prove that $b + c = 2a\cos\left(\frac{B-C}{2}\right)$

$$b + c = 2a\cos\frac{B-C}{2}$$

$$\frac{b+c}{2a} = \cos\frac{B-C}{2}$$

$$L.H.S = \frac{b+c}{2a} = \frac{2R\sin B + 2R\sin C}{2 \times 2R\sin A}$$

$$= \frac{\sin B + \sin C}{2\sin A} = \frac{\cancel{2}\sin\frac{B+C}{2}.\cos\frac{B-C}{2}}{\cancel{2}(2\sin A/2.\cos A/2)}$$

$$= \frac{\sin\left(\frac{\pi}{2} - \frac{A}{2}\right).\cos\left(\frac{B-C}{2}\right)}{2\sin A/2\ \cos A/2}$$

$$= \frac{\cancel{\cos\frac{A}{2}}\cdot\cos\frac{B-C}{2}}{2\sin A/2\cdot\cancel{\cos A/2}} = \frac{\cos\frac{B-C}{2}}{2\sin\frac{60^\circ}{2}}$$

$$= \frac{\cos\frac{B-C}{2}}{2\cdot\sin 30^\circ} = \frac{\cos\left(\frac{B-C}{2}\right)}{2\left(1/2\right)}$$

$$\frac{b+c}{2a} = \cos\left(\frac{B-C}{2}\right)$$

$$\therefore\ b+c = 2a\cdot\cos\left(\frac{B-C}{2}\right)$$

Hence Proved.

$\frac{7}{143}$ In ΔABC, Prove the following:

(i) $a\sin\left(\frac{A}{2}+B\right) = (b+c)\sin\frac{A}{2}$

$$\frac{b+c}{a}\sin\frac{A}{2} = \sin\left(\frac{A}{2}+B\right)$$

$$L.H.S = \frac{b+c}{a}\sin\frac{A}{2} = \frac{2R\sin B+2R\sin C}{2R\sin A}\cdot\sin\frac{A}{2}$$

$$= \frac{\sin B+\sin C}{\sin A}\cdot\sin\frac{A}{2}$$

$$= \frac{\cancel{2}\sin\left(\frac{B+C}{2}\right)\cdot\cos\left(\frac{B-C}{2}\right)}{\cancel{2}\,\cancel{\sin A/2}\cdot\cos A/2}\cdot\cancel{\sin A/2}$$

$$= \frac{\sin\left(\frac{\pi}{2}-\frac{A}{2}\right)\cdot\cos\frac{B-C}{2}}{\cos A/2}$$

$$= \frac{\cos A/2\cdot\cos\frac{B-C}{2}}{\cos A/2} = \cos\left(\frac{B-C}{2}\right) = \cos\left(\frac{C-B}{2}\right)$$

$$= \sin\left(\frac{\pi}{2}-\left(\frac{C-B}{2}\right)\right] \qquad \left[\because A+B+C=\pi,\ \frac{A+B+C}{2}=\frac{\pi}{2}\right.$$

$$= \sin\left[\frac{A+B+C}{2}-\frac{C-B}{2}\right]$$

$$= \sin\left[\frac{A}{2}+\frac{B}{2}+\cancel{\frac{C}{2}}-\cancel{\frac{C}{2}}+\frac{B}{2}\right]$$

$$= \sin\left(\frac{A}{2}+B\right) = R.H.S$$

$$\frac{b+c}{a}\sin A/2 = \sin\left(\frac{A}{2}+B\right)$$

$$(b+c)\sin A/2 = a\sin\left(\frac{A}{2}+B\right)$$

Hence Proved.

(ii) $a(\cos B + \cos C) = 2(b+c)\cdot\sin^2\frac{A}{2}$

$$\frac{a}{2(b+c)} = \frac{\sin^2 A/2}{\cos B + \cos C}$$

$$L.H.S = \frac{a}{2(b+c)} = \frac{2R\sin A}{2(2R\sin B + 2R\sin C)}$$

$$= \frac{\sin A}{2(\sin B + \sin C)} = \frac{\cancel{2}\sin A/2\cdot\cos A/2}{\cancel{2}\left(2\sin\frac{B+C}{2}\cdot\cos\frac{B-C}{2}\right)}$$

$$= \frac{\sin A/2\cdot\cancel{\cos A/2}}{2\cancel{\cos\frac{A}{2}}\cdot\cos\frac{B-C}{2}} = \frac{\sin A/2}{2\cos\frac{B-C}{2}}$$

$$= \frac{\sin^2 A/2}{2\sin A/2\cdot\cos\frac{B-C}{2}}$$

$\frac{A+B+C}{2} = \frac{\pi}{2}$

$\frac{B+C}{2} = \frac{\pi}{2} - A/2$

$\frac{A}{2} = \frac{\pi}{2} - \frac{B+C}{2}$

$$= \frac{\sin^2 A/2}{2\cos\left(\frac{B+C}{2}\right)\cdot\cos\frac{B-C}{2}}$$

$$= \frac{\sin^2 A/2}{\cos B + \cos C} = R.H.S$$

$$\therefore \frac{a}{2(b+c)} = \frac{\sin^2 A/2}{\cos B + \cos C}$$

$$a(\cos B + \cos C) = 2(b+c)\sin^2 A/2.$$

(iii) $\frac{a^2 - c^2}{b^2} = \frac{\sin(A-C)}{\sin(A+C)}$

$$L.H.S = \frac{a^2 - c^2}{b^2} = \frac{(2R\sin A)^2 - (2R\sin C)^2}{(2R\sin B)^2}$$

$$= \frac{4R^2\sin^2 A - 4R^2\sin^2 C}{4R^2\sin^2 B} = \frac{\sin^2 A - \sin^2 C}{\sin^2 B}$$

$$= \frac{\sin(A+C)\cdot\sin(A-C)}{\sin^2 B}.$$

$$= \frac{\sin(\pi - B)\cdot\sin(A-C)}{\sin^2 B}$$

$$= \frac{\cancel{\sin B}\cdot\sin(A-C)}{\sin^{\cancel{2}} B} = \frac{\sin(A-C)}{\sin(\pi - \overline{A+C})}$$

$$= \frac{\sin(A-C)}{\sin(A+C)} = R.H.S.$$

(iv) $\dfrac{a\sin(B-C)}{b^2-c^2} = \dfrac{b\sin(C-A)}{c^2-a^2} = \dfrac{c\sin(A-B)}{a^2-b^2}$

$$\frac{a\sin(B-C)}{b^2-c^2} = \frac{2R\sin A\,\sin(B-C)}{4R^2\sin^2 B - 4R^2\sin^2 C}$$

$$= \frac{2R\sin(\pi-\overline{B+C})\cdot\sin(B-C)}{4R^2(\sin^2 B - \sin^2 C)}$$

$$= \frac{2R\,\sin(\cancel{B+C})\,\sin(\cancel{B-C})}{4R^2\,\sin(\cancel{B+C})\cdot\sin(\cancel{B-C})} = \frac{1}{2R} \quad \text{——— } (1)$$

$$\frac{b\sin(C-A)}{c^2-a^2} = \frac{2R\sin B\cdot\sin(C-A)}{4R^2\sin^2 C - 4R^2\sin^2 A}$$

$$= \frac{2R\sin(\pi-\overline{A+C})\cdot\sin(C-A)}{4R^2\left[\sin^2 C - \sin^2 A\right]}$$

$$= \frac{1}{2R}\cdot\frac{\sin(A+C)\cdot\sin(C-A)}{\sin(C+A)\cdot\sin(C-A)} = \frac{1}{2R} \quad \text{——— } (2)$$

lly $\dfrac{c\sin(A-B)}{a^2-b^2} = \dfrac{1}{2R}$ ——— (3)

from (1), (2) & (3)

$$\frac{a\sin(B-C)}{b^2-c^2} = \frac{b\sin(C-A)}{c^2-a^2} = \frac{c\sin(A-B)}{a^2-b^2}$$

(v) $\dfrac{a+b}{a-b} = \tan\dfrac{A+B}{2}\cdot\cot\dfrac{A-B}{2}$

$$\text{L.H.S} = \frac{a+b}{a-b} = \frac{2R\sin A + 2R\sin B}{2R\sin A - 2R\sin B}$$

$$= \frac{\sin A + \sin B}{\sin A - \sin B}$$

$$= \frac{\cancel{2}\sin\frac{A+B}{2}\cdot\cos\frac{A-B}{2}}{\cancel{2}\cos\frac{A+B}{2}\cdot\sin\frac{A-B}{2}}$$

$$= \tan\frac{A+B}{2}\cdot\cot\frac{A-B}{2}$$

$$= \text{R.H.S.}$$

$\frac{8}{143}$ In a ΔABC, Prove that

$(a^2-b^2+c^2)\tan B = (a^2+b^2-c^2)\tan C$

$$\frac{a^2+b^2-c^2}{a^2+c^2-b^2} = \frac{\tan B}{\tan C}$$

$\left[\because \begin{array}{l} a^2+b^2-c^2 = 2ab\cos C \\ a^2+c^2-b^2 = 2ac\cos B \end{array}\right.$

$$L.H.S = \frac{\cancel{2}\cancel{a}b\cos C}{\cancel{2}\cancel{a}c\cos B} = \frac{b\cos C}{c\cos B}$$

$$= \frac{(2R\sin B)\cos C}{(2R\sin C)\cdot\cos B} = \frac{\sin B\cos C}{\sin C\cos B}$$

$$= \tan B \cdot \cot C$$

$$= \frac{\tan B}{\tan C} = R.H.S$$

$$\frac{a^2+b^2-c^2}{a^2+c^2-b^2} = \frac{\tan B}{\tan C}$$

$$(a^2+c^2-b^2)\tan B = (a^2+b^2-c^2)\tan C$$

Hence Proved.

$\frac{11}{143.}$ Derive Projection formula from (i) Law of Sines (ii) Law of Cosines.

$$\boxed{\begin{array}{c}\text{Projection formula} \\ a = b\cos C + c\cos B\end{array}}$$

(i) Law of Sines

$$R.H.S = b\cos C + c\cos B$$

$$= 2R\sin B\cdot\cos C + 2R\sin C\cos B$$

$$= 2R[\sin B\cos C + \sin C\cos B]$$

$$= 2R[\sin(B+C)] = 2R[\sin(\pi - A)]$$

$$= 2R\sin A$$

$$= a = L.H.S$$

(ii) Law of Cosines:

$$R.H.S = b\cos C + c\cos B$$

$$= b\left\{\frac{a^2+b^2-c^2}{2ab}\right\} + c\left\{\frac{a^2+c^2-b^2}{2ac}\right\}$$

$$= \frac{a^2+b^2-c^2}{2a} + \frac{a^2+c^2-b^2}{2a} = \frac{1}{2a}\left[a^2+\cancel{b^2}-\cancel{c^2}+a^2+\cancel{c^2}-\cancel{b^2}\right]$$

$$= \frac{2a^2}{2a} = a = L.H.S$$

Hence Proved.

Theorem: 3.7:

In ΔABC, $\Delta = \sqrt{s(s-A)(s-B)(s-C)}$ where s is the Semi-Perimeter of ΔABC.

W.K.T

$$\text{Area of } \Delta ABC = \frac{1}{2}ab\sin C$$

$$= \frac{1}{2}ab \cdot 2\sin\frac{C}{2} \cdot \cos\frac{C}{2}$$

$$= ab\sin\frac{C}{2} \cdot \cos\frac{C}{2}$$

$$= ab\sqrt{\frac{(s-a)(s-b)}{ab}}\sqrt{\frac{s(s-c)}{ab}}$$

$$= \cancel{ab}\,\frac{\sqrt{s(s-a)(s-b)(s-c)}}{\cancel{ab}}$$

$$\therefore \Delta = \sqrt{s(s-a)(s-b)(s-c)} \text{ sq. units.}$$

where $s = \frac{a+b+c}{2}$

It is called Heron's Formula.

3.63 / 142 Using Heron's Formula, show that the equilateral triangle has the maximum area for any fixed Perimeter

Let ABC be a triangle with constant Perimeter $2s$.

$$\Delta = \sqrt{s(s-a)(s-b)(s-c)}$$

If Δ is maximum $s(s-a)(s-b)(s-c)$ is maximum.

$$(s-a)(s-b)(s-c) \le \left[\frac{(s-a)+(s-b)+(s-c)}{3}\right]^3$$

$$(s-a)(s-b)(s-c) \le \frac{s^3}{27} \qquad [\because AM \ge G.M]$$

[$\because$ In $xyz \le k$, maximum occurs when $x=y=z$]

when $s-a = s-b = s-c$. That is $a=b=c$

maximum of $(s-a)(s-b)(s-c)$ is $\frac{s^3}{27}$

$\therefore$ for a fixed Perimeter $2s$,

the area is maximum when $a=b=c$.

$\therefore$ The triangle is an equilateral triangle

area of $\Delta ABC = \sqrt{\frac{s s^3}{27}} = \frac{s^2}{3\sqrt{3}}$ sq. units.

$\frac{9}{143}$ An Engineer has to develop a triangular shaped Park with a Perimeter 120m in a village. The Park to be developed must be of maximum area. Find out the dimensions of the Park.

Since the triangular shaped Park has maximum area, the triangle is an equilateral triangle

$$2s = 120\,m$$

$$s = 60\,m$$

$$a+b+c = 120\,m$$

$$3a = 120$$

$$a = 40\,m.$$

$\therefore$ The dimensions of a triangle are 40m, 40m, 40m.

$\frac{10}{143}$ A rope of length 12m is given. Find the largest area of the triangle formed by this rope and find the dimensions of the triangle so formed.

The largest will be an equilateral triangle.

$$a = \frac{12}{3} = 4\,m$$

$$\text{area of a } \Delta = \frac{s^2}{3\sqrt{3}} \text{ (or) } \frac{\sqrt{3}}{4}a^2$$

$$= \frac{\sqrt{3}}{4}\cdot(4)^2 = 4\sqrt{3}\ m^2.$$

Applications of Triangle :

Working rule to solve an oblique triangle in the following table:

oblique triangle (all angles are acute or one angle is obtuse	Details and application for solution.
SAA	Law Sines
SSA	#(ambiguity arises)
SAS	The given angle must be included angle: Either law of Cosines or tangents.
SSS	Law of cosines; first find the largest angle.
AAA	Infinite number of triangles.

#. Angle is not included. we may have more than one triangle.

Application of law of sines yields three cases: No solution, or one triangle or two triangle.

Suppose a, b, A are known.

Let $h = b \sin A$

If $a < h$, there is no triangle.

If $a = h$ then it is right triangle.

If $a > h$ and $a < b$, we have two triangles.

If $a \geq b$, we have only one triangle.

3.64 / 144 In a ΔABC, $a=3$, $b=5$ and $c=7$ find the values of $\cos A$, $\cos B$, and $\cos C$.

$a=3 \quad b=5 \quad c=7$

By Cosine formula,

$$\cos A = \frac{b^2+c^2-a^2}{2bc} = \frac{5^2+7^2-3^2}{2(5)(7)} = \frac{25+49-9}{70}$$

$$= \frac{65}{70} = \frac{13}{14}.$$

$$\cos B = \frac{c^2+a^2-b^2}{2ac} = \frac{7^2+3^2-5^2}{2(3)(7)} = \frac{49+9-25}{42}$$

$$= \frac{33}{42} = \frac{11}{14}$$

$$\cos C = \frac{a^2+b^2-c^2}{2ab} = \frac{9+25-49}{2(3)(5)} = \frac{-15}{30} = -\frac{1}{2}$$

$$\therefore \cos A = \frac{13}{14} \qquad \cos B = \frac{11}{14} \qquad \cos C = \frac{-1}{2}.$$

3.65 / 144 In ΔABC, $A=30^\circ$ $B=60^\circ$ and $c=10$ find a and b

$A=30^\circ \quad B=60^\circ \quad c=10$

By sine formulae,

$A+B+C=180^\circ$

$90+C=180^\circ$

$\boxed{C=90^\circ}$

$$\frac{a}{\sin A} = \frac{b}{\sin B} = \frac{c}{\sin C}$$

$$\frac{a}{\sin 30^\circ} = \frac{b}{\sin 60^\circ} = \frac{c}{\sin 90^\circ}$$

$$\frac{a}{1/2} = \frac{b}{\sqrt{3}/2} = \frac{10}{1}$$

$$2a = \frac{2b}{\sqrt{3}} = 10$$

$2a = 10 \qquad \frac{2b}{\sqrt{3}} = 10 \Rightarrow \boxed{b = 5\sqrt{3}}$

$\boxed{a=5}$

$\therefore a=5 \qquad b=5\sqrt{3}$

3.66 / 144 In a ΔABC, if $a = 2\sqrt{2}$ $b = 2\sqrt{3}$ and $C = 75^\circ$ find the other side and the angles.

$a = 2\sqrt{2}$ $b = 2\sqrt{3}$ $C = 75^\circ$

By cosine formulae

$$c^2 = a^2 + b^2 - 2ab\cos C$$
$$= 8 + 12 - 2(2\sqrt{2})(2\sqrt{3}) \cdot \cos 75^\circ$$
$$= 20 - \cancel{8}^{4}\sqrt{6}\left(\frac{\sqrt{3}-1}{\cancel{2}\sqrt{2}}\right)$$

$\cos 75^\circ = \sin 15^\circ = \frac{\sqrt{3}-1}{2\sqrt{2}}$

$$= 20 - 4\sqrt{3}(\sqrt{3}-1)$$
$$= 20 - 12 + 4\sqrt{3} = 8 + 4\sqrt{3}$$
$$= 6 + 2 + 4\sqrt{3}$$
$$= \sqrt{6}^2 + \sqrt{2}^2 + 2(\sqrt{6})(\sqrt{2}) = (\sqrt{6}+\sqrt{2})^2$$

$c = \sqrt{6} + \sqrt{2} = \sqrt{2}(\sqrt{3}+1)$ $\boxed{c = \sqrt{2}(\sqrt{3}+1)}$

$$\cos A = \frac{b^2 + c^2 - a^2}{2bc} = \frac{(2\sqrt{3})^2 + 2(\sqrt{3}+1)^2 - (2\sqrt{2})^2}{2(2\sqrt{3})(\sqrt{2}(\sqrt{3}+1))}$$
$$= \frac{12 + 2(3 + 1 + 2\sqrt{3}) - 8}{4\sqrt{3}(\sqrt{2}(\sqrt{3}+1))}$$
$$= \frac{12 + \cancel{8} + 4\sqrt{3} - \cancel{8}}{4\sqrt{6}(\sqrt{3}+1)} = \frac{12 + 4\sqrt{3}}{4\sqrt{18} + 4\sqrt{6}}$$
$$= \frac{12 + 4\sqrt{3}}{12\sqrt{2} + 4\sqrt{6}} = \frac{4(\sqrt{3}+3)}{4\sqrt{2}(\sqrt{3}+3)} = \frac{4}{4\sqrt{2}} = \frac{1}{\sqrt{2}}$$

$$\cos A = \frac{1}{\sqrt{2}}$$
$$A = 45^\circ$$

$$A + B + C = 180^\circ$$
$$45 + B + 75 = 180^\circ$$
$$120 + B = 180^\circ$$
$$\boxed{B = 60^\circ}$$

$\therefore$ $A = 45^\circ$ $B = 60^\circ$ $c = \sqrt{2}(\sqrt{3}+1)$

3.67 / 145 Find the area of the triangle whose sides are 13cm, 14 cm and 15 cm

$$S = \frac{a+b+c}{2} = \frac{13+14+15}{2} = \frac{42}{2} = 21 \text{ cm}$$

$$A = \sqrt{S(S-a)(S-b)(S-c)} = \sqrt{21(21-13)(21-14)(21-15)}$$
$$= \sqrt{21(8)(7)(6)} = \sqrt{3\times7\times2\times2\times2\times7\times2\times3}$$
$$= 3\times7\times2\times2 = 84 \text{ sq. cm.}$$

3.68 / 145

In any ΔABC, Prove that $a\cos A + b\cos B + c\cos C = \frac{8\Delta^2}{abc}$

W.K.T

$a\cos A + b\cos B + c\cos C = 2a\sin B\sin C$

$\Delta = \frac{1}{2}ab\sin C = \frac{1}{2}ac\sin B$ [5/143]

$\sin C = \frac{2\Delta}{ab}$ $\quad$ $\sin B = \frac{2\Delta}{ac}$

$a\cos A + b\cos B + c\cos C = 2a\sin B\sin C$

$= 2a\left(\frac{2\Delta}{ac}\right)\left(\frac{2\Delta}{ab}\right)$

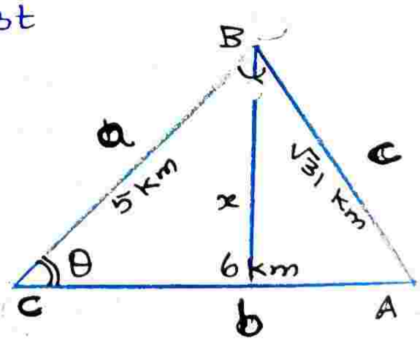
$= \frac{8\Delta^2}{abc}$

Hence Proved.

3.69 / 145.

Suppose that there are two cell phone towers within range of a cell phone. The two towers are located at 6km apart along a straight highway, running east to west and the cell phone is north of the highway. The signal is 5 km from the first tower and $\sqrt{31}$ km from the second tower. Determine the position of the cell phone north and east of the first tower and how far it is from the highway.

Let θ be the position of the cell phone from north to east of the first tower.



By cosine formula

$c^2 = a^2 + b^2 - 2ab\cos\theta$

$\sqrt{31}^2 = 25 + 36 - 2(5)(6)\cos\theta$

$31 = 61 - 60\cos\theta$

$60\cos\theta = 61 - 31$

$60\cos\theta = 30$

$\cos\theta = \frac{1}{2}$ $\quad$ $\boxed{\theta = 60^\circ}$

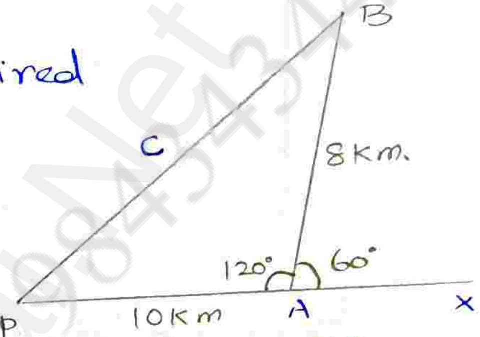
Let x be the distance of the cell phone's' position from the highway.

$$\sin\theta = \frac{x}{5}$$

$$x = 5\sin\theta = 5\sin 60^\circ = \frac{5\sqrt{3}}{2} \text{ Km}$$

3.70 / 146 Suppose that a boat travels 10 km from the port towards east and then turns 60° to its left. If the boat travels further 8 km, how far from the port is the boat?

Let PB be the required distance.



$$PB = c$$

By Cosine formula,

$$c^2 = a^2 + b^2 - 2ab\cos\theta$$

$$c^2 = 10^2 + 8^2 - 2(10)(8)\cos 120^\circ$$

$\angle BAX = 60^\circ$

$\angle PAB = 180 - 60^\circ = 120^\circ$

$$= 100 + 64 - 160\cos(180 - 60^\circ)$$

$$= 164 + 160\cos 60^\circ$$

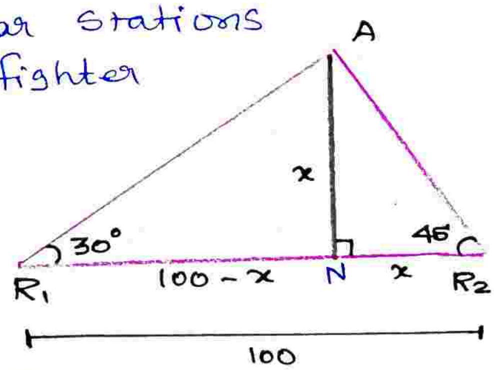
$$= 164 + 160\left(\tfrac{1}{2}\right) = 164 + 80 = 244 \text{ km}$$

$$c^2 = 244$$

$$c = \sqrt{244} = 2\sqrt{61} \text{ km.}$$

3.71 / 146 Suppose two radar stations located 100 km apart, each detect a fighter aircraft between them. The angle of elevation measured by the first station is 30°, whereas the angle of elevation measured by the second station is 45°. Find the altitude of the aircraft at that instant.

Let R_1, R_2 be the two radar stations, A be the position of the fighter aircraft.



Consider the $\triangle ANR_2$

$$\tan 45^\circ = \frac{x}{NR_2} \Rightarrow \boxed{NR_2 = x}$$

Consider the $\triangle AR_1N$

$$\tan 30^\circ = \frac{x}{100 - x} \Rightarrow \frac{1}{\sqrt{3}} = \frac{x}{100 - x}$$

$$100 - x = \sqrt{3}x$$

$$\sqrt{3}x + x = 100 \Rightarrow x = \frac{100}{\sqrt{3}+1} \times \frac{\sqrt{3}-1}{\sqrt{3}-1} = 50(\sqrt{3}-1) \text{ Km}$$

$\therefore$ The height of the fighter aircraft is $50(\sqrt{3}-1)$ Km.

Ex : 3.10

$\frac{1}{146}$ Determine whether the following measurement Produce one triangle, two triangles or no triangles:

$\angle B = 88^\circ$ $a = 23$ $b = 2$.

Solve if solution exists.

By Sine formula

$$\frac{a}{\sin A} = \frac{b}{\sin B}$$

$$\frac{23}{\sin A} = \frac{2}{\sin 88^\circ} = \frac{2}{(1)} \qquad \sin 90^\circ \simeq \sin 88^\circ \simeq 1$$

$\sin A \neq \frac{23}{2}$ which is not Possible.

$\therefore$ no such triangle exist.

$\frac{2}{146}$ If the sides of a ΔABC are $a = 4$ $b = 6$ and $c = 8$ then show that $4\cos B + 3\cos C = 2$

By Projection formula

$$a = b\cos C + c\cos B$$

$$4 = 6\cos C + 8\cos B$$

$\div 2$ $\quad 2 = 3\cos C + 4\cos B$

$\therefore 3\cos C + 4\cos B = 2$

Hence Proved.

$\frac{3}{146}$. In a ΔABC, if $a = \sqrt{3}-1$ $b = \sqrt{3}+1$ and $C = 60^\circ$ find the other side and other two angles.

$a = \sqrt{3}-1$ $\quad b = \sqrt{3}+1$ $\quad \angle C = 60^\circ$

By cosine formula.

$$c^2 = a^2 + b^2 - 2ab\cos C$$

$$= (\sqrt{3}-1)^2 + (\sqrt{3}+1)^2 - 2(\sqrt{3}-1)(\sqrt{3}+1)\cos 60^\circ$$

$$= 2(3+1) - 2(3-1)\tfrac{1}{2}$$

$$= 8 - 2 = 6$$

$\boxed{c = \sqrt{6}}$ $\qquad \left[(a+b)^2 + (a-b)^2 = 2(a^2+b^2)\right]$

By sine formula

$$\frac{a}{\sin A} = \frac{c}{\sin C} \Rightarrow \frac{\sqrt{3}-1}{\sin A} = \frac{\sqrt{6}}{\sin 60^\circ}$$

$$\frac{\sqrt{3}-1}{\sin A} = \frac{\sqrt{6}}{\sqrt{3}/2} = \cancel{\sqrt{6}}^{\sqrt{2}} \times \frac{2}{\cancel{\sqrt{3}}}$$

$$\frac{\sqrt{3}-1}{\sin A} = 2\sqrt{2} \Rightarrow \sin A = \frac{\sqrt{3}-1}{2\sqrt{2}} = \sin 15^\circ$$

$$\boxed{A = 15^\circ}$$

$A = 15^\circ \quad C = 60^\circ$

$A + B + C = 180^\circ$

$B = 180 - 75^\circ = 105^\circ$

$\therefore A = 15^\circ \quad B = 105^\circ \quad C = 60^\circ$

$a = \sqrt{3}-1 \quad b = \sqrt{3}+1 \quad c = \sqrt{6}.$

$\frac{4}{146}$ In any ΔABC, Prove that $\Delta = \frac{b^2+c^2-a^2}{4\cot A}$

W.K.T $\quad \frac{b^2+c^2-a^2}{2bc} = \cos A$

$b^2+c^2-a^2 = 2bc\cos A$

$$R.H.S = \frac{b^2+c^2-a^2}{4\cot A} = \frac{\cancel{2}bc\cos A}{\cancel{4}_2\cot A}$$

$= \frac{1}{2}bc.\sin A = \Delta = L.H.S$

Hence Proved

$\frac{5}{146}$ In a ΔABC, if $a = 12$ cm $b = 8$cm and $C = 30^\circ$ then show that its area is 24 cm^2

$a = 12$ cm $\quad b = 8$ cm $\quad \angle C = 30^\circ$

area of $\Delta ABC = \frac{1}{2}ab\sin C.$

$= \frac{1}{2} \times 12 \times 8 \times \sin 30^\circ$

$= \frac{1}{\cancel{2}} \times 12 \times \cancel{8}^{\cancel{4}\,2} \times \frac{1}{\cancel{2}}$

$= 24\ cm^2.$

$\frac{6}{146}$ In ΔABC, if $a = 18cm$ $b = 24cm$ and $c = 30cm$ then show that its area is 216 sq. cm

$a = 18cm \quad b = 24cm \quad c = 30cm$

$$S = \frac{a+b+c}{2} = \frac{18+24+30}{2} = \frac{72}{2} = 36$$

$$\text{area of } \Delta ABC = \sqrt{S(S-a)(S-b)(S-c)}$$

$$= \sqrt{36(36-18)(36-24)(36-30)}$$

$$= \sqrt{36 \times 18 \times 12 \times 6}$$

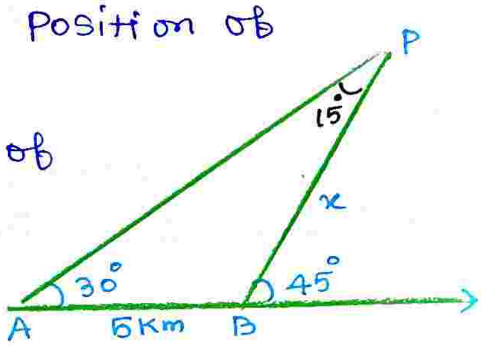
$$= \sqrt{6 \times 6 \times 9 \times 2 \times 4 \times 3 \times 2 \times 3}$$

$$= 6 \times 3 \times 2 \times 2 \times 3 = 216 \text{ cm}^2.$$

$\frac{7}{146}$ Two soldiers A and B in two different underground bunkers on a straight road, spot an intruder at the top of a hill. The angle of elevation of the intruder from A and B to the ground level in the eastern direction 30° and 45° respectively. If A and B stand 5Km apart. Find the distance of the intruder from B.

Let A and B be the position of two soldiers.

P be the position of the intruder.



$\angle A = 30^\circ$

$\angle B = 180^\circ - 45^\circ = 135^\circ$

$\angle P = 180^\circ - (\angle A + \angle B) = 15^\circ$

(or)

$\angle P = 45^\circ - 30^\circ = 15^\circ$

By sine formula

$$\frac{a}{\sin A} = \frac{b}{\sin B} = \frac{c}{\sin C}$$

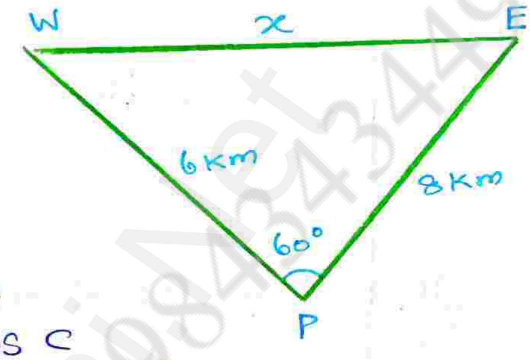
$$\frac{x}{\sin 30^\circ} = \frac{5}{\sin 15^\circ} \Rightarrow x = \frac{5 \sin 30^\circ}{\sin 15^\circ}$$

$$x = \frac{5(1/2)}{\frac{\sqrt{3}-1}{2\sqrt{2}}} = \frac{5\sqrt{2}}{\sqrt{3}-1} \text{ km.}$$

$\therefore$ The distance of the intruder from B is $5\sqrt{2}/\sqrt{3}-1$ km

8
146

A researcher wants to determine the width of a Pond from east to west, which Cannot be done by actual measurement. From a Point P, he finds the distance to the eastern most Point of the Pond to be 8 km, while the distance to the western most Point from P to be 6 km. If the angle between the two lines of Sight is 60°, find the width of the Pond.



Let $PW = 6\,km$

$PE = 8\,km$

$WE = x\,km$

$\angle WPE = 60^\circ$

By cosine formula

$$c^2 = a^2 + b^2 - 2ab\cos C$$

$$x^2 = 6^2 + 8^2 - 2(6)(8).\cos 60^\circ$$

$$= 100 - 96.\tfrac{1}{2} = 100 - 48$$

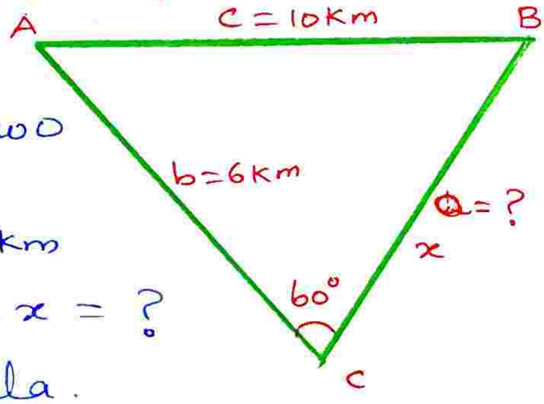
$$x^2 = 52$$

$$x = \sqrt{52} = 2\sqrt{13}\ km.$$

$\therefore$ The width of the Pond is $2\sqrt{13}$ km.

9
147

Two Navy helicopters A and B are flying over the Bay of Bengal at same altitude from the sea level to search a missing boat. Pilots of both the helicopters sight the boat at the same time while they are 10 km from each other. If the distance of the boat from A is 6 km and if the line segment AB subtends 60° at the boat, find the distance of the boat from B.



Let AB be the distance between the two helicopters.

$AB = 10\,km$ $\quad AC = 6\,km$

$\angle ACB = 60^\circ$ $\quad BC = x = ?$

By Cosine formula.

$$c^2 = a^2 + b^2 - 2ab\cos C$$

$$10^2 = x^2 + 6^2 - 2(6)x \cos 60^\circ$$

$$100 = x^2 + 36 - 12x \cdot \frac{1}{2}$$

$$x^2 + 36 - 6x - 100 = 0$$

$$x^2 - 6x - 64 = 0$$

$$x = \frac{6 \pm \sqrt{36+256}}{2} = \frac{6 \pm \sqrt{292}}{2}$$

$$2 \,|\, 292,\quad 2 \,|\, 146,\quad 73$$

$$= \frac{6 \pm 2\sqrt{73}}{2} = 3 \pm \sqrt{73}$$

$$x = 3 + \sqrt{73} \qquad x \neq 3 - \sqrt{73}$$

∴ The distance of the boat from B is $3+\sqrt{73}$ km.

---

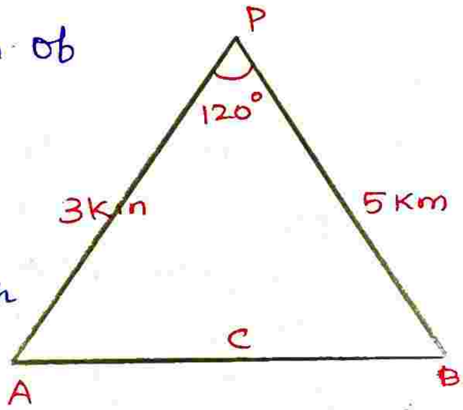
10 / 147. A straight tunnel is to be made through a mountain. A surveyor observes the two extremities A and B of the tunnel to be built from a Point P in front of the mountain. If AP = 3 km, BP = 5 km and $\angle APB = 120^\circ$, then find the length of the tunnel to be built.

Let P be the Position of the surveyor.

PA = 3 Km   PB = 5 Km

$\angle APB = 120^\circ$

Let AB be the length of the tunnel to be built.



By Cosine formula,

$$c^2 = a^2 + b^2 - 2ab\cos C$$

$$c^2 = 3^2 + 5^2 - 2(3)(5)\cos 120^\circ$$

$$= 9 + 25 - 30(\cos(180 - 60^\circ))$$

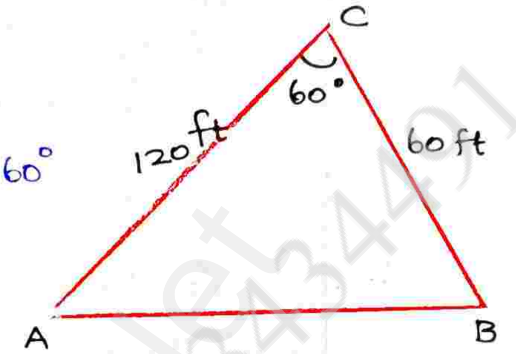
$$= 9 + 25 + 30 \cdot \cos 60^\circ$$

$$= 9 + 25 + 30\left(\tfrac{1}{2}\right)$$

$$c^2 = 34 + 15 = 49$$

$$\boxed{c = 7 \text{ Km}}$$

$\frac{11}{147}$. A farmer wants to Purchase a triangular shaped land with sides 120 ft and 60 ft and the angle included between these two sides is 60°. If the land cost ₹500 Per sq.ft. find the amount he needed to purchase the land. Also find the Perimeter of the land.



area of ΔABC

$= \frac{1}{2} ab \sin C$

$= \frac{1}{2} \times 60 \times 120 \times \sin 60^\circ$

$= 3600 \cdot \frac{\sqrt{3}}{2}$

$= 1800\sqrt{3}$

$= 1800(1.732)$

$= 3117.6$ sq. ft.

$\therefore$ Total cost of the land $= 500(3117.6)$

$= ₹\,155880.$

(ii) $c^2 = a^2 + b^2 - 2ab\cos 60^\circ$

$= 120^2 + 60^2 - \cancel{2}(120)(60) \cdot \frac{1}{\cancel{2}}$

$= 14400 + 3600 - 7200$

$c^2 = 18000 - 7200 = 10800$

$c = 10\sqrt{108} = 10 \times 2 \times 3\sqrt{3}$

$c = 60\sqrt{3}$ ft.

$2\,|\,108$
$2\,|\,54$
$3\,|\,27$
$3\,|\,9$
3

Perimeter $= a + b + c$

$= 120 + 60 + 60\sqrt{3} = 180 + 60\sqrt{3}$

$= 60(3 + \sqrt{3})$ ft

$= 60(4.732) = 283.92$ ft.

$\frac{12}{147}$ A fighter jet has to hit a small target by flying a horizontal distance. When the target is sighted the Pilot measures the angle of depression to be 30°. If after 100 km the target has an angle of depression of 60°, how far is the target from the fighter jet at that instant?

From the given data,

$AB = 100$ km $\quad \angle BAT = \angle ATC = 30^\circ$

$\angle BTC = \angle XBT = 60^\circ \quad \angle ABT = 120^\circ$

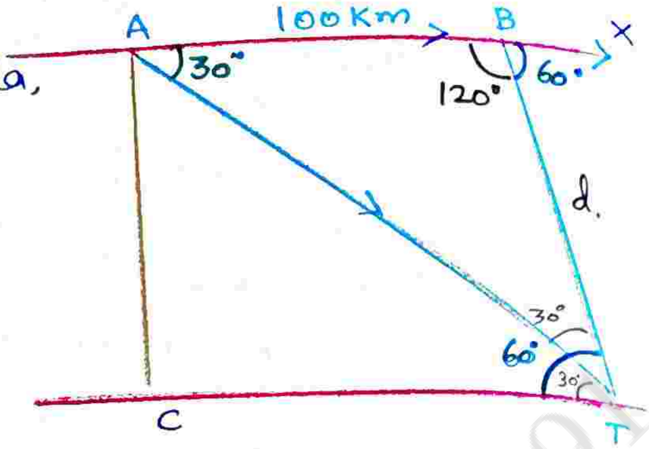
Consider the ΔABT

By sine formula,

$$\frac{a}{\sin A} = \frac{b}{\sin B} = \frac{c}{\sin C}$$

$$\frac{100}{\sin 30^\circ} = \frac{d}{\sin 30^\circ}$$

$$\boxed{d = 100\,km}$$

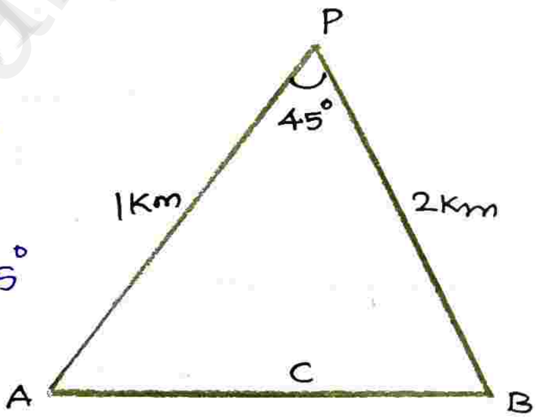


13. A Plane is 1 km from one landmark and 2 km from another. From the Planes Points of view the land between them subtends an angle of 45°. How far apart are the land marks ?

$PA = 1\,Km \quad PB = 2\,km \quad \angle APB = 45^\circ$

$AB = ?$

By cosine formula,



$$c^2 = a^2 + b^2 - 2ab\cos\theta$$

$$c^2 = 1^2 + 2^2 - 2(1)(2)\cos 45^\circ$$

$$= 1 + 4 - 4 \cdot \frac{1}{\sqrt{2}}$$

$$c^2 = 5 - 2\sqrt{2}$$

$$c = \sqrt{5 - 2\sqrt{2}}\ \ km.$$

The distance between the landmarks is $\sqrt{5 - 2\sqrt{2}}$ km.

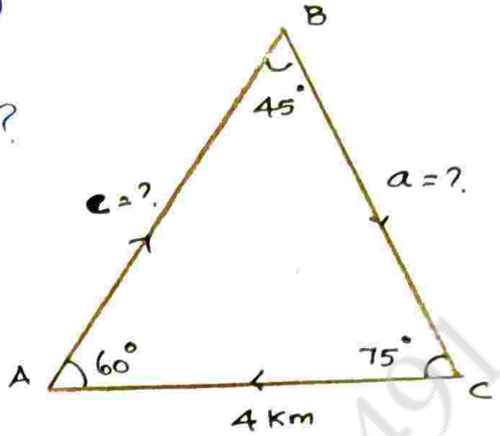
14/147 A man starts his morning walk at a Point A reaches two Points B and C and finally back to A such that $\angle A = 60^\circ$ and $\angle B = 45^\circ$, $AC = 4\,km$ in the ΔABC. Find the total distance he covered during his morning walk.

$\angle A = 60^\circ \quad \angle B = 45^\circ$

$\angle C = 180 - (60^\circ + 45^\circ)$

$\angle C = 75^\circ$

$a = ? \quad b = 4 \text{ km} \quad c = ?$



By Sine formula.

$$\frac{a}{\sin A} = \frac{b}{\sin B} = \frac{c}{\sin C}$$

$$\frac{a}{\sin 60^\circ} = \frac{4}{\sin 45^\circ} = \frac{c}{\sin 75^\circ}$$

$$\frac{a}{\sqrt{3}/2} = \frac{4}{\frac{1}{\sqrt{2}}} = \frac{c}{\frac{\sqrt{3}+1}{2\sqrt{2}}}$$

$$\frac{2a}{\sqrt{3}} = 4\sqrt{2} = \frac{2\sqrt{2}c}{\sqrt{3}+1}$$

$$\frac{2a}{\sqrt{3}} = 4\sqrt{2} \qquad \frac{\cancel{2\sqrt{2}}c}{\sqrt{3}+1} = \cancel{4\sqrt{2}}^{\,2}$$

$$\boxed{a = 2\sqrt{6} \text{ km}} \qquad c = 2(\sqrt{3}+1) = (2\sqrt{3}+2) \text{ km}$$

$\therefore$ Perimeter of the triangle $= a+b+c$

$= 2\sqrt{6} + 4 + 2\sqrt{3} + 2$

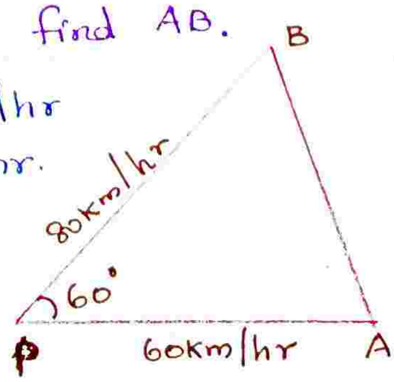
$= 2\sqrt{6} + 2\sqrt{3} + 6$

$= 2(\sqrt{6} + \sqrt{3} + 3)$ km.

$\frac{15}{147}$ Two vehicles leave the same place P at the same time moving along two different roads. One vehicle moves at an average speed of 60 km/hr and other vehicle moves at an average speed of 80 km/hr. After half an hour the vehicle reach the distinations A and B. If AB subtends 60° at the initial point P, then find AB.

Speed of the vehicle A = 60 km/hr

Speed of the vehicle B = 80 km/hr.



distance travelled in half an hour $(AP) = \frac{60}{2}$

$= 30$ km.

$BP = \frac{80}{2} = 40$ km

$\angle APB = 60^\circ$

By Cosine formula

$$c^2 = a^2 + b^2 - 2ab\cos C$$

$$AB^2 = AP^2 + BP^2 - 2AP.PB.\cos 60^\circ$$

$$= 30^2 + 40^2 - \cancel{2}(30)(40)\cdot\frac{1}{\cancel{2}}$$

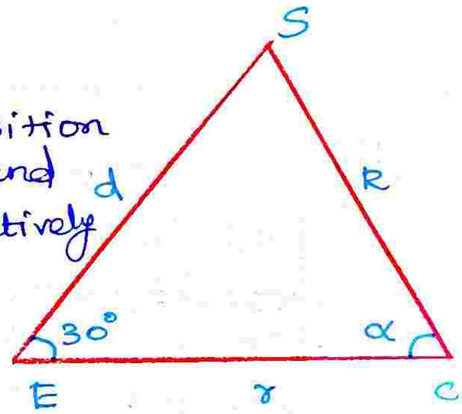
$$= 900 + 1600 - 1200$$

$$AB^2 = 1300$$

$$AB = 10\sqrt{13}\text{ km}$$

16/265 Suppose that a satellite in space, an earth station and the centre of earth all lie in the same plane. Let r be the radius of earth and R be the distance from the centre of earth to the satellite. Let d be the distance from the earth station to the satellite. Let 30° be the angle of elevation from the earth station to the satellite. If the line segment connecting earth station and satellite subtends angle α at the centre of earth, then prove that

$$d = R\sqrt{1 + \left(\frac{r}{R}\right)^2 - \frac{2r}{R}\cos\alpha}.$$



Let S, E, C be the position of satellite, earth station and centre of the earth respectively

$ES = d$ $EC = r$ $SC = R$

$\angle SEC = 30^\circ$ $\angle SCE = \alpha$.

By Cosine formula,

$$d^2 = r^2 + R^2 - 2Rr.\cos\alpha$$

$$= R^2\left[\left(\frac{r}{R}\right)^2 + 1 - \frac{2r}{R}.\cos\alpha\right]$$

$$d = R\sqrt{\left(\frac{r}{R}\right)^2 + 1 - \frac{2r}{R}\cos\alpha}$$

Hence Proved.

Inverse Trigonometric Functions:

function	domain	Range
$\sin x$	$[-\frac{\pi}{2}, \frac{\pi}{2}]$	$[-1, 1]$
$\cos x$	$[0, \pi]$	$[-1, 1]$
$\tan x$	$(-\frac{\pi}{2}, \frac{\pi}{2})$	R (or) $(-\infty, \infty)$
$\operatorname{cosec} x$	$[-\frac{\pi}{2}, \frac{\pi}{2}] - \{0\}$	$R - \{(-1,1)\}$
$\sec x$	$[0, \pi] - \{\frac{\pi}{2}\}$	$R - \{(-1,1)\}$
$\cot x$	$(0, \pi)$	R (or) $(-\infty, \infty)$

function	Domain	Range
$\sin^{-1} x$	$[-1, 1]$	$[-\frac{\pi}{2}, \pi/2]$
$\cos^{-1} x$	$[-1, 1]$	$[0, \pi]$
$\tan^{-1} x$	R (or) $(-\infty, \infty)$	$(-\frac{\pi}{2}, \pi/2)$
$\operatorname{cosec}^{-1} x$	$R - (-1,1)$	$[-\frac{\pi}{2}, \frac{\pi}{2}] - \{0\}$
$\sec^{-1} x$	$R - (-1,1)$	$[0, \pi] - \{\pi/2\}$
$\cot^{-1} x$	R or $(-\infty, \infty)$	$(0, \pi)$

3.72
148

Find the Principal value of

(i) $\sin^{-1}\left(\frac{\sqrt{3}}{2}\right)$

Let $y = \sin^{-1} \frac{\sqrt{3}}{2}$

$\sin y = \frac{\sqrt{3}}{2} = \sin \frac{\pi}{3}$

$y = \frac{\pi}{3} \in [-\frac{\pi}{2}, \frac{\pi}{2}]$.

(ii) $\operatorname{cosec}^{-1}\left(\frac{2}{\sqrt{3}}\right)$

$\operatorname{cosec}^{-1}\left(\frac{2}{\sqrt{3}}\right) = \sin^{-1}\left(\frac{\sqrt{3}}{2}\right)$

Let $y = \sin^{-1}\frac{\sqrt{3}}{2}$

$\sin y = \frac{\sqrt{3}}{2} = \sin\frac{\pi}{3}$

$y = \frac{\pi}{3} \in \left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$

(iii) $\tan^{-1}\left(-\frac{1}{\sqrt{3}}\right)$

Let $y = \tan^{-1}\left(\frac{-1}{\sqrt{3}}\right)$

$\tan y = -\frac{1}{\sqrt{3}} = \tan(-\pi/6)$

$y = -\frac{\pi}{6} \in \left(-\frac{\pi}{2}, \frac{\pi}{2}\right).$

Ex: 3.11

$\frac{1}{149}$ Find the Principal value of

(i) $\sin^{-1}\left(\frac{1}{\sqrt{2}}\right)$

Let $y = \sin^{-1}\left(\frac{1}{\sqrt{2}}\right)$

$\sin y = \frac{1}{\sqrt{2}} = \sin\frac{\pi}{4}$

$y = \frac{\pi}{4} \in \left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$

(ii) $\cos^{-1}\left(\frac{\sqrt{3}}{2}\right)$

$y = \cos^{-1}\left(\frac{\sqrt{3}}{2}\right)$

$\cos y = \frac{\sqrt{3}}{2} = \cos\frac{\pi}{6}$

$y = \frac{\pi}{6} \in [0, \pi]$

(iii) $\operatorname{cosec}^{-1}(-1)$

$y = \operatorname{cosec}^{-1}(-1)$

$\operatorname{cosec} y = -1 \Rightarrow \sin y = -1$

$\sin y = \sin\left(-\frac{\pi}{2}\right)$

$y = -\frac{\pi}{2} \in \left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$

(iv) $\sec^{-1}(-\sqrt{2})$

$$\sec^{-1}(-\sqrt{2}) = \cos^{-1}\left(\frac{-1}{\sqrt{2}}\right)$$

Let $y = \cos^{-1}\left(\frac{-1}{\sqrt{2}}\right)$

$$\cos y = -\frac{1}{\sqrt{2}} = \cos\left(\pi - \frac{\pi}{4}\right)$$

$$\cos y = \cos\frac{3\pi}{4}$$

$$y = 3\pi/4 \in [0, \pi]$$

(v) $\tan^{-1}(\sqrt{3})$

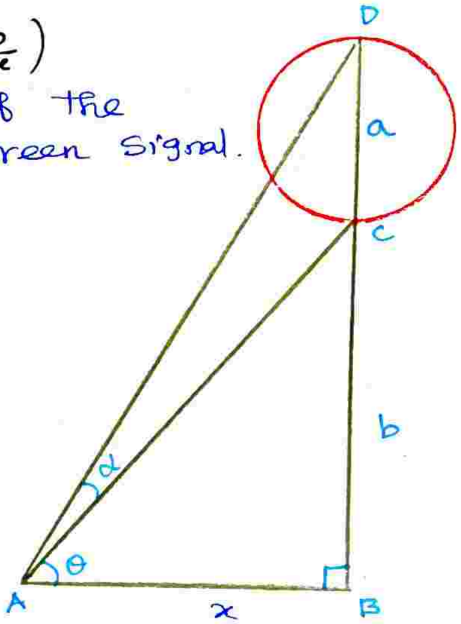
Let $y = \tan^{-1}\sqrt{3}$

$$\tan y = \sqrt{3} = \tan \pi/3$$

$$y = \pi/3 \in \left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$$

$\frac{2}{149}$ A man standing directly opposite to one side of a road of width x metre views a circular shaped traffic green signal of diameter 'a' metere on the other side of the road. The bottom of the green signal is 'b' metre height from the horizontal level of viewer's eye. If α denotes the angle subtended by the diameter of the green signal at the viewer's eye. Then prove that

$$\alpha = \tan^{-1}\left(\frac{a+b}{x}\right) - \tan^{-1}\left(\frac{b}{x}\right)$$



Let CD be the diameter of the circular shaped traffic green signal.

$CD = a$ $\quad BC = b$

$\angle DAC = \alpha$ $\quad$ Let $\angle CAB = \theta$

Consider the Δ ABC

$$\tan\theta = \frac{BC}{AB} \Rightarrow \tan\theta = \frac{b}{AB}$$

$$\tan\theta = \frac{b}{x} \Rightarrow \boxed{\theta = \tan^{-1}\frac{b}{x}}$$

Consider the Δ ABD

$$\tan(\theta+\alpha) = \frac{BD}{AB}$$

$$\tan(\theta+\alpha) = \frac{a+b}{x} \Rightarrow \theta + \alpha = \tan^{-1}\left(\frac{a+b}{x}\right)$$

$$\alpha = \tan^{-1}\left(\frac{a+b}{x}\right) - \theta$$

$$\alpha = \tan^{-1}\left(\frac{a+b}{x}\right) - \tan^{-1}\left(\frac{b}{x}\right)$$

Hence Proved.

5. BINOMIAL THEOREM, SEQUENCES AND SERIES

Binomial Theorem :

Binomial theorem for Positive index:-

$$(a+b)^n = {}^nC_0 a^n b^0 + {}^nC_1 a^{n-1} b^1 + {}^nC_2 a^{n-2} b^2 + \dots + \dots {}^nC_r a^{n-r} \cdot b^r + \dots + {}^nC_n a^0 b^n.$$

Remarks :

* The expansion of $(a+b)^n$, $n \in N$ can also be written as

$$\boxed{(a+b)^n = \sum_{K=0}^{n} {}^nC_K a^{n-K} \cdot b^K \quad (\text{or}) \quad \sum_{K=0}^{n} {}^nC_K a^K b^{n-K}}$$

* The expansion of $(a+b)^n$, $n \in N$ contains exactly $(n+1)$ terms.

* The $(r+1)^{th}$ term in the expansion of $(a+b)^n$, $n \in N$ is

$$\boxed{T_{r+1} = {}^nC_r a^{n-r} b^r, \quad r = 0, 1, 2 \dots n}$$

* In the expansion of $(a+b)^n$, $n \in N$ the Co.efficients at equidistant from the beginning and from the end are equal due to the fact that ${}^nC_r = {}^nC_{n-r}$

* In the expansion of $(a+b)^n$, $n \in N$ the greatest Co.efficient is ${}^nC_{\frac{n}{2}}$ if n is even and the greatest Co.efficients are ${}^nC_{\frac{n-1}{2}}$, ${}^nC_{\frac{n+1}{2}}$ if n is odd.

* In the expansion of $(a+b)^n$ $n \in N$ if n is even, the middle term is

$$T_{\frac{n}{2}+1} = {}^nC_{\frac{n}{2}} a^{n-\frac{n}{2}} \cdot b^{\frac{n}{2}}$$

If n is odd, then the middle terms are $T_{\frac{n-1}{2}+1}$, $T_{\frac{n+1}{2}+1}$

5.3 Particular cases of Binomial Theorem:

$$* \quad (a-b)^n = nC_0\, a^n b^0 - nC_1\, a^{n-1} b^1 + nC_2\, a^{n-2} b^2 - \cdots + (-1)^r\, nC_r\, a^{n-r} \cdot b^r + \cdots + (-1)^n \cdot nC_n\, b^n$$

$$* \quad (1+x)^n = nC_0 + nC_1\, x + nC_2\, x^2 + nC_3\, x^3 + \cdots + nC_n\, x^n$$

(or)

$$(1+x)^n = nC_0\, x^n + nC_1\, x^{n-1} + nC_2\, x^{n-2} + \cdots + nC_n.$$

Pascal Triangle:

								1							
$a+b$							1		1						
$(a+b)^2$						1		2		1					
$(a+b)^3$					1		3		3		1				
$(a+b)^4$				1		4		6		4		1			
$(a+b)^5$			1		5		10		10		5		1		
$(a+b)^6$		1		6		15		20		15		6		1	
$(a+b)^7$	1		7		21		35		35		21		7		1

Remarks:

* Sum of the binomial co.efficients is 2^n.

$$nC_0 + nC_1 + nC_2 + \cdots + nC_n = 2^n$$

* Sum of the co.efficients of odd terms = Sum of the co.efficients of even terms $= 2^{n-1}$

* $(a+b)^n + (a-b)^n = 2\{n_{C_0} a^n + n_{C_2} a^{n-2} b^2 + \cdots\}$
$= 2\{\text{Sum of odd terms}\}$

* $(a+b)^n - (a-b)^n = 2\{n_{C_1} a^{n-1} b^1 + n_{C_3} a^{n-3} b^3 + \cdots\}$
$= 2\{\text{Sum of all even terms}\}$.

* If n is odd
$(a+b)^n + (a-b)^n$ and $(a+b)^n - (a-b)^n$ both have $\frac{n+1}{2}$ terms.

* If n is even
$(a+b)^n + (a-b)^n$ has $\frac{n}{2}+1$ terms whereas
$(a+b)^n - (a-b)^n$ has $\frac{n}{2}$ terms.

5.1 Find the expansion of $(2x+3)^5$.

$$(a+b)^5 = a^5 + 5a^4b + 10a^3b^2 + 10a^2b^3 + 5ab^4 + b^5$$

$$(2x+3)^5 = (2x)^5 + 5(2x)^4(3) + 10(2x)^3(3)^2 + 10(2x)^2(3)^3 + 5(2x)(3)^4 + 3^5$$

$$= 32x^5 + 240x^4 + 720x^3 + 1080x^2 + 810x + 243.$$

5.2 Evaluate : 98^4

$$98^4 = (100-2)^4$$

$$(a-b)^4 = a^4 - 4a^3b + 6a^2b^2 - 4ab^3 + b^4$$

$$(100-2)^4 = 100^4 - 4(100)^3(2) + 6(100)^2(2^2) - 4(100)2^3 + 2^4$$

$$= 100000000 - 8000000 + 240000 - 3200 + 16$$

$$= 92236816.$$

5.3 Find the middle term in the expansion $(x+y)^6$

$n=6$

n is even

It has only one middle term.

$T_{\frac{n}{2}+1}$ is a middle term

$T_{3+1} = T_4$ is a middle term

$T_{r+1} = {}^nC_r\, a^{n-r}\, b^r$ $\qquad a=x$, $b=y$, $n=6$

$T_4 = T_{3+1} = {}^6C_3\, x^{6-3}\, y^3$

$= {}^6C_3\, x^3 y^3$

$= \frac{6\times5\times4}{3\times2\times1} x^3y^3 = 20x^3y^3$

$\therefore$ middle term $= 20x^3y^3$.

5.4 Find the middle terms in the expansion $(x+y)^7$.

$n=7$ $\qquad$ n is odd, it has 2 middle terms.

$T_{\frac{n-1}{2}+1}$, $T_{\frac{n+1}{2}+1}$ are the middle terms

T_4, T_5 are the middle terms.

$T_{r+1} = {}^nC_r\, a^{n-r}\, b^r$ $\qquad a=x$, $b=y$, $n=7$

$T_4 = T_{3+1} = {}^7C_3\, x^4y^3$

$= \frac{7\times6\times5}{3\times2\times1} x^4y^3 = 35x^4y^3$.

$T_5 = T_{4+1} = {}^7C_4\, x^3y^4$

$= 35x^3y^4$

$\therefore$ The middle terms are $35x^4y^3$ and $35x^3y^4$.

5.5 Find the co.efficient of x^6 in the expansion of $(3+2x)^{10}$

$$(3+2x)^{10}$$

$a=3$
$b=2x$
$n=10$

$$T_{r+1} = {}^{n}C_{r}\, a^{n-r} \cdot b^{r}$$

$$T_{r+1} = {}^{10}C_{r}\, 3^{10-r} (2x)^{r}$$

$$= {}^{10}C_{r}\, 3^{10-r} \cdot 2^{r} \cdot x^{r}$$

$$x^{r} = x^{6} \Rightarrow \boxed{r=6}$$

$$T_{6+1} = {}^{10}C_{6}\, 3^{10-6}\, 2^{6} \cdot x^{6}$$

$$= {}^{10}C_{4}\, 3^{4} \cdot 2^{6} \cdot x^{6}$$

$$= \frac{10 \times \overset{3}{\cancel{9}} \times \cancel{8} \times 7}{\cancel{4} \times \cancel{3} \times \cancel{2} \times 1} \cdot 3^{4} \cdot 2^{6} \cdot x^{6}$$

$$= 210 \cdot 3^{4} \cdot 2^{6} \cdot x^{6}$$

$\therefore$ The co.eff of x^6 is $210 \cdot 3^4 \cdot 2^6$

5.6. Find the co.efficient of x^3 in the expansion $(2-3x)^7$

$a=2$
$b=-3x$
$n=7$

$$T_{r+1} = {}^{n}C_{r}\, a^{n-r}\, b^{r}$$

$$T_{r+1} = {}^{7}C_{r}\, 2^{7-r} \cdot (-3x)^{r}$$

$$= {}^{7}C_{r}\, 2^{7-r} (-3)^{r} \cdot x^{r}$$

$$x^{r} = x^{3} \Rightarrow \boxed{r=3}$$

$$T_{3+1} = {}^{7}C_{3}\, 2^{7-3} (-3)^{3} \cdot x^{3}$$

$$= \frac{7 \times 6 \times 5}{3 \times 2 \times 1}\, 2^{4} (-3)^{3} \cdot x^{3}$$

$$= 35 (16) (-27)\, x^{3} = -15120 x^{3}.$$

$\therefore$ The Co.eff of $x^3 = -15120$.

5.8 Expand : $\left(2x-\frac{1}{2x}\right)^4$

$$\boxed{(a-b)^4 = a^4-4a^3b+6a^2b^2-4ab^3+b^4}$$

$a=2x$

$b=\frac{1}{2x}$

$n=4$

$$\left(2x-\frac{1}{2x}\right)^4 = (2x)^4-4(2x)^3\left(\frac{1}{2x}\right)+6(2x)^2\left(\frac{1}{2x}\right)^2-4(2x)\left(\frac{1}{2x}\right)^3+\left(\frac{1}{2x}\right)^4$$

$$=16x^4-16x^2+6-\frac{1}{x^2}+\frac{1}{16x^4}$$

5.9 Expand : $(x^2+\sqrt{1-x^2})^5+(x^2-\sqrt{1-x^2})^5$

$$\boxed{(a+b)^5+(a-b)^5 = 2\{a^5+10a^3b^2+5ab^4\}}$$

$a=x^2$ $\quad$ $b=\sqrt{1-x^2}$ $\quad$ $n=5$

$$(x^2+\sqrt{1-x^2})^5+(x^2-\sqrt{1-x^2})^5$$

$$=2\left\{(x^2)^5+10(x^2)^3(\sqrt{1-x^2})^2+5x^2(\sqrt{1-x^2})^4\right\}$$

$$=2\left[x^{10}+10x^6(1-x^2)+5x^2(1-x^2)^2\right]$$

$$=2\left[x^{10}+10x^6-10x^8+5x^2(1+x^4-2x^2)\right]$$

$$=2\left[x^{10}+10x^6-10x^8+5x^2+5x^6-10x^4\right]$$

$$=2\left[x^{10}+10x^6-10x^8+5x^2+5x^6-10x^4\right]$$

$$=2\left[x^{10}-10x^8+15x^6-10x^4+5x^2\right]$$

5.10 Using binomial theorem, Prove that 6^n-5n always leaves remainder 1 when divided by 25 for all Positive integer 'n'.

$$\boxed{\begin{array}{l}\text{W.K.T}\\ (1+x)^n = nC_0+nC_1x+nC_2x^2+nC_3x^3+\dots+nC_nx^n\end{array}}$$

Put $x=5$

$$(1+5)^n = nC_0+nC_1 5+nC_2 5^2+nC_3 5^3+\dots+nC_n 5^n$$

$$6^n = 1+5n+5^2.nC_2+5^3 nC_3+\dots+5^n nC_n$$

$$6^n-5n = 1+5^2.nC_2+5^3.nC_3+5^4.nC_4+\dots+5^n.nC_n$$

$$6^n-5n = 1+5^2\left[nC_2+5.nC_3+5^2.nC_4+\dots+5^{n-2}.nC_n\right]$$

$6^n - 5n = 1 + 25k$

$k = {}^nC_2 + 5 \cdot {}^nC_3 + 5^2 \cdot {}^nC_4 + \cdots + {}^nC_n \cdot 5^{n-2}]$

$6^n - 5n = 1 + 25k$

Thus $6^n - 5n$ always leaves remainder 1 when divided by 25 for all positive integer n.

5.11 Find the last two digits of the number 7^{400}

$7^{400} = (7^2)^{200} = 49^{200} = (50-1)^{200}$

$\boxed{(a-b)^n = a^n + {}^nC_1 a^{n-1} b + {}^nC_2 a^{n-2} b^2 - \cdots + (-1)^n \cdot b^n}$

$(50-1)^{200} = 50^{200} - 200C_1 (50)^{199} (1) + 200C_2 (50)^{198} - \cdots 200C_{199} 50^1 (1)^{199} + (-1)^{200} \cdot (1)^{200}$

$= 50^2 \{ 50^{198} - 200C_1\, 50^{197} + 200C_2\, 50^{196} - \cdots - 200C_{198} \} - 200(50) + 1$

As 50^2 and 200 are divisible by 100

$\therefore$ The last two digits : 01.

Ex : 5.1

1. Expand : $\left(2x^2 - \frac{3}{x}\right)^3$

$(a-b)^3 = a^3 - 3a^2b + 3ab^2 - b^3$

$\left(2x^2 - \frac{3}{x}\right)^3 = (2x^2)^3 - 3(2x^2)^2\left(\frac{3}{x}\right) + 3(2x^2)\left(\frac{3}{x}\right)^2 - \left(\frac{3}{x}\right)^3$

$= 8x^6 - 3(4x^4)\left(\frac{3}{x}\right) + 6x^2\left(\frac{9}{x^2}\right) - \frac{27}{x^3}$

$= 8x^6 - 36x^3 + 54 - \frac{27}{x^3}$

(ii) $(2x^2-3\sqrt{1-x^2})^4+(2x^2+3\sqrt{1-x^2})^4$

$$\boxed{\begin{aligned}(a+b)^4 &= a^4+4a^3b+6a^2b^2+4ab^3+b^4\\ (a-b)^4 &= a^4-4a^3b+6a^2b^2-4ab^3+b^4\\ (a+b)^4+(a-b)^4 &= 2\{a^4+6a^2b^2+b^4\}\end{aligned}}$$

$a=2x^2 \qquad b=3\sqrt{1-x^2}$

$$(2x^2-3\sqrt{1-x^2})^4+(2x^2+3\sqrt{1-x^2})^4$$

$$=2\left\{(2x^2)^4+6(2x^2)^2(3\sqrt{1-x^2})^2+(3\sqrt{1-x^2})^4\right\}$$

$$=2\{16x^8+6(4x^4)(9)(1-x^2)+81(1-x^2)^2\}$$

$$=2[16x^8+216x^4(1-x^2)+81(1-x^2)^2]$$

2. Compute: (i) 102^4 (ii) 99^4 (iii) 9^7

(i) $(102)^4=(100+2)^4$

$$(a+b)^4=a^4+4a^3b+6a^2b^2+4ab^3+b^4$$

$$(100+2)^4=100^4+4(100)^3(2)+6(100)^2(2)^2+4(100)2^3+2^4$$

$$=100000000+8000000+240000+3200+16$$

$$=108243216$$

(ii) $99^4=(100-1)^4$

$$(a-b)^4=a^4-4a^3b+6a^2b^2-4ab^3+b^4$$

$$(100-1)^4=100^4-4(100)^3(1)+6(100)^2(1)^2-4(100)(1)^3+1^4$$

$$=100000000-4000000+60000-400+1$$

$$=96059601$$

(iii) $9^7=(10-1)^7$

$$(a-b)^7=a^7-7a^6b+21a^5b^2-35a^4b^3+35a^3b^4-21a^2b^5+7ab^6-b^7$$

$$(10-1)^7=10^7-7(10)^6(1)+21(10)^5(1)^2-35(10)^4(1)^3+35(10)^3(1)^4-21(10)^2(1)^5+7(10)(1)^6-1^7$$

$$=10000000-7000000+2100000-350000+35000-2100+70-1$$

$$=4782969.$$

3. Using binomial theorem, indicate which of the following two number is larger: $(1.01)^{1000000}$ and 10000.

$(1.01)^{1000000} = (1+0.01)^{1000000}$

$$\boxed{(1+x)^n = 1+nx+n_{C_1}x^2+n_{C_2}x^2+n_{C_3}x^3+\dots+n_{C_n}x^n}$$

$(1+0.01)^{1000000} = 1+1000000(0.01)+1000000_{C_1}(0.01)^2+$

$1000000_{C_2}(0.01)^3+\dots+1000000_{C_{1000000}}(0.01)^{1000000}$

$= 1+1000000+$ some decimal values

$= 100001 +$ some decimal values > 10000

$\therefore (1.01)^{1000000} > 10000.$

---

4. Find the co.efficient of x^{15} in $\left(x^2+\frac{1}{x^3}\right)^{10}$

$\left(x^2+\frac{1}{x^3}\right)^{10}$ $\quad a = x^2 \quad b = \frac{1}{x^3} \quad n = 10$

$T_{r+1} = n_{C_r} \cdot a^{n-r}\, b^r$

$T_{r+1} = 10_{C_r}(x^2)^{10-r}\left(\frac{1}{x^3}\right)^r$

$= 10_{C_r}\, x^{20-2r} \cdot x^{-3r}$

$= 10_{C_r}\, x^{20-2r-3r}$

$= 10_{C_r}\, x^{20-5r}$

$20-5r = 15$

$5r = 5$

$\boxed{r=1}$

$T_{1+1} = 10_{C_1}\, x^{15} = 10x^{15}$

$\therefore$ The co.efficient of x^{15} is 10.

5. Find the Co.efficient of x^6 and the Co.efficient of x^2 in $\left(x^2 - \frac{1}{x^3}\right)^6$

$$\left(x^2 - \frac{1}{x^3}\right)^6 \qquad a = x^2, \quad b = \frac{-1}{x^3} \qquad n = 6$$

$$T_{r+1} = n_{C_r}\, a^{n-r}\, b^r$$

$$T_{r+1} = 6_{C_r}\, (x^2)^{6-r} \left(\frac{-1}{x^3}\right)^r$$

$$= 6_{C_r}\, x^{12-2r}\, (-1)^r \cdot x^{-3r}$$

$$= 6_{C_r}\, (-1)^r \cdot x^{12-2r-3r}$$

$$= 6_{C_r}\, (-1)^r \cdot x^{12-5r}$$

(i) $12 - 5r = 6$

$5r = 6$

$r \neq \frac{6}{5}$

$\therefore$ x^6 term is not Possible.

(ii) $12 - 5r = 2$

$5r = 10$ $\quad \boxed{r = 2}$

$$T_{2+1} = 6_{C_2}\, (-1)^2\, x^2 = \frac{6 \times 5}{2 \times 1} x^2 = 15x^2$$

$\therefore$ The Co.efficient of $x^2 = 15$.

6. Find the Co.efficient of x^4 in the expansion $(1+x^3)^{50}\left(x^2 + \frac{1}{x}\right)^5$.

$$(1+x^3)^{50}\left(\frac{x^3+1}{x}\right)^5 = (1+x^3)^{55} \cdot x^{-5}$$

$$T_{r+1} = n_{C_r}\, a^{n-r}\, b^r \qquad a = 1, \quad b = x^3, \quad n = 55$$

$$T_{r+1} = 55_{C_r}\, (1)^{55-r} (x^3)^r \cdot x^{-5}$$

$$= 55_{C_r}\, x^{3r} \cdot x^{-5}$$

$$= 55_{C_r}\, x^{3r-5}$$

$$3r - 5 = 4 \Rightarrow 3r = 9 \quad \boxed{r = 3}$$

$$T_{3+1} = 55_{C_3} \cdot x^4 = \frac{55 \times 54 \times 53}{3 \times 2 \times 1} x^4$$

$$= 26235 x^4$$

$\therefore$ The Co.efficient of x^4 is 26235.

7. Find the constant term of $\left(2x^3 - \frac{1}{3x^2}\right)^5$

$$\left(2x^3 - \frac{1}{3x^2}\right)^5 \qquad a = 2x^3 \quad b = \frac{-1}{3x^2} \quad n = 5.$$

$$T_{r+1} = n_{C_r} a^{n-r} b^r$$

$$T_{r+1} = 5_{C_r} (2x^3)^{5-r} \left(\frac{-1}{3x^2}\right)^r$$

$$= 5_{C_r} 2^{5-r} x^{3(5-r)} (-1)^r (3)^{-r} \cdot x^{-2r}$$

$$= 5_{C_r} 2^{5-r} 3^{-r} (-1)^r \cdot x^{15-3r} \cdot x^{-2r}$$

$$= 5_{C_r} \cdot 2^{5-r} \cdot 3^{-r} (-1)^r \cdot x^{15-3r-2r}$$

$$= 5_{C_r} 2^{5-r} \cdot 3^{-r} (-1)^r \cdot x^{15-5r}$$

$$15 - 5r = 0$$

$$5r = 15$$

$$\boxed{r = 3}$$

$$T_{3+1} = 5_{C_3} \cdot 2^{5-3} 3^{-3} (-1)^3 x^0$$

$$= \frac{5 \times 4 \times \cancel{3}}{\cancel{3} \times 2 \times 1} \cdot 2^2 \cdot 3^{-3} (-1)$$

$$= 10 (4) \left(\frac{1}{27}\right) (-1) = \frac{-40}{27}$$

Constant term is $\frac{-40}{27}$.

8. Find the last two digits of the number 3^{600}.

Consider $3^{600} = (3^2)^{300} = 9^{300}$

$$= (10-1)^{300}$$

$$(10-1)^{300} = 300_{C_0} (10)^{300} - 300_{C_1} (10)^{299} (1) + 300_{C_2} (10)^{298} (1)^2 - \dots - 300_{C_{299}} (10)^1 + 1$$

$$= 10^{300} - 300 (10)^{299} + \dots - 300 (10) + 1$$

Hence it is clear that the last two digits in 3^{600} are 01.

9. If n is a Positive integer. show that $9^{n+1}-8n-9$ is always divisible by 64.

$$\boxed{(1+x)^n = {}^nC_0 + {}^nC_1 x + {}^nC_2 x^2 + \cdots + {}^nC_n x^n}$$

$$9^{n+1} = (1+8)^{n+1}$$

$$= {}^{n+1}C_0 + {}^{n+1}C_1(8) + {}^{n+1}C_2(8)^2 + \cdots + {}^{n+1}C_{n+1}8^{n+1}$$

$$= 1 + 8(n+1) + {}^{n+1}C_2 8^2 + {}^{n+1}C_3 8^3 + \cdots + {}^{n+1}C_{n+1}8^{n+1}.$$

$$9^{n+1} = 1+8n+8 + {}^{n+1}C_2(8)^2 + {}^{n+1}C_3(8)^3 + \cdots + {}^{n+1}C_{n+1}8^{n+1}$$

$$9^{n+1}-8n-9 = 8^2\left\{{}^{n+1}C_2 + {}^{n+1}C_3(8) + {}^{n+1}C_4(8^2) + \cdots {}^{n+1}C_{n+1}(8)^{n-1}\right\}$$

$$9^{n+1}-8n-9 = 64k$$

$\therefore$ $9^{n+1}-8n-9$ is divisible by 64.

10. If n is an odd Positive integer, Prove that the co.efficients of the middle terms in the expansion of $(x+y)^n$ are equal.

Since n is an odd integer,

$\therefore$ $(x+y)^n$ has two middle terms.

$T_{\frac{n-1}{2}}$, $T_{\frac{n+1}{2}}$ are the middle terms

The co.efficients of the middle terms are

$${}^nC_{\frac{n-1}{2}} \text{ and } {}^nC_{\frac{n+1}{2}}$$

To Prove : ${}^nC_{\frac{n-1}{2}} = {}^nC_{\frac{n+1}{2}}$ $\boxed{{}^nC_r = {}^nC_{n-r}}$

Proof : ${}^nC_{\frac{n-1}{2}} = {}^nC_{n-\frac{n-1}{2}}$

$$= {}^nC_{\frac{2n-n+1}{2}}$$

$${}^nC_{\frac{n-1}{2}} = {}^nC_{\frac{n+1}{2}}$$

Hence Proved.

11. If n is a Positive integer and r is a non-negative integer. Prove that the Co-efficients of x^r and x^{n-r} in the expansion $(1+x)^n$ are equal.

$(1+x)^n$

Co-efficient of $x^r = nC_r$

Co-efficient of $x^{n-r} = nC_{n-r}$

To Prove : $nC_r = nC_{n-r}$

Proof : $nC_{n-r} = \dfrac{n!}{(n-r)!\,(n-\overline{n-r})!}$

$= \dfrac{n!}{(n-r)!\,r!} = nC_r$

$nC_{n-r} = nC_r$

Hence proved.

12. If a and b are distinct integers, Prove that $a-b$ is a factor of a^n-b^n, whenever n is a Positive integer.

Proof :

$a^n = (a-b+b)^n = [(a-b)+b]^n$

$= nC_0(a-b)^n + nC_1(a-b)^{n-1}\cdot b + nC_2(a-b)^{n-2}\cdot b^2 + \dots + nC_{n-1}(a-b)\,b^{n-1} + nC_n\,b^n$

$a^n = (a-b)^n + nC_1(a-b)^{n-1}\cdot b + nC_2(a-b)^{n-2}b^2 + \dots + nC_{n-1}(a-b)\cdot b^{n-1} + b^n$

$a^n - b^n = (a-b)^n + nC_1(a-b)^{n-1}b + nC_2(a-b)^{n-2}b^2 + \dots + nC_{n-1}(a-b)\,b^{n-1}$

$a^n - b^n = (a-b)\left\{(a-b)^{n-1} + nC_1(a-b)^{n-2}b + nC_2(a-b)^{n-3}b^2 + \dots + nC_{n-1}\,b^{n-1}\right\}$

$a^n - b^n = (a-b)\,k.$

$\therefore (a-b)$ is a factor of $a^n - b^n$

[n is a Positive integer]

13 In the binomial expansion of $(a+b)^n$, the co.efficients of the 4^{th} and 13^{th} terms are equal to each other find 'n'

$$\boxed{(a+b)^n = {}^nC_0\, a^n b^0 + {}^nC_1\, a^{n-1} b^1 + {}^nC_2\, a^{n-2} b^2 \dots + {}^nC_r\, a^{n-r} b^r + \dots {}^nC_n\, b^n}$$

General term :

$$T_{r+1} = {}^nC_r\, a^{n-r} \cdot b^r$$

$T_4 = T_{3+1} = {}^nC_3\, a^{n-3} b^3$

$T_{13} \quad T_{12+1} = {}^nC_{12}\, a^{n-12} b^{12}$

Since the coeff are equal

$${}^nC_3 = {}^nC_{12}$$

$$\boxed{\begin{array}{l} {}^nC_x = {}^nC_y \\ \Rightarrow x = y \text{ (or)} \\ x+y = n \end{array}}$$

$n = x + y$

$n = 3 + 12$

$\boxed{n = 15}$

14. If the binomial co.efficients of three consecutive terms in the expansion of $(a+x)^n$ are in the ratio 1:7:42 then find 'n'

Let the three consecutive terms be r^{th}, $(r+1)^{th}$ and $(r+2)^{th}$ terms

Co. eff of r^{th} term $= {}^nC_{r-1}$

Co. eff of $r+1^{th}$ term $= {}^nC_r$

Co. eff of $r+2^{th}$ term $= {}^nC_{r+1}$

Given that

$$1:7:42 = {}^nC_{r-1} : {}^nC_r : {}^nC_{r+1}$$

$$\frac{{}^nC_{r-1}}{{}^nC_r} = \frac{1}{7} \Rightarrow \frac{\dfrac{\cancel{n!}}{(r-1)!\,(n-r+1)!}}{\dfrac{\cancel{n!}}{r!\,(n-r)!}} = \frac{1}{7}$$

$$\frac{r(\cancel{r-1})!\,(\cancel{n-r})!}{(\cancel{r-1})!\,(n-r+1)\,(\cancel{n-r})!} = \frac{1}{7}$$

$$\frac{r}{n-r+1} = \frac{1}{7}$$

$$7r = n - r + 1$$

$$n - 8r + 1 = 0 \quad \text{——— (1)}$$

$$\frac{n_{C_r}}{n_{C_{r+1}}} = \frac{1}{42} \Rightarrow \frac{\frac{\cancel{n!}}{r!\,(n-r)!}}{\frac{\cancel{n!}}{(r+1)!\,(n-r-1)!}} = \frac{1}{6}$$

$$\frac{(r+1)\cancel{r!}\,\cancel{(n-r-1)!}}{\cancel{r!}\,(n-r)\cancel{(n-r-1)!}} = \frac{1}{6}$$

$$\frac{r+1}{n-r} = \frac{1}{6} \Rightarrow 6r + 6 = n - r$$

$$n - 7r - 6 = 0 \quad \text{——— (2)}$$

$$\begin{array}{r} n - 8r + 1 = 0 \\ n - 7r - 6 = 0 \\ (-)\quad (+)\quad (+) \\ \hline -r + 7 = 0 \end{array}$$

$$\boxed{r = 7}$$

Sub $r = 7$ in (1)

$$n - 56 + 1 = 0$$

$$\boxed{n = 55}$$

---

15. In the binomial co.efficients of $(1+x)^n$ the co.efficients of the 5^{th}, 6^{th} and 7^{th} terms are in A.P. Find all values of 'n'

$$(1+x)^n$$

$$T_{r+1} = n_{C_r}(1)^{n-r} \cdot x^r$$

The co.eff of 5^{th}, 6^{th} and 7^{th} terms are n_{C_4}, n_{C_5} and n_{C_6}

Since n_{C_4}, n_{C_5} and n_{C_6} are in A.P

$$\boxed{2b = a + c}$$

$$\boxed{\frac{n_{C_r}}{n_{C_{r-1}}} = \frac{n-r+1}{r} \qquad \frac{n_{C_{r+1}}}{n_{C_r}} = \frac{n-r}{r+1}}$$

$$2(n_{C_5}) = n_{C_4} + n_{C_6}$$

$$\frac{n_{C_4}}{n_{C_5}} + \frac{n_{C_6}}{n_{C_5}} = 2$$

$$\frac{4+1}{n-4} + \frac{n-5}{5+1} = 2$$

$$\frac{5}{n-4} + \frac{n-5}{6} = 2$$

$30 + (n-4)(n-5) = 12(n-4)$

$30 + n^2 - 9n + 20 = 12n - 48$

$n^2 - 21n + 98 = 0$

$(n-14)(n-7) = 0$

$n = 7, 14.$

98

$\frac{-14}{n} \quad \frac{-7}{n}$

---

16. Prove that $C_0^2 + C_1^2 + C_2^2 + C_3^2 + \dots + C_n^2 = \frac{2n!}{(n!)^2}$

$(1+x)^n = C_0 + C_1 x + C_2 x^2 + C_3 x^3 + \dots + C_n x^n$ —— ①

$(1+x)^n = C_0 x^n + C_1 x^{n-1} + C_2 x^{n-2} + \dots + C_n$ —— ②

Multiplying ① & ② and equating the co. eff of x^n of 'x'

$(1+x)^n (1+x)^n = (C_0 + C_1 x + C_2 x^2 + C_3 x^3 + \dots C_n x^n)$
$(C_0 x^n + C_1 x^{n-1} + C_2 x^{n-2} + \dots + C_n)$

$2n_{C_n} = C_0^2 + C_1^2 + C_2^2 + C_3^2 + \dots + C_n^2$

$\frac{2n!}{n!(2n-n)!} = C_0^2 + C_1^2 + C_2^2 + C_3^2 + \dots + C_n^2$

$C_0^2 + C_1^2 + C_2^2 + \dots + C_n^2 = \frac{2n!}{(n!)^2}$

5.7 The 2^{nd}, 3^{rd} and 4^{th} terms in the binomial expansion of $(x+a)^n$ are 240, 720 and 1080 for a suitable value of x. Find x, a and 'n'

$$\boxed{T_{r+1} = n_{C_r} x^{n-r} \cdot a^r}$$

$T_2 = T_{1+1} = n_{C_1} x^{n-1} \cdot a = 240$ —— ①

$T_3 = T_{2+1} = n_{C_2} x^{n-2} a^2 = 720$ —— ②

$T_4 = T_{3+1} = n_{C_3} x^{n-3} a^3 = 1080$ —— ③

$$\frac{(2)}{(1)} \Rightarrow \frac{{}^nC_2\, x^{n-2}.\, a^2}{{}^nC_1\, x^{n-1}.\, a} = \frac{720}{240}$$

$$\frac{\frac{\cancel{n}(n-1)}{2} \cdot x^{n-2-n+1} \cdot a}{\cancel{n}} = \frac{72}{24} = 3$$

$$\frac{n-1}{2} \cdot \frac{a}{x} = 3$$

$$\frac{a}{x} = \frac{6}{n-1} \quad \text{———} \quad (4)$$

$$\frac{(3)}{(2)} \quad \frac{{}^nC_3\, x^{n-3}\, a^3}{{}^nC_2\, x^{n-2} a^2} = \frac{1080}{720}$$

$$\frac{\frac{\cancel{n}(\cancel{n-1})(n-2)}{6\ 3}}{\frac{\cancel{n}(\cancel{n-1})}{\cancel{2}}}\ x^{n-3-n+2} \cdot a = \frac{108}{72} = \frac{3}{2}$$

$$\frac{(n-2)}{3}\, x^{-1} a = \frac{3}{2}$$

$$\frac{a}{x} = \frac{9}{2(n-2)} \quad \text{———} \quad (5)$$

$$(4) = (5)$$

$$\frac{6}{n-1} = \frac{9}{2(n-2)} \Rightarrow \frac{2}{n-1} = \frac{3}{2(n-2)}$$

$$4(n-2) = 3(n-1)$$

$$4n - 8 = 3n - 3$$

$$\boxed{n = 5}$$

Sub $n = 5$ in (1) & (4)

$${}^5C_1\, x^4.\, a = 240 \qquad \frac{a}{x} = \frac{6}{4}$$

$$5ax^4 = 240 \qquad \frac{a}{x} = \frac{3}{2}$$

$$ax^4 = 48 \quad \text{———} \quad (6) \qquad a = \frac{3}{2}x \quad \text{———} \quad (7)$$

Sub (7) in (6)

$$\frac{3}{2}x^5 = 48$$

$$x^5 = 32 \Rightarrow \boxed{x = 2}$$

Sub $x = 2$ in (7)

$$a = \frac{3}{2}(2) = 3 \qquad \boxed{a = 3}$$

$\therefore \quad n = 5 \qquad a = 3 \qquad x = 2.$

5.4 Finite Sequences :

Arithmetic Progression :

$a, a+d, a+2d, a+3d, \dots$

General term $t_n = a+(n-1)d$

$d = t_2 - t_1$ = common difference.

number of terms $= \frac{l-a}{d} + 1$.

Geometric Progression :

$a, ar, ar^2, ar^3, \dots ar^{n-1}$

$a \to$ first term

$r = \frac{t_2}{t_1}$ = common ratio.

General term $t_n = ar^{n-1}$

Arithmetic-Geometric Progression :

$a, (a+d)r, (a+2d)r^2, \dots [a+(n-1)d]r^{n-1}$

$a \to$ first term

$d \to$ common difference

$r \to$ common ratio

General term :

$t_n = [a+(n-1)d]r^{n-1}$.

Harmonic Progression :

$\frac{1}{a}, \frac{1}{a+d}, \frac{1}{a+2d}, \dots \frac{1}{a+(n-1)d}$.

If a, b, c are in

(i) A.P $\Rightarrow 2b = a+c$

(ii) G.P $\Rightarrow b^2 = ac$

(iii) H.P $\Rightarrow \frac{2ab}{a+b}$

Arithmetic Mean:

Let n be any Positive integer. Let $a_1, a_2, a_3 \ldots a_n$ be n numbers. Then the number

$$\frac{a_1+a_2+a_3+\cdots+a_n}{n}$$ is called the arithmetic mean of the numbers $a_1, a_2, a_3 \ldots a_n$.

Geometric Mean:-

Let n be any Positive integer. Let $a_1, a_2, a_3, \ldots a_n$ be n, non-negative numbers. Then the number

$$\sqrt[n]{a_1 a_2 a_3 \ldots a_n}$$ is called the Geometric mean of the numbers $a_1, a_2, a_3 \ldots a_n$.

Harmonic Mean:

The harmonic mean of a set $\{h_1, h_2, h_3 \ldots h_n\}$ of Positive numbers is defined as

$$\frac{n}{\frac{1}{h_1}+\frac{1}{h_2}+\frac{1}{h_3}+\cdots+\frac{1}{h_n}}$$

---

* A is called the arithmetic mean of the numbers a and b if and only if a, A, b are in A P

$$\boxed{A = \frac{a+b}{2} \quad (or) \quad 2A = a+b}$$

* G is called geometric mean of the numbers a and b if and only if a, G, b are in G.P

$$\boxed{G^2 = ab \quad (or) \quad G = \pm\sqrt{ab}}$$

* Harmonic Mean :

H is called the harmonic mean between a and b if a, H, b are in H. P.

$$\boxed{H = \frac{2ab}{a+b}}$$

Theorem : 5.2

If G.M and AM denote the Geometric mean and arithmetic mean of two nonnegative numbers, then

$$AM \geq G.M.$$

Let a and b be any two nonnegative numbers, Then

$$AM = \frac{a+b}{2} \qquad G.M = \sqrt{ab}$$

$$AM - GM = \frac{a+b}{2} - \sqrt{ab}$$

$$= \frac{a+b-2\sqrt{ab}}{2}$$

$$= \frac{\sqrt{a}^2 + \sqrt{b}^2 - 2\sqrt{a}\sqrt{b}}{2}$$

$$= \frac{(\sqrt{a}-\sqrt{b})^2}{2} \geq 0 \text{ ——— } (1)$$

If $a = b$

$$AM - GM = 0 \text{ ——— } (2)$$

from (1) & (2)

$$AM \geq GM.$$

Theorem : 5.3

If G.M and HM denote the Geometric mean and Harmonic mean of two non negative numbers, then

$$GM \geq HM$$

Let a and b be any two non negative numbers then

$GM = \sqrt{ab}$, $HM = \frac{2ab}{a+b}$

$$G.M - HM = \sqrt{ab} - \frac{2ab}{a+b}$$

$$= \frac{(a+b)\sqrt{ab} - 2ab}{a+b}$$

$$= \frac{\sqrt{ab}\{a+b-2\sqrt{ab}\}}{a+b}$$

$$= \frac{\sqrt{ab}(\sqrt{a}^2+\sqrt{b}^2-2\sqrt{a}\sqrt{b})}{a+b}$$

$$= \frac{\sqrt{ab}(\sqrt{a}-\sqrt{b})^2}{a+b} > 0$$

$GM - HM > 0$

$GM > HM$ ——— (1)

If a **and** b are equal

$GM - HM = 0$

$GM = H.M$ ——— (2)

from (1) & (2)

$GM \geq HM.$

For any two Positive numbers, the three means AM, GM and HM are in G.P.

Let a and **b** be any two Positive real numbers.

$AM = \frac{a+b}{2}$ $G.M = \sqrt{ab}$ $HM = \frac{2ab}{a+b}$

$$\frac{G.M}{AM} = \frac{\sqrt{ab}}{\frac{a+b}{2}} = \frac{2\sqrt{ab}}{a+b}$$

$$\frac{HM}{G.M} = \frac{\frac{2ab}{a+b}}{\sqrt{ab}} = \frac{2ab}{a+b} \times \frac{1}{\sqrt{ab}} = \frac{2\sqrt{ab}}{a+b}$$

$$\frac{GM}{AM} = \frac{HM}{G.M}$$

$$GM^2 = AM \times H.M$$

$\therefore$ AM, GM, HM are in G.P

5.12 Prove that if a, b, c are in H.P if and only if $\frac{a}{c} = \frac{a-b}{b-c}$.

a, b, c are in H.P

$\Leftrightarrow \frac{1}{a}, \frac{1}{b}, \frac{1}{c}$ are in A.P

$\Leftrightarrow \frac{2}{b} = \frac{1}{a} + \frac{1}{c}$

$\Leftrightarrow \frac{1}{b} + \frac{1}{b} = \frac{1}{a} + \frac{1}{c}$

$\Leftrightarrow \frac{1}{b} - \frac{1}{a} = \frac{1}{c} - \frac{1}{b}$

$\Leftrightarrow \frac{a-b}{ab} = \frac{b-c}{bc}$

$\Leftrightarrow \frac{a-b}{a} = \frac{b-c}{c}$

$\Leftrightarrow \frac{a}{c} = \frac{a-b}{b-c}$

Hence Proved.

5.13 If the 5^{th} and 9^{th} terms of a HP are $\frac{1}{19}$ and $\frac{1}{35}$. find the 12^{th} term of the sequence.

In H.P $t_5 = \frac{1}{19}$ $t_9 = \frac{1}{35}$

In A.P $t_5 = 19$ $t_9 = 35$

$\cancel{a} + 4d = 19$ ——— (1)

$\cancel{a} + 8d = 35$ ——— (2)

$(-) \quad (-) \quad (-)$

$-4d = -16$

$\boxed{d = 4}$

Sub $d = 4$ in (1)

$a + 16d = 19$

$a = 19 - 16$

$\boxed{a = 3}$

$t_{12} = a + 11d$

$= 3 + 11(4) = 3 + 44 = 47$

$\therefore$ In A.P $t_{12} = 47$

In H.P $t_{12} = \frac{1}{47}$.

5.14
215

Find 7 numbers $A_1, A_2 \ldots A_7$ so that the sequence $4, A_1, A_2, \ldots A_7, 7$ is an A.P. also 4 numbers G_1, G_2, G_3, G_4 so that the sequence $12, G_1, G_2, G_3, G_4, \frac{3}{8}$ is in G.P.

Solution : $4, A_1, A_2, A_3, A_4 \ldots A_7, 7$ are in A.P

$a = 4 \qquad a+8d = 7$

$4 + 8d = 7$

$8d = 3 \qquad \boxed{d = \frac{3}{8}}$

$A_1 = a+d = 4 + \frac{3}{8} = 4\frac{3}{8}$

$A_2 = a+2d = 4 + \frac{6}{8} = 4\frac{6}{8}$

$A_3 = a+3d = 4 + \frac{9}{8} = 5\frac{1}{8}$

$A_4 = a+4d = 4 + \frac{12}{8} = 5\frac{4}{8}$

$A_5 = a+5d = 4 + \frac{15}{8} = 5\frac{7}{8}$

$A_6 = a+6d = 4 + \frac{18}{8} = 6\frac{2}{8}$

$A_7 = a+7d = 4 + \frac{21}{8} = 6\frac{5}{8}$.

(ii) $12, G_1, G_2, G_3, G_4, \frac{3}{8}$ are in G.P

$a = 12 \qquad ar^5 = \frac{3}{8}$

$12r^5 = \frac{3}{8}$

$r^5 = \frac{3}{8} \times \frac{1}{12} = \frac{1}{32}$

$\boxed{r = \frac{1}{2}}$

$G_1 = ar = 12\left(\frac{1}{2}\right) = 6 \qquad G_2 = ar^2 = 12\left(\frac{1}{4}\right) = 3$

$G_3 = ar^3 = 12\left(\frac{1}{8}\right) = \frac{3}{2} = 1\frac{1}{2} \qquad G_4 = ar^3 = 12\left(\frac{1}{16}\right) = \frac{3}{4}$

$\therefore$ The numbers are $6, 3, 1\frac{1}{2}, \frac{3}{4}$.

5.15 If the Product of the 4^{th}, 5^{th} and 6^{th} terms of a Geometric Progression is 4096 and if the Product of the 5^{th}, 6^{th} and 7^{th} terms of it is 32768, find the sum of first 8 terms of the geometric Progression.

$$t_4 . t_5 . t_6 = 4096$$
$$ar^3 . ar^4 . ar^5 = 4096$$
$$a^3 r^{12} = 4096 \quad \text{——— (1)}$$

$$t_5 . t_6 . t_7 = 32768$$
$$ar^4 . ar^5 . ar^6 = 32768$$
$$a^3 r^{15} = 32768 \quad \text{——— (2)}$$

$$\frac{(2)}{(1)} \qquad \frac{a^3 r^{15}}{a^3 r^{12}} = \frac{32768}{4096}$$
$$r^3 = 8 \Rightarrow \boxed{r=2}$$

Sub $r = 2$ in (1)
$$a^3 . 2^{12} = 4096 = 2^{12}$$
$$a^3 = 1$$
$$\boxed{a=1}$$

$$S_n = \frac{a(r^n - 1)}{r-1} \qquad \boxed{r>1}$$
$$S_8 = \frac{1(2^8-1)}{2-1}$$
$$= \frac{256-1}{1} = 255.$$

Ex: 5.2

1. write the first 6 terms of the sequences whose n^{th} terms are given below. and classify them as A.P, G.P, A.G.P and H.P and none of them.

(i) $a_n = \frac{1}{2^{n+1}}$

$a_1 = \frac{1}{2^2}$ $\quad$ $a_2 = \frac{1}{2^3}$ $\quad$ $a_3 = \frac{1}{2^4}$

$a_4 = \frac{1}{2^5}$ $\quad$ $a_5 = \frac{1}{2^6}$ $\quad$ $a_6 = \frac{1}{2^7}$

$\therefore \frac{1}{2^2}, \frac{1}{2^3}, \frac{1}{2^4}, \frac{1}{2^5}, \frac{1}{2^6}, \frac{1}{2^7}$

It is a G.P

(ii) $a_n = \frac{(n+1)(n+2)}{(n+3)(n+4)}$

$a_n = \frac{n+1)(n+2)}{n+3(n+4)}$ $\quad a_1 = \frac{2 \cdot 3}{1+3 \cdot 5} = \frac{6}{16} = \frac{3}{8}$

$a_2 = \frac{3 \cdot 4}{2+3(6)} = \frac{12}{20} = \frac{3}{5}$

$a_3 = \frac{4 \cdot 5}{3+3(7)} = \frac{20}{24} = \frac{5}{6}$

$a_4 = \frac{5 \cdot 6}{4+3(8)} = \frac{30}{28} = \frac{15}{14}$

$a_5 = \frac{6 \cdot 7}{5+3(9)} = \frac{42}{32} = \frac{21}{16}$

$a_6 = \frac{7 \cdot 8}{6+3(10)} = \frac{56}{36} = \frac{14}{18} = \frac{7}{9}.$

$\therefore$ The six terms are $\frac{3}{8}, \frac{3}{5}, \frac{5}{6}, \frac{15}{14}, \frac{21}{16}$ & $\frac{7}{9}$

$\therefore$ The sequence is not an AP, G.P, AGP and H.P.

(iii) $a_n = 4\left(\frac{1}{2}\right)^n$

$a_1 = 4\left(\frac{1}{2}\right) = 2$ $\quad a_2 = 4\left(\frac{1}{2}\right)^2 = 1$

$a_3 = 4\left(\frac{1}{2}\right)^3 = \frac{4}{8} = \frac{1}{2}$ $\quad a_4 = 4\left(\frac{1}{2}\right)^4 = \frac{4}{16} = \frac{1}{4}$

$a_5 = 4\left(\frac{1}{2}\right)^5 = \frac{4}{32} = \frac{1}{8}$ $\quad a_6 = 4\left(\frac{1}{2}\right)^6 = \frac{4}{64} = \frac{1}{16}$

$\therefore$ The Six terms are

$2, 1, \frac{1}{2}, \frac{1}{4}, \frac{1}{8}, \frac{1}{16}$

It forms a G.P

(iv) $a_n = \frac{(-1)^n}{n}$

$a_1 = \frac{(-1)^1}{1} = -1$ $\quad a_2 = \frac{(-1)^2}{2} = \frac{1}{2}$ $\quad a_3 = \frac{(-1)^3}{3} = \frac{-1}{3}$

$a_4 = \frac{(-1)^4}{4} = \frac{1}{4}$ $\quad a_5 = \frac{(-1)^5}{5} = \frac{-1}{5}$ $\quad a_6 = \frac{(-1)^6}{6} = \frac{1}{6}$

$\therefore$ The first Six terms are

$-1, \frac{1}{2}, -\frac{1}{3}, \frac{1}{4}, -\frac{1}{5}, \frac{1}{6}$

The Sequence is an A.G.P

$d = 1 \quad r = -1$

(v) $a_n = \frac{2n+3}{3n+4}$

$a_1 = \frac{2(1)+3}{3(1)+4} = \frac{5}{7}$ $\quad a_2 = \frac{2(2)+3}{3(2)+4} = \frac{7}{10}$

$a_3 = \frac{2(3)+3}{3(3)+4} = \frac{9}{13}$ $\quad a_4 = \frac{2(4)+3}{3(4)+4} = \frac{11}{16}$

$a_5 = \frac{2(5)+3}{3(5)+4} = \frac{13}{19}$ $\quad a_6 = \frac{2(6)+3}{3(6)+4} = \frac{15}{22}$

$\therefore$ The first 6 terms are

$\frac{5}{7}, \frac{7}{10}, \frac{9}{13}, \frac{11}{16}, \frac{13}{19}$ and $\frac{15}{22}$

The Sequence is not an A.P, G.P, A.G.P and H.P

(vi) $a_n = 2018$

$a_1 = 2018 \quad a_2 = 2018 \quad a_3 = 2018 \quad a_4 = 2018$

$a_5 = 2018 \quad a_6 = 2018$

$\therefore$ The first 6 terms are

2018, 2018, 2018, 2018, 2018, 2018

This is a constant sequence which has Same common difference, common ratio.

$\therefore$ This is an A.P, G.P and A.G.P

(vii) $a_n = \frac{3n-2}{3^{n-1}}$

$a_1 = \frac{3(1)-2}{3^{1-1}} = \frac{1}{1} = 1$ $a_2 = \frac{3(2)-2}{3^{2-1}} = \frac{4}{3}$

$a_3 = \frac{3(3)-2}{3^{3-1}} = \frac{7}{9}$ $a_4 = \frac{3(4)-2}{3^{4-1}} = \frac{10}{27}$

$a_5 = \frac{3(5)-2}{3^{5-1}} = \frac{13}{81}$ $a_6 = \frac{3(6)-2}{3^{6-1}} = \frac{16}{243}$

$\therefore$ The first six terms are

$1, \frac{4}{3}, \frac{7}{9}, \frac{10}{27}, \frac{13}{81}, \frac{16}{243}$

$a=1$ $d = 7-\cancel{4} = 3$ $r = \frac{1}{3}$

$\therefore$ The sequence is an A.G.P.

$\frac{2}{217}$ write the first 6 terms of the sequences whose n^{th} term a_n is given below:

(i) $a_n = \begin{cases} n+1 & \text{if } n \text{ is odd} \\ n & \text{if } n \text{ is even.} \end{cases}$

$a_1 = 1+1 = 2$ $a_2 = 2$ $a_3 = 3+1 = 4$

$a_4 = 4$ $a_5 = 5+1 = 6$ $a_6 = 6$

$\therefore$ The first 6 terms are

$2, 2, 4, 4, 6, 6$

(ii) $a_n = \begin{cases} 1 & \text{if } n=1 \\ 2 & \text{if } n=2 \\ a_{n-1} + a_{n-2} & \text{if } n>2 \end{cases}$

$a_1 = 1$ $a_2 = 2$ $a_3 = a_{3-1} + a_{3-2}$

$= a_2 + a_1 = 1+2 = 3$

$a_4 = a_3 + a_2 = 3+2 = 5$

$a_5 = a_4 + a_3 = 5+3 = 8$

$a_6 = a_5 + a_4 = 8+5 = 13$

$\therefore$ The first 6 terms are

$1, 2, 3, 5, 8, 13$

(iii) $a_n = \begin{cases} n & \text{if } n = 1, 2, 3 \\ a_{n-1} + a_{n-2} + a_{n-3} & \text{if } n > 3 \end{cases}$

$a_1 = 1 \quad a_2 = 2 \quad a_3 = 3$

$a_4 = a_3 + a_2 + a_1 = 3 + 2 + 1 = 6$

$a_5 = a_4 + a_3 + a_2 = 6 + 3 + 2 = 11$

$a_6 = a_5 + a_4 + a_3 = 11 + 6 + 3 = 20$

$\therefore$ The first 6 terms are

1, 2, 3, 6, 11, 20

$\frac{3}{218}$ write the n^{th} term of the following sequences:

(i) 2, 2, 4, 4, 6, 6 ...

$a_1 = 2 \quad a_2 = 2 \quad a_3 = 4 \quad a_4 = 4$

$= 1+1 \quad\quad = 1+3$

$a_5 = 1+4 \quad a_6 = 6$

$a_n = \begin{cases} n+1 & \text{if } n \text{ is odd} \\ n & \text{if } n \text{ is even.} \end{cases}$

(ii) $\frac{1}{2}, \frac{2}{3}, \frac{3}{4}, \frac{4}{5} \ldots$

Nr : 1, 2, 3, 4, ... n

Dr : 2, 3, 4, ... n+1

$a_n = \frac{n}{n+1} \quad n \in N.$

(iii) $\frac{1}{2}, \frac{3}{4}, \frac{5}{6}, \frac{7}{8}, \frac{9}{10} \ldots$

Nr : 1, 3, 5, 7, 9 ... 2n−1

Dr : 2, 4, 6, 8, 10 ... 2n

$a_n = \frac{2n-1}{2n}. \quad n \in N.$

(iv) 6, 10, 4, 12, 2, 14, 0, 16, −2, ...

odd terms : 6, 4, 2, 0, −2, ...

$6 = 7-1$ $\quad 4 = 7-3$ $\quad 2 = 7-5$ $\quad 0 = 7-7$

General term $a_n = 7-n$

Even terms : 10, 12, 14, 16, ...

$10 = 8+2$ $\quad 12 = 8+4$ $\quad 14 = 8+6$ $\quad 16 = 8+8$

General term $a_n = 8+n$

$$\therefore a_n = \begin{cases} 7-n & n \text{ is odd} \\ 8+n & n \text{ is even.} \end{cases}$$

$\frac{4}{218}$ The Product of three increasing numbers in G.P is 5832. If we add 6 to the second number and 9 to the third number, Then resulting numbers form an A.P. Find the numbers in G.P.

Let the numbers be

$\frac{a}{r}, a, ar$

$$\text{Product} = \frac{a}{r} \cdot a \cdot ar = 5832$$

$$a^3 = 5832$$

$$\boxed{a = 18}$$

$\frac{a}{r}, a+6, ar+9$ form an A.P

$$2(a+6) = \frac{a}{r} + ar + 9$$

$$2(24) = \frac{18}{r} + 18r + 9$$

$$48 - 9 = \frac{18}{r} + 18r$$

$$39 = \frac{18}{r} + 18r$$

$$18r^2 - 39r + 18 = 0$$

$\div 3$ $$6r^2 - 13r + 6 = 0$$

$$(3r-2)(2r-3) = 0$$

$$r = \frac{2}{3} \qquad r = \frac{3}{2}$$

36

$-4 \quad -9$

$\frac{-4}{6r} \quad \frac{-9}{6r}$

when $a = 18$ $\quad r = \frac{2}{3}$

$$\frac{a}{r}, a, ar \Rightarrow \frac{18}{2/3}, 18, 18\left(\frac{2}{3}\right)$$

$$= 27, 18, 12$$

when $a = 18$ $\quad r = \frac{3}{2}$

12, 18, 27

Since the numbers are increasing order

∴ The numbers are 12, 18, 27

5/218 Write n^{th} term of the sequence $\frac{3}{1^2 2^2}, \frac{5}{2^2 3^2}, \frac{7}{3^2 4^2}, \ldots$ as a difference of two terms

$$\frac{3}{1^2 2^2}, \frac{5}{2^2 3^2}, \frac{7}{3^2 4^2}, \ldots$$

Nr : 3, 5, 7, ... It is an A.P

$$t_n = 3 + (n-1)2 = 3 + 2n - 2$$
$$= 2n+1$$

Dr : $1^2 2^2, 2^2 3^2, 3^2 4^2 \ldots$

$$t_n = n^2 (n+1)^2$$

$\therefore$ General term of the sequence is

$$t_n = \frac{2n+1}{n^2(n+1)^2} = \frac{n^2+2n+1-n^2}{n^2(n+1)^2}$$
$$= \frac{(n+1)^2 - n^2}{n^2(n+1)^2} = \frac{1}{n^2} - \frac{1}{(n+1)^2}$$

---

6. If t_k is the k^{th} term of a G.P then show that t_{n-k}, t_n, t_{n+k} also form a G.P for any positive integer k.

$$t_k = ar^{k-1}$$

$$t_{n-k} = ar^{n-k-1} \qquad t_n = ar^{n-1}$$

$$t_{n+k} = ar^{n+k-1}$$

Suppose t_{n-k}, t_n, t_{n+k} are in G.P

$$(t_n)^2 = (t_{n-k})(t_{n+k})$$

(or)

$$\frac{t_n}{t_{n-k}} = \frac{t_{n+k}}{t_n}$$

$$\frac{t_n}{t_{n-k}} = \frac{ar^{n-1}}{ar^{n-k-1}} = \frac{\cancel{a}\, r^{\cancel{n}-\cancel{1}-\cancel{n}+k+\cancel{1}}}{\cancel{a}} = r^k$$

$$\frac{t_{n+k}}{t_n} = \frac{ar^{n+k-1}}{ar^{n-1}} = \frac{\cancel{a}\, r^{\cancel{n}+k-\cancel{1}-\cancel{n}+\cancel{1}}}{\cancel{a}} = r^k$$

$\therefore$ t_{n-k}, t_n, t_{n+k} are in G.P

7. If a, b, c are in G.P. and if $a^x = b^y = c^z$ then Prove that x, y, z are in A.P

$$a^x = b^y = c^z = k$$

$$a^x = k \qquad b^y = k \qquad c^z = k$$

$$a = k^{\frac{1}{x}} \qquad b = k^{\frac{1}{y}} \qquad c = k^{\frac{1}{z}}$$

Since a, b, c are in G.P

$$b^2 = ac$$

$$\left[k^{\frac{1}{y}}\right]^2 = k^{\frac{1}{x}} \cdot k^{\frac{1}{z}}$$

$$k^{\frac{2}{y}} = k^{\frac{1}{x} + \frac{1}{z}}$$

$$\frac{2}{y} = \frac{1}{x} + \frac{1}{z}$$

$\therefore \frac{1}{x}, \frac{1}{y}, \frac{1}{z}$ are in A.P.

---

8 The AM of two numbers exceeds their G.M by 10. and HM by 16. Find the numbers.

Let the numbers be a & b

$$AM = \frac{a+b}{2} \qquad G.M = \sqrt{ab} \qquad HM = \frac{2ab}{a+b}.$$

$$A - G = 10 \qquad A - H = 16$$

$$G = A - 10 \qquad H = A - 16$$

$$G^2 = AH$$

$$(A-10)^2 = A(A-16)$$

$$\cancel{A^2} - 20A + 100 = \cancel{A^2} - 16A$$

$$-4A = -100$$

$$\boxed{A = 25}$$

$$\frac{a+b}{2} = 25 \qquad\qquad G = 25 - 10$$

$$a + b = 50 \qquad\qquad G = 15$$

$$\sqrt{ab} = 15$$

$$b = 50 - a \quad \text{——} \; (1) \qquad ab = 225 \; \text{——} \; (2)$$

sub (1) in (2)

$$a(50-a) = 225$$

$$-a^2 + 50a = 225$$

$$a^2 - 50a + 225 = 0$$

$$(a-45)(a-5) = 0$$

$$\boxed{a = 5} \qquad \boxed{a = 45}$$

when $a = 45$, $b = 50 - 45 = 5$

when $a = 5$ $b = 50 - 5 = 45$

$\therefore$ The numbers are 45, 5.

9
218 If the roots of the equation $(q-r)x^2 + (r-p)x + (p-q) = 0$ are equal, then show that p, q, r are in A.P

$$(q-r)x^2 + (r-p)x + (p-q) = 0$$

$a = q - r$ $b = r - p$ $c = p - q$

Since the roots are real & equal

$$\Delta = 0$$

$$b^2 - 4ac = 0$$

$$(r-p)^2 - 4(q-r)(p-q) = 0$$

$$r^2 + p^2 - 2pr - 4pq + 4q^2 + 4pr - 4qr = 0$$

$$r^2 + p^2 + 4q^2 + 2pr - 4pq - 4qr = 0$$

$$p^2 + (-2q)^2 + r^2 + 2p(-2q) + 2(-2q)r + 2pr = 0$$

$$(p - 2q + r)^2 = 0$$

$$p - 2q + r = 0$$

$$2q = p + r$$

$\therefore$ p, q, r are in A.P.

---

10. If a, b, c are respectively the p^{th}, q^{th} and r^{th} terms of a G.P, show that $(q-r)\log a + (r-p)\log b + (p-q)\log c = 0$

$t_p = AR^{p-1} = a$ $t_q = AR^{q-1} = b$

$t_r = AR^{r-1} = c$

$$L.H.S = (q-r)\log a + (r-p)\log b + (p-q)\log c$$

$$= \log a^{q-r} + \log b^{r-p} + \log c^{p-q}$$

$$= \log\left\{a^{q-r} \cdot b^{r-p} \cdot c^{p-q}\right\}$$

$$= \log\left\{(AR^{p-1})^{q-r} \cdot (AR^{q-1})^{r-p} \cdot (AR^{r-1})^{p-q}\right\}$$

$$= \log\left\{A^{q-r} \cdot R^{(p-1)(q-r)} \cdot A^{r-p} \cdot R^{(q-1)(r-p)} \cdot A^{p-q} \cdot R^{(r-1)(p-q)}\right\}$$

$$= \log\left\{A^{q-r+r-p+p-q} \cdot R^{(p-1)(q-r)+(q-1)(r-p)+(r-1)(p-q)}\right\}$$

$$= \log\{A^0 \cdot R^0\} = \log 1 = 0 = R.H.S.$$

Finite Series :

Arithmetic Series :

$$S_n = \frac{n}{2}[2a+(n-1)d]$$

$$S_n = \frac{n}{2}[a+l]$$

Geometric Series :

$$S_n = \begin{cases} \dfrac{a(r^n-1)}{r-1} & r>1 \\ \dfrac{a(1-r^n)}{1-r} & r<1 \\ na & r=1 \end{cases}$$

Arithmetico - Geometric Series:

$$S_n = \frac{a-(a+(n-1)d)r^n}{1-r} + dr\left(\frac{1-r^{n-1}}{(1-r)^2}\right)$$

$r \neq 1$.

5.16 / 219 Find the sum up to n terms of the series:

$1+\frac{6}{7}+\frac{11}{49}+\frac{16}{343}+\cdots$

$a=1 \quad d=6-1=5 \quad r=\frac{1}{7}$

$$S_n = \frac{a-(a+(n-1)d)r^n}{1-r} + \frac{dr(1-r^{n-1})}{(1-r)^2} \quad r\neq 1$$

$$= \frac{1-(1+(n-1)5)\left(\frac{1}{7}\right)^n}{1-\frac{1}{7}} + \frac{\frac{5}{7}\left(1-\left(\frac{1}{7}\right)^{n-1}\right)}{\left(1-\frac{1}{7}\right)^2}$$

$$= \frac{1-(1+5n-5)\frac{1}{7^n}}{6/7} + \frac{\frac{5}{7}\left(1+\left(\frac{1}{7}\right)^{n-1}\right)}{\frac{36}{49}}$$

$$= \frac{7}{6}\left[1-(5n-4)\frac{1}{7^n}\right] + \frac{5}{7}\times\frac{49}{36}\left(\frac{7^{n-1}-1}{7^{n-1}}\right)$$

$$= \frac{7}{6}\left[\frac{7^n - 5n + 4}{7^n}\right] + \frac{5}{36} \times 7 \left(\frac{7^{n-1} - 1}{7^{n-1}}\right)$$

$$S_n = \frac{7^n - 5n + 4}{6 \cdot 7^{n-1}} + \frac{5}{36}\left(\frac{7^{n-1} - 1}{7^{n-2}}\right)$$

$\frac{5.17}{219.}$ Find the sum of the first n terms of the series

$$\frac{1}{1+\sqrt{2}} + \frac{1}{\sqrt{2}+\sqrt{3}} + \frac{1}{\sqrt{3}+\sqrt{4}} + \cdots$$

General term $t_K = \frac{1}{\sqrt{K} + \sqrt{K+1}}$

$$t_K = \frac{1}{\sqrt{K}+\sqrt{K+1}} \times \frac{\sqrt{K}-\sqrt{K+1}}{\sqrt{K}-\sqrt{K+1}} = \frac{\sqrt{K}-\sqrt{K+1}}{K-K+1}$$

$$= \sqrt{K} - \sqrt{K+1}$$

$$\therefore t_1 + t_2 + t_3 + \cdots + t_n = (\sqrt{2}-1) + (\sqrt{3}-\sqrt{2}) + (\sqrt{4}-\sqrt{3}) + \cdots \sqrt{n+1} - \sqrt{n}$$

$$S_n = \sqrt{n+1} - 1.$$

$\frac{5.18}{219.}$ Find $\sum_{K=1}^{n} \frac{1}{K(K+1)}$

General term $t_K = \frac{1}{K(K+1)}$

$$\frac{1}{K(K+1)} = \frac{1}{K} - \frac{1}{K+1} \quad [\text{By Partial fraction}]$$

$$t_1 + t_2 + t_3 + \cdots + t_n = \left(1 - \frac{1}{2}\right) + \left(\frac{1}{2} - \frac{1}{3}\right) + \left(\frac{1}{3} - \frac{1}{4}\right) + \cdots + \left(\frac{1}{n} - \frac{1}{n+1}\right)$$

$$S_n = 1 - \frac{1}{n+1}.$$

Ex : 5.3.

$\frac{1}{220}$ Find the sum of the first 20 terms of the A.P having the sum of first 10 term as 52 and sum of the first 15 terms as 77.

$S_{20} = ?$ $\quad S_{10} = 52$

$S_{15} = 77$

$$S_n = \frac{n}{2}[2a + (n-1)d]$$

$S_{10} = 52$

$\frac{10}{2}[2a+9d] = 52$

$2a+9d = \frac{52}{5}$ — (1)

$S_{15} = 77$

$\frac{15}{2}[2a+14d] = 77$

$2a+14d = \frac{77\times2}{15}$

$2a+14d = \frac{154}{15}$ — (2)

$$\cancel{2a}+9d = \frac{52}{5} \quad \text{—— (1)}$$

$$\cancel{2a}+14d = \frac{154}{15} \quad \text{—— (2)}$$

$$(-) \quad (-) \quad (-)$$

$$-5d = \frac{52}{5} - \frac{154}{15}$$

$$-5d = \frac{156-154}{15}$$

$$-5d = \frac{2}{15} \Rightarrow \boxed{d = \frac{-2}{75}}$$

Sub $d = -\frac{2}{75}$ in (1)

$$2a - \frac{18}{75} = \frac{52}{5} \Rightarrow 2a = \frac{52}{5} + \frac{18}{75}$$

$$2a = \frac{780+18}{75} = \frac{798}{75} = \frac{266}{25}$$

$$\boxed{a = \frac{133}{25}}$$

$$S_{20} = \frac{20}{2}[2a+19d]$$

$$= 10\left[\frac{266}{25} - \frac{38}{75}\right]$$

$$= 10\left[\frac{798-38}{75}\right] = 10\left(\frac{760}{75}\right) = \frac{304}{3}$$

$$\therefore \boxed{S_{20} = \frac{304}{3}}$$

$\frac{2}{220}$ Find the sum up to the 17^{th} term of the series $\frac{1^3}{1} + \frac{1^3+2^3}{1+3} + \frac{1^3+2^3+3^3}{1+3+5} + \cdots$

$$t_n = \frac{1^3+2^3+3^3+\cdots+n^3}{1+3+5+7+\cdots+(2n-1)} = \frac{\left[\frac{n(n+1)}{2}\right]^2}{n^2}$$

$$= \frac{\frac{n^2(n+1)^2}{4}}{n^2} = \frac{1}{4}(n+1)^2.$$

$$= \frac{1}{4}[n^2+2n+1]$$

$$S_n = \sum_{k=1}^{n} t_k$$

$$= \frac{1}{4}\sum_{k=1}^{n} (k^2+2k+1)$$

$$= \frac{1}{4}\left\{ \sum_{k=1}^{n} k^2 + \sum_{k=1}^{n} 2k + \sum_{k=1}^{n} 1 \right\}$$

$$= \frac{1}{4}\left[\frac{n(n+1)(2n+1)}{6} + \frac{\cancel{2}(n(n+1)}{\cancel{2}} + n \right]$$

$$= \frac{1}{24}\left[n(n+1)(2n+1) + 6n(n+1) + 6n \right]$$

$$= \frac{1}{24}\left[(n^2+n)(2n+1) + 6n^2+6n+6n \right]$$

$$= \frac{1}{24}\left[2n^3+n^2+2n^2+n+6n^2+12n \right]$$

$$= \frac{1}{24}\left[2n^3+9n^2+13n \right]$$

$$S_n = \frac{n}{24}\left[2n^2+9n+13 \right]$$

$$S_{17} = \frac{17}{24}\left[2(17)^2+9(17)+13 \right]$$

$$= \frac{17}{24}\left[578+153+13 \right]$$

$$= \frac{17}{24}(744) = 17(31) = 527$$

$$\therefore \boxed{S_{17} = 527}$$

$\frac{3}{220}$ Compute the sum of first n terms of the following Series:

(i) $8+88+888+\cdots$

$$S_n = 8+88+888+\cdots \text{ up to } n \text{ terms}$$

$$= 8(1+11+111+\cdots \text{ up to } n \text{ terms})$$

$$= \frac{8}{9}(9+99+999+\cdots \text{ up to } n \text{ terms})$$

$$= \frac{8}{9}\left[(10-1)+(100-1)+(1000-1)+\cdots \text{ up to } n \text{ terms}\right]$$

$$= \frac{8}{9}\left\{(10+100+1000+\cdots \text{ up to } n \text{ terms}) - (1+1+1+\cdots n \text{ times})\right\}$$

$$= \frac{8}{9}\left[10+10^2+10^3+\cdots \text{ up to } n \text{ terms} - n\right]$$

$$= \frac{8}{9}\left[\frac{10(10^n-1)}{10-1} - n\right] \qquad S_n = \frac{a(r^n-1)}{r-1} \quad r>1$$

$$= \frac{8}{9}\left[\frac{10(10^n-1)}{9} - n\right] = \frac{80}{81}(10^n-1) - \frac{8}{9}n.$$

(ii) $6+66+666+\dots$ upto n terms :-

$S_n = 6+66+666+\dots$ upto n terms

$= 6(1+11+111+\dots$ upto n terms$)$

$= \frac{6}{9}(9+99+999+\dots$ upto n terms$)$

$= \frac{2}{3}[(10-1)+(100-1)+(1000-1)+\dots$ upto n times$]$

$= \frac{2}{3}[10+100+1000+\dots$ up to n times $-(1+1+1+\dots$ upto n term$)]$

$= \frac{2}{3}[10+10^2+10^3+10^4+\dots$ up to n times $-n]$

$= \frac{2}{3}\left[\frac{10(10^n-1)}{10-1} - n\right]$ $S_n = \frac{a(r^n-1)}{r-1}$; $r>1$

$= \frac{2}{3}\left(\frac{10}{9}(10^n-1) - n\right)$

$= \frac{20}{27}(10^n-1) - \frac{2n}{3}$

4. Compute the sum of first n terms of $1+(1+4)+(1+4+4^2)+(1+4+4^2+4^3)+\dots$

$t_n = 1+4+4^2+4^3+\dots$ up to n times.

$t_n = \frac{1(4^n-1)}{4-1}$ $a=1$, $r=4$ $r>1$

$= \frac{4^n-1}{3}$

$S_n = \Sigma t_n = \Sigma \frac{4^n-1}{3}$

$= \frac{1}{3}\Sigma(4^n-1) = \frac{1}{3}\{\Sigma 4^n - \Sigma 1\}$

$= \frac{1}{3}\{4^1+4^2+4^3+\dots$ upto n times $-n\}$

$= \frac{1}{3}\left\{\frac{4(4^n-1)}{4-1} - n\right\}$

$= \frac{1}{3}\left\{\frac{4(4^n-1)}{3} - n\right\}$

$S_n = \frac{4}{9}(4^n-1) - \frac{1}{3}n$

$\frac{5}{220}$ Find the general term and sum of n terms of the sequence $1, \frac{4}{3}, \frac{7}{9}, \frac{10}{27}, \ldots$

Nr : $1, 4, 7, 10, \ldots$ $a = 1$ $d = 4 - 1 = 3$

Dr : $1, \frac{1}{3}, \frac{1}{9} \ldots$ $a = 1$ $r = \frac{t_2}{t_1} = \frac{\frac{1}{3}}{1} = \frac{1}{3}$

$\therefore$ The given sequence is an A.G.P.

$$t_n = [a + (n-1)d]\, r^{n-1}$$

$$t_n = [1 + (n-1)3] \left(\frac{1}{3}\right)^{n-1}$$

$$= (1 + 3n - 3)\left(\frac{1}{3}\right)^{n-1}$$

$$t_n = \frac{(3n-2)}{3^{n-1}}$$

$\therefore$ General term $t_n = \frac{3n-2}{3^{n-1}}$.

$$S_n = \frac{a - [a+(n-1)d]\, r^n}{1-r} + dr\left(\frac{1 - r^{n-1}}{(1-r)^2}\right)$$

$$S_n = \frac{1 - (1 + (n-1)3)\left(\frac{1}{3}\right)^n}{1 - \frac{1}{3}} + 3 \cdot \frac{1}{3}\left(\frac{1 - \left(\frac{1}{3}\right)^{n-1}}{\left(1 - \frac{1}{3}\right)^2}\right)$$

$$= \frac{1 - [1 + 3n - 3]\left(\frac{1}{3}\right)^n}{2/3} + \frac{\left(1 - \left(\frac{1}{3}\right)^{n-1}\right)}{(2/3)^2}$$

$$= \left[1 - \frac{(3n-2)}{3^n}\right] \times \frac{3}{2} + \frac{3^{n-1} - 1}{3^{n-1}} \times \frac{9}{4}$$

$$= \frac{3}{2}\left[\frac{3^n - 3n + 2}{3^n}\right] + \frac{3^{n-1} - 1}{4 \times 3^{n-3}}$$

$$S_n = \frac{3}{2}\left(\frac{3^n - (3n-2)}{3^n}\right) + \frac{1}{4}\left(\frac{3^{n-1} - 1}{3^{n-3}}\right)$$

$\frac{6}{220}$ Find the value of n, if the sum of n terms of the series $\sqrt{3} + \sqrt{75} + \sqrt{243} + \cdots$ is $435\sqrt{3}$.

$$\sqrt{3} + \sqrt{75} + \sqrt{243} + \cdots + x = 435\sqrt{3}$$

$$\sqrt{3} + 5\sqrt{3} + 9\sqrt{3} + \cdots + x = 435\sqrt{3}$$

$$S_n = 435\sqrt{3}$$

It forms an A.P

$a = \sqrt{3}$ $\quad d = 5\sqrt{3} - \sqrt{3} = 4\sqrt{3}$

$$S_n = \frac{n}{2}[2a+(n-1)d]$$

$$\frac{n}{2}[2a+(n-1)d] = 435\sqrt{3}$$

$$\frac{n}{2}[2\sqrt{3}+(n-1)4\sqrt{3}] = 435\sqrt{3}$$

$\div \sqrt{3}$

$$\frac{n}{2}[2+4(n-1)] = 435$$

$$\frac{2n}{2}(1+2(n-1)) = 435$$

$$n(1+2n-2) = 435$$

$$n(2n-1) = 435$$

$$2n^2 - n - 435 = 0$$

$$(n-15)(n+29) = 0$$

$\boxed{n=15}$ $\quad n \neq -29$

2	2×435
3	435
5	145
	29

-15, $\frac{-30}{2n}$ $\quad$ $\frac{29}{2n}$

$\therefore S_{15} = 435\sqrt{3}$.

$\frac{7}{220}$ Show that the sum of $(m+n)^{th}$ and $(m-n)^{th}$ term of an A.P is equal to twice the m^{th} term.

To Prove: $t_{m+n} + t_{m-n} = t_m$

$$t_{m+n} + t_{m-n} = a+(m+n-1)d + a+(m-n-1)d$$

$$= 2a + (m+\cancel{n}-1+m-\cancel{n}-1)d$$

$$= 2a+(2m-2)d$$

$$= 2[a+(m-1)d]$$

$$= 2t_m$$

Hence Proved.

$\frac{8}{220}$ A man repays an amount of Rs 3250 by paying Rs 20 in the first month and then increases the payment by Rs 15 per month. How long will it take him to clear the amount?

The amount forms an A.P

$S_n = 3250$ $\quad a = 20$ $\quad d = 15$

$$S_n = \frac{n}{2}[2a+(n-1)d]$$

$S_n = 3250$

$\frac{n}{2}\left[2(20)+(n-1)15\right] = 3250$

$\frac{\cancel{5}n}{2}\left[8+3(n-1)\right] = \overset{650}{\cancel{3250}}$

$n(8+3n-3) = 1300$

$3n^2+5n-1300=0$

$(n-20)(3n+65)=0$

$\boxed{n=20}$ $\quad n \neq -\frac{65}{3}$

3 | 1300 × 3
2 | 1300
2 | 650
5 | 325
65

$-20 \quad 65$

$-\frac{60}{3n} \quad \frac{65}{3n}$

∴ The amount will clear in 20 months.

9/220 In a race, 20 balls are placed in a line at intervals of 4m. with the first ball 24 m away from the starting point. A contestant is required to bring the balls back to the starting place one at ~~from~~ a time. How far would the ~~the starting~~ contestant run to bring back all balls?

Distance travelled to bring the first ball $= 24+24 = 48m$

Distance travelled to bring the second ball $2(24+4) = 56m$

∴ The series is

$48, 56, 64, \ldots$

$a = 48 \quad d = 8 \quad n = 20$

$S_n = \frac{n}{2}\left[2a+(n-1)d\right]$

$= \frac{20}{2}\left[2(48)+19(8)\right]$

$= 10(96+152) = 10(248)$

$= 2480\,m.$

∴ Total distance $= 2480\,m.$

$\frac{10}{220}$ The number of bacteria in a certain culture doubles every year. If there were 30 bacteria Present in the culture Originally, how many bacteria will be Present at the end of 2nd hour, 4th hour. and nth hour?

No of bacteria at the beginning = 30

No of bacteria after 1 hour $= 30 \times 2 = 60$

" " " 2 hours $= 30 \times 2^2 = 120$

" " " 4 hours $= 30 \times 2^4 = 480$

∴ No of bacteria after nth hour

$= 30 \times 2^n$.

$\frac{11}{220}$ what will ₹500 amounts to in 10 years its deposit in a bank which Pays annual interest rate ~~of~~ 10% Compounded annually?

$P = ₹500$ C.I = 10%.

Amount after 1 year $= P(1 + \frac{r}{100})$

$= P(1 + \frac{10}{100}) = 500(\frac{11}{10})$

Amount after 2 years $= 500(\frac{11}{10})^2$

" " 3 years $= 500(\frac{11}{10})^3$

∴ The Amount after 10 years $= ₹500(\frac{11}{10})^{10}$

$= ₹1296.87$

$\frac{12}{220}$ In a certain town, a viral disease caused severe health hazards upon its People disturbing their normal life. It was found that on each day, the virus which caused the disease spread in G.P. The amount of infections virus Particle gets doubled each day, being 5 Particles on the first day. Find the day when the infectious virus Particles just grow over 1,50,000 units.

$a=5 \quad r=2 \quad t_n > 150000$

To find : 'n'

$$t_n = ar^{n-1} > 150000$$

$$5(2)^{n-1} > 150000$$

$$2^{n-1} > 30000$$

$$2^{n-1} > 2^{14} + \text{some decimals}$$

$$n-1 > 14 + \text{some decimals}$$

$$n > 15 + \text{some decimals}$$

$\therefore$ On the 15^{th} day it will grow over 150000 units.

5.6. Infinite Sequences and Series :-

*** Sum of infinite G.P**

$$S_\infty = \begin{cases} \dfrac{a}{1-r} & |r|<1 \\ \infty & |r| \geq 1 \end{cases}$$

*** Infinite Arithmetico - Geometric Series:**

$$S = \frac{a}{1-r} + \frac{dr}{(1-r)^2} \qquad |r|<1$$

5.19
225

Find the Sum : $1 + \frac{4}{5} + \frac{7}{25} + \frac{19}{125} + \cdots$

$a=1 \quad d = 4-1 = 3 \quad r = \frac{1}{5}$

$$S = \frac{a}{1-r} + \frac{dr}{(1-r)^2}$$

$$= \frac{1}{1-\frac{1}{5}} + \frac{3/5}{(1-1/5)^2} = \frac{1}{4/5} + \frac{3/5}{(4/5)^2}$$

$$= \frac{5}{4} + \frac{3}{5}\left(\frac{25}{16}\right) = \frac{5}{4} + \frac{15}{16}$$

$$= \frac{20+15}{16} = \frac{35}{16}$$

$$S_\infty = \frac{35}{16}$$

5.20
225.

Find $\sum_{n=1}^{\infty} \frac{1}{n^2+5n+6}$

$$a_n = \frac{1}{n^2+5n+6} = \frac{1}{(n+2)(n+3)} = \frac{1}{n+2} - \frac{1}{n+3}$$

$$S_n = a_1 + a_2 + a_3 + \dots + a_n$$

$$= \left(\frac{1}{3}-\frac{1}{4}\right) + \left(\frac{1}{4}-\frac{1}{5}\right) + \left(\frac{1}{5}-\frac{1}{6}\right) + \dots + \left(\frac{1}{n+2}-\frac{1}{n+3}\right)$$

$$= \frac{1}{3} - \cancel{\frac{1}{4}} + \cancel{\frac{1}{4}} - \cancel{\frac{1}{5}} + \cancel{\frac{1}{5}} - \cancel{\frac{1}{6}} + \dots + \cancel{\frac{1}{n+2}} - \frac{1}{n+3}$$

$$S_n = \frac{1}{3} - \frac{1}{n+3} = \frac{n+3-3}{3(n+3)} = \frac{n}{3(n+3)}$$

$$\sum_{n=1}^{\infty} S_n = \sum_{n=1}^{\infty} \frac{n}{3(n+3)} = \lim_{n\to\infty} \frac{\cancel{n}}{3\cancel{n}\left(1+\frac{3}{n}\right)} = \frac{1}{3}$$

$$\therefore \sum_{n=1}^{\infty} \frac{1}{n^2+5n+6} = \frac{1}{3}.$$

Binomial Series:

Binomial Theorem for rational Exponent:

* $(1+x)^n = 1 + nx + \frac{n(n-1)}{2!}x^2 + \frac{n(n-1)(n-2)}{3!}x^3 + \dots \quad |x|<1$
* $(1-x)^n = 1 - nx + \frac{n(n-1)}{2!}x^2 - \frac{n(n-1)(n-2)}{3!}x^3 + \dots \quad |x|<1$
* $(1+x)^{-n} = 1 - nx + \frac{n(n+1)}{2!}x^2 - \frac{n(n+1)(n+2)}{3!}x^3 + \dots \quad |x|<1$
* $(1-x)^{-n} = 1 + nx + \frac{n(n+1)}{2!}x^2 + \frac{n(n+1)(n+2)}{3!}x^3 + \dots \quad |x|<1.$

$$(1+x)^{-1} = 1 - x + x^2 - x^3 + \dots$$

$$(1-x)^{-1} = 1 + x + x^2 + x^3 + \dots$$

$$(1+x)^{-2} = 1 - 2x + 3x^2 - 4x^3 + \dots$$

$$(1-x)^{-2} = 1 - 2x + 3x^2 - 4x^3 + 5x^4 - \dots$$

All the above expressions are valid only when $|x| < 1$.

5.21 / 224. Expand $(1+x)^{2/3}$ upto 4 terms for $|x|<1$

$(1+x)^{2/3}$

$$\boxed{(1+x)^n = 1+\frac{nx}{1!}+\frac{n(n-1)}{2!}x^2+\frac{n(n-1)(n-2)}{3!}x^3+\dots \quad |x|<1}$$

$$(1+x)^{2/3} = 1+\frac{2}{3}x+\frac{\frac{2}{3}\left(\frac{2}{3}-1\right)}{2!}x^2+\frac{\frac{2}{3}\left(\frac{2}{3}-1\right)\left(\frac{2}{3}-2\right)}{3!}x^3+\dots$$

$$= 1+\frac{2x}{3}+\frac{\frac{\cancel{2}}{3}\left(\frac{-1}{3}\right)}{\cancel{2}!}x^2+\frac{\frac{2}{3}\left(\frac{-1}{3}\right)\left(\frac{-4}{3}\right)}{3!}x^3+\dots$$

$$= 1+\frac{2}{3}x-\frac{1}{9}x^2+\frac{4}{81}x^3+\dots \quad |x|<1.$$

5.22 / 227 Expand $\frac{1}{(1+3x)^2}$ in Powers of x. find the condition on x for which the expansion is valid.

$$\frac{1}{(1+3x)^2} = (1+3x)^{-2}$$

$|3x|<1$

$$\boxed{|x|<\frac{1}{3}}$$

$$\boxed{(1+x)^{-n} = 1-nx+\frac{n(n-1)}{2!}x^2+\frac{n(n-1)(n-2)}{3!}x^3+\dots \quad |x|<1}$$

$$(1+3x)^{-2} = 1-2(3x)+\frac{2(2+1)}{2!}(3x)^2-\frac{2(2+1)(2+2)}{3!}(3x)^3+\dots$$

$$= 1-6x+\frac{6}{2}(9x)^2-\frac{2(3)(4)}{6}27x^3+\dots$$

$$= 1-6x+27x^2-108x^3+\dots \quad |x|<\frac{1}{3}.$$

[OR]

$$\boxed{(1+x)^{-2} = 1-2x+3x^2-4x^3+5x^4-6x^5+\dots \quad |x|<1}$$

$$(1+3x)^{-2} = 1-2(3x)+3(3x)^2-4(3x)^3+5(3x)^4-\dots$$

$$= 1-6x+27x^2-108x^3+405x^4-\dots$$

$$|x|<\frac{1}{3}.$$

5.23
228

Expand $\frac{1}{(3+2x)^2}$ in Powers x. Find the Condition on x for which the expansion is valid.

$$\frac{1}{(3+2x)^2} = (3+2x)^{-2} = 3^{-2}\left(1+\frac{2x}{3}\right)^{-2}$$

$$\left|\frac{2x}{3}\right| < 1$$
$$|x| < \frac{3}{2}$$

$$3^{-2}\left(1+\frac{2x}{3}\right)^{-2} = \frac{1}{9}\left(1+\frac{2x}{3}\right)^{-2}$$

$$\boxed{(1+x)^{-2} = 1-2x+3x^2-4x^3+5x^4-\cdots \quad |x|<1}$$

$$\frac{1}{9}\left(1+\frac{2x}{3}\right)^{-2} = 1-2\left(\frac{2x}{3}\right)+3\left(\frac{2x}{3}\right)^2-4\left(\frac{2x}{3}\right)^3+\cdots$$

$$= \frac{1}{9}\left[1-\frac{4x}{3}+\frac{4}{3}x^2-\frac{32}{27}x^3+\cdots\right]$$

$$= \frac{1}{9}-\frac{4x}{27}+\frac{4x^2}{27}-\frac{32}{243}x^3+\cdots \qquad |x| < \frac{3}{2}$$

5.24
228

Find $\sqrt[3]{65}$

$$\sqrt[3]{65} = (65)^{1/3} = (1+64)^{1/3}$$

we know that $|x|<1$

$$(1+64)^{1/3} = 64^{1/3}\left[\frac{1}{64}+1\right]^{1/3} = 4\left(1+\frac{1}{64}\right)^{1/3}$$

$$\boxed{(1+x)^n = 1+\frac{nx}{1!}+\frac{n(n-1)}{2!}x^2+\frac{n(n-1)(n-2)}{3!}x^3+\cdots \quad |x|<1}$$

$$4\left(1+\frac{1}{64}\right)^{1/3} = 4\left\{1+\frac{1}{3}\left(\frac{1}{64}\right)+\frac{\frac{1}{3}\left(\frac{1}{3}-1\right)}{2!}\left(\frac{1}{64}\right)^2+\cdots\right\}$$

$$= 4+\frac{1}{3}\cdot\frac{1}{16}+\cdots \quad \text{[Omitting the higher Powers]}$$

$$= 4+\frac{1}{48} = 4+0.02$$

$$= 4.02$$

$$\therefore \sqrt[3]{65} \simeq 4.02$$

5.25
229

Prove that $\sqrt[3]{x^3+7} - \sqrt[3]{x^3+4}$ is approximately equal to $\frac{1}{x^2}$ when x is large.

$\sqrt[3]{x^3+7} - \sqrt[3]{x^3+4} = (x^3+7)^{1/3} - (x^3+4)^{1/3}$

$= (x^3)^{1/3}\left[1+\frac{7}{x^3}\right]^{1/3} - (x^3)^{1/3}\left[1+\frac{4}{x^3}\right]^{1/3}$

$= x\left\{\left(1+\frac{7}{x^3}\right)^{1/3} - \left(1+\frac{4}{x^3}\right)^{1/3}\right\}$

$$\boxed{(1+x)^n = 1+\frac{nx}{1!}+\frac{n(n-1)}{2!}x^2+\frac{n(n-1)(n-2)}{3!}x^3+\dots \quad |x|<1}$$

$= x\left\{\left[1+\frac{1}{3}\left(\frac{7}{x^3}\right)+\frac{\frac{1}{3}\left(\frac{1}{3}-1\right)}{2!}\left(\frac{7}{x^3}\right)^2+\dots\right] - \left[1+\frac{1}{3}\left(\frac{4}{x^3}\right)+\frac{\frac{1}{3}\left(\frac{1}{3}-1\right)}{2!}\left(\frac{4}{x^3}\right)^2+\dots\right]\right\}$

$= x\left\{\cancel{1}+\frac{7}{3x^3} - \cancel{1} - \frac{4}{3x^3}\right\}$

Since x is large, $\frac{1}{x}$ is very small and hence higher powers of $\frac{1}{x}$ are negligible.

$= x\left[\frac{7}{3x^3}-\frac{4}{3x^3}\right] = x\left(\frac{3}{3x^3}\right) \simeq \frac{1}{x^2}$

$\therefore \sqrt[3]{x^3+7} - \sqrt[3]{x^3+4} \simeq \frac{1}{x^2}$.

Hence Proved.

Ex: 5.4

1
231

Expand the following in ascending powers of x and find the condition on x for which the binomial expansion is valid.

(i) $\frac{1}{5+x}$

$\frac{1}{5+x} = (5+x)^{-1} = 5^{-1}\left(1+\frac{x}{5}\right)^{-1}$ $\quad \left|\frac{x}{5}\right|<1$

$= \frac{1}{5}\left(1+\frac{x}{5}\right)^{-1}$ $\quad |x|<5$

$$\boxed{(1+x)^{-1} = 1-x+x^2-x^3+x^4-\dots \quad |x|<1}$$

$$\frac{1}{5}\left(1+\frac{x}{5}\right)^{-1} = \frac{1}{5}\left[1-\frac{x}{5}+\left(\frac{x}{5}\right)^2-\left(\frac{x}{5}\right)^3+\cdots\right]$$

$$= \frac{1}{5}\left(1-\frac{x}{5}+\frac{x^2}{25}-\frac{x^3}{125}+\cdots\right)$$

$$= \frac{1}{5}-\frac{x}{25}+\frac{x^2}{125}-\frac{x^3}{625}+\cdots \qquad \left|\frac{x}{5}\right|<1.$$

(ii) $\frac{2}{(3+4x)^2}$

$$= 2(3+4x)^{-2}$$

$$= 2\times 3^{-2}\left(1+\frac{4x}{3}\right)^{-2}$$

$$= \frac{2}{9}\left(1+\frac{4x}{3}\right)^{-2}$$

$$\boxed{(1+x)^{-2} = 1-2x+3x^2-4x^3+5x^4-\cdots \qquad |x|<1}$$

$$\frac{2}{9}\left(1+\frac{4}{3}x\right)^{-2} = \frac{2}{9}\left[1-2\left(\frac{4}{3}x\right)+3\left(\frac{4}{3}x^2\right)-4\left(\frac{4}{3}x^3\right)+\cdots\right]$$

$$= \frac{2}{9}\left[1-\frac{8x}{3}+\frac{16x^2}{3}-\frac{256}{27}x^3+\cdots\right]$$

$$\therefore \frac{2}{(3+4x)^2} = \frac{2}{9}\left[1-\frac{8}{3}x+\frac{16}{3}x^2-\frac{256}{27}x^3+\cdots\right]$$

$$\left|\frac{4x}{3}\right|<1$$
$$|x|<\frac{3}{4}$$

(iii) $(5+x^2)^{\frac{2}{3}}$

$$(5+x^2)^{2/3} = 5^{2/3}\left(1+\frac{x^2}{5}\right)^{2/3}$$

$$\boxed{(1+x)^n = 1-nx+\frac{n(n-1)}{2!}x^2+\frac{n(n-1)(n-2)}{3!}x^3+\cdots \qquad |x|<1}$$

$$5^{2/3}\left(1+\frac{x^2}{5}\right)^{2/3} = 5^{2/3}\left[1+\frac{2}{3}\left(\frac{x^2}{5}\right)+\frac{\frac{2}{3}\left(\frac{2}{3}-1\right)}{2!}\left(\frac{x^2}{5}\right)^2+\frac{\frac{2}{3}\left(\frac{2}{3}-1\right)\left(\frac{2}{3}-2\right)}{3!}\left(\frac{x^2}{5}\right)^3+\cdots\right]$$

$$= 5^{2/3}\left[1+\frac{2}{15}x^2+\frac{\cancel{\frac{2}{3}}\left(\frac{-1}{3}\right)}{\cancel{2}}\left(\frac{x^4}{25}\right)+\frac{\cancel{\frac{2}{3}}\left(\frac{-1}{3}\right)\left(\frac{-4}{3}\right)}{\cancel{6}3}\left(\frac{x^6}{125}\right)+\cdots\right]$$

$$= 5^{2/3}\left[1+\frac{2}{15}x^2-\frac{x^4}{225}+\frac{4x^6}{81\times125}+\cdots\right]$$

$$\left|\frac{x^2}{5}\right|<1$$
$$|x^2|<5$$
$$x^2<5.$$

(iv) $(x+2)^{-2/3}$

$$(x+2)^{-2/3} = 2^{-2/3}\left(1+\frac{x}{2}\right)^{-2/3}$$

$$\boxed{(1+x)^{-n} = 1 - nx + \frac{n(n+1)}{2!}x^2 - \frac{n(n+1)(n+2)}{3!}x^3 + \dots \quad |x|<1}$$

$$2^{-2/3}\left(1+\frac{x}{2}\right)^{-2/3} = 2^{-2/3}\left[1 - \frac{2}{3}\left(\frac{x}{2}\right) + \frac{\frac{2}{3}\left(\frac{2}{3}+1\right)}{2!}\left(\frac{x}{2}\right)^2 - \frac{\frac{2}{3}\left(\frac{2}{3}+1\right)\left(\frac{2}{3}+2\right)}{3!}\left(\frac{x}{2}\right)^3 + \dots\right]$$

$$= 2^{-2/3}\left[1 - \frac{x}{3} + \frac{\cancel{2}\frac{x}{3}\left(\frac{5}{3}\right)}{\cancel{2}}\left(\frac{x^2}{4}\right) - \frac{\cancel{2}\left(\frac{5}{3}\right)\left(\frac{\cancel{8}}{3}\right)}{\cancel{6}\,3}\left(\frac{x^3}{\cancel{8}}\right) + \dots\right]$$

$$= 2^{-2/3}\left[1 - \frac{x}{3} + \frac{5x^2}{36} - \frac{5x^3}{81} + \dots\right]$$

$$\therefore (x+2)^{-2/3} = 2^{-2/3}\left[1 - \frac{x}{3} + \frac{5x^2}{36} - \frac{5x^3}{81} + \dots\right]$$

$$\left|\frac{x}{2}\right| < 1 \Rightarrow |x| < 2.$$

$\frac{2}{231}$ Find the approximate value of $\sqrt[3]{1001}$

$$\sqrt[3]{1001} = (1000+1)^{1/3} = (1000)^{\frac{1}{3}}\left(1+\frac{1}{1000}\right)^{\frac{1}{3}} \qquad |x|<1$$

$$= 10\left(1+\frac{1}{1000}\right)^{1/3}$$

$$\boxed{(1+x)^{n} = 1 + nx + \frac{n(n-1)}{2!}x^2 + \frac{n(n-1)(n-2)}{3!}x^3 + \dots \quad |x|<1}$$

$$\sqrt[3]{1001} = 10\left(1+\frac{1}{1000}\right)^{1/3}$$

$$= 10\left[1 + \frac{1}{3}\left(\frac{1}{1000}\right) + \frac{\frac{1}{3}\left(\frac{1}{3}-1\right)}{2!}\left(\frac{1}{1000}\right)^2 + \dots\right]$$

$$= 10\left(1 + \frac{1}{3000}\right)$$

$$= 10 + \frac{1}{300} = 10 + \frac{1}{3}\left(\frac{1}{100}\right)$$

$$= 10 + \frac{0.33}{100} = 10 + 0.0033$$

$$\sqrt[3]{1001} \simeq 10.0033$$

$\frac{3}{231}$. Prove that $\sqrt[3]{x^3+6} - \sqrt[3]{x^3+3}$ is approximately equal to $\frac{1}{x^2}$ when x is sufficently large.

$$\sqrt[3]{x^3+6} - \sqrt[3]{x^3+3} = (x^3+6)^{1/3} - (x^3+3)^{1/3}$$

$$= (x^3)^{1/3}\left(1+\frac{6}{x^3}\right)^{1/3} - (x^3)^{1/3}\left[1+\frac{3}{x^3}\right)^{1/3}$$

$$= x\left[\left(1+\frac{6}{x^3}\right)^{1/3} - \left(1+\frac{3}{x^3}\right)^{1/3}\right]$$

$$= x\left\{\left[1+\frac{1}{3}\left(\frac{6}{x^3}\right)+\frac{\frac{1}{3}\left(\frac{1}{3}-1\right)}{2!}\left(\frac{6}{x^3}\right)^2+\dots\right] - \left[1+\frac{1}{3}\left(\frac{3}{x^3}\right)+\frac{\frac{1}{3}\left(\frac{1}{3}-1\right)}{2!}\left(\frac{3}{x^3}\right)^2+\dots\right]\right.$$

$$= x\left[\left(1+\frac{6}{3x^3}+\dots\right) - \left(1+\frac{3}{3x^3}+\dots\right)\right]$$

$$= x\left[1+\frac{2}{x^3}-1-\frac{1}{x^3}\right] = x\left(\frac{1}{x^3}\right) = \frac{1}{x^2}$$

$$\therefore \sqrt[3]{x^3+6} - \sqrt[3]{x^3+3} \simeq \frac{1}{x^2}$$

$\frac{4}{231}$ Prove that $\sqrt{\frac{1-x}{1+x}}$ is approximately equal to $1-x+\frac{x^2}{2}$. when x is very Small.

$$\sqrt{\frac{1-x}{1+x}} = \frac{\sqrt{1-x}}{\sqrt{1+x}} = (1-x)^{1/2}.(1+x)^{-1/2}$$

$|x|<1$

$$(1-x)^n = 1-nx+\frac{n(n-1)}{2!}x^2 - \frac{n(n-1)(n-2)}{3!}x^3+\dots$$

$$(1+x)^{-n} = 1-nx+\frac{n(n+1)}{2!}x^2 - \frac{n(n+1)(n+2)}{3!}x^3+\dots$$

$|x|<1$

$$(1-x)^{\frac{1}{2}}.(1+x)^{-1/2}$$

$$= \left\{1-\frac{1}{2}x+\frac{\frac{1}{2}\left(\frac{1}{2}-1\right)}{2!}x^2 - \frac{\frac{1}{2}\left(\frac{1}{2}-1\right)\left(\frac{1}{2}-2\right)}{3!}x^3+\dots\right\}$$

$$\left\{1-\frac{1}{2}x+\frac{\frac{1}{2}\left(\frac{1}{2}+1\right)}{2!}x^2 - \frac{\frac{1}{2}\left(\frac{1}{2}+1\right)\left(\frac{1}{2}+2\right)}{3!}x^3+\dots\right\}$$

$$= \left\{1-\frac{x}{2}+\frac{\frac{1}{2}\left(-\frac{1}{2}\right)}{2!}x^2\dots\right\}\left\{1-\frac{x}{2}+\frac{\frac{1}{2}\cdot\frac{3}{2}}{2}x^2\dots\right\}$$

$$= \left(1-\frac{x}{2}-\frac{x^2}{8}-\dots\right)\left(1-\frac{x}{2}+\frac{3x^2}{8}-\dots\right)$$

$$= 1-\frac{x}{2}+\frac{3x^2}{8}-\frac{x}{2}+\frac{x^2}{4}-\frac{x^2}{8}$$

$$= 1-x+\frac{x^2}{4}+\frac{2x^2}{8} = 1-x+\frac{x^2}{2}$$

$$\therefore \sqrt{\frac{1-x}{1+x}} \simeq 1-x+\frac{x^2}{2} \quad \text{(approximately)}$$

Exponential Series :-

The Series $\sum_{n=0}^{\infty} \frac{x^n}{n!}$ is called an exponential Series.

For any real number x,

$$\sum_{n=0}^{\infty} \frac{x^n}{n!} = e^x$$

$$e^x = 1 + \frac{x}{1!} + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \cdots$$

$$e^{-x} = 1 - \frac{x}{1!} + \frac{x^2}{2!} - \frac{x^3}{3!} + \frac{x^4}{4!} - \cdots$$

$$\frac{e^x + e^{-x}}{2} = 1 + \frac{x^2}{2!} + \frac{x^4}{4!} + \frac{x^6}{6!} + \cdots$$

$$\frac{e^x - e^{-x}}{2} = \frac{x}{1!} + \frac{x^3}{3!} + \frac{x^5}{5!} + \cdots$$

$$e = 1 + \frac{1}{1!} + \frac{1}{2!} + \frac{1}{3!} + \frac{1}{4!} + \cdots$$

$$e^{-1} = 1 - \frac{1}{1!} + \frac{1}{2!} - \frac{1}{3!} + \frac{1}{4!} + \cdots$$

$$\frac{e + e^{-1}}{2} = 1 + \frac{1}{2!} + \frac{1}{4!} + \frac{1}{6!} + \cdots$$

$$\frac{e - e^{-1}}{2} = \frac{1}{1!} + \frac{1}{3!} + \frac{1}{5!} + \cdots$$

$\frac{5}{231}$ write the first 6 terms of the exponential Series :

(i) e^{5x}

$$e^x = 1 + \frac{x}{1!} + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \cdots$$

$$e^{5x} = 1 + \frac{5x}{1!} + \frac{(5x)^2}{2!} + \frac{(5x)^3}{3!} + \frac{(5x)^4}{4!} + \frac{(5x)^5}{5!} + \cdots$$

$$= 1 + 5x + \frac{25x^2}{2} + \frac{125x^3}{6} + \frac{625}{24}x^4 + \frac{625}{24}x^5 + \cdots$$

(ii) e^{-2x}

$$e^{-x} = 1 - \frac{x}{1!} + \frac{x^2}{2!} - \frac{x^3}{3!} + \frac{x^4}{4!} - \cdots$$

$$e^{-2x} = 1 - \frac{2x}{1!} + \frac{(2x)^2}{2!} - \frac{(2x)^3}{3!} + \frac{(2x)^4}{4!} - \frac{(2x)^5}{5!} + \cdots$$

$$= 1 - 2x + \frac{4x^2}{2} - \frac{8x^3}{6} + \frac{16x^4}{24} - \frac{32x^5}{120} + \cdots$$

$$= 1 - 2x + 2x^2 - \frac{4}{3}x^3 + \frac{2x^4}{3} - \frac{4}{15}x^5 + \cdots$$

(iii) $e^{\frac{1}{2}x}$

$$e^{x} = 1 + \frac{x}{1!} + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \cdots$$

$$e^{\frac{1}{2}x} = 1 + \frac{\frac{x}{2}}{1!} + \frac{\left(\frac{x}{2}\right)^2}{2!} + \frac{\left(\frac{x}{2}\right)^3}{3!} + \frac{\left(\frac{x}{2}\right)^4}{4!} + \frac{\left(\frac{x}{2}\right)^5}{5!} + \cdots$$

$$= 1 + \frac{x}{2} + \frac{x^2}{4} + \frac{x^3}{48} + \frac{x^4}{384} + \cdots$$

Logarithmic Series:-

The series $\sum_{n=1}^{\infty} (-1)^{n+1} \frac{x^n}{n}$ is called a logarithmic series.

This series converges for all values of x satisfying $|x| < 1$. This series converges when $x = 1$ also.

For all values of x satisfying $|x| < 1$, the sum of the series is $\log(1+x)$.

Thus

$$\boxed{\begin{aligned} \log(1+x) &= x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \cdots \quad |x|<1 \\ \log(1-x) &= -x - \frac{x^2}{2} - \frac{x^3}{3} - \frac{x^4}{4} - \cdots \quad |x|<1. \\ \log(1+x) - \log(1-x) &= \log\left(\frac{1+x}{1-x}\right) \\ &= 2\left[x + \frac{x^3}{3} + \frac{x^5}{5} + \cdots\right] \end{aligned}}$$

$\frac{6}{231}$ Write the first 4 terms of the logarithmic series.

(i) $\log(1+4x)$

$$\log(1+x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \cdots \quad |x|<1$$

$$\log(1+4x) = 4x - \frac{(4x)^2}{2} + \frac{(4x)^3}{3} - \frac{(4x)^4}{4} + \cdots$$

$$= 4x - \frac{16x^2}{2} + \frac{64x^3}{3} - \frac{256}{4}x^4 + \cdots$$

$$= 4x - 8x^2 + \frac{64x^3}{3} - 64x^4 + \cdots \quad |x| < \frac{1}{4}$$

(ii) $\log(1-2x)$

$$\log(1-x) = -x - \frac{x^2}{2} - \frac{x^3}{3} - \frac{x^4}{4} - \dots \quad |x| < 1$$

$$\log(1-2x) = -2x - \frac{(2x)^2}{2} - \frac{(2x)^3}{3} - \frac{(2x)^4}{4} - \dots \quad |x| < \frac{1}{2}$$

$$= -2x - 2x^2 - \frac{8x^3}{3} - 4x^4 - \dots \quad |x| < \frac{1}{2}$$

(iii) $\log\left(\frac{1+3x}{1-3x}\right)$

$$\log\left(\frac{1+x}{1-x}\right) = 2\left[x + \frac{x^3}{3} + \frac{x^5}{5} + \dots\right]$$

$$\log\left(\frac{1+3x}{1-3x}\right) = 2\left[3x + \frac{(3x)^3}{3} + \frac{(3x)^5}{5} + \frac{(3x)^7}{7}\right]$$

$$= 2\left[3x + 9x^3 + \frac{243x^5}{5} + \frac{(3x)^7}{7} + \dots\right] \quad |x| < \frac{1}{3}$$

8/231 If $p-q$ is small compared to either p or q then show that $\sqrt[n]{\frac{p}{q}} = \frac{(n+1)p + (n-1)q}{(n-1)p + (n+1)q}$.

$$\text{R.H.S} = \frac{(n+1)p + (n-1)q}{(n-1)p + (n+1)q} = \frac{np + p + nq - q}{np - p + nq + q}$$

$$= \frac{n(p+q) + (p-q)}{n(p+q) - (p-q)} = \frac{1 + \frac{1}{n}\left(\frac{p-q}{p+q}\right)}{1 - \frac{1}{n}\left(\frac{p-q}{p+q}\right)}$$

$$1 + nx + \frac{n(n-1)}{2!}x^2 + \dots = (1+x)^n$$
$$1 - nx + \frac{n(n-1)}{2}x^2 + \dots = (1-x)^n$$

$$= \frac{\left(1 + \frac{p-q}{p+q}\right)^{1/n}}{\left(1 - \frac{p-q}{p+q}\right)^{1/n}} = \left(\frac{p+\cancel{q}+p-\cancel{q}}{p+q-p+q}\right)^{1/n} = \left(\frac{2p}{2q}\right)^{1/n}$$

$$= \left(\frac{p}{q}\right)^{1/n} = \sqrt[n]{\frac{p}{q}} = \text{L.H.S}$$

To find : $\sqrt[8]{\frac{15}{16}}$ let $n = 8$ $p = 15$ $q = 16$

$$\sqrt[8]{\frac{15}{16}} = \frac{(n+1)p + (n-1)p}{(n-1)p + (n+1)p} = \frac{9 \times 15 + 7(16)}{7(15) + 9(16)} = \frac{135 + 112}{105 + 144}$$

$$= \frac{247}{249} = 0.99196.$$

$\frac{9}{231.}$ Find the Co-efficient of x^4 in the expansion of $\dfrac{3-4x+x^2}{e^{2x}}$

$$\frac{3-4x+x^2}{e^{2x}} = [3-4x+x^2]\, e^{-2x}$$

$$= (3-4x+x^2)\left\{1+\frac{(-2x)}{1!}+\frac{(-2x)^2}{2!}+\frac{(-2x)^3}{3!}+\frac{(-2x)^4}{4!}+\cdots\right\}$$

$$= (3-4x+x^2)\left(1-2x+\frac{4x^2}{2}-\frac{8x^3}{6}+\frac{16}{24}x^4-\cdots\right\}$$

Co-efficient of x^4

$$= \cancel{3}\left(\frac{16}{\cancel{24}_{8}}\right)-4\left(\frac{-8}{6}\right)+1\left(\frac{4}{2}\right)$$

$$= 2+\frac{32}{6}+2 = 4+\frac{16}{3} = \frac{28}{3}.$$

$\frac{10}{231.}$ Find the value of $\displaystyle\sum_{n=1}^{\infty}\frac{1}{2n-1}\left(\frac{1}{9^{n-1}}+\frac{1}{9^{2n-1}}\right)$.

$$S_\infty = \frac{1}{1}\left(1+\frac{1}{9}\right)+\frac{1}{3}\left(\frac{1}{9}+\frac{1}{9^2}\right)+\frac{1}{5}\left(\frac{1}{9^2}+\frac{1}{9^5}\right)+\frac{1}{7}\left(\frac{1}{9^3}+\frac{1}{9^7}\right)+\cdots$$

$$= \left(1+\frac{1}{3}\cdot\frac{1}{9}+\frac{1}{5}\cdot\frac{1}{9^2}+\frac{1}{7}\cdot\frac{1}{9^3}+\cdots\right)+\left(\frac{1}{9}+\frac{1}{3}\cdot\frac{1}{9^3}+\frac{1}{5}\cdot\frac{1}{9^5}+\frac{1}{7}\cdot\frac{1}{9^7}+\cdots\right)$$

$$= \left(1+\frac{1}{3}\cdot\frac{1}{3^2}+\frac{1}{5}\cdot\frac{1}{3^4}+\frac{1}{7}\cdot\frac{1}{3^6}+\cdots\right)+\left[\frac{1}{9}+\frac{1}{3}\left(\frac{1}{9}\right)^3+\frac{1}{5}\left(\frac{1}{9}\right)^5+\frac{1}{7}\left(\frac{1}{9}\right)^7+\cdots\right]$$

$$= 3\left[\frac{1}{3}+\frac{1}{3}\cdot\left(\frac{1}{3}\right)^3+\frac{1}{5}\left(\frac{1}{3}\right)^5+\cdots\right]+\left[\frac{1}{9}+\frac{1}{3}\left(\frac{1}{9}\right)^3+\frac{1}{5}\left(\frac{1}{9}\right)^5+\cdots\right]$$

$$\boxed{\begin{gathered}\left[\log\left(\frac{1+x}{1-x}\right)=\frac{2}{1}\left[x+\frac{x^3}{3}+\frac{x^5}{5}+\cdots\right]\right.\\ \frac{1}{2}\log\left(\frac{1+x}{1-x}\right) = x+\frac{x^3}{3}+\frac{x^5}{5}+\cdots\end{gathered}}$$

$$= 3 \cdot \frac{1}{2} \log \left(\frac{1+\frac{1}{3}}{1-1/3} \right) + \frac{1}{2} \log \left(\frac{1+\frac{1}{9}}{1-\frac{1}{9}} \right)$$

$$= \frac{3}{2} \log \left(\frac{4/3}{2/3} \right) + \frac{1}{2} \log \left(\frac{10/9}{8/9} \right)$$

$$= \frac{3}{2} \log 2 + \frac{1}{2} \log \left(\frac{5}{4} \right)$$

$$= \frac{1}{2} \log 2^3 + \frac{1}{2} \log \frac{5}{4}$$

$$= \frac{1}{2} \log \left(8 * \frac{5}{4} \right)$$

$$= \frac{1}{2} \log_e 10$$

7. MATRICES AND DETERMINANTS

Matrices:

Definition :-

A matrix is a rectangular array or arrangement of entries or elements displayed in rows and columns put within a square bracket [ ].

In general, the entries of a matrix may be real or complex numbers or functions of one variable or more variables or any other object.

Usually, matrices are denoted by capital letters A, B, C . . .

General form of a matrix :

If a matrix A has m rows and n columns, then it is written as

$$A = [a_{ij}]_{m\times n} \quad 1 \le i \le m, \quad 1 \le j \le n.$$

$$A = [a_{ij}] = \begin{bmatrix} a_{11} & a_{12} & a_{13} & \cdots & a_{1n} \\ a_{21} & a_{22} & a_{23} & \cdots & a_{2n} \\ \vdots & & & & \\ a_{m1} & a_{m2} & a_{m3} & \cdots & a_{mn} \end{bmatrix}_{m\times n}.$$

Eg: $A = \begin{bmatrix} 1 & 2 \\ 0 & 5 \end{bmatrix}_{2\times 2}$ $\quad B = \begin{bmatrix} 0.1 & 2 & 3 & 4 \\ 5 & -1 & x & 5 \end{bmatrix}_{2\times 4}$.

Order or Side of a matrix :

If a matrix A has m rows and n columns then the order or side of the matrix A is defined to be $m\times n$ [read as m by n]

N KARTHIKEYAN, M.Sc, B.Ed
AVP TRUST NATIONAL
MATRIC HR SEC SCHOOL
TIRUPUR-9843434491

Types of Matrices:

- Row matrix
- Column matrix
- Zero matrix or null or void matrix.
- Square matrix
- diagonal matrix
- Scalar matrix
- Unit matrix
- Upper Triangular matrix
- Lower Triangular matrix
- Triangular matrix.

* Row Matrix:-

A matrix having only one row is called a row matrix $A = [a_{ij}]_{1 \times n}$

Eg: $A = \begin{bmatrix} 1 & 0 & 5 & 4 \end{bmatrix}_{1 \times 4}$

* Column Matrix:-

A matrix having only on column is called a Column matrix.

$A = [A_{ij}]_{m \times 1} = [a_{ij}]_{m \times 1}$

* Zero or null or Void Matrix:-

A matrix $A = [a_{ij}]_{m \times n}$ is said to be a zero matrix or null matrix or void matrix denoted by 0 if $a_{ij} = 0$ for all values of $1 \leq i \leq m$ and $1 \leq j \leq n$.

Eg: $A = [0 \ 0]$ $B = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$.

* Square matrix :-

A matrix in which number of rows is equal to the number of columns, is called a square matrix. That is a matrix of order $n \times n$ is often referred to as a square matrix of order n.

$$A = [a_{ij}]_{n\times n} \quad i = j$$

Eg: $A = \begin{bmatrix} 0 & 2 \\ 4 & 6 \end{bmatrix}_{2\times 2}$ $\quad B = \begin{bmatrix} -1 & -2 & 5 \\ 5 & 0 & -2 \\ 3 & 8 & 9 \end{bmatrix}_{3\times 3}$

* Diagonal matrix :-

In a square matrix $A = [a_{ij}]_{n\times n}$ of order n, the elements $a_{11}, a_{22}, a_{33} \dots a_{nn}$ are called the principal diagonal or simply the diagonal or main diagonal or leading diagonal elements.

A square matrix $A = [a_{ij}]_{n\times n}$ is called a diagonal matrix if $a_{ij} = 0$ whenever $i \neq j$ Eg: $A = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 5 & 0 \\ 0 & 0 & 2 \end{bmatrix}$ $\quad B = \begin{bmatrix} 2 & 0 \\ 0 & -5 \end{bmatrix}$

* Scalar matrix :-

A diagonal matrix whose entries along the principal diagonal are equal is called a scalar matrix.

$A = [a_{ij}]_{n\times n}$ said to be a scalar matrix. if

$$a_{ij} = \begin{cases} c & \text{if } i = j \\ 0 & \text{if } i \neq j \end{cases}$$

* Unit matrix :-

A square matrix in which all the diagonal entries are 1. and the rest are all zero is called a unit matrix.

We represent the unit matrix of order n is I_n,

Eg: $I_2 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ $\quad I_3 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$

* Upper Triangular Matrix:-

A Square matrix is said to be an upper triangular matrix if all the elements below the main diagonal are zero.

Eg: $A = \begin{bmatrix} -5 & 2 \\ 0 & 1 \end{bmatrix}$ $B = \begin{bmatrix} 4 & 3 & 0 \\ 0 & 2 & 5 \\ 0 & 0 & 2 \end{bmatrix}$

* Lower triangular Matrix:-

A Square matrix is said to be a lower triangular matrix if all the elements above the main diagonal are zero.

Eg: $A = \begin{bmatrix} -2 & 0 \\ 5 & 3 \end{bmatrix}$ $B = \begin{bmatrix} 2 & 0 & 0 \\ 2 & 10 & 0 \\ -1 & -5 & 2 \end{bmatrix}$

* Triangular matrix:-

A Square matrix which is either upper triangular or lower triangular is called a triangular matrix.

Equality of Matrices:

Two matrices $A = [a_{ij}]$ and $B = [b_{ij}]$ are equal [written as $A = B$] if and only if

(i) both A and B are of the same order.

(ii) The corresponding entries of A and B are equal.

That is $a_{ij} = b_{ij}$ for all i and j

N KARTHIKEYAN, M.Sc, B.Ed
AVP TRUST NATIONAL
MATRIC HR SEC SCHOOL
TIRUPUR-9843434491

Algebraic Operations on Matrices:

* Multiplication of a matrix by a Scalar.
* Addition / subtraction of two matrices
* multiplication of two matrices.

* Multiplication of a matrix by a Scalar:

For a given matrix $A = [a_{ij}]_{m\times n}$ and a Scalar K, we define a new matrix $KA = [b_{ij}]_{m\times n}$ where $b_{ij} = Ka_{ij}$ for all i and j

In Particular if $K = -1$, we obtain $-A = [-a_{ij}]_{m\times n}$. This $-A$ is called negative of a matrix A.

* Addition and Subtraction of two matrices:

If A and B are two matrices of the Same order, then their sum denoted by $A+B$, is again a matrix of Same Order obtained by adding the corresponding entries of A and B.

More Precisely if $A = [a_{ij}]_{m\times n}$ and $B = [b_{ij}]_{m\times n}$ are two matrices, then the Sum $A+B$ of A and B is a matrix given by

$$A+B = [c_{ij}]_{m\times n} \qquad c_{ij} = a_{ij} + b_{ij} \text{ for all } i \text{ and } j$$

Similarly subtraction $A-B$ is defined as $A-B = A+(-1)B$.

$$A-B = [d_{ij}]_{m\times n} \quad \text{where } d_{ij} = a_{ij} - b_{ij} \ \forall\ i \,\&\, j$$

Note :-

* If A and B are not of the Same order, then $A+B$ and $A-B$ are not defined.
* The addition and Subtraction can be extended to any finite number of matrices.

* Multiplication of Matrices :

A matrix A is Said to be Comformable for multiplication with a matrix B if the number of columns of A is equal to the number of rows of B.

If $A = [a_{ij}]_{m \times n}$ and $B = [b_{ij}]_{n \times p}$ are two matrices, then the Product of matrices A and B is denoted by AB and its order is $m \times p$.

$$\begin{array}{cc} A & B \\ m \times \boxed{n \quad n} \times p \end{array}$$

Note :

(i) If $A = [a_{ij}]_{m \times n}$ and $B = [b_{ij}]_{n \times p}$ and $m \neq p$ then the Product AB is defined but not BA.

(ii) Even if AB and BA are defined, then $AB = BA$ is not necessarily true.

(iii) In general for any two matrices A and B which are Conformable for addition and multiplication, for the below operations we have

* $(A \pm B)^2$ need not be equal to $A^2 \pm 2AB + B^2$
* $A^2 - B^2$ need not be equal to $(A+B)(A-B)$.

Properties of Matrix Addition, scalar multiplication and Product of matrices:-

Let A, B, C be three matrices of Same order which are conformable for addition and a, b be two Scalars. Then we have the following :

1. A+B yields a matrix of the Same order.

2. $A+B=B+A$ [Commutative]

3. $A+(B+C)=(A+B)+C$ [associative]

4. $A+O=O+A=A$ [O is identity]

5. $A+(-A)=(-A)+A=O$ [$-A$ is the additive inverse of A]

6. $(a+b)A=aA+bA$ and $a(A+B)=aA+aB$.

7. $a(bA)=abA$, $1A=A$ and $0A=0$

multiplication :-

1. $AB \neq BA$ [Not Commutative]

2. $A(BC)=(AB)C$ [Associative]

3. $A(B+C)=AB+AC$ [distributive]
$(A+B)C=AB+BC$

4. $AI=IA=A$ [multiplicative identity I]

5. $\alpha(AB)=A(\alpha B)=(\alpha A)B$

Operation of Transpose of a matrix :-

The transpose of a matrix is obtained by interchanging rows and Columns of A and it is denoted by A^T.

Properties :

(i) $(A^T)^T=A$

(ii) $(KA)^T=KA^T$

(iii) $(A\pm B)^T=A^T\pm B^T$

(iv) $(AB)^T=B^T.A^T$

7.1
4

Suppose that a matrix has 12 elements. What are the possible orders it can have? What if it has 7 elements?

The number of elements is the product of number of rows and number of columns.

(i) The product is 12.

$\therefore$ The orders are 1×12, 2×6, 3×4, 4×3, 6×2, 12×1.

(ii) The product is 7.

$\therefore$ The orders are 1×7, 7×1.

7.2
4.

Construct a 2×3 matrix whose $(i,j)^{th}$ element is given by

$$a_{ij} = \frac{\sqrt{3}}{2}|2i-3j|$$

$$A = [a_{ij}]_{2\times3} = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \end{bmatrix}$$

$$a_{ij} = \frac{\sqrt{3}}{2}|2i-3j|$$

$a_{11} = \frac{\sqrt{3}}{2}|2-3| = \frac{\sqrt{3}}{2}$ $\quad$ $a_{12} = \frac{\sqrt{3}}{2}|2-6| = \frac{4\sqrt{3}}{2} = 2\sqrt{3}$

$a_{13} = \frac{\sqrt{3}}{2}|2-9| = \frac{7\sqrt{3}}{2}$ $\quad$ $a_{21} = \frac{\sqrt{3}}{2}|4-3| = \frac{\sqrt{3}}{2}$

$a_{22} = \frac{\sqrt{3}}{2}|4-6| = \frac{2\sqrt{3}}{2} = \sqrt{3}$ $\quad$ $a_{23} = \frac{\sqrt{3}}{2}|4-9| = \frac{5\sqrt{3}}{2}$

$$\therefore A = \begin{bmatrix} \frac{\sqrt{3}}{2} & 2\sqrt{3} & \frac{7\sqrt{3}}{2} \\ \frac{\sqrt{3}}{2} & \sqrt{3} & \frac{5\sqrt{3}}{2} \end{bmatrix}$$

7.3
7

Find x, y, a and b if $\begin{bmatrix} 3x+4y & 6 & x-2y \\ a+b & 2a-b & -3 \end{bmatrix} = \begin{bmatrix} 2 & 6 & 4 \\ 5 & -5 & -3 \end{bmatrix}$

$$\begin{bmatrix} 3x+4y & 6 & x-2y \\ a+b & 2a-b & -3 \end{bmatrix} = \begin{bmatrix} 2 & 6 & 4 \\ 5 & -5 & -3 \end{bmatrix}$$

Since the matrices are equal

$3x+4y = 2$ ——— (1) $\qquad$ $a+b=5$ ——— (3)

$x-2y = 4$ ——— (2) $\qquad$ $2a-b=-5$ ——— (4)

$$② \times 2 \qquad \begin{aligned} 2x - \cancel{4y} &= 8 \\ 3x + \cancel{4y} &= 2 \\ \hline 5x &= 10 \\ \boxed{x = 2} \end{aligned} \qquad\qquad \begin{aligned} a + \cancel{b} &= 5 \quad —③ \\ 2a - \cancel{b} &= -5 \quad —④ \\ \hline 3a &= 0 \\ \boxed{a = 0} \end{aligned}$$

Sub $x = 2$ in ①

$$3 + 4y = 2$$
$$4y = -4$$
$$\boxed{y = -1}$$

Sub $a = 0$ in ③

$$a + b = 5$$
$$\boxed{b = 5}$$

$\therefore x = 2,\ y = -1,\ a = 0$ and $b = 5$.

$\frac{7.4}{8}$ Compute $A+B$ and $A-B$ if

$$A = \begin{bmatrix} 4 & \sqrt{5} & 7 \\ -1 & 0 & 0.5 \end{bmatrix} \qquad B = \begin{bmatrix} \sqrt{3} & \sqrt{5} & 7.3 \\ 1 & \frac{1}{3} & \frac{1}{4} \end{bmatrix}$$

$$A + B = \begin{bmatrix} 4 & \sqrt{5} & 7 \\ -1 & 0 & 0.5 \end{bmatrix} + \begin{bmatrix} \sqrt{3} & \sqrt{5} & 7.3 \\ 1 & \frac{1}{3} & \frac{1}{4} \end{bmatrix}$$

$$= \begin{bmatrix} 4+\sqrt{3} & \sqrt{5}+\sqrt{5} & 7+7.3 \\ -1+1 & 0+\frac{1}{3} & 0.5+\frac{1}{4} \end{bmatrix} = \begin{bmatrix} 4+\sqrt{3} & 2\sqrt{5} & 14.3 \\ 0 & \frac{1}{3} & \frac{3}{4} \end{bmatrix}$$

$$A - B = \begin{bmatrix} 4 & \sqrt{5} & 7 \\ -1 & 0 & 0.5 \end{bmatrix} - \begin{bmatrix} \sqrt{3} & \sqrt{5} & 7.3 \\ 1 & \frac{1}{3} & \frac{1}{4} \end{bmatrix}$$

$$= \begin{bmatrix} 4-\sqrt{3} & 0 & -0.3 \\ -2 & -\frac{1}{3} & \frac{1}{4} \end{bmatrix}$$

$\frac{7.5}{9}$ Find the sum $A+B+C$ if A, B, C are given by

$$A = \begin{bmatrix} \sin^2\theta & 1 \\ \cot^2\theta & 0 \end{bmatrix} \qquad B = \begin{bmatrix} \cos^2\theta & 0 \\ -\csc^2\theta & 1 \end{bmatrix} \qquad C = \begin{bmatrix} 0 & -1 \\ -1 & 0 \end{bmatrix}$$

$$A+B+C = \begin{bmatrix} \sin^2\theta & 1 \\ \cot^2\theta & 0 \end{bmatrix} + \begin{bmatrix} \cos^2\theta & 0 \\ -\csc^2\theta & 1 \end{bmatrix} + \begin{bmatrix} 0 & -1 \\ -1 & 0 \end{bmatrix}$$

$$= \begin{bmatrix} \sin^2\theta + \cos^2\theta + 0 & 1+0-1 \\ \cot^2\theta - \csc^2\theta - 1 & 0+1-0 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ -1-1 & 1 \end{bmatrix}$$

$$A+B+C = \begin{bmatrix} 1 & 0 \\ -2 & 1 \end{bmatrix}.$$

N KARTHIKEYAN, M.Sc, B.Ed
AVP TRUST NATIONAL
MATRIC HR SEC SCHOOL
TIRUPUR-9843434491

7.6 / 9

Determine $3B+4C-D$ if B, C and D are given by

$B=\begin{bmatrix}2&3&0\\1&-1&5\end{bmatrix}$ $C=\begin{bmatrix}-1&-2&3\\-1&0&2\end{bmatrix}$ $D=\begin{bmatrix}0&4&-1\\5&6&-5\end{bmatrix}$

$$3B+4C-D=3\begin{bmatrix}2&3&0\\1&-1&5\end{bmatrix}+4\begin{bmatrix}-1&-2&3\\-1&0&2\end{bmatrix}-\begin{bmatrix}0&4&-1\\5&6&-5\end{bmatrix}$$

$$=\begin{bmatrix}6&9&0\\3&-3&15\end{bmatrix}+\begin{bmatrix}-4&-8&12\\-4&0&8\end{bmatrix}-\begin{bmatrix}0&4&-1\\5&6&-5\end{bmatrix}$$

$$=\begin{bmatrix}6-4-0&9-8-4&0+12+1\\3-4-5&-3+0-6&15+8+5\end{bmatrix}=\begin{bmatrix}2&-3&13\\-6&-9&28\end{bmatrix}$$

$$\therefore\ 3B+4C-D=\begin{bmatrix}2&-3&13\\-6&-9&28\end{bmatrix}.$$

7.7 / 9

Simplify: $\sec\theta\begin{bmatrix}\sec\theta&\tan\theta\\\tan\theta&\sec\theta\end{bmatrix}-\tan\theta\begin{bmatrix}\tan\theta&\sec\theta\\\sec\theta&\tan\theta\end{bmatrix}$

$$\sec\theta\begin{bmatrix}\sec\theta&\tan\theta\\\tan\theta&\sec\theta\end{bmatrix}-\tan\theta\begin{bmatrix}\tan\theta&\sec\theta\\\sec\theta&\tan\theta\end{bmatrix}$$

$$=\begin{bmatrix}\sec^2\theta&\sec\theta\tan\theta\\\sec\theta\tan\theta&\sec^2\theta\end{bmatrix}-\begin{bmatrix}\tan^2\theta&\tan\theta\sec\theta\\\tan\theta\sec\theta&\tan^2\theta\end{bmatrix}$$

$$=\begin{bmatrix}\sec^2\theta-\tan^2\theta&0\\0&\sec^2\theta-\tan^2\theta\end{bmatrix}=\begin{bmatrix}1&0\\0&1\end{bmatrix}.$$

7.8 / 10

If $A=\begin{bmatrix}0&c&b\\c&0&a\\b&a&0\end{bmatrix}$ Compute A^2

$$A=\begin{bmatrix}0&c&b\\c&0&a\\b&a&0\end{bmatrix}$$

$$A^2=A\times A=\begin{bmatrix}0&c&b\\c&0&a\\b&a&0\end{bmatrix}\begin{bmatrix}0&c&b\\c&0&a\\b&a&0\end{bmatrix}$$

$$=\begin{bmatrix}0+c^2+b^2&0+0+ab&0+ac+0\\0+0+ab&c^2+0+a^2&bc+0+0\\0+ac+0&bc+0+0&b^2+a^2+0\end{bmatrix}$$

$$=\begin{bmatrix}b^2+c^2&ab&ac\\ab&a^2+c^2&bc\\ac&bc&a^2+b^2\end{bmatrix}$$

N KARTHIKEYAN, M.Sc, B.Ed
AVP TRUST NATIONAL
MATRIC HR SEC SCHOOL
TIRUPUR-9843434491

T.9 / 11. Solve for x if $\begin{bmatrix} x & 2 & -1 \end{bmatrix}\begin{bmatrix} 1 & 1 & 2 \\ -1 & -4 & 1 \\ -1 & -1 & -2 \end{bmatrix}\begin{bmatrix} x \\ 2 \\ 0 \end{bmatrix} = 0$

$$\begin{bmatrix} x & 2 & -1 \end{bmatrix}\begin{bmatrix} 1 & 1 & 2 \\ -1 & -4 & 1 \\ -1 & -1 & -2 \end{bmatrix}\begin{bmatrix} x \\ 2 \\ 1 \end{bmatrix} = 0$$

$$\begin{bmatrix} x-2+1 & x-8+1 & 2x+2+2 \end{bmatrix}\begin{bmatrix} x \\ 2 \\ 1 \end{bmatrix} = 0$$

$$\begin{bmatrix} x-1 & x-7 & 2x+4 \end{bmatrix}\begin{bmatrix} x \\ 2 \\ 1 \end{bmatrix} = 0$$

$$x(x-1) + 2(x-7) + 1(2x+4) = 0$$

$$x^2 - x + 2x - 14 + 2x + 4 = 0$$

$$x^2 + 3x - 10 = 0$$

$$(x-2)(x+5) = 0$$

$$x = 2 \qquad x = -5.$$

-10

$\frac{-2}{x}$ $\frac{5}{x}$

T.10 / 12 If $A = \begin{bmatrix} 1 & -1 & 2 \\ -2 & 1 & 3 \\ 0 & -3 & 4 \end{bmatrix}$ and $B = \begin{bmatrix} 1 & -3 \\ -1 & 1 \\ 1 & 2 \end{bmatrix}$ find AB and BA if they exists.

$$A = \begin{bmatrix} 1 & -1 & 2 \\ -2 & 1 & 3 \\ 0 & -3 & 4 \end{bmatrix} \quad B = \begin{bmatrix} 1 & -3 \\ -1 & 1 \\ 1 & 2 \end{bmatrix}$$

A B

$3 \times \boxed{3 \quad 3} \times 2$

Product of AB exists. and its order 3×2.

$$AB = \begin{bmatrix} 1 & -1 & 2 \\ -2 & 1 & 3 \\ 0 & -3 & 4 \end{bmatrix}\begin{bmatrix} 1 & -3 \\ -1 & 1 \\ 1 & 2 \end{bmatrix}$$

$$= \begin{bmatrix} 1+1+2 & -3-1+4 \\ -2-1+3 & +6+1+6 \\ 0+3+4 & 0-3+8 \end{bmatrix} = \begin{bmatrix} 4 & 0 \\ 0 & 13 \\ 7 & 5 \end{bmatrix}$$

B $\times$ A

$3 \times \boxed{2 \quad 3} \times 3$

The Product of BA does not exists. because the number of Columns in B is not equal to no of rows in A.

N KARTHIKEYAN, M.Sc, B.Ed

AVP TRUST NATIONAL

MATRIC HR SEC SCHOOL

TIRUPUR-9843434491

7.11 / 13. A fruit shop keeper prepares 3 different varieties of gift packages. Pack-I contains 6 apples, 3 oranges, and 3 pomegrantates. Pack-II contains 5 apples 4 oranges and 4 pomegranates and Pack III contains 6 apples 6 oranges and 6 pomegrantates. The cost of an apple, an orange and a pomegranate respectively are ₹30, ₹15 and ₹45. What is the cost of preparing each package of fruits?

Let cost matrix $A = \begin{bmatrix} 30 & 15 & 45 \end{bmatrix}$

Fruit matrix
$$B = \begin{array}{c} \begin{array}{ccc} P_I & P_{II} & P_{III} \end{array} \\ \begin{bmatrix} 6 & 5 & 6 \\ 3 & 4 & 6 \\ 3 & 4 & 6 \end{bmatrix} \end{array} \begin{array}{l} \text{Apples} \\ \text{oranges} \\ \text{Pomegramatis} \end{array}$$

$$AB = \begin{bmatrix} 30 & 15 & 45 \end{bmatrix} \begin{bmatrix} 6 & 5 & 6 \\ 3 & 4 & 6 \\ 3 & 4 & 6 \end{bmatrix}$$

$$= \begin{bmatrix} 180+45+135 & 150+60+180 & 180+60+360 \end{bmatrix}$$

$$= \begin{bmatrix} 360 & 390 & 540 \end{bmatrix}$$

$\therefore$ Pack I Cost = ₹ 360

Pack II Cost = ₹ 390

Pack III Cost = ₹ 540

T.12 / 14 If $A = \begin{bmatrix} 4 & 6 & 2 \\ 0 & 1 & 5 \\ 0 & 3 & 2 \end{bmatrix}$ and $B = \begin{bmatrix} 0 & 1 & -1 \\ 3 & -1 & 4 \\ -1 & 2 & 1 \end{bmatrix}$

Verify (i) $(AB)^T = B^T A^T$ (ii) $(A+B)^T = A^T + B^T$

(iii) $(A-B)^T = A^T - B^T$ (iv) $(3A)^T = 3A^T$

$$A = \begin{bmatrix} 4 & 6 & 2 \\ 0 & 1 & 5 \\ 0 & 3 & 2 \end{bmatrix} \qquad B = \begin{bmatrix} 0 & 1 & -1 \\ 3 & -1 & 4 \\ -1 & 2 & 1 \end{bmatrix}$$

$$AB = \begin{bmatrix} 4 & 6 & 2 \\ 0 & 1 & 5 \\ 0 & 3 & 2 \end{bmatrix} \begin{bmatrix} 0 & 1 & -1 \\ 3 & -1 & 4 \\ -1 & 2 & 1 \end{bmatrix} = \begin{bmatrix} 0+18-2 & 4-6+4 & -4+24+2 \\ 0+3-5 & 0-1+10 & 0+4+5 \\ 0+9-2 & 0-3+4 & 0+12+2 \end{bmatrix}$$

$$= \begin{bmatrix} 16 & 2 & 22 \\ -2 & 9 & 9 \\ 7 & 1 & 14 \end{bmatrix}$$

$$(AB)^T = \begin{bmatrix} 16 & -2 & 7 \\ 2 & 9 & 1 \\ 22 & 9 & 14 \end{bmatrix} \text{ ——— } (1)$$

$$B^T = \begin{bmatrix} 0 & 3 & -1 \\ 1 & -1 & 2 \\ -1 & 4 & 1 \end{bmatrix} \qquad A^T = \begin{bmatrix} 4 & 0 & 0 \\ 6 & 1 & 3 \\ 2 & 5 & 2 \end{bmatrix}$$

$$B^T A^T = \begin{bmatrix} 0 & 3 & -1 \\ 1 & -1 & 2 \\ -1 & 4 & 1 \end{bmatrix} \begin{bmatrix} 4 & 0 & 0 \\ 6 & 1 & 3 \\ 2 & 5 & 2 \end{bmatrix} = \begin{bmatrix} 0+18-2 & 0+3-5 & 0+9-2 \\ 4-6+4 & 0+1+10 & 0-3+4 \\ -4+24+2 & 0+4+5 & 0+12+2 \end{bmatrix}$$

$$= \begin{bmatrix} 16 & -2 & 7 \\ 2 & 9 & 1 \\ 22 & 9 & 14 \end{bmatrix} \text{ ——— } (2)$$

from (1) & (2) $(AB)^T = B^T A^T$

Hence verified.

(ii) $(A+B)^T = A^T + B^T$

$$A+B = \begin{bmatrix} 4 & 6 & 2 \\ 0 & 1 & 5 \\ 0 & 3 & 2 \end{bmatrix} + \begin{bmatrix} 0 & 1 & -1 \\ 3 & -1 & 4 \\ -1 & 2 & 1 \end{bmatrix} = \begin{bmatrix} 4 & 7 & 1 \\ 3 & 0 & 9 \\ -1 & 5 & 3 \end{bmatrix}$$

$$(A+B)^T = \begin{bmatrix} 4 & 3 & -1 \\ 7 & 0 & 5 \\ 1 & 9 & 3 \end{bmatrix} \text{ ——— } (1)$$

$$A^T + B^T = \begin{bmatrix} 4 & 0 & 0 \\ 6 & 1 & 3 \\ 2 & 5 & 2 \end{bmatrix} + \begin{bmatrix} 0 & 3 & -1 \\ 1 & -1 & 2 \\ -1 & 4 & 1 \end{bmatrix} = \begin{bmatrix} 4 & 3 & -1 \\ 7 & 0 & 5 \\ 1 & 9 & 3 \end{bmatrix} \text{ ——— } (2)$$

from (1) & (2) $(A+B)^T = A^T + B^T$

Hence verified.

(iii) $(A-B)^T = A^T - B^T$

$$A-B = \begin{bmatrix} 4 & 6 & 2 \\ 0 & 1 & 5 \\ 0 & 3 & 2 \end{bmatrix} - \begin{bmatrix} 0 & 1 & -1 \\ 3 & -1 & 4 \\ -1 & 2 & 1 \end{bmatrix} = \begin{bmatrix} 4 & 5 & 3 \\ -3 & 2 & 1 \\ 1 & 1 & 1 \end{bmatrix}$$

$$(A-B)^T = \begin{bmatrix} 4 & -3 & 1 \\ 5 & 2 & 1 \\ 3 & 1 & 1 \end{bmatrix} \text{ ——— } (1)$$

$$A^T - B^T = \begin{bmatrix} 4 & 0 & 0 \\ 6 & 1 & 3 \\ 2 & 5 & 2 \end{bmatrix} - \begin{bmatrix} 0 & 3 & -1 \\ 1 & -1 & 2 \\ -1 & 4 & 1 \end{bmatrix} = \begin{bmatrix} 4 & -3 & 1 \\ 5 & 2 & 1 \\ 3 & 1 & 1 \end{bmatrix} \text{ ——— } (2)$$

from (1) & (2) $(A-B)^T = A^T - B^T$

Hence verified.

(iv) $(3A)^T = 3A^T$

$$A = \begin{bmatrix} 4 & 6 & 2 \\ 0 & 1 & 5 \\ 0 & 3 & 2 \end{bmatrix} \qquad 3A = \begin{bmatrix} 12 & 18 & 6 \\ 0 & 3 & 15 \\ 0 & 9 & 6 \end{bmatrix}$$

$$(3A)^T = \begin{bmatrix} 12 & 0 & 0 \\ 18 & 3 & 9 \\ 6 & 15 & 6 \end{bmatrix} \text{ ———— } (1)$$

$$3A^T = 3\begin{bmatrix} 4 & 0 & 0 \\ 6 & 1 & 3 \\ 2 & 5 & 2 \end{bmatrix} = \begin{bmatrix} 12 & 0 & 0 \\ 18 & 3 & 9 \\ 6 & 15 & 6 \end{bmatrix} \text{ ———— } (2)$$

from (1) & (2)

$(3A)^T = 3A^T$

Hence verified.

Ex : 7.1

1/17. Construct an $m \times n$ matrix $A = [a_{ij}]$ where a_{ij} is given by

(i) $a_{ij} = \dfrac{(i-2j)^2}{2}$ with $m = 2$ $n = 3$

$$A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \end{bmatrix}$$

$a_{ij} = \dfrac{(i-2j)^2}{2}$ $\quad a_{11} = \dfrac{(1-2)^2}{2} = \dfrac{1}{2}$ $\quad a_{12} = \dfrac{(1-4)^2}{2} = \dfrac{9}{2}$

$a_{13} = \dfrac{(1-6)^2}{2} = \dfrac{25}{2}$ $\quad a_{21} = \dfrac{(2-2)^2}{2} = 0$ $\quad a_{22} = \dfrac{(2-4)^2}{2} = \dfrac{4}{2} = 2$

$a_{23} = \dfrac{(2-6)^2}{2} = \dfrac{16}{2} = 4$

$$\therefore A = \begin{bmatrix} \frac{1}{2} & \frac{9}{2} & \frac{25}{2} \\ 0 & 2 & 4 \end{bmatrix}$$

(ii) $a_{ij} = \dfrac{|3i-4j|}{4}$ with $m = 3$ $n = 4$

$$A = \begin{bmatrix} a_{11} & a_{12} & a_{13} & a_{14} \\ a_{21} & a_{22} & a_{23} & a_{24} \\ a_{31} & a_{32} & a_{33} & a_{34} \end{bmatrix}$$

$$a_{ij} = \frac{|3i-4j|}{4}$$

$a_{11} = \dfrac{|3-4|}{4} = \dfrac{1}{4}$ $\quad a_{12} = \left|\dfrac{3-8}{4}\right| = \dfrac{5}{4}$

$a_{13} = \frac{|3-12|}{4} = \frac{9}{4}$ $\quad a_{14} = \left|\frac{3-16}{4}\right| = \frac{13}{4}$ $\quad a_{21} = \frac{|6-4|}{4} = \frac{1}{2}$

$a_{22} = \frac{|6-8|}{4} = \frac{2}{4} = \frac{1}{2}$ $\quad a_{23} = \frac{|6-12|}{4} = \frac{6}{4} = \frac{3}{2}$

$a_{24} = \frac{|6-16|}{4} = \frac{10}{4} = \frac{5}{2}$ $\quad a_{31} = \frac{|9-4|}{4} = \frac{5}{4}$

$a_{32} = \frac{|9-8|}{4} = \frac{1}{4}$ $\quad a_{33} = \frac{|9-12|}{4} = \frac{3}{4}$

$a_{34} = \frac{|9-16|}{4} = \frac{7}{4}.$

$$\therefore A = \begin{bmatrix} \frac{1}{4} & \frac{5}{4} & \frac{9}{4} & \frac{13}{4} \\ \frac{1}{2} & \frac{1}{2} & \frac{3}{2} & \frac{5}{2} \\ \frac{5}{4} & \frac{1}{4} & \frac{3}{4} & \frac{7}{4} \end{bmatrix} = \frac{1}{4}\begin{bmatrix} 1 & 5 & 9 & 13 \\ 2 & 2 & 6 & 10 \\ 5 & 1 & 3 & 7 \end{bmatrix}$$

2/18 Find the values of p, q, r and s if

$$\begin{bmatrix} p^2-1 & 0 & -31-q^3 \\ 7 & r+1 & 9 \\ -2 & 8 & s-1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & -4 \\ 7 & \frac{3}{2} & 9 \\ -2 & 8 & -\pi \end{bmatrix}$$

$p^2 - 1 = 1$

$p^2 = 2$

$p = \pm\sqrt{2}$

$-31 - q^3 = -4$

$-q^3 = 27$

$q^3 = -27$

$\boxed{q = -3}$

$r + 1 = \frac{3}{2}$

$\boxed{r = \frac{1}{2}}$

$s - 1 = -\pi$

$\boxed{s = 1-\pi}$

---

3. Determine the value of $x+y$ if

$$\begin{bmatrix} 2x+y & 4x \\ 5x-7 & 4x \end{bmatrix} = \begin{bmatrix} 7 & 7y-13 \\ y & x+6 \end{bmatrix}$$

$4x = x+6$

$4x - x = 6$

$3x = 6$

$\boxed{x = 2}$

$y = 5x - 7$

$y = 10 - 7$

$\boxed{y = 3}$

N KARTHIKEYAN, M.Sc, B.Ed

AVP TRUST NATIONAL

MATRIC HR SEC SCHOOL

TIRUPUR-9843434491

4/18 Determine the matrices A and B if they satisfy

$$2A - B + \begin{bmatrix} 6 & -6 & 0 \\ -4 & 2 & 1 \end{bmatrix} = 0 \text{ and } A - 2B = \begin{bmatrix} 3 & 2 & 8 \\ -2 & 1 & -7 \end{bmatrix}$$

$$2A - B = \begin{bmatrix} -6 & 6 & 0 \\ 4 & -2 & -1 \end{bmatrix} \quad \text{——— (1)}$$

$$A - 2B = \begin{bmatrix} 3 & 2 & 8 \\ -2 & 1 & -7 \end{bmatrix} \quad \text{——— (2)}$$

① × 2 $\quad 4A - 2B = \begin{bmatrix} -12 & 12 & 0 \\ 8 & -4 & -2 \end{bmatrix}$ ——— ③

② × 1 $\quad A - 2B = \begin{bmatrix} 3 & 2 & 8 \\ -2 & 1 & -7 \end{bmatrix}$ ——— ④

(−) (+) (−)

$$3A = \begin{bmatrix} -15 & 10 & -8 \\ 10 & -5 & 5 \end{bmatrix}$$

$$A = \frac{1}{3}\begin{bmatrix} -15 & 10 & -8 \\ 10 & -5 & 5 \end{bmatrix}$$

Sub $A = \frac{1}{3}\begin{bmatrix} -15 & 10 & -8 \\ 10 & -5 & 5 \end{bmatrix}$ in ①

$$2A - B = \begin{bmatrix} -6 & 6 & 0 \\ 4 & -2 & -1 \end{bmatrix}$$

$$\frac{2}{3}\begin{bmatrix} -15 & 10 & -8 \\ 10 & -5 & 5 \end{bmatrix} - B = \begin{bmatrix} -6 & 6 & 0 \\ 4 & -2 & -1 \end{bmatrix}$$

$$B = \frac{2}{3}\begin{bmatrix} -15 & 10 & -8 \\ 10 & -5 & 5 \end{bmatrix} - \begin{bmatrix} -6 & 6 & 0 \\ 4 & -2 & -1 \end{bmatrix}$$

$$= \begin{bmatrix} -10 & \frac{20}{3} & \frac{-16}{3} \\ \frac{20}{3} & \frac{-10}{3} & \frac{10}{3} \end{bmatrix} - \begin{bmatrix} -6 & 6 & 0 \\ 4 & -2 & -1 \end{bmatrix}$$

$$= \begin{bmatrix} -4 & 2/3 & -16/3 \\ 8/3 & -4/3 & \frac{13}{3} \end{bmatrix} = \frac{1}{3}\begin{bmatrix} -12 & 2 & -16 \\ 8 & -4 & 13 \end{bmatrix}$$

$\therefore A = \frac{1}{3}\begin{bmatrix} -15 & 10 & -8 \\ 10 & -5 & 5 \end{bmatrix} \qquad B = \frac{1}{3}\begin{bmatrix} -12 & 2 & -16 \\ 8 & -4 & 13 \end{bmatrix}$

$\frac{5}{18}$ If $A = \begin{bmatrix} 1 & a \\ 0 & 1 \end{bmatrix}$ then Compute A^4.

$A = \begin{bmatrix} 1 & a \\ 0 & 1 \end{bmatrix} \qquad A^2 = A \cdot A = \begin{bmatrix} 1 & a \\ 0 & 1 \end{bmatrix}\begin{bmatrix} 1 & a \\ 0 & 1 \end{bmatrix}$

$$= \begin{bmatrix} 1+0 & a+a \\ 0+0 & 0+1 \end{bmatrix} = \begin{bmatrix} 1 & 2a \\ 0 & 1 \end{bmatrix}$$

$$A^4 = A^2 \cdot A^2 = \begin{bmatrix} 1 & 2a \\ 0 & 1 \end{bmatrix}\begin{bmatrix} 1 & 2a \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 1+0 & 2a+2a \\ 0+0 & 0+1 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 4a \\ 0 & 1 \end{bmatrix}.$$

$\frac{6}{18}$ Consider the matrix $A_\alpha = \begin{bmatrix} \cos\alpha & -\sin\alpha \\ \sin\alpha & \cos\alpha \end{bmatrix}$

(i) Show that $A_\alpha \cdot A_\beta = A_{\alpha+\beta}$.

(ii) Find all possible real values of α satisfying the condition $A_\alpha + A_\alpha^T = 1$.

$$A_\alpha = \begin{bmatrix} \cos\alpha & -\sin\alpha \\ \sin\alpha & \cos\alpha \end{bmatrix} \qquad A_\beta = \begin{bmatrix} \cos\beta & -\sin\beta \\ \sin\beta & \cos\beta \end{bmatrix}$$

$$A_\alpha A_\beta = \begin{bmatrix} \cos\alpha & -\sin\alpha \\ \sin\alpha & \cos\alpha \end{bmatrix} \begin{bmatrix} \cos\beta & -\sin\beta \\ \sin\beta & \cos\beta \end{bmatrix}$$

$$= \begin{bmatrix} \cos\alpha\cos\beta - \sin\alpha\sin\beta & -\cos\alpha\sin\beta - \sin\alpha\cos\beta \\ \sin\alpha\cos\beta + \cos\alpha\sin\beta & -\sin\alpha\sin\beta + \cos\alpha\cos\beta \end{bmatrix}$$

$$= \begin{bmatrix} \cos(\alpha+\beta) & -\sin(\alpha+\beta) \\ \sin(\alpha+\beta) & \cos(\alpha+\beta) \end{bmatrix}$$

$$= A_{(\alpha+\beta)}$$

Hence Proved.

(ii) $A_\alpha + A_\alpha^T = I$

$$\begin{bmatrix} \cos\alpha & -\sin\alpha \\ \sin\alpha & \cos\alpha \end{bmatrix} + \begin{bmatrix} \cos\alpha & \sin\alpha \\ -\sin\alpha & \cos\alpha \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

$$\begin{bmatrix} 2\cos\alpha & 0 \\ 0 & 2\cos\alpha \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

$$2\cos\alpha = 1$$

$$\cos\alpha = \frac{1}{2}$$

$\cos\theta = \cos\alpha$

$\theta = 2n\pi \pm \alpha$, $n \in z$

$$\boxed{\alpha = \frac{\pi}{3}}$$

$$\alpha = 2n\pi \pm \frac{\pi}{3} \quad n \in z$$

7/18 If $A = \begin{bmatrix} 4 & 2 \\ -1 & x \end{bmatrix}$ and such that $(A-2I)(A-3I)=0$ find the value of x.

$$A = \begin{bmatrix} 4 & 2 \\ -1 & x \end{bmatrix}$$

$$(A-2I)(A-3I) = 0$$

$$\left\{\begin{bmatrix} 4 & 2 \\ -1 & x \end{bmatrix} - \begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix}\right\}\left\{\begin{bmatrix} 4 & 2 \\ -1 & x \end{bmatrix} - \begin{bmatrix} 3 & 0 \\ 0 & 3 \end{bmatrix}\right\} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$$

$$\begin{bmatrix} 2 & 2 \\ -1 & x-2 \end{bmatrix}\begin{bmatrix} 1 & 2 \\ -1 & x-3 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$$

$$\begin{bmatrix} 2-2 & 4-2x-6 \\ -1+2-x & -2+(x-2)(x-3) \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$$

$$\begin{bmatrix} 0 & -2x-2 \\ 1-x & -2+x^2-5x+6 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$$

$$1-x = 0$$

$$\boxed{x = 1}$$

$\frac{8}{18}$ If $A = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ a & b & -1 \end{bmatrix}$ show that A^2 is a unit matrix.

$$A^2 = AA = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ a & b & -1 \end{bmatrix}\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ a & b & -1 \end{bmatrix}$$

$$= \begin{bmatrix} 1+0+0 & 0+0+0 & 0+0+0 \\ 0+0+0 & 0+1+0 & 0+0+0 \\ a+0-a & 0+b-b & 0+0+1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$A^2 = I$

Hence Proved.

$\frac{9}{18}$ If $A = \begin{bmatrix} 1 & 0 & 2 \\ 0 & 2 & 1 \\ 2 & 0 & 3 \end{bmatrix}$ and $A^3 - 6A^2 + 7A + kI = 0$ find 'k'

$$A^2 = A \cdot A = \begin{bmatrix} 1 & 0 & 2 \\ 0 & 2 & 1 \\ 2 & 0 & 3 \end{bmatrix}\begin{bmatrix} 1 & 0 & 2 \\ 0 & 2 & 1 \\ 2 & 0 & 3 \end{bmatrix}$$

$$= \begin{bmatrix} 1+0+4 & 0+0+0 & 2+0+6 \\ 0+0+2 & 0+4+0 & 0+2+3 \\ 2+0+6 & 0+0+0 & 4+0+9 \end{bmatrix}$$

$$= \begin{bmatrix} 5 & 0 & 8 \\ 2 & 4 & 5 \\ 8 & 0 & 13 \end{bmatrix}$$

$$A^3 = A^2 \cdot A = \begin{bmatrix} 5 & 0 & 8 \\ 2 & 4 & 5 \\ 8 & 0 & 13 \end{bmatrix}\begin{bmatrix} 1 & 0 & 2 \\ 0 & 2 & 1 \\ 2 & 0 & 3 \end{bmatrix}$$

$$= \begin{bmatrix} 5+0+16 & 0+0+0 & 10+0+24 \\ 2+0+10 & 0+8+0 & 4+4+15 \\ 8+0+26 & 0+0+0 & 16+0+39 \end{bmatrix}$$

$$= \begin{bmatrix} 21 & 0 & 34 \\ 12 & 8 & 23 \\ 34 & 0 & 55 \end{bmatrix}$$

$A^3 - 6A^2 + 7A + kI = 0$

$$\begin{bmatrix} 21 & 0 & 34 \\ 12 & 8 & 23 \\ 34 & 0 & 55 \end{bmatrix} - 6\begin{bmatrix} 5 & 0 & 8 \\ 2 & 4 & 5 \\ 8 & 0 & 13 \end{bmatrix} + 7\begin{bmatrix} 1 & 0 & 2 \\ 0 & 2 & 1 \\ 2 & 0 & 3 \end{bmatrix} + k\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

$$\begin{bmatrix} 21 & 0 & 34 \\ 12 & 8 & 23 \\ 34 & 0 & 55 \end{bmatrix} - \begin{bmatrix} 30 & 0 & 48 \\ 12 & 24 & 30 \\ 48 & 0 & 78 \end{bmatrix} + \begin{bmatrix} 7 & 0 & 14 \\ 0 & 14 & 7 \\ 14 & 0 & 21 \end{bmatrix} + \begin{bmatrix} k & 0 & 0 \\ 0 & k & 0 \\ 0 & 0 & k \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

$$\begin{bmatrix} -2 & 0 & 0 \\ 0 & -2 & 0 \\ 0 & 0 & -2 \end{bmatrix} + \begin{bmatrix} k & 0 & 0 \\ 0 & k & 0 \\ 0 & 0 & k \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

$k - 2 = 0$

$\boxed{k = 2}$

N KARTHIKEYAN, M.Sc, B.Ed
AVP TRUST NATIONAL
MATRIC HR SEC SCHOOL
TIRUPUR-9843434491

$\frac{10}{18}$ Give your own examples of matrices satisfying the following conditions in each case:

(i) A and B such that $AB \neq BA$

Let $A = \begin{bmatrix} 1 & 2 \\ 0 & 2 \end{bmatrix}$ $\quad B = \begin{bmatrix} 0 & 1 \\ 2 & 1 \end{bmatrix}$

$$AB = \begin{bmatrix} 1 & 2 \\ 0 & 2 \end{bmatrix}\begin{bmatrix} 0 & 1 \\ 2 & 1 \end{bmatrix} = \begin{bmatrix} 0+4 & 1+2 \\ 0+4 & 0+2 \end{bmatrix} = \begin{bmatrix} 4 & 3 \\ 4 & 2 \end{bmatrix}$$

$$BA = \begin{bmatrix} 0 & 1 \\ 2 & 1 \end{bmatrix}\begin{bmatrix} 1 & 2 \\ 0 & 2 \end{bmatrix} = \begin{bmatrix} 0+0 & 0+2 \\ 2+0 & 4+2 \end{bmatrix} = \begin{bmatrix} 0 & 2 \\ 2 & 6 \end{bmatrix}$$

$$AB \neq BA$$

(ii) $AB = BA = O$, $A \neq O$, $B \neq O$

Let $A = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}$ $\quad B = \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}$

$$AB = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} = \begin{bmatrix} 1-1 & -1+1 \\ 1-1 & -1+1 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} = O$$

$$BA = \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}\begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} = \begin{bmatrix} 1-1 & 1-1 \\ -1+1 & -1+1 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} = O$$

$$AB = BA = O, \quad A \neq O \quad B \neq O$$

(iii) A and B such that $AB = O$ and $BA \neq O$

Let $A = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$ $\quad B = \begin{bmatrix} 0 & 0 \\ 3 & 0 \end{bmatrix}$

$$AB = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} 0 & 0 \\ 3 & 0 \end{bmatrix} = \begin{bmatrix} 0+0 & 0+0 \\ 0+0 & 0+0 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$$

$$BA = \begin{bmatrix} 0 & 0 \\ 3 & 0 \end{bmatrix}\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} 0+0 & 0+0 \\ 3+0 & 0+0 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 3 & 0 \end{bmatrix}$$

$$AB = O, \quad BA \neq O$$

$\frac{11}{18}$ Show that $f(x) \cdot f(y) = f(x+y)$ where

$$f(x) = \begin{bmatrix} \cos x & -\sin x & 0 \\ \sin x & \cos x & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$f(x) \cdot f(y) = \begin{bmatrix} \cos x & -\sin x & 0 \\ \sin x & \cos x & 0 \\ 0 & 0 & 1 \end{bmatrix}\begin{bmatrix} \cos y & -\sin y & 0 \\ \sin y & \cos y & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} \cos x \cos y - \sin x \sin y & -\cos x \sin y - \sin x \cos y & 0 \\ \sin x \cos x + \cos x \sin y & -\sin x \sin y + \cos x \cos y & 0 \\ 0 & 0 & 0+0+1 \end{bmatrix}$$

$$= \begin{bmatrix} \cos(x+y) & -\sin(x+y) & 0 \\ \sin(x+y) & \cos(x+y) & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$= f(x+y)$$

Hence Proved.

12/18

If A is a square matrix such that $A^2 = A$ find the value of $7A - (I+A)^3$.

$$7A - (I+A)^3 = 7A - [I^3 + 3I^2A + 3IA^2 + A^3)$$

$$= 7A - (I + 3A + 3A^2 + A^3) \qquad [\because A^2 = A]$$

$$= 7A - (I + 3A + 3A + A^2 \cdot A]$$

$$= 7A - (I + 6A + A \cdot A)$$

$$= 7A - (I + 6A + A^2)$$

$$= 7A - (I + 6A + A)$$

$$= 7A - 7A - I = -I.$$

13/18

Verify the property $A(B+C) = AB + AC$, when the matrices A, B and C are given by

$$A = \begin{bmatrix} 2 & 0 & -3 \\ 1 & 4 & 5 \end{bmatrix} \quad B = \begin{bmatrix} 3 & 1 \\ -1 & 0 \\ 4 & 2 \end{bmatrix} \quad C = \begin{bmatrix} 4 & 7 \\ 2 & 1 \\ 1 & -1 \end{bmatrix}$$

L.H.S :

$$B+C = \begin{bmatrix} 3 & 1 \\ -1 & 0 \\ 4 & 2 \end{bmatrix} + \begin{bmatrix} 4 & 7 \\ 2 & 1 \\ 1 & -1 \end{bmatrix} = \begin{bmatrix} 7 & 8 \\ 1 & 1 \\ 5 & 1 \end{bmatrix}$$

$$A(B+C) = \begin{bmatrix} 2 & 0 & -3 \\ 1 & 4 & 5 \end{bmatrix} \begin{bmatrix} 7 & 8 \\ 1 & 1 \\ 5 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 14+0-15 & 16+0-3 \\ 7+4+25 & 8+4+5 \end{bmatrix} = \begin{bmatrix} -1 & 13 \\ 36 & 17 \end{bmatrix} \text{ — (1)}$$

R.H.S :

$$AB = \begin{bmatrix} 2 & 0 & -3 \\ 1 & 4 & 5 \end{bmatrix} \begin{bmatrix} 3 & 1 \\ -1 & 0 \\ 4 & 2 \end{bmatrix} = \begin{bmatrix} 6+0-12 & 2+0-6 \\ 3-4+20 & 1+0+10 \end{bmatrix}$$

$$= \begin{bmatrix} -6 & -4 \\ 19 & 11 \end{bmatrix}$$

$$AC = \begin{bmatrix} 2 & 0 & -3 \\ 1 & 4 & 5 \end{bmatrix} \begin{bmatrix} 4 & 7 \\ 2 & 1 \\ 1 & -1 \end{bmatrix} = \begin{bmatrix} 8+0-3 & 14+0+3 \\ 4+8+5 & 7+4-5 \end{bmatrix}$$

$$= \begin{bmatrix} 5 & 17 \\ 17 & 6 \end{bmatrix}$$

$$AB + AC = \begin{bmatrix} -6 & -4 \\ 19 & 11 \end{bmatrix} + \begin{bmatrix} 5 & 17 \\ 17 & 6 \end{bmatrix} = \begin{bmatrix} -1 & 13 \\ 36 & 17 \end{bmatrix} \text{ — (2)}$$

N KARTHIKEYAN, M.Sc, B.Ed
AVP TRUST NATIONAL
MATRIC HR SEC SCHOOL
TIRUPUR-9843434491

from (1) & (2)

$A(B+C) = AB + AC$

Hence verified.

$\frac{14}{19}$ Find the matrix A which satisfies the matrix relation $A\begin{bmatrix}1 & 2 & 3\\ 4 & 5 & 6\end{bmatrix} = \begin{bmatrix}-7 & -8 & -9\\ 2 & 4 & 6\end{bmatrix}$

let $B = \begin{bmatrix}1 & 2 & 3\\ 4 & 5 & 6\end{bmatrix}$ $C = \begin{bmatrix}-7 & -8 & -9\\ 2 & 4 & 6\end{bmatrix}$

$A_{m\times n}\, B_{2\times 3} = C_{2\times 3}$

$\boxed{n=2}$

$\boxed{m=2}$

$\therefore$ The order of A is 2×2

$$AB = C$$

$$\begin{bmatrix}a & b\\ c & d\end{bmatrix}\begin{bmatrix}1 & 2 & 3\\ 4 & 5 & 6\end{bmatrix} = \begin{bmatrix}-7 & -8 & -9\\ 2 & 4 & 6\end{bmatrix}$$

$$\begin{bmatrix}a+4b & 2a+5b & 3a+6b\\ c+4d & 2c+5d & 3c+6d\end{bmatrix} = \begin{bmatrix}-7 & -8 & -9\\ 2 & 4 & 6\end{bmatrix}$$

$a+4b = -7$ —— ①

$2a+5b = -8$ —— ②

$c+4d = 2$ —— ③

$2c+5d = 4$ —— ④

① × 2

$2a+8b = -14$

$2a+5b = -8$

$(-)\quad (-)\quad (+)$

$3b = -6$

$\boxed{b=-2}$

Sub $b=-2$ in ①

$a-8=-7$

$\boxed{a=1}$

③ × 2

$2c+8d = 4$ —— ⑤

$2c+5d = 4$

$(-)\quad (-)\quad (-)$

$3d = 0$

$\boxed{d=0}$

Sub $d=0$ in ③

$c+0=2$

$\boxed{c=2}$

$$\therefore A = \begin{bmatrix}1 & -2\\ 2 & 0\end{bmatrix}$$

$\frac{15}{19}$ If $A^T = \begin{bmatrix}4 & 5\\ -1 & 0\\ 2 & 3\end{bmatrix}$ and $B = \begin{bmatrix}2 & -1 & 1\\ 7 & 5 & -2\end{bmatrix}$ verify the following:

(i) $(A+B)^T = A^T + B^T = B^T + A^T$ (ii) $(A-B)^T = A^T - B^T$

(iii) $(B^T)^T = B$.

(i) $(A+B)^T = A^T + B^T = B^T + A^T$

$A^T = \begin{bmatrix}4 & 5\\ -1 & 0\\ 2 & 3\end{bmatrix}$ $A = \begin{bmatrix}4 & -1 & 2\\ 5 & 0 & 3\end{bmatrix}$ $B = \begin{bmatrix}2 & -1 & 1\\ 7 & 5 & -2\end{bmatrix}$

$B^T = \begin{bmatrix}2 & 7\\ -1 & 5\\ 1 & -2\end{bmatrix}$

$$A+B = \begin{bmatrix}4 & -1 & 2\\ 5 & 0 & 3\end{bmatrix} + \begin{bmatrix}2 & -1 & 1\\ 7 & 5 & -2\end{bmatrix} = \begin{bmatrix}6 & -2 & 3\\ 12 & 5 & 1\end{bmatrix}$$

$(A+B)^T = \begin{bmatrix}6 & 12\\ -2 & 5\\ 3 & 1\end{bmatrix}$ —— ①

$$A^T + B^T = \begin{bmatrix} 4 & 5 \\ -1 & 0 \\ 2 & 3 \end{bmatrix} + \begin{bmatrix} 2 & 7 \\ -1 & 5 \\ 1 & -2 \end{bmatrix} = \begin{bmatrix} 6 & 12 \\ -2 & 5 \\ 3 & 1 \end{bmatrix} \text{ ——— } (2)$$

$$B^T + A^T = \begin{bmatrix} 2 & 7 \\ -1 & 5 \\ 1 & -2 \end{bmatrix} + \begin{bmatrix} 4 & 5 \\ -1 & 0 \\ 2 & 3 \end{bmatrix} = \begin{bmatrix} 6 & 12 \\ -2 & 5 \\ 3 & 1 \end{bmatrix} \text{ ——— } (3)$$

from (1), (2) & (3)

$(A+B)^T = A^T + B^T = B^T + A^T$

Hence verified.

(ii) $(A-B)^T = A^T - B^T$

$$A - B = \begin{bmatrix} 4 & -1 & 2 \\ 5 & 0 & 3 \end{bmatrix} - \begin{bmatrix} 2 & -1 & 1 \\ 7 & 5 & -2 \end{bmatrix}$$

$$= \begin{bmatrix} 2 & 0 & 1 \\ -2 & -5 & 5 \end{bmatrix}$$

$$(A-B)^T = \begin{bmatrix} 2 & -2 \\ 0 & -5 \\ 1 & 5 \end{bmatrix} \text{ ——— } (1)$$

$$A^T - B^T = \begin{bmatrix} 4 & 5 \\ -1 & 0 \\ 2 & 3 \end{bmatrix} - \begin{bmatrix} 2 & 7 \\ -1 & 5 \\ 1 & -2 \end{bmatrix} = \begin{bmatrix} 2 & -2 \\ 0 & -5 \\ 1 & 5 \end{bmatrix} \text{ ——— } (2)$$

from (1) & (2)

$(A-B)^T = B^T - A^T$

Hence verified.

(iii) $(B^T)^T = B$

$$B = \begin{bmatrix} 2 & -1 & 1 \\ 7 & 5 & -2 \end{bmatrix} \qquad B^T = \begin{bmatrix} 2 & 7 \\ -1 & 5 \\ 1 & -2 \end{bmatrix}$$

$$(B^T)^T = \begin{bmatrix} 2 & -1 & 1 \\ 7 & 5 & -2 \end{bmatrix} = B$$

$\therefore (B^T)^T = B$

Hence verified.

16/19 If A is a 3×4 matrix and B is a matrix such that both A^TB and BA^T are defined, what is the order of the matrix B?

A is a 3×4 matrix

A^T is a 4×3 matrix

Since A^TB is defined when B is a matrix of order $3\times n$.

BA^T is also defined when B is of order $m\times4$.

$\therefore$ The order of B is 3×4.

Symmetric and Skew - Symmetric matrices:

* A Square matrix A is said to be Symmetric if $A = A^T$

* A Square matrix A is said to be Skew - Symmetric if $A = -A^T$

Theorem : T.1

For any Square matrix A with real numbers entries, $A + A^T$ is a Symmetric matrix and $A - A^T$ is a Skew Symmetric matrix

* Let $B = A + A^T$

Taking Transpose on both sides.

$$B^T = (A + A^T)^T$$
$$= A^T + (A^T)^T$$
$$= A^T + A$$
$$= A + T^A$$
$$B^T = B$$

This implies $A + A^T$ is symmetric matrix.

* Let $C = A - A^T$

Taking Transpose on both sides,

$$C^T = (A - A^T)^T$$
$$= A^T - (A^T)^T$$
$$= A^T - A$$
$$C^T = -(A - A^T)$$
$$C^T = -C$$

$\therefore$ $A - A^T$ is a Skew Symmetric matrix.

N KARTHIKEYAN, M.Sc, B.Ed
AVP TRUST NATIONAL
MATRIC HR SEC SCHOOL
TIRUPUR-9843434491

Theorem : T.2

A Square matrix can be expressed as the Sum of a Symmetric matrix and a Skew-Symmetric matrix

Let $A = \frac{1}{2}(A+A^T) + \frac{1}{2}(A-A^T)$

We know that $A+A^T$ is symmetric and $A-A^T$ is a skew symmetric.

$\frac{1}{2}(A+A^T)$ is symmetric and $\frac{1}{2}(A-A^T)$ is skew symmetric

$\therefore A = \frac{1}{2}(A+A^T) + \frac{1}{2}(A-A^T)$

Note :

A matrix which is both Symmetric and Skew Symmetric is a zero matrix.

7.13
16

Express the matrix $A = \begin{bmatrix} 1 & 3 & 5 \\ -6 & 8 & 3 \\ -4 & 6 & 5 \end{bmatrix}$ as the sum of a Symmetric and Skew Symmetric matrices.

$$A = \begin{bmatrix} 1 & 3 & 5 \\ -6 & 8 & 3 \\ -4 & 6 & 5 \end{bmatrix} \qquad A^T = \begin{bmatrix} 1 & -6 & -4 \\ 3 & 8 & 6 \\ 5 & 3 & 5 \end{bmatrix}$$

Let $P = \frac{1}{2}[A + A^T]$

$$= \frac{1}{2}\left\{\begin{bmatrix} 1 & 3 & 5 \\ -6 & 8 & 3 \\ -4 & 6 & 5 \end{bmatrix} + \begin{bmatrix} 1 & -6 & -4 \\ 3 & 8 & 6 \\ 5 & 3 & 5 \end{bmatrix}\right\}$$

$$P = \frac{1}{2}\begin{bmatrix} 2 & -3 & 1 \\ -3 & 16 & 9 \\ 1 & 9 & 10 \end{bmatrix}$$

$$P^T = \frac{1}{2}\begin{bmatrix} 2 & -3 & 1 \\ -3 & 16 & 9 \\ 1 & 9 & 10 \end{bmatrix} = P$$

$P^T = P$

$\therefore P$ is a Symmetric matrix.

Let $Q = \frac{1}{2}(A - A^T)$

$$= \frac{1}{2}\left\{\begin{bmatrix} 1 & 3 & 5 \\ -6 & 8 & 3 \\ -4 & 6 & 5 \end{bmatrix} - \begin{bmatrix} 1 & -6 & -4 \\ 3 & 8 & 6 \\ 5 & 3 & 5 \end{bmatrix}\right\}$$

$$= \frac{1}{2}\begin{bmatrix} 0 & 9 & 9 \\ -9 & 0 & 9 \\ -9 & -3 & 0 \end{bmatrix}$$

$$Q^T = \frac{1}{2}\begin{bmatrix} 0 & -9 & -9 \\ 9 & 0 & -3 \\ 9 & 9 & 0 \end{bmatrix} = -Q$$

$$Q^T = -Q$$

$\therefore$ Q is a Skew Symmetric matrix

$$A = P + Q = \frac{1}{2}\begin{bmatrix} 2 & -3 & 1 \\ -3 & 16 & 9 \\ 1 & 9 & 10 \end{bmatrix} + \frac{1}{2}\begin{bmatrix} 0 & 9 & 9 \\ -9 & 0 & -3 \\ -9 & 3 & 0 \end{bmatrix}$$

Thus A is expressed as the Sum of Symmetric and Skew Symmetric matrices.

17/19 Express the following matrices as the Sum of a Symmetric matrix and a Skew Symmetric matrix.

(i) $A = \begin{bmatrix} 4 & -2 \\ 3 & -5 \end{bmatrix}$ $\quad A^T = \begin{bmatrix} 4 & 3 \\ -2 & -5 \end{bmatrix}$

Let $P = \frac{1}{2}(A + A^T)$

$$P = \frac{1}{2}\left\{\begin{bmatrix} 4 & -2 \\ 3 & -5 \end{bmatrix} + \begin{bmatrix} 4 & 3 \\ -2 & -5 \end{bmatrix}\right\} = \frac{1}{2}\begin{bmatrix} 8 & 1 \\ 1 & -10 \end{bmatrix}$$

$$P^T = \frac{1}{2}\begin{bmatrix} 8 & 1 \\ 1 & -10 \end{bmatrix} = P$$

$$P^T = P$$

$\therefore$ P is a Symmetric matrix.

Let $Q = \frac{1}{2}(A - A^T) = \frac{1}{2}\left\{\begin{bmatrix} 4 & -2 \\ 3 & -5 \end{bmatrix} - \begin{bmatrix} 4 & 3 \\ -2 & -5 \end{bmatrix}\right\}$

$$= \frac{1}{2}\begin{bmatrix} 0 & -5 \\ 5 & 0 \end{bmatrix}$$

$$Q^T = \frac{1}{2}\begin{bmatrix} 0 & 5 \\ -5 & 0 \end{bmatrix} = -Q$$

$$Q^T = -Q$$

$\therefore$ Q is a Skew Symmetric matrix.

$$\therefore A = P + Q = \frac{1}{2}\begin{bmatrix} 8 & 1 \\ 1 & -10 \end{bmatrix} + \frac{1}{2}\begin{bmatrix} 0 & -5 \\ 5 & 0 \end{bmatrix}$$

(ii) $A = \begin{bmatrix} 3 & 3 & -1 \\ -2 & -2 & 1 \\ -4 & -5 & 2 \end{bmatrix}$ $A^T = \begin{bmatrix} 3 & -2 & -4 \\ 3 & -2 & -5 \\ -1 & 1 & 2 \end{bmatrix}$

Let $P = \frac{1}{2}(A + A^T)$

$= \frac{1}{2}\left\{\begin{bmatrix} 3 & 3 & -1 \\ -2 & -2 & 1 \\ -4 & -5 & 2 \end{bmatrix} + \begin{bmatrix} 3 & -2 & -4 \\ 3 & -2 & -5 \\ -1 & 1 & 2 \end{bmatrix}\right\}$

$= \frac{1}{2}\begin{bmatrix} 6 & 1 & -5 \\ 1 & -4 & -4 \\ -5 & -4 & 4 \end{bmatrix}$

$P^T = \frac{1}{2}\begin{bmatrix} 6 & 1 & -5 \\ 1 & -4 & -4 \\ -5 & -4 & 4 \end{bmatrix} = P$

$P^T = P$

$\therefore$ P is a symmetric matrix.

Let $Q = \frac{1}{2}(A - A^T)$

$= \frac{1}{2}\left\{\begin{bmatrix} 3 & 3 & -1 \\ -2 & -2 & 1 \\ -4 & -5 & 2 \end{bmatrix} - \begin{bmatrix} 3 & -2 & -4 \\ 3 & -2 & -5 \\ -1 & 1 & 2 \end{bmatrix}\right\}$

$= \frac{1}{2}\begin{bmatrix} 0 & 5 & 3 \\ -5 & 0 & 6 \\ -3 & -6 & 0 \end{bmatrix}$

$Q^T = \frac{1}{2}\begin{bmatrix} 0 & -5 & -3 \\ 5 & 0 & -6 \\ 3 & 6 & 0 \end{bmatrix} = -Q$

$Q^T = -Q$

$\therefore$ Q is a skew-symmetric matrix

$\therefore$ $A = P + Q = \frac{1}{2}(A + A^T) + \frac{1}{2}(A - A^T)$

$A = \frac{1}{2}\begin{bmatrix} 6 & 1 & -5 \\ 1 & -4 & -4 \\ -5 & -4 & 4 \end{bmatrix} + \frac{1}{2}\begin{bmatrix} 0 & 5 & 3 \\ -5 & 0 & 6 \\ -3 & -6 & 0 \end{bmatrix}$

N KARTHIKEYAN, M.Sc, B.Ed
AVP TRUST NATIONAL
MATRIC HR SEC SCHOOL
TIRUPUR-9843434491

$\frac{18}{19}$ Find the matrix A such that

$$\underset{3\times 2}{\begin{bmatrix} 2 & -1 \\ 1 & 0 \\ -3 & 4 \end{bmatrix}} A^T_{m\times n} = \begin{bmatrix} -1 & -8 & -10 \\ 1 & 2 & -5 \\ 9 & 22 & 15 \end{bmatrix}_{3\times 3}$$

$m=2 \quad n=3$

$\therefore$ Order of A^T is 2×3

$$\begin{bmatrix} 2 & -1 \\ 1 & 0 \\ -3 & 4 \end{bmatrix}\begin{bmatrix} a & b & c \\ d & e & f \end{bmatrix} = \begin{bmatrix} -1 & -8 & -10 \\ 1 & 2 & -5 \\ 9 & 22 & 15 \end{bmatrix}$$

$$\begin{bmatrix} 2a-d & 2b-e & 2c-f \\ a+0 & b+0 & c+0 \\ -3a+4d & -3b+4e & -3c+4f \end{bmatrix} = \begin{bmatrix} -1 & -8 & -10 \\ 1 & 2 & -5 \\ 9 & 22 & 15 \end{bmatrix}$$

$a=1 \qquad b=2 \qquad c=-5$

$2a-d=-1 \qquad 2b-e=-8 \qquad 2c-f=-10$

$2-d=-1 \qquad 4-e=-8 \qquad -10-f=-10$

$\boxed{d=+3} \qquad -e=-12 \qquad \boxed{f=0}$

$\boxed{e=12}$

$$\therefore A^T = \begin{bmatrix} 1 & 2 & -5 \\ 3 & 12 & 0 \end{bmatrix}$$

$$\therefore A = \begin{bmatrix} 1 & 3 \\ 2 & 12 \\ -5 & 0 \end{bmatrix}$$

N KARTHIKEYAN, M.Sc, B.Ed

AVP TRUST NATIONAL

MATRIC HR SEC SCHOOL

TIRUPUR-9843434491

$\frac{19}{19}$ If $A = \begin{bmatrix} 1 & 2 & 2 \\ 2 & 1 & -2 \\ x & 2 & y \end{bmatrix}$ is a matrix such that $AA^T = 9I$, find the values of x and y.

$$A = \begin{bmatrix} 1 & 2 & 2 \\ 2 & 1 & -2 \\ x & 2 & y \end{bmatrix} \qquad A^T = \begin{bmatrix} 1 & 2 & x \\ 2 & 1 & 2 \\ 2 & -2 & y \end{bmatrix}$$

$$AA^T = 9I$$

$$\begin{bmatrix} 1 & 2 & 2 \\ 2 & 1 & -2 \\ x & 2 & y \end{bmatrix}\begin{bmatrix} 1 & 2 & x \\ 2 & 1 & 2 \\ 2 & -2 & y \end{bmatrix} = \begin{bmatrix} 9 & 0 & 0 \\ 0 & 9 & 0 \\ 0 & 0 & 9 \end{bmatrix}$$

$$\begin{bmatrix} 1+4+4 & 2+2-4 & x+4+2y \\ 2+2-4 & 4+1+4 & 2x+2-2y \\ x+4+2y & 2x+2-2y & x^2+4+y^2 \end{bmatrix} = \begin{bmatrix} 9 & 0 & 0 \\ 0 & 9 & 0 \\ 0 & 0 & 9 \end{bmatrix}$$

$$\begin{bmatrix} 9 & 0 & x+2y+4 \\ 0 & 9 & 2x-2y+2 \\ x+2y+4 & 2x+2-2y & x^2+y^2+4 \end{bmatrix} = \begin{bmatrix} 9 & 0 & 0 \\ 0 & 9 & 0 \\ 0 & 0 & 9 \end{bmatrix}$$

$x+\cancel{2y}+4=0$

$2x-\cancel{2y}+2=0$

$3x=-6 \Rightarrow \boxed{x=-2}$

Sub $x=-2$ in (1)

$-2+2y+4=0$

$2y=-2$

$\boxed{y=-1}$

$\therefore (x,y) = (-2,-1)$

$\frac{20}{19}$ (i) For what value of x, the matrix $A = \begin{bmatrix} 0 & 1 & -2 \\ -1 & 0 & x^3 \\ 2 & -3 & 0 \end{bmatrix}$ is skew symmetric.

$$A = \begin{bmatrix} 0 & 1 & -2 \\ -1 & 0 & x^3 \\ 2 & -3 & 0 \end{bmatrix} \quad A^T = \begin{bmatrix} 0 & -1 & 2 \\ 1 & 0 & -3 \\ -2 & x^3 & 0 \end{bmatrix}$$

if A is Skew Symmetric

$$A = -A^T$$

$$\begin{bmatrix} 0 & 1 & -2 \\ -1 & 0 & x^3 \\ 2 & -3 & 0 \end{bmatrix} = -\begin{bmatrix} 0 & -1 & 2 \\ 1 & 0 & -x^3 \\ -2 & x^3 & 0 \end{bmatrix}$$

$$-x^3 = -3$$

$$x^3 = 3$$

$$x = \sqrt[3]{3}.$$

(ii) If $\begin{bmatrix} 0 & p & 3 \\ 2 & q^2 & -1 \\ r & 1 & 0 \end{bmatrix}$ is Skew Symmetric, find the values of p, q and r.

$$A = \begin{bmatrix} 0 & p & 3 \\ 2 & q^2 & -1 \\ r & 1 & 0 \end{bmatrix} \quad A^T = \begin{bmatrix} 0 & 2 & r \\ p & q^2 & 1 \\ 3 & -1 & 0 \end{bmatrix}$$

Since A is Skew Symmetric

$$A = -A^T$$

$$\begin{bmatrix} 0 & p & 3 \\ 2 & q^2 & -1 \\ r & 1 & 0 \end{bmatrix} = -\begin{bmatrix} 0 & 2 & r \\ p & q^2 & 1 \\ 3 & -1 & 0 \end{bmatrix}$$

$$\boxed{p = -2} \quad \boxed{r = -3}$$

$$\boxed{q = 0}$$

$\frac{21}{19}$ Construct the matrix $A = [a_{ij}]_{3\times 3}$ where $a_{ij} = i - j$. State whether A is Symmetric or Skew Symmetric.

Let $A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$

$a_{ij} = i - j$

$a_{11} = 1-1 = 0$ $\quad a_{12} = 1-2 = -1$ $\quad a_{13} = 1-3 = -2$

$a_{21} = 2-1 = 1$ $\quad a_{22} = 2-2 = 0$ $\quad a_{23} = 2-3 = -1$

$a_{31} = 3-1 = 2$ $\quad a_{32} = 3-2 = 1$ $\quad a_{33} = 3-3 = 0$

$$\therefore A = \begin{bmatrix} 0 & -1 & -2 \\ 1 & 0 & -1 \\ 2 & 1 & 0 \end{bmatrix}$$

$$A^T = \begin{bmatrix} 0 & 1 & 2 \\ -1 & 0 & 1 \\ -2 & -1 & 0 \end{bmatrix}$$

$$A^T = -A$$

$\therefore$ A is Skew symmetric matrix.

22/19. Let A and B be two symmetric matrices. Prove that $AB = BA$ if and only if AB is symmetric matrix.

Since A and B are Symmetric

$A = A^T \qquad B = B^T$

Given that $AB = BA$

$\Leftrightarrow (AB)^T = B^T A^T$

$\Leftrightarrow (AB)^T = BA$

$\Leftrightarrow (AB)^T = AB$

$\therefore$ AB is a symmetric matrix.

23/19. If A and B are symmetric matrices of same order, Prove that

(i) $AB + BA$ is symmetric matrix.

(ii) $AB - BA$ is Skew Symmetric matrix

Since A and B are Symmetric matrices

$A = A^T \qquad B = B^T$

(i) $(AB + BA)^T = (AB)^T + (BA)^T$

$= B^T A^T + A^T B^T$

$(AB + BA)^T = BA + AB$

$\therefore AB + BA$ is a Symmetric matrix.

(ii) $(AB-BA)^T = (AB)^T - (BA)^T$

$$= B^TA^T - A^TB^T$$

$$= BA - AB$$

$$(AB-BA)^T = -(AB-BA)$$

$\therefore AB-BA$ is a Skew Symmetric matrix.

$\frac{24}{19}$ A shopkeeper in a Nuts and spices shop makes gift packs of cashew nuts, raisins and almonds.

Pack I : Contains 100gm of cashew nuts 100gm of raisians and 50gm of almonds

Pack II : Contains 200gm of cashew nuts 100gm of raisins and 100gm of almonds

Pack III : Contains 250gm of cashew nuts 250gm of raisions and 150gm of almonds.

The cost of 50gm of cashew nuts is ₹50 50gm of raisins is ₹10 and 50gm of almonds is ₹60. What is the cost of each gift pack?

	Cashew	Raisins	Almonds
Pack I	100gm	100gm	50gm
Pack II	200gm	100gm	100gm
Pack III	250gm	250gm	150gm

Cost per gm :-

Cashew : 1gm $= ₹50/50 = ₹1$

Raisins : 1gm $= ₹\frac{10}{50} = ₹1/5$

Almonds : 1gm $= ₹\frac{60}{50} = ₹6/5$

Cost of each pack :

$$\begin{bmatrix} 100 & 100 & 50 \\ 200 & 100 & 100 \\ 250 & 250 & 150 \end{bmatrix}\begin{bmatrix} 1 \\ 1/5 \\ 6/5 \end{bmatrix} = \begin{bmatrix} 100+20+60 \\ 200+20+120 \\ 250+50+180 \end{bmatrix} = \begin{bmatrix} 180 \\ 340 \\ 480 \end{bmatrix}$$

$\therefore$ Cost of Pack I = ₹180

Cost of Pack II = ₹340

Cost of Pack III = ₹480.

7.3 Determinants :

To every square matrix $A = [a_{ij}]$ of order n, we can associate a number called determinant of the matrix A.

If $A = \begin{bmatrix} a_{11} & a_{12} & a_{13} & \cdots & a_{1n} \\ a_{12} & a_{13} & \cdots & \cdots & a_{2n} \\ \vdots & & & & \\ a_{n1} & a_{n2} & a_{13} & \cdots & a_{nn} \end{bmatrix}_{n \times n}$ then

determinant of A is written as

$$|A| = \begin{vmatrix} a_{11} & a_{12} & a_{13} & \cdots & a_{1n} \\ a_{12} & a_{13} & \cdots & \cdots & a_{2n} \\ \vdots & & & & \\ a_{n1} & a_{n2} & a_{n3} & \cdots & a_{nn} \end{vmatrix}$$

Note :

* If $A = [a_{ij}]_{n \times n}$ then determinant of A can also be denoted as $\det(A)$ or $\det A$ or Δ.
* Determinants can be defined only for square matrices.
* For a square matrix A, $|A|$ is read as determinant of A.
* Matrix is only a represantation whereas determinant is a value of a matrix.

Determinants of Matrices of different order :

* Determinant of a matrix of order 1.

Let $A = [a_{ij}]$ be the matrix of order 1, then the determinant of A is defined as 'a'

$$|A| = |a| = a.$$

N KARTHIKEYAN, M.Sc, B.Ed
AVP TRUST NATIONAL
MATRIC HR SEC SCHOOL
TIRUPUR-9843434491

* Determinant of a matrix of order 2.

Let $A = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}$ be a matrix of order 2. Then the determinant of A is defined as

$$|A| = \begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix} = a_{11}a_{12} - a_{12}a_{21}$$

Minor of an element:

Let $A = [a_{ij}]_{3\times3}$ be a given Square matrix of order 3. The minor of an arbitrary element a_{ij} is the determinant obtained by deleting the i^{th} row and j^{th} column in which the element a_{ij} stands. The minor of a_{ij} is usually denoted by M_{ij}.

Co-factor of an element:

The co-factor is a signed minor. The co-factor of a_{ij} usually denoted by A_{ij} and is defined as $A_{ij} = (-1)^{i+j} M_{ij}$

Evaluation of determinant of order 3.

* Laplace Expansion :-

For a given matrix $A = [a_{ij}]_{3\times3}$ the sum of the Product of elements of the first row with their corresponding co-factors is the determinant of A.

That is $|A| = a_{11}A_{11} + a_{12}A_{12} + a_{13}A_{13}$

* Sarrus Rule :-

Let $A = [a_{ij}]_{3\times3} = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$

Write the entries of matrix A as follows

$$\begin{array}{ccccc} a_{11} & a_{12} & a_{13} & a_{11} & a_{12} \\ a_{21} & a_{22} & a_{23} & a_{21} & a_{22} \\ a_{31} & a_{32} & a_{33} & a_{31} & a_{32} \end{array}$$

$$|A| = [a_{11} \cdot a_{22} a_{33} + a_{12} a_{23} a_{31} + a_{13} a_{21} a_{32}] - [a_{31} a_{22} a_{13} + a_{32} a_{23} a_{11} + a_{33} a_{21} a_{12}]$$

Properties of Determinants :-

* The determinant of a matrix remains unaltered if its rows are changed into columns and columns into rows.

(ie) $|A| = |A^T|$

* If any two rows / columns of a determinants are interchanged, then the determinant changes in sign but its absolute value remains unaltered.

* If there are n interchanges of rows (columns) of a matrix A then the determinant of the resulting matrix is $(-1)^n |A|$.

* If two rows / columns of a matrix are identical, then its determinant is zero.

* If a row of a matrix (column) A is a scalar multiple of another row (column) of A. then its determinant is zero.

→ If all the entries of a row or a column are zero, then the determinant is zero

→ determinant of a triangular matrix is obtained by the Product of the Principal diagonal elements.

* If each element in a row / column of a matrix is multiplied by a scalar 'k', then the determinant is multiplied by the same scalar k.

* If each element of a row/column of a determinant is expressed as sum of two or more terms then the whole determinant is expressed as sum of two or more determinants.

* If, to each element of any row/column of a determinant the equi-multiples of the corresponding entries of one or more rows/column are added or subtracted then the value of the determinant remains unchanged.

If A is a square matrix of order n then,

(i) $|AB| = |A||B|$

(ii) If $AB = 0$ then either $|A| = 0$ (or) $|B| = 0$

(iii) $|A^n| = [|A|]^n$.

7.14 / 20 Evaluate : (i) $\begin{vmatrix} 2 & 4 \\ -1 & 2 \end{vmatrix}$ (ii) $\begin{vmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{vmatrix}$

(i) $\begin{vmatrix} 2 & 4 \\ -1 & 2 \end{vmatrix} = 4 + 4 = 8$

(ii) $\begin{vmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{vmatrix} = \cos^2\theta + \sin^2\theta = 1.$

7.15 / 21. Compute all minors, cofactors of A and hence compute $|A|$ if $A = \begin{bmatrix} 1 & 3 & -2 \\ 4 & -5 & 6 \\ -3 & 5 & 2 \end{bmatrix}$. Also check that $|A|$ remains unaltered by expanding along any row or any column.

$$A = \begin{bmatrix} 1 & 3 & -2 \\ 4 & -5 & 6 \\ -3 & 5 & 2 \end{bmatrix}$$

$M_{11} = \begin{vmatrix} -5 & 6 \\ 5 & 2 \end{vmatrix} = -10-30 = -40$

$M_{12} = \begin{vmatrix} 4 & 6 \\ -3 & 2 \end{vmatrix} = 8+18 = 26$

$M_{13} = \begin{vmatrix} 4 & -5 \\ -3 & 5 \end{vmatrix} = 20-15 = 5$

$M_{21} = \begin{vmatrix} 3 & -2 \\ 5 & 2 \end{vmatrix} = 6+10 = 16$

$M_{22} = \begin{vmatrix} 1 & -2 \\ -3 & 2 \end{vmatrix} = 2-6 = -4$

$M_{23} = \begin{vmatrix} 1 & 3 \\ -3 & 5 \end{vmatrix} = 5+9 = 14$

$M_{31} = \begin{vmatrix} 3 & -2 \\ -5 & 6 \end{vmatrix} = 18-10 = 8$

$M_{32} = \begin{vmatrix} 1 & -2 \\ 4 & 6 \end{vmatrix} = 6+8 = 14$

$M_{33} = \begin{vmatrix} 1 & 3 \\ 4 & -5 \end{vmatrix} = -5-12 = -17.$

N KARTHIKEYAN, M.Sc, B.Ed
AVP TRUST NATIONAL
MATRIC HR SEC SCHOOL
TIRUPUR-9843434491

Cofactor $A_{ij} = (-1)^{i+j} M_{ij}$

$A_{11} = (-1)^{1+1}(-40) = -40$

$A_{12} = (-1)^{1+2}(26) = -26$

$A_{13} = (-1)^{1+3}(5) = 5$

$A_{21} = (-1)^{2+1}(16) = -16$

$A_{22} = (-1)^{2+2}(-4) = -4$

$A_{23} = (-1)^{2+3}(14) = -14$

$A_{31} = (-1)^{3+1}(8) = 8$

$A_{32} = (-1)^{3+2}(14) = -14$

$A_{33} = (-1)^{3+3}(-17) = -17.$

$|A| = \begin{vmatrix} 1 & 3 & -2 \\ 4 & -5 & 6 \\ -3 & 5 & 2 \end{vmatrix}$ Expanding along R_1

$|A| = 1\begin{vmatrix} -5 & 6 \\ 5 & 2 \end{vmatrix} - 3\begin{vmatrix} 4 & 6 \\ -3 & 2 \end{vmatrix} - 2\begin{vmatrix} 4 & -5 \\ -3 & 5 \end{vmatrix}$

$= 1(-40) - 3(26) - 2(5) = -40-78-10$

$= -128$ ——— ①

Expanding along C_1

$|A| = 1\begin{vmatrix} -5 & 6 \\ 5 & 2 \end{vmatrix} - 4\begin{vmatrix} 3 & -2 \\ 5 & 2 \end{vmatrix} - 3\begin{vmatrix} 3 & -2 \\ -5 & 6 \end{vmatrix}$

$= 1(-10-30) - 4(6+10) - 3(18-10)$

$= -40-64-24 = -128$ ——— ②

$\therefore$ $|A|$ obtained by expanding along R_1 is equal to expanding along C_1.

7.16 / 23

Find $|A|$ if $A = \begin{bmatrix} 0 & \sin\alpha & \cos\alpha \\ \sin\alpha & 0 & \sin\beta \\ \cos\alpha & -\sin\beta & 0 \end{bmatrix}$

$$|A| = \begin{vmatrix} 0 & \sin\alpha & \cos\alpha \\ \sin\alpha & 0 & \sin\beta \\ \cos\alpha & -\sin\beta & 0 \end{vmatrix}$$

$$= 0 - \sin\alpha \begin{vmatrix} \sin\alpha & \sin\beta \\ \cos\alpha & 0 \end{vmatrix} + \cos\alpha \begin{vmatrix} \sin\alpha & 0 \\ \cos\alpha & -\sin\beta \end{vmatrix}$$

$$= -\sin\alpha(0 - \sin\beta\cos\alpha) + \cos\alpha(-\sin\alpha\sin\beta - 0)$$

$$= \sin\alpha\sin\beta\cos\alpha - \sin\alpha\sin\beta\cos\alpha$$

$$|A| = 0$$

7.17 / 23

Compute $|A|$ using Sarrus rule if

$$A = \begin{bmatrix} 3 & 4 & 1 \\ 0 & -1 & 2 \\ 5 & -2 & 6 \end{bmatrix}$$

$$\begin{matrix} 3 & 4 & 1 & 3 & 4 \\ 0 & -1 & 2 & 0 & -1 \\ 5 & -2 & 6 & 5 & -2 \end{matrix}$$

$$|A| = (3(-1)(6) + 4(2)(5) + (1)(0)(-2)) - [(5)(-1)(1) + (-2)(2)(3) + 6(0)(4)]$$

$$= (-18 + 40 - 0) - (-5 - 12 + 0)$$

$$= 22 + 17 = 39$$

$$|A| = 39$$

7.18 / 27

If a, b, c and x are positive real numbers then show that $\begin{vmatrix} (a^x+a^{-x})^2 & (a^x-a^{-x})^2 & 1 \\ (b^x+b^{-x})^2 & (b^x-b^{-x})^2 & 1 \\ (c^x+c^{-x})^2 & (c^x-c^{-x})^2 & 1 \end{vmatrix} = 0$

$$\Delta = \begin{vmatrix} (a^x+a^{-x})^2 & (a^x-a^{-x})^2 & 1 \\ (b^x+b^{-x})^2 & (b^x-b^{-x})^2 & 1 \\ (c^x+c^{-x})^2 & (c^x-c^{-x})^2 & 1 \end{vmatrix}$$

$$= \begin{vmatrix} 4 & (a^x-a^{-x})^2 & 1 \\ 4 & (b^x-b^{-x})^2 & 1 \\ 4 & (c^x-c^{-x})^2 & 1 \end{vmatrix} \quad C_1 \to C_1 - C_2$$

$$= 0 \quad [C_1 \text{ is pro to } C_3]$$

Hence Proved.

$$\{(a+b)^2 - (a-b)^2 = 4ab\}$$

7.19 / 27 without expanding the determinants, show that $|B| = 2|A|$.

where $B = \begin{bmatrix} b+c & c+a & a+b \\ c+a & a+b & b+c \\ a+b & b+c & c+a \end{bmatrix}$ and $A = \begin{bmatrix} a & b & c \\ b & c & a \\ c & a & b \end{bmatrix}$

$$B = \begin{bmatrix} b+c & c+a & a+b \\ c+a & a+b & c+a \\ a+b & b+c & a+b \end{bmatrix}$$

$$|B| = \begin{vmatrix} b+c & c+a & a+b \\ c+a & a+b & c+a \\ a+b & b+c & a+b \end{vmatrix}$$

$$= \begin{vmatrix} 2(a+b+c) & 2(a+b+c) & 2(a+b+c) \\ c+a & a+b & c+a \\ a+b & b+c & a+b \end{vmatrix} \quad R_1 \to R_1+R_2+R_3$$

$$= 2\begin{vmatrix} a+b+c & a+b+c & a+b+c \\ c+a & a+b & c+a \\ a+b & b+c & a+b \end{vmatrix}$$

$$= 2\begin{vmatrix} a+b+c & a+b+c & a+b+c \\ -b & -c & -a \\ -c & -a & -b \end{vmatrix} \quad \begin{matrix} R_2 \to R_2-R_1 \\ R_3 \to R_3-R_1 \end{matrix}$$

$$= 2\begin{vmatrix} a & b & c \\ -b & -c & -a \\ -c & -a & -b \end{vmatrix} \quad R_1 \to R_1+R_2+R_3$$

$$= 2(-1)(-1)\begin{vmatrix} a & b & c \\ b & c & a \\ c & a & b \end{vmatrix} = 2\begin{vmatrix} a & b & c \\ b & c & a \\ c & a & b \end{vmatrix}$$

$|B| = 2|A|$

Hence Proved.

T.20 / 28 Evaluate : $\begin{vmatrix} 2014 & 2017 & 0 \\ 2020 & 2023 & 1 \\ 2023 & 2026 & 0 \end{vmatrix}$

$$\Delta = \begin{vmatrix} 2014 & 2017 & 0 \\ 2020 & 2023 & 1 \\ 2023 & 2026 & 0 \end{vmatrix} = \begin{vmatrix} 2014 & 3 & 0 \\ 2020 & 3 & 1 \\ 2023 & 3 & 0 \end{vmatrix} \quad C_2 \to C_2 - C_1$$

$$= 3\begin{vmatrix} 2014 & 1 & 0 \\ 2020 & 1 & 1 \\ 2023 & 1 & 0 \end{vmatrix} = 3\{0 - 1(2014 - 2023) + 0\}$$

$$= 3\{9\} = 27$$

$\therefore \Delta = 27.$

N KARTHIKEYAN, M.Sc, B.Ed
AVP TRUST NATIONAL
MATRIC HR SEC SCHOOL
TIRUPUR-9843434491

T.21 / 28 Find the value of x if $\begin{vmatrix} x-1 & x & x-2 \\ 0 & x-2 & x-3 \\ 0 & 0 & x-3 \end{vmatrix} = 0$

$$\begin{vmatrix} x-1 & x & x-2 \\ 0 & x-2 & x-3 \\ 0 & 0 & x-3 \end{vmatrix} = 0$$

Since, it is in the form of triangular

$\therefore \Delta =$ Product of the leading diagonal elements

$$(x-1)(x-2)(x-3) = 0$$

$$x = 1, 2, 3.$$

7.22 / 28 Prove that $\begin{vmatrix} 1 & 1 & 1 \\ x & y & z \\ x^2 & y^2 & z^2 \end{vmatrix} = (x-y)(y-z)(z-x)$

$$\text{L.H.S} = \begin{vmatrix} 1 & 1 & 1 \\ x & y & z \\ x^2 & y^2 & z^2 \end{vmatrix}$$

$$= \begin{vmatrix} 0 & 0 & 1 \\ x-y & y-z & z \\ x^2-y^2 & y^2-z^2 & z^2 \end{vmatrix} \quad \begin{matrix} C_1 \to C_1 - C_2 \\ C_2 \to C_2 - C_3 \end{matrix}$$

$$= (x-y)(y-z) \begin{vmatrix} 0 & 0 & 1 \\ 1 & 1 & z \\ x+y & y+z & z^2 \end{vmatrix}$$

$$= (x-y)(y-z)\{0 - 0 + 1(\cancel{y} + z - x - \cancel{y})\}$$

$$= (x-y)(y-z)(z-x)$$

$$= \text{R.H.S}$$

Ex: 7.2

1) without expanding the determinant, Prove that $\begin{vmatrix} s & a^2 & b^2+c^2 \\ s & b^2 & c^2+a^2 \\ s & c^2 & a^2+b^2 \end{vmatrix} = 0$

$$\text{L.H.S} = \begin{vmatrix} s & a^2 & b^2+c^2 \\ s & b^2 & c^2+a^2 \\ s & c^2 & a^2+b^2 \end{vmatrix}$$

$$= \begin{vmatrix} s & a^2 & a^2+b^2+c^2 \\ s & b^2 & a^2+b^2+c^2 \\ s & c^2 & a^2+b^2+c^2 \end{vmatrix} \quad C_3 \to C_2 + C_3$$

$$= 5(a^2+b^2+c^2)\begin{vmatrix} 1 & a^2 & 1 \\ 1 & b^2 & 1 \\ 1 & c^2 & 1 \end{vmatrix} \quad \because C_1 \equiv C_3$$

$$= 5(a^2+b^2+c^2)(0)$$

$$= 0$$

$$= R.H.S$$

2/28 Show that $\begin{vmatrix} b+c & bc & b^2c^2 \\ c+a & ca & c^2a^2 \\ a+b & ab & a^2b^2 \end{vmatrix} = 0$

$$L.H.S = \begin{vmatrix} b+c & bc & b^2c^2 \\ c+a & ca & c^2a^2 \\ a+b & ab & a^2b^2 \end{vmatrix}$$

$$= \frac{1}{abc}\begin{vmatrix} a(b+c) & abc & ab^2c^2 \\ b(c+a) & abc & a^2bc^2 \\ c(a+b) & abc & a^2b^2c \end{vmatrix}$$

$$= \frac{(abc)(abc)}{abc}\begin{vmatrix} ab+ac & 1 & bc \\ bc+ab & 1 & ca \\ ac+bc & 1 & ab \end{vmatrix}$$

$$= abc\begin{vmatrix} ab+bc+ca & 1 & bc \\ ab+bc+ca & 1 & ca \\ ab+bc+ca & 1 & ab \end{vmatrix} \quad C_1 \to C_1 + C_3$$

$$= abc(ab+bc+ca)\begin{vmatrix} 1 & 1 & bc \\ 1 & 1 & ca \\ 1 & 1 & ab \end{vmatrix} \quad C_1 \equiv C_3$$

$$= 0 = R.H.S$$

3/29 Prove that $\begin{vmatrix} a^2 & bc & ac+c^2 \\ a^2+ab & b^2 & ac \\ ab & b^2+bc & c^2 \end{vmatrix} = 4a^2b^2c^2$

$$L.H.S = \begin{vmatrix} a^2 & bc & ac+c^2 \\ a^2+ab & b^2 & ac \\ ab & b^2+bc & c^2 \end{vmatrix}$$

$$= abc\begin{vmatrix} a & c & a+c \\ a+b & b & a \\ b & b+c & c \end{vmatrix}$$

$$= abc\begin{vmatrix} 0 & c & a+c \\ 2b & b & a \\ 2b & b+c & c \end{vmatrix} \quad C_1 \to C_1 + C_2 - C_3$$

$$= 2ab^2c\begin{vmatrix} 0 & c & a+c \\ 1 & b & a \\ 1 & b+c & c \end{vmatrix}$$

Expanding along C_1

$$= 2ab^2c\left\{0 - 1(c^2 - (b+c)(a+c)) + 1(ac - b(a+c))\right\}$$

$$= 2ab^2c\left\{-\cancel{c^2} + \cancel{ab} + \cancel{bc} + ac + \cancel{c^2} + ac - \cancel{ab} - \cancel{bc}\right\}$$

$$= 2ab^2c\,(2ac)$$

$$= 4a^2b^2c^2$$

$$= R.H.S$$

$\frac{4}{29}$ Prove that $\begin{vmatrix} 1+a & 1 & 1 \\ 1 & 1+b & 1 \\ 1 & 1 & 1+c \end{vmatrix} = abc\left(1+\frac{1}{a}+\frac{1}{b}+\frac{1}{c}\right)$

$$L.H.S = \begin{vmatrix} 1+a & 1 & 1 \\ 1 & 1+b & 1 \\ 1 & 1 & 1+c \end{vmatrix}$$

$$= \begin{vmatrix} a & 0 & 1 \\ -b & b & 1 \\ 0 & -c & 1+c \end{vmatrix} \quad \begin{matrix} C_1 \to C_1 - C_2 \\ C_2 \to C_2 - C_3 \end{matrix}$$

$$= a\{b(1+c)+c\} - 0 + 1\{bc - 0\}$$

$$= a\{b + bc + c\} + bc$$

$$= ab + abc + ac + bc$$

$$= abc\left(1+\frac{1}{a}+\frac{1}{b}+\frac{1}{c}\right)$$

$$= R.H.S.$$

$\frac{5}{29.}$ Prove that $\begin{vmatrix} \sec^2\theta & \tan^2\theta & 1 \\ \tan^2\theta & \sec^2\theta & -1 \\ 38 & 36 & 2 \end{vmatrix} = 0$

$$L.H.S = \begin{vmatrix} \sec^2\theta & \tan^2\theta & 1 \\ \tan^2\theta & \sec^2\theta & -1 \\ 38 & 36 & 2 \end{vmatrix}$$

$$= \begin{vmatrix} \sec^2\theta - \tan^2\theta & \tan^2\theta & 1 \\ \tan^2\theta - \sec^2\theta & \sec^2\theta & -1 \\ 38 - 36 & 36 & 2 \end{vmatrix} \quad C_1 \to C_1 - C_2$$

$$= \begin{vmatrix} 1 & \tan^2\theta & 1 \\ -1 & \sec^2\theta & -1 \\ 2 & 36 & 2 \end{vmatrix} \quad C_1 \equiv C_3$$

$$= 0 = R.H.S$$

6/29 Show that $\begin{vmatrix} x+2a & y+2b & z+2c \\ x & y & z \\ a & b & c \end{vmatrix} = 0$

$$L.H.S = \begin{vmatrix} x+2a & y+2b & z+2c \\ x & y & z \\ a & b & c \end{vmatrix}$$

$$= \begin{vmatrix} x & y & z \\ x & y & z \\ a & b & c \end{vmatrix} + \begin{vmatrix} 2a & 2b & 2c \\ x & y & z \\ a & b & c \end{vmatrix}$$

$R_1 \equiv R_2$ $\quad$ R_1 is Pro to R_3

$$= 0 + 0$$

$$= 0 = R.H.S$$

7/29 write the general form of a 3×3 Skew Symmetric matrix and Prove that its determinant is 0.

Let $A = \begin{bmatrix} 0 & a & b \\ -a & 0 & -c \\ -b & c & 0 \end{bmatrix}$

$$A^T = \begin{bmatrix} 0 & -a & -b \\ a & 0 & c \\ b & -c & 0 \end{bmatrix}$$

$$A^T = -A$$

$\therefore$ A is Skew Symmetric matrix.

$$|A| = \begin{vmatrix} 0 & a & b \\ -a & 0 & -c \\ -b & c & 0 \end{vmatrix}$$

$$= 0 - a(0-bc) + b(-ac+0)$$

$$= abc - abc = 0$$

$\therefore$ $|A| = 0$.

8/29 If $\begin{vmatrix} a & b & a\alpha+b \\ b & c & b\alpha+c \\ a\alpha+b & b\alpha+c & 0 \end{vmatrix} = 0$, Prove that

a, b, c are in G.P or α is a root of

$ax^2 + 2bx + c = 0$.

N KARTHIKEYAN, M.Sc, B.Ed
AVP TRUST NATIONAL
MATRIC HR SEC SCHOOL
TIRUPUR-9843434491

$$\begin{vmatrix} a & b & a\alpha+b \\ b & c & b\alpha+c \\ a\alpha+b & b\alpha+c & 0 \end{vmatrix} = 0$$

$$\begin{vmatrix} a & b & a\alpha+b \\ b & c & b\alpha+c \\ 0 & 0 & 0-\alpha(a\alpha+b)-(b\alpha+c) \end{vmatrix} = 0 \quad R_3 \to R_3 - \alpha R_1 - R_2$$

$$\begin{vmatrix} a & b & a\alpha+b \\ b & c & b\alpha+c \\ 0 & 0 & -(a\alpha^2+2b\alpha+c) \end{vmatrix} = 0$$

$\alpha(a\alpha+b)+b\alpha+c$

$a\alpha^2+b\alpha+b\alpha+c$

$a\alpha^2+2b\alpha+c$

Expanding along R_3

$$0-0-(a\alpha^2+2b\alpha+c)\{ac-b^2\}=0$$

$$(a\alpha^2+2b\alpha+c)(b^2-ac)=0$$

$a\alpha^2+2b\alpha+c=0$ $\qquad$ $b^2=ac$

$\therefore a, b, c$ are in G.P

α is a root of $ax^2+2bx+c=0$.

$\frac{9}{29}$. Prove that $\begin{vmatrix} 1 & a & a^2-bc \\ 1 & b & b^2-ac \\ 1 & c & c^2-ab \end{vmatrix} = 0$

$$\text{L.H.S} = \begin{vmatrix} 1 & a & a^2-bc \\ 1 & b & b^2-ac \\ 1 & c & c^2-ab \end{vmatrix}$$

$$= \begin{vmatrix} 0 & a-b & a^2-b^2-bc+ac \\ 0 & b-c & b^2-c^2-ac+ab \\ 1 & c & c^2-ab \end{vmatrix} \quad \begin{matrix} R_1 \to R_1-R_2 \\ R_2 \to R_2-R_3 \end{matrix}$$

$$= \begin{vmatrix} 0 & a-b & (a+b)(a-b)+c(a-b) \\ 0 & b-c & (b+c)(b-c)+a(b-c) \\ 1 & c & c^2-ab \end{vmatrix}$$

$$=(a-b)(b-c)\begin{vmatrix} 0 & 1 & a+b+c \\ 0 & 1 & b+c+a \\ 1 & c & c^2-ab \end{vmatrix}$$

$=(a-b)(b-c)\,0$ $\qquad$ $[R_1 \equiv R_2]$

$=0$

$=$ R.H.S

10/29 If a, b, c are p^{th}, q^{th} and r^{th} terms of an A.P find the value of $\begin{vmatrix} a & b & c \\ p & q & r \\ 1 & 1 & 1 \end{vmatrix}$

$t_p = a \qquad t_q = b \qquad t_r = c$

$A+(p-1)d = a \qquad A+(q-1)d = b \qquad A+(r-1)d = c$

$$\Delta = \begin{vmatrix} a & b & c \\ p & q & r \\ 1 & 1 & 1 \end{vmatrix}$$

$$= \begin{vmatrix} A+(p-1)d & A+(q-1)d & A+(r-1)d \\ p & q & r \\ 1 & 1 & 1 \end{vmatrix}$$

$$= \begin{vmatrix} A & A & A \\ p & q & r \\ 1 & 1 & 1 \end{vmatrix} + d\begin{vmatrix} p-1 & q-1 & r-1 \\ p & q & r \\ 1 & 1 & 1 \end{vmatrix}$$

$$= A\begin{vmatrix} 1 & 1 & 1 \\ p & q & r \\ 1 & 1 & 1 \end{vmatrix} + d\begin{vmatrix} 0 & 0 & 0 \\ p & q & r \\ 1 & 1 & 1 \end{vmatrix} \quad R_1 \to R_1 - R_2 + R_3$$

$$= A(0) + d(0) = 0.$$

11/29 Show that $\begin{vmatrix} a^2+x^2 & ab & ac \\ ab & b^2+x^2 & bc \\ ac & bc & c^2+x^2 \end{vmatrix}$ is divisible by x^4.

Let $\Delta = \begin{vmatrix} a^2+x^2 & ab & ac \\ ab & b^2+x^2 & bc \\ ac & bc & c^2+x^2 \end{vmatrix}$

multiplying R_1, R_2 and R_3 by a, b, c respectively

$$\Delta = \begin{vmatrix} a(a^2+x^2) & a^2b & a^2c \\ ab^2 & b(b^2+x^2) & b^2c \\ ac^2 & bc^2 & c(c^2+x^2) \end{vmatrix}$$

$$= abc\begin{vmatrix} a^2+x^2 & a^2 & a^2 \\ b^2 & b^2+x^2 & b^2 \\ c^2 & c^2 & c^2+x^2 \end{vmatrix}$$

$$= abc\begin{vmatrix} x^2+a^2+b^2+c^2 & x^2+a^2+b^2+c^2 & x^2+a^2+b^2+c^2 \\ b^2 & b^2+x^2 & b^2 \\ c^2 & c^2 & c^2+x^2 \end{vmatrix}$$

$R_1 \to R_1 + R_2 + R_3$

$$= abc(x^2+a^2+b^2+c^2)\begin{vmatrix} 1 & 1 & 1 \\ b^2 & b^2+x^2 & b^2 \\ c^2 & c^2 & c^2+x^2 \end{vmatrix}$$

$$= abc(x^2+a^2+b^2+c^2)\begin{vmatrix} 0 & 0 & 1 \\ -x^2 & x^2 & b^2 \\ 0 & -x^2 & c^2+x^2 \end{vmatrix}$$

$C_1 \to C_1 - C_2$

$C_2 \to C_2 - C_3$

$$= abc(x^2+a^2+b^2+c^2)\, x^2 \cdot x^2 \begin{vmatrix} 0 & 0 & 1 \\ -1 & 1 & b^2 \\ 0 & -1 & c^2+x^2 \end{vmatrix}$$

$$= x^4(abc)(x^2+a^2+b^2+c^2)\{0-0+1(1-0)\}$$

$$= x^4 abc(x^2+a^2+b^2+c^2)$$

$\therefore$ The given determinant is divisible by x^4.

12/29 If a, b, c are all positive and are p^{th}, q^{th} and r^{th} terms of a G.P. Show that

$$\begin{vmatrix} \log a & p & 1 \\ \log b & q & 1 \\ \log c & r & 1 \end{vmatrix} = 0$$

$t_p = a \qquad t_q = b \qquad t_r = c$

$AR^{p-1} = a \qquad AR^{q-1} = b \qquad AR^{r-1} = c$

$$\Delta = \begin{vmatrix} \log a & p & 1 \\ \log b & q & 1 \\ \log c & r & 1 \end{vmatrix} = \begin{vmatrix} \log(AR^{p-1}) & p & 1 \\ \log(AR^{q-1}) & q & 1 \\ \log(AR^{r-1}) & r & 1 \end{vmatrix}$$

$$= \begin{vmatrix} \log A + (p-1)\log R & p & 1 \\ \log A + (q-1)\log R & q & 1 \\ \log A + (r-1)\log R & r & 1 \end{vmatrix}$$

$$= \begin{vmatrix} \log A & p & 1 \\ \log A & q & 1 \\ \log A & r & 1 \end{vmatrix} + \log R \begin{vmatrix} p-1 & p & 1 \\ q-1 & q & 1 \\ r-1 & r & 1 \end{vmatrix}$$

$$= \log A \begin{vmatrix} 1 & p & 1 \\ 1 & q & 1 \\ 1 & r & 1 \end{vmatrix} + \log R \begin{vmatrix} 0 & p & 1 \\ 0 & q & 1 \\ 0 & r & 1 \end{vmatrix}$$

$C_1 \to C_1 - C_2 + C_3$

$$= \log A\,(0) + \log R\,(0) = 0 + 0 = 0$$

Hence Proved.

$\frac{13}{29}$ Find the value of $\begin{vmatrix} 1 & \log_x y & \log_x z \\ \log_y x & 1 & \log_y z \\ \log_z x & \log_z y & 1 \end{vmatrix}$ if $x, y, z \neq 1$

$$\Delta = \begin{vmatrix} 1 & \log_x y & \log_x z \\ \log_y x & 1 & \log_y z \\ \log_z x & \log_z y & 1 \end{vmatrix}$$

$\left[\because \log_n m = \frac{\log m}{\log n}\right]$

$$= \begin{vmatrix} 1 & \frac{\log y}{\log x} & \frac{\log z}{\log x} \\ \frac{\log x}{\log y} & 1 & \frac{\log z}{\log y} \\ \frac{\log x}{\log z} & \frac{\log y}{\log z} & 1 \end{vmatrix}$$

$$= \frac{1}{\log x \log y \log z} \begin{vmatrix} \log x & \log y & \log z \\ \log x & \log y & \log z \\ \log x & \log y & \log z \end{vmatrix}$$

$$= \frac{1}{\log x \log y \log z} (0) = 0$$

$\frac{14}{30}$ If $A = \begin{bmatrix} \frac{1}{2} & \alpha \\ 0 & \frac{1}{2} \end{bmatrix}$ Prove that $\sum_{k=1}^{n} |A|^k = \frac{1}{3}\left(1 - \frac{1}{4^n}\right)$

$$|A| = \begin{vmatrix} \frac{1}{2} & \alpha \\ 0 & \frac{1}{2} \end{vmatrix} = \frac{1}{4} - 0 = \frac{1}{4}$$

$|A^k| = |A|^k$

$|A|^2 = \left(\frac{1}{4}\right)^2 = \frac{1}{4^2}$ $\quad |A|^3 = \frac{1}{4^3}$

$|A|^4 = \frac{1}{4^4} \dots$

$$\sum_{k=1}^{n} |A|^k = \frac{1}{4} + \frac{1}{4^2} + \frac{1}{4^3} + \dots + \frac{1}{4^n}$$

$S_n = \frac{a(1-r^n)}{1-r}$ $\quad r < 1$

$$\sum_{k=1}^{n} |A^k| = \frac{\frac{1}{4}\left(1 - \frac{1}{4^n}\right)}{1 - 1/4} = \frac{\frac{1}{4}\left(1 - \frac{1}{4^n}\right)}{3/4}$$

$$= \frac{1}{3}\left(1 - \frac{1}{4^n}\right)$$

Hence Proved.

N KARTHIKEYAN, M.Sc, B.Ed
AVP TRUST NATIONAL
MATRIC HR SEC SCHOOL
TIRUPUR-9843434491

15/30 without expanding. Evaluate the determinants

(i) $|A| = \begin{vmatrix} 2 & 3 & 4 \\ 5 & 6 & 8 \\ 6x & 9x & 12x \end{vmatrix}$

$= 3x \begin{vmatrix} 2 & 3 & 4 \\ 5 & 6 & 8 \\ 2 & 3 & 4 \end{vmatrix} = 3x(0)$ $[R_1 \equiv R_3]$

$= 0$

(ii) $|A| = \begin{vmatrix} x+y & y+z & z+x \\ z & x & y \\ 1 & 1 & 1 \end{vmatrix}$

$= \begin{vmatrix} x+y+z & x+y+z & x+y+z \\ z & x & y \\ 1 & 1 & 1 \end{vmatrix}$ $R_1 \to R_1 + R_2$

$= (x+y+z) \begin{vmatrix} 1 & 1 & 1 \\ z & x & y \\ 1 & 1 & 1 \end{vmatrix}$ $[R_1 \equiv R_3]$

$= (x+y+z)(0)$

$= 0.$

16/30 If A is a square matrix and $|A| = 2$ find the value of $|AA^T|$

$|A| = 2$

$|AA^T| = |A||A^T|$

$= 2 \cdot 2 = 4$

$\boxed{\begin{array}{l} |AB| = |A||B| \\ |A| = |A^T| \end{array}}$

17/30 If A and B are square matrices of order 3 such that $|A| = -1$ and $|B| = 3$. find the value of $|3AB|$

$|A| = -1$ $\quad |B| = 3$

$|3AB| = 3^3 |AB|$

$= 3^3 |A||B|$

$= 27(-1)(3)$

$= -81$

$\boxed{\begin{array}{l} |KA| = K^n |A| \\ |AB| = |A||B| \end{array}}$

$\frac{18}{30}$ If $\lambda = -2$ determine the value of

$$\begin{vmatrix} 0 & 2\lambda & 1 \\ \lambda^2 & 0 & 3\lambda^2+1 \\ -1 & 6\lambda-1 & 0 \end{vmatrix}$$

$$\Delta = \begin{vmatrix} 0 & -4 & 1 \\ 4 & 0 & 13 \\ -1 & -13 & 0 \end{vmatrix} \quad \because \lambda = -2$$

$$= 0(0+169) + 4(0+13) + 1(-52-0)$$

$$= 0 + 52 - 52 = 0$$

$$\therefore \boxed{\Delta = 0}$$

$\frac{19}{30}$ Determine the roots of the equation

$$\begin{vmatrix} 1 & 4 & 20 \\ 1 & -2 & 5 \\ 1 & +2x & 5x^2 \end{vmatrix} = 0$$

$$\begin{vmatrix} 1 & 4 & 20 \\ 1 & -2 & 5 \\ 1 & 2x & 5x^2 \end{vmatrix} = 0$$

$$2(5)\begin{vmatrix} 1 & 2 & 4 \\ 1 & -1 & 1 \\ 1 & x & x^2 \end{vmatrix} = 0$$

$$10\left\{1(-x^2-x) - 2(x^2-1) + 4(x+1)\right\} = 0$$

$$-x(x+1) - 2(x+1)(x-1) + 4(x+1) = 0$$

$$(x+1)[-x - 2(x-1) + 4] = 0$$

$$(x+1)[-x - 2x + 2 + 4] = 0$$

$$(x+1)(-3x+6) = 0$$

$$\boxed{x = -1} \quad \boxed{x = 2}$$

$\frac{20}{30}$ verify that $|AB| = |A||B|$ for $A = \begin{bmatrix} 4 & 3 & -2 \\ 1 & 0 & 7 \\ 2 & 3 & -5 \end{bmatrix}$

and $B = \begin{bmatrix} 1 & 3 & 3 \\ -2 & 4 & 0 \\ 9 & 7 & 5 \end{bmatrix}$

$$|A| = \begin{vmatrix} 4 & 3 & -2 \\ 1 & 0 & 7 \\ 2 & 3 & -5 \end{vmatrix} = 4(0-21) - 3(-5-14) - 2(3)$$

$$= -84 + 57 - 6 = -90 + 57$$

$$|A| = -33$$

$$|B| = \begin{vmatrix} 1 & 3 & 3 \\ -2 & 4 & 0 \\ 9 & 7 & 5 \end{vmatrix} = 1(20-0) - 3(-10-0) + 3(-14-36)$$

$$= 20 + 30 - 150 = -100$$

$$AB = \begin{bmatrix} 4 & 3 & -2 \\ 1 & 0 & 7 \\ 2 & 3 & -5 \end{bmatrix} \begin{bmatrix} 1 & 3 & 3 \\ -2 & 4 & 0 \\ 9 & 7 & 5 \end{bmatrix}$$

$$= \begin{bmatrix} 4-6-18 & 12+12-14 & 12+0-10 \\ 1-0+63 & 3+0+49 & 3+0+35 \\ 2-6-45 & 6+12-35 & 6+0-25 \end{bmatrix}$$

$$= \begin{bmatrix} -20 & 10 & 2 \\ 64 & 52 & 38 \\ -49 & -17 & -19 \end{bmatrix}$$

$$|AB| = \begin{vmatrix} -20 & 10 & 2 \\ 64 & 52 & 38 \\ -49 & -17 & -19 \end{vmatrix} = 2(2) \begin{vmatrix} -10 & 10 & 1 \\ 32 & 26 & 19 \\ -49 & -17 & -19 \end{vmatrix}$$

$$= 4 \begin{vmatrix} -10 & 5 & 1 \\ -17 & 9 & 0 \\ -49 & -17 & -19 \end{vmatrix} \quad R_2 \to R_2 + R_3$$

$$= 4\{-10(-171-0) - 5(323-0) + 1(289+441)\}$$

$$= 4\{+1710 - 1615 + 730)$$

$$= 4(825) = 3300 \text{ ——— } (1)$$

$$|A||B| = -33(-100) = 3300 \text{ ——— } (2)$$

from (1) & (2)

$$|AB| = |A|.|B|$$

Hence verified.

21/30 Using the Co-factors of second row Evaluate:

$$A = \begin{bmatrix} 5 & 3 & 8 \\ 2 & 0 & 1 \\ 1 & 2 & 3 \end{bmatrix}$$

$$|A| = \begin{vmatrix} 5 & 3 & 8 \\ 2 & 0 & 1 \\ 1 & 2 & 3 \end{vmatrix}$$

$$= -2 \begin{vmatrix} 3 & 8 \\ 2 & 3 \end{vmatrix} + 0 \begin{vmatrix} 5 & 8 \\ 1 & 3 \end{vmatrix} - 1 \begin{vmatrix} 5 & 3 \\ 1 & 2 \end{vmatrix}$$

$$= -2(9-16) + 0 - 1(10-3)$$

$$= -2(-7) - 1(7)$$

$$= 14 - 7 = 7$$

$$\therefore |A| = 7.$$

7.3.3 Application of Factor Theorem to Determinants :-

Factor Theorem :-

If each element of a matrix A is a Polynomial in x and if $|A|$ vanishes for $x=a$ then $x-a$ is a factor of $|A|$.

Note :-

* This theorem is very much useful when we have to obtain the value of the determinant in "factors" form
* If we substitute b for a in the determinant $|A|$. any two of its rows or columns become identical then $|A|=0$ and hence by factor theorem $(a-b)$ is a factor of $|A|$.
* If 'r' rows (columns) are identical in a determinant of order n $(n \geq r)$ when we put $x=a$ then $(x-a)^{r-1}$ is a factor of $|A|$.
* A Square matrix (or its determinant) is said to be in cyclic symmetric form if each row is obtained from the first row by changing the variables cyclically.

N KARTHIKEYAN, M.Sc, B.Ed
AVP TRUST NATIONAL
MATRIC HR SEC SCHOOL
TIRUPUR-9843434491

If the determinant is in cyclic symmetric form and if 'm' is the difference between the degree of the Product of the factors (obtained by substitution) and the degree of the Product of the leading diagonal elements and if

(i) $m=0$, then the required factor is a constant k.

(ii) $m=1$, then the required factor is $k(a+b+c)$

(iii) $m=2$, then the required factor is $k(a^2+b^2+c^2)+l(ab+bc+ca)$.

7.23 / 31 Using factor theorem, Prove that

$$\begin{vmatrix} x+1 & 3 & 5 \\ 2 & x+2 & 5 \\ 2 & 3 & x+4 \end{vmatrix} = (x-1)^2(x+9)$$

Let $\Delta = \begin{vmatrix} x+1 & 3 & 5 \\ 2 & x+2 & 5 \\ 2 & 3 & x+4 \end{vmatrix}$

Put $x=1$

$$\Delta = \begin{vmatrix} 2 & 3 & 5 \\ 2 & 3 & 5 \\ 2 & 3 & 5 \end{vmatrix} = 0 \qquad R_1 \equiv R_2 \equiv R_3$$

$\therefore (x-1)^2$ is a factor

N KARTHIKEYAN, M.Sc, B.Ed
AVP TRUST NATIONAL
MATRIC HR SEC SCHOOL
TIRUPUR-9843434491

Put $x=-9$

$$\Delta = \begin{vmatrix} -8 & 3 & 5 \\ 2 & -7 & 5 \\ 2 & 3 & -5 \end{vmatrix}$$

$$= \begin{vmatrix} 0 & 3 & 5 \\ 0 & -7 & 5 \\ 0 & 3 & -5 \end{vmatrix} = 0 \qquad C_1 \to C_1 + C_2 + C_3$$

$\therefore (x+9)$ is a factor.

$\therefore m = 3 - 3 = 0$

$\therefore$ The remaining factor is constant 'k'

$$\begin{vmatrix} x+1 & 3 & 5 \\ 2 & x+2 & 5 \\ 2 & 3 & x+4 \end{vmatrix} = k\,(x-1)^2 (x+9)$$

Equating the coeff of x^3

$$1 = k$$

$$\therefore \boxed{k = 1}$$

$$\therefore \begin{vmatrix} x+1 & 3 & 5 \\ 2 & x+2 & 5 \\ 2 & 3 & x+4 \end{vmatrix} = (x-1)^2 (x+9)$$

Hence Proved.

7.24
31. Prove that $\begin{vmatrix} 1 & x^2 & x^3 \\ 1 & y^2 & y^3 \\ 1 & z^2 & z^3 \end{vmatrix} = (x-y)(y-z)(z-x)(xy+yz+zx)$

Let $\Delta = \begin{vmatrix} 1 & x^2 & x^3 \\ 1 & y^2 & y^3 \\ 1 & z^2 & z^3 \end{vmatrix}$

Put $x = y$

$$\Delta = \begin{vmatrix} 1 & y^2 & y^3 \\ 1 & y^2 & y^3 \\ 1 & z^2 & z^3 \end{vmatrix} = 0 \quad [R_1 \equiv R_2]$$

$\therefore (x-y)$ is a factor of Δ

lly $y-z$ and $z-x$ are the factors of Δ

$$m = 5 - 3 = 2$$

$\therefore$ Other factor is $k(a^2+b^2+c^2) + l(ab+bc+ca)$

$$\therefore \begin{vmatrix} 1 & x^2 & x^3 \\ 1 & y^2 & y^3 \\ 1 & z^2 & z^3 \end{vmatrix} = (x-y)(y-z)(z-x)\left\{k(a^2+b^2+c^2) + l(ab+bc+ca)\right\}$$

Put $x = 0 \quad y = 1 \quad z = -1$

$$\begin{vmatrix} 1 & 0 & 0 \\ 1 & 1 & 1 \\ 1 & 1 & -1 \end{vmatrix} = (-1)(2)(-1)\left\{k(2) + l(0-1+0)\right\}$$

$$1(-1-1) = 2(2k - l)$$

$$2(2k - l) = -2$$

$$2k - l = -1 \quad \text{——— } (1)$$

Put $x=0 \quad y=1 \quad z=2$

$$\begin{vmatrix} 1 & 0 & 0 \\ 1 & 1 & 1 \\ 1 & 4 & 8 \end{vmatrix} = (-1)(-1)(2)\{k(5)+l(0+2)\}$$

$$1(8-4) = 2(5k+2l)$$

$$2(5k+2l) = 4$$

$$5k+2l = 2 \quad \text{———} \quad (2)$$

Solving (1) & (2)

$$2k-l=-1 \quad \text{———} \quad (1)$$

$$5k+2l = 2 \quad \text{———} \quad (2)$$

(1) × 2

$$4k-\cancel{2l} = -\cancel{2}$$

$$5k+\cancel{2l} = \cancel{2}$$

$$\boxed{k=0}$$

Sub $k=0$ in (1)

$$-l=-1 \Rightarrow \boxed{l=1}$$

$$\therefore \begin{vmatrix} 1 & x^2 & x^3 \\ 1 & y^2 & y^3 \\ 1 & z^2 & z^3 \end{vmatrix} = (x-y)(y-z)(z-x)(xy+yz+zx)$$

Hence Proved.

7.25
32. Prove that $\begin{vmatrix} (q+r)^2 & p^2 & p^2 \\ q^2 & (r+p)^2 & q^2 \\ r^2 & r^2 & (p+q)^2 \end{vmatrix} = 2pqr(p+q+r)^3.$

Let $\Delta = \begin{vmatrix} (q+r)^2 & p^2 & p^2 \\ q^2 & (r+p)^2 & q^2 \\ r^2 & r^2 & (p+q)^2 \end{vmatrix}$

Put $p=0$

$$\Delta = \begin{vmatrix} (q+r)^2 & 0 & 0 \\ q^2 & r^2 & q^2 \\ r^2 & r^2 & q^2 \end{vmatrix} = 0 \qquad C_2 \equiv C_3$$

$\therefore$ $p-0$ is a factor of Δ

||ly $(q-0)$ and $(r-0)$ are the factors of

Put $p+q+r=0$

$$p+q=-r \qquad q+r=-p \qquad r+p=-q$$

$$\Delta = \begin{vmatrix} (-p)^2 & p^2 & p^2 \\ q^2 & (-q)^2 & q^2 \\ r^2 & r^2 & (-r)^2 \end{vmatrix} = 0 \qquad c_1 \equiv c_2 \equiv c_3$$

$\therefore (p+q+r)^2$ is a factor of Δ

$$m = 6-5 = 1$$

$\therefore$ Other factor factor is $k(p+q+r)$

$$\therefore \begin{vmatrix} (q+r)^2 & p^2 & p^2 \\ q^2 & (p+r)^2 & q^2 \\ r^2 & r^2 & (p+q)^2 \end{vmatrix} = k\,pqr(p+q+r)^3$$

Put $p=1 \quad q=1 \quad r=1$

$$\begin{vmatrix} 4 & 1 & 1 \\ 1 & 4 & 1 \\ 1 & 1 & 4 \end{vmatrix} = k(1)(1)(1)(3)^3$$

$$4(16-1) - 1(4-1) + 1(1-4) = 27k$$

$$27k = 54$$

$$\boxed{k=2}$$

$$\therefore \begin{vmatrix} (q+r)^2 & p^2 & p^2 \\ q^2 & (p+r)^2 & q^2 \\ r^2 & r^2 & (p+q)^2 \end{vmatrix} = 2pqr(p+q+r)^3$$

Hence Proved.

$\frac{T.26}{33.}$ In a ΔABC, if $\begin{vmatrix} 1 & 1 & 1 \\ 1+\sin A & 1+\sin B & 1+\sin C \\ \sin A(1+\sin A) & \sin B(1+\sin B) & \sin C(1+\sin C) \end{vmatrix} = 0$

Prove that ΔABC is an isosceles triangle.

Let $\Delta = \begin{vmatrix} 1 & 1 & 1 \\ 1+\sin A & 1+\sin B & 1+\sin C \\ \sin A(1+\sin A) & \sin B(1+\sin B) & \sin C(1+\sin C) \end{vmatrix}$

Put $\sin A = \sin B$

$$\Delta = \begin{vmatrix} 1 & 1 & 1 \\ 1+\sin A & 1+\sin A & 1+\sin C \\ \sin A(1+\sin A) & \sin A(1+\sin A) & \sin C(1+\sin C) \end{vmatrix} = 0$$

$c_1 \equiv c_2$

Ily $\sin B = \sin C$, $\sin C = \sin A$

we have $A=B$ or $B=C$ or $C=A$

In all cases atleast two angles are equal. $\therefore$ The triangle is isoscales.

Ex : 7.3.

Solve the following by using Factor theorem:

1.
34 Show that $\begin{vmatrix} x & a & a \\ a & x & a \\ a & a & x \end{vmatrix} = (x-a)^2(x+2a)$

Let $\Delta = \begin{vmatrix} x & a & a \\ a & x & a \\ a & a & x \end{vmatrix}$

Put $x = a$

$\Delta = \begin{vmatrix} a & a & a \\ a & a & a \\ a & a & a \end{vmatrix} = 0 \quad c_1 \equiv c_2 \equiv c_3$

$\therefore (x-a)^2$ is a factor

Put $x = -2a$

$\Delta = \begin{vmatrix} -2a & a & a \\ a & -2a & a \\ a & a & -2a \end{vmatrix} = \begin{vmatrix} 0 & a & a \\ 0 & -2a & a \\ 0 & a & -2a \end{vmatrix} = 0$

$\therefore (x+2a)$ is a factor of Δ.

$m = 3 - 3 = 0$

$\therefore$ The other factor is k.

$\begin{vmatrix} x & a & a \\ a & x & a \\ a & a & x \end{vmatrix} = k(x-a)^2(x+2a)$

Equating the co.eff of x^3

$1 = k \Rightarrow \boxed{k=1}$

$\therefore \begin{vmatrix} x & a & a \\ a & x & a \\ a & a & x \end{vmatrix} = (x-a)^2(x+2a)$

Hence Proved.

2.
34 Show that $\begin{vmatrix} b+c & a-c & a-b \\ b-c & c+a & b-a \\ c-b & c-a & a+b \end{vmatrix} = 8abc$

$\Delta = \begin{vmatrix} b+c & a-c & a-b \\ b-c & c+a & b-a \\ c-b & c-a & a+b \end{vmatrix}$

Put $a = 0$

$\Delta = \begin{vmatrix} b+c & -c & -b \\ b-c & c & b \\ c-b & c & b \end{vmatrix}$ $\quad$ c_2 is Proportional to c_3

$= 0$

$\therefore (a-0)$ is a factor of Δ

Ily $(b-0)$ and $(c-0)$ are the factors of Δ

$$m = 3-3 = 0$$

$\therefore$ The other factor is 'k'

$$\begin{vmatrix} b+c & a-c & a-b \\ b-c & c+a & b-a \\ c-b & c-a & a+b \end{vmatrix} = kabc$$

Put $a=b=c=1$

$$\begin{vmatrix} 2 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \end{vmatrix} = k(1)(1)(1)$$

$$\boxed{k=8}$$

$$\therefore \begin{vmatrix} b+c & a-c & a-b \\ b-c & c+a & b-a \\ c-b & c-a & a+b \end{vmatrix} = 8abc.$$

Hence Proved.

$\frac{3}{35}$ Solve : $\begin{vmatrix} x+a & b & c \\ a & x+b & c \\ a & b & x+c \end{vmatrix} = 0$

Let $\Delta = \begin{vmatrix} x+a & b & c \\ a & x+b & c \\ a & b & x+c \end{vmatrix}$

Put $x=0$

$$\Delta = \begin{vmatrix} a & b & c \\ a & b & c \\ a & b & c \end{vmatrix} = 0 \qquad R_1 \equiv R_2 \equiv R_3$$

$\therefore (x-0)^2$ is a factor.

Put $x = -(a+b+c)$

$$\Delta = \begin{vmatrix} -b-c & b & c \\ a & -a-c & c \\ a & b & -a-b \end{vmatrix} = \begin{vmatrix} 0 & b & c \\ 0 & -a-c & c \\ 0 & b & -a-b \end{vmatrix} = 0$$

$C_1 \to C_1 + C_2 + C_3$

$\therefore (x+a+b+c)$ is a factor of Δ.

$$m = 3-3 = 0$$

$\therefore$ Other factor is 'k'

$$\therefore \begin{vmatrix} x+a & b & c \\ a & x+b & c \\ a & b & x+c \end{vmatrix} = k(x+a+b+c).x^2$$

Equating the co.effs of x^3

$$\boxed{k=1}$$

$$\therefore \begin{vmatrix} x+a & b & c \\ a & x+b & c \\ a & b & x+c \end{vmatrix} = x^2(x+a+b+c)$$

$$\begin{vmatrix} x+a & b & c \\ a & x+b & c \\ a & b & x+c \end{vmatrix} = 0$$

$$x^2(x+a+b+c) = 0$$

$$x = 0,\ 0,\ -(a+b+c)$$

$\frac{4}{34}$ Show that $\begin{vmatrix} b+c & a & a^2 \\ c+a & b & b^2 \\ a+b & c & c^2 \end{vmatrix} = (a+b+c)(a-b)(b-c)(c-a)$

Let $\Delta = \begin{vmatrix} b+c & a & a^2 \\ c+a & b & b^2 \\ a+b & c & c^2 \end{vmatrix}$

Put $a=b$

$$\Delta = \begin{vmatrix} a+c & a & a^2 \\ a+c & a & a^2 \\ 2b & c & c^2 \end{vmatrix} = 0 \qquad R_1 \equiv R_2$$

$\therefore$ $(a-b)$ is a factor of Δ.

Illy $b-c$ and $c-a$ are the factors of Δ.

$$m = 4-3 = 1$$

$\therefore$ $k(a+b+c)$ is a other factor.

$$\begin{vmatrix} b+c & a & a^2 \\ c+a & b & b^2 \\ a+b & c & c^2 \end{vmatrix} = k(a+b+c)(a-b)(b-c)(c-a)$$

Put $a=0 \quad b=1 \quad c=2$

$$\begin{vmatrix} 3 & 0 & 0 \\ 2 & 1 & 1 \\ 1 & 2 & 4 \end{vmatrix} = k(3)(-1)(-1)(2)$$

$$3(4-2) = 6k \Rightarrow 6k = 6 \Rightarrow \boxed{k=1}$$

$$\therefore \begin{vmatrix} b+c & a & a^2 \\ c+a & b & b^2 \\ a+b & c & c^2 \end{vmatrix} = (a+b+c)(a-b)(b-c)(c-a)$$

Hence Proved

$\frac{5}{34}$ Solve : $\begin{vmatrix} 4-x & 4+x & 4+x \\ 4+x & 4-x & 4+x \\ 4+x & 4+x & 4-x \end{vmatrix} = 0$

Let $\Delta = \begin{vmatrix} 4-x & 4+x & 4+x \\ 4+x & 4-x & 4+x \\ 4+x & 4+x & 4-x \end{vmatrix}$

Put $x = 0$

$\Delta = \begin{vmatrix} 4 & 4 & 4 \\ 4 & 4 & 4 \\ 4 & 4 & 4 \end{vmatrix} = 0 \quad C_1 \equiv C_2 \equiv C_3$

$\therefore (x-0)^2$ is a factor.

Put $x = -12$

$\Delta = \begin{vmatrix} 16 & 8 & 8 \\ 8 & 16 & 8 \\ 8 & 8 & 16 \end{vmatrix} = \begin{vmatrix} 0 & 8 & 8 \\ 0 & 16 & 8 \\ 0 & 8 & 16 \end{vmatrix} = 0 \quad C_1 \to C_1 + C_2 + C_3$

$\therefore (x+12)$ is a factor

$m = 3 - 3 = 0$

$\therefore$ Other factor is 'k'

$\begin{vmatrix} 4-x & 4+x & 4+x \\ 4+x & 4-x & 4+x \\ 4+x & 4+x & 4+x \end{vmatrix} = x^2(x+12) = 0$

$x^2 = 0 \quad x+12 = 0$

$x = 0, 0, -12.$

---

6. Show that $\begin{vmatrix} 1 & 1 & 1 \\ x & y & z \\ x^2 & y^2 & z^2 \end{vmatrix} = (x-y)(y-z)(z-x)$

$\Delta = \begin{vmatrix} 1 & 1 & 1 \\ x & y & z \\ x^2 & y^2 & z^2 \end{vmatrix}$

Put $x = y$

$\Delta = \begin{vmatrix} 1 & 1 & 1 \\ y & y & z \\ y^2 & y^2 & z^2 \end{vmatrix} = 0 \quad R_1 \equiv R_2$

$\therefore (x-y)$ is a factor of Δ.

IIly $y-z$ and $z-x$ are the factors of Δ.

$m = 3 - 3 = 0$

$\therefore$ Other factor is 'k'

$$\begin{vmatrix} 1 & 1 & 1 \\ x & y & z \\ x^2 & y^2 & z^2 \end{vmatrix} = k(x-y)(y-z)(z-x)$$

Put $x = 0 \quad y = 1 \quad z = -1$

$$\begin{vmatrix} 1 & 1 & 1 \\ 0 & 1 & -1 \\ 0 & 1 & 1 \end{vmatrix} = k(-1)(2)(-1)$$

$1(2) = 2k$

$\boxed{k = 1}$

$$\therefore \begin{vmatrix} 1 & 1 & 1 \\ x & y & z \\ x^2 & y^2 & z^2 \end{vmatrix} = (x-y)(y-z)(z-x)$$

Hence Proved.

7.3.4 Product of Determinants :

* Row by Column multiplication Rule.
* Row by Row multiplication Rule.
* Column by Column multiplication Rule.
* Column by Row multiplication Rule.

Note :-

* If A and B are Square matrices of the Same order n then $|AB| = |A||B|$ holds.

* In matrices, although $AB \neq BA$ in general, we do have $|AB| = |BA|$ always.

Relationship between a determinant and its cofactor determinant.

Let $|A| = \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}$

Co factor determinant

$$|A_{ij}| = \begin{vmatrix} A_1 & B_1 & C_1 \\ A_2 & B_2 & C_2 \\ A_3 & B_3 & C_3 \end{vmatrix}$$

* The Sum of the Product of elements of any row (or Column) with their Corresponding Co-factors is the value of determinant.

* If elements of a row (or column) are multiplied with Corresponding co-factors of any other row (or column) then their sum is zero.

7.27
34

Verify that $|AB| = |A||B|$ if $A = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}$ and $B = \begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix}$

$A = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}$ $\quad B = \begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix}$

$$|A| = \begin{vmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{vmatrix} = \cos^2\theta + \sin^2\theta = 1$$

$$|B| = \begin{vmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{vmatrix} = \cos^2\theta + \sin^2\theta = 1$$

$$AB = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}\begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix}$$

$$= \begin{bmatrix} \cos^2\theta + \sin^2\theta & \sin\theta\cos\theta - \sin\theta\cos\theta \\ \sin\theta\cos\theta - \sin\theta\cos\theta & \sin^2\theta + \cos^2\theta \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

$|AB| = \begin{vmatrix} 1 & 0 \\ 0 & 1 \end{vmatrix} = 1$ ——— ①

$|A| . |B| = 1 \times 1 = 1$ ——— ②

from ① & ②

$|AB| = |A||B|$

Hence verified.

7.28
35

Show that $\begin{vmatrix} 0 & c & b \\ c & 0 & a \\ b & a & 0 \end{vmatrix}^2 = \begin{vmatrix} b^2+c^2 & ab & ac \\ ab & c^2+a^2 & bc \\ ac & bc & a^2+b^2 \end{vmatrix}$

$$L.H.S = \begin{vmatrix} 0 & c & b \\ c & 0 & a \\ b & a & 0 \end{vmatrix}^2 = \begin{vmatrix} 0 & c & b \\ c & 0 & a \\ b & a & 0 \end{vmatrix}\begin{vmatrix} 0 & c & b \\ c & 0 & a \\ b & a & 0 \end{vmatrix}$$

$$= \begin{vmatrix} 0+c^2+b^2 & 0+0+ab & 0+ac+0 \\ 0+0+ab & c^2+0+a^2 & bc+0+0 \\ 0+ac+0 & bc+0+0 & b^2+a^2+0 \end{vmatrix}$$

$$= \begin{vmatrix} b^2+c^2 & ab & ac \\ ab & c^2+a^2 & bc \\ ac & bc & a^2+b^2 \end{vmatrix} = R.H.S$$

Hence Proved.

7.29
35.

Show that $\begin{vmatrix} 2bc-a^2 & c^2 & b^2 \\ c^2 & 2ac-b^2 & a^2 \\ b^2 & a^2 & 2ab-c^2 \end{vmatrix} = \begin{vmatrix} a & b & c \\ b & c & a \\ c & a & b \end{vmatrix}^2$

$$R.H.S = \begin{vmatrix} a & b & c \\ b & c & a \\ c & a & b \end{vmatrix}^2 = \begin{vmatrix} a & b & c \\ b & c & a \\ c & a & b \end{vmatrix}\begin{vmatrix} a & b & c \\ b & c & a \\ c & a & b \end{vmatrix}$$

$$= \begin{vmatrix} a & b & c \\ b & c & a \\ c & a & b \end{vmatrix}(-1)\begin{vmatrix} a & b & c \\ c & a & b \\ b & c & a \end{vmatrix} \qquad R_2 \leftrightarrow R_3$$

$$= \begin{vmatrix} a & b & c \\ b & c & a \\ c & a & b \end{vmatrix}\begin{vmatrix} -a & -b & -c \\ c & a & b \\ b & c & a \end{vmatrix}$$

$$= \begin{vmatrix} -a^2+bc+bc & -ab+ab+c^2 & -ac+b^2+ac \\ -ab+c^2+ab & -b^2+ac+ac & -bc+bc+a^2 \\ -ac+ac+b^2 & -bc+a^2+bc & -c^2+ab+ab \end{vmatrix}$$

$$= \begin{vmatrix} 2bc-a^2 & c^2 & b^2 \\ c^2 & 2ca-b^2 & a^2 \\ b^2 & a^2 & 2ab-c^2 \end{vmatrix} = L.H.S$$

Hence Proved

7.30
36

Prove that $\begin{vmatrix} 1 & x & x \\ x & 1 & x \\ x & x & 1 \end{vmatrix}^2 = \begin{vmatrix} 1-2x^2 & -x^2 & -x^2 \\ -x^2 & -1 & x^2-2x \\ -x^2 & x^2-2x & -1 \end{vmatrix}$

$$L.H.S = \begin{vmatrix} 1 & x & x \\ x & 1 & x \\ x & x & 1 \end{vmatrix}^2 = \begin{vmatrix} 1 & x & x \\ x & 1 & x \\ x & x & 1 \end{vmatrix}\begin{vmatrix} 1 & x & x \\ x & 1 & x \\ x & x & 1 \end{vmatrix}$$

$$= \begin{vmatrix} 1 & x & x \\ x & 1 & x \\ x & x & 1 \end{vmatrix} \begin{vmatrix} 1 & x & x \\ -x & -1 & -x \\ -x & -x & -1 \end{vmatrix}$$

$$= \begin{vmatrix} 1-x^2-x^2 & x-x-x^2 & x-x^2-x \\ x-x-x^2 & x^2-1-x^2 & x^2-x-x \\ x-x^2-x & x^2-x-x & x^2-x^2-1 \end{vmatrix}$$

$$= \begin{vmatrix} 1-2x^2 & -x^2 & -x^2 \\ -x^2 & -1 & x^2-2x \\ -x^2 & x^2-2x & -1 \end{vmatrix} = R.H.S$$

Hence Proved.

7.31 / 37. If A_i, B_i, C_i are the co factors of a_i, b_i, c_i respectively $i = 1$ to 3 in $|A| = \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}$, show that $\begin{vmatrix} A_1 & B_1 & C_1 \\ A_2 & B_2 & C_2 \\ A_3 & B_3 & C_3 \end{vmatrix} = |A|^2$.

$$|A| = \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}$$

$$\begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} \begin{vmatrix} A_1 & B_1 & C_1 \\ A_2 & B_2 & C_2 \\ A_3 & B_3 & C_3 \end{vmatrix}$$

$$= \begin{vmatrix} a_1A_1+b_1B_1+C_1C_1 & A_1a_2+b_1B_2+C_1C_2 & a_1A_3+b_1B_3+C_1C_3 \\ a_2A_1+b_2B_1+C_2C_1 & a_2A_2+b_2B_2+C_2C_2 & a_2A_3+b_2B_3+C_2C_3 \\ a_3A_1+B_3B_1+C_3C_1 & a_3A_2+b_3B_2+C_3C_2 & a_3B_3+b_3B_3+C_3C_3 \end{vmatrix}$$

$$= \begin{vmatrix} |A| & 0 & 0 \\ 0 & |A| & 0 \\ 0 & 0 & |A| \end{vmatrix} = |A|^3$$

$$\cancel{|A|} \begin{vmatrix} A_1 & B_1 & C_1 \\ A_2 & B_2 & C_2 \\ A_3 & B_3 & C_3 \end{vmatrix} = |A|^{\cancel{3}}$$

$$\begin{vmatrix} A_1 & B_1 & C_1 \\ A_2 & B_2 & C_2 \\ A_3 & B_3 & C_3 \end{vmatrix} = |A|^2$$

Hence Proved.

7.3.6. Area of a triangle:

let (x_1, y_1) (x_2, y_2) and (x_3, y_3) be the vertices of of triangle ABC.

$$\text{area of } \Delta ABC = \text{ab. value of } \frac{1}{2}\begin{vmatrix} x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \\ x_3 & y_3 & 1 \end{vmatrix}$$

Singular and non Singular matrices:-

A Square matrix A is Said to be Singular if $|A| = 0$. A Square matrix A is Said to be non Singular if $|A| \neq 0$

Note:

If A and B are two non-singular matrices of the Same order then AB and BA are also non-Singular matrices

$$|AB| = |A||B| = |BA|.$$

7.32 / 38 If the area of the triangle with vertices $(-3,0)$ $(3,0)$ and $(0,k)$ is 9 sq.units find the values of k.

Area of Δ = 9 sq. units.

$$\left|\frac{1}{2}\begin{vmatrix} x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \\ x_3 & y_3 & 1 \end{vmatrix}\right| = 9$$

$$\begin{vmatrix} x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \\ x_3 & y_3 & 1 \end{vmatrix} = \pm 18$$

$$\begin{vmatrix} -3 & 0 & 1 \\ 3 & 0 & 1 \\ 0 & k & 1 \end{vmatrix} = \pm 18$$

$$-3(-k) - 0 + 1(3k) = \pm 18$$

$$6k = \pm 18$$

$$\boxed{k = \pm 3}$$

N KARTHIKEYAN, M.Sc, B.Ed
AVP TRUST NATIONAL
MATRIC HR SEC SCHOOL
TIRUPUR-9843434491

7.33
38

Find the area of the triangle whose vertices are $(-2,-3)$, $(3,2)$ and $(-1,-8)$

area of the triangle $= \left|\frac{1}{2}\begin{vmatrix} x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \\ x_3 & y_3 & 1 \end{vmatrix}\right|$

$= \left|\frac{1}{2}\begin{vmatrix} -2 & -3 & 1 \\ 3 & 2 & 1 \\ -1 & -8 & 1 \end{vmatrix}\right|$

$= \left|\frac{1}{2}\left[-2(2+8)+3(3+1)+1(-24+2)\right]\right|$

$= \left|\frac{1}{2}\left[-20+12-22\right]\right| = \left|\frac{1}{2}(-30)\right| = |-15|$

$= 15$ Sq. units.

7.34
38

Show that the Points $(a, b+c)$ $(b, c+a)$ and $(c, a+b)$ are collinear.

If the Points are Collinear

area of the triangle $=0$

$\begin{vmatrix} x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \\ x_3 & y_3 & 1 \end{vmatrix} = 0$

$\begin{vmatrix} a & b+c & 1 \\ b & c+a & 1 \\ c & a+b & 1 \end{vmatrix}$ $\quad C_2 \to C_1 + C_2$

$= \begin{vmatrix} a & a+b+c & 1 \\ b & a+b+c & 1 \\ c & a+b+c & 1 \end{vmatrix} = (a+b+c)\begin{vmatrix} a & 1 & 1 \\ b & 1 & 1 \\ c & 1 & 1 \end{vmatrix}$

$= 0$

$\therefore$ The Points are collinear.

Ex : 7.4

1
39

Find the area of the triangle whose vertices $(0,0)$ $(1,2)$ $(4,3)$

area of the triangle $= \left|\frac{1}{2}\begin{vmatrix} x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \\ x_3 & y_3 & 1 \end{vmatrix}\right|$

$= \left|\frac{1}{2}\begin{vmatrix} 0 & 0 & 1 \\ 1 & 2 & 1 \\ 4 & 3 & 1 \end{vmatrix}\right| = \left|\frac{1}{2}(3-8)\right|$

$= \frac{5}{2}$ Sq. units

2/39 If $(k,2)$, $(2,4)$ and $(3,2)$ are the vertices of the triangle of area 4 sq. units then determine the value of 'k'.

area of the triangle = 4

$$\left|\frac{1}{2}\begin{vmatrix} k & 2 & 1 \\ 2 & 4 & 1 \\ 3 & 2 & 1 \end{vmatrix}\right| = 4$$

$$\begin{vmatrix} k & 2 & 1 \\ 2 & 4 & 1 \\ 3 & 2 & 1 \end{vmatrix} = \pm 8$$

$$k(4-2)-2(2-3)+1(4-12) = \pm 8$$

$$2k+2-8 = \pm 8$$

$$2k-6 = \pm 8$$

$2k-6 = 8$ | $2k-6=-8$

$2k = 14$ | $2k=-2$

$\boxed{k=7}$ | $\boxed{k=-1}$

3/39 Identify the Singular and non Singular matrices :-

(i) $A = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{bmatrix}$

$$|A| = \begin{vmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{vmatrix} = 1(45-48)-2(36-42)+3(32-35)$$

$$= -3+12-9 = 12-12 = 0$$

$\therefore$ A is a Singular matrix.

(ii) $A = \begin{bmatrix} 2 & -3 & 5 \\ 6 & 0 & 4 \\ 1 & 5 & -7 \end{bmatrix}$

N KARTHIKEYAN, M.Sc, B.Ed
AVP TRUST NATIONAL
MATRIC HR SEC SCHOOL
TIRUPUR-9843434491

$$|A| = \begin{vmatrix} 2 & -3 & 5 \\ 6 & 0 & 4 \\ 1 & 5 & -7 \end{vmatrix}$$

$$= 2(0-20)+3(-42-4)+5(30-0)$$

$$= -40+3(-46)+150$$

$$= -40-138+150 = -178+150 = -28 \neq 0$$

$\therefore$ A is non Singular matrix.

(iii) $A = \begin{bmatrix} 0 & a-b & k \\ b-a & 0 & 5 \\ -k & -5 & 0 \end{bmatrix}$

$|A| = \begin{vmatrix} 0 & a-b & k \\ b-a & 0 & 5 \\ -k & -5 & 0 \end{vmatrix}$

$= 0 - (a-b)(0+5k) + k(-5(b-a))$

$= -5k(a-b) + 5k(a-b) = 0$

$|A| = 0$

$\therefore$ A is a Singular matrix.

[OR]

A is a Skew symmetric matrix.

If A is a skew Symmetric

$|A| = 0$

$\therefore$ A is Singular matrix.

$\frac{4}{40}$ Determine the values of a and b so that the following matrices are singular.

(i) $A = \begin{bmatrix} 7 & 3 \\ -2 & a \end{bmatrix}$

Since A is Singular,

$|A| = 0 \Rightarrow \begin{vmatrix} 7 & 3 \\ -2 & a \end{vmatrix} = 0$

$7a + 6 = 0$

$7a = -6$

$\boxed{a = -6/7}$

N KARTHIKEYAN, M.Sc, B.Ed
AVP TRUST NATIONAL
MATRIC HR SEC SCHOOL
TIRUPUR-9843434491

(ii) $B = \begin{bmatrix} b-1 & 2 & 3 \\ 3 & 1 & 2 \\ 1 & -2 & 4 \end{bmatrix}$

Since B is Singular, $|B| = 0$

$\begin{vmatrix} b-1 & 2 & 3 \\ 3 & 1 & 2 \\ 1 & -2 & 4 \end{vmatrix} = 0$

$(b-1)(4+4) - 2(12-2) + 3(-6-1) = 0$

$8b - 8 - 20 - 21 = 0$

$8b = 48$

$\boxed{b = 6}$

$\frac{5}{40}$ If $\cos 2\theta = 0$ determine $\begin{vmatrix} 0 & \cos\theta & \sin\theta \\ \cos\theta & \sin\theta & 0 \\ \sin\theta & 0 & \cos\theta \end{vmatrix}^2$

$\cos 2\theta = 0$

$2\theta = \cos^{-1}(0) = \frac{\pi}{2}$

$\theta = \frac{\pi}{4}$

$$\begin{vmatrix} 0 & \cos\theta & \sin\theta \\ \cos\theta & \sin\theta & 0 \\ \sin\theta & 0 & \cos\theta \end{vmatrix} = \begin{vmatrix} 0 & \cos\frac{\pi}{4} & \sin\frac{\pi}{4} \\ \cos\frac{\pi}{4} & \sin\frac{\pi}{4} & 0 \\ \sin\frac{\pi}{4} & 0 & \cos\frac{\pi}{4} \end{vmatrix}$$

$$= \begin{vmatrix} 0 & 1/\sqrt{2} & \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} & 0 \\ \frac{1}{\sqrt{2}} & 0 & \frac{1}{\sqrt{2}} \end{vmatrix} = \frac{1}{\sqrt{2}} \cdot \frac{1}{\sqrt{2}} \cdot \frac{1}{\sqrt{2}} \begin{vmatrix} 0 & 1 & 1 \\ 1 & 1 & 0 \\ 1 & 0 & 1 \end{vmatrix}$$

$$= \frac{1}{2\sqrt{2}} \left\{ 0 - 1(1-0) + 1(0-1) \right\}$$

$$= \frac{1}{2\sqrt{2}} (-1-1) = \frac{-1}{\sqrt{2}}$$

$$\therefore \begin{vmatrix} 0 & \cos\theta & \sin\theta \\ \cos\theta & \sin\theta & 0 \\ \sin\theta & 0 & \cos\theta \end{vmatrix}^2 = \left(\frac{-1}{\sqrt{2}}\right)^2 = \frac{1}{2}$$

$\frac{6}{40}$ Find the Product $\begin{vmatrix} \log_3 64 & \log_4 3 \\ \log_3 8 & \log_4 9 \end{vmatrix} \begin{vmatrix} \log_2 3 & \log_8 3 \\ \log_3 4 & \log_3 4 \end{vmatrix}$

$$= \begin{vmatrix} \log_3 64 \log_2 3 + \log_4 3 . \log_3 4 & \log_3 64 . \log_8 3 + \log_4 3 . \log_3 4 \\ \log_3 8 . \log_2 3 + \log_4 9 . \log_3 4 & \log_3 8 \log_8 3 + \log_4 9 . \log_3 4 \end{vmatrix}$$

$$= \begin{vmatrix} \log_2 64 + \log_3 3 & \log_8 64 + \log_3 3 \\ \log_2 8 + \log_3 9 & \log_8 8 + \log_3 9 \end{vmatrix}$$

$$= \begin{vmatrix} 6+1 & 2+1 \\ 3+2 & 1+2 \end{vmatrix}$$

$$= \begin{vmatrix} 7 & 3 \\ 5 & 3 \end{vmatrix} = 21 - 15$$

$= 6$

N KARTHIKEYAN, M.Sc, B.Ed

AVP TRUST NATIONAL

MATRIC HR SEC SCHOOL

TIRUPUR-9843434491