vector calculus

practice problems.

Q). $\nabla \times \nabla \times P$, where P is a vector, is equal,

a). $P \times \nabla \times P - \nabla^2 P$

b). $\nabla^2 P + \nabla(\nabla \cdot P)$

c) $\nabla^2 P + \nabla \times P$

d). $\nabla(\nabla \cdot P) - \nabla^2 P$

Sol:- $A \times B \times C = B(A \cdot C) - C(A \cdot B)$

$$\nabla \times \nabla \times P = \nabla(\nabla \cdot P) - P(\nabla \cdot \nabla)$$

$$\nabla \times \nabla \times P = \nabla(\nabla \cdot P) - \nabla^2 P$$

Q). The gauss divergence theorem relates certain,

a). surface integrals to volume integrals

b). surface to line

c). vector quantities to other vector quantity

d). line integral to volume integral.

Sol:- gauss divergence theorem:-

$$\oint_S \vec{A}\cdot\hat{n}\,d\vec{s} = \int_V div\ \vec{A}\ d\vec{v} = \int_V \nabla\cdot\vec{A}\cdot d\vec{v}$$

$$\underbrace{\oint_S \vec{A}\cdot d\vec{s}}_{\text{surface}} = \underbrace{\int_V \nabla\cdot\vec{F}\ d\vec{v}}_{\text{volume,}}$$

" Relationship b/w surface and volume integral

" only for closed surface Integral "

Q). The vector Field $\vec{F} = x\hat{i} - y\hat{j}$

find nature about irrotational,

Sol:- For irrotational vector,

$$\nabla \times \vec{A} = curl(\vec{A}) = 0$$

$$\nabla\cdot\vec{A} = div(\vec{A}) = 0 \rightarrow \text{divergence free}$$

Sol:- $\nabla\cdot\vec{A} = \left(\partial/\partial x\,\hat{i} + \partial/\partial y\,\hat{j} + \partial/\partial z\,\hat{k}\right)\cdot\vec{A}$

$\Rightarrow \left(\partial/\partial x\,\hat{i} + \partial/\partial y\,\hat{j} + \partial/\partial z\,\hat{k}\right)\cdot(x\hat{i} - y\hat{j})$

$\nabla \cdot \vec{A} \Rightarrow 1 + (-1) = 0 \rightarrow$ divergence free.

$$\nabla \times \vec{A} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ x & -y & 0 \end{vmatrix} = \hat{i}\left(+\partial y/\partial z\right) - \hat{j}\left(-\partial x/\partial z\right) + \hat{k}\left(-\partial y/\partial x - \partial x/\partial y\right)$$

$$\Rightarrow \hat{i}\, \cancel{\partial y/\partial z} + \hat{j}\, \cancel{\partial x/\partial z} - \hat{k}\, \cancel{\partial y/\partial x} - \hat{k}\, \cancel{\partial x/\partial y}$$

$\nabla \times \vec{A} = 0$

So the vector $\vec{A}$ is irrotational,

Q) Find the unit vector parallel to the resultant of vectors $r_1 = 2i + 4j - 5k$, $r_2 = i + 2j + 3k$.

Sol: Resultant $= r_1 + r_2 = (2i + 4j - 5k) +$

$$\begin{array}{r} (2i + 4j - 5k) \\ (i + 2j + 3k) \\ \hline +3i + 6j - 2k \\ \hline \end{array}$$

$R = |R| = |3i + 6j - 2k| = \sqrt{3^2 + 6^2 + (-2)^2}$

$= \sqrt{9 + 36 + 4} = \sqrt{49}$

$= 7$

Unit vector parallel to R is, $\vec{R}/R$

$= \frac{3i+6j-2k}{7} = \frac{3}{7}i + \frac{6}{7}j - \frac{2}{7}k$

If check |unit vector|, $\sqrt{\left(\frac{3}{7}\right)^2+\left(\frac{6}{7}\right)^2+\left(\frac{-2}{7}\right)^2}$

$= \frac{\sqrt{9+36+4}}{7} = \frac{\sqrt{49}}{7} = \frac{\cancel{7}}{\cancel{7}} = 1$

Q). Determine the value of a, so that

$A = 2i + aj + k$, $B = 4i - 2j - 2k$, are $\perp r$,

Sol:- A & B are $\perp r$, if $A \cdot B = 0$

$A \cdot B = |A|\,|B| \cos\theta$, $\perp r$, $\theta = 90°$

$A \cdot B = 0$

$(2i + aj + k) \cdot (4i - 2j - 2k) = 0$

$8 - a2 - 2 = 0$, $6 - 2a = 0$

$6 = 2a$, $a = 3$,

Q). Find the projection of vector

H-W $A = i - 2j + k$ on the vector $B = 4i - 4j + 7k$

Hint:- projection of A on B = A . b

b = unit vector in the direction of B.

Q) Find the work done in moving an object along a vector $r = 3i + 2j - 5k$, if the applied force is $F = 2i - j - k$

Sol:- work done = (magnitude of force in direction of motion) (distance moved)

$= F \cdot r$

$= (2i - j - k) \cdot (3i + 2j - 5k)$

$= 6 - 2 + 5 = 9.$

Q) $A = 3i - j + 2k$, $B = 2i + j - k$, $C = i - 2j + 2k$

find $(A \times B) \times C$

Sol:- $A \times B = \begin{vmatrix} i & j & k \\ 3 & -1 & 2 \\ 2 & 1 & -1 \end{vmatrix} = i(1 - 2) - j(-3 - 4) + k(3 + 2)$

$= i(-1) - j(-7) + k(5)$

$= -i + 7j + 5k$

$$(A \times B) \times C = \begin{vmatrix} i & j & k \\ -1 & 7 & 5 \\ 1 & -2 & 2 \end{vmatrix} = i(14+10) - j(-2-5) + k(2-7)$$

$$= i(24) - j(-7) + k(-5)$$

$$= i(24) + j7 - 5k$$

Q). Area of the triangle having vertices at $P(1,3,2)$, $Q(2,-1,1)$, $R(-1,2,3)$

$$PQ = Q - P = (2-1)i + (-1-3)j + (1-2)k$$

$$= i - 4j - k$$

$$PR = R - P = (-1-1)i + (2-3)j + (3-2)k$$

$$= (-2)i + (-1)j + (1)k$$

$$= -2i - j + k$$

$$\text{Area of triangle} = \tfrac{1}{2} |PQ \times PR)$$

$$= \tfrac{1}{2} \left| (i - 4j - k) \times (-2i - j + k) \right|$$

$$= \tfrac{1}{2} \begin{vmatrix} i & j & k \\ 1 & -4 & -1 \\ -2 & -1 & 1 \end{vmatrix}$$

$$= \frac{1}{2}\left| i(-4-1) - j(1-2) + k(-1-8) \right|$$

$$= \frac{1}{2}\left| -5i + j - 9k \right|$$

$$= \frac{1}{2}\sqrt{(-5)^2 + (1)^2 + (-9)^2} = \frac{1}{2}\sqrt{25+1+81}$$

$$= \frac{1}{2}\sqrt{107}$$

Q). Find a unit vector normal to the surface $x^2y + 2xz = 4$ at the point $(2,-2,3)$

sol:- $\nabla(x^2y + 2xz) = \frac{\partial}{\partial x}(x^2y + 2xz) + \frac{\partial}{\partial y}(x^2y + 2xz) + \frac{\partial}{\partial z}(x^2y + 2xz)$

$$= (2xy + 2z)i + x^2 j + 2xk$$

$(2,-2,3)$

$$= (2\times 2\times -2 + 2(3))i + (2)^2 j + 2(2)k$$

$$= (-8+6)i + 4j + 4k$$

$$= -2i + 4j + 4k$$

Unit normal to the surface $= \dfrac{-2i + 4j + 4k}{\sqrt{(-2)^2 + (4)^2 + 4^2}}$

$$= \frac{-2i + 4j + k}{\sqrt{4 + 16 + 16}} = \frac{-2i + 4j + k}{\sqrt{36}} = \frac{-2i + 4j + k}{6}$$

Important formulas:-

1. Unit normal vector ϕ at $P(x, y, z)$

$$= \frac{\nabla \phi}{|\nabla \phi|}$$

2. Component of the velocity in the direction of $\vec{a} = \vec{v} \cdot \dfrac{\vec{a}}{|a|}$

3. Component of the acceleration in the direction of $\vec{a} = \vec{f} \cdot \dfrac{\vec{a}}{|\vec{a}|}$

4. The tangential component of acceleration $X : \vec{f} \cdot \frac{\vec{v}}{|\vec{v}|}$

5. normal component of acceleration,

$$Y = \sqrt{R^2 - X^2} \quad , \quad R = |\vec{f}|$$

6. Magnitude of tangential component of acceleration $: \frac{v \cdot \vec{f}}{|v|}$

7. Magnitude of normal component of acceleration $: \frac{|\vec{v} \times \vec{f}|}{|\vec{v}|}$

Q). What is curl of the vector field,

$$2x^2y\,\hat{i} + 5x^2\,\hat{j} - 4yz\,\hat{k}$$

a). $-14x\,\hat{i} - 2x^2 k$ c) $-14x\,\hat{i} + 6y\hat{j} + 2x^2\hat{k}$

b). $6x\hat{i} + 4x^2\hat{j} - 2x^2\hat{k}$ d) $6x\,\hat{i} - 8xy\hat{j} + 2x^2y\hat{k}$

Sol:-