Sun Tuition Center

Poon Thotta Pathai Hindu Mission Hospital Opposite

Villupuram

2022-12th-2023

Mathematics

All Units	Price
5 Marks	Rs.
Solution	200

Only Maths Tuition

9th to 12th

Contact

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Example 1.1 If $A = \begin{bmatrix} 8 & -6 & 2 \\ -6 & 7 & -4 \\ 2 & -4 & 3 \end{bmatrix}$ verify the result $A(adj\ A) = (adj\ A)A = |A|\ I_3$

Solution:

$$|A| = \begin{vmatrix} 8 & -6 & 2 \\ -6 & 7 & -4 \\ 2 & -4 & 3 \end{vmatrix}$$

$= 8(21-16) - (-6)(-18+8) + 2(24-14)$

$= 40 - 60 + 20 = 0.$

$$adjA = \begin{bmatrix} 5 & 10 & 10 \\ 10 & 20 & 20 \\ 10 & 20 & 20 \end{bmatrix}^T = \begin{bmatrix} 5 & 10 & 10 \\ 10 & 20 & 20 \\ 10 & 20 & 20 \end{bmatrix}$$

$$A(adj\ A) = \begin{bmatrix} 8 & -6 & 2 \\ -6 & 7 & -4 \\ 2 & -4 & 3 \end{bmatrix}\begin{bmatrix} 5 & 10 & 10 \\ 10 & 20 & 20 \\ 10 & 20 & 20 \end{bmatrix}$$

$$= \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} = 0\ I_3 = |A|\ I_3$$

$$(adj\ A)A = \begin{bmatrix} 5 & 10 & 10 \\ 10 & 20 & 20 \\ 10 & 20 & 20 \end{bmatrix}\begin{bmatrix} 8 & -6 & 2 \\ -6 & 7 & -4 \\ 2 & -4 & 3 \end{bmatrix}$$

$$= \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} = 0\ I_3 = |A|\ I_3$$

Hence, $A(adj\ A) = (adj\ A)A = |A|\ I_3$.

Example 1.10 If $A = \begin{bmatrix} 4 & 3 \\ 2 & 5 \end{bmatrix}$, find x and y such that $A^2 + xA + y\ I_2 = O_2$. Hence, find A^{-1}.

Solution:

Since, $A^2 = A \cdot A = \begin{bmatrix} 4 & 3 \\ 2 & 5 \end{bmatrix}\begin{bmatrix} 4 & 3 \\ 2 & 5 \end{bmatrix} = \begin{bmatrix} 22 & 27 \\ 18 & 31 \end{bmatrix}$

$A^2 + xA + y\ I_2 = O_2$

$$\Rightarrow \begin{bmatrix} 22 & 27 \\ 18 & 31 \end{bmatrix} + \begin{bmatrix} 4x & 3x \\ 2x & 5x \end{bmatrix} + \begin{bmatrix} y & 0 \\ 0 & y \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$$

$$\begin{bmatrix} 22+4x+y & 27+3x+0 \\ 18+2x+0 & 31+5x+y \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$$

$27 + 3x = 0 \quad \Rightarrow \quad x = -9$

$31 + 5x + y = 0 \quad \Rightarrow \quad y = 14$

$A^2 + xA + y\ I_2 = O_2 \Rightarrow A^2 - 9A + 14\ I_2 = O_2$

Post-multiplying this equation by A^{-1}

$A - 9\ I_2 + 14A^{-1} = O_2$

$A^{-1} = \frac{1}{14}(9I_2 - A)$

$$A^{-1} = \frac{1}{14}\left(\begin{bmatrix} 9 & 0 \\ 0 & 9 \end{bmatrix} - \begin{bmatrix} 4 & 3 \\ 2 & 5 \end{bmatrix}\right) = \frac{1}{14}\begin{bmatrix} 5 & -3 \\ -2 & 4 \end{bmatrix}$$

Example 1.12 If $A = \frac{1}{7}\begin{bmatrix} 6 & -3 & a \\ b & -2 & 6 \\ 2 & c & 3 \end{bmatrix}$ is orthogonal, find a, b and c, and hence A^{-1}.

Solution:

A is **orthogonal** if and only if A is non-singular and $A^{-1} = A^T$

If A is orthogonal, then $AA^T = A^TA = I_3$

$A\,A^T = I_3$

$$\Rightarrow \frac{1}{7}\begin{bmatrix} 6 & -3 & a \\ b & -2 & 6 \\ 2 & c & 3 \end{bmatrix}\frac{1}{7}\begin{bmatrix} 6 & -3 & a \\ b & -2 & 6 \\ 2 & c & 3 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$\begin{bmatrix} 45+a^2 & 6b+6+6a & 12-3c+3a \\ 6b+6+6a & b^2+40 & 2b-2c+18 \\ 12-3c+3a & 2b-2c+18 & c^2+13 \end{bmatrix} = 49\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$\begin{bmatrix} 45+a^2 & 6b+6+6a & 12-3c+3a \\ 6b+6+6a & b^2+40 & 2b-2c+18 \\ 12-3c+3a & 2b-2c+18 & c^2+13 \end{bmatrix} = \begin{bmatrix} 49 & 0 & 0 \\ 0 & 49 & 0 \\ 0 & 0 & 49 \end{bmatrix}$$

Comparing the corresponding elements,

$45 + a^2 = 49$	$b^2 + 40 = 49$	$c^2 + 13 = 49$
$a^2 = 4$	$b^2 = 9$	$c^2 = 36$
$a = 2$	$b = -3$	$c = 6$

$$A = \frac{1}{7}\begin{bmatrix} 6 & -3 & 2 \\ -3 & -2 & 6 \\ 2 & 6 & 3 \end{bmatrix} \Longrightarrow A^{-1} = A^T = \frac{1}{7}\begin{bmatrix} 6 & -3 & 2 \\ -3 & -2 & 6 \\ 2 & 6 & 3 \end{bmatrix}$$

EXERCISE 1.1

3. If $F(\alpha) = \begin{bmatrix} \cos\alpha & 0 & \sin\alpha \\ 0 & 1 & 0 \\ -\sin\alpha & 0 & \cos\alpha \end{bmatrix}$ show that $[F(\alpha)]^{-1} = F(-\alpha)$.

Solution:

$$F(\alpha) = \begin{bmatrix} \cos\alpha & 0 & \sin\alpha \\ 0 & 1 & 0 \\ -\sin\alpha & 0 & \cos\alpha \end{bmatrix}$$

$$|F(\alpha)| = cos^2\alpha - 0 + sin^2\alpha = 1 \neq 0$$

$$adj\ F(\alpha) = \begin{bmatrix} \cos\alpha & 0 & -\sin\alpha \\ 0 & 1 & 0 \\ \sin\alpha & 0 & \cos\alpha \end{bmatrix}$$

$$[F(\alpha)]^{-1} = \frac{1}{|F(\alpha)|}adj\ F(\alpha) = \begin{bmatrix} \cos\alpha & 0 & -\sin\alpha \\ 0 & 1 & 0 \\ \sin\alpha & 0 & \cos\alpha \end{bmatrix} \ldots(1)$$

$$F(\alpha) = \begin{bmatrix} \cos\alpha & 0 & \sin\alpha \\ 0 & 1 & 0 \\ -\sin\alpha & 0 & \cos\alpha \end{bmatrix}$$

$$\Rightarrow F(-\alpha) = \begin{bmatrix} \cos\alpha & 0 & -\sin\alpha \\ 0 & 1 & 0 \\ \sin\alpha & 0 & \cos\alpha \end{bmatrix} \ldots(2)$$

From (1) and (2) $[F(\alpha)]^{-1} = F(-\alpha)$

4. If $A = \begin{bmatrix} 5 & 3 \\ -1 & -2 \end{bmatrix}$, show that $A^2 - 3A - 7\ I_2 = O_2$. Hence find A^{-1}.

Solution:

$$A^2 = A \cdot A = \begin{bmatrix} 5 & 3 \\ -1 & -2 \end{bmatrix}\begin{bmatrix} 5 & 3 \\ -1 & -2 \end{bmatrix} = \begin{bmatrix} 22 & 9 \\ -3 & 1 \end{bmatrix}$$

$$A^2 - 3A - 7I_2 = O_2$$

$$\Rightarrow \begin{bmatrix} 22 & 9 \\ -3 & 1 \end{bmatrix} + \begin{bmatrix} -15 & -9 \\ 3 & 6 \end{bmatrix} + \begin{bmatrix} -7 & 0 \\ 0 & -7 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$$

Post-multiplying this equation by A^{-1}

$$A - 3I_2 - 7A^{-1} = O_2$$

$$A^{-1} = \frac{1}{7}(A - 3\ I_2)$$

$$A^{-1} = \frac{1}{7}\left(\begin{bmatrix} 5 & 3 \\ -1 & -2 \end{bmatrix} - \begin{bmatrix} 3 & 0 \\ 0 & 3 \end{bmatrix}\right)$$

$$A^{-1} = \frac{1}{7}\begin{bmatrix} 2 & 3 \\ -1 & -5 \end{bmatrix}$$

14. If $A = \begin{bmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{bmatrix}$, show that

$A^{-1} = \frac{1}{2}(A^2 - 3I)$.

Solution:

$$|A| = 0 - 1(-1) + 1(1) = 2$$

$$adj\ A = \begin{bmatrix} -1 & 1 & 1 \\ 1 & -1 & 1 \\ 1 & 1 & -1 \end{bmatrix}$$

$$A^{-1} = \frac{1}{2}\begin{bmatrix} -1 & 1 & 1 \\ 1 & -1 & 1 \\ 1 & 1 & -1 \end{bmatrix} \text{.......................(1)}$$

$$A^2 = A\,A = \begin{bmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{bmatrix}\begin{bmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{bmatrix} = \begin{bmatrix} 2 & 1 & 1 \\ 1 & 2 & 1 \\ 1 & 1 & 2 \end{bmatrix}$$

$$\frac{1}{2}(A^2 - 3I) = \frac{1}{2}\left(\begin{bmatrix} 2 & 1 & 1 \\ 1 & 2 & 1 \\ 1 & 1 & 2 \end{bmatrix} - \begin{bmatrix} 3 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 3 \end{bmatrix}\right)$$

$$\frac{1}{2}(A^2 - 3I) = \frac{1}{2}\begin{bmatrix} -1 & 1 & 1 \\ 1 & -1 & 1 \\ 1 & 1 & -1 \end{bmatrix} \text{............(2)}$$

From (1) and (2), $A^{-1} = \frac{1}{2}(A^2 - 3I)$

Example 1.21 Find the inverse of $A = \begin{bmatrix} 2 & 1 & 1 \\ 3 & 2 & 1 \\ 2 & 1 & 2 \end{bmatrix}$ by Gauss-Jordan method.

Solution:

$$(A|I_3) = \left(\begin{array}{ccc|ccc} 2 & 1 & 1 & 1 & 0 & 0 \\ 3 & 2 & 1 & 0 & 1 & 0 \\ 2 & 1 & 2 & 0 & 0 & 1 \end{array}\right)$$

$$\sim \left(\begin{array}{ccc|ccc} 1 & 1 & 0 & -1 & 1 & 0 \\ 3 & 2 & 1 & 0 & 1 & 0 \\ 0 & 0 & 1 & -1 & 0 & 1 \end{array}\right) \begin{matrix} R_1 \to R_2 - R_1 \\ R_3 \to R_3 - R_1 \end{matrix}$$

$$\sim \left(\begin{array}{ccc|ccc} 1 & 1 & 0 & -1 & 1 & 0 \\ 0 & -1 & 1 & 3 & -2 & 0 \\ 0 & 0 & 1 & -1 & 0 & 1 \end{array}\right) R_2 \to R_2 - 3R_1$$

$$\sim \left(\begin{array}{ccc|ccc} 1 & 1 & 0 & -1 & 1 & 0 \\ 0 & -1 & 0 & 4 & -2 & -1 \\ 0 & 0 & 1 & -1 & 0 & 1 \end{array}\right) R_2 \to R_2 - R_3$$

$$(I_3|A^{-1}) = \left(\begin{array}{ccc|ccc} 1 & 0 & 0 & 3 & -1 & -1 \\ 0 & 1 & 0 & -4 & 2 & 1 \\ 0 & 0 & 1 & -1 & 0 & 1 \end{array}\right) \begin{matrix} R_1 \to R_1 + R_2 \\ R_2 \to -R_2 \end{matrix}$$

$$A^{-1} = \begin{bmatrix} 3 & -1 & -1 \\ -4 & 2 & 1 \\ -1 & 0 & 1 \end{bmatrix}$$

EXERCISE 1.2

3. Find the inverse of each of the following by Gauss-Jordan method:

(ii) $\begin{bmatrix} 1 & -1 & 0 \\ 1 & 0 & -1 \\ 6 & -2 & -3 \end{bmatrix}$ **(iii)** $\begin{bmatrix} 1 & 2 & 3 \\ 2 & 5 & 3 \\ 1 & 0 & 8 \end{bmatrix}$

Solution:

(ii) Let $A = \begin{bmatrix} 1 & -1 & 0 \\ 1 & 0 & -1 \\ 6 & -2 & -3 \end{bmatrix}$

$$[A|\,I_3] = \left[\begin{array}{ccc|ccc} 1 & -1 & 0 & 1 & 0 & 0 \\ 1 & 0 & -1 & 0 & 1 & 0 \\ 6 & -2 & -3 & 0 & 0 & 1 \end{array}\right]$$

$$\sim \left[\begin{array}{ccc|ccc} 1 & -1 & 0 & 1 & 0 & 0 \\ 0 & 1 & -1 & -1 & 1 & 0 \\ 0 & 4 & -3 & -6 & 0 & 1 \end{array}\right] \begin{matrix} R_2 \to R_2 - R_1 \\ R_3 \to R_3 - 6R_1 \end{matrix}$$

$$\sim \left[\begin{array}{ccc|ccc} 1 & -1 & 0 & 1 & 0 & 0 \\ 0 & 1 & -1 & -1 & 1 & 0 \\ 0 & 0 & 1 & -2 & -4 & 1 \end{array}\right] R_3 \to R_3 - 4R_2$$

$$\sim \left[\begin{array}{ccc|ccc} 1 & -1 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & -3 & -3 & 1 \\ 0 & 0 & 1 & -2 & -4 & 1 \end{array}\right] R_2 \to R_2 + R_3$$

$$(I_3|A^{-1}) = \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & -2 & -3 & 1 \\ 0 & 1 & 0 & -3 & -3 & 1 \\ 0 & 0 & 1 & -2 & -4 & 1 \end{array}\right] R_1 \to R_1 + R_2$$

$$A^{-1} = \begin{bmatrix} -2 & -3 & 1 \\ -3 & -3 & 1 \\ -2 & -4 & 1 \end{bmatrix}$$

Do it your self
Ex :1.1 - 11

5. The prices of three commodities A, B and C are $Rs. x, y$ and z per units respectively. A person P purchases 4 units of B and sells two units of A and 5 units of C. Person Q purchases 2 units of C and sells 3 units of A and one unit of B. Person R purchases one unit of A and sells 3 unit of B and one unit of C. In the process, P, Q and R earn $Rs.$ 15,000, $Rs.$ 1,000 and $Rs.$ 4,000 respectively. Find the prices per unit of A, B and C. (Use matrix inversion method to solve the problem.)

Solution:

Let the prices of one unit of A, B and C are Rs.x, y and z respectively.

From the given data,

$2x - 4y + 5z = 15000 \rightarrow (1)$

$3x + y - 2z = 1000 \rightarrow (2)$

$-x + 3y + z = 4000 \rightarrow (3)$

$$\begin{bmatrix} 2 & -4 & 5 \\ 3 & 1 & -2 \\ -1 & 3 & 1 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 15000 \\ 1000 \\ 4000 \end{bmatrix} \quad AX = B$$

$$A = \begin{bmatrix} 2 & -4 & 5 \\ 3 & 1 & -2 \\ -1 & 3 & 1 \end{bmatrix}, \quad X = \begin{bmatrix} x \\ y \\ z \end{bmatrix}, \quad B = \begin{bmatrix} 15000 \\ 1000 \\ 4000 \end{bmatrix}$$

$|A| = 2(1+6) - 4(3+2) + 5(9+1) = 68 \neq 0$

$$adjA = [A_{ij}]^T = \begin{bmatrix} 7 & 19 & 3 \\ -1 & 7 & 19 \\ 10 & -2 & 14 \end{bmatrix}$$

$$A^{-1} = \frac{1}{|A|}(adj\ A) = \frac{1}{68}\begin{bmatrix} 7 & 19 & 3 \\ -1 & 7 & 19 \\ 10 & -2 & 14 \end{bmatrix}$$

$$X = A^{-1}B \implies X = \frac{1}{68}\begin{bmatrix} 7 & 19 & 3 \\ -1 & 7 & 19 \\ 10 & -2 & 14 \end{bmatrix}\begin{bmatrix} 15000 \\ 1000 \\ 4000 \end{bmatrix}$$

$$X = \frac{1000}{68}\begin{bmatrix} 7 & 19 & 3 \\ -1 & 7 & 19 \\ 10 & -2 & 14 \end{bmatrix}\begin{bmatrix} 15 \\ 1 \\ 4 \end{bmatrix}$$

$$= \frac{1000}{68}\begin{bmatrix} 105+19+12 \\ -15+7+76 \\ 150-2+56 \end{bmatrix}$$

$$X = \frac{1000}{68}\begin{bmatrix} 136 \\ 68 \\ 204 \end{bmatrix} = \begin{bmatrix} 2000 \\ 1000 \\ 3000 \end{bmatrix}$$

Hence, $x = 2000$, $y = 1000$, $z = 3000$.

The price of one unit of A, B and C are Rs.2000, 1000 and 3000 respectively.

Example 1.25 Solve, by Cramer's rule, the system of equations $x_1 - x_2 = 3$,

$2x_1 + 3x_2 + 4x_3 = 17$, $x_2 + 2x_3 = 7$.

Solution:

$$\Delta = \begin{vmatrix} 1 & -1 & 0 \\ 2 & 3 & 4 \\ 0 & 1 & 2 \end{vmatrix}$$

$\Delta = 1(6-4) - (-1)(4-0) + 0 = 6$

$$\Delta_{x_1} = \begin{vmatrix} 3 & -1 & 0 \\ 17 & 3 & 4 \\ 7 & 1 & 2 \end{vmatrix}$$

$\Delta_{x_1} = 3(6-4) - (-1)(34-28) + 0 = 12$

$$\Delta_{x_2} = \begin{vmatrix} 1 & 3 & 0 \\ 2 & 17 & 4 \\ 0 & 7 & 2 \end{vmatrix}$$

$\Delta_{x_2} = 1(34-28) - 3(4-0) + 0 = -6$

$$\Delta_{x_3} = \begin{vmatrix} 1 & -1 & 3 \\ 2 & 3 & 17 \\ 0 & 1 & 7 \end{vmatrix}$$

$\Delta_{x_3} = 1(21-17) - (-1)(14-0) + 3(2-0) = 24$

B Cramer's rule, we get

$x_1 = \frac{\Delta_{x_1}}{\Delta}$	$x_2 = \frac{\Delta_{x_2}}{\Delta}$	$x_3 = \frac{\Delta_{x_3}}{\Delta}$
$= \frac{12}{6} = 2$	$= \frac{-6}{6} = -1$	$= \frac{24}{6} = 4$

So, the solution is ($x_1 = 2$, $x_2 = -1$, $x_3 = 4$)

To Achieve your Target Plan Well

4. A boy is walking along the path $y = ax^2 + bx + c$ through the points $(-6, 8), (-2, -12)$ and $(3, 8)$. He wants to meet his friend at $P(7, 60)$. Will he meet his friend? (Use Gaussian elimination method.)

Solution:

$$y = ax^2 + bx + c$$

The points $(-6,8)$

$\Rightarrow 8 = a(-6)^2 + b(-6) + c$

$\Rightarrow \quad 36a - 6b + c = 8 \ldots\ldots\ldots(1)$

The points $(-2,-12)$

$\Rightarrow -12 = a(-2)^2 + b(-2) + c$

$\Rightarrow \quad 4a - 2b + c = -12 \ldots\ldots(2)$

The points $(3,8)$

$\Rightarrow 8 = a(\ 3)^2 + b(3) + c$

$\Rightarrow \quad 9a + 3b + c = 8 \ldots\ldots\ldots(3)$

The matrix form of the system is AX=B, where

$$A = \begin{bmatrix} 36 & -6 & 1 \\ 4 & -2 & 1 \\ 9 & 3 & 1 \end{bmatrix}, \quad X = \begin{bmatrix} a \\ b \\ c \end{bmatrix}, \quad B = \begin{bmatrix} 8 \\ -12 \\ 8 \end{bmatrix}$$

Transforming the augmented matrix to echelon form, we get

$$[A, B] = \left[\begin{array}{ccc|c} 36 & -6 & 1 & 8 \\ 4 & -2 & 1 & -12 \\ 9 & 3 & 1 & 8 \end{array}\right]$$

$$\sim \left[\begin{array}{ccc|c} 36 & -6 & 1 & 8 \\ 0 & -12 & 8 & -116 \\ 0 & 18 & 3 & 24 \end{array}\right] \begin{matrix} R_2 \to 9R_2 - R_1 \\ R_3 \to 4R_3 - R_1 \end{matrix}$$

$$\sim \left[\begin{array}{ccc|c} 36 & -6 & 1 & 8 \\ 0 & 3 & -2 & 29 \\ 0 & 6 & 1 & 8 \end{array}\right] \begin{matrix} R_2 \to \frac{R_2}{-4} \\ R_3 \to \frac{R_3}{3} \end{matrix}$$

$$\sim \left[\begin{array}{ccc|c} 36 & -6 & 1 & 8 \\ 0 & 3 & -2 & 29 \\ 0 & 0 & 5 & -50 \end{array}\right] R_3 \to R_3 - 2R_1$$

$36a - 6b + c = 8 \ldots\ldots\ldots(1)$

$3b - 2c = 29 \quad \ldots\ldots \quad (2)$

$5c = -50 \ldots\ldots(3)$

From (3) $\quad c = -10$

From (2) $\quad b = \frac{29-20}{3} = 3$

From (1) $\quad a = \frac{8+10+18}{36} = 1$

So, the solution is $(a = 1, \ b = 3, \ c = -10)$

$y = ax^2 + bx + c \quad \Rightarrow \quad y = x^2 + 3x - 10$

$x = 7 \quad \Rightarrow \quad y = 49 + 21 - 10 = 60$

The point (7,60) satisfies the equation

$y = x^2 + 3x - 10$, hence the boy will meet friend at (7, 60).

Here $\Delta = \begin{vmatrix} 100 & 10 & 1 \\ 400 & 20 & 1 \\ 1600 & 40 & 1 \end{vmatrix} = 100\times 10\begin{vmatrix} 1 & 1 & 1 \\ 4 & 2 & 1 \\ 16 & 4 & 1 \end{vmatrix} = 1000\,\{1[2-4]-1[4-16]+1[16-32]\}$

$= 1000\,\{1[-2]-1[-12]+1[-16]\} = 1000\,\{-2+12-16\} = 1000\{-6\} = -6000 \neq 0$

$\Delta_1 = \begin{vmatrix} 8 & 10 & 1 \\ 16 & 20 & 1 \\ 22 & 40 & 1 \end{vmatrix} = 10\begin{vmatrix} 8 & 1 & 1 \\ 16 & 2 & 1 \\ 22 & 4 & 1 \end{vmatrix} = 10\,\{8[2-4]-1[16-22]+1[64-44]\}$

$= 10\,\{8[-2]-1[-6]+1[20]\} = 10\,\{-16+6+20\} = 10\{10\} = 100$

$\Delta_2 = \begin{vmatrix} 100 & 8 & 1 \\ 40 & 16 & 1 \\ 1600 & 22 & 1 \end{vmatrix} = 100\times 2\begin{vmatrix} 1 & 4 & 1 \\ 4 & 8 & 1 \\ 16 & 11 & 1 \end{vmatrix} = 200\,\{1[8-11]-4[4-16]+1[44-128]\}$

$= 200\,\{1[-3]-4[-12]+1[-84]\} = 200\,\{-3+48-84\} = 200\{-39\} = -7800$

$\Delta_3 = \begin{vmatrix} 100 & 10 & 8 \\ 400 & 20 & 16 \\ 1600 & 40 & 22 \end{vmatrix} = 100\times 10\times 2\begin{vmatrix} 1 & 1 & 4 \\ 4 & 2 & 8 \\ 16 & 4 & 11 \end{vmatrix} = 2000\,\{1[22-32]-1[44-128]+4[16-32]\}$

$= 2000\,\{1[-10]-1[-84]+4[-16]\} = 2000\,\{-10+84-64\} = 2000\{10\} = 20000$

By Cramer's rule, we get $a = \frac{\Delta_1}{\Delta} = \frac{100}{-6000} = -\frac{1}{60}$; $b = \frac{\Delta_2}{\Delta} = \frac{-7800}{-6000} = \frac{13}{10}$; $c = \frac{\Delta_3}{\Delta} = \frac{20000}{-6000} = -\frac{10}{3}$.

So the equation of the path is $y = -\frac{1}{60}x^2 + \frac{13}{10}x - \frac{10}{3}$.

When $x = 70$; $y = -\frac{1}{60}\times(70)^2 + \frac{13}{10}\times 70 - \frac{10}{3} = -\frac{1}{60}\times 4900 + 91 - \frac{10}{3}$

$= -\frac{490}{6} + 91 - \frac{10}{3} = \frac{-490-20}{6} + 91 = -85 + 91 = 6$

So the ball went by 6 metres high over the boundary line and it is impossible for a fielder standing even just before the boundary line to jump and catch the ball. Hence the ball went for a super six and the Chennai Super Kings won the match.

"*Your Positive Action Combined with Positive Thinking Result in sucuess*"

Let $x_4 = k$. Then from (3), $2x_3 - 3x_4 = 0 \Rightarrow 3x_3 - 3k = 0 \Rightarrow 2x_3 = 3k \Rightarrow x_3 = \frac{3k}{2}$

From (2), $2x_2 - x_3 - 2x_4 = 0 \Rightarrow 2x_2 - \frac{3k}{2} - 2k = 0 \Rightarrow 2x_2 = \frac{3k+4k}{2} \Rightarrow 2x_2 = \frac{7k}{2} \Rightarrow x_2 = \frac{7k}{4}$

From (3), $x_1 - x_3 + x_4 = 0 \Rightarrow x_1 - \frac{3k}{2} + k = 0 \Rightarrow x_1 = \frac{3k}{2} - k \Rightarrow x = \frac{3k-2k}{2} \Rightarrow x = \frac{k}{2}$

The solution is $(x_1, x_2, x_3, x_4) = \left(\frac{k}{2}, \frac{7k}{4}, \frac{3k}{2}, k\right), k \in R$.

If $k = 4$, then $(x_1, x_2, x_3, x_4) = (2, 7, 6, 4)$

So the balanced equation is $2C_2H_6 + 7O_2 \rightarrow 6H_2O + 4CO_2$.

COMPLEX NUMBERS

Example 2.8 (*i*) Show that

$(2+i\sqrt{3})^{10}+(2-i\sqrt{3})^{10}$ is real and (*ii*)

$\left(\frac{19+9i}{5-3i}\right)^{15}-\left(\frac{8+i}{1+2i}\right)^{15}$ is purely imaginary.

Solution:

Let $Z=(2+i\sqrt{3})^{10}+(2-i\sqrt{3})^{10}$

Then $\bar{Z}=\overline{(2+i\sqrt{3})^{10}+(2-i\sqrt{3})^{10}}$

$=\overline{(2+i\sqrt{3})^{10}}+\overline{(2-i\sqrt{3})^{10}}$

$=\left(\overline{2+i\sqrt{3}}\right)^{10}+\left(\overline{2-i\sqrt{3}}\right)^{10}$

$=(2-i\sqrt{3})^{10}+(2+i\sqrt{3})^{10}$

$=Z$, so

$(2+i\sqrt{3})^{10}+(2-i\sqrt{3})^{10}$ is real.

(*ii*) Let $Z=\left(\frac{19+9i}{5-3i}\right)^{15}-\left(\frac{8+i}{1+2i}\right)^{15}$

$\frac{19+9i}{5-3i}=\frac{19+9i}{5-3i}\times\frac{5+3i}{5+3i}=\frac{95+57i+45i+27i^2}{25+9}$

$=\frac{95+102i+27(-1)}{34}=\frac{95+102i-27}{34}$

$=\frac{68+102i}{34}=\frac{34(2+3i)}{34}=2+3i$

$\frac{8+i}{1+2i}=\frac{8+i}{1+2i}\times\frac{1-2i}{1-2i}=\frac{8-16i+i-2i^2}{1+4}$

$=\frac{8-15i-2(-1)}{5}=\frac{8-15i+2}{5}$

$=\frac{10-15i}{5}=\frac{5(2-3i)}{5}=2-3i$

So $Z=\left(\frac{19+9i}{5-3i}\right)^{15}-\left(\frac{8+i}{1+2i}\right)^{15}$ gives

$Z=(2+3i)^{15}-(2-3i)^{15}$

So, $\bar{Z}=\overline{(2+3i)^{15}-(2-3i)^{15}}$

$=\overline{(2+3i)^{15}}-\overline{(2-3i)^{15}}$

$=(\overline{2+3i})^{15}-(\overline{2-3i})^{15}$

$=(2-3i)^{15}-(2+3i)^{15}$

$=-[(2+3i)^{15}-(2-3i)^{15}]$

$\bar{Z}=-Z$

$Z=\left(\frac{19+9i}{5-3i}\right)^{15}-\left(\frac{8+i}{1+2i}\right)^{15}$

is purely imaginary.

4. The complex numbers u, v, and w are related by $\frac{1}{u}=\frac{1}{v}+\frac{1}{w}$.
If $v=3-4i$ and $w=4+3i$, find u in rectangular form.

Solution: $\frac{1}{u}=\frac{1}{v}+\frac{1}{w}=\frac{1}{3-4i}+\frac{1}{4+3i}$

$=\frac{4+3i+3-4i}{(3-4i)(4+3i)}=\frac{7-i}{12+9i-16i-12i^2}$

$=\frac{7-i}{12-7i-12(-1)}=\frac{7-i}{12-7i+12}=\frac{7-i}{24-7i}$

$u=\frac{24-7i}{7-i}\times\frac{7+i}{7+i}=\frac{168+24i-49i-7i^2}{49+1}$

$=\frac{168-25i-7(-1)}{50}=\frac{168-25i+7}{50}$

$=\frac{175-25i}{50}=\frac{25(7-i)}{50}=\frac{(7-i)}{2}$

$u=\frac{1}{2}(7-i)$

Example 2.11 Which one of the points $i, -2+i$, and 3 is farthest from the origin?

Solution:

The distance from the origin to

$Z=i, -2+i$, and 3 are

$|Z|=|i|=\sqrt{1^2}=\sqrt{1}=1$

$|Z|=|-2+i|=\sqrt{(-2)^2+1^2}$

$=\sqrt{4+1}=\sqrt{5}$

$|Z|=|3|=3$

The farthest point from the origin is 3

Example 2.13 If $|Z|=2$ show that $3\le|z+3+4i|\le 7$

Solution:

We know that $|Z_1+Z_2|\le|Z_1|+|Z_2|$

Hence, $|Z+(3+4i)|\le|Z|+|3+4i|$

$\le 2+|3+4i|$

$\le 2+\sqrt{3^2+4^2}$

$\le 2+\sqrt{9+16}$

$\le 2+\sqrt{25}$

$\le 2+5$

≤ 7 ....(1)

5. If $\cos\alpha+\cos\beta+\cos\gamma=0=\sin\alpha+\sin\beta+\sin\gamma$, prove that

(i) $\cos 3\alpha+\cos 3\beta+\cos 3\gamma=3\cos(\alpha+\beta+\gamma)$

(ii) $\sin 3\alpha+\sin 3\beta+\sin 3\gamma=3\sin(\alpha+\beta+\gamma)$

Solution: Given

$\cos\alpha+\cos\beta+\cos\gamma=0=\sin\alpha+\sin\beta+\sin\gamma$

$(\cos\alpha+\cos\beta+\cos\gamma)+i(\sin\alpha+\sin\beta+\sin\gamma)=0$

$(\cos\alpha+i\sin\alpha)+(\cos\beta+i\sin\beta)+(\cos\gamma+i\sin\gamma)=0$

If $a+b+c=0$, then $a^3+b^3+c^3=3abc$

where $a=\cos\alpha+i\sin\alpha$, $b=\cos\beta+i\sin\beta$ and $c=\cos\gamma+i\sin\gamma$

$\therefore (\cos\alpha+i\sin\alpha)^3+(\cos\beta+i\sin\beta)^3+(\cos\gamma+i\sin\gamma)^3$

$=3(\cos\alpha+i\sin\alpha)(\cos\alpha+i\sin\alpha)(\cos\alpha+i\sin\alpha)$

$(\cos 3\alpha+i\sin 3\alpha)+(\cos 3\beta+i\sin 3\beta)+(\cos 3\gamma+i\sin 3\gamma)$

$=3[\cos(\alpha+\beta+\gamma)+i\sin(\alpha+\beta+\gamma)]$

$(\cos 3\alpha+\cos 3\beta+\cos 3\gamma)+i(\sin 3\alpha+\sin 3\beta+\sin 3\gamma)$

$=[3\cos(\alpha+\beta+\gamma)+i3\sin(\alpha+\beta+\gamma)]$

Equating real and imaginary parts,

(i) $\cos 3\alpha+\cos 3\beta+\cos 3\gamma=3\cos(\alpha+\beta+\gamma)$

(ii) $\sin 3\alpha+\sin 3\beta+\sin 3\gamma=3\sin(\alpha+\beta+\gamma)$

is proved.

6. If $Z=x+iy$ and $\arg\left(\frac{z-i}{z+2}\right)=\frac{\pi}{4}$, then show that $x^2+y^2+3x-3y+2=0$.

Solution: Given $Z=x+iy$

$\frac{z-i}{z+2}=\frac{x+iy-i}{x+iy+2}=\frac{x+(y-1)i}{(x+2)+iy}$

$\arg(Z)=\frac{bc-ad}{ac+bd}$

$a=x$, $b=y-1$, $c=x+2$ and $d=y$

$\arg\left(\frac{z-1}{z+1}\right)=\frac{\pi}{4}$ $\quad \frac{bc-ad}{ac+bd}=\tan\left(\frac{\pi}{4}\right)$

$\frac{bc-ad}{ac+bd}=1$ $\quad bc-ad=ac+bd$

$(y-1)(x+2)-xy=x(x+2)+(y-1)y$

$xy+2y-x-2-xy=x^2+2x+y^2-y$

$2y-x-2=x^2+2x+y^2-y$

$x^2+2x+y^2-y-2y+x+2=0$

$x^2+y^2+3x-3y+2=0$, is proved.

If $Z=(\cos\theta+i\sin\theta)$, show that

(i) $z^n+\frac{1}{z^n}=2\cos\theta$ (ii) $z^n-\frac{1}{z^n}=2i\sin\theta$.

Solution: Given $Z=(\cos\theta+i\sin\theta)$

$\therefore Z^n=(\cos\theta+i\sin\theta)^n$

$=(\cos n\theta+i\sin n\theta)$

$\frac{1}{z^n}=z^{-n}=(\cos n\theta-i\sin n\theta)$

(i) $z^n+\frac{1}{z^n}$

$=(\cos n\theta+i\sin n\theta)+(\cos n\theta-i\sin n\theta)$

$=\cos n\theta+i\sin n\theta+\cos n\theta-i\sin n\theta$

$=2\cos n\theta$

(ii) $z^n-\frac{1}{z^n}$

$=(\cos n\theta+i\sin n\theta)-(\cos n\theta-i\sin n\theta)$

$=\cos n\theta+i\sin n\theta-\cos n\theta+i\sin n\theta$

$=2i\sin n\theta$

Example 2.30 Simplify $\left(\frac{1+\cos 2\theta+i\sin 2\theta}{1+\cos 2\theta-i\sin 2\theta}\right)^{30}$

Solution: Let $(\cos 2\theta+i\sin 2\theta)=Z$

$\therefore \frac{1}{z}=(\cos 2\theta-i\sin 2\theta)$

$\because \left(\frac{1+\cos 2\theta+i\sin 2\theta}{1+\cos 2\theta-i\sin 2\theta}\right)=\left(\frac{1+Z}{1+\frac{1}{Z}}\right)$

$=\left(\frac{1+Z}{\frac{Z+1}{Z}}\right)=\left(\frac{1+Z}{1+Z}\times z\right)=Z$

$\left(\frac{1+\cos 2\theta+i\sin 2\theta}{1+\cos 2\theta-i\sin 2\theta}\right)^{30}=z^{30}$

$=(\cos 2\theta+i\sin 2\theta)^{30}$

$=(\cos 60\theta+i\sin 60\theta)$

Example 2.32 Find the cube roots of unity.

Solution: Let $x=(1)^{\frac{1}{3}}=(\cos 0+i\sin 0)^{\frac{1}{3}}$

$=(\cos 2k\pi+i\sin 2k\pi)^{\frac{1}{3}}$

$=\left(\cos\frac{1}{3}(2k\pi)+i\sin\frac{1}{3}(2k\pi)\right)$

Substituting k=0, 1, 2 we get 3 values $cis\, 0$, $cis\frac{2\pi}{3}$, $cis\frac{4\pi}{3}$ ....

Example 3.6 Form the equation whose roots are the squares of the roots of the cubic equation $\boldsymbol{x^3 + ax^2 + bx + c = 0}$.

Solution:8

Since α , β , and γ are the roots of the equation $x^3 + ax^2 + bx + c = 0$

$\sum_1 = \alpha + \beta + \gamma = -a \qquad \sum_2 = \alpha\beta + \beta\gamma + \gamma\alpha = b \qquad \sum_3 = \alpha\beta\gamma = -c$

We have to form the equation whose roots are α^2, β^2 and γ^2

$$\sum_1 = \alpha^2 + \beta^2 + \gamma^2 = (\alpha + \beta + \gamma)^2 - 2(\alpha\beta + \beta\gamma + \gamma\alpha)$$

$$= (-a)^2 - 2b = a^2 - 2b$$

$$\sum_2 = \alpha^2\beta^2 + \beta^2\gamma^2 + \gamma^2\alpha^2 = (\alpha\beta + \beta\gamma + \gamma\alpha)^2 - 2[(\alpha\beta)(\beta\gamma) + (\beta\gamma)(\gamma\alpha) + (\gamma\alpha)(\alpha\beta)]$$

$$= b^2 - 2(-c)(-a) = b^2 - 2ac$$

$$\sum_3 = \alpha^2\beta^2\gamma^2 = (\alpha\beta\gamma)^2 = c^2$$

Hence, the required equation is

$$x^3 - (\alpha^2 + \beta^2 + \gamma^2)x^2 + (\alpha^2\beta^2 + \beta^2\gamma^2 + \gamma^2\alpha^2)x - \alpha^2\beta^2\gamma^2 = 0$$

$$x^3 - (a^2 - 2b)x^2 + (b^2 - 2ac)x - c^2 = 0$$

EXERCISE 3.1

4. Solve the equation $\boldsymbol{3x^3 - 16x^2 + 23x - 6 = 0}$. If the product of two roots is 1.

Solution:

Let α, β and γ are the roots of the cubic equation

$3x^3 - 16x^2 + 23x - 6 = 0 \qquad (3 \div) \Rightarrow \quad x^3 - \frac{16}{3}x^2 + \frac{23}{3}x - 2 = 0$

The product of two roots is 1. $\quad \alpha\beta = 1 \Rightarrow \quad \beta = \frac{1}{\alpha}$

$\sum_3 = \alpha\beta\gamma$ $= -(-2) = 2$	$\sum_2 = \alpha\beta + \beta\gamma + \gamma\alpha = \frac{23}{3}$	$\sum_1 = \alpha + \beta + \gamma = -\left(-\frac{16}{3}\right) = \frac{16}{3}$
$\alpha\beta\gamma = 2$ $\Rightarrow \alpha\left(\frac{1}{\alpha}\right)\gamma = 2$ $\Rightarrow \quad \gamma = 2$	$\alpha + \beta + \gamma = \frac{16}{3}$ $\alpha + \frac{1}{\alpha} = \frac{16}{3} - 2 = \frac{10}{3}$ $\frac{\alpha^2 + 1}{\alpha} = \frac{10}{3}$ $3\alpha^2 - 10\alpha + 3 = 0$ $(\alpha - 3)(3\alpha - 1) = 0$ $\alpha = \frac{1}{3}$ and $\alpha = 3$	When $\alpha = \frac{1}{3} \quad \beta = \frac{1}{\alpha} = 3$ When $\alpha = 3 \quad \beta = \frac{1}{\alpha} = \frac{1}{3}$ Do it your self Ex : 3.2 - 4

Thus the roots are 3, $\frac{1}{3}$ and 2 (or) $\frac{1}{3}$, 3, 2.

$$x(x-8)+7(x-8)=0$$
$$(x-8)(x+7)=0$$
$$x-8=0 \text{ (or) } x+7=0$$
$$x=8 \text{ (or) } x=-7$$

Therefore the roots of the given equation are : 3, −2, 8, −7.

(ii) The given polynomial equation is

$$(x-4)(x-7)(x-2)(x+1)=16$$
$$(x-4)(x-2)(x-7)(x+1)=16$$
$$(x^2-2x-4x+8)(x^2+x-7x-7)=16$$
$$(x^2-6x+8)(x^2-6x-7)=16$$

By taking $y=x^2-6x$, the above equation becomes,

$$(y+8)(y-7)=16$$
$$y^2-7y+8y-56-16=0$$
$$y^2+y-72=0$$
$$y^2-8y+9y-72=0$$
$$y(y-8)+9(y-8)=0$$
$$(y-8)(y+9)=0$$
$$y-8=0 \text{ (or) } y+9=0$$
$$y=8 \text{ (or) } y=-9$$

Substituting the values of y in $y=x^2-6x$, we get,

$$x^2-6x=8 \text{ (or) } x^2-6x=-9$$
$$x^2-6x-8=0 \text{ (or) } x^2-6x+9=0$$

From $x^2-6x-8=0$

$$x=\frac{-(-6)\pm\sqrt{(-6)^2-4(1)(-8)}}{2(1)}$$
$$=\frac{6\pm\sqrt{68}}{2}=\frac{6\pm2\sqrt{17}}{2}=\frac{2(3\pm\sqrt{17})}{2}$$
$$x=3\pm\sqrt{17}$$

From $x^2-6x+9=0$

$$x^2-3x-3x+9=0$$
$$x(x-3)-3(x-3)=0$$
$$(x-3)(x-3)=0$$
$$x-3=0 \text{ (or) } x-3=0$$
$$x=3 \text{ (or) } x=3$$

Therefore the roots of the given equation are :3, 3, $3+\sqrt{17}$, $3-\sqrt{17}$.

2. Solve : $(2x-1)(x+3)(x-2)(2x+3)+20=0$

Solution :

The given polynomial equation is

$$(2x-1)(x+3)(x-2)(2x+3)+20=0$$
$$(2x-1)(2x+3)(x+3)(x-2)+20=0$$
$$(4x^2+6x-2x-3)(x^2-2x+3x-6)+20=0$$
$$(4x^2+4x-3)(x^2+x-6)+20=0$$
$$(4(x^2+x)-3)(x^2+x-6)+20=0$$

By taking $y=x^2+x$, the above equation becomes,

$$(4y-3)(y-6)+20=0$$
$$4y^2-24y-3y+18+20=0$$
$$4y^2-27y+38=0$$
$$4y^2-8y-19y+38=0$$
$$4y(y-2)-19(y-2)=0$$
$$(y-2)(4y-19)=0$$
$$y-2=0 \text{ (or) } 4y-19=0$$
$$y=2 \text{ (or) } y=\frac{19}{4}$$

Substituting the values of y in $y=x^2+x$, we get,

$$x^2+x=2 \text{ (or) } x^2+x=\frac{19}{4}$$

Example 5.2 Find the equation of the circle described on the chord $3x + y + 5 = 0$ of the circle $x^2 + y^2 = 16$ as diameter.

Solution:

Equation of the circle passing through the points of intersection of the chord

$$x^2 + y^2 - 16 + \lambda(3x + y + 5) = 0$$
$$x^2 + y^2 + 3\lambda x + \lambda y + 5\lambda - 16 = 0$$

Centre $(-g, -f) = \left(\frac{-3\lambda}{2}, \frac{-\lambda}{2}\right)$ lies on the chord.

$$3\left(\frac{-3\lambda}{2}\right) - \frac{\lambda}{2} + 5 = 0$$
$$\frac{-9\lambda - \lambda}{2} + 5 = 0$$

$-10\lambda = -10$ $\qquad \lambda = 1$

The equation of the circle is

$x^2 + y^2 + 3x + y - 11 = 0.$

Example 5.10 Find the equation of the circle passing through the points (1,1), (2,-1), and (3,2).

Solution:

Let the general equation of the circle be

$x^2 + y^2 + 2gx + 2fy + c = 0 \ldots\ldots\ldots (1)$

It passes through (1, 1)

$1^2 + 1^2 + 2g(1) + 2f(1) + c = 0$

$$2g + 2f + c + 2 = 0 \ldots\ldots (2)$$

It passes through (2, -1)

$$2^2 + (-1)^2 + 2g(2) + 2f(-1) + c = 0$$
$$4g - 2f + c + 5 = 0 \ldots\ldots (3)$$

It passes through (3, 2)

$3^2 + 2^2 + 2g(3) + 2f(2) + c = 0$

$$6g + 4f + c + 13 = 0 \ldots\ldots (4)$$

(2) – (3) gives $-2g + 4f - 3 = 0 \ldots (5)$

(4) – (3) gives $2g + 6f + 8 = 0 \ldots (6)$

(5) + (6) gives $f = -\frac{1}{2}$

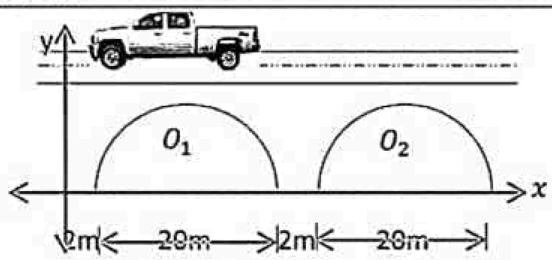
Substituting $f = -\frac{1}{2}$ in eqn(6) $g = -\frac{5}{2}$

Substituting $f = -\frac{1}{2}$ and $g = -\frac{5}{2}$ in eqn(2), $c = 4$

The required equation is

$$x^2 + y^2 + 2gx + 2fy + c = 0$$
$$x^2 + y^2 + 2\left(-\frac{5}{2}\right)x + 2\left(-\frac{1}{2}\right)y + 4 = 0$$
$$x^2 + y^2 - 5x - y + 4 = 0$$

Example 5.13 A road bridge over an irrigation canal have two semi-circular vents each with a span of 20m and the supporting pillars of width 2m. Use to write the equations that model the arches.



Solution:

Let O_1, O_2 be the centres of the two semi circular vents.

First vent with centre $O_1(12, 0)$ and r =10 yields equation to first semicircle as

$$(x - 12)^2 + (y - 0)^2 = 10^2$$
$$x^2 - 24x + 144 + y^2 = 100$$
$$x^2 + y^2 - 24x + 44 = 0$$

Second vent with centre $O_2(34, 0)$and r =10 yields equation to second vent as

$$(x - 34)^2 + (y - 0)^2 = 10^2$$
$$x^2 - 68x + 11156 + y^2 = 100$$
$$x^2 + y^2 - 68x + 1056 = 0$$

EXERCISE 5.1

4. Find the equation of the circle with centre (2, 3) and passing through the intersection of the lines $3x - 2y - 1 = 0$ and $4x + y - 27 = 0$.

Solution:

Given: Centre $(h, h) = (2, 3)$

The required equation is, $(x - h)^2 + (y - k)^2 = r^2$

$$(x - 2)^2 + (y - 3)^2 = r^2$$

Point of intersection:

$3x - 2y - 1 = 0 \qquad 3x - 2y - 1 = 0$

$4x + y - 27 = 0 \qquad 8x + 2y - 54 = 0$

$11x - 55 = 0 \qquad x = 5$

Substitution $x = 5$ in eqn(1) $3(5) - 2y - 1 = 0$

$-2y = -14 \qquad y = 7$

Point of intersection $(x, y) = (5, 7)$

The required equation passes through (5, 7)

$(5 - 2)^2 + (7 - 3)^2 = r^2 \qquad r = 5$

The equation of the circle is $(x - 2)^2 + (y - 3)^2 = 5^2$

$$x^2 + y^2 - 4x - 6y + 13 = 25$$
$$x^2 + y^2 - 4x - 6y - 12 = 0$$

6. Find the equation of the circle through the points (1,0), (-1,0) and (0,1).

Solution:

Given: centre (0,0) radius $r = 1$

The required equation is

$(x-0)^2 + (y-0)^2 = 1^2 \qquad x^2 + y^2 = 1$

Example 5.17 Find the vertex, focus, directrix, and length of the lactus rectum of the parabola $x^2 - 4x - 5y - 1 = 0$.

Solution:

The parabola $x^2 - 4x - 5y - 1 = 0$

$x^2 - 4x = 5y + 1 \qquad (x-2)^2 = 5y + 1 + (2)^2$

$(x-2)^2 = 5(y+1)$ is open upwards.

	X, Y	$x = X + 2$, $y = Y - 1$
Vertex (0,0)	(0, 0)	(2, -1)
Focus $(0, a)$	$\left(0, \frac{5}{4}\right)$	$\left(2, \frac{1}{4}\right)$
Directrix $y = -a$	$Y = -\frac{5}{4}$	$y = \frac{-5}{4} - 1 = \frac{-9}{4}$
Length of lactus rectum	$4a = 5$	$4a = 5$

Example 5.19 Find the equation of the ellipse whose eccentricity is $\frac{1}{2}$, one of the foci is $(2, 3)$ and a directrix is $x = 7$. Also find the length of the major and minor axes of the ellipse.

Solution:

$\frac{FM}{PM} = e \qquad FM^2 = PM^2 e^2$

$$(x - ae)^2 + (y-0)^2 = e^2\left[\left(x - \frac{a}{e}\right)^2 + (0-0)^2\right]$$

Foci $(ae, 0) = (2, 3)$ $\quad e = \frac{1}{2}$ $\quad$ Directrix $x = \frac{a}{e} = 7$

$$(x-2)^2 + (y-3)^2 = \left(\frac{1}{2}\right)^2 (x-7)^2$$

$$3x^2 - 2x + 4y^2 - 24y + 3 = 0$$

$$3\left(x^2 - \frac{2}{3}x\right) + 4(y^2 - 6y) = -3$$

$$3\left(x - \frac{1}{3}\right)^2 + 4(y-3)^2 = 3\left(\frac{1}{9}\right) + 4(9) - 3$$

$$3\left(x - \frac{1}{3}\right)^2 + 4(y-3)^2 = \frac{100}{3}$$

$$\frac{\left(x - \frac{1}{3}\right)^2}{\frac{100}{9}} + \frac{(y-3)^2}{\frac{100}{12}} = 1$$

Therefore, the length of major axis = 2a

$$= 2\sqrt{\frac{100}{9}} = \frac{20}{3}$$

Excercise	Examples
5.4 -3,4,5,6	5.23,5.29, 5.31,5.33, 5.22,5.26

8. Identify the type of conic and find centre, foci, vertices, and directrices of each of the following :

(i) $\frac{(x-3)^2}{225}+\frac{(y-4)^2}{289}=1$

$$\frac{(x-3)^2}{225}+\frac{(y-4)^2}{289}=1$$

$$\Rightarrow \quad \frac{X^2}{225}+\frac{Y^2}{289}=1$$

It is an ellipse. Type II. Major axis : Y- axis

$a^2=289 \quad b^2=225$ $a=17$		$X=x-3$ $\Rightarrow x=X+3$ $Y=y-4$ $\Rightarrow y=Y+4$
$e=\sqrt{\frac{a^2-b^2}{a^2}}$ $=\sqrt{\frac{289-225}{289}}$ $=\frac{8}{17}$	ae $=17\left(\frac{8}{17}\right)$ $=8$	

ELLIPSE	Referred to X , Y	Referred to x, y	
		$x=X+3$	$y=Y+4$
Centre	$C(0,0)$	$C(3,4)$	
Foci $(0,\ \pm ae)$	$(0,8)$	$F_1(3,12)$	
	$(0,-8)$	$F_2(3,-4)$	
Vertices $(0, \pm a)$	$(0,17)$	$A(3,21)$	
	$(0,-17)$	$A'(3,-13)$	
Directrices $Y=\pm\frac{a}{e}$	$Y=\pm\frac{289}{8}$	$y=\pm\frac{289}{8}+4$	

(ii) $\frac{(x+1)^2}{100}+\frac{(y-2)^2}{64}=1$

$$\frac{(x+1)^2}{100}+\frac{(y-2)^2}{64}=1 \quad \Rightarrow \quad \frac{X^2}{100}+\frac{Y^2}{64}=1$$

It is an ellipse. Type I. Major axis : X- axis

$a^2=100 \quad a=10$ $b^2=64$		$\frac{a}{e}$ $=\frac{10}{6/10}$ $=\frac{100}{6}$ $\frac{a}{e}=\frac{50}{3}$	$X=x+1$ $\Rightarrow x=X-1$ $Y=y-2$ $\Rightarrow y=Y+2$
$e=\sqrt{\frac{a^2-b^2}{a^2}}$ $=\sqrt{\frac{100-64}{100}}$ $e=\frac{6}{10}=\frac{3}{5}$	ae $=10\left(\frac{6}{10}\right)$ $ae=6$		

ELLIPSE	Referred to X , Y	Referred to x, y	
		$x=X-1$	$y=Y+2$
Centre	$C(0,0)$	$C(-1,2)$	
Foci $(\pm ae,\ 0)$	$(6,0)$	$F_1(5,2)$	
	$(-6,0)$	$F_2(-7,2)$	
Vertices $(\pm a,0)$	$(10,0)$	$A(9,2)$	
	$(-10,0)$	$A'(-11,2)$	
Directrices $X=\pm\frac{a}{e}$	$X=\frac{50}{3}$	$x=\frac{50}{3}-1=\frac{47}{3}$	
	$X=\frac{-50}{3}$	$x=\frac{-50}{3}-1=\frac{-53}{3}$	

Life is Like Riding a Bicycle to Keep your Balance, you Must Keep Moving

Example 5.30 A semielliptical archway over a one-way road has a height of 3m and a width of 12m. The truck has a width of 3m and a height of 27.m. Will the truck clear the opening of the archway?

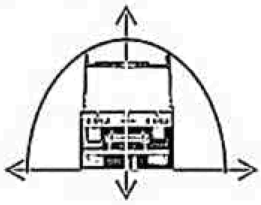
Solution:

Width of the road is 12. $a = 6$

The height of the truck is 3. $b = 3$

From the given data, the ellipse is $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$

$$\frac{(1.5)^2}{36} + \frac{y^2}{9} = 1$$



$$y^2 = 9\left(1 - \frac{3^2}{144}\right)$$

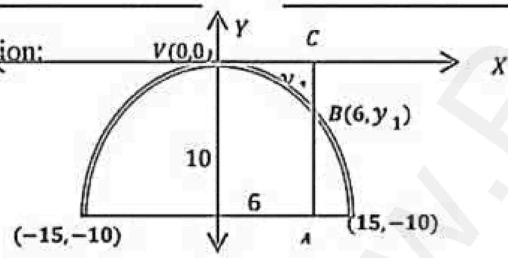
$$y^2 = \frac{9}{144}(135) = \frac{135}{16} = 2.90$$

Thus the height of arch way 15.m from the centre is approximately 2.90m . Since the truck's height is 2.7 m, the truck will clear the archway.

EXERCISE 5.5

1. A bridge has a parabolic arch that is 10m high in the centre and 30m wide at the bottom. Find the height of the arch 6m from the centre, on either sides.

Solution:



Let the equation of the parabola be $x^2 = -4ay$.

$B(15, -10)$ is a point on the parabola $x^2 = -4ay$.

$15^2 = -4a(-10)$ $\quad 4a = \frac{225}{10}$

The parabola is $x^2 = -\frac{225}{10}y$.

$(6, y_1)$ lies on the parabola $\quad 36 = -\frac{225}{10}y_1$

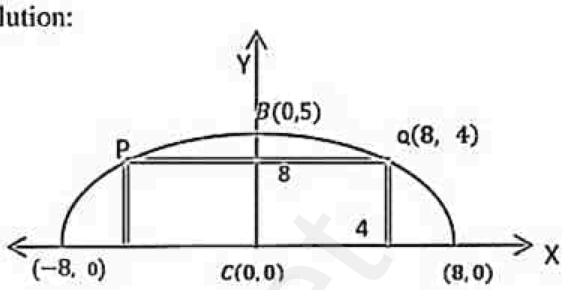
$y_1 = \frac{-360}{225} = -1.6$

Height of the arch 6m from the centre is

CD=CE-DE=10-1.6=8.4m

2. A tunnel through a mountain for a four lane highway is to have a elliptical opening. The total width of the highway (not the opening) is to be 16m, and the height at the edge of the road must be sufficient for a truck 4m high to clear if the highest point of the opening is to be 5m approximately . How wide must the opening be?

Solution:



Let the equation of the ellipse be $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$

$b = 5$

(8, 4) lies on the ellipse. $\frac{8^2}{a^2} + \frac{4^2}{5^2} = 1$

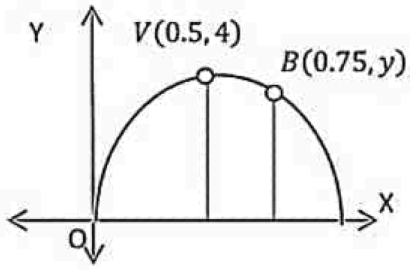
$$\frac{64}{a^2} = 1 - \frac{16}{25}$$

$a^2 = \frac{25}{9}(64)$ $\qquad a = \frac{40}{3}$

Width $2a = 2\left(\frac{40}{3}\right) = \frac{80}{3} = 26.66m$

3. At a water fountain, water attains a maximum height of 4m at horizontal distance of 0.5m from its origin. If the path of water is a parabola, find the height of water at a horizontal distance of 0.75m from the point of origin.

Solution:



From the data the parabola is open downwards.

$$(x-h)^2 = -4a(y-k)$$

Vertex (0.5, 4) $\quad (x-0.5)^2 = -4a(y-4)$

It passes through (0,0) $\quad 4a = \frac{0.25}{4}$

$$(x-0.5)^2 = -\frac{0.25}{4}(y-4)$$

(0.75, y) lies on the parabola

$$(0.75-0.5)^2 = -\frac{0.25}{4}(y-4)$$

$$(0.25)^2 = -\frac{0.25}{4}(y-4)$$

$0.25 \times (-4) = y - 4$

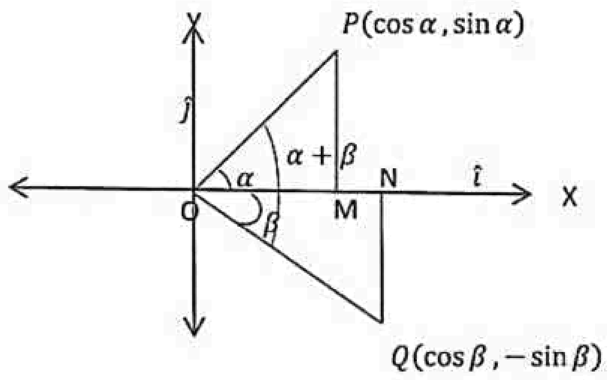
$-1 = y - 4 \qquad y = 3$

The required height is 3mm.

Example 6.3 By vector method, prove that

$\cos(\alpha+\beta) = \cos\alpha\cos\beta - \sin\alpha\sin\beta$.

Proof:



Take the unit vectors $\hat{a} = \overrightarrow{OP}$ *and* $\hat{b} = \overrightarrow{OQ}$ which make angles α and β respectively, with positive x – axis. Draw MP and QN perpendicular to the x -axis.

$\lfloor XOP = \alpha$, $\lfloor XOQ = \beta$, $\lfloor POQ = \alpha+\beta$

Take the unit vectors $\hat{\imath}$ *and* $\hat{\jmath}$ along the X and Y axes.

$$\overrightarrow{OP} = \cos\alpha\,\hat{\imath} + \sin\alpha\,\hat{\jmath}$$

$$\overrightarrow{OQ} = \cos\beta\,\hat{\imath} - \sin\beta\,\hat{\jmath}$$

$\overrightarrow{OQ}\cdot\overrightarrow{OP} = \cos\alpha\cos\beta - \sin\alpha\sin\beta$……..(1)

By the definition,

$\overrightarrow{OQ}\cdot\overrightarrow{OP} = |\overrightarrow{OQ}|\ |\overrightarrow{OP}|\cos(\alpha+\beta)$

$\overrightarrow{OQ}\cdot\overrightarrow{OP} = \cos(\alpha+\beta)$ …..(2)

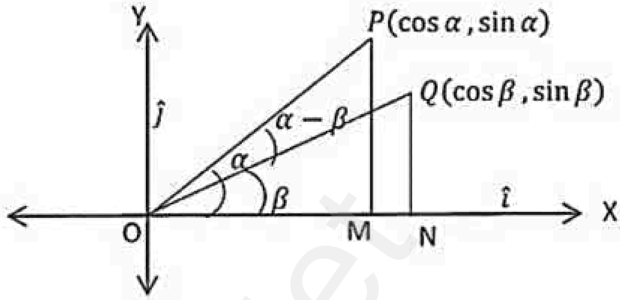
From (1) and (2),

$$\cos(\alpha+\beta) = \cos\alpha\cos\beta - \sin\alpha\sin\beta$$

Example 6.5 Prove by vector method that

$\sin(\alpha-\beta) = \sin\alpha\cos\beta - \cos\alpha\sin\beta$**.**

Proof:



Take the unit vectors $\hat{a} = \overrightarrow{OP}$ *and* $\hat{b} = \overrightarrow{OQ}$ which make angles α and β respectively, with positive x – axis.

Draw MP and QN perpendicular to the x -axis.

$\lfloor XOP = \alpha$, $\lfloor XOQ = \beta$, $\lfloor POQ = \alpha-\beta$

$$\overrightarrow{OP} = \cos\alpha\,\hat{\imath} + \sin\alpha\,\hat{\jmath}$$

$$\overrightarrow{OQ} = \cos\beta\,\hat{\imath} + \sin\beta\,\hat{\jmath}$$

Take the unit vectors $\hat{\imath}$ *and* $\hat{\jmath}$ along the X and Y axes.

$$\overrightarrow{OQ}\times\overrightarrow{OP} = \begin{vmatrix} \hat{\imath} & \hat{\jmath} & \hat{k} \\ \cos\beta & \sin\beta & 0 \\ \cos\alpha & \sin\alpha & 0 \end{vmatrix}$$

$\overrightarrow{OQ}\times\overrightarrow{OP} = \hat{k}\,(\sin\alpha\cos\beta - \cos\alpha\ \sin\beta)$ ……..(1)

By the definition,

$\overrightarrow{OQ}\times\overrightarrow{OP} = |\overrightarrow{OQ}|\ |\overrightarrow{OP}|\sin(\alpha-\beta)\,\hat{k}$

$\overrightarrow{OQ}\times\overrightarrow{OP} = \hat{k}\sin(\alpha-\beta)$ … ….(2)

From (1) and (2),

$$\sin(\alpha-\beta) = \sin\alpha\cos\beta - \cos\alpha\sin\beta$$

Example 6.23 If $\vec{a} = \hat{\imath} - \hat{\jmath}$, $\vec{b} = \hat{\imath} - \hat{\jmath} - 4\hat{k}$, $\vec{c} = 3\hat{\jmath} - \hat{k}$ and $\vec{d} = 2\hat{\imath} + 5\hat{\jmath} + \hat{k}$, verify that

(i) $(\vec{a}\times\vec{b})\times(\vec{c}\times\vec{d}) = [\vec{a},\ \vec{b},\ \vec{d}]\vec{c} - [\vec{a},\ \vec{b},\ \vec{c}]\vec{d}$

(ii) $(\vec{a}\times\vec{b})\times(\vec{c}\times\vec{d}) = [\vec{a},\ \vec{c},\ \vec{d}]\vec{b} - [\vec{b},\ \vec{c},\ \vec{d}]\vec{a}$

Solution:

(i) $(\vec{a}\times\vec{b})\times(\vec{c}\times\vec{d}) = [\vec{a},\ \vec{b},\ \vec{d}]\vec{c} - [\vec{a},\ \vec{b},\ \vec{c}]\vec{d}$

LHS $(\vec{a}\times\vec{b})\times(\vec{c}\times\vec{d})$	RHS $[\vec{a},\ \vec{b},\ \vec{d}]\vec{c} - [\vec{a},\ \vec{b},\ \vec{c}]\vec{d}$
$\vec{a}\times\vec{b} = \begin{vmatrix} \hat{\imath} & \hat{\jmath} & \hat{k} \\ 1 & -1 & 0 \\ 1 & -1 & -4 \end{vmatrix} = 4\hat{\imath} + 4\hat{\jmath}$	$[\vec{a},\ \vec{b},\ \vec{d}] = \begin{vmatrix} 1 & -1 & 0 \\ 1 & -1 & -4 \\ 2 & 5 & 1 \end{vmatrix} = 28$
$\vec{c}\times\vec{d} = \begin{vmatrix} \hat{\imath} & \hat{\jmath} & \hat{k} \\ 0 & 3 & -1 \\ 2 & 5 & 1 \end{vmatrix} = 8\hat{\imath} - 2\hat{\jmath} - 6\hat{k}$	$[\vec{a},\ \vec{b},\ \vec{c}] = \begin{vmatrix} 1 & -1 & 0 \\ 1 & -1 & -4 \\ 0 & 3 & -1 \end{vmatrix} = 12$
$(\vec{a}\times\vec{b})\times(\vec{c}\times\vec{d}) = \begin{vmatrix} \hat{\imath} & \hat{\jmath} & \hat{k} \\ 4 & 4 & 0 \\ 8 & -2 & -6 \end{vmatrix}$	$[\vec{a},\ \vec{b},\ \vec{d}]\vec{c} - [\vec{a},\ \vec{b},\ \vec{c}]\vec{d}$ $= 28(3\hat{\jmath} - \hat{k}) - 12(2\hat{\imath} + 5\hat{\jmath} + \hat{k})$
$= -24\hat{\imath} + 24\hat{\jmath} - 40\hat{k}$ ……..(1)	$= -24\hat{\imath} + 24\hat{\jmath} - 40\hat{k}$ ……..(2)

From (1) and (2) $(\vec{a}\times\vec{b})\times(\vec{c}\times\vec{d}) = [\vec{a},\ \vec{b},\ \vec{d}]\vec{c} - [\vec{a},\ \vec{b},\ \vec{c}]\vec{d}$

(ii) $(\vec{a}\times\vec{b})\times(\vec{c}\times\vec{d}) = [\vec{a},\ \vec{c},\ \vec{d}]\vec{b} - [\vec{b},\ \vec{c},\ \vec{d}]\vec{a}$

LHS $(\vec{a}\times\vec{b})\times(\vec{c}\times\vec{d})$	RHS $[\vec{a},\ \vec{c},\ \vec{d}]\vec{b} - [\vec{b},\ \vec{c},\ \vec{d}]\vec{a}$
$\vec{a}\times\vec{b} = \begin{vmatrix} \hat{\imath} & \hat{\jmath} & \hat{k} \\ 1 & -1 & 0 \\ 1 & -1 & -4 \end{vmatrix} = 4\hat{\imath} + 4\hat{\jmath}$	$[\vec{a},\ \vec{c},\ \vec{d}] = \begin{vmatrix} 1 & -1 & 0 \\ 0 & 3 & -1 \\ 2 & 5 & 1 \end{vmatrix} = 10$
$\vec{c}\times\vec{d} = \begin{vmatrix} \hat{\imath} & \hat{\jmath} & \hat{k} \\ 0 & 3 & -1 \\ 2 & 5 & 1 \end{vmatrix} = 8\hat{\imath} - 2\hat{\jmath} - 6\hat{k}$	$[\vec{b},\ \vec{c},\ \vec{d}] = \begin{vmatrix} 1 & -1 & -4 \\ 0 & 3 & -1 \\ 2 & 5 & 1 \end{vmatrix} = 34$
$(\vec{a}\times\vec{b})\times(\vec{c}\times\vec{d}) = \begin{vmatrix} \hat{\imath} & \hat{\jmath} & \hat{k} \\ 4 & 4 & 0 \\ 8 & -2 & -6 \end{vmatrix}$	$[\vec{a},\ \vec{c},\ \vec{d}]\vec{b} - [\vec{b},\ \vec{c},\ \vec{d}]\vec{a} = 10(\hat{\imath} - \hat{\jmath} - 4\hat{k}) - 34(\hat{\imath} - \hat{\jmath})$
$= -24\hat{\imath} + 24\hat{\jmath} - 40\hat{k}$ ……..(3)	$= -24\hat{\imath} + 24\hat{\jmath} - 40\hat{k}$ ……..(4)

From (3) and (4) $(\vec{a}\times\vec{b})\times(\vec{c}\times\vec{d}) = [\vec{a},\ \vec{c},\ \vec{d}]\vec{b} - [\vec{b},\ \vec{c},\ \vec{d}]\vec{a}$

" An Equation Means Nothing to Us Unless It Express a Thought of God "

4. If the straight lines $\frac{x-1}{2}=\frac{y+1}{\lambda}=\frac{z}{2}$ and $\frac{x+1}{5}=\frac{y+1}{2}=\frac{z}{\lambda}$ are coplanar, find λ and equations of the planes containing these two lines.

Solution:

$\frac{x-1}{2}=\frac{y+1}{\lambda}=\frac{z}{2}$ and $\frac{x+1}{5}=\frac{y+1}{2}=\frac{z}{\lambda}$

Let $\vec{a}=\hat{\imath}-\hat{\jmath}+0\hat{k}$ $\quad\vec{b}=2\hat{\imath}+\lambda\hat{\jmath}+2\hat{k}$,

$\vec{c}=-\hat{\imath}-\hat{\jmath}+0\hat{k}$ and $\vec{d}=5\hat{\imath}+2\hat{\jmath}+\lambda\hat{k}$

$\vec{c}-\vec{a}=-2\hat{\imath}+0\hat{\jmath}+0\hat{k}$

$$(\vec{c}-\vec{a})\cdot(\vec{b}\times\vec{d})=\begin{vmatrix}-2&0&0\\2&\lambda&2\\5&2&\lambda\end{vmatrix}$$

$=(\lambda^2-4)(-2)-(2\lambda-10)(0)+(4-5\lambda)(0)$

$(\vec{c}-\vec{a})\cdot(\vec{b}\times\vec{d})=0$

$(-2)(\lambda^2-4)=0 \qquad \lambda^2=4 \qquad \lambda=\pm2$

Cartesian equation:

$$\begin{vmatrix}x-1&y+1&z-0\\2&\lambda&2\\5&2&\lambda\end{vmatrix}=0$$

When $\lambda=-2$

$$\begin{vmatrix}x-1&y+1&z-0\\2&-2&2\\5&2&-2\end{vmatrix}=0$$

$(x-1)(4-4)-(y+1)(-4-10)+z(4+10)=0$

$14(y+1)+14z=0$

$\div(14) \qquad y+1+z=0$

When $\lambda=2$

$$\begin{vmatrix}x-1&y+1&z-0\\2&2&2\\5&2&2\end{vmatrix}=0$$

$(x-1)(4-4)-(y+1)(4-10)+z(4-10)=0$

$6(y+1)-6z=0$

$\div(6) \qquad y+1-z=0$

Cartesian equation is

$y+z+1=0$ and $y-z+1=0$

Example 6.50 Find the distance of the point (5, -5, -10) from the point of intersection of a straight line passing through the points A(4, 1, 2) and B(7, 5, 4) with the plane $x-y+z=5$.

Solution:

The Cartesian equation of the straight line joining A and B is

$$\frac{x-x_1}{x_2-x_1}=\frac{y-y_1}{y_2-y_1}=\frac{z-z_1}{z_2-z_1}$$

The required line is $\frac{x-4}{3}=\frac{y-1}{4}=\frac{z-2}{2}$

$$\frac{x-4}{3}=\frac{y-1}{4}=\frac{z-2}{2}=t$$

$(x,y,z)=(3t+4,4t+1,2t+2)\ \ldots\ldots\ldots(1)$

$x-y+z=5 \Rightarrow 3t+4-4t-1+2t+2=5$

$\Rightarrow t=0$

Point of intersection of the straight line is $(4,1,2)$.

Now, the distance between the two points $(4,1,2)$ and $(5,-5,-10)$ is

$d=\sqrt{(5-4)^2+(-5-1)^2+(-10-2)^2}$

$=\sqrt{1+36+144}=\sqrt{181}$ units.

Do it Your Self Excercise 6.9 - 8

Mathematics is Queen of Science

Example 7.1 For the function $f(x) = x^2$, $x \in [0, 2]$ compute the average rate of changes in the subintervals $[0, 0.5], [0.5, 1], [1, 1.5], [1.5, 2]$ and the instantaneous rate of changes at the points $x = 0.5, 1, 1.5, 2$.

Solution:

The average rate of change in an interval $[a, b]$ is $\frac{f(b)-f(a)}{b-a}$ whereas, the instantaneous rate of change at a point x is $f'(x)$ for the given function. They are respectively, $b + a$ and 2x.

Rate of changes

a	b	x	Average rate is $\frac{f(b)-f(a)}{b-a} = b+a$	Instantaneous rate is $f'(x) = 2x$
0	0.5	0.5	0.5	1
0.5	1	1	1.5	2
1	1.5	1.5	2.5	3
1.5	2	2	3.5	4

Example 7.5 A particle is fired straight up from the ground to reach a height of s feet in t seconds, where $s(t) = 128t - 16t^2$.
(1) Compute the maximum height of the particle reached.
(2) What is the velocity when the particle hits the ground?

Solution:

(1) At the maximum height, the velocity $v(t)$ of the particle is zero.

$V(t) = \frac{ds}{dt} = 128 - 32t$

$v(t) = 0$

$128 - 32t = 0 \qquad t = 4.$

At $t = 4$ is

$s(4) = 128(4) - 16(4)^2 = 512 - 256 = 256$ft.

(2) When the particle hits the ground then $s = 0$.

$s(t) = 128t - 16t^2 = 0$

$t = 0, 8$ seconds.

The particle hits the ground at $t = 8$ seconds.

The velocity when it hits the ground is

$v(8) = 128 - 32(8) = 128 - 256 = -128\ ft/s.$

Example 7.6 A particle moves along a horizontal line such that its position at any time $t \geq 0$ is given by $s(t) = t^3 - 6t^2 + 9t + 1$, where s is measured in metres and t in seconds?
(1) At what time the particle is at rest?
(2) At what time the particle changes direction?
(3) Find the total distance travelled by the particle in the first 2 seconds.

Solution:

$s(t) = t^3 - 6t^2 + 9t + 1$	S(0) = 0+1=1
$v(t) = 3t^2 - 12t^2 + 9$	S(1) = 1-6+9+1=5
$a(t) = 6t - 24$	S(2) = 8-24+18+1=3

(1) when $v = 0$, $3t^2 - 12t^2 + 9 = 0$

$t = 1, \quad t = 3$

(2) The particle changes direction when $v(t)$ changes its sign. Now.

If $0 \leq t < 1$ then both $(t-3) < 0$ *and* $(t-1) < 0$ and hence $v(t) > 0$.

If $1 \leq t < 3$ then $(t-3) < 0$ *and* $(t-1) > 0$. and hence $v(t) < 0$.

If $t > 3$ then both $(t-3) > 0$ *and* $(t-1) > 0$ and hence $v(t) > 0$.

Therefore, the particle changes direction when $t = 1$ and $t = 3$.

(3) The total distance travelled by the particle from time $t = 0$ to $t = 2$ is given by,

$|s(0) - s(1)| + |s(1) - s(2)|$

$= |1 - 5| + |5 - 3| = 6$ metres.

Example 7.7 If we blow air into a balloon of spherical shape at a rate of $1000\ cm^3$ per second. At what rate the radius of the baloon changes when the radius is 7cm? Also compute the rate at which the surface area changes.

Solution:

Given: $\frac{dV}{dt} = 1000, \quad r = 7$

The volume of the baloon of radius r is

$V = \frac{4}{3}\pi r^3.$

$\frac{dV}{dt} = 4\pi r^2 \times \frac{dr}{dt}$

$\frac{dr}{dt} = \frac{1000}{4\pi r^2} = \frac{1000}{4\pi \times 49} = \frac{250}{49\pi}$

The surface area S of the baloon is $S = 4\pi r^2$

$\frac{ds}{dt} = 8\pi \times r \times \frac{dr}{dt} = 8\pi \times 7 \times \frac{250}{49\pi} = \frac{2000}{7}$

$\frac{dr}{dt} = \frac{250}{49\pi}$ cm/sec and $\frac{ds}{dt} = \frac{2000}{7}\ cm^2/sec.$

Example 7.60 Find the local extrema of the function $f(x) = 4x^6 - 6x^4$.

Solution:

$f(x) = 4x^6 - 6x^4$

$f'(x) = 24x^5 - 24x^3 \qquad f'(x) = 0$

$24x^3(x^2 - 1) = 0 \qquad x = 0 \ \ and\ x = \pm 1$

$f''(x) = 120x^4 - 72x^2$

When $x = 0 \qquad f''(x) = 120(0)^4 - 72(0)^2 = 0$

Does not give any information about local extrema at $x = 0$.

When $x = 1 \qquad f''(x) = 120(1)^4 - 72(1)^2 > 0$

When $x = -1 \quad f''(x) = 120(-1)^4 - 72(-1)^2 > 0$

The intervals are

$(-\infty, -1), (-1, 0), (0, 1), (1, \infty)$.

Intervals	Sign of $f'(x)$	Monotonicity
$(-\infty, -1)$	−	Strictly decreasing
$(-1,0)$	+	Strictly increasing
$(0,1)$	−	Strictly decreasing
$(1, \infty)$	+	Strictly increasing

When x = 0 $\qquad f(0) = 4x^6 - 6x^4 = 0$

When x = −1 $\qquad f(-1) = 4x^6 - 6x^4 = -2$

When x = 1 $\qquad f(1) = 4x^6 - 6x^4 = -2$

Local minimum value is -2.

Local maximum value is 0.

Example 7.61 Find the local maximum and minimum of the function x^2y^2 on the line $x + y = 10$.

Solution:

Let $f(x) = x^2y^2$	$x + y = 10$ $y = 10 - x$

$f(x) = x^2(10 - x)^2 = x^2(100 - 20x + x^2)$

$f(x) = 100x^2 - 20x^3 + x^4$

$f'(x) = 200x - 60x^2 + 4x^3$

$f'(x) = 4x(x^2 - 15x + 50)$

$f'(x) = 4x(x - 10)(x - 5)$

$f'(x) = 0 \qquad x = 0, 10\ and\ 5$

$f''(x) = 200 - 120x + 12x^2$

When $x = 0 \qquad f''(x) = 200 > 0$

When $x = 5 \quad f''(x) = 12(5)^2 - 120(5) + 200 < 0$

When $x = 10 \quad f''(x) = 12(10)^2 - 120(10) + 200 > 0$

The intervals are $(-\infty, 0), (0, 5), (5, 10), (10, \infty)$.

Intervals	Sign of $f'(x)$	Monotonicity
$(-\infty, 0)$	−	Strictly decreasing
$(0,5)$	+	Strictly increasing
$(5,10)$	−	Strictly decreasing
$(10, \infty)$	+	Strictly increasing

When x = 0 $\quad f(0) = 0^2(10 - 0)^2 = 100$

When x = 5 $\qquad f(5) = 5^2(10 - 5)^2 = 625$

When x = 10 $\qquad f(10) = 10^2(10 - 10)^2 = 0$

Local minimum value is 0.

Local maximum value is 625.

EXERCISE 7.7

1. Find intervals of concavity and points of inflexion for the following functions:

(i) $f(x) = x(x - 4)^3$

(ii) $f(x) = \sin x + \cos x,\ 0 < x < 2\pi$

(iii) $f(x) = \frac{1}{2}(e^x - e^{-x})$

Solution:

(i) $f(x) = x(x - 4)^3 = (x - 4 + 4)(x - 4)^3$

$f(x) = (x - 4)^4 + 4(x - 4)^3$

$f'(x) = 4(x - 4)^3 + 12(x - 4)^2$

$f''(x) = 12(x - 4)^2 + 24(x - 4)$

$f''(x) = 12(x - 4)(x - 4 + 2)$

$f''(x) = 12(x - 4)(x - 2)$

$f''(x) = 0 \qquad x = 4\ and\ 2.$

Intervals are $(-\infty, 2), (2, 4)\ and\ (4, \infty)$.

Intervals	Sign of $f''(x)$	Concavity
$(-\infty, 2)$	+	Up ward
$(2, 4)$	−	Down ward
$(4, \infty)$	+	Up ward

When x = 2 $\qquad f(2) = 2(2 - 4)^3 = -16$

When x = 4 $\qquad f(5) = 4(4 - 4)^3 = 0$

Inflection points $(2, -16)\ and\ (4, 0)$.

(ii) $f(x) = \sin x + \cos x,\ 0 < x < 2\pi$

$f'(x) = \cos x - \sin x$

$f''(x) = -\sin x - \cos x$

$f''(x) = 0 \qquad \sin x = -\cos x \qquad \tan x = -1$

$x = n\pi - \frac{\pi}{4} \qquad x = \frac{3\pi}{4}, \frac{7\pi}{4} \in (0, 2\pi).$

Intervals are $\left(0, \frac{3\pi}{4}\right), \left(\frac{3\pi}{4}, \frac{7\pi}{4}\right)\ and\ \left(\frac{7\pi}{4}, 2\pi\right)$.

Intervals	Sign of $f''(x)$	Concavity
$\left(0, \frac{3\pi}{4}\right)$	−	Down ward
$\left(\frac{3\pi}{4}, \frac{7\pi}{4}\right)$	+	Up ward
$\left(\frac{7\pi}{4}, 2\pi\right)$	−	Down ward

When x = $\frac{3\pi}{4}$ $\qquad f\left(\frac{3\pi}{4}\right) = \sin\frac{3\pi}{4} + \cos\frac{3\pi}{4} = 0$

When x = $\frac{7\pi}{4}$ $\qquad f\left(\frac{7\pi}{4}\right) = \sin\frac{7\pi}{4} + \cos\frac{7\pi}{4} = 0$

Inflection points $\left(\frac{3\pi}{4}, 0\right)\ and\ \left(\frac{7\pi}{4}, 0\right)$.

5. A rectangular page is to contain 24 cm^2 of print. The margins at the top and bottom of the page are 1.5 cm and the margins at other sides of the page is 1 cm. What should be the dimensions of the page so that the area of the paper used is minimum.

Solution:

Let x and y be the length and breadth of the rectangle paper in printed area.

Area $A = xy = 24$ $\qquad y = \frac{24}{x}$

Dimensions of the paper length and breadth is $x + 2$ *and* $y + 3$.

Area of the paper

$A(x) = (x+2)(y+3) = xy + 3x + 2y + 6$

$A(x) = 24 + 6 + 3x + 2\left(\frac{24}{x}\right) = 30 + 3x + \frac{48}{x}$

$A'(x) = 3 - \frac{48}{x^2}$ $\qquad A''(x) = \frac{96}{x^3}$

$A'(x) = 0$ $\qquad 3x^2 = 48$ $\qquad x = \pm 4$

At $x = 4$ $\qquad A''(4) = \frac{96}{4^3} > 0.$

Area minimum at $x = 4$.

When $x = 4$ $\qquad y = \frac{24}{4} = 6$

Dimensions of the paper length and breadth are 4+2=6 and 6+3=9.

6. A farmer plans to fence a rectangular pasture adjacent to a river. The pasture must contain 1,80,000 sq.mtrs in order to provide enough grass for herds. No fencing is needed along the river. What is the length of the minimum needed fencing material?

Solution:

Let x and y be thee length and breadth of the rectangular pasture.

Given: Total area = 180000 sq.m

$Area = xy = 180000$ $\qquad y = \frac{180000}{x}$

Length of the perimeter (or) fencing

$L(x) = x + 2y = x + 2\left(\frac{180000}{x}\right) = x + \frac{360000}{x}$

$L'(x) = 1 - \frac{360000}{x^2}$

$L'(x) = 0$ $\qquad x^2 = 360000$ $\qquad x = \pm 600$

$L''(x) = \frac{720000}{x^3}$

At $x = 600$ $\qquad L''(600) = \frac{720000}{(600)^3} > 0$

Minimum length at $x = 600$.

When $x = 600$ $\qquad y = \frac{180000}{600} = 300$

Minimum length of the fencing

$x + 2y = 600 + 600 = 1200m$

7. Find the dimensions of the rectangle with maximum area that can be inscribed in a circle of radius 10 cm.

Solution:

Let $2x$ and $2y$ be the length and breadth of the rectangle.

Let θ be made by OP in x-axis.

$\cos\theta = \frac{x}{r}$	$\sin\theta = \frac{y}{r}$
$x = r\cos\theta$	$y = r\sin\theta$
When $r = 10$	When $r = 10$
$x = 10\cos\theta$	$y = 10\sin\theta$

Area of the rectangle

$A = 2x \times 2y = 2(10\cos\theta) \times 2(10\sin\theta)$

$A(\theta) = 400\cos\theta\sin\theta = 200\sin 2\theta$

$A'(\theta) = 400\cos 2\theta$

$A'(\theta) = 0$ $\qquad \cos 2\theta = 0$ $\qquad \theta = \frac{\pi}{4} \in \left(0, \frac{\pi}{2}\right)$

$A''(\theta) = -800\sin 2\theta$

At $\theta = \frac{\pi}{4}$ $\qquad A''\left(\frac{\pi}{4}\right) = -800\sin 2\left(\frac{\pi}{4}\right) < 0$

Area is maximum at $\theta = \frac{\pi}{4}$

Length $2x = 2\left(10\cos\frac{\pi}{4}\right) = 20 \times \frac{1}{\sqrt{2}} = 10\sqrt{2}$ cm

Breadth $2y = 2\left(10\sin\frac{\pi}{4}\right) = 20 \times \frac{1}{\sqrt{2}} = 10\sqrt{2}$ cm

8. Prove that among all the rectangles of the given perimeter, the square has the maximum area.

Solution:

Let x and y be the length and breadth of the rectangle.

Let L be the perimeter of the rectangle.

Perimeter $L(x) = 2x + 2y$ $\qquad y = \frac{L-2x}{2}$

Let A be the area of the rectangle. $\qquad A = xy$

$A(x) = x\left(\frac{L-2x}{2}\right) = \frac{xL - 2x^2}{2} = \frac{xL}{2} - x^2$

$$A'(x) = \frac{L}{2} - 2x$$

$A'(x) = 0$ $\qquad \frac{L}{2} - 2x = 0$ $\qquad x = \frac{L}{4}$

$A''(x) = -2$ $\qquad$ At $x = \frac{L}{4}$ $\qquad A''\left(\frac{L}{4}\right) = -2 < 0$

Area is maximum at $x = \frac{L}{4}$ $\qquad y = \frac{4x - 2x}{2} = x$

It is square.

The rectangle is square has the maximum area.

EXERCISE 8.2

5. The trunk of a tree has diameter 30 cm. During the following year, the circumference grew 6 cm.
(i) Approximately, how much did the tree's diameter grow?
(ii) What is the percentage increase in area of the tree's cross-section?

Solution:

Given: $r = 15$ cm and $D = 30$ cm

Circumference of the circle $S(r) = 2\pi r$

$= 2\pi(15) = 30\pi$ cm

Differentiating $= 6$ cm

$2\pi r_2 - 2\pi r_1 = 6 \qquad dr = r_2 - r_1 = \frac{3}{\pi}$

(i)

Approximate error

$= A(r + \Delta r) + A(r)$

$\approx A'(r)dr$

$= 90\ cm^2 (increasing)$

Area of the circle

$A(r) = \pi r^2$

$dA = 2\pi r\, dr$

$dA = 30\pi \left(\frac{3}{\pi}\right) = 90$

(ii) Percentage increasing:

Actual value (Area) $= \pi r^2 = 225\pi$

Actual error $= \frac{Approximate\ value}{Actual\ value} \times 100 = \frac{90}{225\pi} \times 100 = \frac{40}{\pi}$ %

The Mathematics has Reached Highest Step On the Ladder Of Each and Every Human Thought "

EXERCISE 8.6

1. If $u(x,y)=x^2y+3xy^4, x=e^t$ and $y=\sin t$, find $\frac{du}{dt}$ and evaluate it at $t=0$

Solution: $\frac{du}{dt}=\frac{\partial u}{\partial x}\frac{dx}{dt}+\frac{\partial u}{\partial y}\frac{dy}{dt}$

$u(x,y)=x^2y+3xy^4$

$\frac{\partial u}{\partial x}=2xy+3y^4$

$\frac{\partial u}{\partial y}=x^2+12xy^3$

$x(t)=e^t$

$\frac{dx}{dt}=e^t$

$y(t)=\sin t$

$\frac{dy}{dt}=\cos t$

$\frac{du}{dt}=\left(\frac{\partial u}{\partial x}\right)\left(\frac{dx}{dt}\right)+\left(\frac{\partial u}{\partial y}\right)\left(\frac{dy}{dt}\right)$

$=(2xy+3y^4)(e^t)+(x^2+12xy^3)(\cos t)$

$=[2(e^t)(\sin t)+3(\sin t)^4](e^t)$

$+[(e^t)^2+12(e^t)(\sin t)^3](\cos t)$

$=2e^{2t}\sin t+3e^t\sin^4 t+e^{2t}\cos t+12e^t\cos t\sin^3 t$

$\frac{du}{dt}=e^t(2e^t\sin t+3\sin^4 t+e^t\cos t+12\cos t\sin^3 t)$

at $t=0$,

$\frac{du}{dt}=e^0(2e^0\sin 0+3\sin^4 0+e^0\cos 0+12\cos 0\sin^3 0)$

$=1(0+0+1+0)$

$=1(1)$

$\frac{du}{dt}=1$

---

2. If $u(x,y,z)=xy^2z^3$, $x=\sin t, y=\cos t, z=1+e^{2t}$ find $\frac{du}{dt}$

Solution: $\frac{du}{dt}=\frac{\partial u}{\partial x}\frac{dx}{dt}+\frac{\partial u}{\partial y}\frac{dy}{dt}+\frac{\partial u}{\partial z}\frac{dz}{dt}$

$u(x,y,z)=xy^2z^3$

$\frac{\partial u}{\partial x}=y^2z^3$

$\frac{\partial u}{\partial y}=x(2y)z^3=2xyz^3$

$\frac{\partial u}{\partial z}=xy^2(3z^2)=3xy^2z^2$

$x=\sin t$

$\frac{dx}{dt}=\cos t$

$x=\cos t$

$\frac{dy}{dt}=-\sin t$

$z=1+2e^{2t}$

$\frac{dz}{dt}=2(e^{2t})=2e^{2t}$

$\frac{du}{dt}=(y^2z^3)(\cos t)+(2xyz^3)(-\sin t)$

$+(3xy^2z^2)(2e^{2t})$

$=y^2z^3\cos t-2xyz^3\sin t+6xy^2z^2e^{2t}$

$=yz^2(yz\cos t-2xz\sin t+6xye^{2t})$

$=\cos t(1+e^{2t})^2$

$[\cos^2 t(1+e^{2t})-2\sin^2 t(1+e^{2t})+6\sin t\cos t\, e^{2t}]$

---

3. If $w(x,y,z)=x^2+y^2+z^2$, $x=e^t, y=e^t\sin t, z=e^t\cos t$, find $\frac{dw}{dt}$

Solution: $\frac{dw}{dt}=\frac{\partial w}{\partial x}\frac{dx}{dt}+\frac{\partial w}{\partial y}\frac{dy}{dt}+\frac{\partial w}{\partial z}\frac{dz}{dt}$

$w(x,y,z)=x^2+y^2+z^2$

$\frac{\partial w}{\partial x}=2x$; $\frac{\partial w}{\partial y}=2y$; $\frac{\partial w}{\partial z}=2z$

$x=e^t$; $\frac{dx}{dt}=e^t$,

$y=e^t\sin t$; $\frac{dy}{dt}=e^t\cos t+\sin t\, e^t$

$z=e^t\cos t$; $\frac{dz}{dt}=e^t(-\sin t)+\cos t\, e^t$

$\frac{\partial w}{\partial x}\frac{dx}{dt}=(2x)(e^t)$

$=(2e^t)e^t$

'Mathematics is Most Beautiful and Most Power ful Creation in the Universe it is Accepted Globally'

$$\frac{\partial w}{\partial y} = x + \cos(xy)x$$

$$= x + x\cos(xy)$$

$$\frac{\partial^2 w}{\partial x \partial y} = 1 + x[-\sin(xy)y] + \cos(xy)(1)$$

$$= 1 - xy\sin(xy) + \cos(xy) \ldots\ldots (ii)$$

From (i) and (ii) $\frac{\partial^2 w}{\partial y \partial x} = \frac{\partial^2 w}{\partial x \partial y}$ is proved.

10. A firm produces two types of calculators each week. x number of type A and y number of type B. The weekly revenue and cost functions (in rupees) are $R(x,y) = 80x + 90y + 0.04xy - 0.05x^2 - 0.05y^2$ and $C(x,y) = 8x + 6y + 2000$ respectively.
(i) Find the profit function $P(x,y)$.
(ii) Find $\frac{\partial P}{\partial x}$ (1200,1800) and $\frac{\partial P}{\partial y}$ at (1200,1800) and interpret these results.

Solution:

Given Revenue $= R(x,y)$ and

Cost $= C(x,y)$

So, Profit $P(x,y) = R(x,y) - C(x,y)$

$$= (80x + 90y + 0.04xy - 0.05x^2 - 0.05y^2) - (8x + 6y + 2000)$$

(i) $P(x,y) = 72x + 84y + 0.04xy - 0.05x^2 - 0.05y^2 - 2000$

$$\frac{\partial P}{\partial x} = 72 + 0.04y - 0.1x$$

At (1200,1800)

$$\frac{\partial P}{\partial x} = 72 + 0.04(1800) - 0.1(1200)$$

$$= 72 + 72.00 - 120.0$$

$$= 144 - 120$$

$$\frac{\partial P}{\partial x} = 24$$

$$\frac{\partial P}{\partial y} = 84 + 0.04x - 0.1y$$

At (1200,1800)

$$\frac{\partial P}{\partial y} = 84 + 0.04(1200) - 0.1(1800)$$

$$= 84 + 48.00 - 180.0$$

$$= 132 - 180$$

$$\frac{\partial P}{\partial y} = -48$$

(ii) $\frac{\partial P}{\partial x} = 24$ and $\frac{\partial P}{\partial y} = -48$ At (1200,1800), shows Profit increases when keeping y as constant.

'Mathematics is not About Numbers, Euations, Coputation or Algorithms. It is Understanding and Working Out Perfectly'

Example 9.10 Evaluate : $\int_1^2 \frac{x}{(x+1)(x+2)}dx$

Solution:

$$\frac{x}{(x+1)(x+2)} = \frac{A}{x+1} + \frac{B}{x+2}$$

$$x = A(x+2) + B(x+1)$$

When $x = -1$ $\quad A = -1$

When $x = -2$ $\quad B = 2$

$$\frac{x}{(x+1)(x+2)} = \frac{-1}{x+1} + \frac{2}{x+2}$$

$$\int_1^2 \frac{x}{(x+1)(x+2)}dx = \int_1^2 \left[\frac{-1}{x+1} + \frac{2}{x+2}\right]dx$$

$$= [-\log(x+1) + 2\log(x+2)]_1^2$$

$$= \left[\log\left|\frac{(x+2)^2}{x+1}\right|\right]_1^2 = \log\left(\frac{16}{3}\right) - \log\left(\frac{9}{2}\right)$$

$$= \log\left(\frac{16}{3} \times \frac{2}{9}\right) = \log\frac{32}{27}$$

Example 9.11 Evaluate : $\int_0^{\frac{\pi}{2}} \frac{\cos\theta}{(1+\sin\theta)(2+\sin\theta)}d\theta.$

Solution:

$$\int_0^{\frac{\pi}{2}} \frac{\cos\theta}{(1+\sin\theta)(2+\sin\theta)}d\theta = \int_1^2 \frac{1}{u(u+1)}du$$

$u = 1 + \sin\theta$	When $x = 0$ $\quad u = 1$
$u + 1 = 2 + \sin\theta$	$x = \frac{\pi}{2}$ $\quad u = 2$
$du = \cos\theta \; d\theta$	

$$= \int_1^2 \frac{u+1-u}{u(u+1)}du$$

$$= \int_1^2 \left[\frac{1}{u} - \frac{1}{(u+1)}\right]du$$

$$= [\log|u| - \log|u+1|]_1^2$$

$$= \left[\log\left|\frac{u}{u+1}\right|\right]_1^2$$

$$= \log\left(\frac{2}{3}\right) - \log\left(\frac{1}{2}\right) = \log\left(\frac{2}{3} \times 2\right)$$

$$= \log\frac{4}{3}$$

Example 9.12 Evaluate : $\int_0^{\frac{1}{\sqrt{2}}} \frac{\sin^{-1}x}{(1-x^2)^{3/2}}dx$

Solution:

$u = \sin^{-1}x$	$du = \frac{1}{\sqrt{1-x^2}}dx$
$x = \sin u$	$x = 0 \quad u = 0$
$1 - x^2 = 1 - \sin^2 u = \cos^2 u$	$x = \frac{1}{\sqrt{2}} \quad u = \frac{\pi}{4}$

$$= \int_0^{\frac{\pi}{4}} \frac{u}{\cos^2 u}du = \int_0^{\frac{\pi}{4}} u\sec^2 u \, du$$

$$= [u \tan u]_0^{\pi/4} - \int_0^{\frac{\pi}{4}} \tan u \, du$$

$$= [u \tan u]_0^{\pi/4} - [\log|\cos u|]_0^{\pi/4}$$

$$= \frac{\pi}{4} + \log\frac{1}{\sqrt{2}} = \frac{\pi}{4} - \frac{1}{2}\log 2.$$

Example 9.13

Evaluate : $\int_0^{\frac{\pi}{2}}(\sqrt{\tan x} + \sqrt{\cot x})dx.$

Solution:

$$I = \int_0^{\frac{\pi}{2}}(\sqrt{\tan x} + \sqrt{\cot x})dx$$

$$= \int_0^{\frac{\pi}{2}}\left(\sqrt{\frac{\sin x}{\cos x}} + \sqrt{\frac{\cos x}{\sin x}}\right)dx$$

$$= \int_0^{\frac{\pi}{2}}\left(\frac{\sqrt{\sin x}}{\sqrt{\cos x}} + \frac{\sqrt{\cos x}}{\sqrt{\sin x}}\right)dx$$

$$I = \int_0^{\frac{\pi}{2}}\left(\frac{\sin x + \cos x}{\sqrt{\sin x \cos x}}\right)dx$$

$$= \sqrt{2}\int_0^{\frac{\pi}{2}}\left(\frac{\sin x + \cos x}{\sqrt{2\sin x \cos x}}\right)dx$$

$$= \sqrt{2}\int_0^{\frac{\pi}{2}}\left(\frac{\sin x + \cos x}{\sqrt{1-(\sin x - \cos x)^2}}\right)dx$$

$$= \sqrt{2}\int_{-1}^{1}\frac{du}{\sqrt{1-u^2}} = \sqrt{2}[\sin^{-1}u]_{-1}^{1}$$

$$= \sqrt{2}\left[\frac{\pi}{2} - \left(\pi - \frac{\pi}{2}\right)\right] = \pi\sqrt{2}$$

5. Find the area of the region bounded between the curves $y = \sin x$ and $y = \cos x$ and the lines $x = 0$ and $x = \pi$.

Solution:

$y = \cos x$	$y = \sin x$	$\cos x = \sin x$ $\frac{\pi}{4} \le x \le \frac{5\pi}{4}$

$$Area \ \ A = \int_a^b [Y_U - Y_L] \ dx$$

$$= \int_0^{\frac{\pi}{4}} [\cos x - \sin x]\, dx + \int_{\frac{\pi}{4}}^{\pi} [\sin x - \cos x]\, dx$$

$$= [\sin x + \cos x]_0^{\frac{\pi}{4}} + [-\cos x - \sin x]_{\frac{\pi}{4}}^{\pi}$$

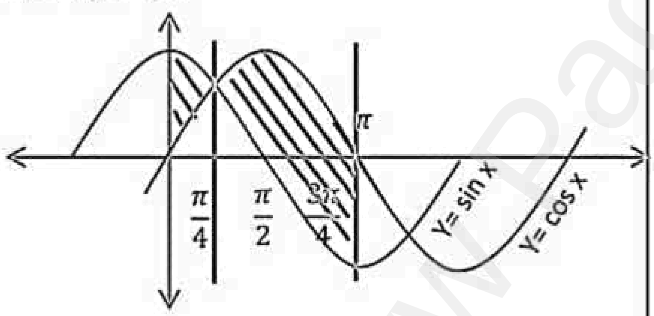
$$= [\sin x + \cos x]_0^{\frac{\pi}{4}} - [\cos x + \sin x]_{\frac{\pi}{4}}^{\pi}$$

$$= \left[\frac{1}{\sqrt{2}} + \frac{1}{\sqrt{2}} - 0 - 1\right] - \left[-1 - 0 - \frac{1}{\sqrt{2}} - \frac{1}{\sqrt{2}}\right]$$

$$= \frac{2}{\sqrt{2}} + \frac{2}{\sqrt{2}}$$

$$= [\sqrt{2} + \sqrt{2}]$$

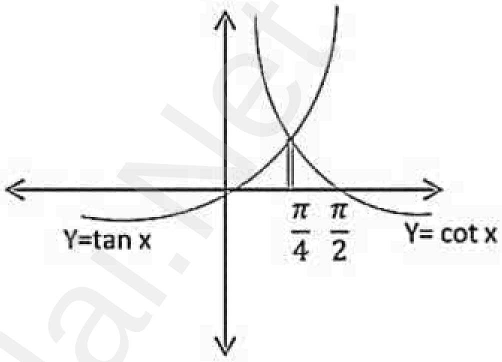
$$= 2\sqrt{2} \text{ sq. units.}$$



6. Find the area of the region bounded by $y = \tan x$, $y = \cot x$ and the lines $x = 0, x = \frac{\pi}{2}$.

Solution:

$y = \tan x$ $y = \cot x$	$\tan x = \cot x$ $x = \frac{\pi}{4} \in \left[0, \frac{\pi}{2}\right]$



$$Area = \int_0^{\frac{\pi}{4}} y \ dx + \int_{\frac{\pi}{4}}^{\frac{\pi}{2}} y\, dx$$

$$= \int_0^{\frac{\pi}{4}} \tan x \ dx + \int_{\frac{\pi}{4}}^{\frac{\pi}{2}} \cot x\, dx$$

$$= [\log \sec x]_0^{\pi/4} + [\log \sin x]_{\pi/4}^{\pi/2}$$

$$= \log\sqrt{2} - \log 1 + \log 1 - \log\frac{1}{\sqrt{2}}$$

$$= \log\sqrt{2} - \log\frac{1}{\sqrt{2}}$$

$$= \log\sqrt{2} - [\log 1 - \log\sqrt{2}]$$

$$= \log\sqrt{2} + \log\sqrt{2}$$

$$= 2\log\sqrt{2}$$

$$= \log 2 \text{ sq.units.}$$

Only Maths Tuition

EXERCISE 10.6

2. $(x^3+y^3)dy - x^2y\,dx = 0$

Solution:

$$(x^3+y^3)dy - x^2y\,dx = 0$$
$$(x^3+y^3)dy = x^2y\,dx$$
$$\frac{dx}{dy} = \frac{x^3+y^3}{x^2y} = \frac{x}{y} + \frac{y^2}{x^2}$$

Put $x = vy \quad v = \frac{x}{y} \quad \frac{dx}{dy} = v + y\frac{dv}{dy}$

$v + y\frac{dv}{dy} = v + \frac{1}{v^2} \qquad y\frac{dv}{dy} = \frac{1}{v^2}$

$$v^2\,dv = \frac{dy}{y}$$

Integrating on both sides,

$$\frac{v^3}{3} = \log|y| + \log|c|$$

$\frac{v^3}{3} = \log|yc| \qquad \frac{x^3}{3y^3} = \log|yc|$

$yc = e^{\frac{x^3}{3y^3}} \qquad y = k\,e^{\frac{x^3}{3y^3}}$

4. $2xy\,dx + (x^2+2y^2)dy = 0$

Solution:

$$(x^2+2y^2)dy = -2xy\,dx$$
$$2xy\,dx = -(x^2+2y^2)dy$$
$$\frac{dx}{dy} = -\frac{(x^2+2y^2)}{2xy}$$

Put $x = vy \quad v = \frac{x}{y} \quad \frac{dx}{dy} = v + y\frac{dv}{dy}$

$$v + y\frac{dv}{dy} = -\frac{(v^2y^2+2y^2)}{2(vy)y} = \frac{(v^2+2)}{2v}$$
$$y\frac{dv}{dy} = -\frac{(v^2+2)}{2v} - v$$
$$y\frac{dv}{dy} = \frac{-v^2-2-2v^2}{2v} = \frac{-3v^2-2}{2v}$$
$$\frac{2v\,dv}{3v^2+2} = -\frac{dy}{y}$$
$$\frac{1}{3}\frac{6v\,dv}{3v^2+2} = \frac{-dy}{y}$$

Integrating on both sides,

$$\frac{1}{3}\log|3v^2+2| = -\log|y| + \log|c|$$
$$3v^2+2 = \left(\frac{c}{y}\right)^3$$
$$3\left(\frac{x}{y}\right)^2 + 2 = \left(\frac{c}{y}\right)^3$$

$3x^2y + 2y^3 = c$

5. $(y^2-2xy)dx = (x^2-2xy)dy$

Solution:

$$(y^2-2xy)dx = (x^2-2xy)dy$$
$$\frac{dx}{dy} = \frac{y^2-2xy}{x^2-2xy}$$

Put $y = vx \quad v = \frac{y}{x} \quad \frac{dy}{dx} = v + x\frac{dv}{dx}$

$$v + x\frac{dv}{dx} = \frac{v^2x^2-2x(vx)}{x^2-2x(vx)} = \frac{v^2-2v}{1-2v}$$
$$x\frac{dv}{dx} = \frac{v^2-2v}{1-2v} - v$$
$$x\frac{dv}{dx} = \frac{v^2-2v-v+2v^2}{1-2v} = \frac{3(v^2-v)}{1-2v}$$
$$\frac{(2v-1)dv}{(v^2-v)} = -3\frac{dx}{x}$$
$$\log|v^2-v| = -3\log|x| + \log|c|$$
$$\log|v^2-v| = \log\left|\frac{c}{x^3}\right|$$
$$v^2 - v = \frac{c}{x^3}$$

$\left(\frac{y}{x}\right)^2 - \frac{y}{x} = \frac{c}{x^3}$ or $xy^2 - x^2y = c$

7. $\left(1+3e^{\frac{y}{x}}\right)dy + 3e^{\frac{y}{x}}\left(1-\frac{y}{x}\right)dx = 0$, given that $y = 0$ when $x = 1$.

Solution:

$$\left(1+3e^{\frac{y}{x}}\right)dy + 3e^{\frac{y}{x}}\left(1-\frac{y}{x}\right)dx = 0$$

$$\frac{dy}{dx} = -\frac{3e^{\frac{y}{x}}\left(1-\frac{x}{y}\right)}{\left(1+3e^{\frac{y}{x}}\right)} \quad \ldots\ldots\ldots(1)$$

Put $y = vx \quad v = \frac{y}{x} \qquad \frac{dy}{dx} = v + x\frac{dv}{dx} \quad \ldots\ldots(2)$

Sub (2) in (1) $\qquad v + x\frac{dv}{dx} = -\frac{3e^v(1-v)}{(1+3e^v)}$

$$x\frac{dv}{dx} = -\frac{3e^v(1-v)}{(1+3e^v)} - v$$

$$x\frac{dv}{dx} = \frac{-3e^v+3ve^v-v-3ve^v}{(1+3e^v)}$$

$$\int\frac{(1+3e^v)}{(v+3e^v)}dv = -\int\frac{dx}{x}$$

$$\log(v+3e^v) = -\log x + \log c$$

EXERCISE 10.7

3. $\frac{dy}{dx}+\frac{y}{x}=\sin x$

Solution:

$$\frac{dy}{dx}+\frac{y}{x}=\sin x$$

Linear in y. $P=\frac{1}{x}$ and $Q=\sin x$

I.F $= e^{\int Pdx} = e^{\int \frac{1}{x}dx} = e^{\log|x|} = x$

Differential equation is

$y(I.F)=\int Q(I.F)dx+c$

$$yx=\int (\sin x)\, x\, dx+c$$

$$yx=\int \frac{1}{\sqrt{1-x^2}}\, dx+c$$

$$xy=-x\cos x+\sin x+c$$

7. $\left(y-e^{\sin^{-1}x}\right)\frac{dx}{dy}+\sqrt{1-x^2}=0$

Solution:

$$\left(y-e^{\sin^{-1}x}\right)\frac{dx}{dy}+\sqrt{1-x^2}=0$$

$$\left(y-e^{\sin^{-1}x}\right)\frac{dx}{dy}=-\sqrt{1-x^2}$$

$$\frac{dx}{dy}=\frac{-\sqrt{1-x^2}}{y-e^{\sin^{-1}x}}$$

$$\frac{dy}{dx}=\frac{y-e^{\sin^{-1}x}}{-\sqrt{1-x^2}}=\frac{-y}{\sqrt{1-x^2}}+\frac{e^{\sin^{-1}x}}{\sqrt{1-x^2}}$$

$$\frac{dy}{dx}+\frac{1}{\sqrt{1-x^2}}y=\frac{e^{\sin^{-1}x}}{\sqrt{1-x^2}}$$

Linear in y. $P=\frac{1}{\sqrt{1-x^2}}$ and $Q=\frac{e^{\sin^{-1}x}}{\sqrt{1-x^2}}$

I.F $= e^{\int Pdx} = e^{\int \frac{1}{\sqrt{1-x^2}}dx} = e^{\sin^{-1}x}$

Differential equation is $y(I.F)=\int Q(I.F)dx+c$

$$y\left(e^{\sin^{-1}x}\right)=\int\left(\frac{e^{\sin^{-1}x}}{\sqrt{1-x^2}}\right)\left(e^{\sin^{-1}x}\right)dx+c$$

$$y\left(e^{\sin^{-1}x}\right)=\int\frac{\left(e^{2\sin^{-1}x}\right)}{\sqrt{1-x^2}}dx+c$$

$$y\left(e^{\sin^{-1}x}\right)=\int e^{2\sin^{-1}x}\, d\left(\sin^{-1}x\right)+c$$

$$y\left(e^{\sin^{-1}x}\right)=\frac{e^{2\sin^{-1}x}}{2}+c$$

8. $\frac{dy}{dx}+\frac{y}{(1-x)\sqrt{x}}=1-\sqrt{x}$

Solution:

$$\frac{dy}{dx}+\frac{y}{(1-x)\sqrt{x}}=1-\sqrt{x}$$

Linear in y. $P=\frac{1}{(1-x)\sqrt{x}}$ and $Q=1-\sqrt{x}$

I.F $= e^{\int Pdx} = e^{\int\frac{1}{(1-x)\sqrt{x}}dx}$

$$=e^{\int\frac{1}{(\sqrt{x}-x)}dx}=\frac{1+\sqrt{x}}{1-\sqrt{x}}$$

Differential equation is $y(I.F)=\int Q(I.F)dx+c$

$$y\left(\frac{1+\sqrt{x}}{1-\sqrt{x}}\right)=\int(1-\sqrt{x})\left(\frac{1+\sqrt{x}}{1-\sqrt{x}}\right)dx+c$$

$$y\left(\frac{1+\sqrt{x}}{1-\sqrt{x}}\right)=\int(1+\sqrt{x})\, dx+c$$

$$y\left(\frac{1+\sqrt{x}}{1-\sqrt{x}}\right)=x+\frac{2}{3}x\sqrt{x}+c$$

9. $(1+x+xy^2)\frac{dy}{dx}+(y+y^3)=0$

Solution:

$$(1+x+xy^2)\frac{dy}{dx}+(y+y^3)=0$$

$$(1+x+xy^2)\frac{dy}{dx}=-(y+y^3)$$

$$\frac{dy}{dx}=-\frac{(y+y^3)}{(1+x+xy^2)}$$

$$\frac{dx}{dy}=-\frac{1+x(1+y^2)}{(y+y^3)}=-\frac{1}{y+y^3}-\frac{x(1+y^2)}{y(1+y^2)}$$

$$\frac{dx}{dy}=-\frac{1}{y+y^3}-\frac{x}{y}$$

$$\frac{dx}{dy}+\frac{x}{y}=-\frac{1}{y+y^3}$$

Linear in x. $P=\frac{1}{y}$ and $Q=\frac{-1}{y+y^3}=\frac{-1}{y(1+y^2)}$

I.F $= e^{\int\frac{1}{y}dy}=e^{\log y}=y$

Differential equation is $x(I.F)=\int Q(I.F)\, dy+c$

$$x(y)=\int\frac{-1}{y(1+y^2)}(y)\, dy+c$$

$$xy=-\int\frac{1}{1+y^2}dy+c$$

$$xy=-\tan^{-1}y+c$$

6. Assume that the rate at which radioactive nuclei decay is proportional to the number of such nuclei that are present in a given sample. In a certain sample 10% of the original number of radioactive nuclei have undergone disintegration in a period of 100 years. What percentage of the original radioactive nuclei will remain after 1000 years?

Solution:

Let A be the amount of radioactive nuclei present at any time t.

$$\frac{dA}{dt} \propto A$$

$$\frac{dA}{dt} = -kA \qquad k > 0$$

$$A = Ce^{-kt}$$

When $t = 0$, $A = A_0 \rightrightarrows c = A_0$

When $t = 100$, $A = 90\%\ Of\ A_0$

$$\frac{90}{100}A_0 = A_0e^{-100k}$$

$$e^{-100k} = \frac{9}{10}$$

When $t = 1000$, $A = ?$

$$A = A_0e^{-1000k}$$

$$A = A_0\left(e^{-10k}\right)^{10}$$

$$A = A_0\left(\frac{9}{10}\right)^{10}\%$$

Hence the radioactive nuclei after 1000 years is

$$A = A_0\left(\frac{9}{10}\right)^{10}(100)\%$$

$$A = A_0\left(\frac{9^{10}}{10^{8}}\right)\%$$

7. Water at temperature 100^0C cools in 10 minutes to 80^0C in a room temperature of 25^0C. Find

(i) The temperature of water after 20 minutes

(ii) The time when the temperature is 40^0C

$$\left[\log_e\frac{11}{15} = -0.3101;\ \log_e 5 = 1.6094\right]$$

Solution:

Let T be the temperature of water at any time t.

$$\frac{dT}{dt} \propto (T-S)$$

$$\frac{dT}{dt} = k(T-S)$$

$$T - S = Ce^{kt}$$

When $t = 0$, $T = 100$, $s = 25 \rightrightarrows c = 75$

When $t = 10$, $T = 80$, $s = 25$

$e^{10k} = \frac{55}{75} = \frac{11}{15}$ $\qquad k = \frac{1}{10}log\left(\frac{11}{15}\right)$

(i) When $t = 20$, $T = ?$, $s = 25$

$$T - 25 = 75\left(e^{10t}\right)^2$$

$$T = 25 + 75\left(\frac{11}{15}\right)^2$$

$$T = 25 + 75\frac{11\times 11}{15\times 15} = 25 + \frac{121}{3}$$

$$= 25 + 40.33 = 65.33^0C$$

The temperature of water after 20 minutes is 65.33^0C (approximately)

(ii) when $T = 40$ $\qquad 40 - 25 = 75e^{kt}$

$$75e^{kt} = 15$$

$$e^{kt} = \frac{1}{5}$$

$$kt = \log\left(\frac{1}{5}\right)$$

$$t = \frac{\log\left(\frac{1}{5}\right)}{k} = 10\frac{\log\left(\frac{1}{5}\right)}{log\left(\frac{11}{15}\right)}$$

$$t = -10\frac{1.6094}{-0.3101} \approx 51.899$$

The time when the temperature 40^0C is 51.9 minutes (approximately).

Do it your Self Examples

10.15,

10.20,

10.26.

Example 11.8 A six sided die is marked '1' on one face, '2' on two of its faces, and '3' on remaining three faces. The die is rolled twice. If X denotes the total score in two throws.

(i) Find the probability mass function. (ii) Find the cumulative distribution function.

(iii) Find $P(3 \le X < 6)$ (iv) Find $(X \ge 4)$.

Solution:

Number of Sample space n(S) = **36**

$$S = \begin{Bmatrix} (1,1),(1,2),(1,2),(1,3),(1,3),(1,3) \\ (2,1),(2,2),(2,2),(2,3),(2,3),(2,3) \\ (2,1),(2,2),(2,2),(2,3),(2,3),(2,3) \\ (3,1),(3,2),(3,2),(3,3),(3,3),(3,3) \\ (3,1),(3,2),(3,2),(3,3),(3,3),(3,3) \\ (3,1),(3,2),(3,2),(3,3),(3,3),(3,3) \end{Bmatrix}$$

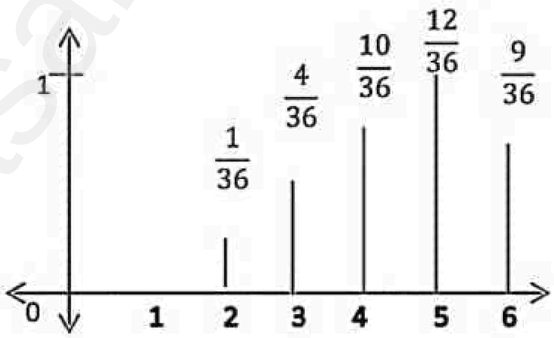
Let X is a random variables takes the values 2, 3, 4, 5 and 6.

Let X (ω) denotes the total score in two throws, this gives

Values of Random Variables	2	3	4	5	6	Total
Number of points in inverse image	1	4	10	12	9	36

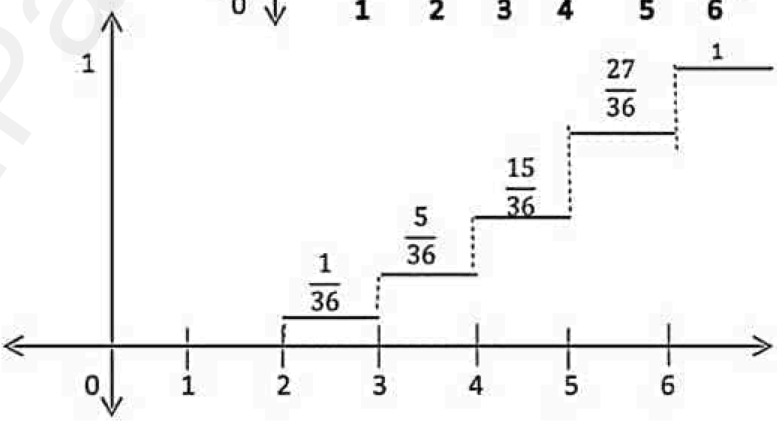
(i) The probability mass function is given by

x	2	3	4	5	6
$f(x)$	$\frac{1}{36}$	$\frac{4}{36}$	$\frac{10}{36}$	$\frac{12}{36}$	$\frac{9}{36}$



(ii) Cumulative distribution function is

$$F(x) = \begin{cases} 0; & -\infty < x < 2 \\ \frac{1}{36}; & 2 \le x < 3 \\ \frac{5}{36}; & 3 \le x < 4 \\ \frac{15}{36}; & 4 \le x < 5 \\ \frac{27}{36}; & 5 \le x < 6 \\ 1; & 6 < x < \infty \end{cases}$$



(iii) $P(3 \le X < 6) = P(X = 3) + P(X = 4) + P(X = 5) = \frac{26}{36}$

(iv) $P(X \ge 4) = P(X = 4) + P(X = 5) + P(X = 6) = \frac{31}{36}$

CHAPTER –12

Verify the (i) closure property, (ii) commutative property, (iii) associative property (iv) existence of identity and (v) existence of inverse for the arithmetic operation + on ℤ Solution:

(i) Sum of two integer is again an integer , closure property is true

Hence + is a binary operation on ℤ .

(ii) Also m + n = n + m ,_ $\forall m,n \in \mathbb{Z}$.

So the commutative property is satisfied

(iii)∀a,b,c∈ℤ , a + (b + c) = (a + b) + c
Hence the associative property is satisfied
(iv) a + e = e + a = a⇒ e = 0.
Thus ∃ 0 ∈ ℤ, ∍ (a + 0) = (0 + a) = a.
Hence the existence of identity is assured.

(v) ∀ a ∈ ℤ, ∃ – a∈ℤ, such that a+ (–a) = (–a) + a = 0. **Hence, the existence of inverse property is also assured.**
Thus we see that the usual addition + on ℤ satisfies all the above five properties. Note that the additive identity is 0 and the additive inverse of any integer m is $-m$.

....

......Verify the (i) closure property, (ii) commutative property, (iii) associative property (iv) existence of identity and (v)existence of inverse for the arithmetic operation – *on*ℤ Solution:

(i)Closure property :
Though – is not binary on ℕ; it is binary on ℤ . To check the validity of any more properties satisfied by – on ℤ , it is better to check them for some particular simple values.

(ii) Take m = 4, n = 5 and
(m – n) = (4 – 5) = –1and
(n – m) = (5 – 4) =1.
Hence (m – n) ≠ (n – m). So the operation – is not commutative on ℤ.
(iii) In order to check the associative property, let us put m = 4, n = 5 and p = 7 in both (m – n) – p and m – (n – p) .

(m – n)– p = (4 –5)–7=(–1–7)=–8 ...(1)

m –(n – p)= 4 –(5 –7)=(4+2)=6 . ... (2)

From (1) and (2), it follows that

(m – n) – p ≠ m – (n – p).

Hence – is not associative on ℤ .

(iv) Identity does not exist

(v) So, Inverse does not exist .

Verify the (i) closure property, (ii) commutative property, (iii) associative property (iv) existence of identity and (v) existence of inverse for the arithmetic operation + on ℤ.Z = the set of all even integers. **Solution** : Consider the set of all even integers ℤ= {2 k | k ∈ℤ} = {...,–6,–4,–2,0,2,4,6,...}. Let us verify the properties satisfied by + on ℤ

(i) The sum of any two even integers is also an even integer. Because x, ∈ℤe⇒x = 2m and y = 2 n , m,n∈ℤ .
So (x + y) = 2m + 2= 2(m + n) ∈ℤ
= 2(n + m) = 2n + 2m = (y + x).
Hence + is a binary operation on ℤ.

(ii)∀x, ∈ℤ,
(x + y) = 2m + 2n= 2(m + n) =2(n + m)
= 2n + 2m = (y + x).
So + has commutative property.**(iii)**
Similarly it can be seen that ∀x,,z∈ℤ,
(x + y) + z = x + (y + z) . Hence the associative property is true.
(iv) Now take x = 2k, then
2k + e = e + 2k = 2k⇒e = 0 . Thus
∀x∈ℤ , ∃ 0 ℤ , ∍x + 0 = 0 + x = x. So,

So, $A * B \in M = B * A \in M$

$\therefore *$ is commutative on M

(3) Associative

Matrix multiplication is always associative.

That is $A * (B * C) = (A * B) * C, \forall A, B, C \in M$.

$\therefore *$ is also associative on M

Solution: (ii)

(1) Closure

Let $A = \begin{pmatrix} x & x \\ x & x \end{pmatrix}$ and $B = \begin{pmatrix} y & y \\ y & y \end{pmatrix} : x, y \in R-(0)$

Hence $M = \{A, B\}$

Now, $AB = \begin{pmatrix} x & x \\ x & x \end{pmatrix}\begin{pmatrix} y & y \\ y & y \end{pmatrix}$

$= \begin{pmatrix} xy+xy & xy+xy \\ xy+xy & xy+xy \end{pmatrix}$

$= \begin{pmatrix} 2xy & 2xy \\ 2xy & 2xy \end{pmatrix} \in M$

Since, $x, y \in R-(0)$ gives $xy\ also \in R-(0)$

So, $AB \in M \Rightarrow A * B \in M$

$\therefore *$ is closed on M

(2) Existence of Identity

Let $A = \begin{pmatrix} x & x \\ x & x \end{pmatrix}$ and

$E = \begin{pmatrix} e & e \\ e & e \end{pmatrix}$ be the identity,

such that : $a, e \in R-(0)$

Hence $M = \{A, E\}$

Now, $A * E = E * A = A$

$\begin{pmatrix} x & x \\ x & x \end{pmatrix}\begin{pmatrix} e & e \\ e & e \end{pmatrix} = \begin{pmatrix} x & x \\ x & x \end{pmatrix}$

$\begin{pmatrix} 2xe & 2xe \\ 2xe & 2xe \end{pmatrix} = \begin{pmatrix} x & x \\ x & x \end{pmatrix}$

$2xe = x$

$2e = 1$

$e = \frac{1}{2} \in R-(0)$

$\therefore E = \begin{pmatrix} \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & \frac{1}{2} \end{pmatrix}$ is the identity $\in M$

$\therefore *$ has identity on M

(3) Existence of Inverse

Let $A = \begin{pmatrix} x & x \\ x & x \end{pmatrix}$ and

$A^{-1} = \begin{pmatrix} x^{-1} & x^{-1} \\ x^{-1} & x^{-1} \end{pmatrix}$ be the inverse of A.

Then $A * A^{-1} = A^{-1} * A = E$

$\begin{pmatrix} x & x \\ x & x \end{pmatrix}\begin{pmatrix} x^{-1} & x^{-1} \\ x^{-1} & x^{-1} \end{pmatrix} = \begin{pmatrix} e & e \\ e & e \end{pmatrix}$

$\begin{pmatrix} 2xx^{-1} & 2xx^{-1} \\ 2xx^{-1} & 2xx^{-1} \end{pmatrix} = \begin{pmatrix} \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & \frac{1}{2} \end{pmatrix}$

$2xx^{-1} = \frac{1}{2}$

$x^{-1} = \frac{1}{4x}, \in R-(0)$

$\therefore A^{-1} = \begin{pmatrix} \frac{1}{4x} & \frac{1}{4x} \\ \frac{1}{4x} & \frac{1}{4x} \end{pmatrix}$ is the inverse of $A \in M$

$\therefore *$ has inverse on M

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10. (i) Let $A\ be\ \mathbb{Q} \setminus \{1\}$. Define $*$ on A by $x * y = x + y - xy$. Is $*$ binary on A? If so, examine the commutative and associative properties satisfied by $*$ on A.

(ii) Let $A\ be\ \mathbb{Q} \setminus \{1\}$. Define $*$ on A by $x * y = x + y - xy$. Is $*$ binary on A? If so, examine the existence of identity, existence of inverse properties for the operation $*$ on A.

Solution: (i) (1) Closure

Given $A = \{\mathbb{Q} \setminus \{1\}\}$

Let $x, y \in A$. That is $x \neq 1, and\ y \neq 1$

$*$ is defined on A by

$x * y = x + y - xy$

Let us assume that $x * y = 1$. Then,

$x + y - xy = 1$

$y - xy = 1 - x$

$y(1-x) = 1 - x$

$y = \frac{1-x}{(1-x)}$

It Gives, $y = 1$, which is contradiction to our assumption.

Hence, $x * y \neq 1 \in A$

$\therefore *$ is closed on A

Solution:

$p \to (q \to r) \equiv p \to (\neg q \vee r)$

$\equiv \neg p \vee (\neg q \vee r)$

$\equiv (\neg p \vee \neg q) \vee r$

$\equiv \neg (p \wedge q) \vee r$

$\equiv (p \wedge q) \to r$ Hence proved.

15. Prove that $p \to (\neg q \vee r) \equiv \neg p \vee (\neg q \vee r)$ using truth table.

LHS : $p \to (\neg q \vee r)$

p	q	r	$\neg q$	$(\neg q \vee r)$	$p \to (\neg q \vee r)$
T	T	T	F	T	T
T	T	F	F	F	F
T	F	T	T	T	T
T	F	F	T	T	T
F	T	T	F	T	T
F	T	F	F	F	T
F	F	T	T	T	T
F	F	F	T	T	T

RHS : $\neg p \vee (\neg q \vee r)$

p	q	r	$\neg q$	$(\neg q \vee r)$	$\neg p$	$\neg p \vee (\neg q \vee r)$
T	T	T	F	T	F	T
T	T	F	F	F	F	F
T	F	T	T	T	F	T
T	F	F	T	T	F	T
F	T	T	F	T	T	T
F	T	F	F	F	T	T
F	F	T	T	T	T	T
F	F	F	T	T	T	T

From the above tables, the last columns are identical. Hence $p \to (\neg q \vee r) \equiv \neg p \vee (\neg q \vee r)$.

Do it your self
Example
12.13, 12.15