CHAPTER 1

APPLICATIONS OF MATRICES AND DETERMINANTS

MUST KNOW DEFINITIONS

- If $|A| \neq 0$, then A is a non-singular matrix and if $|A| = 0$, then A is a singular matrix.
- The adjoint matrix of A is defined as the transpose of the matrix of co-factors of A.
- If $AB = BA = I_n$, then the matrix B is called the inverse of A.
- If a square matrix has an inverse, then it is unique.
- A^{-1} exists if and only if A is non-singular.
- Singular matrix has no inverse.
- If A is non – singular and AB = AC, then B = C (left cancellation law).
- If A is non – singular and BA = CA then B = C (Right cancellation law).
- If A and B are any two non-singular square matrices of order n, then adj (AB) = (adj B) (adj A)
- A square matrix A is called orthogonal if $AA^T = A^TA = I$
- Two matrices A and B of same order are said to the **equivalent** if one can be obtained from the other by the applications of elementary transformations (A~B).
- A non – zero matrix is in a **row - echelon form** if all zero rows occur as bottom rows of the matrix and if the first non – zero element in any lower row occurs to the right of the first non – zero entry in the higher row.
- The **rank** of a matrix A is defined as the order of a highest order non – vanishing minor of the matrix A [$\rho(A)$].
- The **rank** of a non – zero matrix is equal to the number of non – zero rows in a row – echelon form of the matrix.
- An **elementary matrix** is a matrix which is obtained from an identity matrix by applying only one elementary transformation. Every non-singular matrix can be transformed to an identity matrix by a sequence of elementary row operations.
- A system of linear equations having atleast one solution is said to be **consistent**.
- A system of linear equations having no solutions is said to be **inconsistent**.

Public Exam Frequently Asked Questions

1 MARK

1. If A and B are orthogonal, then $(AB)^T$ (AB) is [PTA - 1]

(1) A (2) B (3) I (4) A^T

[Ans: (3) I]

Hint :
$$AA^T = A^TA = I$$
$$BB^T = B^TB = I$$
$$(AB)^T(AB) = B^TA^T(AB) = B^T(A^TA)B = B^T(IB) = I$$

2. The adjoint of 3×3 matrix P is $\begin{vmatrix} -1 & 2 & 2 \\ 1 & 1 & 2 \\ 2 & 2 & 1 \end{vmatrix}$, then the possible value(s) of the determinant P is (are) [PTA - 4]

(1) 3 (2) –3 (3) ± 3 (4) $\pm\sqrt{3}$

[Ans: (3) ±3]

Hint : $\begin{vmatrix} -1 & 2 & 2 \\ 1 & 1 & 2 \\ 2 & 2 & 1 \end{vmatrix} = -1[1-4]-2[1-4]+2[2-2]$

$= -1(-3)-2(-3)+0 = 3+6 = 9$

$|P| = \pm\sqrt{9} = \pm 3$

3. If A is a 3 × 3 matrix such that |3*adj* A| = 3 then |A| is equal to [PTA - 5]

(1) $\frac{1}{3}$ (2) $-\frac{1}{3}$ (3) $\pm\frac{1}{3}$ (4) ± 3

[Ans: (3) $\pm\frac{1}{3}$]

Hint : $|3\, adj\, A| = 27\,|adj\, A|$
$$3 = 27|A|^2$$
$$|A|^2 = \frac{1}{9}$$
$$|A| = \pm\frac{1}{3}$$

4. Let A be a non-singular matrix then which one of the following is false [PTA - 6]

(1) $(adjA)^{-1} = \frac{A}{|A|}$
(2) I is an orthogonal matrix
(3) $adj(adjA) = |A|^n A$
(4) If A is symmetric then *adj* A is symmetric

[Ans: (3) $adj(adjA) = |A|^n A$]

Hint : $adj\,(adjA) = |A|^{n-2} A$

5. If the inverse of the matrix $\begin{bmatrix} 1 & 2 \\ 3 & -5 \end{bmatrix}^n$ is $\frac{1}{11}\begin{bmatrix} a & b \\ c & d \end{bmatrix}$, then the ascending order of *a, b, c, d* is [Govt. MQP-2019]

(1) *a, b, c, d* (2) *d, b, c, a*
(3) *c, a, b, d* (4) *b, a, c, d*

[Ans: (2) *d, b, c, a*]

Hint : Inverse matrix $= \frac{1}{-5-6}\begin{bmatrix} -5 & -2 \\ -3 & 1 \end{bmatrix}$

$= \frac{-1}{11}\begin{bmatrix} -5 & -2 \\ -3 & 1 \end{bmatrix} = \begin{bmatrix} \frac{5}{11} & \frac{2}{11} \\ \frac{3}{11} & \frac{-1}{11} \end{bmatrix} = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$

6. If A is an orthogonal matrix, then |A| is [Qy - 2019]

(1) 1 (2) –1 (3) ± 1 (4) 0

[Ans: (3) ± 1]

Hint : The determinant of an orthogonal matrix is equal to 1 or –1.

7. The system of linear equations $x + y + z = 2$, $2x + y - z = 3$, $3x + 2y + kz = 4$ has a unique solution if [Qy - 2019]

(1) $k \neq 0$ (2) $-1 < k < 1$
(3) $-2 < k < 2$ (4) $k = 0$ [Ans: (1) $k \neq 0$]

Hint : $\begin{vmatrix} 1 & 1 & 1 \\ 2 & 1 & -1 \\ 3 & 2 & k \end{vmatrix} \neq 0$

$\Rightarrow 1[k+2]-1[2k+3]+1[4-3] \neq 0$

$\Rightarrow k+2-2k-3+1 \neq 0 \Rightarrow -k+0 \neq 0 \Rightarrow k \neq 0$

8. The inverse of $\begin{bmatrix} 3 & 1 \\ 5 & 2 \end{bmatrix}$ is : [Aug, - 2021]

(1) $\begin{bmatrix} 3 & -1 \\ -5 & -3 \end{bmatrix}$ (2) $\begin{bmatrix} 2 & -1 \\ -5 & 3 \end{bmatrix}$

(3) $\begin{bmatrix} -3 & 5 \\ 1 & -2 \end{bmatrix}$ (4) $\begin{bmatrix} -2 & 5 \\ 1 & -3 \end{bmatrix}$

[Ans: (2) $\begin{bmatrix} 2 & -1 \\ -5 & 3 \end{bmatrix}$]

Hint : $A^{-1} = \frac{1}{|A|} adj\, A = \frac{1}{1}\begin{bmatrix} 2 & -1 \\ -5 & 3 \end{bmatrix} = \begin{bmatrix} 2 & -1 \\ -5 & 3 \end{bmatrix}$

9. The adjoint of $\begin{bmatrix} -2 & 1 \\ -1 & 4 \end{bmatrix}$ is [FRT - 2022]

(1) $\begin{bmatrix} 4 & -1 \\ 1 & -2 \end{bmatrix}$ (2) $\begin{bmatrix} -2 & -1 \\ 1 & 4 \end{bmatrix}$

(3) $\begin{bmatrix} -4 & 1 \\ -1 & 2 \end{bmatrix}$ (4) $\begin{bmatrix} 4 & 1 \\ -1 & -2 \end{bmatrix}$

[**Ans:** (1) $\begin{bmatrix} 4 & -1 \\ 1 & -2 \end{bmatrix}$]

10. Which one of the following is incorrect? [May - 2022]

(1) If A is a square matrix of order n, and λ is a scalar, then Adj $(\lambda A) = \lambda^n$ (Adj A).
(2) Adjoint of a symmetric matrix is also a symmetric matrix.
(3) A(Adj A) = (Adj A) A = |A|I.
(4) Adjoint of a diagonal matrix is also a diagonal matrix.

[**Ans:** (1) If A is a square matrix of order n, and λ is a scalar, then Adj $(\lambda A) = \lambda^n$ (Adj A).]

2 MARKS

1. Prove that $\begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}$ is orthogonal. [PTA - 1; FRT -2022]

Sol. Let $A = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}$

Then, $A^T = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}^T = \begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix}$

$= \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}\begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix}$

$= \begin{bmatrix} \cos^2\theta + \sin^2\theta & \cos\theta\sin\theta - \sin\theta\cos\theta \\ \sin\theta\cos\theta - \cos\theta\sin\theta & \sin^2\theta + \cos^2\theta \end{bmatrix}$

$= \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = I_2.$

Similarly, we get $A^TA = I_2$.

Hence $AA^T = A^TA = I_2 \Rightarrow$ A is orthogonal.

2. If $A = \begin{bmatrix} 2 & 3 \\ 1 & 2 \end{bmatrix}$, $B = \begin{bmatrix} 4 & 0 \\ 2 & 5 \end{bmatrix}$, find *adj* (AB). [PTA - 3]

Sol. $AB = \begin{bmatrix} 2 & 3 \\ 1 & 2 \end{bmatrix}\begin{bmatrix} 4 & 0 \\ 2 & 5 \end{bmatrix} = \begin{bmatrix} 8+6 & 0+15 \\ 4+4 & 0+10 \end{bmatrix} = \begin{bmatrix} 14 & 15 \\ 8 & 10 \end{bmatrix}$

$adj\,(AB) = \begin{bmatrix} 10 & -15 \\ -8 & 14 \end{bmatrix}$

3. Solve the following system of linear equations by Cramer's rule $2x - y = 3$, $x + 2y = -1$. [Govt. MQP-2019]

Sol.

$$2x - y = 3$$
$$x + 2y = -1$$

$$\Delta = \begin{vmatrix} 2 & -1 \\ 1 & 2 \end{vmatrix} = 2 \times 2 - (1)(-1) = 4 + 1 = 5$$

$$\Delta_1 = \begin{vmatrix} 3 & -1 \\ -1 & 2 \end{vmatrix} = 3 \times 2 - (-1) \times -1 = 6 - 1 = 5$$

$$\Delta_2 = \begin{vmatrix} 2 & 3 \\ 1 & -1 \end{vmatrix} = -2 - 3 = -5$$

$$x = \frac{\Delta_1}{\Delta} = \frac{5}{5} = 1$$

$$y = \frac{\Delta_2}{\Delta} = \frac{-5}{5} = -1$$

4. If adj A = $\begin{bmatrix} -1 & 2 & 2 \\ 1 & 1 & 2 \\ 2 & 2 & 1 \end{bmatrix}$, find A^{-1}. [Qy - 2019]

Sol. We compute |adj A| = 9

So, we get $A^{-1} = \pm \frac{1}{\sqrt{|adj\ A|}}$ adj (A)

$$= \pm \frac{1}{\sqrt{9}}\begin{bmatrix} -1 & 2 & 2 \\ 1 & 1 & 2 \\ 2 & 2 & 1 \end{bmatrix} = \pm \frac{1}{3}\begin{bmatrix} -1 & 2 & 2 \\ 1 & 1 & 2 \\ 2 & 2 & 1 \end{bmatrix}$$

3 MARKS

1. If $A = \begin{bmatrix} 3 & -2 \\ \lambda & -2 \end{bmatrix}$, find the value of λ so that $A^2 = \lambda A - 2I$. [PTA - 2]

Sol. $A^2 = \begin{bmatrix} 3 & -2 \\ \lambda & -2 \end{bmatrix}\begin{bmatrix} 3 & -2 \\ \lambda & -2 \end{bmatrix}$

$= \begin{bmatrix} 9-2\lambda & -6+4 \\ 3\lambda-2\lambda & -2\lambda+4 \end{bmatrix} = \begin{bmatrix} 9-2\lambda & -2 \\ \lambda & -2\lambda+4 \end{bmatrix}$

$\lambda A - 2I = \lambda\begin{bmatrix} 3 & -2 \\ \lambda & -2 \end{bmatrix} - 2\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$

$= \begin{bmatrix} 3\lambda & -2\lambda \\ \lambda^2 & -2\lambda \end{bmatrix} - \begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix} = \begin{bmatrix} 3\lambda-2 & -2\lambda \\ \lambda^2 & -2\lambda-2 \end{bmatrix}$

$\therefore -2\lambda = -2$

$\lambda = 1$

2. Find the rank of the matrix $\begin{bmatrix} 4 & -2 & 6 & 8 \\ 1 & 1 & -3 & -1 \\ 15 & -3 & 9 & 21 \end{bmatrix}$ **[PTA - 4]**

Sol. $\begin{bmatrix} 4 & -2 & 6 & 8 \\ 1 & 1 & -3 & -1 \\ 15 & -3 & 9 & 21 \end{bmatrix} \xrightarrow{R_1 \leftrightarrow R_2} \begin{pmatrix} 1 & 1 & -3 & -1 \\ 4 & -2 & 6 & 8 \\ 15 & -3 & 9 & 21 \end{pmatrix}$

$\xrightarrow[R_3 \to R_3 + (-15)R_1]{R_2 \to R_2 + (4)R_1} \begin{bmatrix} 1 & 1 & -3 & -1 \\ 0 & -6 & 18 & 12 \\ 0 & -18 & 54 & 36 \end{bmatrix}$

$\xrightarrow[R_3 \to \frac{R_3}{-18}]{R_2 \to \frac{R_2}{-6}} \begin{bmatrix} 1 & 1 & -3 & -1 \\ 0 & 1 & -3 & -2 \\ 0 & 1 & -3 & -2 \end{bmatrix} \xrightarrow{R_3 \to R_3 - R_2} \begin{bmatrix} 1 & 1 & -3 & -1 \\ 0 & 1 & -3 & -2 \\ 0 & 0 & 0 & 0 \end{bmatrix}$

3. Solve by matrix inversion method : $5x + 2y = 4$, $7x + 3y = 5$.

Sol. Matrix form AX = B **[PTA - 5]**

$A = \begin{bmatrix} 5 & 2 \\ 7 & 3 \end{bmatrix}, X = \begin{bmatrix} x \\ y \end{bmatrix}, B = \begin{bmatrix} 4 \\ 5 \end{bmatrix}$

$|A| = \begin{vmatrix} 5 & 2 \\ 7 & 3 \end{vmatrix} = 15 - 14 = 1$

$A^{-1} = \begin{bmatrix} 3 & -2 \\ -7 & 5 \end{bmatrix}$

$X = A^{-1}B$

$\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 3 & -2 \\ -7 & 5 \end{bmatrix}\begin{bmatrix} 4 \\ 5 \end{bmatrix}$

$\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 12 & -10 \\ -28 & +25 \end{bmatrix} = \begin{bmatrix} 2 \\ -3 \end{bmatrix}$

$x = 2, y = -3$

4. Find the adjoint of the matrix $A = \begin{bmatrix} 1 & 3 \\ 2 & -5 \end{bmatrix}$ and verify that $A(adjA) = (adjA)A = |A|I$. **[PTA - 6]**

Sol. $A = \begin{bmatrix} 1 & 3 \\ 2 & -5 \end{bmatrix}$

$\text{adj } A = \begin{bmatrix} -5 & -3 \\ -2 & 1 \end{bmatrix}$

$A(\text{adj}A) = \begin{bmatrix} 1 & 3 \\ 2 & -5 \end{bmatrix}\begin{bmatrix} -5 & -3 \\ -2 & 1 \end{bmatrix}$

$= \begin{bmatrix} -5-6 & -3+3 \\ -10+10 & -6-5 \end{bmatrix} = \begin{bmatrix} -11 & 0 \\ 0 & -11 \end{bmatrix}$ ...(1)

$(\text{adj } A)A = \begin{bmatrix} -5 & -3 \\ -2 & 1 \end{bmatrix}\begin{bmatrix} 1 & 3 \\ 2 & -5 \end{bmatrix}$

$= \begin{bmatrix} -5-6 & -15+15 \\ -2+2 & -6-5 \end{bmatrix} = \begin{bmatrix} -11 & 0 \\ 0 & -11 \end{bmatrix}$ ...(2)

$|A| = \begin{vmatrix} 1 & 3 \\ 2 & -5 \end{vmatrix} = -5 - 6 = -11$

$|A|I = -11\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} -11 & 0 \\ 0 & -11 \end{bmatrix}$ ...(3)

From (1), (2) and (3)

$A(adjA) = (adj\,A)A = |A|I$

5. Verify $(AB)^{-1} = B^{-1}A^{-1}$ with $A = \begin{bmatrix} 0 & -3 \\ 1 & 4 \end{bmatrix}$, $B = \begin{bmatrix} -2 & -3 \\ 0 & -1 \end{bmatrix}$ **[Sep.- 2020; July - 2022]**

Sol. We get

$AB = \begin{bmatrix} 0 & -3 \\ 1 & 4 \end{bmatrix}\begin{bmatrix} -2 & -3 \\ 0 & -1 \end{bmatrix}$

$= \begin{bmatrix} 0+0 & 0+3 \\ -2+0 & -3-4 \end{bmatrix}\begin{bmatrix} 0 & 3 \\ -2 & -7 \end{bmatrix}$

$(AB)^{-1} = \frac{1}{(0+6)}\begin{bmatrix} -7 & -3 \\ 2 & 0 \end{bmatrix} = \frac{1}{6}\begin{bmatrix} -7 & -3 \\ 2 & 0 \end{bmatrix}$ ...(1)

$A^{-1} = \frac{1}{(0+3)}\begin{bmatrix} 4 & 3 \\ -1 & 0 \end{bmatrix} = \frac{1}{3}\begin{bmatrix} 4 & 3 \\ -1 & 0 \end{bmatrix}$

$B^{-1} = \frac{1}{(2-0)}\begin{bmatrix} -1 & 3 \\ 0 & -2 \end{bmatrix} = \frac{1}{2}\begin{bmatrix} -1 & 3 \\ 0 & -2 \end{bmatrix}$

$B^{-1}A^{-1} = \frac{1}{2}\begin{bmatrix} -1 & 3 \\ 0 & -2 \end{bmatrix}\frac{1}{3}\begin{bmatrix} 4 & 3 \\ -1 & 0 \end{bmatrix}$

$= \frac{1}{6}\begin{bmatrix} -4-3 & -3+0 \\ 0+2 & 0+0 \end{bmatrix}\frac{1}{6}\begin{bmatrix} -7 & -3 \\ 2 & 0 \end{bmatrix}$ ...(2)

As the matrices in (1) and (2) are same,

$(AB)^{-1} = B^{-1}A^{-1}$ is verified.

6. Solve the following system of linear equations, using matrix inversion method : $5x + 2y = 3$, $3x + 2y = 5$. **[May - 2022]**

Sol. The matrix form of the system is AX = B, where

$A = \begin{bmatrix} 5 & 2 \\ 3 & 2 \end{bmatrix}, X = \begin{bmatrix} x \\ y \end{bmatrix}, B = \begin{bmatrix} 3 \\ 5 \end{bmatrix}$

We find $|A| = \begin{vmatrix} 5 & 2 \\ 3 & 2 \end{vmatrix} = 10 - 6 = 4 \neq 0$. So A^{-1} exists and $A^{-1} = \frac{1}{4}\begin{bmatrix} 2 & -2 \\ -3 & 5 \end{bmatrix}$.

Then, applying the formula $X = A^{-1}B$, we get

$$\begin{bmatrix}x\\y\end{bmatrix}=\frac{1}{4}\begin{bmatrix}2&-2\\-3&5\end{bmatrix}\begin{bmatrix}3\\5\end{bmatrix}=\frac{1}{4}\begin{bmatrix}6&-10\\-9&+25\end{bmatrix}$$

$$=\frac{1}{4}\begin{bmatrix}-4\\16\end{bmatrix}\begin{bmatrix}\frac{-4}{4}\\\frac{16}{4}\end{bmatrix}=\begin{bmatrix}-1\\4\end{bmatrix}$$

So the solution is $x = -1, y = 4$.

7. If A = $\begin{pmatrix}5&2\\7&3\end{pmatrix}$ and B = $\begin{pmatrix}2&-1\\-1&1\end{pmatrix}$ then verify $(AB)^{-1} = B^{-1}A^{-1}$. **[FRT - 2022]**

Sol.

$$A = \begin{pmatrix}5&2\\7&3\end{pmatrix}\ B=\begin{pmatrix}2&-1\\-1&1\end{pmatrix}$$

$$AB = \begin{pmatrix}5&2\\7&3\end{pmatrix}\begin{pmatrix}2&-1\\-1&1\end{pmatrix}$$

$$= \begin{pmatrix}10-2&-5+2\\14-3&-7+3\end{pmatrix}=\begin{pmatrix}8&-3\\11&-4\end{pmatrix}$$

$$|AB| = \begin{vmatrix}8&-3\\11&-4\end{vmatrix}=-32+33=1$$

$$(AB)^{-1} = \frac{1}{|AB|}\text{adj }AB=\frac{1}{1}\begin{pmatrix}-4&3\\-11&8\end{pmatrix} \quad \ldots (1)$$

$$B^{-1} = \frac{1}{|B|}\text{adj }B=\frac{1}{(2-1)}\begin{bmatrix}1&1\\1&2\end{bmatrix}=\begin{bmatrix}1&1\\1&2\end{bmatrix}$$

$$A^{-1} = \frac{1}{|A|}\text{adj }A$$

$$A^{-1} = \frac{1}{15-14}\begin{pmatrix}3&-2\\-7&5\end{pmatrix}=\begin{pmatrix}3&-2\\-7&5\end{pmatrix}$$

$$B^{-1}A^{-1} = \begin{bmatrix}1&1\\1&2\end{bmatrix}\begin{bmatrix}3&-2\\-7&5\end{bmatrix}$$

$$= \begin{bmatrix}3-7&-2+5\\3-14&-2+10\end{bmatrix}=\begin{bmatrix}-4&3\\-11&8\end{bmatrix} \ldots (2)$$

From (1) and (2) $(AB)^{-1} = B^{-1}A^{-1}$.

Hence proved.

5 MARKS

1. Examine the consistency of the system of equation $4x + 3y + 6z = 25$, $x + 5y + 7z = 13$. $2x + 9y + z = 1$. It it is consistent then solve. **[PTA - 1]**

Sol. Transforming the augmented matrix to echelon form, we get

$$\left[\begin{array}{ccc|c}4&3&6&25\\1&5&7&13\\2&9&1&1\end{array}\right]\xrightarrow{R_1\to R_2}\left[\begin{array}{ccc|c}1&5&7&13\\4&3&6&25\\2&9&1&1\end{array}\right]$$

$$\xrightarrow[R_3\to R_3-2R_1]{R_2\to R_2-4R_1,}\left[\begin{array}{ccc|c}1&5&7&13\\0&-17&-22&-27\\0&-1&-13&-25\end{array}\right]$$

$$\xrightarrow[R_3\to R_3\div(-1)]{R_2\to R_2\div(-1),}\left[\begin{array}{ccc|c}1&5&7&13\\0&17&22&27\\0&1&13&25\end{array}\right]$$

$$\xrightarrow{R_3\to 17R_3-R_2}\left[\begin{array}{ccc|c}1&5&7&13\\0&17&22&27\\0&0&199&398\end{array}\right]$$

The equivalent system is written by using the echelon form:

$$x + 5y + 7z = 13, \quad \ldots (1)$$
$$17y + 22z = 27, \quad \ldots (2)$$
$$199z = 398. \quad \ldots (3)$$

From (3), we get $z = \frac{398}{199} = 2$.

Substituting $z = 2$ in (2), we get $\frac{27-22\times 2}{17}=\frac{-17}{17}=-1$

Substituting $z = 2, y = -1$ in (1), we get

$x = 13 - 5 \times (-1) - 7 \times 2 = 4$.

So, the solution is $(x = 4, y = -1, z = 2)$.

Note. The above method of going from the last equation to the first equation is called the method of back substitution.

2. Find the inverse of the non-singular matrix A = $\begin{bmatrix}2&1&1\\3&2&1\\2&1&2\end{bmatrix}$, by elimentary transformations. **[PTA – 6]**

Applying Gauss-Jordan method, we get

$$[A|I_3] = \left[\begin{array}{ccc|ccc}2&1&1&1&0&0\\3&2&1&0&1&0\\2&1&2&0&0&1\end{array}\right]$$

$$\xrightarrow{R_2\to\frac{1}{2}R_1}\left[\begin{array}{ccc|ccc}1&\left(\frac{1}{2}\right)&\left(\frac{1}{2}\right)&\left(\frac{1}{2}\right)&0&0\\3&2&1&0&1&0\\2&1&2&0&0&1\end{array}\right]$$

$$\xrightarrow[R_3 \to R_3 \to 2R_1]{R_2 \to R_2 \to 3R_1} \left[\begin{array}{ccc|ccc} 1 & \left(\frac{1}{2}\right) & \left(\frac{1}{2}\right) & \left(\frac{1}{2}\right) & 0 & 0 \\ 3 & \left(\frac{1}{2}\right) & \left(\frac{1}{2}\right) & -\left(\frac{3}{2}\right) & 1 & 0 \\ 0 & 0 & 1 & -1 & 0 & 1 \end{array}\right]$$

$$\xrightarrow{R_2 \to 2R_2} \left[\begin{array}{ccc} 1 & \left(\frac{1}{2}\right) & \left(\frac{1}{2}\right) \\ 0 & 1 & -1 \\ 0 & 0 & 1 \end{array}\right]$$

$$\xrightarrow{R_2 \to R_1 - \frac{1}{2}R_2} \left[\begin{array}{ccc|ccc} 1 & 0 & 1 & 2 & -1 & 0 \\ 0 & 1 & -1 & -3 & 2 & 0 \\ 0 & 0 & 1 & -1 & 0 & 1 \end{array}\right]$$

$$\xrightarrow[R_2 \to R + R_3]{R_1 \to R_1 - R_3} \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 3 & -1 & -1 \\ 0 & 1 & 0 & -4 & 2 & 1 \\ 0 & 0 & 1 & -1 & 0 & 1 \end{array}\right]$$

So, $A^{-1} = \begin{bmatrix} 3 & -1 & -1 \\ -4 & 2 & 1 \\ -1 & 0 & 1 \end{bmatrix}$

3. Investigate for what values of λ and μ the system of linear equations $x + 2y + z = 7$, $x + y + \lambda z = \mu$ has (i) no solution (ii) a unique solution (iii) an infinite number of solutions. **[PTA - 2]**

Sol. Here the number of unknowns is 3.

The matrix form of the system is AX = B, where

$$A = \begin{bmatrix} 1 & 2 & 1 \\ 1 & 1 & \lambda \\ 1 & 3 & -5 \end{bmatrix}, X = \begin{bmatrix} x \\ y \\ z \end{bmatrix}, B = \begin{bmatrix} 7 \\ \mu \\ 5 \end{bmatrix}$$

Applying elementary row operations on the augmented matrix [A|B], we get

$$[A|B] = \left[\begin{array}{ccc|c} 1 & 2 & 1 & 7 \\ 1 & 1 & \lambda & \mu \\ 1 & 3 & -5 & 5 \end{array}\right] \xrightarrow{R_2 \leftrightarrow R_3} \left[\begin{array}{ccc|c} 1 & 2 & 1 & 7 \\ 1 & 3 & -5 & 5 \\ 1 & 1 & \lambda & \mu \end{array}\right]$$

$$\xrightarrow[R_3 \to R_3 - R_1,]{R_2 \to R_2 - R_1,} \left[\begin{array}{ccc|c} 1 & 2 & 1 & 7 \\ 0 & 1 & -6 & -2 \\ 0 & -1 & \lambda - 1 & \mu - 7 \end{array}\right]$$

$$\xrightarrow{R_3 \to R_3 + R_2} \left[\begin{array}{ccc|c} 1 & 2 & 1 & 7 \\ 0 & 1 & -6 & -2 \\ 0 & 0 & \lambda - 7 & \mu - 9 \end{array}\right]$$

(i) If $\lambda = 7$ and $\mu \neq 9$, then $\rho(A) = 2$ and $([A|B]) = 3$ and So $\rho(A) \neq 2([A|B])$. Hence the given system is inconsistent and has no solution.

(ii) If $\lambda = 7$ and μ is any real number, then $\rho(A) = 3$ and $([A|B]) = 3$.

So $\rho(A) = ([A|B]) = 3$ = Number of unknowns Hence the given system is consistent and has a unique solution.

(iii) If $\lambda = 7$ and $\mu = 9$, then $\rho(A) = 2$ and $\rho([A|B]) = 2$.

So, $\rho(A) = ([A|B]) = 2$ Number of unknowns. Hence the given system is consistent and has infinite number of solutions.

4. Test the consistency of the following system of linear equations by rank method.

$x - y + z = -9$; $2x - y + z = 4$

$3x - y + z = 6$; $4x - y + 2z = 7$ **[Mar. - 2020]**

Sol. $[A/B] = \left[\begin{array}{ccc|c} 1 & -1 & 1 & -9 \\ 2 & -1 & 1 & 4 \\ 3 & -1 & 1 & 6 \\ 4 & -1 & 2 & 7 \end{array}\right]$

$$= \sim \left[\begin{array}{ccc|c} 1 & -1 & 1 & -9 \\ 0 & 1 & -1 & 22 \\ 0 & 2 & -2 & 33 \\ 0 & 3 & -2 & 43 \end{array}\right] \begin{array}{l} \\ R_2 \to R_2 - 2R_1 \\ R_3 \to R_3 - 3R_1 \\ R_4 \to R_4 - 4R_1 \end{array}$$

$$= \sim \left[\begin{array}{ccc|c} 1 & -1 & 1 & -9 \\ 0 & 1 & -1 & 22 \\ 0 & 0 & 0 & -11 \\ 0 & 0 & 1 & -23 \end{array}\right] \begin{array}{l} \\ R_3 \to R_3 - 2R_2 \\ R_4 \to R_4 - 3R_2 \\ \\ \end{array}$$

$R_3 \leftrightarrow R_4$

$\rho(A) \neq \rho(A/B)$ Inconsistent and no solution.