Chapter 6 Application of Derivatives

Applications in Optimization

Optimization is a process of finding an extreme value (either maximum or minimum) under certain conditions.

A procedure for solving for an extremum or optimization problems.

Step 1 : Draw an appropriate figure and label the quantities relevant to the problem.
Step 2 : Find a experssion for the quantity to be maximized or minimized.
Step 3 : Using the given conditions of the problem, the quantity to be extremized .
Step 4 : Determine the interval of possible values for this variable from the conditions given in the problem.
Step 5 : Using the techniques of extremum (absolute extremum, first derivative test or second derivative test) obtain the maximum or minimum.

Theorem 4 (The Second Derivative Test)

Suppose that c is **a critical point** at which $f'(c) = 0$, that $f'(x)$ exists in a neighborhood of c, and that $f'(c)$ exists. Then $f(x)$ has
(i) a relative **maximum value** at c **if** $f''(c) < 0$ and
(ii) a relative **minimum value** at c **if** $f''(c) > 0$.
(iii) **If** $f''(c) = 0$ **, the test is not informative.**

EXERCISE 6.5

13. Find two positive numbers whose sum is 24 and whose product is as large as possible.

SOLUTION:

Let two positive numbers be x and y.

Given $x + y = 24$

$\Rightarrow y = 24 - x$ …….(1)

To find x and y when their product xy is maximum.

$\therefore$ Product = xy

$= x(24 - x)$ [$\because$ By (1)]

$P(x) = 24x - x^2$

$P'(x) = 24 - 2x$

$P''(x) = -2$ ……….(2)

To find the critical point for $P'(x)=0$

$24 - 2x = 0$

$\Rightarrow 2x = 24$

$\Rightarrow x = 12$

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When $x = 12$, $P''(12) = -2 < 0$ [∵by(2)]
By Second Derivative Test
P(x) is maximum when $x = 12$
(i.e) product is maximum when $x=12$
when $x = 12$, (1)⇒ $y = 24 - 12 = 12$
∴Required two positive numbers are 12 and 12

14. Find two positive numbers x and y such that $x + y = 60$ and xy^3 is maximum.

SOLUTION:

Let two positive numbers be x and y.
Given $x + y = 60$
⇒ $x = 60 - y$ …….(1)
To find x and y when their product xy^3 is maximum.
∴ Product $= xy^3$
$= (60 - y)\, y^3$ [∵By (1)]
$P(y) = 60y^3 - y^4$
$P'(y) = 180y^2 - 4y^3$
$P''(y) = 360y - 4y^3 = 4y(90 - y^2)$……….(2)

To find the critical point for $P'(y)=0$
⇒ $180y^2 - 4y^3 = 0$
⇒ $4y^2(45 - y) = 0$
⇒ $y = 0$ and $y = 45$
⇒we tale $y = 45$ since $y>0$

When $y = 45$, $P''(45) = 4(45)(90 - 45^2) = 180(90 - 2025) < 0$ [∵by(2)]

By Second Derivative Test
P(y) is maximum when $y = 45$
(i.e) product $= xy^3$ is maximum when $y=45$

when $y = 45$,
(1)⇒ $x = 60 - 45 = 15$
∴Required two positive numbers are 15 and 45

Chapter 6 Application of Derivatives

15. Find two positive numbers x and y such that their sum is 35 and the product $x^2 y^5$ is a maximum.

SOLUTION:

Let two positive numbers be x and y.

Given $x + y = 35$

$\Rightarrow x = 35 - y$ …….(1)

To find x and y when their product $x^2 y^5$ is maximum.

$\therefore$ Product $= x^2 y^5$

$= (35 - y)^2 y^5$ [$\because$ By (1)

$P(y) = (1225 - 70y + y^2)\, y^5$

$= 1225y^5 - 70y^6 + y^7$

$P'(y) = 6125y^4 - 420y^5 + 7y^6 = 7y^4 (875 - 60y + y^2)$

$P''(y) = 24500y^3 - 2100y^4 + 42y^5$

$= 14y^3 (1750 - 150y + 3y^2)$ ………(2)

To find the critical point for $P'(y)=0$

$\Rightarrow 7y^4 (875 - 60y + y^2)=0$

$\Rightarrow 7y^4 (y^2 - 60y + 875) = 0$

$\Rightarrow 7y^4 (y - 25)(y - 35) = 0$

$\Rightarrow y = 0,\ y = 25$ and $y = 35$

$\Rightarrow$ we tale $y = 25$ and $y = 35$ since $y>0$

When $y = 35$, $x = 0$, is not possible as $x>0$

$\Rightarrow$ we tale $y = 25$ only

When $y = 25$, $P''(25) = 14(25)^3 (1750 - 150(25) + 3(25)^2)$ [$\because$by(2)

$= 14(15625)(1750 - 3750 + 1875)$

$= 14(15625)(3625 - 3750) < 0$

By Second Derivative Test

P(y) is maximum when y = 25

(i.e) product $= x^2 y^5$ is maximum when y=25

when y= 25,

(1)$\Rightarrow x = 35 - 25 = 10$

$\therefore$Required two positive numbers are 25 and 10

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Example 34

Find two positive numbers whose sum is 15 and the sum of whose squares is minimum.

SOLUTION:

Let two positive numbers be x and y.

Given $x + y = 15$

$\Rightarrow y = 15 - x$ ....(1)

To find x and y when sum of whose squares is minimum.

$\therefore$ Sum of squares $= x^2 + y^2$

$= x^2 + (15 - x)^2$ [$\because$By (1)]

$S(x) = x^2 + (15 - x)^2$

$S'(x) = 2x + 2(15 - x)(-1)$

$= 2x - 30 + 2x$

$= 4x - 30 = 2(2x - 15)$

$S''(x) = 4$ ....(2)

To find the critical point for $S'(x)=0$

$2(2x - 15) = 0$

$\Rightarrow 2x = 15$

$\Rightarrow x = \frac{15}{2} = 7.5$

When $x = 7.5$, $S''(7.5) = 4 > 0$ [$\because$by(2)]

By Second Derivative Test

S(x) is minimum when x = 7.5

(i.e) Sum of squares is minimum when x= 7.5

when x= 7.5, (1)$\Rightarrow y = 15 - 7.5 = 7.5$

$\therefore$Required two positive numbers are 7.5 and 7.5

Remark

Proceeding as in above Example 34 one may prove that the two positive numbers, whose sum is k and the sum of whose squares is minimum, are $\frac{k}{2}$ and $\frac{k}{2}$.

Exercise 6.5

16. Find two positive numbers whose sum is 16 and the sum of whose cubes is minimum.

SOLUTION:

Let two positive numbers be x and y.

Given $x + y = 16$

$\Rightarrow y = 16 - x$ …….(1)

To find x and y when sum of whose cubes is minimum.

$\therefore$ Sum of cubes $= x^3 + y^3$

$= x^3 + (16 - x)^3$ [$\because$ By (1)

$S(x) = x^3 + (16 - x)^3$

$S'(x) = 3x^2 + 3(16 - x)^2(-1)$

$= 3x^2 - 3(16 - x)^2$

$S''(x) = 6x - 6(16 - x)(-1)$

$= 6x + 6(16 - x)$

$S''(x) = 6x + 96 - 6x = 96$ ……….(2)

To find the critical point for $S'(x)=0$

$3x^2 - 3(16 - x)^2 = 0$

$3x^2 - 3(256 - 32x + x^2) = 0$

$3x^2 - 768 + 96x - 3x^2 = 0$

$\Rightarrow 96x = 768$

$\Rightarrow x = 8$

When $x = 8$, $S''(8) = 96 > 0$ [$\because$ by(2)

By Second Derivative Test

S(x) is minimum when x = 8

(i.e) Sum of cubes is minimum when x= 8

when x= 8, (1)$\Rightarrow y = 16 - 8 = 8$

$\therefore$Required two positive numbers are 8 and 8

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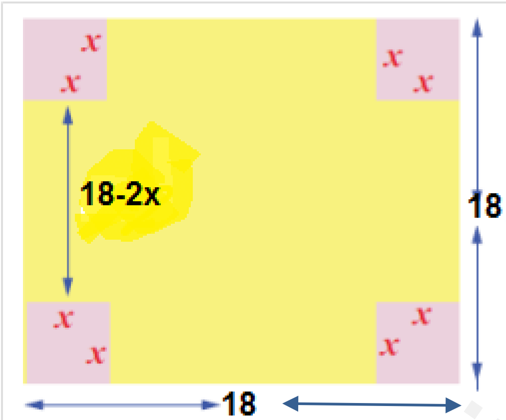
17. A square piece of tin of side 18 cm is to be made into a box without top, by cutting a square from each corner and folding up the flaps to form the box. What should be the side of the square to be cut off so that the volume of the box is the maximum possible.

SOLUTION:



Let x = length of the cut on each side of the little squares

V = the volume of the folded box

∴The length of the square base after two cuts = 18 − 2x

The depth of the box after folding = x

To find x lies between 0 and 9 when volume of the box is maximum

∴ volume of the box is V = length × breadth × height

$$= (18 - 2x) \times (18 - 2x) \times x$$

$$V(x) = x(18 - 2x)^2 \quad \ldots\ldots(1)$$

$$= x(324 - 72x + 4x^2)$$

$$V(x) = 4x^3 - 72x^2 + 324x$$

$$V'(x) = 12x^2 - 144x + 324$$

$$V''(x) = 24x - 144 = 24(x - 6) \quad \ldots\ldots(2)$$

To find the critical point for $V'(x)=0$

$$\Rightarrow 12x^2 - 144x + 324 = 0$$

$$\Rightarrow 12(x^2 - 12x + 27) = 0$$

$$\Rightarrow 12(x - 3)(x - 9) = 0$$

$$\Rightarrow x = 3 \text{ and } x = 9$$

Since the value of x lies between 0 and 6, we take $x = 3$

When x = 3 $V''(3) = 24(3 - 6) = -72 < 0$ [∵By (2)]

By Second Derivative Test

V(x) is maximum when x = 3

(i.e) volume is maximum when x=3

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Hence the maximum cut of side can only be 3 units.

when x = 3

the maximum volume of the box V $= x(18-2x)^2$ [∵By (1)]

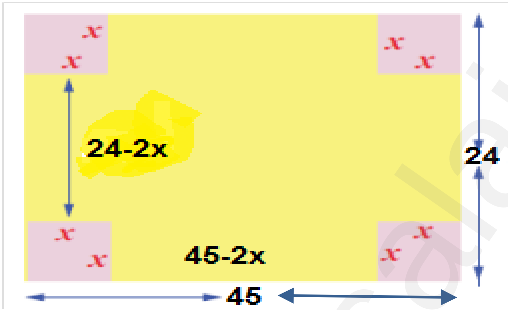
$= 3(18-6)^2$

$= 3(144)$

$= 432$ cubic units

18. A rectangular sheet of tin 45 cm by 24 cm is to be made into a box without top, by cutting off square from each corner and folding up the flaps. What should be the side of the square to be cut off so that the volume of the box is maximum ?

SOLUTION:



Let x = length of the cut on each side of the little squares

V = the volume of the folded box

∴The length of the rectangle base after two cuts = $45-2x$

and breadth of the rectangle base after two cuts = $24-2x$

The depth of the box after folding = x

To find x lies between 0 and 12 when volume of the box is maximum

∴ volume of the box is V = length × breadth × height

$= (45-2x)\times(24-2x)\times x$

$V(x) = 2x(45-2x)(12-x)$ ……..(1)

$= 2x(540-69x+2x^2)$

$V(x) = 4x^3-138x^2+1080x$

$V'(x) = 12x^2-276x+1080$

$V''(x) = 24x-276 = 12(2x-23)$ ……(2)

To find the critical point for $V'(x)=0$

$\Rightarrow 12x^2-276x+1080=0$

$\Rightarrow 12(x^2-23x+90)=0$

$\Rightarrow 12(x-5)(x-18)=0$

$\Rightarrow x = 5$ and $x = 18$

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Since the value of x lies between 0 and 12, we take $x = 5$

When $x = 5$ $V''(5) = 12(10 - 23) = 12(-13) < 0$ [∵By (2)]

By Second Derivative Test

V(x) is maximum when $x = 5$

(i.e) volume is maximum when $x=5$

Hence the maximum cut of side can only be 5 units.

when $x = 5$

the maximum volume of the box V $= 2x(45 - 2x)(12 - x)$ [∵By (1)]

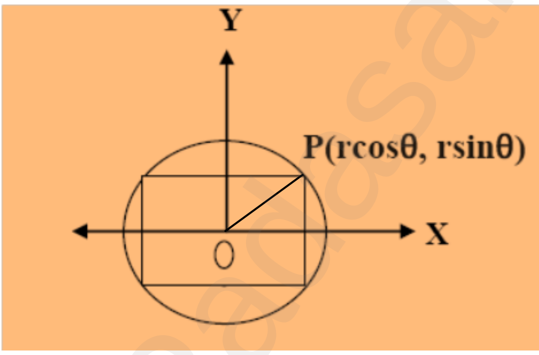
$= 15(45 - 10)(12 - 5)$

$= 15(35)(7)$

$= 3675$ cubic units

19. Show that of all the rectangles inscribed in a given fixed circle, the square has the maximum area.

SOLUTION:



Let r be the radius of the semicircle with centre at O.

Let a rectangle is inscribed in the semicircle.

To find the dimensions of the rectangle when its area is maximum

Let the equation of the semicircle is $x^2 + y^2 = r^2$.

We have the point $P(r\cos\theta, r\sin\theta)$

∴ length of the rectangle is $= 2r\cos\theta$

breadth $= 2r\sin\theta$ …………..(1)

Area of the rectangle A $=$ length × breadth

$= 2r\cos\theta \times 2r\sin\theta$

$= 4r^2\cos\theta\sin\theta$

$A(\theta) = 2r^2\sin 2\theta$

$A'(\theta) = 4r^2\cos 2\theta$

$A''(\theta) = -8r^2\sin 2\theta$ ……….(2)

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To find the critical point for $A'(\theta)=0$

$$\Rightarrow 4r^2 \cos 2\theta = 0$$
$$\Rightarrow \cos 2\theta = 0$$
$$\Rightarrow 2\theta = (2n+1)\frac{\pi}{2},\ n\epsilon\ Z$$
$$\Rightarrow 2\theta = \frac{\pi}{2}, \frac{3\pi}{2}, \frac{5\pi}{2}, \ldots$$
$$\Rightarrow \theta = \frac{\pi}{4}, \frac{3\pi}{4}, \frac{5\pi}{4}, \ldots$$

Since $0 < \theta < \frac{\pi}{2}$, we have $\theta = \frac{\pi}{4}$

When $\theta = \frac{\pi}{4}$, $A''(\frac{\pi}{4}) = -8r^2 \sin 2(\frac{\pi}{4})$ [∵By (2)]

$$= -8r^2 \sin(\frac{\pi}{2})$$
$$= -8r^2 < 0$$

By Second Derivative Test

A(θ) is maximum when $\theta = \frac{\pi}{4}$

(i.e) Area of the rectangle is maximum when $\theta = \frac{\pi}{4}$

when $\theta = \frac{\pi}{4}$

(1)⇒ length = $2r\cos(\frac{\pi}{4}) = 2r\left(\frac{1}{\sqrt{2}}\right) = r\sqrt{2}$ cm

breadth = $2r\sin(\frac{\pi}{4}) = 2r\left(\frac{1}{\sqrt{2}}\right) = r\sqrt{2}$ cm

∴ the dimension of the largest rectangle are $r\sqrt{2}$ cm and $r\sqrt{2}$ cm.

∴ length = breadth

∴ the rectangle is a square.

∴ all the rectangles inscribed in a given fixed circle, the square has the maximum area.

20. Show that the right circular cylinder of given surface and maximum volume is such that its height is equal to the diameter of the base.

SOLUTION:

Let the radius of the base of the cylinder = r units

and its height = h units

Given Surface area of the cylinder = $2\pi r(h+r)$ sq.units

$$S = 2\pi r(h+r)$$
$$\Rightarrow h+r = \frac{S}{2\pi r}$$
$$\Rightarrow h = \frac{S}{2\pi r} - r \quad \ldots(1)$$

∴ Volume of the cylinder $= \pi r^2 h$

$$= \pi r^2\left(\frac{S}{2\pi r} - r\right)$$

$$= \frac{S\pi r^2}{2\pi r} - \pi r^3$$

$$V(r) = \frac{Sr}{2} - \pi r^3$$

$$V'(r) = \frac{S}{2} - 3\pi r^2$$

$$V''(x) = -6\pi r \text{ ....(2)}$$

To find the critical point for $V'(x)=0$

$$\Rightarrow \frac{S}{2} - 3\pi r^2 = 0$$

$$\Rightarrow 3\pi r^2 = \frac{S}{2}$$

$$\Rightarrow r^2 = \frac{S}{6\pi}$$

$$\Rightarrow r = \left(\frac{S}{6\pi}\right)^{1/2}$$

When $r = \left(\frac{S}{6\pi}\right)^{1/2}$, $V''\left(\left(\frac{S}{6\pi}\right)^{1/2}\right) = -- 6\pi\left(\frac{S}{6\pi}\right)^{1/2} < 0$ [∵By (2)

By Second Derivative Test

V(x) is maximum when $r = \left(\frac{S}{6\pi}\right)^{1/2}$

(i.e) Volume of the cylinder is maximum when $r = \left(\frac{S}{6\pi}\right)^{1/2}$

when $r = \left(\frac{S}{6\pi}\right)^{1/2}$,

(1)⇒ $h = \frac{S}{2\pi r} - r = \frac{S-2\pi r^2}{2\pi r}$ $= \frac{S-2\pi\left(\frac{S}{6\pi}\right)}{2\pi\left(\frac{S}{6\pi}\right)^{1/2}}$

$$= \frac{S-\left(\frac{S}{3}\right)}{2\pi} \times \left(\frac{6\pi}{S}\right)^{1/2}$$

$$= \frac{\frac{2S}{3}}{2\pi} \times \left(\frac{6\pi}{S}\right)^{1/2}$$

$$= \frac{S}{3\pi} \times \frac{(6\pi)^{1/2}}{(S)^{1/2}}$$

$$= \frac{(S)^{1/2}}{(3\pi)^{1/2}} \times \frac{(2)^{1/2}}{1}$$

$$h = \left(\frac{2S}{3\pi}\right)^{1/2}$$

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$\therefore$ radius r $= \left(\frac{S}{6\pi}\right)^{1/2}$ and height h $= \left(\frac{2S}{3\pi}\right)^{1/2}$

$\therefore$ diameter $= 2r = 2\left(\frac{S}{6\pi}\right)^{1/2} = \left(\frac{4S}{6\pi}\right)^{1/2} = \left(\frac{2S}{3\pi}\right)^{1/2} =$ height

$\therefore$maximum volume is such that its height is equal to the diameter of the base.

21. Of all the closed cylindrical cans (right circular), of a given volume of 100 cubic centimeters, find the dimensions of the can which has the minimum surface area?

SOLUTION:

Let the radius of the base of the cylinder = r units

and its height = h units

Given Volume of the cylinder $= 100$ c.c

$$\pi r^2 h = 100$$

$$\Rightarrow h = \frac{100}{\pi r^2} \quad .....(1)$$

$\therefore$ Surface area of the cylinder $= 2\pi r(h+r)$ sq.units

$$S(r) = 2\pi r\left(\frac{100}{\pi r^2} + r\right)$$

$$S(r) = \frac{200}{r} + 2\pi r^2$$

$$S'(r) = -\frac{200}{r^2} + 4\pi r$$

$$S''(r) = \frac{400}{r^3} + 4 \quad ....(2)$$

To find the critical point for S′(x)=0

$$-\frac{200}{r^2} + 4\pi r = 0$$

$$\Rightarrow 4\pi r = \frac{200}{r^2}$$

$$\Rightarrow r^3 = \frac{50}{\pi}$$

$$\Rightarrow r = \left(\frac{50}{\pi}\right)^{1/3}$$

When $r = \left(\frac{50}{\pi}\right)^{1/3}$, $\quad S''\left[\left(\frac{50}{\pi}\right)^{1/3}\right] = \frac{400}{\frac{50}{\pi}} + 4\pi \quad$ [∵By (2)]

$$= 8\pi + 4\pi$$

$$= 12\pi > 0$$

By Second Derivative Test

S(r) is minimum when $r = \left(\frac{50}{\pi}\right)^{1/3}$

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(i.e) Surface Area of the cylinder is maximum when r $= \left(\frac{50}{\pi}\right)^{1/3}$

when r $= \left(\frac{50}{\pi}\right)^{1/3}$

$$\Rightarrow h = \frac{100}{\pi\left(\frac{50}{\pi}\right)^{2/3}}$$

$$= \frac{100}{\pi} \times \left(\frac{\pi}{50}\right)^{2/3}$$

$$= \frac{50\times 2}{\pi} \times \frac{\pi^{2/3}}{50^{2/3}}$$

$$h = 2\left(\frac{50}{\pi}\right)^{1/3}$$

$\therefore$ radius r $= \left(\frac{50}{\pi}\right)^{1/3}$ and height h $= 2\left(\frac{50}{\pi}\right)^{1/3}$.

22. A wire of length 28 m is to be cut into two pieces. One of the pieces is to be made into a square and the other into a circle. What should be the length of the two pieces so that the combined area of the square and the circle is minimum?

SOLUTION:

The length of the wire = 28m
It is to be cut into two pieces.
Let the length of the two pieces are x m and (28 – x)m.
One of the pieces is to be made into a square and
the other into a circle.

Let the Perimeter of the square = x m
4 × side of the square = x m
$\therefore$ side of the square = $\frac{x}{4}$ m
$\therefore$ area of the square is = $\frac{x^2}{16}$ sq.m

Also the Perimeter of the circle = (28 – x) m
$2\pi r = (28 - x)$ m
$\therefore$ the radius of the circle, r = $\frac{1}{2\pi}(28 - x)$ m
$\therefore$ area of the circle is $= \pi r^2$ sq.m
$= \pi \frac{1}{4\pi^2}(28 - x)^2$
$= \frac{1}{4\pi}(28 - x)^2$

$\therefore$ the combined area of the square and the circle is

$$A(x) = \frac{x^2}{16} + \frac{1}{4\pi}(28 - x)^2$$

$$A'(x) = \frac{2x}{16} + \frac{1}{4\pi}(2)(28-x)(-1)$$
$$= \frac{x}{8} - \frac{1}{2\pi}(28-x)$$
$$A''(x) = \frac{1}{8} - \frac{1}{2\pi}(-1)$$
$$= \frac{1}{8} + \frac{1}{2\pi} \ldots\ldots. (2)$$

To find the critical point for $A'(x)=0$

$$\Rightarrow \frac{x}{8} - \frac{1}{2\pi}(28-x)=0$$
$$\Rightarrow \frac{x}{8} = \frac{1}{2\pi}(28-x)$$
$$\Rightarrow 2\pi x = 8(28-x)$$
$$\Rightarrow x = 4(28-x)$$
$$\Rightarrow x = 112 - 4x$$
$$\Rightarrow x + 4x = 112$$
$$\Rightarrow (\pi + 4)x = 112$$
$$\Rightarrow x = \frac{112}{\pi+4}$$

When $x = \frac{112}{\pi+4}$, $A''\left(\frac{112}{\pi+4}\right) = \frac{1}{8} + \frac{1}{2\pi} > 0$ **[∵By (2)**

By Second Derivative Test

A(x) is minimum when $x = \frac{112}{\pi+4}$

(i.e) Combined area is minmum when $x = \frac{112}{\pi+4}$

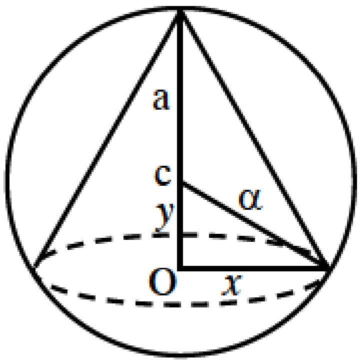
when $x = \frac{112}{\pi+4}$,

$$28 - x = 28 - \frac{112}{\pi+4}$$
$$= \frac{28(\pi+4)-112}{\pi+4}$$
$$= \frac{28\pi+112-112}{\pi+4}$$
$$= \frac{28\pi}{\pi+4}\text{ m}$$

∴The length of the pieces are x and (28 – x) are $\frac{112}{\pi+4}$ **m and** $\frac{28\pi}{\pi+4}$ **m**

23. Prove that the volume of the largest cone that can be inscribed in a sphere of radius R is $\frac{8}{27}$ of the volume of the sphere.

SOLUTION:



Let the radius of the sphere be 'a'
and let the base radius of the cone be 'x'.

If *h* is the height of the cone,
then its volume is $V = \frac{1}{3}\pi x^2 h$

$= \frac{1}{3}\pi x^2 (a + y)$ ....(1)

where OC = y so that height h = a + y.
From the diagram $x^2 + y^2 = a^2 \Rightarrow x^2 = a^2 - y^2$

(1) ⇒ volume of the cone is $V = \frac{1}{3}\pi (a^2 - y^2)(a + y)$ ......(2)

$V(y) = \frac{1}{3}\pi (a^3 + a^2y - ay^2 - y^3)$

$V'(y) = \frac{1}{3}\pi (a^2 - 2ay - 3y^2)$

$V''(y) = \frac{1}{3}\pi (-2a - 6y)$

$= \frac{-2}{3}\pi (a + 3y)$ ...... (3)

To find the critical point for $V'(y)=0$

$\Rightarrow \frac{1}{3}\pi (a^2 - 2ay - 3y^2)=0$

$\Rightarrow a^2 - 2ay - 3y^2 = 0$

$\Rightarrow 3y^2 + 2ay - a^2 = 0$

$\Rightarrow (3y - a)(y + a) = 0$

$\Rightarrow y = \frac{a}{3}$ and $y = -a$

$\Rightarrow$ we take $y = \frac{a}{3}$ [$\because y>0$]

When $y = \frac{a}{3}$, $V''(\frac{a}{3}) = \frac{-2}{3}\pi (a + a) < 0$ [∵By (2)]

By Second Derivative Test

V(x) is maximum when $y = \frac{a}{3}$

(i.e) volume of the cone is maximum when $y = \frac{a}{3}$

when $y = \frac{a}{3}$, (1)⇒ volume of the cone is

$$V = \frac{1}{3}\pi\left[a^2 - \left(\frac{a}{3}\right)^2\right]\left(a + \frac{a}{3}\right)$$

$$= \frac{1}{3}\pi\left[\frac{8a^2}{9}\right]\left(\frac{4a}{3}\right)$$

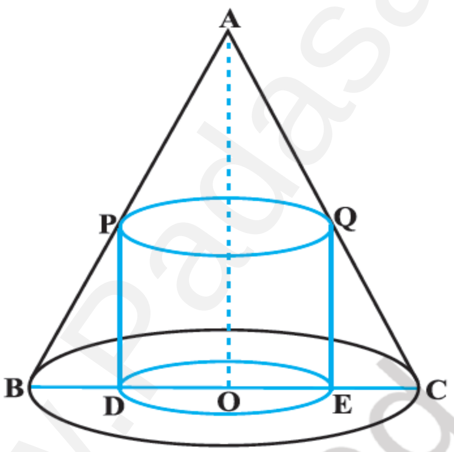
$$= \frac{8}{27}\left(\frac{4}{3}\pi a^3\right)$$

$$= \frac{8}{27}(\text{ volume of the sphere })$$

Example 38

Prove that the radius of the right circular cylinder of greatest curved surface area which can be inscribed in a given cone is half of that of the cone.

SOLUTION:



Let the radius of the cone be OC = r and its height be OA = h.

Let a cylinder inscribed in the given cone with radius OE = x

To find the height of the cylinder QE.

Since ΔQEC ~ΔAOC

$$\frac{QE}{AO} = \frac{EC}{OC}$$

$$\frac{QE}{h} = \frac{r-x}{r}$$

$$QE = \frac{h(r-x)}{r}$$

∴ Curved Surface Area of the cylinder $= 2\pi RH$

$$= 2\pi x\frac{h(r-x)}{r}$$

$$S(x) = \frac{2\pi h}{r}(rx - x^2)$$

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$$S'(x) = \frac{2\pi h}{r}(r - 2x)$$
$$S''(x) = \frac{2\pi h}{r}(-2) = \frac{-4\pi h}{r} \quad .....(2)$$

To find the critical point for $S'(x)=0$

$$\Rightarrow \frac{2\pi h}{r}(r - 2x) = 0$$
$$\Rightarrow r - 2x = 0$$
$$\Rightarrow 2x = r$$
$$\Rightarrow x = \frac{r}{2}$$

When $x = \frac{r}{2}$, $S''(\frac{r}{2}) = \frac{-4\pi h}{r} = <0$ [∵By (2)]

By Second Derivative Test

S(x) is maximum when $x = \frac{r}{2}$

(i.e) Curved surface Area is maximum when $x = \frac{r}{2}$

∴ The radius of the cylinder $x = \frac{r}{2}$

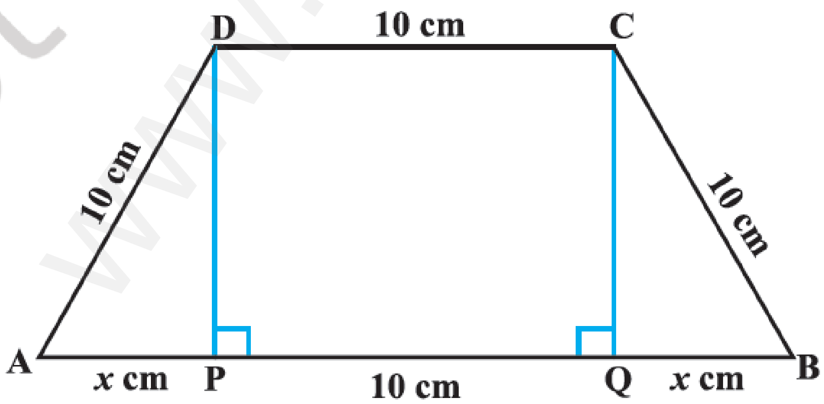
The radius of the cone = r

∴ when curved surface area of the cylinder is maximum, radius of the right circular cylinder is = half of that of the cone.

Example 37

If length of three sides of a trapezium other than base are equal to 10cm, then find the area of the trapezium when it is maximum.

SOLUTION:



Let AP = x cm.

Here ΔAPD ~ ΔBQC.

Therefore, QB = x cm.

By Pythagoras theorem,

height of the trapezium, DP = QC = $\sqrt{100 - x^2}$.

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∴ the area of the trapezium $= \frac{1}{2}$ (sum of parallel sides) (height)

$= \frac{1}{2}(10 + 2x + 10)(\sqrt{100 - x^2})$

$= \frac{1}{2}(2x + 20)(\sqrt{100 - x^2})$

$$A(x) = (x + 10)(\sqrt{100 - x^2}) \ldots\ldots (1)$$

$$A'(x) = (x + 10)\frac{-2x}{2\sqrt{100-x^2}} + \sqrt{100 - x^2}$$

$$= \frac{-x^2-10x}{\sqrt{100-x^2}} + \sqrt{100 - x^2}$$

$$= \frac{-x^2-10x+100-x^2}{\sqrt{100-x^2}}$$

$$A'(x) = \frac{-2x^2-10x+100}{\sqrt{100-x^2}}$$

$$A''(x) = \frac{\sqrt{100-x^2}\,(-4x-10)-(-2x^2-10x+100)\times\left[\frac{-2x}{2\sqrt{100-x^2}}\right]}{\left(\sqrt{100-x^2}\right)^2}$$

$$= \frac{\sqrt{100-x^2}\,(-4x-10)+(-2x^2-10x+100)\times\left[\frac{x}{\sqrt{100-x^2}}\right]}{\left(\sqrt{100-x^2}\right)^2}$$

$$= \frac{\frac{(100-x^2)(-4x-10)+(-2x^2-10x+100)x}{\sqrt{100-x^2}}}{100-x^2}$$

$$= \frac{-400x-1000+4x^3+10x^2-2x^3-10x^2+100x}{(100-x^2)\sqrt{100-x^2}}$$

$$A''(x) = \frac{2x^3-300x-1000}{(100-x^2)^{3/2}} \ldots\ldots(2)$$

To find the critical point for $A'(x)=0$

$$\Rightarrow \frac{-2x^2-10x+100}{\sqrt{100-x^2}} = 0$$

$$\Rightarrow -2x^2 - 10x + 100 = 0$$

$$\Rightarrow x^2 + 5x - 50 = 0$$

$$\Rightarrow (x+10)(x-5) = 0$$

$$\Rightarrow x = -10 \text{ and } x = 5$$

$\Rightarrow$ we take $x = 5$ [∵ $x>0$

When $x = 5$, $A''(5) = \frac{2(125)-300(5)-1000}{(100-25)^{3/2}}$ [∵By (2)

$$= \frac{250-1500-1000}{(75)^{3/2}}$$

$$= \frac{-2250}{(75)^{3/2}} < 0$$

By Second Derivative Test

A(x) is maximum when x = 5

(i.e) Area of the trapezium is maximum when x = 5

∴ when x = 5

Area of the trapezium is $A(x) = (x+10)(\sqrt{100-x^2})$ [∵by (1)]

$A(5) = (15)(\sqrt{75})$

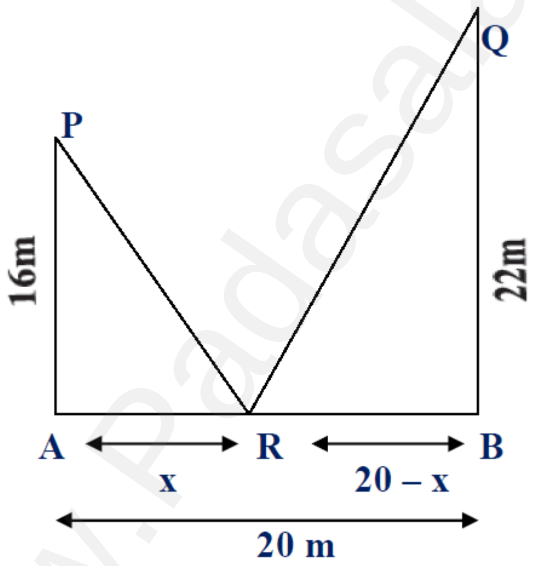
$= (15)(5\sqrt{3})$

$= 75\sqrt{3}$ sq. cm

Example 36

Let AP and BQ be two vertical poles at points A and B, respectively. If AP = 16 m, BQ = 22 m and AB = 20 m, then find the distance of a point R on AB from the point A such that $RP^2 + RQ^2$ is minimum.

Solution:



Let R be a point on AB such that AR = *x* m.

Then RB = (20 – *x*) m (as AB = 20 m).

By Pythagoras Theorem,

$RP^2 = AP^2 + AR^2$

$= 16^2 + x^2$

$RP^2 = 256 + x^2$ ....(1)

$RQ^2 = BQ^2 + BR^2$

$= 22^2 + (20-x)^2$

$= 484 + (20-x)^2$

$= 484 + 400 - 40x + x^2$

$RQ^2 = 884 - 40x + x^2$ .... (2)

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∴ (1) + (2) ⇒

$RP^2 + RQ^2 = (256 + x^2) + (884 - 40x + x^2)$

$S(x) = 2x^2 - 40x + 1140$

$S'(x) = 4x - 40$

$S''(x) = 4$ .....(2)

To find the critical point for $S'(x)=0$

$\Rightarrow 4x - 40 = 0$

$\Rightarrow 4x = 40$

$\Rightarrow x = 10$

When $x = 10$, $S''(10) = 4 > 0$ [∵By (2)]

By Second Derivative Test

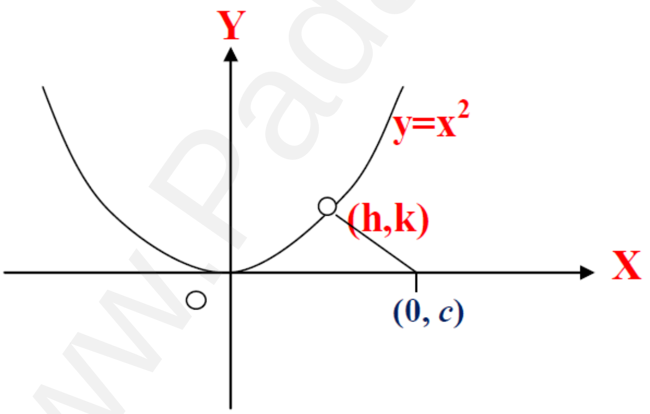
S(x) is minimum when $x = 10$

(i.e) $RP^2 + RQ^2$ is minimum when $x = 10$

Example 35

Find the shortest distance of the point (0, *c*) from the parabola $y = x^2$, where $\frac{1}{2} \le c \le 5$.

Solution:



Let (*h*, *k*) be any point on the parabola $y = x^2$.

∴ we have $k = h^2$ .....(1)

Let D be the required distance between (*h*, *k*) and (0, *c*).

Then D $= \sqrt{(h-0)^2 + ((k-c)^2}$

$= \sqrt{h^2 + ((k-c)^2}$

$D(k) = \sqrt{k + ((k-c)^2}$ .... (2) [∵ $k = h^2$

$D'(k) = \frac{1}{2\sqrt{k+((k-c)^2}}[1 + 2(k-c)]$

$D'(k) = \frac{1+2(k-c)}{2\sqrt{k+((k-c)^2}}$ ...... (3)

To find the critical point for D′(k) =0

$$\Rightarrow \frac{1+2(k-c)}{2\sqrt{k+((k-c)^2}} = 0$$

$$\Rightarrow 1 + 2(k-c) = 0$$

$$\Rightarrow 2(k-c) = -1$$

$$\Rightarrow k = \frac{-1}{2} + c$$

$$\Rightarrow k = \frac{2c-1}{2} \text{ where } \frac{1}{2} \le c \le 5$$

since $\frac{1}{2} \le c \le 5,\ k = \frac{2c-1}{2} \ge 0$

when $k < \frac{2c-1}{2}$ **we have** $1 + 2(k-c) < 0$

D′(k) <0 [∵by (3)

when $k > \frac{2c-1}{2}$ **we have** $1 + 2(k-c) > 0$

D′(k) > 0 [∵by (3)

∴ D(k) changes negatives to positive at $k = \frac{2c-1}{2}$

Using First Derivative Test

D(k) is minimum when $k = \frac{2c-1}{2}$

(i.e) the Distance D is the shortest when $k = \frac{2c-1}{2}$

when $k = \frac{2c-1}{2}$

(2) ⇒the Shortest distance $D\left(\frac{2c-1}{2}\right) = \sqrt{\frac{2c-1}{2} + \left(\left(\frac{2c-1}{2} - c\right)^2\right.}$

$$= \sqrt{\frac{2c-1}{2} + \left(\frac{2c-1-2c}{2}\right)^2}$$

$$= \sqrt{\frac{2c-1}{2} + \frac{1}{4}}$$

$$= \sqrt{\frac{4c-2+1}{4}}$$

$$= \frac{\sqrt{4c-1}}{2} \text{ units}$$

24. Show that the right circular cone of least curved surface and given volume has an altitude equal to $\sqrt{2}$ times the radius of the base.

SOLUTION:

Let the radius of the base of the cone = r units

Let the height of the cone = h units

and its slant height = ℓ units

To prove $h = \sqrt{2}\, r$

Given Volume of the cone = V cu. units [here V is constant

$$\frac{1}{3}\pi r^2 h = V$$

$$\Rightarrow h = \frac{3V}{\pi r^2}$$

$$\Rightarrow h = \frac{k}{r^2} \quad \ldots\ldots (1) \quad [\text{ where } k = \frac{3V}{\pi}$$

$\therefore$Slant height of the cone, $\ell = \sqrt{h^2 + r^2}$

$$= \sqrt{\left(\frac{k}{r^2}\right)^2 + r^2}$$

$$= \sqrt{\frac{k^2}{r^4} + r^2}$$

$$= \sqrt{\frac{k^2 + r^6}{r^4}}$$

$$\ell = \frac{\sqrt{k^2 + r^6}}{r^2}$$

$\therefore$ Curved Surface area of the cone $= \pi r \ell$ sq.units

$$S = \pi r \frac{\sqrt{k^2 + r^6}}{r^2}$$

$$S = \frac{\pi}{r}\sqrt{k^2 + r^6}$$

$$S^2 = \frac{\pi^2}{r^2}(k^2 + r^6)$$

$$f(r) = \frac{\pi^2 k^2}{r^2} + \pi^2 r^4$$

$$f'(r) = \frac{-2\pi^2 k^2}{r^3} + 4\pi^2 r^3$$

$$f''(r) = \frac{6\pi^2 k^2}{r^4} + 12\pi^2 r^2 \quad \ldots\ldots\ldots\ldots(2)$$

To find the critical point for $f'(x)=0$

$$\frac{-2\pi^2 k^2}{r^3} + 4\pi^2 r^3 = 0$$

$$\Rightarrow 4\pi^2 r^3 = \frac{2\pi^2 k^2}{r^3}$$

$$\Rightarrow 2r^6 = k^2 \quad \ldots\ldots(3)$$

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When r is any with $2r^6 = k^2$, $f''(r) = \frac{6\pi^2 k^2}{r^4} + 12\pi^2 r^2 > 0$ [∵By (2)

By Second Derivative Test

f(r) is minimum when $2r^6 = k^2$

S^2 is minimum when $2r^6 = k^2$

S is minimum when $2r^6 = k^2$

∴Curved Surface Area of the cone is minimum when $2r^6 = k^2$

when $2r^6 = k^2$

(1) $\Rightarrow h = \frac{k}{r^2}$

$\Rightarrow h^2 = \frac{k^2}{r^4} = \frac{2r^6}{r^4}$ [∵by (3)

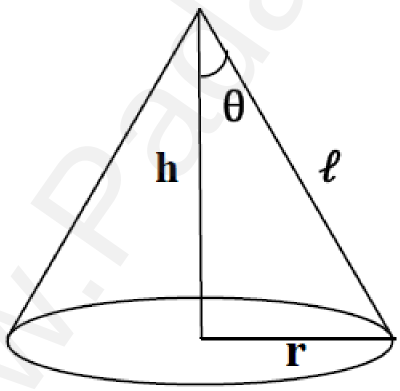
$\Rightarrow h^2 = 2r^2$

$\Rightarrow h = \sqrt{2}\, r$

∴ altitude equal to $\sqrt{2}$ times the radius of the base.

25. Show that the semi-vertical angle of the cone of the maximum volume and of given slant height is $\tan^{-1}(\sqrt{2})$.

SOLUTION:



Let the radius of the base of the cone = r units

Let the height of the cone = h units

Let the semi-vertical angle of the cone = θ

To prove $\theta = \tan^{-1}(\sqrt{2})$ when volume of the cone is maximum.

Given its slant height = ℓ units [where ℓ is constant

∴ $\ell^2 = h^2 + r^2$ .... (1)

$\Rightarrow r^2 = \ell^2 - h^2$

Given Volume of the cone $V = \frac{1}{3}\pi r^2 h$ cu. units

$= \frac{1}{3}\pi(\ell^2 - h^2)h$

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$$\Rightarrow \quad V(h) = \frac{1}{3}\pi(\ell^2 h - h^3)$$
$$V'(h) = \frac{1}{3}\pi(\ell^2 - 3h^2)$$
$$V''(h) = \frac{1}{3}\pi(0 - 6h) = -2\pi h$$
$$V''(h) = -2\pi h \; ..... (2)$$

To find the critical point for V′(h)=0

$$\frac{1}{3}\pi(\ell^2 - 3h^2) = 0$$
$$\Rightarrow \quad \ell^2 - 3h^2 = 0$$
$$\Rightarrow \quad \ell^2 = 3h^2$$
$$\Rightarrow \quad \ell = \sqrt{3}h \; ....(3)$$
$$\Rightarrow \quad h = \frac{\ell}{\sqrt{3}}$$

When $h = \frac{\ell}{\sqrt{3}}$, $\quad V''(h) = -2\pi h < 0$ **[∵By (2)]**

By Second Derivative Test

V(h) is maximum when $h = \frac{\ell}{\sqrt{3}}$

∴Volume of the cone is minimum when $h = \frac{\ell}{\sqrt{3}}$

when $h = \frac{\ell}{\sqrt{3}}, \ell = \sqrt{3}h$ [∵by (3)]

(1) $\Rightarrow r^2 = \ell^2 - h^2$

$\Rightarrow r^2 = 3h^2 - h^2$

$\Rightarrow r^2 = 2h^2$

$\Rightarrow r = \sqrt{2}h$

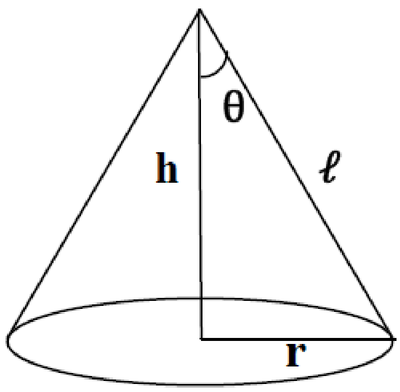
From the diagram,

$\tan\theta = \frac{r}{h} = \frac{\sqrt{2}h}{h} = \sqrt{2}$

$\Rightarrow \theta = \tan^{-1}(\sqrt{2})$

26. Show that semi-vertical angle of right circular cone of given surface area and maximum volume is $\sin^{-1}\left(\frac{1}{3}\right)$.

SOLUTION:



Let the radius of the base of the cone = r units
Let the height of the cone = h units
Let the slant height = ℓ units
Let the semi-vertical angle of the cone = θ

To prove $\theta = \sin^{-1}(\frac{1}{3})$ when volume of the cone is maximum.

Given Surface area of the cone $S = \pi r\ell + \pi r^2$ sq. units ...... (1)

$$S - \pi r^2 = \pi r\ell$$

$$\therefore \ell = \frac{S - \pi r^2}{\pi r}$$

$\therefore \ell^2 = h^2 + r^2$ .... (1)

$$\Rightarrow h^2 = \ell^2 - r^2$$

$$= \left(\frac{S - \pi r^2}{\pi r}\right)^2 - r^2$$

$$= \frac{(S - \pi r^2)^2}{\pi^2 r^2} - r^2$$

$$= \frac{S^2 - 2S\pi r^2 + \pi^2 r^4 - \pi^2 r^4}{\pi^2 r^2}$$

$$h^2 = \frac{S^2 - 2S\pi r^2}{\pi^2 r^2}$$

$$\Rightarrow h = \frac{\sqrt{S^2 - 2S\pi r^2}}{\pi r}$$

Given Volume of the cone $V = \frac{1}{3}\pi r^2 h$ cu. units

$$= \frac{1}{3}\pi r^2 \left(\frac{\sqrt{S^2 - 2S\pi r^2}}{\pi r}\right)$$

$$\Rightarrow V = \frac{r}{3}\sqrt{S^2 - 2S\pi r^2}$$

$$\Rightarrow V^2 = \frac{r^2}{9}(S^2 - 2S\pi r^2)$$

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$$\Rightarrow \quad f(r) = \frac{r^2S^2}{9} - \frac{2S\pi r^4}{9}$$
$$f'(r) = \frac{2rS^2}{9} - \frac{8S\pi r^3}{9}$$
$$f''(r) = \frac{2S^2}{9} - \frac{24S\pi r^2}{9} \quad \text{........(2)}$$

To find the critical point for f ′(r)=0

$$\frac{2rS^2}{9} - \frac{8S\pi r^3}{9} = 0$$
$$\frac{2rS^2}{9} = \frac{8S\pi r^3}{9}$$
$$\Rightarrow \quad S = 4\pi r^2$$
$$\Rightarrow \quad r^2 = \frac{S}{4\pi}$$

when $r^2 = \frac{S}{4\pi}$, $f''(r) = \frac{2S^2}{9} - \frac{24S\pi \times \frac{S}{4\pi}}{9}$ [∵by (2)]

$$= \frac{2S^2}{9} - \frac{6S^2}{9}$$
$$= -\frac{4S^2}{9} < 0$$

By Second Derivative Test

f(r) is maximum when $r^2 = \frac{S}{4\pi}$

V^2 is maximum when $r^2 = \frac{S}{4\pi}$

V is maximum when $r^2 = \frac{S}{4\pi}$

∴Volume of the cone is maximum when $r^2 = \frac{S}{4\pi}$

when $r^2 = \frac{S}{4\pi}$

$$\Rightarrow S = 4\pi r^2$$
$$\Rightarrow \pi r\ell + \pi r^2 = 4\pi r^2 \qquad [\because \text{by (1)}]$$
$$\Rightarrow \pi r\ell = 3\pi r^2$$
$$\Rightarrow \ell = 3r$$
$$\Rightarrow \frac{r}{\ell} = \frac{1}{3}$$

From the diagram,

$$\sin\theta = \frac{r}{\ell} = \frac{1}{3}$$
$$\Rightarrow \theta = \sin^{-1}\left(\frac{1}{3}\right)$$

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