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NOTE FROM PUBLISHER

It gives me great pride and pleasure in bringing to you **Sura's Business Mathematics Guide** for +1 Standard. A deep understanding of the text and exercises is rudimentary to have an insight into Business Mathematics. The students and teachers have to carefully understand the topics and exercises.

Sura's Business Mathematics +1 Standard Guide encompasses all the requirements of the students to comprehend the text and the evaluation of the textbook.

- It will be a teaching companion to teachers and a learning companion to students.
- Exhaustive additional Questions and Answers have been given to help students practice and learn effectively all the sections of the textbook.
- In order to learn effectively, I advise students to learn the subject sectionwise and practice the exercises given.

Though these salient features are available in our Sura's Business Mathematics Guide 11th Standard, I cannot negate the indispensable role of the teachers in assisting the student to understand the Business Mathematics.

I sincerely believe this guide satisfies the needs of the students and bolsters the teaching methodologies of the teachers.

I pray the almighty to bless the students for consummate success in their examinations.

Subash Raj, B.E., M.S.
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CONTENTS

01 MATRICES AND DETERMINANTS

TEXTUAL QUESTIONS

EXERCISE 1.1

1. Find the minors and cofactors of all the elements of the following determinants.

(i) $\begin{vmatrix} 5 & 20 \\ 0 & -1 \end{vmatrix}$ [CRT - 2022] **(ii)** $\begin{vmatrix} 1 & -3 & 2 \\ 4 & -1 & 2 \\ 3 & 5 & 2 \end{vmatrix}$

Sol : **(i)** $\begin{vmatrix} 5 & 20 \\ 0 & -1 \end{vmatrix}$

Let A $= \begin{vmatrix} 5 & 20 \\ 0 & -1 \end{vmatrix}$

Minor of 5 $= M_{11} = -1$

Minor of 20 $= M_{12} = 0$

Minor of 0 $= M_{21} = 20$

Minor of –1 $= M_{22} = 5$

Co-factor of 5 $= A_{11} = (-1)^{1+1} M_{11}$

$= (-1)^2 (-1) = -1$

Co-factor of 20 $= A_{12} (-1)^{1+2} M_{12} = -0 = 0$

Co-factor of 0 $= A_{21} (-1)^{2+1} M_{21} = 20$

Co-factor of –1 $= A_{22} (-1)^{2+2} M_{22} = 5$

(ii) $\begin{vmatrix} 1 & -3 & 2 \\ 4 & -1 & 2 \\ 3 & 5 & 2 \end{vmatrix}$

Let B $= \begin{vmatrix} 1 & -3 & 2 \\ 4 & -1 & 2 \\ 3 & 5 & 2 \end{vmatrix}$

Minor of 1 $= M_{11} = \begin{vmatrix} -1 & 2 \\ 5 & 2 \end{vmatrix} = -2 -10 = -12$

Minor of –3 $= M_{12} = \begin{vmatrix} 4 & 2 \\ 3 & 2 \end{vmatrix} = 8 - 6 = 2$

Minor of 2 $= M_{13} = \begin{vmatrix} 4 & -1 \\ 3 & 5 \end{vmatrix} = 20 + 3 = 23$

Minor of 4 $= M_{21} = \begin{vmatrix} -3 & 2 \\ 5 & 2 \end{vmatrix} = -6 -10 = -16$

Minor of –1 $= M_{22} = \begin{vmatrix} 1 & 2 \\ 3 & 2 \end{vmatrix} = 2 - 6 = -4$

Minor of 2 $= M_{23} = \begin{vmatrix} 1 & -3 \\ 3 & 5 \end{vmatrix} = 5 + 9 = 14$

Minor of 3 $= M_{31} = \begin{vmatrix} -3 & 2 \\ -1 & 2 \end{vmatrix} = -6 + 2 = -4$

Minor of 5 $= M_{32} = \begin{vmatrix} 1 & 2 \\ 4 & 2 \end{vmatrix} = 2 - 8 = -6$

Minor of 2 $= M_{33} = \begin{vmatrix} 1 & -3 \\ 4 & -1 \end{vmatrix} = -1 + 12 = 11$

Co-factor of 1 = $A_{11} = (-1)^{1+1} M_{11} = -12$

Co-factor of –3 = $A_{12} = (-1)^{1+2} M_{12} = 2$

Co-factor of 2 = $A_{13} = (-1)^{1+3} M_{13} = 23$

Co-factor of 4 = $A_{21} = (-1)^{2+1} M_{21} = -16$

Co-factor of –1 = $A_{22} = (-1)^{2+2} M_{22} = -4$

Co-factor of 2 = $A_{23} = (-1)^{2+3} M_{23} = 14$

Co-factor of 3 = $A_{31} = (-1)^{3+1} M_{31} = -4$

Co-factor of 5 = $A_{32} = (-1)^{3+2} M_{32} = -6$

Co-factor of 2 = $A_{33} = (-1)^{3+3} M_{33} = 11$

2. Evaluate : $\begin{vmatrix} 3 & -2 & 4 \\ 2 & 0 & 1 \\ 1 & 2 & 3 \end{vmatrix}$

Sol :

Let A = $\begin{vmatrix} 3 & -2 & 4 \\ 2 & 0 & 1 \\ 1 & 2 & 3 \end{vmatrix}$

Expanding along R_1 we get,

$$|A| = 3\begin{vmatrix} 0 & 1 \\ 2 & 3 \end{vmatrix} - (-2)\begin{vmatrix} 2 & 1 \\ 1 & 3 \end{vmatrix} + 4\begin{vmatrix} 2 & 0 \\ 1 & 2 \end{vmatrix}$$

$= 3(0-2) + 2(6-1) + 4(4-0)$

$= 3(-2) + 2(5) + 4(4)$

$= -6 + 10 + 16 = 20$

3. Solve : $\begin{vmatrix} 2 & x & 3 \\ 4 & 1 & 6 \\ 1 & 2 & 7 \end{vmatrix} = 0$ **[Sep. - 2021]**

Sol : Expanding along R_1 we get,

$$2\begin{vmatrix} 1 & 6 \\ 2 & 7 \end{vmatrix} - x\begin{vmatrix} 4 & 6 \\ 1 & 7 \end{vmatrix} + 3\begin{vmatrix} 4 & 1 \\ 1 & 2 \end{vmatrix} = 0$$

$\Rightarrow 2(7-12) - x(28-6) + 3(8-1) = 0$

$\Rightarrow 2(-5) - x(22) + 3(7) = 0$

$\Rightarrow -10 - 22x + 21 = 0$

$\Rightarrow -22x = -11$

$\Rightarrow x = \dfrac{11}{22} = \dfrac{1}{2}$

4. Find |AB| if A = $\begin{bmatrix} 3 & -1 \\ 2 & 1 \end{bmatrix}$ **and B =** $\begin{bmatrix} 3 & 0 \\ 1 & -2 \end{bmatrix}$. **[Sep. - 2021]**

Sol :

$$AB = \begin{bmatrix} 3 & -1 \\ 2 & 1 \end{bmatrix}\begin{bmatrix} 3 & 0 \\ 1 & -2 \end{bmatrix}$$

$$= \begin{bmatrix} 9-1 & 0+2 \\ 6+1 & 0-2 \end{bmatrix} = \begin{bmatrix} 8 & 2 \\ 7 & -2 \end{bmatrix}$$

$\therefore |AB| = \begin{vmatrix} 8 & 2 \\ 7 & -2 \end{vmatrix} = -16 - 14 = -30$

$\therefore |AB| = -30$

5. Solve : $\begin{vmatrix} 7 & 4 & 11 \\ -3 & 5 & x \\ -x & 3 & 1 \end{vmatrix} = 0$

Sol : Expanding the given determinant along R_1 we get

$$7\begin{vmatrix} 5 & x \\ 3 & 1 \end{vmatrix} - 4\begin{vmatrix} -3 & x \\ -x & 1 \end{vmatrix} + 11\begin{vmatrix} -3 & 5 \\ -x & 3 \end{vmatrix} = 0$$

$\Rightarrow 7(5-3x) - 4(-3+x^2) + 11(-9+5x) = 0$

$\Rightarrow 35 - 21x + 12 - 4x^2 - 99 + 55x = 0$

$\Rightarrow -4x^2 + 34x - 52 = 0$

Dividing by – 2 we get,

$2x^2 - 17x + 26 = 0$

Using factorization, the factors are $2x^2 - 17x + 26 = 0$

$2x^2 - 4x - 13x + 26 = 0$

$2x(x-2) - 13(x-2) = 0$

$(x-2)(2x-13) = 0$

$\Rightarrow x = 2$ or $x = \dfrac{13}{2}$

The values of x are 2, $\dfrac{13}{2}$

6. Evaluate : $\begin{vmatrix} 1 & a & a^2-bc \\ 1 & b & b^2-ca \\ 1 & c & c^2-ab \end{vmatrix}$

Sol :

Let A = $\begin{vmatrix} 1 & a & a^2-bc \\ 1 & b & b^2-ca \\ 1 & c & c^2-ab \end{vmatrix}$

$$A = \begin{vmatrix} 1 & a & a^2 \\ 1 & b & b^2 \\ 1 & c & c^2 \end{vmatrix} - \begin{vmatrix} 1 & a & bc \\ 1 & b & ca \\ 1 & c & ab \end{vmatrix}$$ [Using property (6)]... (1)

Consider $\begin{vmatrix} 1 & a & a^2 \\ 1 & b & b^2 \\ 1 & c & c^2 \end{vmatrix} = \begin{vmatrix} 0 & a-b & a^2-b^2 \\ 0 & b-c & b^2-c^2 \\ 1 & c & c^2 \end{vmatrix} \begin{matrix} R_1 \to R_1 \to R_2 \\ R_2 \to R_2 - R_3 \end{matrix}$

Taking $(a-b)(b-c)$ common from R_1 and R_2,

$$= \begin{vmatrix} 0 & a-b & (a-b)(a+b) \\ 0 & b-c & (b-c)(b+c) \\ 1 & c & c^2 \end{vmatrix}$$

$$= (a-b)(b-c) \begin{vmatrix} 0 & 1 & a+b \\ 0 & 1 & b+c \\ 1 & c & c^2 \end{vmatrix}$$

$= (a-b)(b-c)\,[0 - 0 + \{b + c - (a + b)\}]$ [Expanding along C_1]

$= (a-b)(b-c)(c-a)$

Let B $= \begin{vmatrix} 1 & a & bc \\ 1 & b & ca \\ 1 & c & ab \end{vmatrix}$

Applying the elementary transformations.

$R_1 \to R_1 - R_2$ and $R_2 \to R_2 - R_3$ we get,

$$B = \begin{vmatrix} 0 & a-b & bc-ca \\ 0 & b-c & ca-ab \\ 1 & c & ab \end{vmatrix} = \begin{vmatrix} 0 & a-b & -c(a-b) \\ 0 & b-c & -a(b-c) \\ 1 & c & ab \end{vmatrix}$$

Taking $(a-b)$ and $(b-c)$ common from R_1 and R_2 we get

$$B = (a-b)(b-c) \begin{vmatrix} 0 & 1 & -c \\ 0 & 1 & -a \\ 1 & c & ab \end{vmatrix}$$

Expanding along C_1 we get,

$$B = (a-b)(b-c)(1) \begin{vmatrix} 1 & -c \\ 1 & -a \end{vmatrix}$$

$$= (a-b)(b-c)(-a+c)$$

$$= (a-b)(b-c)(c-a) \quad \ldots (2)$$

Substituting (2) in (1) we get,

$A = (a-b)(b-c)(c-a) - (a-b)(b-c)(c-a) = 0$

7. Prove that $\begin{vmatrix} \frac{1}{a} & bc & b+c \\ \frac{1}{b} & ca & c+a \\ \frac{1}{c} & ab & a+b \end{vmatrix} = 0.$ **[Aug. - 2022]**

Sol : LHS $= \begin{vmatrix} \frac{1}{a} & bc & b+c \\ \frac{1}{b} & ca & c+a \\ \frac{1}{c} & ab & a+b \end{vmatrix}$

Multiplying R_1 by a, R_2 by b, and R_3 by c respectively and dividing the determinant by abc we get.

$$\text{LHS} = \frac{1}{abc} \begin{vmatrix} 1 & abc & ab+ac \\ 1 & abc & bc+ab \\ 1 & abc & ac+bc \end{vmatrix} = \frac{1}{abc}(abc) \begin{vmatrix} 1 & 1 & ab+ac \\ 1 & 1 & bc+ab \\ 1 & 1 & ac+bc \end{vmatrix}$$

[Taking abc common from C_2]

$= 0$ [$\because C_1 \equiv C_2$] $=$ RHS.

Hence Proved.

8. Prove that $\begin{vmatrix} -a^2 & ab & ac \\ ab & -b^2 & bc \\ ac & bc & -c^2 \end{vmatrix} = 4a^2 b^2 c^2$

[First Mid - 2018, Sep-2020]

Sol : LHS $= \begin{vmatrix} -a^2 & ab & ac \\ ab & -b^2 & bc \\ ac & bc & -c^2 \end{vmatrix}$

Taking a, b, c common from R_1, R_2 and R_3 respectively we get.

$$\text{LHS} = abc \begin{vmatrix} -a & b & c \\ a & -b & c \\ a & b & -c \end{vmatrix}$$

Again taking a, b, c common from C_1 C_2 and C_3 respectively.

$$\text{LHS} = a^2 b^2 c^2 \begin{vmatrix} -1 & 1 & 1 \\ 1 & -1 & 1 \\ 1 & 1 & -1 \end{vmatrix}$$

Applying the elementary formation $R_1 \to R_1 + R_2$

$$= a^2b^2c^2\begin{vmatrix} 0 & 0 & 2 \\ 1 & -1 & 1 \\ 1 & 1 & -1 \end{vmatrix}$$

Now, expanding along R_1 we get

$$\text{LHS} = a^2b^2c^2(2)\begin{vmatrix} 1 & -1 \\ 1 & 1 \end{vmatrix}$$

$$= 2a^2b^2c^2(1+1)$$

$$= 4a^2b^2c^2 = \text{RHS}$$

Hence proved.

Exercise 1.2

1. Find the adjoint of the matrix $A = \begin{bmatrix} 2 & 3 \\ 1 & 4 \end{bmatrix}$

Sol : Given $A = \begin{bmatrix} 2 & 3 \\ 1 & 4 \end{bmatrix}$

[Interchange the elements in the leading diagonal and change the sign in the off diagonal elements]

$$[A_{ij}] = \begin{bmatrix} 4 & -1 \\ -3 & 2 \end{bmatrix}$$

$$\text{adj } A = [A_{ij}]^T = \begin{bmatrix} 4 & -3 \\ -1 & 2 \end{bmatrix}$$

2. If $A = \begin{bmatrix} 1 & 3 & 3 \\ 1 & 4 & 3 \\ 1 & 3 & 4 \end{bmatrix}$ **then verify that A (*adj* A) = |A| I and also find** A^{-1}**.** **[GMQP - 2019]**

Sol : Given $A = \begin{bmatrix} 1 & 3 & 3 \\ 1 & 4 & 3 \\ 1 & 3 & 4 \end{bmatrix}$

$$\text{Co-factor matrix } Aij = \begin{bmatrix} +\begin{vmatrix} 4 & 3 \\ 3 & 4 \end{vmatrix} & -\begin{vmatrix} 1 & 3 \\ 1 & 4 \end{vmatrix} & +\begin{vmatrix} 1 & 4 \\ 1 & 3 \end{vmatrix} \\ -\begin{vmatrix} 3 & 3 \\ 3 & 4 \end{vmatrix} & +\begin{vmatrix} 1 & 3 \\ 1 & 4 \end{vmatrix} & -\begin{vmatrix} 1 & 3 \\ 1 & 3 \end{vmatrix} \\ +\begin{vmatrix} 3 & 3 \\ 4 & 3 \end{vmatrix} & -\begin{vmatrix} 1 & 3 \\ 1 & 3 \end{vmatrix} & +\begin{vmatrix} 1 & 3 \\ 1 & 4 \end{vmatrix} \end{bmatrix}$$

$$= \begin{bmatrix} 16-9 & -(4-3) & +(3-4) \\ -(12-9) & +(4-3) & -(3-3) \\ +(9-12) & -(3-3) & +(4-3) \end{bmatrix} = \begin{bmatrix} 7 & -1 & -1 \\ -3 & 1 & 0 \\ -3 & 0 & 1 \end{bmatrix}$$

Now, $\text{adj } A = [A_{ij}]^T$

$$= \begin{bmatrix} 7 & -1 & -1 \\ -3 & 1 & 0 \\ -3 & 0 & 1 \end{bmatrix}^T = \begin{bmatrix} 7 & -3 & -3 \\ -1 & 1 & 0 \\ -1 & 0 & 1 \end{bmatrix}$$

$$|A| = \begin{vmatrix} 1 & 3 & 3 \\ 1 & 4 & 3 \\ 1 & 3 & 4 \end{vmatrix}$$

$$= 1\begin{vmatrix} 4 & 3 \\ 3 & 4 \end{vmatrix} - 3\begin{vmatrix} 1 & 3 \\ 1 & 4 \end{vmatrix} + 3\begin{vmatrix} 1 & 4 \\ 1 & 3 \end{vmatrix}$$

$$= 16 - 9 - 3(4-3) + 3(3-4)$$

$$= 7 - 3 - 3 = 7 - 6$$

$$= 1$$

$$\text{Now, } A(\text{adj } A) = \begin{bmatrix} 1 & 3 & 3 \\ 1 & 4 & 3 \\ 1 & 3 & 4 \end{bmatrix}\begin{bmatrix} 7 & -3 & -3 \\ -1 & 1 & 0 \\ -1 & 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 7-3-3 & -3+3+0 & -3+0+3 \\ 7-4-3 & -3+4+0 & -3+0+3 \\ 7-3-4 & -3+3+0 & -3+0+4 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = 1\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = |A|\, I$$

$[\because |A| = 1]$

Hence A (adj A) = |A|. I

Since $|A| \neq 0$, A^{-1} exists.

We know that

$$A^{-1} = \frac{1}{|A|}\text{adj } A = \frac{1}{1}\begin{bmatrix} 7 & -3 & -3 \\ -1 & 1 & 0 \\ -1 & 0 & 1 \end{bmatrix}$$

$$A^{-1} = \begin{bmatrix} 7 & -3 & -3 \\ -1 & 1 & 0 \\ -1 & 0 & 1 \end{bmatrix}$$

3. Find the inverse of each of the following matrices.

(i) $\begin{bmatrix} 1 & -1 \\ 2 & 3 \end{bmatrix}$ **(ii)** $\begin{bmatrix} 3 & 1 \\ -1 & 3 \end{bmatrix}$

(iii) $\begin{bmatrix} 1 & 2 & 3 \\ 0 & 2 & 4 \\ 0 & 0 & 5 \end{bmatrix}$ **(iv)** $\begin{bmatrix} -3 & -5 & 4 \\ -2 & 3 & -1 \\ 1 & -4 & -6 \end{bmatrix}$

Sol :

(i) $\begin{bmatrix} 1 & -1 \\ 2 & 3 \end{bmatrix}$

$$\text{Let } A = \begin{bmatrix} 1 & -1 \\ 2 & 3 \end{bmatrix}$$

$$\text{adj } A = \begin{bmatrix} 3 & 1 \\ -2 & 1 \end{bmatrix}$$

[Interchange the places of leading diagonal elements and change the sign of off diagonal elements]

$$|A| = \begin{vmatrix} 1 & -1 \\ 2 & 3 \end{vmatrix} = 3 + 2 = 5$$

Since $|A| \neq 0$, A^{-1} exists.

$$\therefore A^{-1} = \frac{1}{|A|}\text{adj } A = \frac{1}{5}\begin{bmatrix} 3 & 1 \\ -2 & 1 \end{bmatrix}$$

(ii) $\begin{bmatrix} 3 & 1 \\ -1 & 3 \end{bmatrix}$

$$\text{Let } B = \begin{bmatrix} 3 & 1 \\ -1 & 3 \end{bmatrix}$$

$$|B| = \begin{vmatrix} 3 & 1 \\ -1 & 3 \end{vmatrix} = 9 + 1 = 10$$

Since $|B| \neq 0$, B^{-1} exists.

$$\text{adj } B = \begin{bmatrix} 3 & -1 \\ 1 & 3 \end{bmatrix}$$

[Interchange the places of leading diagonal elements and change the sign of off diagonal elements]

$$\text{Now, } B^{-1} = \frac{1}{|B|}\text{ adj } B = \frac{1}{10}\begin{bmatrix} 3 & -1 \\ 1 & 3 \end{bmatrix}$$

(iii) $\begin{bmatrix} 1 & 2 & 3 \\ 0 & 2 & 4 \\ 0 & 0 & 5 \end{bmatrix}$

$$\text{Let } C = \begin{bmatrix} 1 & 2 & 3 \\ 0 & 2 & 4 \\ 0 & 0 & 5 \end{bmatrix}$$

$$|C| = 1\begin{vmatrix} 2 & 4 \\ 0 & 5 \end{vmatrix} - 2\begin{vmatrix} 0 & 4 \\ 0 & 5 \end{vmatrix} + 3\begin{bmatrix} 0 & 2 \\ 0 & 0 \end{bmatrix}$$

[Expanding along R_1]

$$= 1\,(10 - 0) - 2\,(0 - 0) + 3\,(0 - 0)$$
$$= 10$$

Since $|C| \neq 0$, C^{-1} exists.

$$\text{adj } C = \begin{bmatrix} +\begin{vmatrix} 2 & 4 \\ 0 & 5 \end{vmatrix} & -\begin{vmatrix} 0 & 4 \\ 0 & 5 \end{vmatrix} & +\begin{vmatrix} 0 & 2 \\ 0 & 0 \end{vmatrix} \\ -\begin{vmatrix} 2 & 3 \\ 0 & 5 \end{vmatrix} & +\begin{vmatrix} 1 & 3 \\ 0 & 5 \end{vmatrix} & -\begin{vmatrix} 1 & 2 \\ 0 & 0 \end{vmatrix} \\ +\begin{vmatrix} 2 & 3 \\ 2 & 4 \end{vmatrix} & -\begin{vmatrix} 1 & 3 \\ 0 & 4 \end{vmatrix} & +\begin{vmatrix} 1 & 2 \\ 0 & 2 \end{vmatrix} \end{bmatrix}^T$$

$$= \begin{bmatrix} 10-0 & -(0-0) & +(0-0) \\ -(10-0) & +(5-0) & -(0-0) \\ +(8-6) & -(4-0) & +(2-0) \end{bmatrix}^T$$

$$= \begin{bmatrix} 10 & 0 & 0 \\ -10 & 5 & 0 \\ 2 & -4 & 2 \end{bmatrix}^T = \begin{bmatrix} 10 & -10 & 2 \\ 0 & 5 & -4 \\ 0 & 0 & 2 \end{bmatrix}$$

$$\text{Now, } C^{-1} = \frac{1}{|C|}\text{adj } C = \frac{1}{10}\begin{bmatrix} 10 & -10 & 2 \\ 0 & 5 & -4 \\ 0 & 0 & 2 \end{bmatrix}$$

(iv) $\begin{bmatrix} -3 & -5 & 4 \\ -2 & 3 & -1 \\ 1 & -4 & -6 \end{bmatrix}$

$$\text{Let } D = \begin{bmatrix} -3 & -5 & 4 \\ -2 & 3 & -1 \\ 1 & -4 & -6 \end{bmatrix}$$

$$|D| = \begin{vmatrix} -3 & -5 & 4 \\ -2 & 3 & -1 \\ 1 & -4 & -6 \end{vmatrix}$$

$$= -3\begin{vmatrix} 3 & -1 \\ -4 & -6 \end{vmatrix} + 5\begin{vmatrix} -2 & -1 \\ 1 & -6 \end{vmatrix} + 4\begin{vmatrix} -2 & 3 \\ 1 & -4 \end{vmatrix}$$

[Expanding along R_1]

$$= -3(-18-4) + 5\,(12 + 1) + 4\,(8 - 3)$$
$$= -3\,(-22) + 5\,(13) + 4\,(5)$$
$$= 66 + 65 + 20 = 151$$

Since $|D| \neq 0$, D^{-1} exists.

$$\text{adj } D = \begin{bmatrix} +\begin{vmatrix} 3 & -1 \\ -4 & -6 \end{vmatrix} & -\begin{vmatrix} -2 & -1 \\ 1 & -6 \end{vmatrix} & +\begin{vmatrix} -2 & 3 \\ 1 & -4 \end{vmatrix} \\ -\begin{vmatrix} -5 & 4 \\ -4 & -6 \end{vmatrix} & +\begin{vmatrix} -3 & 4 \\ 1 & -6 \end{vmatrix} & -\begin{vmatrix} -3 & -5 \\ 1 & -4 \end{vmatrix} \\ +\begin{vmatrix} -5 & 4 \\ 3 & -1 \end{vmatrix} & -\begin{vmatrix} -3 & 4 \\ -2 & -1 \end{vmatrix} & +\begin{vmatrix} -3 & -5 \\ -2 & 3 \end{vmatrix} \end{bmatrix}^T$$

$$= \begin{bmatrix} (-18-4) & -(12+1) & +(8-3) \\ -(30+16) & +(18-4) & -(12+5) \\ +(5-12) & -(3+8) & +(-9-10) \end{bmatrix}^T$$

$$= \begin{bmatrix} -22 & -13 & 5 \\ -46 & 14 & -17 \\ -7 & -11 & -19 \end{bmatrix}^T = \begin{bmatrix} -22 & -46 & -7 \\ -13 & 14 & -11 \\ 5 & -17 & -19 \end{bmatrix}$$

$$\therefore D^{-1} = \frac{1}{|D|}\text{adj } D = \frac{1}{151}\begin{bmatrix} -22 & -46 & -7 \\ -13 & 14 & -11 \\ 5 & -17 & -19 \end{bmatrix}$$

4. If $A = \begin{bmatrix} 2 & 3 \\ 1 & -6 \end{bmatrix}$ and $B = \begin{bmatrix} -1 & 4 \\ 1 & -2 \end{bmatrix}$, then verify adj (AB) = (adj B) (adj A). **[CRT - 2022]**

Sol :

$$AB = \begin{bmatrix} 2 & 3 \\ 1 & -6 \end{bmatrix}\begin{bmatrix} -1 & 4 \\ 1 & -2 \end{bmatrix}$$

$$= \begin{bmatrix} -2+3 & 8-6 \\ -1-6 & 4+12 \end{bmatrix} = \begin{bmatrix} 1 & 2 \\ -7 & 16 \end{bmatrix}$$

Now, LHS

$$\text{adj } AB = \begin{bmatrix} 16 & -2 \\ 7 & 1 \end{bmatrix}$$ [Interchange the places of leading diagonal elements and change the sign of off diagonal elements] ... (1)

$$\text{Given } B = \begin{bmatrix} -1 & 4 \\ 1 & -2 \end{bmatrix}$$

$$\text{adj } B = \begin{bmatrix} -2 & -4 \\ -1 & -1 \end{bmatrix} \text{ and } A = \begin{bmatrix} 2 & 3 \\ 1 & -6 \end{bmatrix}$$

$$\text{adj } A = \begin{bmatrix} -6 & -3 \\ -1 & 2 \end{bmatrix}$$

RHS = (adj B) (adj A)

$$= \begin{bmatrix} -2 & -4 \\ -1 & -1 \end{bmatrix}\begin{bmatrix} -6 & -3 \\ -1 & 2 \end{bmatrix}$$

$$= \begin{bmatrix} 12+4 & 6-8 \\ 6+1 & 3-2 \end{bmatrix} = \begin{bmatrix} 16 & -2 \\ 7 & 1 \end{bmatrix} \quad \text{... (2)}$$

From (1) and (2)

adj (AB) = (adj B) (adj A)

5. If $A = \begin{bmatrix} 2 & -2 & 2 \\ 2 & 3 & 0 \\ 9 & 1 & 5 \end{bmatrix}$ then, show that (adj A)A = 0.

Sol : $$\text{Given } A = \begin{bmatrix} 2 & -2 & 2 \\ 2 & 3 & 0 \\ 9 & 1 & 5 \end{bmatrix}$$

$$\text{adj } A = \begin{bmatrix} +\begin{vmatrix} 3 & 0 \\ 1 & 5 \end{vmatrix} & -\begin{vmatrix} 2 & 0 \\ 9 & 5 \end{vmatrix} & +\begin{vmatrix} 2 & 3 \\ 9 & 1 \end{vmatrix} \\ -\begin{vmatrix} -2 & 2 \\ 1 & 5 \end{vmatrix} & +\begin{vmatrix} 2 & 2 \\ 9 & 5 \end{vmatrix} & -\begin{vmatrix} 2 & -2 \\ 9 & 1 \end{vmatrix} \\ +\begin{vmatrix} -2 & 2 \\ 3 & 0 \end{vmatrix} & -\begin{vmatrix} 2 & 2 \\ 2 & 0 \end{vmatrix} & +\begin{vmatrix} 2 & -2 \\ 2 & 3 \end{vmatrix} \end{bmatrix}^T$$

$$= \begin{bmatrix} (15-0) & -(10-0) & +(2-27) \\ -(-10-2) & +(10-18) & -(2+18) \\ +(0-6) & -(0-4) & +(6+4) \end{bmatrix}^T$$

$$= \begin{bmatrix} 15 & -10 & -25 \\ 12 & -8 & -20 \\ -6 & 4 & 10 \end{bmatrix}^T = \begin{bmatrix} 15 & 12 & -6 \\ -10 & -8 & 4 \\ -25 & -20 & 10 \end{bmatrix}$$

Consider (adj A)A

$$= \begin{bmatrix} 15 & 12 & -6 \\ -10 & -8 & 4 \\ -25 & -20 & 10 \end{bmatrix}\begin{bmatrix} 2 & -2 & 2 \\ 2 & 3 & 0 \\ 9 & 1 & 5 \end{bmatrix}$$

$$= \begin{bmatrix} 30+24-54 & -30+36-6 & 30+0-30 \\ -20-16+36 & 20-24+4 & -20+0+20 \\ -50-40+90 & 50-60+10 & -50+0+50 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

Hence (adj A)A = 0

6. If $A = \begin{bmatrix} -1 & 2 & -2 \\ 4 & -3 & 4 \\ 4 & -4 & 5 \end{bmatrix}$ then, show that the inverse of A is A itself. **[First Mid - 2018]**

Sol : $$\text{Given } A = \begin{bmatrix} -1 & 2 & -2 \\ 4 & -3 & 4 \\ 4 & -4 & 5 \end{bmatrix}$$

$$|A| = \begin{vmatrix} -1 & 2 & -2 \\ 4 & -3 & 4 \\ 4 & -4 & 5 \end{vmatrix}$$

$$= -1\begin{vmatrix} -3 & 4 \\ -4 & 5 \end{vmatrix} - 2\begin{vmatrix} 4 & 4 \\ 4 & 5 \end{vmatrix} - 2\begin{vmatrix} 4 & -3 \\ 4 & -4 \end{vmatrix}$$

$$= -1(-15+16) - 2(20-16) - 2(-16+12)$$

$$= -1(1) - 2(4) - 2(-4)$$

$$= -1 - 8 + 8 = -1$$

Since $|A| \neq 0$, A^{-1} exists.

$$\text{adj } A = \begin{bmatrix} +\begin{vmatrix} -3 & 4 \\ -4 & 5 \end{vmatrix} & -\begin{vmatrix} 4 & 4 \\ 4 & 5 \end{vmatrix} & +\begin{vmatrix} 4 & -3 \\ 4 & -4 \end{vmatrix} \\ -\begin{vmatrix} 2 & -2 \\ -4 & 5 \end{vmatrix} & +\begin{vmatrix} -1 & -2 \\ 4 & 5 \end{vmatrix} & -\begin{vmatrix} -1 & 2 \\ 4 & -4 \end{vmatrix} \\ +\begin{vmatrix} 2 & -2 \\ -3 & 4 \end{vmatrix} & -\begin{vmatrix} -1 & -2 \\ 4 & 4 \end{vmatrix} & +\begin{vmatrix} -1 & 2 \\ 4 & -3 \end{vmatrix} \end{bmatrix}^T$$

$$= \begin{bmatrix} (-15+16) & -(20-16) & +(-16+12) \\ -(10-8) & +(-5+8) & -(4-8) \\ +(8-6) & -(-4+8) & +(3-8) \end{bmatrix}^T$$

$$= \begin{bmatrix} 1 & -4 & -4 \\ -2 & 3 & 4 \\ 2 & -4 & -5 \end{bmatrix}^T = \begin{bmatrix} 1 & -2 & 2 \\ -4 & 3 & -4 \\ -4 & 4 & -5 \end{bmatrix}$$

$$A^{-1} = \frac{1}{|A|} \text{adj } A = \frac{1}{-1}\begin{bmatrix} 1 & -2 & 2 \\ -4 & 3 & -4 \\ -4 & 4 & -5 \end{bmatrix}$$

$$= \begin{bmatrix} -1 & 2 & -2 \\ 4 & -3 & 4 \\ 4 & -4 & 5 \end{bmatrix} = A \Rightarrow A^{-1} = A \text{ itself}$$

7. If $A^{-1} = \begin{bmatrix} 1 & 0 & 3 \\ 2 & 1 & -1 \\ 1 & -1 & 1 \end{bmatrix}$ then, find A.

Sol : Given $A^{-1} = \begin{bmatrix} 1 & 0 & 3 \\ 2 & 1 & -1 \\ 1 & -1 & 1 \end{bmatrix}$

We know that $(A^{-1})^{-1} = A$

$\therefore$ We have to find $(A^{-1})^{-1}$

$$|A^{-1}| = 1\begin{vmatrix} 1 & -1 \\ -1 & 1 \end{vmatrix} - 0\begin{vmatrix} 2 & -1 \\ 1 & 1 \end{vmatrix} + 3\begin{vmatrix} 2 & 1 \\ 1 & -1 \end{vmatrix}$$

$$= 1(1-1) + 3(-2-1) = -9$$

Since $|A^{-1}| \neq 0$, $(A^{-1})^{-1}$ exists.

$$\text{Now, adj } A^{-1} = \begin{bmatrix} +\begin{vmatrix} 1 & -1 \\ -1 & 1 \end{vmatrix} & -\begin{vmatrix} 2 & -1 \\ 1 & 1 \end{vmatrix} & +\begin{vmatrix} 2 & 1 \\ 1 & -1 \end{vmatrix} \\ -\begin{vmatrix} 0 & 3 \\ -1 & 1 \end{vmatrix} & +\begin{vmatrix} 1 & 3 \\ 1 & 1 \end{vmatrix} & -\begin{vmatrix} 1 & 0 \\ 1 & -1 \end{vmatrix} \\ +\begin{vmatrix} 0 & 3 \\ 1 & -1 \end{vmatrix} & -\begin{vmatrix} 1 & 3 \\ 2 & -1 \end{vmatrix} & +\begin{vmatrix} 1 & 0 \\ 2 & 1 \end{vmatrix} \end{bmatrix}^T$$

$$= \begin{bmatrix} +(1-1) & -(2+1) & +(-2-1) \\ -(0+3) & +(1-3) & -(-1-0) \\ +(0-3) & -(-1-6) & +(1-0) \end{bmatrix}^T = \begin{bmatrix} 0 & -3 & -3 \\ -3 & -2 & 1 \\ -3 & 7 & 1 \end{bmatrix}^T$$

$$= \begin{bmatrix} 0 & -3 & -3 \\ -3 & -2 & 7 \\ -3 & 1 & 1 \end{bmatrix}$$

$$(A^{-1})^{-1} = \frac{1}{|A^{-1}|} \text{adj } (A^{-1}) = \frac{-1}{9}\begin{bmatrix} 0 & -3 & -3 \\ -3 & -2 & 7 \\ -3 & 1 & 1 \end{bmatrix}$$

$$\Rightarrow A = \frac{-1}{9}\begin{bmatrix} 0 & -3 & -3 \\ -3 & -2 & 7 \\ -3 & 1 & 1 \end{bmatrix} \Rightarrow A = \frac{1}{9}\begin{bmatrix} 0 & 3 & 3 \\ 3 & 2 & -7 \\ 3 & -1 & -1 \end{bmatrix}$$

8. Show that the matrices $A = \begin{bmatrix} 2 & 2 & 1 \\ 1 & 3 & 1 \\ 1 & 2 & 2 \end{bmatrix}$ and $B = \begin{bmatrix} \frac{4}{5} & -\frac{2}{5} & -\frac{1}{5} \\ -\frac{1}{5} & \frac{3}{5} & -\frac{1}{5} \\ -\frac{1}{5} & -\frac{2}{5} & \frac{4}{5} \end{bmatrix}$ are inverses of each other. **[Mar. -2020]**

Sol : Given $A = \begin{bmatrix} 2 & 2 & 1 \\ 1 & 3 & 1 \\ 1 & 2 & 2 \end{bmatrix}$ and $B = \begin{bmatrix} \frac{4}{5} & \frac{-2}{5} & -\frac{1}{5} \\ -\frac{1}{5} & \frac{3}{5} & -\frac{1}{5} \\ -\frac{1}{5} & -\frac{2}{5} & \frac{4}{5} \end{bmatrix}$

$$\text{Now } AB = \begin{bmatrix} 2 & 2 & 1 \\ 1 & 3 & 1 \\ 1 & 2 & 2 \end{bmatrix}\begin{bmatrix} \frac{4}{5} & -\frac{2}{5} & -\frac{1}{5} \\ -\frac{1}{5} & \frac{3}{5} & -\frac{1}{5} \\ -\frac{1}{5} & -\frac{2}{5} & \frac{4}{5} \end{bmatrix}$$

$$= \begin{bmatrix} 2 & 2 & 1 \\ 1 & 3 & 1 \\ 1 & 2 & 2 \end{bmatrix}\frac{1}{5}\begin{bmatrix} 4 & -2 & -1 \\ -1 & 3 & -1 \\ -1 & -2 & 4 \end{bmatrix}$$

[Taking $\frac{1}{5}$ common from matrix B]

$$= \frac{1}{5}\begin{bmatrix} 2 & 2 & 1 \\ 1 & 3 & 1 \\ 1 & 2 & 2 \end{bmatrix}\begin{bmatrix} 4 & -2 & -1 \\ -1 & 3 & -1 \\ -1 & -2 & 4 \end{bmatrix}$$

$$= \frac{1}{5}\begin{bmatrix} 8-2-1 & -4+6-2 & -2-2+4 \\ 4-3-1 & -2+9-2 & -1-3+4 \\ 4-2-2 & -2+6-4 & -1-2+8 \end{bmatrix}$$

$$= \frac{1}{5}\begin{bmatrix} 5 & 0 & 0 \\ 0 & 5 & 0 \\ 0 & 0 & 5 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \quad \ldots (1)$$

and $BA = \frac{1}{5}\begin{bmatrix} 4 & -2 & -1 \\ -1 & 3 & -1 \\ -1 & -2 & 4 \end{bmatrix}\begin{bmatrix} 2 & 2 & 1 \\ 1 & 3 & 1 \\ 1 & 2 & 2 \end{bmatrix}$

$= \frac{1}{5}\begin{bmatrix} 8-2-1 & 8-6-2 & 4-2-2 \\ -2+3-1 & -2+9-2 & -1+3-2 \\ -2-2+4 & -2-6+8 & -1-2+8 \end{bmatrix}$

$= \frac{1}{5}\begin{bmatrix} 5 & 0 & 0 \\ 0 & 5 & 0 \\ 0 & 0 & 5 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$ ... (2)

From (1) and (2), AB = BA = I

∴ The matrices A and B are inverses of each other.

9. If $A = \begin{bmatrix} 3 & 7 \\ 2 & 5 \end{bmatrix}$ and $B = \begin{bmatrix} 6 & 8 \\ 7 & 9 \end{bmatrix}$ then, verify that $(AB)^{-1} = B^{-1}A^{-1}$. **[Qy-2018; Sep. - 2021]**

Sol : Given $A = \begin{bmatrix} 3 & 7 \\ 2 & 5 \end{bmatrix}$ and $B = \begin{bmatrix} 6 & 8 \\ 7 & 9 \end{bmatrix}$

Then $|A| = \begin{vmatrix} 3 & 7 \\ 2 & 5 \end{vmatrix}$

$= 15 - 14 = 1 \neq 0 \Rightarrow A^{-1}$ exists.

$|B| = \begin{vmatrix} 6 & 8 \\ 7 & 9 \end{vmatrix} = 54 - 56 = -2 \neq 0 \Rightarrow B^{-1}$ exists.

$AB = \begin{bmatrix} 3 & 7 \\ 2 & 5 \end{bmatrix}\begin{bmatrix} 6 & 8 \\ 7 & 9 \end{bmatrix} = \begin{bmatrix} 18+49 & 24+63 \\ 12+35 & 16+45 \end{bmatrix}$

$= \begin{bmatrix} 67 & 87 \\ 47 & 61 \end{bmatrix}$

$|AB| = \begin{vmatrix} 67 & 87 \\ 47 & 61 \end{vmatrix} = 4087 - 4089$

$= -2 \neq 0$ $(AB)^{-1}$ exists.

$\text{adj } AB = \begin{bmatrix} 61 & -87 \\ -47 & 67 \end{bmatrix}$

[Interchange the places of leading diagonal elements and change the sign of off diagonal elements]

$\Rightarrow (AB)^{-1} = \frac{1}{|AB|}\text{adj }(AB)$

$= \frac{-1}{2}\begin{bmatrix} 61 & -87 \\ -47 & 67 \end{bmatrix} = \begin{bmatrix} -\frac{61}{2} & \frac{87}{2} \\ \frac{47}{2} & \frac{-67}{2} \end{bmatrix}$ ... (1)

$B^{-1} = \frac{1}{|B|}\text{adj } B = -\frac{1}{2}\begin{bmatrix} 9 & -8 \\ -7 & 6 \end{bmatrix}$

$A^{-1} = \frac{1}{|A|}\text{adj } A = \frac{1}{1}\begin{bmatrix} 5 & -7 \\ -2 & 3 \end{bmatrix} = \begin{bmatrix} 5 & -7 \\ -2 & 3 \end{bmatrix}$

$B^{-1}A^{-1} = \frac{-1}{2}\begin{bmatrix} 9 & -8 \\ -7 & 6 \end{bmatrix}\begin{bmatrix} 5 & -7 \\ -2 & 3 \end{bmatrix}$

$= \frac{-1}{2}\begin{bmatrix} 45+16 & -63-24 \\ -35-12 & 49+18 \end{bmatrix}$

$= -\frac{1}{2}\begin{bmatrix} 61 & -87 \\ -47 & 67 \end{bmatrix} = \begin{bmatrix} \frac{-61}{2} & \frac{87}{2} \\ \frac{47}{2} & \frac{-67}{2} \end{bmatrix}$ ... (2)

From (1) and (2), $(AB)^{-1} = B^{-1}A^{-1}$.

10. Find λ if the matrix $\begin{bmatrix} 1 & 1 & 3 \\ 2 & \lambda & 4 \\ 9 & 7 & 11 \end{bmatrix}$ has no inverse. **[Qy-2018]**

Sol : Let $A = \begin{bmatrix} 1 & 1 & 3 \\ 2 & \lambda & 4 \\ 9 & 7 & 11 \end{bmatrix}$

Since the matrix A has no-inverse, A is a singular matrix ⇒ |A| = 0.

$|A| = \begin{vmatrix} 1 & 1 & 3 \\ 2 & \lambda & 4 \\ 9 & 7 & 11 \end{vmatrix} = 0$

$\Rightarrow 1\begin{vmatrix} \lambda & 4 \\ 7 & 11 \end{vmatrix} - 1\begin{vmatrix} 2 & 4 \\ 9 & 11 \end{vmatrix} + 3\begin{vmatrix} 2 & \lambda \\ 9 & 7 \end{vmatrix} = 0$

[Expanding along R_1]

$11\lambda - 28 - 1(22-36) + 3(14 - 9\lambda) = 0$

$\Rightarrow 11\lambda - 28 + 14 + 42 - 27\lambda = 0$

$\Rightarrow -16\lambda + 28 = 0$

$\Rightarrow 16\lambda = 28$

$\Rightarrow \lambda = \frac{28}{16} = \frac{14}{8} = \frac{7}{4}$

$\Rightarrow \therefore \lambda = \frac{7}{4}$

11. If $X = \begin{bmatrix} 8 & -1 & -3 \\ -5 & 1 & 2 \\ 10 & -1 & -4 \end{bmatrix}$ and $Y = \begin{bmatrix} 2 & 1 & -1 \\ 0 & 2 & 1 \\ 5 & p & q \end{bmatrix}$ then, find *p*, *q* if $Y = X^{-1}$. **[HY-2019]**

Sol : Given $X = \begin{bmatrix} 8 & -1 & -3 \\ -5 & 1 & 2 \\ 10 & -1 & -4 \end{bmatrix}$ and $Y = \begin{bmatrix} 2 & 1 & -1 \\ 0 & 2 & 1 \\ 5 & p & q \end{bmatrix}$

Also it is given that $Y = X^{-1}$

$\Rightarrow XY = XX^{-1}$ [Pre - multiply by X]

$\Rightarrow XY = I$

$$= \begin{bmatrix} +(3-0) & -(15-0) & +(5-0) \\ -(0+1) & +(6-0) & -(2-0) \\ +(0+1) & -(0+5) & +(2-0) \end{bmatrix}^T = \begin{bmatrix} 3 & -15 & 5 \\ -1 & 6 & -2 \\ 1 & -5 & 2 \end{bmatrix}^T$$

$$= \begin{bmatrix} 3 & -1 & 1 \\ -15 & 6 & -5 \\ 5 & -2 & 2 \end{bmatrix}$$

$$A^{-1} = \frac{1}{|A|} \text{adj } A = \frac{1}{1}\begin{bmatrix} 3 & -1 & 1 \\ -15 & 6 & -5 \\ 5 & -2 & 2 \end{bmatrix}$$

$$\therefore X = A^{-1}B = \begin{bmatrix} 3 & -1 & 1 \\ -15 & 6 & -5 \\ 5 & -2 & 2 \end{bmatrix}\begin{bmatrix} 0 \\ 4 \\ 5 \end{bmatrix}$$

$$= \begin{bmatrix} 0-4+5 \\ 0+24-25 \\ 0-8+10 \end{bmatrix} = \begin{bmatrix} 1 \\ -1 \\ 2 \end{bmatrix}$$

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 1 \\ -1 \\ 2 \end{pmatrix} \quad \therefore x = 1, y = -1, z = 2$$

3. A sales person Ravi has the following record of sales for the month of January, February and March 2009 for three products A, B and C He has been paid a commission at fixed rate per unit but at varying rates for products A, B and C.

Months	Sales in Units			Commission
	A	B	C	
January	9	10	2	800
February	15	5	4	900
March	6	10	3	850

Find the rate of commission payable on A, B and C per unit sold using matrix inversion method.

Sol : Let x, y, z represent the rate of commission payable on A, B and C respectively.

Then $9x + 10y + 2z = 800$

$15x + 5y + 4z = 900$

$6x + 10y + 3z = 850$

The given equations can be written in matrix form as

$$\begin{bmatrix} 9 & 10 & 2 \\ 15 & 5 & 4 \\ 6 & 10 & 3 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 800 \\ 900 \\ 850 \end{bmatrix}$$

$\Rightarrow$ AX = B

Where $A = \begin{bmatrix} 9 & 10 & 2 \\ 15 & 5 & 4 \\ 6 & 10 & 3 \end{bmatrix}$, $X = \begin{bmatrix} x \\ y \\ z \end{bmatrix}$ and $B = \begin{bmatrix} 800 \\ 900 \\ 850 \end{bmatrix}$

$$|A| = \begin{vmatrix} 9 & 10 & 2 \\ 15 & 5 & 4 \\ 6 & 10 & 3 \end{vmatrix} = 9\begin{vmatrix} 5 & 4 \\ 10 & 3 \end{vmatrix} - 10\begin{vmatrix} 15 & 4 \\ 6 & 3 \end{vmatrix} + 2\begin{vmatrix} 15 & 5 \\ 6 & 10 \end{vmatrix}$$

$= 9(15 - 40) - 10(45 - 24) + 2(150 - 30)$

$= 9(-25) - 10(21) + 2(120)$

$= -225 - 210 + 240 = -195 \neq 0 \Rightarrow A^{-1}$ exists.

$$\text{adj } A = \begin{bmatrix} +\begin{vmatrix} 5 & 4 \\ 10 & 3 \end{vmatrix} & -\begin{vmatrix} 15 & 4 \\ 6 & 3 \end{vmatrix} & +\begin{vmatrix} 15 & 5 \\ 6 & 10 \end{vmatrix} \\ -\begin{vmatrix} 10 & 2 \\ 10 & 3 \end{vmatrix} & +\begin{vmatrix} 9 & 2 \\ 6 & 3 \end{vmatrix} & -\begin{vmatrix} 9 & 10 \\ 6 & 10 \end{vmatrix} \\ +\begin{vmatrix} 10 & 2 \\ 5 & 4 \end{vmatrix} & -\begin{vmatrix} 9 & 2 \\ 15 & 4 \end{vmatrix} & +\begin{vmatrix} 9 & 10 \\ 15 & 5 \end{vmatrix} \end{bmatrix}^T$$

$$= \begin{bmatrix} +(15-40) & -(45-24) & +(150-30) \\ -(30-20) & +(27-12) & -(90-60) \\ +(40-10) & -(36-30) & +(45-150) \end{bmatrix}^T$$

$$= \begin{bmatrix} -25 & -21 & 120 \\ -10 & 15 & -30 \\ 30 & -6 & -105 \end{bmatrix}^T = \begin{bmatrix} -25 & -10 & 30 \\ -21 & 15 & -6 \\ 120 & -30 & -105 \end{bmatrix}$$

$$A^{-1} = \frac{1}{|A|} \text{adj } A = \frac{-1}{195}\begin{bmatrix} -25 & -10 & 30 \\ -21 & 15 & -6 \\ 120 & -30 & -105 \end{bmatrix}$$

Also, $X = A^{-1}B$

$$= \frac{-1}{195}\begin{bmatrix} -25 & -10 & 30 \\ -21 & 15 & -6 \\ 120 & -30 & -105 \end{bmatrix}\begin{bmatrix} 800 \\ 900 \\ 850 \end{bmatrix}$$

$$= \frac{-1}{195}\begin{bmatrix} -20,000 - 9,000 + 25,500 \\ -16,800 + 13,500 - 5,100 \\ 96,000 - 27,000 - 89,250 \end{bmatrix}$$

$$= \frac{-1}{195}\begin{bmatrix} -3500 \\ -8400 \\ -20,250 \end{bmatrix} = \begin{bmatrix} \frac{3500}{195} \\ \frac{8400}{195} \\ \frac{20,250}{195} \end{bmatrix}$$

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 17.95 \\ 43.08 \\ 103.85 \end{bmatrix}$$

[Hence the rate of commission payable on A,B and C are 17.95, 43.08 and 103.85]

4. The prices of three commodities A, B and C are ₹ x, ₹ y and ₹ z per unit respectively. P purchases 4 units of C and sells 3 units of A and 5 units of B. Q purchases 3 units of B and sells 2 units of A and 1 unit of C. R purchases 1 unit of A and sells 4 units of B and 6 units of C. In the process P, Q and R earn ₹6,000 , ₹5,000 and ₹13,000 respectively. By using matrix inversion method, find the prices per unit of A, B and C.

Sol : By the given data

$$\begin{aligned} -4z + 3x + 5y &= 6000 \\ -3y + 2x + z &= 5000 \\ -x + 4y + 6z &= 13{,}000 \\ \Rightarrow 3x + 5y - 4z &= +6000 \\ 2x - 3y + z &= +5000 \\ -x + 4y + 6z &= 13{,}000 \end{aligned}$$

It can be written in matrix form as

$$\begin{bmatrix} 3 & 5 & -4 \\ 2 & -3 & 1 \\ -1 & +4 & +6 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} +6000 \\ +5000 \\ 13{,}000 \end{bmatrix}$$

$$\Rightarrow \quad AX = B \text{ Where } A = \begin{bmatrix} 3 & 5 & -4 \\ 2 & -3 & 1 \\ -1 & +4 & +6 \end{bmatrix},$$

$$X = \begin{bmatrix} x \\ y \\ z \end{bmatrix}, B = \begin{bmatrix} +6000 \\ +5000 \\ 13{,}000 \end{bmatrix}$$

$$\Rightarrow \quad X = A^{-1}B$$

$$|A| = \begin{bmatrix} 3 & 5 & -4 \\ 2 & -3 & 1 \\ -1 & +4 & +6 \end{bmatrix}$$

$$= 3\begin{vmatrix} -3 & 1 \\ +4 & +6 \end{vmatrix} - 5\begin{vmatrix} 2 & 1 \\ -1 & +6 \end{vmatrix} - 4\begin{vmatrix} 2 & -3 \\ -1 & +4 \end{vmatrix}$$

$$= 3(-18-4) - 5(12+1) - 4(+8-3)$$

$$= 3(-22) - 5(13) - 4(5)$$

$$= -66 - 65 - 20 = -151 \neq 0 \Rightarrow A^{-1} \text{ exists}$$

$$\text{adj } A = \begin{bmatrix} +\begin{vmatrix} -3 & 1 \\ +4 & +6 \end{vmatrix} & -\begin{vmatrix} 2 & 1 \\ -1 & +6 \end{vmatrix} & +\begin{vmatrix} 2 & -3 \\ -1 & +4 \end{vmatrix} \\ -\begin{vmatrix} 5 & -4 \\ +4 & +6 \end{vmatrix} & +\begin{vmatrix} 3 & -4 \\ -1 & +6 \end{vmatrix} & -\begin{vmatrix} 3 & 5 \\ -1 & +4 \end{vmatrix} \\ +\begin{vmatrix} 5 & -4 \\ -3 & 1 \end{vmatrix} & -\begin{vmatrix} 3 & -4 \\ 2 & 1 \end{vmatrix} & +\begin{vmatrix} 3 & 5 \\ 2 & -3 \end{vmatrix} \end{bmatrix}^T$$

$$= \begin{bmatrix} +(-18-4) & -(+12+1) & +(+8-3) \\ -(+30+16) & +(+18-4) & -(+12+5) \\ +(5-12) & -(3+8) & +(-9-10) \end{bmatrix}^T$$

$$= \begin{bmatrix} -22 & -13 & +5 \\ -46 & +14 & -17 \\ -7 & -11 & -19 \end{bmatrix}^T = \begin{bmatrix} -22 & -46 & -7 \\ -13 & +14 & -11 \\ +5 & -17 & -19 \end{bmatrix}$$

$$A^{-1} = \frac{1}{|A|}\text{adj } A = \frac{-1}{151}\begin{bmatrix} -22 & -46 & -7 \\ -13 & +14 & -11 \\ +5 & -17 & -19 \end{bmatrix}$$

Now, $X = A^{-1}B$

$$X = \frac{-1}{151}\begin{bmatrix} -22 & -46 & -7 \\ -13 & 14 & -11 \\ 5 & -17 & -19 \end{bmatrix}\begin{bmatrix} 6000 \\ 5000 \\ 13000 \end{bmatrix}$$

$$= \frac{-1}{151}\begin{bmatrix} -132000 - 230000 - 91000 \\ -78000 + 70000 \;\; 143000 \\ 30000 - 85000 - 247000 \end{bmatrix}$$

$$X = \frac{-1}{151}\begin{bmatrix} -453000 \\ -151000 \\ -302000 \end{bmatrix} = \begin{bmatrix} 3000 \\ 1000 \\ 2000 \end{bmatrix}$$

⇒ The prices of per unit of A, B, C are ₹ 3000, ₹ 1000 and ₹ 2000 respectively.

5. The sum of three numbers is 20. If we multiply the first by 2 and add the second number and subtract the third we get 23. If we multiply the first by 3 and add second and third to it, we get 46. By using matrix inversion method find the numbers.

Sol : Let the required numbers be x, y and z.

[HY-2019]

$$\begin{aligned} \text{Given } x + y + z &= 20 \\ 2x + y - z &= 23 \\ 3x + y + z &= 46 \end{aligned}$$

These equations can be written in matrix form as

$$\begin{bmatrix} 1 & 1 & 1 \\ 2 & 1 & -1 \\ 3 & 1 & 1 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 20 \\ 23 \\ 46 \end{bmatrix} \Rightarrow AX = B$$

Where $A = \begin{bmatrix} 1 & 1 & 1 \\ 2 & 1 & -1 \\ 3 & 1 & 1 \end{bmatrix}, X = \begin{bmatrix} x \\ y \\ z \end{bmatrix}, B = \begin{bmatrix} 20 \\ 23 \\ 46 \end{bmatrix}$

$$|A| = \begin{vmatrix} 1 & 1 & 1 \\ 2 & 1 & -1 \\ 3 & 1 & 1 \end{vmatrix} = 1\begin{vmatrix} 1 & -1 \\ 1 & 1 \end{vmatrix} - 1\begin{vmatrix} 2 & -1 \\ 3 & 1 \end{vmatrix} + 1\begin{vmatrix} 2 & 1 \\ 3 & 1 \end{vmatrix}$$

$= 1(1+1) - 1(2+3) + 1(2-3)$

$= 2 - 5 - 1 = 2 - 6 = -4 \neq 0 \Rightarrow A^{-1}$ exists.

$$\text{adj } A = \begin{bmatrix} +\begin{vmatrix} 1 & -1 \\ 1 & 1 \end{vmatrix} & -\begin{vmatrix} 2 & -1 \\ 3 & 1 \end{vmatrix} & +\begin{vmatrix} 2 & 1 \\ 3 & 1 \end{vmatrix} \\ -\begin{vmatrix} 1 & 1 \\ 1 & 1 \end{vmatrix} & +\begin{vmatrix} 1 & 1 \\ 3 & 1 \end{vmatrix} & -\begin{vmatrix} 1 & 1 \\ 3 & 1 \end{vmatrix} \\ +\begin{vmatrix} 1 & 1 \\ 1 & -1 \end{vmatrix} & -\begin{vmatrix} 1 & 1 \\ 2 & -1 \end{vmatrix} & +\begin{vmatrix} 1 & 1 \\ 2 & 1 \end{vmatrix} \end{bmatrix}^T$$

$$= \begin{bmatrix} +(1+1) & -(2+3)) & +(2-3) \\ -(1-1) & +(1-3) & -(1-3) \\ +(-1-1) & -(-1-2) & +(1-2) \end{bmatrix}^T$$

$$= \begin{bmatrix} 2 & -5 & -1 \\ 0 & -2 & 2 \\ -2 & 3 & -1 \end{bmatrix}^T = \begin{bmatrix} 2 & 0 & -2 \\ -5 & -2 & 3 \\ -1 & 2 & -1 \end{bmatrix}$$

$$\therefore A^{-1} = \frac{1}{|A|}\text{adj } A = \frac{-1}{4}\begin{bmatrix} 2 & 0 & -2 \\ -5 & -2 & 3 \\ -1 & 2 & -1 \end{bmatrix}$$

$$X = A^{-1}B = \frac{-1}{4}\begin{bmatrix} 2 & 0 & -2 \\ -5 & -2 & 3 \\ -1 & 2 & -1 \end{bmatrix}\begin{bmatrix} 20 \\ 23 \\ 46 \end{bmatrix}$$

$$= \frac{-1}{4}\begin{bmatrix} 40+0-92 \\ -100-46+138 \\ -20+46-46 \end{bmatrix} = \frac{-1}{4}\begin{bmatrix} -52 \\ -8 \\ -20 \end{bmatrix}$$

$X = \begin{bmatrix} 13 \\ 2 \\ 5 \end{bmatrix}$ ∴ The required numbers are 13, 2 and 5.

6. Weekly expenditure in an office for three weeks is given as follows. Assuming that the salary in all the three weeks of different categories of staff did not vary, calculate the salary for each type of staff, using matrix inversion method.

Week	Number of employees			Total weekly Salary (in ₹)
	A	B	C	
1st week	4	2	3	4900
2nd week	3	3	2	4500
3rd week	4	3	4	5800

Sol : Let the salary for each type of staff be x, y and z respectively.

Then, by the given data,

$$4x + 2y + 3z = 4900$$
$$3x + 3y + 2z = 4500$$
$$4x + 3y + 4z = 5800$$

These equations can be converted into matrix form as

$$\begin{bmatrix} 4 & 2 & 3 \\ 3 & 3 & 2 \\ 4 & 3 & 4 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 4900 \\ 4500 \\ 5800 \end{bmatrix} \Rightarrow AX = B$$

Where $A = \begin{bmatrix} 4 & 2 & 3 \\ 3 & 3 & 2 \\ 4 & 3 & 4 \end{bmatrix}, X = \begin{bmatrix} x \\ y \\ z \end{bmatrix}, B = \begin{bmatrix} 4900 \\ 4500 \\ 5800 \end{bmatrix}$

$$|A| = \begin{vmatrix} 4 & 2 & 3 \\ 3 & 3 & 2 \\ 4 & 3 & 4 \end{vmatrix} = 4\begin{vmatrix} 3 & 2 \\ 3 & 4 \end{vmatrix} - 2\begin{vmatrix} 3 & 2 \\ 4 & 4 \end{vmatrix} + 3\begin{vmatrix} 3 & 3 \\ 4 & 3 \end{vmatrix}$$

$= 4(12-6) - 2(12-8) + 3(9-12)$

$= 4(6) - 2(4) + 3(-3) = 24 - 8 - 9 = 7 \neq 0$

∴ A^{-1} exists.

$$\text{adj } A = \begin{bmatrix} +\begin{vmatrix} 3 & 2 \\ 4 & 4 \end{vmatrix} & -\begin{vmatrix} 3 & 2 \\ 4 & 4 \end{vmatrix} & +\begin{vmatrix} 3 & 3 \\ 4 & 3 \end{vmatrix} \\ +\begin{vmatrix} 2 & 3 \\ 3 & 4 \end{vmatrix} & +\begin{vmatrix} 4 & 3 \\ 4 & 4 \end{vmatrix} & -\begin{vmatrix} 4 & 2 \\ 4 & 3 \end{vmatrix} \\ +\begin{vmatrix} 2 & 3 \\ 3 & 2 \end{vmatrix} & -\begin{vmatrix} 4 & 3 \\ 3 & 2 \end{vmatrix} & +\begin{vmatrix} 4 & 2 \\ 3 & 3 \end{vmatrix} \end{bmatrix}$$

$$= \begin{bmatrix} +(12-6)-(12-8)+(9-12) \\ -(8-9)+(16-12)-(12-8) \\ +(4-9)-(8-9)+(12-6) \end{bmatrix}^T$$

$$= \begin{bmatrix} 6 & -4 & -3 \\ 1 & 4 & -4 \\ -5 & 1 & 6 \end{bmatrix}^T = \begin{bmatrix} 6 & 1 & -5 \\ -4 & 4 & 1 \\ -3 & -4 & 6 \end{bmatrix}$$

$$\therefore A^{-1} = \frac{1}{|A|}\text{adj } A = \frac{1}{7}\begin{bmatrix} 6 & 1 & -5 \\ -4 & 4 & 1 \\ -3 & -4 & 6 \end{bmatrix}$$

$$\therefore X = A^{-1}B = \frac{1}{7}\begin{bmatrix} 6 & 1 & -5 \\ -4 & 4 & 1 \\ -3 & -4 & 6 \end{bmatrix}\begin{bmatrix} 4900 \\ 4500 \\ 5800 \end{bmatrix}$$

$$= \frac{1}{7}\begin{bmatrix} 29400+4500-29,000 \\ -19600+18000+5800 \\ -14700-18,000+34800 \end{bmatrix} = \frac{1}{7}\begin{bmatrix} 4900 \\ 4200 \\ 2100 \end{bmatrix} = \begin{bmatrix} 700 \\ 600 \\ 300 \end{bmatrix}$$

Hence, the salary for each type of staff are ₹700, ₹600 and ₹300 respectively.

Exercise 1.4

1. The technology matrix of an economic system of two industries is $\begin{bmatrix} 0.50 & 0.30 \\ 0.41 & 0.33 \end{bmatrix}$. Test whether the system is viable as per Hawkins Simon conditions.

[Qy-2018; Sep. - 2021]

Sol : The technology matrix is $B = \begin{bmatrix} 0.50 & 0.30 \\ 0.41 & 0.33 \end{bmatrix}$

$$I - B = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} - \begin{bmatrix} 0.50 & 0.30 \\ 0.41 & 0.33 \end{bmatrix} = \begin{bmatrix} 0.50 & -0.30 \\ -0.41 & 0.67 \end{bmatrix}$$

$$|I - B| = \begin{vmatrix} 0.50 & -0.30 \\ -0.41 & 0.67 \end{vmatrix}$$

$$= (0.50)(0.67) - (0.30)(0.41) = 0.335 - 0.123$$

$$= 0.212 > 0$$

Since the diagonals elements of I – B are positive and |I – B| is positive. Hawkins - Simon conditions are satisfied.

Hence the given system is viable.

2. The technology matrix of an economic system of two industries is $\begin{bmatrix} 0.6 & 0.9 \\ 0.20 & 0.80 \end{bmatrix}$. Test whether the system is viable as per Hawkins-Simon conditions.

[May - 2022]

Sol : The technology matrix B is $\begin{bmatrix} 0.6 & 0.9 \\ 0.20 & 0.80 \end{bmatrix}$

$$\therefore I - B = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} - \begin{bmatrix} 0.6 & 0.9 \\ 0.20 & 0.80 \end{bmatrix} = \begin{bmatrix} 0.4 & -0.9 \\ -0.20 & 0.20 \end{bmatrix}$$

$$|I - B| = \begin{vmatrix} 0.4 & -0.9 \\ -0.20 & 0.20 \end{vmatrix} = (0.4)(0.20) - (0.20)(0.9)$$

$$= 0.08 - 0.18 = -0.1 < 0$$

Since |I – B| is negative, Hawkins – Simon conditions are not satisfied. Therefore the given system is not viable.

3. The technology matrix of an economic system of two industries is $\begin{bmatrix} 0.50 & 0.25 \\ 0.40 & 0.67 \end{bmatrix}$. Test whether the system is viable as per Hawkins-Simon conditions

Sol : The technology matrix is $B = \begin{bmatrix} 0.50 & 0.25 \\ 0.40 & 0.67 \end{bmatrix}$

$$I - B = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} - \begin{bmatrix} 0.50 & 0.25 \\ 0.40 & 0.67 \end{bmatrix} = \begin{bmatrix} 0.50 & -0.25 \\ -0.40 & 0.33 \end{bmatrix}$$

$$= (0.50)(0.33) - (0.40)(0.25)$$

$$= 0.165 - 0.1 = 0.065 > 0$$

Since the diagonal elements of (I – B) are positive and | I – B| is positive,

Hawkins – Simon conditions are satisfied.

Hence the given system is viable.

4. Two commodities A and B are produced such that 0.4 tonne of A and 0.7 tonne of B are required to produce a tonne of A. Similarly 0.1 tonne of A and 0.7 tonne of B are needed to produce a tonne of B. Write down the technology matrix. If 6.8 tonnes of A and 10.2 tonnes of B are required, find the gross production of both of them. **[QY-2019; GMQP-2018]**

Sol : The technology matrix B is $\begin{bmatrix} 0.4 & 0.1 \\ 0.7 & 0.7 \end{bmatrix}$

$$I - B = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} - \begin{bmatrix} 0.4 & 0.1 \\ 0.7 & 0.7 \end{bmatrix} = \begin{bmatrix} 0.6 & -0.1 \\ -0.7 & 0.3 \end{bmatrix}$$

$$|I - B| = \begin{vmatrix} 0.6 & -0.1 \\ -0.7 & 0.3 \end{vmatrix} = (0.6)(0.3) - (0.7)(0.1)$$

$$= 0.18 - 0.07 = 0.11$$

$$\therefore (I - B)^{-1} = \frac{1}{|I - B|} \text{adj}(I - B) = \frac{1}{0.11}\begin{bmatrix} 0.3 & 0.1 \\ 0.7 & 0.6 \end{bmatrix}$$

Since the diagonal elements of (I – B) are positive and |I – B| is positive, the system is viable.

Now $X = (I - B)^{-1} D$ where $D = \begin{bmatrix} 6.8 \\ 10.2 \end{bmatrix}$

$$X = \frac{1}{0.11}\begin{bmatrix} 0.3 & 0.1 \\ 0.7 & 0.6 \end{bmatrix}\begin{bmatrix} 6.8 \\ 10.2 \end{bmatrix}$$

$$= \frac{1}{0.11}\begin{bmatrix} 0.3 \times 6.8 + 0.1 \times 10.2 \\ 0.7 \times 6.8 + 0.6 \times 10.2 \end{bmatrix}$$

$$= \frac{1}{0.11}\begin{bmatrix} 2.04 + 1.02 \\ 4.76 + 6.12 \end{bmatrix}$$

$$= \frac{1}{0.11}\begin{bmatrix} 3.06 \\ 10.88 \end{bmatrix} = \begin{bmatrix} 27.82 \\ 98.91 \end{bmatrix}$$

∴ Gross production of commodity A and B are 27.82 and 98.91 tonnes.

5. Suppose the inter-industry flow of the product of two industries are given as under.

Production sector	Consumption sector		Domestic demand	Total output
	X	Y		
X	30	40	50	120
Y	20	10	30	60

04 TRIGONOMETRY

POINTS TO REMEMBER

- If in a circle of radius r, an are arc of length l subtends an angle of θ radians, then $l = r\theta$.
- $\sin^2 x + \cos^2 x = 1$
- $\tan^2 x + 1 = \sec^2 x$
- $\cot^2 x + 1 = \text{cosec}^2 x$
- $\sin(\alpha + \beta) = \sin\alpha\cos\beta + \cos\alpha\sin\beta$
- $\sin(\alpha - \beta) = \sin\alpha\cos\beta - \cos\alpha\sin\beta$
- $\cos(\alpha + \beta) = \cos\alpha\cos\beta - \sin\alpha\cos\beta$
- $\cos(\alpha - \beta) = \cos\alpha\cos\beta + \sin\alpha\sin\beta$
- $\tan(\alpha + \beta) = \dfrac{\tan\alpha + \tan\beta}{1 - \tan\alpha\tan\beta}$
- $\tan(\alpha - \beta) = \dfrac{\tan\alpha - \tan\beta}{1 + \tan\alpha\tan\beta}$
- $\sin 2\alpha = 2\sin\alpha\cos\alpha$
- $\cos 2\alpha = \cos^2\alpha - \sin^2\alpha = 1 - 2\sin^2\alpha = 2\cos^2\alpha - 1$
- $\tan 2\alpha = \dfrac{2\tan\alpha}{1 - \tan^2\alpha}$
- $\cos 3\alpha = 4\cos^3\alpha - 3\cos\alpha$
- $\tan 3\alpha = \dfrac{3\tan\alpha - \tan^3\alpha}{1 - 3\tan^2\alpha}$
- $\sin x + \sin y = 2\sin\dfrac{x+y}{2}\cos\dfrac{x-y}{2}$
- $\sin x - \sin y = 2\cos\dfrac{x+y}{2}\sin\dfrac{x-y}{2}$
- $\cos x + \cos y = 2\cos\dfrac{x+y}{2}\cos\dfrac{x-y}{2}$
- $\cos x - \cos y = -2\sin\dfrac{x+y}{2}\sin\dfrac{x-y}{2}$
- $\sin x\cos y = \dfrac{1}{2}[\sin(x+y) + \sin(x-y)]$
- $\cos x\sin y = \dfrac{1}{2}[\sin(x+y) - \sin(x-y)]$
- $\cos x\cos y = \dfrac{1}{2}[\cos(x+y) + \cos(x-y)]$
- $\sin x\sin y = \dfrac{1}{2}[\cos(x-y) - \cos(x+y)]$

TEXTUAL QUESTIONS

Exercise 4.1

1. Convert the following degree measure into radian measure

(i) 60° (ii) 150° (iii) 240° (iv) – 320°

Sol :

(i) 60°

$$60° = 60 \times \frac{\pi}{180} \text{ radians} = \frac{\pi}{3} \text{ radians.}$$

(ii) 150°

$$150° = 150 \times \frac{\pi}{180} \text{ radians}$$
$$= \frac{5\pi}{6} \text{ radians.}$$

(iii) 240°

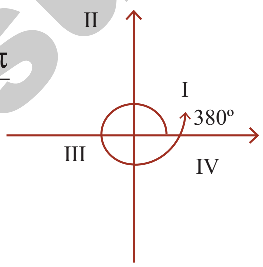
$$240° = 240 \times \frac{\pi}{180} \text{ radians}$$
$$= \frac{4\pi}{3} \text{ radians.}$$

(iv) – 320°

$$-320° = -320° \times \frac{\pi}{180} \text{ radians}$$
$$= \frac{-16\pi}{9} \text{ radians.}$$

2. Find the degree measure corresponding to the following radian measure.

(i) $\frac{\pi}{8}$ (ii) $\frac{9\pi}{5}$ (iii) –3 (iv) $\frac{11\pi}{18}$



Sol :

(i) $\frac{\pi}{8}$

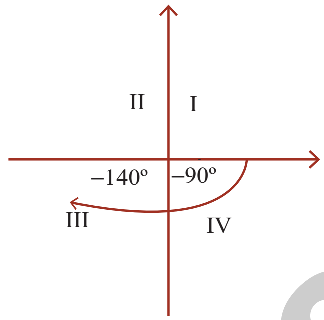
$$\frac{\pi}{8} \text{ radians} = \frac{\pi}{8} \times \frac{180°}{\pi} = 22½° = 22° \, 30'$$

[one degree = 60 minutes (60′) and $\frac{1}{2}$ degree = 30′]

(ii) $\frac{9\pi}{5}$

$$\frac{9\pi}{5} \text{ radians} = \frac{9\pi}{5} \times \frac{180°}{\pi} = 324°$$

(iii) –3



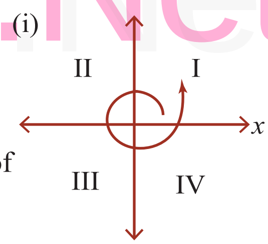
$$-3 \text{ radians} = -3 \times \frac{180}{\pi} \text{ degrees}$$
$$= -3 \times \frac{180}{22} \times 7$$
$$= -\frac{3780°}{22} = -171,81°$$
$$= -171°48' [0.81° = 0.81 \times 60' = 48']$$

(iv) $\frac{11\pi}{18}$

$$\frac{11\pi}{18} \text{ radians} = \frac{11\pi}{18} \times \frac{180}{\pi} \text{ degrees}$$
$$= 11 \times 10 = 110°$$

3. Determine the quadrants in which the following degree lie. (i) 380° (ii) – 140° (iii) 1195°

Sol :

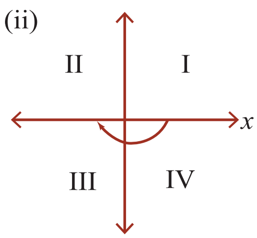


(i) 380°

380° = 360° + 20°

After one completion of round, the angle is 20°

380° lie's in the I quadrant.

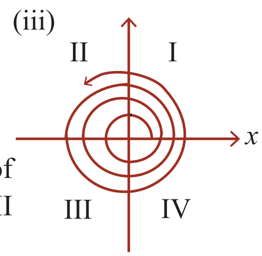


(ii) –140°

140° = – 90° + (–50°)

Since the angle is negative it moves in the anti clockwise direction.

∴ –140° lies in the III quadrant.



(iii) 1195°

1195° = 3 × 360° + 115°

= 3 × 360° + 90° + 25°

∴ After 3 Completion of round, the angle will lie in II quadrant

∴ 1195° lies in the II quadrant.

4. Find the values of each of the following trigonometric ratios.

(i) sin 300° **(ii) cos (–210°)**

(iii) sec 390° **(iv) tan (–855°)**

(v) cosec 1125° **[Aug. - 2022]**

Sol :

(i) sin (300°)

$= \sin(270° + 30°)$

$= -\cos 30°$

[∵ 300° lies in the IV quadrant sin is negative]

$= -\dfrac{\sqrt{3}}{2}$

(ii) cos (–210°)

$= \cos 210°$ [Since cos is an even function]

$= \cos(180° + 30°)$

$= -\cos 30°$ [∵ cos is negative in III quadrant]

$= -\dfrac{\sqrt{3}}{2}$

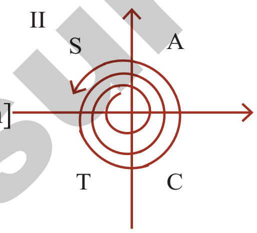
(iii) sec (390°)

$= \sec(360 + 30°)$

$= \sec 30°$ [All the ratios are positive]

$= \dfrac{2}{\sqrt{3}}$

(iv) tan (–855°)



$= -\tan 855°$

[Since tan is an odd function]

$= -\tan(720° + 135°)$

$= -\tan(135°)$

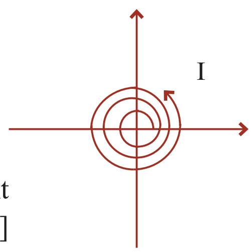
$= -\tan(90° + 45°)$

$= \cot 45°$

[∵ 855° lies in the II quadrant and tan is negative]

$= 1$

(v) cosec (1125°)



$= \text{cosec}(3 \times 360° + 45°)$

$= \text{cosec } 45°$

[1125° lies in the I quadrant and all the ratios are positive]

$= \sqrt{2}$

5. Prove that

(i) tan (– 225°) cot (– 405°) – tan (– 765°) cot (675°) = 0.

(ii) $2\sin^2\dfrac{\pi}{6} + \text{cosec}^2\dfrac{7\pi}{6}\cos^2\dfrac{\pi}{3} = \dfrac{3}{2}$

(iii) $\sec\left(\dfrac{3\pi}{2}-\theta\right)\sec\left(\theta-\dfrac{5\pi}{2}\right) + \tan\left(\dfrac{5\pi}{2}+\theta\right)\tan\left(\theta-\dfrac{5\pi}{2}\right) = -1.$

Sol :

(i) tan (–225°) cot (–405°) – tan (–765°) cot (675°) = 0

$\tan(-225°) = -\tan 225°$

$= -\tan(180 + 45°)$

$= -\tan 45°$

$= -1$ [∵ tan and cot are odd functions]

$\cot(-405°) = -\cot(405°) = -\cot(360 + 45°)$

$= -\cot 45° = -1$

$\tan(-765°) = -\tan(765°)$

$= -\tan(720 + 45°)$

$= -\tan 45° = -1$

$\cot(675°) = \cot(720° - 45°)$

$= -\cot 45° = -1$

$\therefore \text{LHS} = \tan(-225°)\cot(-405°) - \tan(-765°)\cot(675°)$

$= (-1)(-1) - (-1)(-1)$

$= 1 - 1 = 0 = \text{RHS}.$

Hence proved.

(ii) $2\sin^2\dfrac{\pi}{6} + \text{cosec}^2\dfrac{7\pi}{6}\cos^2\dfrac{\pi}{3} = \dfrac{3}{2}$

$\sin^2\dfrac{\pi}{6} = \left[\sin\dfrac{\pi}{6}\right]^2 = [\sin 30°]^2$

$= \left(\dfrac{1}{2}\right)^2 = \dfrac{1}{4}$

$\text{cosec}\dfrac{7\pi}{6} = \text{cosec } 7\times\dfrac{180}{6} = \text{cosec } 210°$

$= \text{cosec}(180 + 30°)$

$= -\text{cosec } 30° = -2$

$\therefore \text{cosec}^2\dfrac{7\pi}{6} = \left(\text{cosec}\dfrac{7\pi}{6}\right)^2 = (-2)^2 = 4$

$$\cos^2\frac{\pi}{3} = \left[\cos\left(\frac{\pi}{3}\right)\right]^2 = [\cos 60°]^2$$
$$= \left(\frac{1}{2}\right)^2 = \frac{1}{4}$$

$\therefore$ LHS $2\sin^2\frac{\pi}{6} + \text{cosec}^2\frac{7\pi}{6}\cos^2\frac{\pi}{3}$

$$= 2\left(\frac{1}{4}\right)+4\left(\frac{1}{4}\right) = \frac{1}{2}+1 = \frac{3}{2}$$
$= $ RHS

Hence Proved.

(iii) $\sec\left(\frac{3\pi}{2}-\theta\right)\sec\left(\theta-\frac{5\pi}{2}\right)+\tan\left(\frac{5\pi}{2}+\theta\right)\tan\left(\theta-\frac{5\pi}{2}\right) = -1$

$$\sec\left(\frac{3\pi}{2}-\theta\right) = \sec(270-\theta) = -\text{cosec}\,\theta$$
$$\sec\left(\theta-\frac{5\pi}{2}\right) = \sec\left[-\left(\frac{5\pi}{2}-\theta\right)\right] = \sec\left(\frac{5\pi}{2}-\theta\right)$$
[$\because$ sec θ is an even function]
$$= \sec(450-\theta)$$
$$= \sec(360+(90-\theta))$$
$$= \sec(90-\theta) = \text{cosec}\,\theta$$
$$\tan\left(\frac{5\pi}{2}+\theta\right) = \tan(450+\theta)$$
$$= \tan(360+(90+\theta))$$
$$= \tan(90+\theta) = -\cot\theta.$$
$$\tan\left(\theta-\frac{5\pi}{2}\right) = \tan\left[-\left(\frac{5\pi}{2}-\theta\right)\right]$$
$$= -\tan\left(\frac{5\pi}{2}-\theta\right)$$
[$\because$ tan is an odd function]
$$= -\tan(450-\theta)$$
$$= -\tan(360+(90-\theta))$$
$$= -\tan(90-\theta) = -\cot\theta$$
$$\therefore \text{LHS} = (-\text{cosec}\,\theta)(\text{cosec}\,\theta) - \cot\theta(-\cot\theta)$$
$$= -\text{cosec}^2\theta + \cot^2\theta$$
$$= -[\text{cosec}^2\theta - \cot^2]$$
$$= -1 \; [\because \text{cosec}^2\theta - \cot^2\theta = 1]$$
$= $ RHS

Hence Proved.

6. If A, B, C, D are angles of a cyclic quadrilateral, prove that cosA + cosB + cosC + cosD = 0.

Sol : Since A, B, C, D are angles of a cyclic quadrilateral, we have A + C=180°

$\Rightarrow$ C = 180 – A and B + D = 180° $\Rightarrow$ D = 180° – B

$\therefore$ LHS = cos A + cos B + cos C + cos D

= cos A + cos B + cos (180 – A) + cos (180 – B)

= cos A + cos B – cos A – cos B

[$\because$ (180–A) & (180 – B) are in the III quadrant and cos is negative]

= 0 = RHS. Hence proved.

7. Prove that :

(i) $\dfrac{\sin(180-\theta)\cos(90+\theta)\tan(270-\theta)\cot(360-\theta)}{\sin(360-\theta)\cot(360+\theta)\sin(270-\theta)\text{cosec}(-\theta)} = -1$ **[Sep. - 2021]**

(ii) $\sin\theta\cos\theta\left\{\sin\left(\frac{\pi}{2}-\theta\right).\text{cosec}\,\theta+\cos\left(\frac{\pi}{2}-\theta\right).\sec\theta\right\}=1$

Sol :

(i) $\dfrac{\sin(180-\theta)\cos(90+\theta)\tan(270-\theta)\cot(360-\theta)}{\sin(360-\theta)\cot(360+\theta)\sin(270-\theta)\text{cosec}(-\theta)} = -1$

sin (180 – θ)	=	sin θ	sin (360 – θ)	=	–sin θ
cos (90 + θ)	=	– sin θ	cos (360 + θ)	=	cos θ
tan (270 – θ)	=	cot θ	sin (270 – θ)	=	–cosθ
cot (360 – θ)	=	–cot θ	cosec (– θ)	=	–cosecθ

$$\therefore \text{LHS} = \frac{(\sin\theta)(-\sin\theta)(\cot\theta)(-\cot\theta)}{(-\sin\theta)(\cos\theta)(-\cos\theta)(-\text{cosec}\theta)}$$
$$= \frac{\sin\theta.\sin\theta.\frac{\cos\theta}{\sin\theta}.\frac{\cos\theta}{\sin\theta}}{\sin\theta.\cos\theta.\cos\theta.\frac{1}{\sin\theta}}$$
$$= \frac{-\cos^2\theta}{\cos^2\theta} = -1 = \text{RHS}$$

Hence proved.

(ii) $\sin\theta\cos\theta\left\{\sin\left(\frac{\pi}{2}-\theta\right).\text{cosec}\,\theta+\cos\left(\frac{\pi}{2}-\theta\right).\sec\theta\right\}=1$

$$\sin\left(\frac{\pi}{2}-\theta\right) = \cos\theta$$
$$\cos\left(\frac{\pi}{2}-\theta\right) = \sin\theta$$
$$\therefore \text{LHS} = \sin\theta\cos\theta\left\{\sin\left(\frac{\pi}{2}-\theta\right)\text{cosec}\theta+\cos\left(\frac{\pi}{2}-\theta\right)\sec\theta\right\}$$
$$= \sin\theta\cos\theta\left[\cos\theta\frac{1}{\sin\theta}+\sin\theta\frac{1}{\cos\theta}\right]$$
$$= \sin\theta\cos\theta\left[\frac{\cos^2\theta+\sin^2\theta}{\sin\theta\cos\theta}\right]$$

$$= \cancel{\sin\theta\cos\theta} \times \frac{1}{\cancel{\sin\theta\cos\theta}}$$

$[\because \sin^2\theta + \cos^2\theta = 1]$

$= 1$ = RHS. Hence proved.

8. Prove that: cos 510° cos 330° + sin 390° cos 120° = –1 **[Mar.-2019]**

Sol :

$\cos 510° = \cos(360° + 150°)$

$= \cos 150° = \cos(180 - 30°)$

$= -\cos 30° = -\frac{\sqrt{3}}{2}$

$\cos 330° = \cos(360 - 30°) = \cos 30° = \frac{\sqrt{3}}{2}$

$\sin 390° = \sin(360 + 30°) = \sin 30° = \frac{1}{2}$

$\cos 120° = \cos(90 + 30°) = -\sin 30° = -\frac{1}{2}$

$\therefore \text{LHS} = \cos 510°\cos 330° + \sin 390°\cos 120°$

$$= \left(-\frac{\sqrt{3}}{2}\right)\left(\frac{\sqrt{3}}{2}\right) + \left(\frac{1}{2}\right)\left(\frac{-1}{2}\right) = -\frac{3}{4} - \frac{1}{4} = \frac{-\cancel{4}}{\cancel{4}} = -1$$

= RHS. Hence Proved.

9. Prove that

(i) $\tan(\pi + x)\cot(x - \pi) - \cos(2\pi - x)\cos(2\pi + x) = \sin^2 x$

(ii) $\dfrac{\sin(180° + A)\cos(90° - A)\tan(270° - A)}{\sec(540° - A)\cos(360° + A)\csc(270° + A)} = -\sin A\cos^2 A.$ **[Qy-2018; May - 2022]**

Sol : **(i)** $\tan(\pi + x)\cot(x - \pi) - \cos(2\pi - x)\cos(2\pi + x) = \sin^2 x$

$\tan(\pi + x) = \tan(180 + x) = \tan x$

$\cot(x - \pi) = \cot[-(\pi - x)]$

$= -\cot(\pi - x) = \cot x$

$\cos(2\pi - x) = \cos(360 - x) = \cos x$

$\cos(2\pi + x) = \cos(360 + x) = \cos x$

$\therefore \text{LHS} = \tan(\pi + x)\cot(x - \pi) - \cos(2\pi - x)\cos(2\pi + x)$

$= \tan x \cot x - \cos x \cos x$

$= \frac{\cancel{\sin x}}{\cancel{\cos x}} \cdot \frac{\cancel{\cos x}}{\cancel{\sin x}} - \cos^2 x$

$= 1 - \cos^2 x = \sin^2 x$

$[\because 1 - \cos^2 x = \sin^2 x]$

= RHS. Hence Proved.

(ii) $\dfrac{\sin(180° + A)\cos(90° - A)\tan(270° - A)}{\sec(540° - A)\cos(360° + A)\csc(270° + A)} = -\sin A\cos^2 A.$

$\sin(180 + A) = -\sin A$

$\cos(90 - A) = +\sin A$

$\tan(270 - A) = \cot A$

$\sec(540 - A) = \sec(360 + 180 - A)$

$\sec(180 - A) = -\sec A$

$\cos(360 + A) = \cos A$

$\csc(270 + A) = -\sec A$

$$\therefore \text{LHS} = \frac{\sin(180 + A)\cos(90 - A)\tan(270 - A)}{\sec(540 - A)\cos(360 + A)\csc(270 + A)}$$

$$= \frac{(-\sin A)(\sin A)(\cot A)}{(-\sec A)(\cos A)(-\sec A)}$$

$$= -\frac{\sin A\ \cancel{\sin A}\ \dfrac{\cos A}{\cancel{\sin A}}}{-\sec A\ \cancel{\cos A}\ \dfrac{-1}{\cancel{\cos A}}}$$

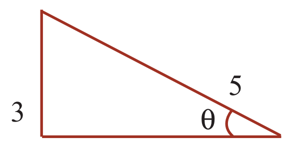
$$= -\frac{\sin A\cos A}{\sec A}$$

$= -\sin A.\cos A.\cos A \left[\because \frac{1}{\sec A} = \cos A\right]$

$= -\sin A\cos^2 A$ = RHS. Hence proved.

10. If $\sin\theta = \frac{3}{5}$, $\tan\varphi = \frac{1}{2}$ **and** $\frac{\pi}{2} < \theta < \pi < \varphi < \frac{3\pi}{2}$, **then find the value of** $8\tan\theta - \sqrt{5}\sec\varphi$.

Sol :



Given $\sin\theta = \frac{3}{5}$

$= \sqrt{5^2 - 3^2}$

and $\frac{\pi}{2} < \theta < \pi$

$= \sqrt{25 - 9}$

$\therefore$ θ is in II quadrant, only sin and its reciprocal is positive

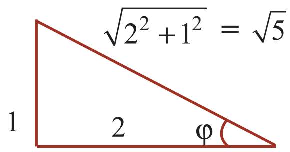
$= 4$

$\cos\theta = \frac{adj}{hyp} = -\frac{4}{5}$, $\tan\theta = \frac{-3}{4}$ ... (1)

Also $\tan\varphi = \frac{1}{2}$ and $\pi < \varphi < \frac{3\pi}{2}$

$\therefore$ φ is in III quadrant, tan φ and its reciprocal alone are positive.

$\therefore \sec\varphi = \frac{hyp}{adj} = -\frac{\sqrt{2}}{2}$ ... (2)



$8 \tan\theta - \sqrt{5}\sec\varphi = \overset{2}{\cancel{8}}\left(\frac{-3}{\cancel{4}}\right) - \sqrt{5}\left(-\frac{\sqrt{5}}{2}\right)$ [using (1) and (2)

$= +2(-3) + \frac{5}{2} = -6 + \frac{5}{2} = \frac{-12+5}{2} = \frac{-7}{2}$

Hence Proved.

Exercise 4.2

1. Find the values of the following :

(i) cosec 15° **(ii) sin (–105°)**

(iii) cot 75° **[May - 2022]**

Sol :

(i) cosec 15°

$\text{cosec } 15° = \frac{1}{\sin 15°}$

$\sin 15° = \sin(45° - 30°)$

[∵ sin (A – B) = sin A cosB – cosA sinB, A = 45°, B = 30°]

$= \sin 45°\cos 30° - \cos 45°\sin 30°$

$= \frac{1}{\sqrt{2}}.\frac{\sqrt{3}}{2} - \frac{1}{\sqrt{2}}.\frac{1}{2}$

$= \frac{\sqrt{3}}{2\sqrt{2}} - \frac{1}{2\sqrt{2}} = \frac{\sqrt{3}-1}{2\sqrt{2}}$

$\therefore \text{cosec } 15° = \frac{1}{\sin 15°} = \frac{2\sqrt{2}}{\sqrt{3}-1}$

(ii) sin (–105)°

$\sin(-105)° = -\sin(105°)$

[sin is an odd function]

$= -\sin(60° + 45°)$

$= -[\sin 60°\cos 45° + \cos 60°\sin 45°]$

[sin (A + B) = sin A cos B + cos A sin B ; A = 60°, B = 45°]

$= -\left[\frac{\sqrt{3}}{2}.\frac{1}{\sqrt{2}} + \frac{1}{2}.\frac{1}{\sqrt{2}}\right]$

$= -\left[\frac{\sqrt{3}}{2\sqrt{2}} + \frac{1}{2\sqrt{2}}\right] = -\left[\frac{\sqrt{3}+1}{2\sqrt{2}}\right]$

(iii) cot 75°

$\cot 75° = \frac{1}{\tan 75°}$

$\tan 75° = \tan(45° + 30°)$

$= \frac{\tan 45° + \tan 30°}{1 - \tan 45.\tan 30}$

$\left[\because \tan(A+B) = \frac{\tan A + \tan B}{1 - \tan A \tan B}\ A = 45°, B = 30°\right]$

$\left[\tan 45° = 1, \tan 30° = \frac{1}{\sqrt{3}}\right]$

$= \frac{1 + \frac{1}{\sqrt{3}}}{1 - 1\left(\frac{1}{\sqrt{3}}\right)} = \frac{\frac{\sqrt{3}+1}{\sqrt{3}}}{\frac{\sqrt{3}-1}{\sqrt{3}}} = \frac{\sqrt{3}+1}{\sqrt{3}-1}$

$\cot 75° = \frac{1}{\tan 75°} = \frac{\sqrt{3}-1}{\sqrt{3}+1}$

2. Find the values of the following.

(i) sin 76° cos 16° + cos 76° sin 16°

(ii) $\sin\frac{\pi}{4}\cos\frac{\pi}{12} + \cos\frac{\pi}{4}\sin\frac{\pi}{12}$

(iii) cos 70° cos 10° – sin 70° sin 10°

(iv) $\cos^2 15° - \sin^2 15°$

Sol :

(i) sin 76° cos 16° + cos 76° sin 16°

$= \sin(76° + 16°)$

[∵ sinA cosB + cosA sinB = sin (A + B)]

$= \sin 92° = \sin 92°$

(ii) $\sin\frac{\pi}{4}\cos\frac{\pi}{12} + \cos\frac{\pi}{4}\sin\frac{\pi}{12}$

$= \sin\left(\frac{\pi}{4} + \frac{\pi}{12}\right)$

[∵ sin A cos B + cos A sin B = sin (A + B)]

$= \sin\left(\frac{3\pi+\pi}{12}\right) = \sin\left(\frac{4\pi}{12}\right)$

$= \sin\left(\frac{\pi}{3}\right) = \sin(60°) = \frac{\sqrt{3}}{2}$

(iii) cos 70° cos 10° – sin 70° sin 10°

$= \cos(70 + 10°)$

[∵ cos A cos B – sin A sin B = cos (A + B]

$= \cos 80°$

$$\text{LHS} = \cot(A-B) = \frac{1+\tan A\tan B}{\tan A - \tan B}$$

$$= \frac{1+\tan A\tan B}{x} \quad \text{...(1)}$$

Given $\cot B - \cot A = y$

$$\Rightarrow \frac{1}{\tan B} - \frac{1}{\tan A} = y \qquad \Rightarrow \frac{\tan A - \tan B}{\tan A\tan B} = y$$

$$\Rightarrow \frac{x}{\tan A\tan B} = y \; [\because \tan A - \tan B = x]$$

$$\Rightarrow \frac{x}{y} = \tan A\tan B \quad \text{...(2)}$$

Substituting (2) in (1) we get,

$$\text{LHS} = \cot(A-B) = \frac{1+\frac{x}{y}}{x} = \frac{y+x}{xy}$$

$$= \frac{\not{y}}{x\not{y}} + \frac{\not{x}}{\not{x}y} = \frac{1}{x}+\frac{1}{y} = \text{RHS.}$$

Hence proved.

13. If $\sin\alpha + \sin\beta = a$ and $\cos\alpha + \cos\beta = b$, then prove that $\cos(\alpha-\beta) = \dfrac{a^2+b^2-2}{2}$

Sol : RHS $= \dfrac{a^2+b^2-2}{2}$

$$= \frac{(\sin\alpha+\sin\beta)^2 + (\cos\alpha+\cos\beta)^2 - 2}{2}$$

$$= \frac{\sin^2\alpha+\sin^2\beta+2\sin\alpha\sin\beta+\cos^2\alpha+\cos^2\beta+2\cos\alpha\cos\beta-2}{2}$$

$$= \frac{(\sin^2\alpha+\cos^2\alpha)+(\sin^2\beta+\cos^2\beta)+2(\sin\alpha+\sin\beta+\cos\alpha\cos\beta)-2}{2}$$

$$= \frac{1+1+2(\sin\alpha\sin\beta+\cos\alpha\cos\beta)-2}{2}$$

$$= \frac{2(\sin\alpha\sin\beta+\cos\alpha\cos\beta)}{2}$$

$= \sin\alpha\sin\beta + \cos\alpha\cos\beta = \cos(\alpha-\beta) = \text{LHS}$

$[\because \cos(A-B) = \cos A\cos B + \sin A\sin B]$

Hence Proved.

14. Find the value of $\tan\dfrac{\pi}{8}$.

Sol : We know the identity, $\dfrac{2\tan A}{1-\tan^2 A} = \tan 2A$

Putting $A = \dfrac{\pi}{8}$, we get,

$$\frac{2\tan\frac{\pi}{8}}{1-\tan^2\frac{\pi}{8}} = \tan \not{2}\times\frac{\pi}{\not{8}} = \tan\frac{\pi}{4} = 1$$

$$\Rightarrow 2\tan\frac{\pi}{8} = 1-\tan^2\frac{\pi}{8}$$

$$\Rightarrow \tan^2\frac{\pi}{8} + 2\tan\frac{\pi}{8} - 1 = 0$$

Here $a = 1$, $b = 2$, $c = -1$

$$\therefore \tan\frac{\pi}{8} = \frac{-2\pm\sqrt{4-4(1)(-1)}}{2(1)} \left[x = \frac{-b\pm\sqrt{b^2-4ac}}{2a}\right]$$

$$= \frac{-2\pm\sqrt{8}}{2} = \frac{-2\pm2\sqrt{2}}{2} = 2\frac{(-1\pm\sqrt{2})}{2}$$

$$= -1\pm\sqrt{2} = -1+\sqrt{2}\; r\; -1-\sqrt{2}$$

Since $\tan\dfrac{\pi}{8}$ is an acute angle,

$\tan\dfrac{\pi}{8} = -1-\sqrt{2}$ is not possible

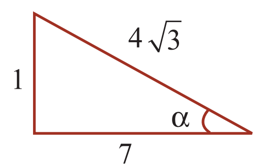
$$\therefore \tan\frac{\pi}{8} = -1+\sqrt{2}$$

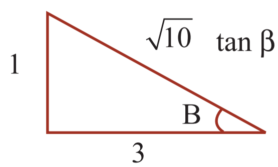
$$\tan\frac{\pi}{8} = \sqrt{2}-1.$$

15. If $\tan\alpha = \dfrac{1}{7}$, $\sin\beta = \dfrac{1}{\sqrt{10}}$ Prove that $\alpha + 2\beta = \dfrac{\pi}{4}$ where $0 < \alpha < \dfrac{\pi}{2}$ and $0 < \beta < \dfrac{\pi}{2}$.

Sol : Since $0 < \alpha < \dfrac{\pi}{2}$ and $0 < \beta < \dfrac{\pi}{2}$

[α, β lies in I quadrant]





$\tan\beta = \dfrac{1}{3}$

⇒ All the trigonometric ratios are positive.

Given $\tan\alpha = \dfrac{1}{7}$

$$\therefore \tan(\alpha+2\beta) = \frac{\tan\alpha+\tan2\beta}{1-\tan\alpha\tan2\beta}$$

$$\tan(A+B) = \frac{\tan A+\tan B}{1-\tan A+\tan B}$$

$$= \frac{\frac{1}{7}+\tan2\beta}{1-\frac{1}{7}(\tan2\beta)} \quad \text{...(1)}$$

$$\text{Now, } \tan2\beta = \frac{2\tan\beta}{1-\tan^2\beta} = \frac{2\left(\frac{1}{3}\right)}{1-\frac{1}{9}} = \frac{\frac{2}{3}}{\frac{8}{9}}$$

$$= \frac{2}{3}\times\frac{9}{8} = \frac{3}{4}$$

Substituting $\tan 2\beta = \frac{3}{4}$ in (1) we get,

$$\tan(\alpha+2\beta) = \frac{\frac{1}{7}+\frac{3}{4}}{1-\frac{1}{7}\left(\frac{3}{4}\right)} = \frac{\frac{4+21}{28}}{1-\frac{3}{28}}$$

$$= \frac{\frac{25}{28}}{\frac{25}{28}} = 1 = \tan\left(\frac{\pi}{4}\right)$$

$$\therefore \tan(\alpha+2\beta) = \tan\left(\frac{\pi}{4}\right)$$

$$\Rightarrow \alpha+2\beta = \frac{\pi}{4}$$

Exercise 4.3

1. Express each of the following as the sum or difference of sine or cosine :

(i) $\sin\frac{A}{8}\sin\frac{3A}{8}$

(ii) $\cos(60°+A)\sin(120°+A)$

(iii) $\cos\frac{7A}{3}\sin\frac{5A}{3}$

(iv) $\cos 7\theta\sin 3\theta$

Sol : **(i) $\sin\frac{A}{8}\sin\frac{3A}{8}$**

Given expression is $\sin\frac{A}{8}.\sin\frac{3A}{8}$ multiplying and dividing by 2, we get,

$$= \frac{1}{2}\left[2\sin\frac{A}{8}.\sin\frac{3A}{8}\right]$$

$$= \frac{1}{2}\left[\cos\left(\frac{A}{8}-\frac{3A}{8}\right)-\cos\left(\frac{A}{8}+\frac{3A}{8}\right)\right]$$

[$\because$ 2sin A sin B = cos (A – B) – cos (A + B]

$$= \frac{1}{2}\left[\cos\left(\frac{-2A}{8}\right)-\cos\left(\frac{4A}{8}\right)\right]$$

$$= \frac{1}{2}\left[\cos\frac{-A}{4}-\cos\frac{A}{2}\right] = \frac{1}{2}\left[\cos\frac{A}{4}-\cos\frac{A}{2}\right]$$

$\left[\because \cos\left(\frac{-2A}{8}\right) = \cos\left(\frac{A}{4}\right)\text{ which is an even function}\right]$

(ii) cos (60° + A) sin (120° + A)

Multiplying and dividing by 2, we get,

$$= \frac{1}{2}[2\cos(60°+A)\sin(120°+A)]$$

$$= \frac{1}{2}[\sin(60°+A+120°+A) - \sin(60°+A-120°-A)]$$

[$\because$ 2cos A sin B = sin (A + B) – sin (A –B)]

$$= \frac{1}{2}[\sin(180°+2A) - \sin(-60°)]$$

$$= \frac{1}{2}\left[-\sin 2A+\frac{\sqrt{3}}{2}\right]\quad\left[\because \sin(-60°) = -\sin 60° = \frac{-\sqrt{3}}{2}\right]$$

(iii) $\cos\frac{7A}{3}\sin\frac{5A}{3}$

multiplying and dividing by 2, we get,

$$= \frac{1}{2}\left[2\cos\frac{7A}{3}\sin\frac{5A}{3}\right]$$

$$= \frac{1}{2}\left[\sin\left(\frac{7A}{3}+\frac{5A}{3}\right)-\sin\left(\frac{7A}{3}-\frac{5A}{3}\right)\right]$$

[$\because$ 2cos A sin B = sin (A + B) – sin (A –B)]

$$= \frac{1}{2}\left[\sin\frac{12A}{3}-\sin\frac{2A}{3}\right] = \frac{1}{2}\left[\sin(4A)-\sin\left(\frac{2A}{3}\right)\right]$$

(iv) cos 7θ sin 3θ

multiplying and dividing by 2 we get,

$$= \frac{1}{2}[2\cos 7\theta\sin 3\theta]$$

$$= \frac{1}{2}[\sin(7\theta+3\theta) - \sin(7\theta-3\theta)]$$

$$= \frac{1}{2}(\sin 10\theta - \sin 4\theta)$$

2. Express each of the following as the product of sine or cosine. (i) sin A + sin 2A (ii) cos 2A + cos 4A (iii) sin 6θ – sin 2θ (iv) cos 2θ – cos θ

Sol : **(i) sin A + sin 2A.**

$$\sin A+\sin 2A = 2\sin\left(\frac{A+2A}{2}\right)\cos\left(\frac{A-2A}{2}\right)$$

$$\left[\because \sin C+\sin D = 2\sin\left(\frac{C+D}{2}\right)\cos\left(\frac{C-D}{2}\right)\right]$$

$$= 2\sin\left(\frac{3A}{2}\right)\cos\left(\frac{-A}{2}\right)$$

$$= 2\sin\left(\frac{3A}{2}\right)\cos\left(\frac{A}{2}\right)$$

$\left[\because \cos\left(\frac{-A}{2}\right) = \cos\frac{A}{2}\text{ which is an even function}\right]$

4. Prove that

(i) $(\cos\alpha - \cos\beta)^2 + (\sin\alpha - \sin\beta)^2 = 4\sin^2\left(\frac{\alpha-\beta}{2}\right)$

(ii) $\sin A \sin(60° + A)\sin(60° - A) = \frac{1}{4}\sin 3A$

Sol :

(i) $(\cos\alpha - \cos\beta)^2 + (\sin\alpha - \sin\beta)^2 = 4\sin^2\left(\frac{\alpha-\beta}{2}\right)$

$$\text{LHS} = (\cos\alpha - \cos\beta)^2 + (\sin\alpha - \sin\beta)^2$$

$$= \left[-2\sin\left(\frac{\alpha+\beta}{2}\right)\sin\left(\frac{\alpha-\beta}{2}\right)\right]^2 + \left[2\cos\left(\frac{\alpha+\beta}{2}\right)\sin\left(\frac{\alpha-\beta}{2}\right)\right]^2$$

$$\left[\because \cos\alpha - \cos\beta = -2\sin\left(\frac{\alpha+\beta}{2}\right)\sin\left(\frac{\alpha-\beta}{2}\right) \text{ and } \sin\alpha - \sin\beta = 2\cos\left(\frac{\alpha+\beta}{2}\right)\sin\left(\frac{\alpha-\beta}{2}\right)\right]$$

$$= 4\sin^2\left(\frac{\alpha+\beta}{2}\right)\sin^2\left(\frac{\alpha-\beta}{2}\right) + 4\cos^2\left(\frac{\alpha+\beta}{2}\right)\sin^2\left(\frac{\alpha-\beta}{2}\right)$$

$$= 4\sin^2\left(\frac{\alpha-\beta}{2}\right)\left[\sin^2\left(\frac{\alpha+\beta}{2}\right) + \cos^2\left(\frac{\alpha+\beta}{2}\right)\right]$$

$[\because \sin^2\theta + \cos^2\theta = 1]$

$$= 4\sin^2\left(\frac{\alpha-\beta}{2}\right) = \text{RHS}$$

(ii) $\sin A \sin(60° + A)\sin(60° - A) = \frac{1}{4}\sin 3A$

LHS $= \sin A \sin(60° + A)\sin(60° - A)$

$= \sin A[\sin^2 60° - \sin^2 A]$

$[\because \sin(A + B)\sin(A - B) = \sin^2 A - \sin^2 B]$

$$= \sin A\left[\left(\frac{\sqrt{3}}{2}\right)^2 - \sin^2 A\right]$$

$$= \sin A\left[\frac{3}{4} - \sin^2 A\right] = \sin A\left[\frac{3 - 4\sin^2 A}{4}\right]$$

$$= \frac{1}{4}[3\sin A - 4\sin^3 A]$$

$$= \frac{1}{4}\sin 3A \quad [\because \sin 3A = 3\sin A - 4\sin^3 A]$$

= RHS. Hence proved.

5. Prove that

(i) $\sin(A-B)\sin C + \sin(B - C)\sin A + \sin(C - A)\sin B = 0$

(ii) $2\cos\frac{\pi}{13}\cos\frac{9\pi}{13} + \cos\frac{3\pi}{13} + \cos\frac{5\pi}{13} = 0$

Sol : (i) $\sin(A-B)\sin C + \sin(B - C)\sin A + \sin(C - A)\sin B = 0$

Since A, B, C are angles of a triangle,

$A + B + C = 180°$

$\Rightarrow C = 180 - (A + B)$

$A = 180 - (B + C)$ and $B = 180 - (A + C)$

LHS $= \sin(A - B)\sin C + \sin(B - C)\sin A + \sin(C - A)\sin B$

$= \sin(A - B)[\sin(180 - (A + B)] + \sin(B - C)\sin[(180 - (B + C)] + \sin(C - A)\sin[(180 - (C + A)]$

$= \sin(A - B)\sin(A + B) + \sin(B - C)\sin(B + C) + \sin(C - A)\sin(C + A)$

$= \sin^2 A - \sin^2 B + \sin^2 B - \sin^2 C + \sin^2 C - \sin^2 A$

$[\therefore \sin(A+B)\sin(A-B) = \sin^2 A - \sin^2 B]$

$= 0 =$ RHS. Hence Proved.

(ii) $2\cos\frac{\pi}{13}\cos\frac{9\pi}{13} + \cos\frac{3\pi}{13} + \cos\frac{5\pi}{13} = 0$

$$\text{LHS} = 2\cos\frac{\pi}{13}\cos\frac{9\pi}{13} + \cos\frac{3\pi}{13} + \cos\frac{5\pi}{13}$$

$$= \cos\left(\frac{\pi}{13} + \frac{9\pi}{13}\right) + \cos\left(\frac{\pi}{13} - \frac{9\pi}{13}\right) + \cos\frac{3\pi}{13} + \cos\frac{5\pi}{13}$$

$[\because 2\cos A\cos B = \cos(A + B) + \cos(A - B)]$

$$= \cos\left(\frac{10\pi}{13}\right) + \cos\left(\frac{-8\pi}{13}\right) + \cos\left(\frac{3\pi}{13}\right) + \cos\left(\frac{5\pi}{13}\right)$$

$$= \cos\left(\frac{10\pi}{13}\right) + \cos\left(\frac{8\pi}{13}\right) + \cos\left(\frac{3\pi}{13}\right) + \cos\left(\frac{5\pi}{13}\right)$$

$$= \cos\left(\pi - \frac{3\pi}{13}\right) + \cos\left(\pi - \frac{5\pi}{13}\right) + \cos\left(\frac{3\pi}{13}\right) + \cos\left(\frac{5\pi}{13}\right)$$

$$= -\cos\left(\frac{3\pi}{13}\right) - \cos\left(\frac{5\pi}{13}\right) + \cos\left(\frac{3\pi}{13}\right) + \cos\left(\frac{5\pi}{13}\right)$$

$[\because \cos(\pi - \theta) = -\cos\theta]$

$= 0 =$ RHS. Hence Proved.

6. Prove that

(i) $\frac{\cos 2A - \cos 3A}{\sin 2 + \sin 3A} = \tan\frac{A}{2}$

(ii) $\frac{\cos 7A + \cos 5A}{\sin 7A - \sin 5A} = \cot A$

Sol : (i) $\frac{\cos 2A - \cos 3A}{\sin 2 + \sin 3A} = \tan\frac{A}{2}$

$$\text{LHS} = \frac{\cos 2A - \cos 3A}{\sin 2A + \sin 3A} \quad ...(1)$$

$$\left[\because \cos C - \cos D = -2\sin\left(\frac{C+D}{2}\right)\sin\left(\frac{C-D}{2}\right)\right]$$

$$= \left[\frac{-2\sin\frac{(2A+3A)}{2}.\sin\frac{(2A-3A)}{2}}{2\sin\frac{(2A+3A)}{2}.\cos\frac{(2A-3A)}{2}}\right]$$

$$\left(\sin C + \sin D = 2\sin\left(\frac{C+D}{2}\right)\cos\left(\frac{C-D}{2}\right)\right)$$

$$= \left[\frac{-2\sin\left(\frac{5A}{2}\right)\sin\left(\frac{-A}{2}\right)}{2\sin\left(\frac{5A}{2}\right)\cos\left(\frac{-A}{2}\right)}\right]$$

$$\left[\therefore \sin\left(\frac{-A}{2}\right) = -\sin\frac{A}{2} \text{ and } \cos\left(\frac{-A}{2}\right) = \cos\frac{A}{2}\right]$$

$$= \frac{\sin\left(\frac{A}{2}\right)}{\cos\left(\frac{A}{2}\right)} = \tan\left(\frac{A}{2}\right) = \text{RHS}$$

Hence proved.

(ii) $\frac{\cos 7A + \cos 5A}{\sin 7A - \sin 5A} = \cot A$

$$\text{LHS} = \frac{\cos 7A + \cos 5A}{\sin 7A - \sin 5A}$$

$$= \frac{2\cos\left(\frac{7A+5A}{2}\right).\cos\left(\frac{7A-5A}{2}\right)}{2\cos\left(\frac{7A+5A}{2}\right).\sin\left(\frac{7A-5A}{2}\right)}$$

$$= \frac{\cos(6A)\cos(A)}{\cos(6A)\sin(A)} = \frac{\cos A}{\sin A} = \cot A = \text{RHS}$$

Hence Proved.

7. Prove that $\cos 20° \cos 40° \cos 60° \cos 80° = \frac{1}{16}$. **[Mar.-2019; Sep.-2020]**

Sol :

LHS $= \cos 20° \cos 40° \cos 60° \cos 80°$

$= \cos 20° \cos 40° \frac{1}{2} \cos 80°$

Let $x = \cos 20° \cos 40° \cos 80°$

Multiply by 2 sin 20° on both sides we get

$2x \sin 20° = 2 \sin 20° \cos 20° \cos 40° \cos 80°$

$= \sin 40° \cos 40° \cos 80°$

$[\because 2 \sin A \cos A = \sin 2A]$

$= \frac{1}{2}(2 \sin 40° \cos 40°) \cos 80°$

$= \frac{1}{2}(\sin 80°) \cos 80°$

$= \frac{1}{4}(2\sin 80° \cos 80°) = \frac{1}{4}(\sin 160°)$

$= \frac{1}{4}\sin(180-20) = \frac{1}{4}\sin(20°)$

$\therefore 2x \sin 20° = \frac{1}{4}\sin 20° \Rightarrow 2x = \frac{1}{4} \Rightarrow x = \frac{1}{8}$

$\therefore \cos 20° \cos 40° \cos 60° \cos 80°$

$= \frac{1}{8}.\cos 60° = \frac{1}{8} \times \frac{1}{2} = \frac{1}{16}$

Hence proved.

8. Evaluate

(i) cos 20° + cos 100° + cos 140°

(ii) sin 50° – sin 70° + sin 10°

Sol :

(i) cos 20° + cos100° + cos140°

$= 2\cos\left(\frac{20°+100°}{2}\right)\cos\left(\frac{20°-100°}{2}\right) + \cos 140°$

$\left[\because \cos C + \cos D = 2\cos\left(\frac{C+D}{2}\right)\cos\left(\frac{C-D}{2}\right)\right]$

$= 2\cos(60°)\cos(-40°) + \cos 140°$

$= 2\left(\frac{1}{2}\right)\cos(40°) + \cos 140°$

$[\because \cos(-\theta) = \cos\theta]$

$= \cos 40° + \cos 140°$

$= 2\cos\left(\frac{40°+140°}{2}\right)\cos\left(\frac{40°-140°}{2}\right)$

$= 2\cos(90°)\cos(-50°)$

$= 2 \times 0 \times \cos(50°) = 0$

(ii) sin 50° – sin 70° + sin 10°

$= 2\cos\left(\frac{50°+70°}{2}\right).\sin\left(\frac{50°-70°}{2}\right) + \sin 10°$

$\left[\because \sin C - \sin D = 2\cos\left(\frac{C+D}{2}\right)\sin\left(\frac{C-D}{2}\right)\right]$

$= 2\cos(60°)\sin(-10°) + \sin 10°$

$= 2\times\frac{1}{2}(-\sin 10°) + \sin 10° [\because \sin(-\theta) = -\sin\theta]$

$= -\sin 10° + \sin 10° = 0.$

06 APPLICATIONS OF DIFFERENTIATION

TEXTUAL QUESTIONS

EXERCISE 6.1

1. A firm produces x tonnes of output at a total cost of $C(x) = \frac{1}{10}x^3 - 4x^2 - 20x + 7$. Find the

(i) average cost function [GMQP - 2019]

(ii) average variable cost function [GMQP - 2019]

(iii) average fixed cost function [GMQP - 2019]

(iv) marginal cost function and

(v) marginal average cost function

Sol : Given $C(x) = \frac{1}{10}x^3 - 4x^2 - 20x + 7$

(i) Average Cost (AC)

$$= \frac{C}{x} = \frac{\frac{1}{10}x^3 - 4x^2 - 20x + 7}{x}$$

$$= \frac{1}{10}x^2 - 4x - 20 + \frac{7}{x}$$

(ii) Average variable cost (AVC)

$$= \frac{\frac{1}{10}x^3 - 4x^2 - 20x}{x}$$

$$= \frac{1}{10}\frac{x^3}{x} - \frac{4x^2}{x} - \frac{20x}{x}$$

$$= \frac{1}{10}x^2 - 4x - 20$$

(iii) Average fixed cost (AFC)

$$= \frac{k}{x} = \frac{7}{x}$$

(iv) Marginal cost (MC)

$$= \frac{dC}{dx} = \frac{d}{dx}\left(\frac{1}{10}x^3 - 4x^2 - 20x + 7\right)$$

$$= \frac{1}{10}(3)x^{3-1} - 4(2)x^{2-1} - 20(1)$$

$$= \frac{1}{10}\left(3x^2\right) - 4\left(2x\right) - 20$$

$$= \frac{3}{10}x^2 - 8x - 20$$

(v) Marginal Average cost (MAC)

$$= \frac{d}{dx}(AC)$$

$$= \frac{d}{dx}\left(\frac{1}{10}x^2 - 4x - 20\right) + \frac{7}{x}$$

$$= \frac{1}{10}\frac{d}{dx}(x^2) - 4\frac{d}{dx}(x) - \frac{d}{dx}(20) + \frac{d}{dx}\left(\frac{7}{x}\right)$$

$$= \frac{1}{10}\left(2x^{2-1}\right) - 4(1) - 0 - \frac{7}{x^2}$$

$$= \frac{1}{5}x - 4 - \frac{7}{x^2}$$

2. The total cost of x units of output of a firm is given by $C = \frac{2}{3}x + \frac{35}{2}$. Find the

(i) cost, when output is 4 units

(ii) average cost, when output is 10 units

(iii) marginal cost, when output is 3 units

Sol : (i) Given $C = \frac{2}{3}x + \frac{35}{2}$

When $x = 4$ units,

$$C = \frac{2}{3}(4) + \frac{35}{2} = \frac{8}{3} + \frac{35}{2}$$

$$= \frac{16+105}{6} = \frac{₹121}{6}$$

(ii) Average cost when the output is 10 units.

Average cost (AC)

$$= \frac{C}{x} = \frac{\frac{2}{3}x + \frac{35}{2}}{x}$$

$$AC = \frac{2}{3} + \frac{35}{2x}$$

When $x = 10$,

$$AC = \frac{2}{3} + \frac{35}{2(10)} = \frac{2}{3} + \frac{35}{20}$$

$$= \frac{2}{3} + \frac{7}{4} = \frac{8+21}{12} = \frac{29}{12}$$

Average cost when output is 10 units is $\frac{₹29}{12}$.

(iii) Marginal cost when output is 3 units

Marginal Cost

$$= \frac{dC}{dx} = \frac{d}{dx}\left(\frac{2}{3}x + \frac{35}{2}\right) = \frac{2}{3}$$

When $x = 3$ units, MC = $\frac{2}{3}$

Marginal cost when output is 3 units will be $\frac{₹2}{3}$

3. Revenue function 'R' and cost function 'C' are $R = 14x - x^2$ and $C = x(x^2 - 2)$. Find the
(i) average cost function,
(ii) marginal cost function,
(iii) average revenue function and
(iv) marginal revenue function.

Sol : Given $C = x(x^2 - 2) = x^3 - 2x$

(i) Average Cost (AC)

$$= \frac{C}{x} = \frac{x^3 - 2x}{x} = x^2 - 2$$

(ii) Marginal Cost (MC)

$$= \frac{dC}{dx} = \frac{d}{dx}(x^3 - 2x) = 3x^2 - 2$$

(iii) Average Reveue (AR)

$$= \frac{R}{x} = \frac{14x - x^2}{x} = 14 - x$$

(iv) Marginal Revenue (MR)

$$= \frac{d}{dx}(R) = \frac{d}{dx}(14x - x^2) = 14 - 2x$$

4. If the demand law is given by $p = 10e^{-\frac{x}{2}}$, then find the elasticity of demand. **[Mar.-2019]**

Sol : Given $p = 10e^{-x/2}$

Differentiating with respect to 'x' we get,

$$\frac{dp}{dx} = 10e^{-x/2}\left(-\frac{1}{2}\right) = -5e^{-x/2}$$

$$\Rightarrow \frac{dx}{dp} = -\frac{1}{5e^{-x/2}}$$

Elasticity of demand

$$(\eta_d) = \frac{-p}{x}\cdot\frac{dx}{dp} = -\frac{10e^{-x/2}}{x}\left(\frac{1}{-5e^{-x/2}}\right)$$

$$\eta_d = \frac{10}{5x} = \frac{2}{x}$$

5. Find the elasticity of demand in terms of x for the following demand laws and also find the output (x), when the elasticity is equal to unity. (i) $p = (a - bx)^2$ (ii) $p = a - bx^2$

Sol : (i) Given $p = (a - bx)^2$

Differentiating with respect to 'x' we get,

$$\frac{dp}{dx} = 2(a - bx)(-b) = -2b(a - bx)$$

$$\Rightarrow \frac{dx}{dp} = \frac{-1}{2b(a - bx)}$$

Elasticity of demand $(\eta_d) = \frac{-p}{x}\cdot\frac{dx}{dp}$

$$\Rightarrow -\frac{(a-bx)^2}{x}\left(\frac{-1}{2b(a-bx)}\right) = \frac{a-bx}{x(2b)}$$

$$\Rightarrow \eta_d = \frac{a-bx}{2bx}$$

When the elasticity of demand equal to one,

$$1 = \frac{a-bx}{2bx}$$

$$\Rightarrow 2bx = a - bx$$

$$\Rightarrow 2bx + bx = a$$

$$\Rightarrow 3bx = a$$

$$\Rightarrow x = \frac{a}{3b}$$

$\therefore$ The value of x when elasticity is equal to unity is $\frac{a}{3b}$

(ii) Given $p = a - bx^2$

Differentiating with respect to 'x' we get,

$$\frac{dp}{dx} = 0 - b\frac{d}{dx}(x^2) = -b(2x) = -2bx$$

Elasticity of demand : $\eta d = -\frac{p}{x}.\frac{dx}{dp}$

$$= \frac{-p}{x} \times \frac{1}{\left(\frac{dp}{dx}\right)} = \frac{-(a-bx)^2}{x} \times \frac{1}{-2bx}$$

$$\eta d = \frac{a-bx^2}{2bx^2}$$

When elasticity is equals to unit,

$$\frac{a-bx^2}{2bx^2} = 1$$

$$a - bx^2 = 2bx^2$$

$$2bx^2 = a - bx^2$$

$$2bx^2 + bx^2 = a$$

$$3bx^2 = a$$

$$x^2 = \frac{a}{3b}$$

$$x = \sqrt{\frac{a}{\sqrt{3b}}}$$

∴ The value of x when elasticity is equal to unity is $\sqrt{\frac{a}{3b}}$.

6. Find the elasticity of supply for the supply function $x = 2p^2 + 5$ when $p = 3$. [Mar.-2020; Sep. - 2021]

Sol : Given $x = 2p^2 + 5$

Differentiating with respect to 'p' we get,

$$\frac{dx}{dp} = 2(2p) = 4p$$

Elasticity of supply

$$\eta_s = \frac{p}{x}.\frac{dx}{dp}$$

$$\eta_s = \frac{p}{2p^2+5}(4p) = \frac{4p^2}{2p^2+5}$$

when $p = 3$, $\eta_s = \frac{4(3^2)}{2(3^2)+5} = \frac{4(9)}{18+5} = \frac{36}{23}$

7. The demand curve of a commodity is given by $p = \frac{50-x}{5}$, find the marginal revenue for any output x and also find marginal revenue at $x = 0$ and $x = 25$?

Sol : Given $p = \frac{50-x}{5}$

Differentiating with respect to 'x' we get,

$$\frac{dp}{dx} = -\frac{1}{5}$$

$$\Rightarrow \frac{dx}{dp} = -5$$

$$\eta_d = \frac{-p}{x}.\frac{dx}{dp} = -\frac{(50-x)}{x}(-5) = \frac{(50-x)}{5x}$$

Marginal Revenue

$$MR = p\left(1-\frac{1}{\eta_d}\right)$$

$$MR = \frac{50-x}{5}\left(1-\frac{x}{50-x}\right)$$

$$\Rightarrow MR = \frac{50-x}{5}\left(\frac{50-x-x}{50-x}\right) = \frac{50-2x}{5}$$

when $x = 0$,

$$MR = \frac{50-0}{5} = \frac{50}{5} = 10$$

when $x = 25$,

$$MR = \frac{50-2(25)}{5} = \frac{50-50}{5} = \frac{0}{5} = 0$$

8. The supply function of certain goods is given by $x = a\sqrt{p-b}$ where p is unit price, a and b are constants with $p > b$. Find the elasticity of supply at $p = 2b$.

Sol : Given $x = a\sqrt{p-b}$

$$\Rightarrow x = a(p-b)^{1/2}$$

Differentiating with respect to 'p' we get,

$$\frac{dx}{dp} = a\cdot\frac{1}{2}(p-b)^{\frac{1}{2}-1}$$

$$= \frac{a}{2}(p-b)^{-1/2}$$

$$= \frac{a}{2\sqrt{p-b}}$$

Elasticity of supply

$$\eta_s = \frac{p}{x} \cdot \frac{dx}{dp}$$

$$\Rightarrow \eta_s = \frac{p}{\cancel{a}\sqrt{p-b}}\left(\frac{\cancel{a}}{2\sqrt{p-b}}\right) = \frac{p}{2(p-b)}$$

when $p = 2b$, $\eta_s = \frac{2b}{2(2b-b)} = \frac{2b}{2(b)} = 1 \Rightarrow \eta_s = 1$

9. Show that MR = $p\left[1-\frac{1}{\eta_d}\right]$ for the demand function $p = 400 - 2x - 3x^2$ where p is unit price and x is quantity demand.

Sol : Given $p = 400 - 2x - 3x^2$

$$\text{we know R} = px$$

$$\Rightarrow R = (400 - 2x - 3x^2)\,x$$

$$\Rightarrow R = 400x - 2x^2 - 3x^3$$

$$\therefore \text{MR} = \frac{dR}{dx} = \frac{d}{dx}\left(400x - 2x^2 - 3x^3\right)$$

$$\text{MR} = 400 - 4x - 9x^2 \quad \dots (1)$$

$$\text{Also, } p = 400 - 2x - 3x^2$$

Differentiating, with respect to 'x' we get,

$$\frac{dp}{dx} = -2 - 6x$$

$$\Rightarrow \frac{dx}{dp} = \frac{1}{-2-6x} = \frac{1}{-2(1+3x)}$$

Elasticity of demand

$$\eta_d = \frac{-p}{x} \cdot \frac{dx}{dp}$$

$$\Rightarrow \eta_d = \frac{-\left(400-2x-3x^2\right)}{x}\left(\frac{-1}{2(1+3x)}\right)$$

$$= \frac{400-2x-3x^2}{2x(1+3x)}$$

$$\Rightarrow \frac{1}{\eta_d} = \frac{2x(1+3x)}{400-2x-3x^2}$$

$$\therefore p\left(1-\frac{1}{\eta_d}\right) = \left(400-2x-3x^2\right)\left(1-\frac{2x(1+3x)}{400-2x-3x^2}\right)$$

$$= \cancel{\left(400-2x-3x^2\right)}\left(\frac{400-2x-3x^2-2x-6x^2}{\cancel{400-2x-3x^2}}\right)$$

$$= 400 - 4x - 9x^2 \quad \dots (2)$$

From (1) and (2),

$$\text{MR} = p\left(1-\frac{1}{\eta_d}\right)$$

10. For the demand function $p = 550 - 3x - 6x^2$ where x is quantity demand and p is unit price. Show that MR = $p\left[1-\frac{1}{\eta_d}\right]$ **[HY - 2019; Aug. - 2022]**

Sol : Given $p = 550 - 3x - 6x^2$

$$\text{we know R} = px$$

$$\Rightarrow R = (550 - 3x - 6x^2)x$$

$$\Rightarrow R = 550x - 3x^2 - 6x^3$$

$$\therefore \text{MR} = \frac{dR}{dx} = \frac{d}{dx}(550x - 3x^2 - 6x^3)$$

$$\text{MR} = 550 - 6x - 18x^2 \quad \dots (1)$$

Differentiating $p = 550 - 3x - 6x^2$ with respect to 'x' we get,

$$\frac{dp}{dx} = -3 - 12x$$

$$\Rightarrow \frac{dx}{dp} = \frac{1}{-3-12x} = \frac{-1}{3+12x}$$

$$\therefore \eta_d = -\frac{p}{x} \cdot \frac{dx}{dp}$$

$$= \frac{-\left(550-3x-6x^2\right)}{x}\left(\frac{-1}{3+12x}\right)$$

$$= \frac{550-3x-6x^2}{x(3+12x)}$$

$$\Rightarrow \frac{1}{\eta d} = \frac{x(3+12x)}{550-3x-6x^2}$$

$$\therefore p\left(1-\frac{1}{\eta_d}\right) = \left(550-3x-6x^2\right)\left(1-\frac{x(3+12x)}{550-3x-6x^2}\right)$$

$$=\left(\cancel{550-3x}-6x^2\right)\frac{\left(550-3x-6x^2-3x-12x^2\right)}{\cancel{550-3x}-6x^2}$$

$$p\left(1-\frac{1}{\eta_d}\right) = 550-6x-18x^2 \quad \ldots (2)$$

From (1) and (2),

$$\text{MR} = p\left(1-\frac{1}{\eta_d}\right)$$

11. For the demand function $x = \frac{25}{p^4}$, $1 \le p \le 5$, determine the elasticity of demand.

Sol : Given $x = \frac{25}{p^4}$, $1 \le p \le 5$.

Differentiating with respect to 'p' we get,

$$\frac{dx}{dp} = 25(-4)p^{-4-1} = -100p^{-5} = \frac{-100}{p^5}$$

$\therefore$ Elasticity of demand

$$\eta_d = -\frac{p}{x}\cdot\frac{dx}{dp} = \frac{-p}{\frac{25}{p^4}}\left(\frac{-100}{p^5}\right)$$

$$= \frac{\overset{4}{\cancel{100}}\cancel{p}}{\cancel{25}\cancel{p}} = 4$$

12. The demand function of a commodity is $p = 200 - \frac{x}{100}$ and its cost is C = 40x + 120 where p is a unit price in rupees and x is the number of units produced and sold. Determine (i) profit function (ii) average profit at an output of 10 units (iii) marginal profit at an output of 10 units and (iv) marginal average profit at an output of 10 units.

Sol : Given $p = 200-\frac{x}{100}$ and

$$\text{C} = 40x + 120$$

(i) $\text{R} = px$

$$\Rightarrow \text{R} = \left(200-\frac{x}{100}\right)x = 200x-\frac{x^2}{100}$$

$$\text{Profit function} = \text{R} - \text{C}$$

$$= 200x-\frac{x^2}{100}-40x-120$$

$$= 160x-\frac{x^2}{100}-120$$

(ii) Average profit function $= \frac{\text{Total profit}}{\text{output}}$

$$= \frac{1}{x}\left(160x-\frac{x^2}{100}-120\right)$$

$$= 160-\frac{x}{100}-\frac{120}{x}$$

When $x = 10$, average profit function

$$= 160-\frac{10}{100}-\frac{120}{10}$$

$$= 160-\frac{1}{10}-12 = 148-\frac{1}{10}$$

$$= 148-0.1$$

$$= ₹147.9$$

(iii) Marginal Profit

$$= \frac{d}{dx}\left(160x-\frac{x^2}{100}-120\right)$$

$$= 160-\frac{\cancel{2}x}{\underset{50}{\cancel{100}}} = 160-\frac{x}{50}$$

When $x = 10$, marginal profit

$$= 160-\frac{10}{50}$$

$$= 160-\frac{1}{5} = 160-0.2 = ₹159.8$$

(iv) Marginal Average Profit

$$= \frac{d}{dx}\left(160-\frac{x}{100}-\frac{120}{x}\right)$$

$$= \frac{-1}{100}+\frac{120}{x^2}$$

When $x = 10$, marginal average profit

$$= \frac{-1}{100}+\frac{120}{100} = \frac{-1}{100}+1.20$$

$$= ₹1.19$$

13. Find the values of x, when the marginal function of $y = x^3 + 10x^2 - 48x + 8$ is twice the x. **[May - 2022]**

Sol : Given $y = x^3 + 10x^2 - 48x + 8$

and $\frac{dy}{dx} = 2x$

$$\Rightarrow 3x^2 + 20x - 48 = 2x$$

$$\Rightarrow 3x^2 + 20x - 48 - 2x = 0$$

$$\Rightarrow 3x^2 + 18x - 48 = 0$$

Dividing by 3 we get,

$$x^2 + 6x - 16 = 0$$

−16
6
8 −2

$\Rightarrow \quad (x+8)(x-2) = 0$

$\Rightarrow \quad x = -8, 2$

14. The total cost function y for x units is given by $y = 3x\left(\frac{x+7}{x+5}\right) + 5$. Show that the marginal cost [MC] decreases continuously as the output (x) increases.

Sol : Given $y = 3x\left(\frac{x+7}{x+5}\right)+5$

$$\Rightarrow \quad y = \frac{3x^2+21x}{x+5}+5$$

Differentiating with respect to 'x' we get,

$$\frac{dy}{dx} = \frac{(x+5)(6x+21)-(3x^2+21x)(1)}{(x+5)^2}+0$$

$$= \frac{6x^2+21x+30x+105-3x^2-21x}{(x+5)^2}$$

$$= \frac{3x^2+30x+105}{(x+5)^2}$$

$$= \frac{3(x^2+10x+35)}{(x+5)^2}$$

$$= \frac{3(x^2+10x+25+10)}{(x+5)^2}$$

$$= 3\left[\frac{(x+5)^2+10}{(x+5)^2}\right]$$

$$= 3\left[\frac{(x+5)^2}{(x+5)^2}+\frac{10}{(x+5)^2}\right]$$

$$\frac{dy}{dx} = 3\left[1+\frac{10}{(x+5)^2}\right]$$

$$\Rightarrow \text{Marginal cost function} = 3\left[1+\frac{10}{(x+5)^2}\right]$$

Since $(x+5)^2$ lies in the denominator, as the output (x) increases, marginal cost function $\frac{dy}{dx}$ decreases, continuously.

15. Find the price elasticity of demand for the demand function $x = 10 - p$ where x is the demand and p is the price. Examine whether the demand is elastic, inelastic or unit elastic at $p = 6$.

Sol : Given $x = 10 - p$

Differentiating with respect to 'p' we get,

$$\frac{dx}{dp} = -1$$

Elasticity of demand

$$\eta_d = \frac{-p}{x}\cdot\frac{dx}{dp}$$

$$\Rightarrow \quad \eta_d = \frac{-p}{10-p}(-1) = \frac{p}{10-p}$$

$$\text{when } p = 6, \ \eta_d = \frac{6}{10-6} = \frac{6}{4} = \frac{3}{2}$$

$$|\eta_d| = \left|\frac{3}{2}\right| = \frac{3}{2} > 1$$

$\therefore \eta_d$ is elastic.

16. Find the equilibrium price and equilibrium quantity for the following functions. Demand: $x = 100 - 2p$ and supply: $x = 3p - 50$

Sol : Given demand $x = 100 - 2p$ **[HY-2019]**

and supply $x = 3p - 50$

At equilibrium, demand = supply

$\Rightarrow \quad 100 - 2p = 3p - 50$

$\Rightarrow \quad 100 + 50 = 3p + 2p$

$\Rightarrow \quad 150 = 5p$

$$\Rightarrow \quad p = \frac{150}{5} = 30$$

When $p = 30$, $x = 100 - 2(30) = 100 - 60$

$x = 40$

$\therefore$ Equilibrium price is p_E = ₹30 and equilibrium quantity is $x_E = 40$ units.

17. The demand and cost functions of a firm are $x = 6000 - 30p$ and $C = 72000 + 60x$ respectively. Find the level of output and price at which the profit is maximum. **[Aug. - 2022]**

Sol : Given $x = 6000 - 30p$

$\Rightarrow \quad 30p = 6000 - x$

$$\Rightarrow \quad p = \frac{6000 - x}{30}$$

$$\therefore R = px = \left(\frac{6000 - x}{30}\right)x = \frac{6000x - x^2}{30}$$

$$MR = \frac{d}{dx}\left(\frac{6000x - x^2}{30}\right)$$

$$= \frac{6000 - 2x}{30} = \frac{2(3000 - x)}{30}$$

$$= \frac{3000 - x}{15} \quad \text{... (1)}$$

Also $C = 72000 + 60x$

$$MC = \frac{dC}{dx} = \frac{d}{dx}(72000 + 60x) = 60 \quad \text{... (2)}$$

We know that profit is maximum when

$$MR = MC$$

$$\Rightarrow \frac{3000 - x}{15} = 60 \text{ [From (1) and (2)]}$$

$$\Rightarrow 3000 - x = 15 \times 60$$

$$\Rightarrow 3000 - x = 900 \Rightarrow 3000 - 900 = x$$

$$\Rightarrow x = 2100 \text{ units}$$

Substituting $x = 2100$ in

$$x = 6000 - 30p \Rightarrow 2100 = 6000 - 30p$$

$$\Rightarrow 30p = 6000 - 2100$$

$$\Rightarrow 30p = 3900$$

$$\Rightarrow p = \frac{3900}{30} = ₹130$$

Hence the level of output and price at which the profit is maximum are 2100 and 130.

18. The cost function of a firm is C = $x^3 - 12x^2 + 48x$. Find the level of output ($x > 0$) at which average cost is minimum.

Sol : Given $C = x^3 - 12x^2 + 48x$

$$\text{Average Cost (AC)} = \frac{C}{x} = \frac{x^3 - 12x^2 + 48x}{x}$$

$$AC = x^2 - 12x + 48 \quad \text{... (1)}$$

$$\text{Marginal Cost (MC)} = \frac{dC}{dx} = 3x^2 - 24x + 48 \quad \text{... (2)}$$

We know that AC is minimum when AC = MC

$$\Rightarrow x^2 - 12x + 48 = 3x^2 - 24x + 48$$

$$3x^2 - 24x + 48 - x^2 + 12x - 48 = 0$$

$$\Rightarrow 2x^2 - 12x = 0$$

$$\Rightarrow 2x(x - 6) = 0$$

$$\Rightarrow x = 0 \text{ or } x = 6$$

$$\Rightarrow x = 6 \ [\because x > 0]$$

Hence the level of output at which average cost is minimum 6 units.

EXERCISE 6.2

1. The average cost function associated with producing and marketing x units of an item is given by AC = $2x - 11 + \frac{50}{x}$. Find the range of values of the output x, for which AC is increasing. **[Sep. - 2021]**

Sol : Given $AC = 2x - 11 + \frac{50}{x}$

$$\frac{d}{dx}(AC) = 2 - \frac{50}{x^2}$$

$$\frac{d}{dx}(AC) = 0$$

$$\Rightarrow 2 - \frac{50}{x^2} = 0 \Rightarrow 2 = \frac{50}{x^2}$$

$$\Rightarrow x^2 = \frac{50}{2} = 25 \Rightarrow x^2 > 25$$

$$x > 5 \ [\because x = -5 \text{ is not possible}]$$

Interval	Sign of $\frac{d(AC)}{dx}$	Nature of function
(0, 5) say x =1	$2 - \frac{50}{1} = -48$ (Negative)	Decreasing function
(5, ∞) say x =6	$2 - \frac{50}{36} = 2 - 1.38 = 0.62$ (Positive)	Increasing function

Hence AC is increasing in the interval (5, ∞)

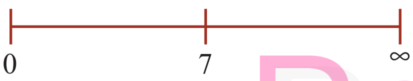
⇒ AC is increasing when $x > 5$.

2. A television manufacturer finds that the total cost for the production and marketing of x number of television sets is $C(x) = 300x^2 + 4200x + 13{,}500$. If each product is sold for ₹ 8,400, show that the profit of the company is increasing.

Sol : Given $C(x) = 300x^2 + 4200x + 13500$

Since the selling price of 1 product is ₹ 8400, let the selling price of x products be $8400x$

$$\therefore \text{Profit} = \text{Selling Price} - \text{Cost Price}$$
$$= 8400x - 300x^2 - 4200x - 13{,}500$$
$$= -300x^2 + 4200x - 13{,}500$$
$$\frac{d}{dx}(\text{Profit}) = -600x + 4200 \quad \ldots (1)$$
$$\frac{d}{dx}(\text{Profit}) = 0$$
$$-600x + 4200 = 0 \Rightarrow 600x = 4200 \Rightarrow x = \frac{4200}{600} = 7$$



Intervals	Sign of (1)	Nature of Function
(0, 7) say $x = 6$	$-600(6) + 4200$ $= -3600 + 4200$ $= 600$ (Positive)	Increasing
(7, ∞) say $x = 8$	$-600(8) + 4200$ $= -4800 + 4200$ $= -600$ (Negative)	Decreasing

Hence, profit of the company is increasing when each product is sold for ₹ 8400.

3. A monopolist has a demand curve $x = 106 - 2p$ and average cost curve $AC = 5 + \frac{x}{50}$, where p is the price per unit output and x is the number of units of output. If the total revenue is $R = px$, determine the most profitable output and the maximum profit.

Sol : Given $x = 106 - 2p$

$$\Rightarrow 2p = 106 - x$$
$$\Rightarrow p = \frac{106 - x}{2}$$

$$\text{Revenue (R)} = px = \left(\frac{106 - x}{2}\right)x = \frac{106x - x^2}{2}$$

$$\text{and given AC} = 5 + \frac{x}{50}$$

$$\frac{C}{x} = 5 + \frac{x}{50} \qquad \left[\because AC = \frac{C}{x}\right]$$

$$\Rightarrow C = 5x + \frac{x^2}{50}$$

We know that

$$\text{Profit function} = R - C$$
$$\Rightarrow \text{Profit} = \frac{106x - x^2}{2} - 5x - \frac{x^2}{50}$$
$$= 53x - \frac{x^2}{2} - 5x - \frac{x^2}{50}$$
$$= 48x - \frac{26x^2}{50}$$
$$\text{Profit} = 48x - \frac{13x^2}{25} \quad \ldots (1)$$
$$\frac{d}{dx}(\text{Profit}) = 48 - \frac{26x}{25}$$
$$\frac{d(\text{Profit})}{dx} = 0$$
$$\Rightarrow 48 = \frac{26x}{25}$$
$$\Rightarrow \frac{48 \times 25}{26} = x$$
$$\therefore x = 46.15 = 46 \text{ (app)}$$
$$\frac{d^2(\text{Profit})}{dx^2} = \frac{-26}{25} < 0$$

$\therefore$ Profit is maximum, when $x = 46$

Maximum profit when $x = 46$ is

$$= 48(46) - \frac{13(46)^2}{25} \text{ [From (1)]}$$
$$= 2208 - \frac{13(2116)}{25} = 2208 - 1100.32$$
$$= ₹1107.68$$

4. A tour operator charges ₹136 per passenger with a discount of 40 paisa for each passenger in excess of 100. The operator requires at least 100 passengers to operate the tour. Determine the number of passenger that will maximize the amount of money the tour operator receives. **[Sep.- 2020]**

Sol :

(i) Let x be the number of passengers and R(x) be the revenue function. The function R(x) should be maximized to find the number of passengers that will maximize the amount of money for the tour operator receives.

The revenue from each passenger

$$r = 136 - \frac{40}{100}(x-100), \text{ for } x \geq 100$$

If R be the total revenue, then

$$R = rx = x\left[136 - \frac{40}{100}(x-100)\right]$$

$$= 136x - \frac{2}{5}x^2 + 40x$$

$$= 176x - \frac{2}{5}x^2$$

For R to be maximum,

$$\frac{dR}{dx} = 0 \text{ and } \frac{d^2R}{dx^2} = -\text{ve}$$

$$\frac{dR}{dx} = 0 \text{ gives } \frac{d}{dx}\left[176x - \frac{2}{5}x^2\right] = 0$$

$$176 - \frac{4}{5}x = 0$$

$$\Rightarrow x = \frac{5 \times \overset{44}{\cancel{176}}}{\cancel{4}} = 220$$

$$\frac{d^2R}{dx^2} = \frac{-4}{5} = -\text{ve for } x = 220$$

Thus R is maximum for $x = 220$

Revenue is maximum when $x = 220$

5. Find the local minimum and local maximum of $y = 2x^3 - 3x^2 - 36x + 10$.

Sol : Given $y = 2x^3 - 3x^2 - 36x + 10$

Differentiating with respect to 'x' we get,

$$\frac{dy}{dx} = 6x^2 - 6x - 36$$

$$\frac{dy}{dx} = 0$$

$$\Rightarrow 6x^2 - 6x - 36 = 0$$

$$\Rightarrow x^2 - x - 6 = 0 \text{ (Divided by 6)}$$

$$\Rightarrow (x-3)(x+2) = 0$$

$$\Rightarrow x = -2; +3$$

$$\frac{d^2y}{dx^2} = 12x - 6$$

when $x = -2$,

$$\frac{d^2y}{dx^2} = 12(-2) - 6 = -24 - 6 = -30 < 0$$

$\therefore y$ is maximum when $x = -2$

$\therefore$ Maximum value

$$= 2(-2)^3 - 3(-2)^2 - 36(-2) + 10$$

$$= -16 - 12 + 72 + 10$$

$$= -28 + 82 = 54$$

when $x = +3$,

$$\frac{d^2y}{dx^2} = 12(+3) - 6 = +36 - 6 > 0$$

$\therefore$ y is minimum when $x = +3$

Minimum value is $2(+3)^3 - 3(+3)^2 - 36(+3) + 10$

$$= 2(+27) - 3(9) - 108 + 10$$

$$= +54 - 27 - 108 + 10$$

$$= 64 - 135 = -71$$

$\therefore$ Local minimum is -71 and local maximum is 54.

6. The total revenue function for a commodity is $R = 15x + \frac{x^2}{3} - \frac{1}{36}x^4$. Show that at the highest point average revenue is equal to the marginal revenue. **[GMQP - 2019;Mar.-2019]**

Sol : Given $R = 15x + \frac{x^2}{3} - \frac{x^4}{36}$

Average Revenue (AR)

08 DESCRIPTIVE STATISTICS AND PROBABILITY

TEXTUAL QUESTIONS

EXERCISE 8.1

1. Find the first quartile and third quartile for the given observations. [Sep.-2020]

2, 4, 6, 8, 10, 12, 14, 16, 18, 20, 22

Sol : Arranging the observations in ascending order, we get

2, 4, 6, 8, 10, 12, 14, 16, 18, 20, 22

Here $n = 11$

First quartile $Q_1 = \text{size of } \frac{(n+1)^{th}}{4} \text{ value}$

$= \text{size of } \left(\frac{11+1}{4}\right)^{th} \text{ value}$

$= \text{size of } 3^{rd} \text{ value}$

$Q_1 = 6$

Third quartile $Q_3 = \text{Size of } \left(\frac{3(n+1)}{4}\right)^{th} \text{ value}$

$= \text{size of } \left(3\left(\frac{11+1}{4}\right)\right)^{th} \text{ value}$

$= \text{size of } 9^{th} \text{ value}$

$Q_3 = 18$

2. Find Q_1, Q_3, D_8 and P_{67} of the following data :

Size of Shares	4	4.5	5	5.5	6	6.5	7	7.5	8
Frequency	10	18	22	25	40	15	10	8	7

Sol :

Size of shares x	Frequency f	Cumulative Frequency Cf
4	10	10
4.5	18	28
5	22	50
5.5	25	75
6	40	115
6.5	15	130
7	10	140
7.5	8	148
8	7	N = 155

$Q_1 = \text{size of } \left(\frac{N+1}{4}\right)^{th} \text{ value} = \text{size of } \left(\frac{155+1}{4}\right)^{th} \text{ value}$

$= \text{size of } 39^{th} \text{ value} = 5$

$Q_3 = \text{size of } \left[3\left(\frac{N+1}{4}\right)^{th}\right] \text{ value} = \text{size of } 3(39)^{th} \text{ value}$

$= \text{size of } 117^{th} \text{ value} = 6.5$

$D_8 = \text{size of } \left[8\left(\frac{N+1}{10}\right)^{th}\right] \text{value} = \text{size of } 8\left(\frac{156}{10}\right)^{th} \text{ value}$

$= \text{size of } 124.8^{th} \text{ value} = 6.5$

$P_{67} = \text{size of } \left(\frac{67(N+1)}{100}\right)^{th} \text{ value} = \text{size of } 67\left(\frac{156}{100}\right)^{th} \text{ value}$

$= \text{size of } 104.52^{th} \text{ value}$

$P_{67} = 6$

3. Find lower quartile, upper quartile, 7^{th} decile, 5^{th} decile and 60^{th} percentile for the following frequency distribution.

Wages	10 - 20	20 - 30	30 - 40	40 - 50	50 - 60	60 - 70	70 - 80
Frequency	1	3	11	21	43	32	9

Sol :

Wages	Frequency	Cumulative Frequency
10 - 20	1	1
20 - 30	3	4
30 - 40	11	15
40 - 50	21	36
50 - 60	43	79
60 - 70	32	111
70 - 80	9	N = 120

Lower Quartile:

$$Q_1 = \text{Size of } \left(\frac{N}{4}\right)^{th} \text{ value}$$

$$= \text{size of } \left(\frac{120}{4}\right)^{th} \text{ value}$$

$$= \text{size of } 30^{th} \text{ value}$$

Thus, Q_1 lies in the class (40 – 50) and its corresponding values are

$$L = 40, \frac{N}{4} = \frac{120}{4} = 30,$$

$c = 10, f = 21, pcf = 15$

$$\therefore Q_1 = L + \left(\frac{\frac{N}{4} - pcf}{f}\right)c$$

$$= 40 + \left(\frac{30-15}{21}\right)10 = 40 + \frac{15}{21}(10)$$

$$Q_1 = 40 + 7.142 = 47.14$$

Upper Quartile:

$$Q_3 = \text{Size of } \left(\frac{3N}{4}\right)^{th} \text{ value}$$

$$= \text{Size of } \left(\frac{3\times120}{4}\right)^{th} \text{ value}$$

$$= \text{size of } 90^{th} \text{ value}$$

So, Q_3 lies in the class (60 – 70) and its corresponding values are

$$L = 60, \frac{3N}{4} = 90, f = 32, pcf = 79, c = 10$$

$$\therefore Q_3 = L + \left(\frac{\frac{3N}{4} - pcf}{f}\right)\times c$$

$$= 60 + \left(\frac{90-79}{32}\right)\times10$$

$$= 60 + \frac{11}{32}\times10$$

$$= 60 + 3.437 = 63.44$$

$$D_7 = L + \left(\frac{\frac{7N}{10} - pcf}{f}\right)\times c$$

$$D_7 = \text{size of } \frac{7N}{10} = \left(\frac{7\times120}{10}\right)^{th} \text{ value}$$

$$= 84^{th} \text{ value}$$

$\therefore D_7$ lies in the class (60 – 70) and its corresponding values are

$$L = 60, \frac{7N}{10} = 84, f = 32, pcf = 79, c = 10$$

$$\therefore D_7 = 60 + \frac{84-79}{32}\times10 = 60 + \frac{50}{32}$$

$$= 60 + 1.56 = 61.56$$

$$D_5 = \text{size of } \left(\frac{5N}{10}\right)^{th} \text{ value} = \text{size of } 60^{th} \text{ value}$$

So, D_5 lies in the class (50 – 60) and its corresponding values are

$$L = 50, \frac{5N}{10} = 60, f = 43, pcf = 36, c = 10$$

$$\therefore D_5 = L + \left(\frac{\frac{5N}{10} - pcf}{f}\right)\times c$$

$$= 50 + \frac{60-36}{43}\times10$$

$$= 50 + \frac{24}{43}\times10 = 50 + 5.58 = 55.58$$

$$\therefore D_5 = 55.58$$

$$P_{60} = \text{size } \left(\frac{60N}{100}\right)^{th} \text{ value}$$

$$= \text{size of } \left(\frac{60\times120}{100}\right)^{th} \text{ value}$$

$$= \text{size of } (72)^{th} \text{ value}$$

$\therefore P_{60}$ lies in the class (50 – 60) and its corresponding values are

$L = 50, \frac{60N}{100} = 72, pcf = 36, f = 43, c = 10$

$P_{60} = L + \frac{\frac{60N}{100} - pcf}{f} \times c = 50 + \frac{72-36}{43} \times 10$

$= 50 + \frac{360}{43} = 50 + 8.37 \Rightarrow P_{60} = 58.37$

4. Calculate GM for the following table gives the weight of 31 persons in sample survey.

Weight (lbs)	130	135	140	145	146	148	149	150	157
Frequency	3	4	6	6	3	5	2	1	1

Sol :

Weight (x)	Frequency (f)	$\log x$	$f \log x$
130	3	2.1139	6.3417
135	4	2.1303	8.5212
140	6	2.1461	12.8766
145	6	2.1614	12.9684
146	3	2.1644	6.4932
148	5	2.1703	10.8515
149	2	2.1732	4.3464
150	1	2.1761	2.1761
157	1	2.1959	2.1959
	31		66.771

Geometric Mean $= \text{Antilog}\left(\frac{\Sigma f \log x}{N}\right)$

$= \text{Antilog}\left(\frac{66.771}{31}\right)$

$= \text{Antilog}(2.1539)$

GM $= 142.5$ lbs

5. The price of a commodity increased by 5% from 2004 to 2005 , 8% from 2005 to 2006 and 77% from 2006 to 2007. Calculate the average increase from 2004 to 2007? **[HY-2019]**

Sol :

Percentage in rise	x	$\log x$
5	105	2.021
8	108	2.0334
77	177	2.2479
		6.3024

Geometric Mean $= \text{Antilog}\left(\frac{\Sigma \log x}{n}\right)$

$= \text{Antilog}\left(\frac{6.3026}{3}\right)$

$= \text{Antilog}(2.1008)$

GM $= 126.12$

$\therefore$ Average increase of the commodity from 2004 to 2007 = 126.12 – 100 = 26.1%

6. An aeroplane flies, along the four sides of a square at speeds of 100, 200, 300 and 400 kilometres per hour respectively. What is the average speed of the plane in its flight around the square. **[June - 2019]**

Sol : Since we are given speed per hour, harmonic mean will give the correct answer.

Harmonic mean $= \frac{n}{1/a + 1/b + 1/c + 1/d}$

$= \frac{4}{1/100 + 1/200 + 1/300 + 1/400} = \frac{4}{\frac{12+6+4+3}{1200}}$

$= \frac{4}{25/1200} = \frac{4 \times 1200}{25} = 4 \times 48$

HM $= 192$ km/hr

7. A man travelled by car for 3 days. He covered 480 km each day. On the first day he drove for 10 hours at 48 km an hour. On the second day, he drove for 12 hours at 40 km an hour and for the last day he drove for 15 hours at 32 km. What is his average speed? **[Mar.-2019]**

Sol : Total distance covered 480 km

The first-day distance covered 48 km

The second-day distance covered 40 km

Third day distance covered 32 km

Number of observations = 3

Average speed = HM

$= \frac{3}{\sum \frac{1}{x}} = \frac{3}{\left(\frac{1}{48} + \frac{1}{40} + \frac{1}{32}\right)} = \frac{3}{\frac{10+12+15}{480}}$

$= \frac{3}{\frac{37}{480}} = \frac{3 \times 480}{37} = 38.92$ km/hr

8. The monthly incomes of 8 families in rupees in a certain locality are given below. Calculate the mean, the geometric mean and the harmonic mean and confirm that the relations among them holds true. Verify their relationships among averages.

Family :	A	B	C	D	E	F	G	H
Income (Rs.):	70	10	50	75	8	25	8	42

Sol :

Family	Income (₹) (x)	log x	1/x
A	70	1.8451	0.0143
B	10	1.000	0.1
C	50	1.6990	0.02
D	75	1.8751	0.0133
E	8	0.9031	0.125
F	25	1.3979	0.04
G	8	0.9031	0.125
H	42	1.6232	0.0239
	288	11.2465	0.4615

(i) Arithimetic Mean

$$AM = \frac{\Sigma x}{n} = \frac{288}{8} = 36$$

(ii) Geometric Mean

$$GM = \text{Antilog}\left(\frac{\Sigma \log x}{n}\right) = \text{Antilog}\left(\frac{11.2465}{8}\right)$$

$$= \text{Antilog}\ (1.4058) = 25.46$$

(iii) Harmonic Mean

$$HM = \frac{n}{\Sigma\left(\frac{1}{x}\right)} = \frac{8}{0.4615} = 17.33$$

Thus, 36 > 25.46 > 17.33

∴ AM > GM > HM

9. Calculate AM, GM and HM and also verify their relations among them for the following data

X	5	15	10	30	25	20	35	40
f	18	16	20	21	22	13	12	16

Sol :

x	f	fx	log x	f log x	f/x
5	18	90	0.6990	12.582	3.6
15	16	240	1.1761	18.8176	1.0667
10	20	200	1.0000	20	2
30	21	630	1.4771	31.0191	0.7
25	22	550	1.3979	30.7538	0.88
20	13	260	1.3010	16.913	0.65
35	12	420	1.5441	18.5292	0.3429
40	16	640	1.6021	25.6336	0.4
	138	3030		174.2483	9.6396

(i) Arithimetic Mean

$$AM = \frac{\Sigma fx}{N} = \frac{3030}{138} = 21.96$$

(ii) Geometric Mean

$$GM = \text{Antilog}\left(\frac{\Sigma f \log x}{N}\right)$$

$$= \text{Antilog}\left(\frac{174.2483}{138}\right)$$

$$= \text{Antilog}\ (1.7425) = 18.31$$

(iii) Harmonic Mean

$$HM = \frac{N}{\Sigma\left(\frac{f}{x}\right)} = \frac{138}{9.6396} = 14.32$$

Since 21.96 > 18.31 > 14.32, we get AM > GM > HM.

10. Calculate AM, GM and HM from the following data and also find its relationship:

Marks	0 - 10	10 - 20	20 - 30	30 - 40	40 - 50	50 - 60
No. of students	5	10	25	30	20	10

Sol :

Marks	No.of students (f)	Mid $x(x)$	fx	log x	f log x	f/x
0 - 10	5	5	25	0.6990	3.495	1.0000
10 - 20	10	15	150	1.1761	11.761	0.6667
20 - 30	25	25	625	1.3979	34.9475	1.00
30 - 40	30	35	1050	1.5441	46.323	0.8571
40 - 50	20	45	900	1.6532	33.064	0.4444
50 - 60	10	55	550	1.7404	17.404	0.1818
	100		3300		146.9945	4.15

(i) Arithmetic Mean $AM = \frac{\Sigma fx}{N} = \frac{3300}{100} = 33$

(ii) Geometric Mean $GM = \text{Antilog}\left(\frac{\Sigma f \log x}{N}\right)$

$$= \text{Antilog}\left(\frac{146.9975}{100}\right)$$

$$= \text{Antilog}\ (1.469975) = \text{Antilog}\ (1.4700)$$

$$= 29.51$$

(iii) Harmonic Mean

$$HM = \frac{N}{\Sigma\left(\frac{f}{x}\right)} = \frac{100}{4.15} = 24.10$$

Since 21.96 > 18.31 > 24.10, we get

AM > GM > HM

10 OPERATIONS RESEARCH

TEXTUAL QUESTIONS

Exercise 10.1

1. A company produces two types of pens A and B. Pen A is of superior quality and pen B is of lower quality. Profits on pens A and B are ₹5 and ₹3 per pen respectively. Raw materials required for each pen A is twice as that of pen B. The supply of raw material is sufficient only for 1000 pens per day. Pen A requires a special clip and only 400 such clips are available per day. For pen B, only 700 clips are available per day . Formulate this problem as a linear programming problem.

Sol :

(i) **Variables:** Let x_1 and x_2 represents the types of pen A and B respectively.

(ii) **Objective function:** Profit on pens of type A = $5x_1$

and profit on pens of type B = $3x_2$

$\therefore$ Total profit = $5x_1 + 3x_2$

Let Z = $5x_1 + 3x_2$ which is the objective function.

Since the total profit is to be maximized, we have to maximize Z = $5x_1 + 3x_2$

(iii) **Constraints:**

Raw materials required is

$(2x_1 + x_2)$ which is sufficient only for 1000 pens

$\therefore 2x_1 + x_2 \le 1000$

Pen A requires 400 clips $\Rightarrow x_1 \le 400$

Pen B requires 700 clips $\Rightarrow x_2 \le 700$

(iv) **Non-negative restrictions:** Since the number of pens cannot be negative, we have $x_1 = 0, x_2 = 0$.

Thus, the mathematical formulation of the LPP is

Maximize Z = $5x_1 + 3x_2$

Subject to the constraints

$$2x_1 + x_2 \le 1000,$$
$$x_1 \le 400,$$
$$x_2 \le 700$$
$$x_1, x_2 \ge 0$$

2. A company produces two types of products say type A and B. Profits on the two types of product are ₹30/- and ₹40/- per kg respectively. The data on resources required and availability of resources are given below. **[HY-2019]**

	Requirements		Capacity available per month
	Product A	Product B	
Raw material (kgs)	60	120	12000
Machining hours / piece	8	5	600
Assembling (man hours)	3	4	500

Formulate this problem as a linear programming problem to maximize the profit.

Sol : (i) **Variables:** Let x_1 and x_2 denote the types of products A and B respectively.

(ii) **Objective function:**

Profit on Type A product = $30x_1$

Profit on Type B product = $40x_2$

$\therefore$ Total profit = $30x_1 + 40x_2$

Let Z = $30x_1 + 40x_2$ which is the objective function.

Since the total profit is to be maximized, we have to maximize Z = $30x_1 + 40x_2$

(iii) Constraints:

Raw material required for type A and B product is $60x_1 + 120x_2$

Since the capacity available is 12,000 per month, we have $60x_1 + 120x_2 \leq 12,000$

Machine hours for type A and B product, with the capacity available 600 per month is

$8x_1 + 5x_2 \leq 600$

Also, required assembling man hours, with the capacity available 500 per month is

$3x_1 + 4x_2 \leq 500$

(iv) Non-negative restructions: Since the number of products on type A and B cannot be negative,

we have $x_1, x_2 \geq 0$

Thus, the mathematical formulation of the LPP is maximize $Z = 30x_1 + 40x_2$

subject to the constraints

$$60x_1 + 120x_2 \leq 12,000$$
$$8x_1 + 5x_2 \leq 600$$
$$3x_1 + 4x_2 \leq 500$$
$$x_1, x_2 \geq 0$$

3. A company manufactures two models of voltage stabilizers viz., ordinary and auto-cut. All components of the stabilizers are purchased from outside sources, assembly and testing is carried out at company's own works. The assembly and testing time required for the two models are 0.8 hour each for ordinary and 1.20 hours each for auto-cut. Manufacturing capacity 720 hours at present is available per week. The market for the two models has been surveyed which suggests maximum weekly sale of 600 units of ordinary and 400 units of auto-cut . Profit per unit for ordinary and auto-cut models has been estimated at ₹100 and ₹150 respectively. Formulate the linear programming problem.

Sol : (i) **Variables:** Let x_1, x_2 denote the number of ordinary and auto-cut voltage stabilizers.

(ii) **Objective function:**

Profit for ordinary stabilizers = $100x_1$

Profit for auto-cut stabilizers = $150x_2$

Total Profit = $100x_1 + 150x_2$

Let $Z = 100x_1 + 150x_2$ which is the objective function.

Since the profit is to be maximized, we have to maximize $z = 100x_1 + 150x_2$.

(iii) **Constraints:**

The assembling and testing time required for ordinary stabilizers = $0.8x_1$ and for auto cut stabilizers = $1.2x_2$

Since the manufacturing capacity is 720 hours / week, we get $0.8x_1 + 1.2x_2 \leq 720$

Maximum sale for both the models are 600 and 400 respectively.

we get, $x_1 \leq 600$, $x_2 \leq 400$

(iv) **Non-negative restrictions:**

Since the number of both the types of stabilizers cannot be negative, we get

$x_1, x_2 \geq 0$

Thus, the mathematical formulation of the LPP is maximize $z = 100x_1 + 150x_2$

subject to the constraints

$$0.8x_1 + 1.2x_2 \leq 720$$
$$x_1 \leq 600$$
$$x_2 \leq 400$$
$$x_1, x_2 \geq 0$$

4. Solve the following linear programming problems by graphical method.

(i) Maximize $Z = 6x_1 + 8x_2$ subject to constraints $30x_1 + 20x_2 \leq 300$; $5x_1 + 10x_2 \leq 110$; and $x_1, x_2 \geq 0$. (GMQP - 2019)

(ii) Maximize $Z = 22x_1 + 18x_2$ subject to constraints $960x_1 + 640x_2 \leq 15360$; $x_1 + x_2 \leq 20$ and $x_1, x_2 \geq 0$.

(iii) Minimize $Z = 3x_1 + 2x_2$ subject to the constraints $5x_1 + x_2 \geq 10$; $x_1 + x_2 \geq 6$; $x_1 + 4x_2 \geq 12$ and $x_1, x_2 \geq 0$.

(iv) Maximize $Z = 40x_1 + 50x_2$ subject to constraints $3x_1 + x_2 \leq 9$; $x_1 + 2x_2 \leq 8$ and $x_1, x_2 \geq 0$

(v) Maximize $Z = 20x_1 + 30x_2$ subject to constraints $3x_1 + 3x_2 \leq 36$; $5x_1 + 2x_2 \leq 50$; $2x_1 + 6x_2 \leq 60$ and $x_1, x_2 \geq 0$

(vi) Minimize $Z = 20x_1 + 40x_2$ subject to the constraints $36x_1 + 6x_2 \geq 108$, $3x_1 + 12x_2 \geq 36$, $20x_1 + 10x_2 \geq 100$ and $x_1, x_2 \geq 0$ [Mar. - 2020]

Sol : (i) Maximize $Z = 6x_1 + 8x_2$

subject to the constraints

$$30x_1 + 20x_2 \leq 300$$

$$5x_1 + 10x_2 = 110$$
$$x_1, x_2 = 0$$

Since both the decision variables x_1 and x_2 are non-negative, the solution lies in the first quadrant of the plane.

Consider the equations,

$30x_1 + 20x_2 = 300$

x_1	0	10
x_2	15	0

$5x_1 + 10x_2 = 110$

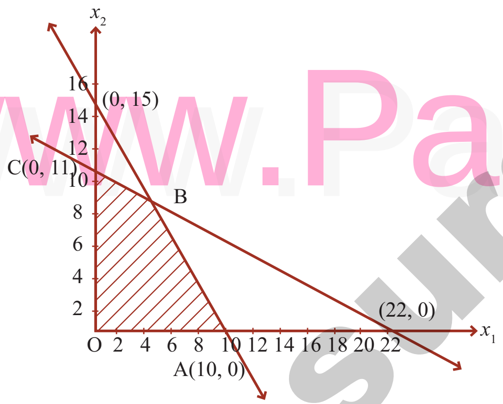
x_1	0	22
x_2	11	0

$30x_1 + 20x_2 = 300$ is a line passing through the points (0, 15) and (10, 0).

Any point lying on or below the line $30x_1 + 20x_2 = 300$ satisfies the constraint $30x_1 + 20x_2 = 300$.

$5x_1 + 10x_2 = 110$ is a line passing through the points (0, 11) and (22, 0)

Any point lying on or below the line $5x_1 + 10x_2 = 110$ satisfies the constraint $5x_1 + 10x_2 = 110$



The feasible region satisfying all the conditions is on BC. The coordinates of the points are O(0, 0), A(10, 0), C(0, 11) and B is the point of intersection of two lines.

$$30x_1 + 20x_2 = 300 \quad \ldots (1)$$
$$5x_1 + 10x_2 = 110 \quad \ldots (2)$$
$$(1) \Rightarrow 30x_1 + 20x_2 = 300$$
$$(-) \quad (-) \quad (-)$$
$$(2)\times 2 \Rightarrow 10x_1 + 20x_2 = 220$$
$$20x_1 = 80$$
$$\Rightarrow x_1 = 4$$

Substituting $x_1 = 4$ in (1) we get,

$$\Rightarrow 30(4) + 20x_2 = 300$$
$$\Rightarrow 120 + 20x_2 = 300$$
$$\Rightarrow 20x_2 = 300 - 120 = 180$$
$$\Rightarrow x_2 = 9$$

$\therefore$ B is (4, 9)

Corner Points	$Z = 6x_1 + 8x_2$
O(0, 0)	0
A(10, 0)	60
B(4, 9)	24 + 72 = 96
C(0, 11)	88

Maximum value of Z occurs at B(4, 9)

$\therefore$ The solution is $x_1 = 4$, $x_2 = 9$ and $Z_{max} = 96$.

(ii) Maximize $Z = 22x_1 + 18x_2$

Subject to the constraints

$$960\,x_1 + 640\,x_2 = 15360$$
$$x_1 + x_2 = 20$$
$$x_1 , x_2 = 0$$

Since both the decision variables x_1 and x_2 are non-negative, the solution lies in the first quadrant of the plane.

Consider the equations

$960x_1 + 640x_2 = 15360$

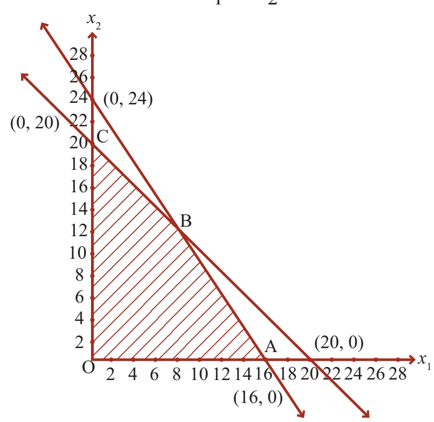
x_1	0	16
x_2	24	0

$x_1 + x_2 = 20$

x_1	0	20
x_2	20	0

Any point lying on or below the line $960x_1 + 640x_2 = 15360$ satisfies the constraint $960x_1 + 640x_2 = 15360$

Any point lying on or below the line $x_1 + x_2 = 20$, satisfies the constraint $x_1 + x_2 = 20$



The feasible region satisfying all the constraints OABC.

Its co-ordinates are O(0, 0), A(16, 0), C(0, 20) and B is the point of intersection of the lines.

$\therefore$ B(8, 12) [From the graph]

Verification for B(8, 12):

$$960x_1 + 640x_2 = 15360 \quad \text{... (1)}$$
$$x_1 + x_2 = 20 \quad \text{... (2)}$$

$(1) \Rightarrow \quad 960x_1 + 640x_2 = 15360$

$\qquad\qquad (-) \quad (-) \quad (-)$

$(2) \times 640 \Rightarrow \quad 640x_1 + 640x_2 = 12800$

$\Rightarrow \quad 320x_1 = 2560$

$$x_1 = \frac{2560}{320} = 8$$

Substituting $x_1 = 8$ in (2) we get,

$$8 + x_2 = 20$$

$\Rightarrow \quad x_2 = 12$

$\therefore$ B(8, 12)

Corner Points	$Z = 22x_1 + 18x_2$
O(0, 0)	0
A(16, 0)	352
B(8, 12)	392
C(0, 20)	360

Maximum value of Z occurs at B(8, 12)

$\therefore$ The solution is $x_1 = 8, x_2 = 12$ and $Z_{max} = 392$.

(iii) Minimize $Z = 3x_1 + 2x_2$

Subject to the constraints

$$5x_1 + x_2 = 10, \qquad x_1 + x_2 = 6$$
$$x_1 + 4x_2 = 12$$
$$x_1, x_2 = 0$$

Since both the decision variables x_1 and x_2 are non-negative the solution lies in the I quadrant. Consider the equations.

$5x_1 + x_2 = 10$

x_1	0	2
x_2	10	0

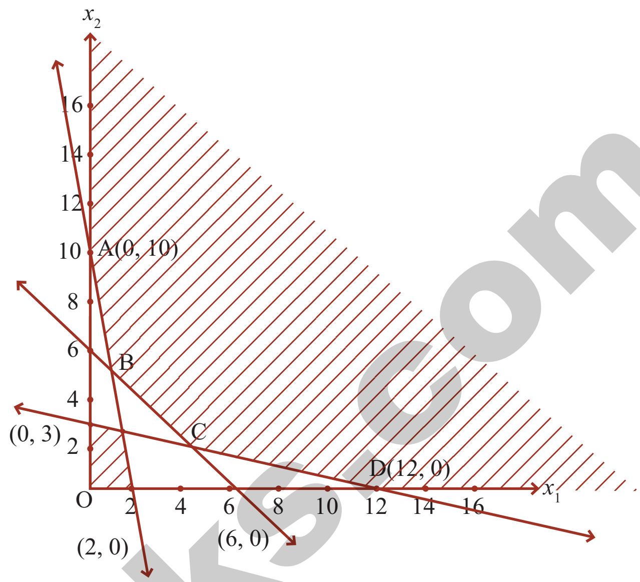
$x_1 + x_2 = 6$

x_1	0	6
x_2	6	0

$x_1 + 4x_2 = 12$

x_1	0	12
x_2	3	0

Any point lying on or above the lines $5x_1 + x_2 = 10$, $x_1 + x_2 = 6$, $x_1 + 4x_2 = 12$ satisfy the constraints $5x_1 + x_2 = 10$, $x_1 + x_2 = 6$ and $x_1 + 4x_2 = 12$ respectively.



The feasible region is ABCD and its co-ordinates as A(0, 10), D(12, 0) and B is the point of the intersection of the lines $5x_1 + x_2 = 10$ and $x_1 + x_2 = 6$.

Also C is the point of intersection of the lines $x_1 + x_2 = 6$ and $x_1 + 4x_2 = 12$.

Verification of B and C:

For B

$$5x_1 + x_2 = 10 \quad \text{... (1)}$$

$\quad (-) \quad (-) \quad (-)$

$$x_1 + x_2 = 6 \quad \text{... (2)}$$

$$4x_1 = 4$$

$\Rightarrow \quad x_1 = 1$

Substituting $x_1 = 1$ in (2) we get,

$$1 + x_2 = 6$$

$\Rightarrow \quad x_2 = 5$

$\therefore$ B is (1, 5)

For C

$$x_1 + x_2 = 6$$

$\quad (-) \quad (-) \quad (-)$

$$x_1 + 4x_2 = 12$$

$$-3x_2 = -6$$

$\Rightarrow \quad x_2 = 2$

Substituting $x_2 = 2$ in (3) we get,

$$x_1 + 2 = 6$$

$\Rightarrow \quad x_1 = 4$

$\therefore$ C is (4, 2)

Corner Points	$z = 3x_1 + 2x_2$
A(0, 10)	20
B(1, 5)	13
C(4, 2)	16
D(12, 0)	36

Minimum value of Z occurs at B(1, 5).

∴ The solution is $x_1 = 1, x_2 = 5$ and $Z_{min} = 13$.

(iv) Maximize $Z = 40x_1 + 50x_2$ subject to the constraints $3x_1 + x_2 = 9$, $x_1 + 2x_2 = 8$ and $x_1, x_2 = 0$

Since both the decision variables x_1 and x_2 are non-negative, the solution lies in the first quadrant of the plane.

Consider the equations

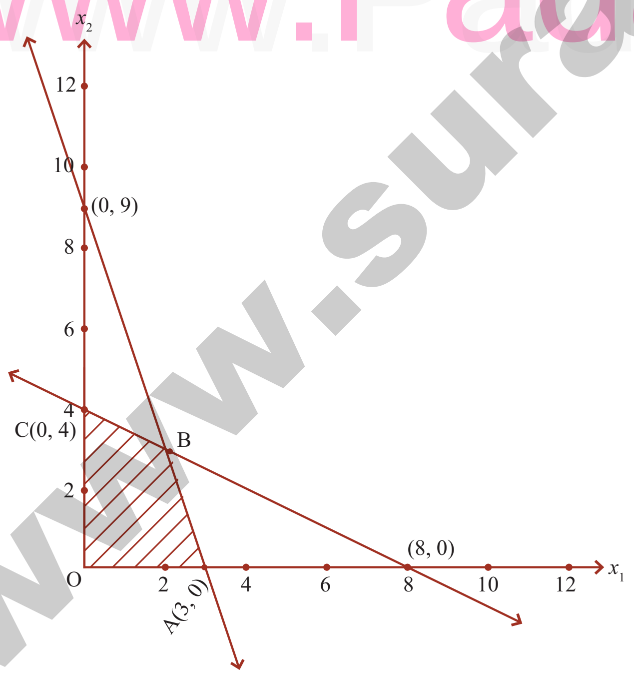
$3x_1 + x_2 = 9$

x_1	0	3
x_2	9	0

$x_1 + 2x_2 = 8$

x_1	0	8
x_2	4	0

Any point lying on or below the lines $3x_1 + x_2 = 9$ and $x_1 + 2x_2 = 8$ satisfies the constraints $3x_1 + x_2 = 9$ and $x_1 + 2x_2 = 8$.



The feasible region is OABC and its co-ordinates are O(0, 0), A(3, 0), C(0, 4) and B is the point of intersection of the lines

$$3x_1 + x_2 = 9 \quad \text{... (1)}$$
$$\text{and } x_1 + 2x_2 = 8 \quad \text{... (2)}$$

Verification of B

$$(1) \Rightarrow \quad 3x_1 + x_2 = 9$$
$$(2) \times 3 \Rightarrow \quad 3x_1 + 6x_2 = 24$$
$$(-) \quad (-) \quad (-)$$
$$-5x_2 = -15 \quad \Rightarrow x_2 = 3$$

Substituting $x_2 = 3$ in (1) we get

$$3x_1 + 3 = 9$$
$$\Rightarrow \quad 3x_1 = 6$$
$$\Rightarrow \quad x_1 = 2$$

∴ B is (2, 3)

Corner Points	$Z = 40x_1 + 50x_2$
O(0, 0)	0
A(3, 0)	120
B(2, 3)	230
C(0, 4)	200

Maximum value occurs at B(2, 3).

∴ The solution is $x_1 = 2, x_2 = 3$ and $Z_{max} = 230$

(v) Maximize $Z = 20x_1 + 30x_2$ subject to the constraints $3x_1 + 3x_2 = 36$, $5x_1 + 2x_2 = 50$, $2x_1 + 6x_2 = 60$ and $x_1, x_2 = 0$.

Since both the decision variables x_1, x_2 are non-negative, the solution lies in the first quadrant of the plane. Consider the equations.

$3x_1 + 3x_2 = 36$

x_1	0	12
x_2	12	0

$5x_1 + 2x_2 = 50$

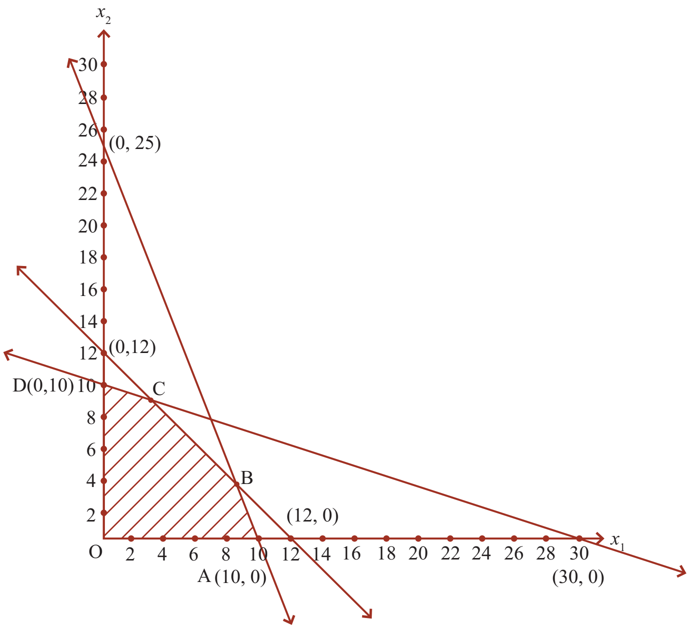
x_1	0	10
x_2	25	0

$2x_1 + 6x_2 = 60$

x_1	0	30
x_2	10	0

The feasible region is OABCD and its co-ordinates are O(0, 0), A(10, 0), D(0, 10), B is the point of intersection of the lines $3x_1 + 3x_2 = 36$ and $5x_1 + 2x_2 = 50$.

Also C is the point of intersection of the lines $2x_1 + 6x_2 = 60$ and $3x_1 + 3x_2 = 36$.



Verification of the points B and C:

For B

$$3x_1 + 3x_2 = 36$$

$$\Rightarrow \quad x_1 + x_2 = 12 \quad \text{... (1)}$$

$$5x_1 + 2x_2 = 50 \quad \text{... (2)}$$

$$(1) \times 2 \Rightarrow 2x_1 + 2x_2 = 24$$

$$(-) \quad (-) \quad (-)$$

$$(2) \Rightarrow 5x_1 + 2x_2 = 50$$

$$-3x_1 = -6$$

$$\Rightarrow \quad x_1 = 2$$

Substituting $x_1 = 2$ in (1) we get

$$2 + x_2 = 12$$

$$\Rightarrow \quad x_2 = 6$$

$\therefore$ B is (2, 6)

For C

$$3x_1 + 3x_2 = 36$$

$$\Rightarrow \quad x_1 + x_2 = 12 \quad \text{... (3)}$$

$$2x_1 + 6x_2 = 60$$

$$\Rightarrow \quad x_1 + 3x_2 = 30 \quad \text{... (4)}$$

$$(3) \Rightarrow x_1 + x_2 = 12$$

$$(-) \quad (-) \quad (-)$$

$$(4) \Rightarrow x_1 + 3x_2 = 30$$

$$-2x_2 = -18 \quad \Rightarrow x_2 = 9$$

Substituting $x_2 = 9$ in (3), we get

$$x_1 + 9 = 12$$

$$\Rightarrow \quad x_1 = 3$$

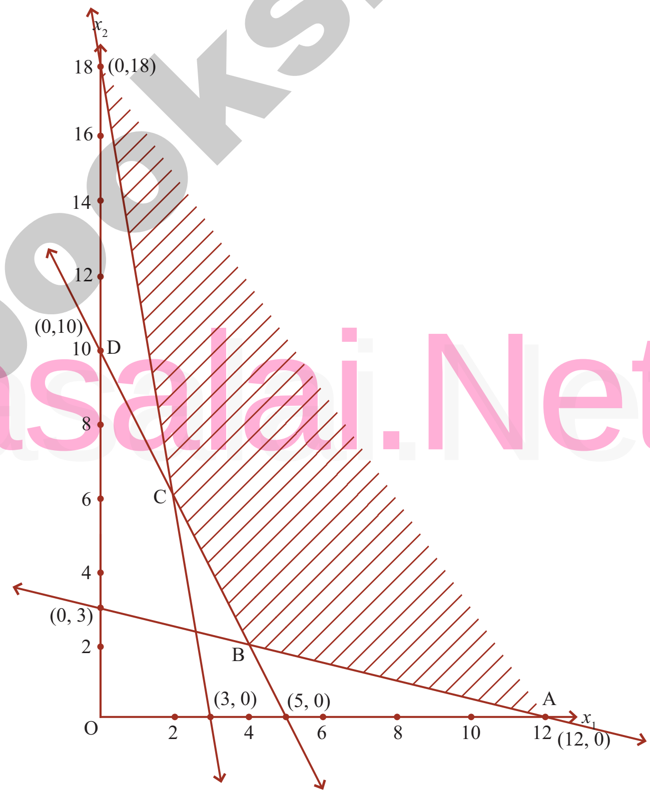
$\therefore$ C is (3,9)

Corner Points	$Z = 20x_1 + 30x_2$
O(0, 0)	0
A(10, 0)	200
B(2, 6)	220
C(3, 9)	330
D(0, 10)	300

The maximum value of Z occurs at C(3, 9).

$\therefore$ The solution is $x_1 = 3$, $x_2 = 9$ and $Z_{max} = 330$.

(vi) Minimize $Z = 20x_1 + 40x_2$ subject to the constraints $36x_1 + 6x_2 = 108$, $3x_1 + 12x_2 = 36$, $20x_1 + 10x_2 = 100$ and $x_1, x_2 = 0$.



Since both the decision variables x_1 and x_2 are non-negative, the solution lies in the first quadrant of the plane.

Consider the equations.

$36x_1 + 6x_2 = 108$

x_1	0	3
x_2	18	0

$3x_1 + 12x_2 = 36$

x_1	0	12
x_2	3	0

$20x_1 + 10x_2 = 100$

x_1	0	5
x_2	10	0

Any point lying on or above the lines $36x_1 + 6x_2 = 108$, $3x_1 + 12x_2 = 36$ and $20x_1 + 10x_2 = 100$ satisfy the constraints $36x_1 + 6x_2 = 108$, $3x_1 + 12x_2 = 36$ and $20x_1 + 10x_2 = 100$.

The feasible region is ABCD and its co-ordinates are A(12, 0) D(0, 10), B is the point of intersection of the lines $3x_1 + 12x_2 = 36$, $20x_1 + 10x_2 = 100$.

Also, C is the point of the intersection of the lines $36x_1 + 6x_2 = 108$ and $20x_1 + 10x_2 = 100$.

Verification of B and C: For B:

$3x_1 + 12x_2 = 36 \Rightarrow x_1 + 4x_2 = 12$ ... (1)

$20x_1 + 10x_2 = 100 \Rightarrow 2x_1 + x_2 = 10$ ... (2)

$$\begin{aligned}(1) \Rightarrow \quad x_1 + 4x_2 &= 12\\ (-) \quad (-) \quad &\quad (-)\\ (2) \times 4 \Rightarrow \quad 8x_1 + 4x_2 &= 40\\ \hline -7x_1 &= -28 \quad \Rightarrow x_1 = 4\end{aligned}$$

Substituting $x_1 = 4$ in (2), we get $8 + x_2 = 10 \Rightarrow x_2 = 2$

$\therefore$ B is (4, 2)

For C:

$36x_1 + 6x_2 = 108 \quad \Rightarrow 6x_1 + x_2 = 18$ ... (3)

$20x_1 + 10x_2 = 100 \quad \Rightarrow 2x_1 + x_2 = 10$ ... (4)

$$\begin{aligned}(3) \Rightarrow \quad 6x_1 + x_2 &= 18\\ (-) \quad (+) \quad &\quad (-)\\ (4) \Rightarrow \quad 2x_1 + x_2 &= 10\\ \hline 4x_1 &= 8 \quad \Rightarrow x_1 = 2\end{aligned}$$

Substituting $x_1 = 2$ in (4) we get,

$4 + x_2 = 10$

$\Rightarrow \quad x_2 = 6$

$\therefore$ C is (2, 6)

Corner Points	$Z = 20x_1 + 40x_2$
A(12, 0)	240
B(4, 2)	160
C(2, 6)	280
D(0, 18)	720

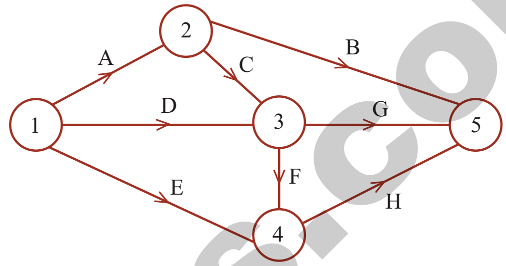
The minimum of Z value occurs at B.

$\therefore$ The solution is $x_1 = 4$, $x_2 = 2$ and $Z_{min} = 160$.

Exercise 10.2

1. Draw the network for the project whose activities with their relationships are given below:
Activities A, D, E can start simultaneously; B, C > A ; G, F > D, C; H > E, F. [Aug. - 2022]

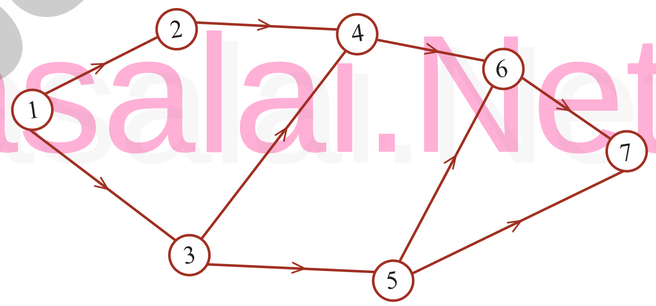
Sol :



2. Draw the event oriented network for the following data: [Mar.-2020; Sep. - 2021]

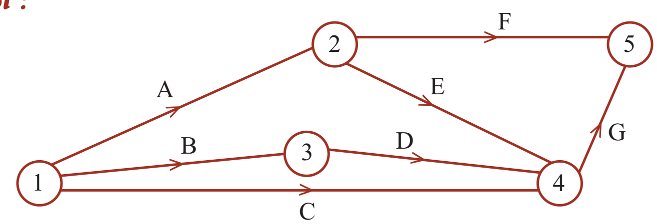
Events	1	2	3	4	5	6	7
Immediate Predecessors	-	1	1	2,3	3	4,5	5,6

Sol :



3. Construct the network for the projects consisting of various activities and their precedence relationships are as given below: A,B,C can start simultaneously A < F, E ; B < D, C ; E, D < G

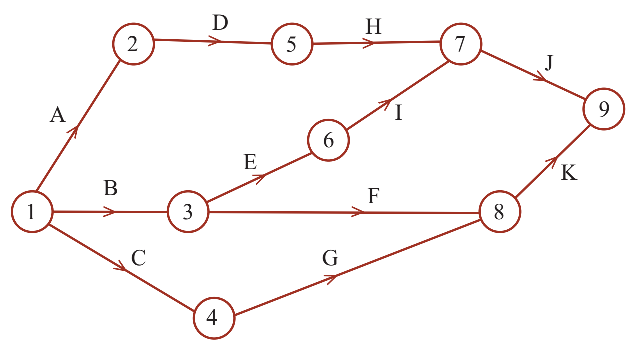
Sol :



4. Construct the network for each the projects consisting of various activities and their precedence relationships are as given below: **[SEP.-2020]**

Activity	A	B	C	D	E	F	G	H	I	J	K
Immediate Predecessors	-	-	-	A	B	B	C	D	E	H,I	F,G

Sol :



$E_1 = 0 + 3 = 3$
$E_2 = E_1 + t_{12} = 8 + 3 = 11$
$E_3 = 3 + 12 = 15$
$E_4 = 15 + 3 = 18$
$E_5 = E_2 + 3 = 11 + 3 = 14$
$E_6 = E_3 + 8 = 15 + 8 = 23$
$E_7 = E_6 + 8 = 23 + 8 = 31$

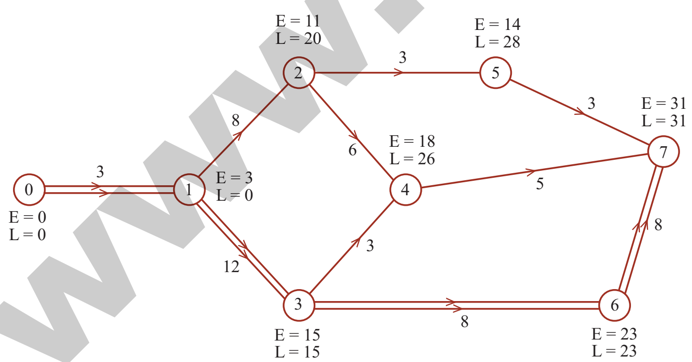
$L_7 = 31$
$L_6 = L_7 - 8 = 31 - 8 = 23$
$L_5 = L_7 - 3 = 31 - 3 = 28$
$L_4 = L_7 - 5 = 31 - 5 = 26$
$L_3 = L_6 - 8 = 23 - 8 = 15$
$L_2 = L_5 - 3$ or $L_4 - 6$ whichever is minimum
$= (28 - 3)$ or $(26 - 6)$
$= 25$ or 20
$= 20$ (which is minimum)
$L_1 = L_2 - 8$ or $L_3 - 12$ whichever is minimum
$= (20 - 8)$ or $(15 - 12) = 12$ or $3 = 3$
$L_0 = 0$

5. Construct the network for the project whose activities are given below. **[May - 2022]**

Activity	0-1	1-2	1-3	2-4	2-5	3-4	3-6	4-7	5-7	6-7
Duration (in week)	3	8	12	6	3	3	8	5	3	8

Calculate the earliest start time, earliest finish time, latest start time and latest finish time of each activity. Determine the critical path and the project completion time.

Sol :



Activity	Duration t_{ij}	EST	EFT = EST + t_{ij}	LST = LFT $- t_{ij}$	LFT
0 - 1	3	0	3	3	3
1 - 2	8	3	11	20 – 8 = 12	20
1 - 3	12	3	15	15 – 12 = 3	15
2 - 4	6	11	17	26 – 6 = 20	26
2 - 5	3	11	14	28 – 3 = 25	28
3 - 4	3	15	18	26 – 3 = 23	26
3 - 6	8	15	23	23 – 8 = 15	23
4 - 7	5	18	23	31 – 5 = 26	31
5 - 7	3	14	17	31 – 3 = 28	31
6 - 7	8	23	31	31 – 8 = 23	31

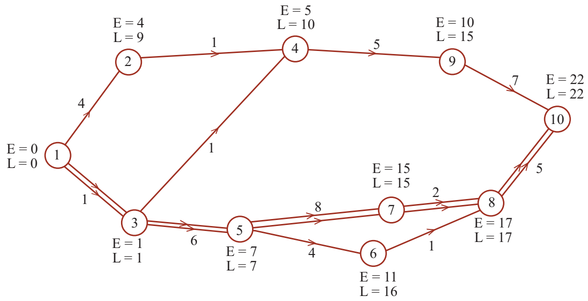
The critical path is 0 – 1 – 3 – 6 – 7 and the project completion time is 31 weeks.

6. A project schedule has the following characteristics

Activity	1-2	1-3	2-4	3-4	3-5	4-9	5-6	5-7	6-8	7-8	8-10	9-10
Time	4	1	1	1	6	5	4	8	1	2	5	7

Construct the network and calculate the earliest start time, earliest finish time, latest start time and latest finish time of each activity and determine the Critical path of the project and duration to complete the project.

Sol :



$E_1 = 0$
$E_2 = 0 + 4 = 4$
$E_3 = 0 + 1 = 1$
$E_4 = 4 + 1 = 5$
$E_5 = 1 + 6 = 7$
$E_6 = 7 + 4 = 11$
$E_7 = 8 + 7 = 15$
$E_8 = 15 + 2 = 17$
$E_9 = 5 + 5 = 10$
$E_{10} = 17 + 5 = 22$
$L_{10} = 22$
$L_9 = 22 - 7 = 15$
$L_8 = 22 - 5 = 17$
$L_7 = 17 - 2 = 15$
$L_6 = 17 - 1 = 16$
$L_5 = (16 - 4)$ or $(15 - 8)$
whichever is minimum $= 7$
$L_4 = 15 - 5 = 10$
$L_3 = (10 - 1)$ or $(7 - 6)$
whichever is minimum $= 1$
$L_2 = 10 - 1 = 9$
$L_1 = 0$

Activity	Duration	EST	EFT = EST + t_{ij}	LST = LFT – t_{ij}	LFT
1 – 2	4	0	4	9 – 4 = 5	9
1 – 3	1	0	1	1 – 1 = 0	1
2 – 4	1	4	5	10 – 1 = 9	10
3 – 4	1	1	2	10 – 1 = 9	10
3 – 5	6	1	7	7 – 6 = 1	7
4 – 9	5	5	10	15 – 5 = 10	15
5 – 6	4	7	11	16 – 4 = 12	16
5 – 7	8	7	15	15 – 8 = 7	15
6 – 8	1	11	12	17 – 1 = 16	17
7 – 8	2	15	17	17 – 2 = 15	17
8 – 10	5	17	22	22 – 5 = 17	22
9 – 10	7	10	17	22 – 7 = 15	22

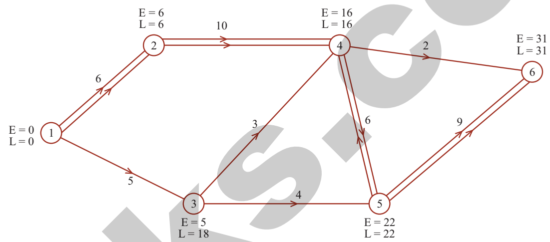
Since EFT and LFT is same on 1 – 3, 3 – 5, 5 – 7 and 7 – 8 and 8 – 10 the critical path is

1 – 3 – 5 – 7 – 8 – 10 and the duration is 22 time units.

7. Draw the network and calculate the earliest start time, earliest finish time, latest start time and latest finish time of each activity and determine the Critical path of the project and duration to complete the project.

Jobs	**1-2**	**1-3**	**2-4**	**3-4**	**3-5**	**4-5**	**4-6**	**5-6**
Duration	**6**	**5**	**10**	**3**	**4**	**6**	**2**	**9**

Sol :



$E_1 = 0$
$E_2 = 0 + 6 = 6$
$E_3 = 0 + 5 = 5$
$E_4 = 6 + 10 = 16$
$E_5 = (5 + 4)$ or $(16 + 6)$
Whichever is maximum
$= 22$
$E_6 = (16 + 2)$ or $(22 + 9)$
Whichever is maximum
$= 31$
$L_6 = 31$
$L_5 = 31 - 9 = 22$
$L_4 = 22 - 6 = 16$ (or) $(31 - 2)$
whichever is minimum
$L_3 = 22 - 4 = 18$
$L_2 = 16 - 10 = 6$
$L_1 = 6 - 6 = 0$

Activity	Duration	EST	EFT = EST + t_{ij}	LST = LFT – t_{ij}	LFT
1 - 2	6	0	6	6 – 6 = 0	6
1 - 3	5	0	5	18 – 5 = 13	18
2 - 4	10	6	16	16 – 10 = 6	16
3 - 4	3	5	8	16 – 3 = 13	16
3 - 5	4	5	9	22 – 4 = 18	22
4 - 5	6	16	22	22 – 6 = 16	22
4 - 6	2	16	18	31 – 2 = 29	31
5 - 6	9	22	31	31 – 9 = 22	31

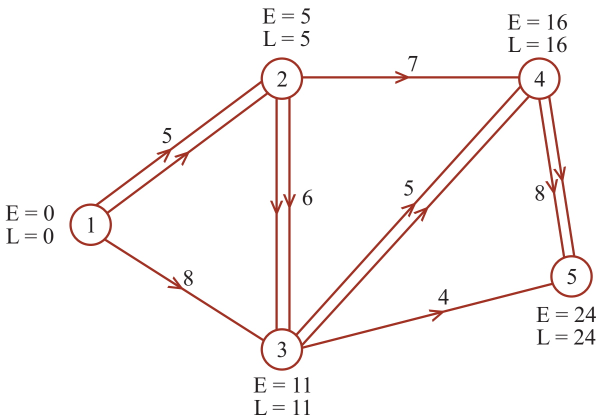
Since EFT and LFT is same on 1 – 2, 2 – 4, 4 – 5 and 5 – 6, the critical path is 1 – 2 – 4 – 5 – 6 and duration time taken is 31 days.

8. The following table gives the activities of a project and their duration in days

Activity	1 – 2	1 – 3	2 – 3	2 – 4	3 – 4	3 – 5	4 – 5
Duration	5	8	6	7	5	4	8

Construct the network and calculate the earliest start time, earliest finish time, latest start time and latest finish time of each activity and determine the Critical path of the project and duration to complete the project. **[Aug. - 2022]**

Sol :



$E_1 = 0$

$E_2 = 0 + 5 = 5$

$E_3 = (0 + 8)$ or $(5 + 6)$ whichever is maximum

$= 11$

$E_4 = (11 + 5)$ or $(5 + 7)$ whichever is maximum

$= 16$

$E_5 = (11 + 4)$ or $(16 + 8)$ Whichever is maximum

$= 24$

$L_5 = 24$

$L_4 = 24 - 8 = 16$

$L_3 = (24 - 4)$ or $(16 - 5)$ whichever is minimum

$= 11$

$L_2 = (16 - 6)$ or $(16 - 7)$ whichever is minimum

$= 5$

$L_1 = (5 - 5)$ or $(11 - 8)$ whichever is minimum

$L_1 = 0$

Activity	Duration	EST	EFT = EST + t_{ij}	LST = LFT – t_{ij}	LFT
1 – 2	5	0	5	5 – 5 = 0	5
1 – 3	8	0	8	11 – 8 = 3	11
2 – 3	6	5	11	11 – 6 = 5	11
2 – 4	7	5	12	16 – 7 = 9	16
3 – 4	5	11	16	16 – 5 = 11	16
3 – 5	4	11	15	24 – 4 = 20	24
4 – 5	8	16	24	24 – 8 = 16	24

EFT and LFT are same in 1 – 2, 2 – 3, 3 – 4 and 4 – 5.

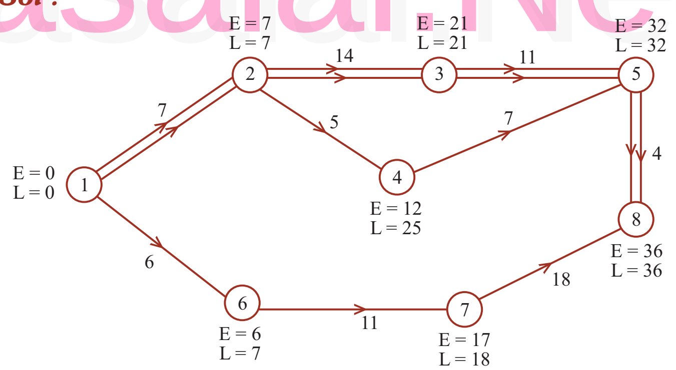
Hence the critical path is 1 – 2 – 3 – 4 – 5 and the duration time taken is 24 days.

9. A Project has the following time schedule **[Mar.-2020]**

Activity	1–2	1–6	2–3	2–4	3–5	4–5	6–7	5–8	7–8
Duration (in days)	7	6	14	5	11	7	11	4	18

Construct the network and calculate the earliest start time, earliest finish time, latest start time and latest finish time of each activity and determine the Critical path of the project and duration to complete the project.

Sol :



$E_1 = 0$

$E_2 = 0 + 7 = 7$

$E_3 = 7 + 14 = 21$

$E_4 = 7 + 5 = 12$

$E_5 = 21 + 11$ or $(12 + 7)$ whichever is maximum

$= 32$

$E_6 = 0 + 6 = 6$

$E_7 = 6 + 11 = 17$

$E_8 = 17 + 18$ (or) $32 + 4$ (whichever is maximum)

	=	36
L_8	=	36
L_7	=	36 – 18 = 18
L_6	=	18 – 11 = 7
L_5	=	36 – 4 = 32
L_4	=	32 – 7 = 25
L_3	=	32 – 11 = 21
L_2	=	(21 – 14) or (25 – 5) whichever is minimum
	=	7
L_1	=	7 – 7 = 0

Activity	Duration	EST	EFT = EST+t_{ij}	LST = LFT–t_{ij}	LFT
1 - 2	7	0	7	7 – 7 = 0	7
1 - 6	6	0	6	7 – 6 = 1	7
2 - 3	14	7	21	21 – 14 = 7	21
2 - 4	5	7	12	25 – 5 = 20	25
3 - 5	11	21	32	32 – 11 = 21	32
4 - 5	7	12	19	32 – 25 = 7	14
6 - 7	11	6	17	18 – 11 = 7	18
5 - 8	4	32	36	36 – 4 = 32	36
7 - 8	18	17	36	36 – 19 = 17	36

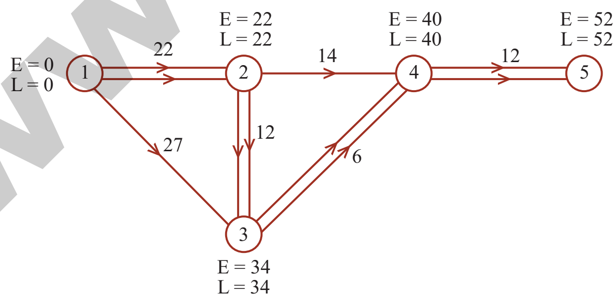
EFT and LFT are same in 1 – 2, 2 – 3, 3 – 5 and 5 – 8. Hence the critical path is 1 – 2 – 3 – 5 – 8 and the duration of time taken is 36 days.

10. The following table use the activities in a construction projects and relevant information **[Mar.- 2019; Sep. - 2020]**

Activity	1 - 2	1 - 3	2 - 3	2 - 4	3 - 4	4 - 5
Duration (in days)	22	27	12	14	6	12

Draw the network for the project, calculate the earliest start time, earliest finish time, latest start time and latest finish time of each activity and find the critical path. Compute the project duration.

Sol :



E_1	=	0
E_2	=	22 + 0 = 22
E_3	=	(0 + 27) or (22 + 12) whichever is maximum
E_3	=	34
E_4	=	(22 + 14) or (34 + 6) whichever is maximum
E_4	=	40
E_5	=	40 + 12 = 52
L_5	=	32
L_4	=	52 – 12 = 40
L_3	=	(40 – 6) = 34
L_2	=	(40 – 14) or (34 – 12) whichever is minimum
	=	22
L_1	=	22 – 22 = 0

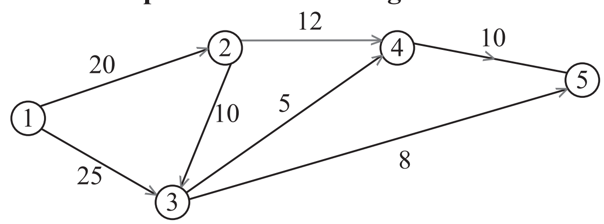
Activity	Duration	EST	EFT = EST+t_{ij}	LST = LFT–t_{ij}	LFT
1 - 2	22	0	22	22 – 22 = 0	22
1 - 3	27	0	27	34 – 27 = 7	34
2 - 3	12	22	34	34 – 12 = 22	34
2 - 4	14	22	36	40 – 14 = 26	40
3 - 4	6	34	40	40 – 6 = 34	40
4 - 5	12	40	52	52 – 12 = 40	52

EFT and LFT are same in 1 – 2, 2 – 3, 3 – 4 and 4 – 5. Hence the critical path is 1 – 2 – 3 – 4 – 5 and the duration time taken is 52 days.

Exercise 10.3

Choose the correct answer:

1. The critical path of the following network is



(a) 1 – 2 – 4 – 5 (b) 1– 3– 5
(c) 1 – 2 – 3 – 5 (d) 1 – 2 – 3 – 4 – 5

Ans: (d) 1 – 2 – 3 – 4 – 5

Hint:

1 – 2 – 4 – 5 ⇒ EFT = 20 + 12 + 10 = 42
1 – 3 – 5 ⇒ EFT = 25 + 8 = 33
1 – 2 – 3 – 5 ⇒ EFT = 20 + 10 + 8 = 38
1 – 2 – 3 – 4 – 5 ⇒ EFT = 20 + 10 + 5 + 10 = 45

2. Maximize: $z = 3x_1 + 4x_2$ subject to $2x_1 + x_2 \leq 40$, $2x_1 + 5x_2 \leq 180$, $x_1, x_2 \geq 0$. In the LPP, which one of the following is feasible corner point?

(a) $x_1 = 18, x_2 = 24$ (b) $x_1 = 15, x_2 = 30$
(c) $x_1 = 2.5, x_2 = 35$ (d) $x_1 = 20.5, x_2 = 19$

Hint: $x_1 = 2.5, x_2 = 35$ which is satisfies the constraints $2x_1 + x_2 \leq 40$, $2x_1 + 5x_2 \leq 180$ and $x_1, x_2 \geq 0$

Ans: (c) $x_1 = 2.5, x_2 = 35$

3. One of the conditions for the activity (i, j) to lie on the critical path is [GMQP-2019; Mar.-2020; May - 2022]

(a) $E_j - E_i = L_j - L_i = t_{ij}$ (b) $E_i - E_j = L_j - L_i = t_{ij}$
(c) $E_j - E_i = L_i - L_j = t_{ij}$ (d) $E_j - E_i = L_j - L_i \neq t_{ij}$

Ans: (a) $E_j - E_i = L_j - L_i - t_{ij}$

4. In constructing the network which one of the following statement is false ?

(a) Each activity is represented by one and only one arrow. (i.e) only one activity can connect any two nodes.

(b) Two activities can be identified by the same head and tail events.

(c) Nodes are numbered to identify an activity uniquely. Tail node (starting point) should be lower than the head node (end point) of an activity.

(d) Arrows should not cross each other.

Ans: (b) Two activities can be identified by the same head and tail events.

5. In a network while numbering the events which one of the following statement is false ? [Aug. - 2022]

(a) Event numbers should be unique.

(b) Event numbering should be carried out on a sequential basis from left to right.

(c) The initial event is numbered 0 or 1.

(d) The head of an arrow should always bear a number lesser than the one assigned at the tail of the arrow.

Ans: (d) Thee head of an arrow should always bear a number lesser than the one assigned at the tail of the arrow.

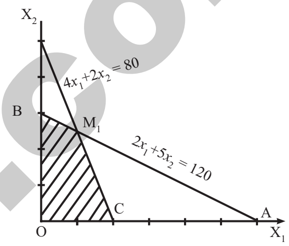
6. A solution which maximizes or minimizes the given LPP is called [MAR.-2019] [JUNE-2019]

(a) a solution (b) a feasible solution
(c) an optimal solution (d) none of these

Ans: (c) an optimal solution

7. In the given graph the coordinates of M_1 are

(a) $x_1 = 5, x_2 = 30$
(b) $x_1 = 20, x_2 = 16$
(c) $x_1 = 10, x_2 = 20$
(d) $x_1 = 20, x_2 = 30$



Hint:

$4x_1 + 2x_2 = 80$

$\Rightarrow \quad 2x_1 + x_2 = 40$

$\quad (-) \quad (-) \quad (-)$

$\quad 2x_1 + 5x_2 = 120$

$\quad -4x_2 = -80$

$\quad x_2 = 20$

$\therefore \quad 4x_1 + 2(20) = 80$

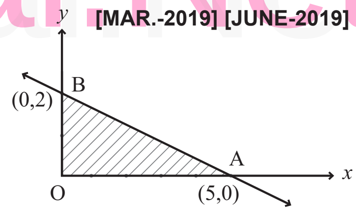
$\quad 4x_1 = 80 - 40 = 40$

$\therefore \quad x_1 = 10$

Ans: (c) $x_1 = 10, x_2 = 20$

8. The maximum value of the objective function $Z = 3x + 5y$ subject to the constraints, $x \geq 0, y \geq 0$ and $2x + 5y \leq 10$ is [MAR.-2019] [JUNE-2019]

(a) 6
(b) 15
(c) 25
(d) 31



Hint: $2x + 5y = 10$

x	0	5
y	2	0

Corner Points	$z = 3x + 5y$
O(0, 0)	0
A(5, 0)	15
B(0, 2)	10

$\therefore$ Maximum value is 15. **Ans: (b) 15**

9. The minimum value of the objective function $Z = x + 3y$ subject to the constraints $2x + y \leq 20$, $x + 2y \leq 20$, $x > 0$ and $y > 0$ is

(a) 10 (b) 20 (c) 0 (d) 5

Hint:

$2x + y = 20$

x	0	10
y	20	0

$x + y = 20$

x	0	20
y	20	0

Corner Points	$z = x + 3y$
O(0, 0)	0
A(0, 20)	60
B(10, 0)	10
C(20, 0)	20

∴ Minimum value is 0 **Ans: (c) 0**

10. Which of the following is not correct?

(a) Objective that we aim to maximize or minimize
(b) Constraints that we need to specify
(c) Decision variables that we need to determine
(d) Decision variables are to be unrestricted.

Ans: (d) Decision variables are to be unrestricted.

11. In the context of network, which of the following is not correct

(a) A network is a graphical representation .
(b) A project network cannot have multiple initial and final nodes
(c) An arrow diagram is essentially a closed network
(d) An arrow representing an activity may not have a length and shape

Ans: (d) An arrow representing an activity may not have a length and shape

12. The objective of network analysis is to **[HY-2019]**

(a) Minimize total project cost
(b) Minimize total project duration
(c) Minimize production delays, interruption and conflicts
(d) All the above

Ans: (b) Minimize total project duration

13. Network problems have advantage in terms of project

(a) Scheduling (b) Planning **[Sep. - 2021]**
(c) Controlling (d) All the above

Ans: (d) All the above

14. In critical path analysis, the word CPM mean

(a) Critical path method
(b) Crash project management
(c) Critical project management
(d) Critical path management

Ans: (a) Critical path method

15. Given an L.P.P maximize $Z = 2x_1 + 3x_2$ subject to the constrains $x_1 + x_2 \leq 1$, $5x_1 + 5x_2 \geq 0$ and $x_1 \geq 0$, $x_2 \geq 0$ using graphical method, we observe

(a) No feasible solution **[Sep.- 2020]**
(b) unique optimum solution
(c) multiple optimum solution
(d) none of these **Ans: (a) No feasible solution**

Hint: Since there is no common area between the lines $x_1 + x_2 \leq 1$ and $5x_1 + 5x_2 \geq 0$

Miscellaneous Problems

1. A firm manufactures two products A and B on which the profits earned per unit are ₹3 and ₹4 respectively. Each product is processed on two machines M_1 and M_2. Product A requires one minute of processing time on M_1 and two minutes on M_2, While B requires one minute on M_1 and one minute on M_2. Machine M_1 is available for not more than 7 hrs 30 minutes while M_2 is available for 10 hrs during any working day. Formulate this problem as a linear programming problem to maximize the profit.

Sol : **(i) Variables:** Let x_1 represents the product A and x_2 represents the product B.

(ii) Objective function:

Profit earned from Product A = $3x_1$
Profit earned from Product B = $4x_2$
Let $Z = 3x_1 + 4x_2$
Since the profit is to be maximized, we have maximize $Z = 3x_1 + 4x_2$

(iii) Constraints

	M_1	M_2
Requirement for A	1 min	2 min
Requirement for B	1 min	1 min

M_1 is available for 7 hrs 30 min = 7 × 60 + 30 = 450 min

M_2 is available for 10 hrs = 10 × 60 = 600 min

∴ $x_1 + x_2 \leq 450$ [for M_1]

$2x_1 + x_2 \leq 600$ [for M_2]

(iv) Non-negative restrictions:

Since the number of products of type A and B cannot be negative, $x_1, x_2 \geq 0$.

Hence, the mathematical formulation of the LPP is maximize

$$Z = 3x_1 + 4x_2$$

Subject to the constraints

$$x_1 + x_2 \leq 450$$
$$2x_1 + x_2 \leq 600$$
$$x_1, x_2 \geq 0$$

2. A firm manufactures pills in two sizes A and B. Size A contains 2 mgs of aspirin, 5 mgs of bicarbonate and 1 mg of codeine. Size B contains 1 mg. of aspirin, 8 mgs. of bicarbonate and 6 mgs. of codeine. It is found by users that it requires atleast 12 mgs. of aspirin, 74 mgs. of bicarbonate and 24 mgs. of codeine for providing immediate relief. It is required to determine the least number of pills a patient should take to get immediate relief. Formulate the problem as a standard LLP.

Sol : (i) **Variables:** Let x_1 and x_2 represent the pills in two sizes A and B.

	A	B	Requirement (atleast)
Aspirin	2 mg	1 mg	12 mg
Bicarbonate	5 mg	8 mg	74 mg
Codeine	1 mg	6 mg	24 mg

Requirement of aspirin $2x_1 + x_2 \geq 12$

Requirement of Bicarbonate $5x_1 + 8x_2 \geq 74$

Requirement of Codeine $x_1 + 6x_2 \geq 24$

(ii) **Objective function:**

Number of pills required for a patient = $x_1 + x_2$

let $Z = x_1 + x_2$

$\therefore$ Minimize $z = x_1 + x_2$ is the objective function.

(iii) **Non - negative restrictions:** Since the number of pills of size A and B cannot be negative, we have $x_1, x_2 \geq 0$

Hence, the mathematical formulation of the LPP minimize $Z = x_1 + x_2$

Subject to the constraints

$$2x_1 + x_2 \geq 12$$
$$5x_1 + 8x_2 \geq 74$$
$$x_1 + 6x_2 \geq 24$$
$$x_1, x_2 \geq 0$$

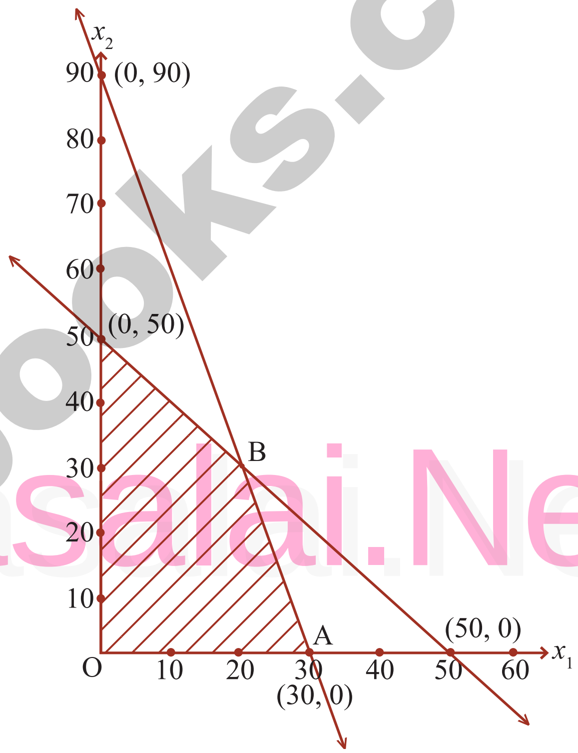
3. Solve the following linear programming problem graphically.

Maximise $Z = 4x_1 + x_2$ subject to the constraints $x_1 + x_2 \leq 50$; $3x_1 + x_2 \leq 90$ and $x_1 \geq 0, x_2 \geq 0$

Sol : Maximize $Z = 4x_1 + x_2$

Subject to the constraints

$$x_1 + x_2 \leq 50$$
$$3x_1 + x_2 \leq 90$$
$$x_1, x_2 \geq 0$$



Since the decision variables x_1 and x_2 are non-negative, the solution lies in the I quadrant of the plane.

Consider the equations

$x_1 + x_2 = 50$

x_1	0	50
x_2	50	0

$3x_1 + x_2 = 90$

x_1	0	30
x_2	90	0

The feasible region is OABC and its co-ordinates are O(0, 0) A(30, 0) C(0, 50) and B is the point of intersection of the lines

$$x_1 + x_2 = 50 \quad \ldots (1)$$
$$\text{and } 3x_1 + x_2 = 90 \quad \ldots (2)$$

11th STD

INSTANT SUPPLEMENTARY EXAM - AUGUST 2022

Reg. No.

PART - III

Business Mathematics And Statistics (with answers)

Time Allowed : 3.00 Hours] **[Maximum Marks : 90**

Instructions :

(1) Check the question paper for fairness of printing. If there is any lack of fairness, inform the Hall Supervisor immediately.

(2) Use **Blue** or **Black** ink to write and underline and pencil to draw diagrams.

PART - I

Note : (i) Answer **all** the questions. **[20 × 1 = 20]**

(ii) Choose the most appropriate answer from the given **four** alternatives and write the option code and the corresponding answer.

1. The inverse matrix of $\begin{bmatrix} \frac{4}{5} & \frac{-5}{12} \\ \frac{-2}{5} & \frac{1}{2} \end{bmatrix}$ is :

(a) $\frac{7}{30}\begin{bmatrix} \frac{1}{2} & \frac{5}{12} \\ \frac{2}{5} & \frac{4}{5} \end{bmatrix}$ (b) $\frac{7}{30}\begin{bmatrix} \frac{1}{2} & \frac{-5}{12} \\ \frac{-2}{5} & \frac{1}{5} \end{bmatrix}$

(c) $\frac{30}{7}\begin{bmatrix} \frac{1}{2} & \frac{5}{12} \\ \frac{2}{5} & \frac{4}{5} \end{bmatrix}$ (d) $\frac{30}{7}\begin{bmatrix} \frac{1}{2} & \frac{-5}{12} \\ \frac{-2}{5} & \frac{4}{5} \end{bmatrix}$

2. The inventor of input - output analysis is :
(a) Sir Francis Galton (b) Fisher
(c) Prof. Wassily W. Leontief
(d) Arthur Caylay

3. The greatest positive integer which divide $+6\ (n + 1)(n + 2)(n + 3)$ for all $n \in \mathbb{N}$ is :
(a) 6 (b) 6 (c) 20 (d) 24

4. The number of parallelograms that can be formed from a set of four parallel lines intersecting another set of three parallel lines is :
(a) 18 (b) 12 (c) 9 (d) 6

5. The centre of the circle $x^2 + y^2 - 2x + 2y - 9 = 0$ is
(a) (1, 1) (b) (–1, –1) (c) (–1, 1) (d) (1, –1)

6. If the equation of the circle $x^2 + y^2 = 16$ then, y intercept(s) is / are :
(a) 4 (b) 16 (c) ± 4 (d) ± 16

7. The value of sin 15° is :
(a) $\frac{\sqrt{3}+1}{2\sqrt{2}}$ (b) $\frac{\sqrt{3}-1}{2\sqrt{2}}$ (c) $\frac{\sqrt{3}}{\sqrt{2}}$ (d) $\frac{\sqrt{3}}{2\sqrt{2}}$

8. The value of cos (– 480°) is :
(a) $\sqrt{3}$ (b) $\frac{-\sqrt{3}}{2}$ (c) $\frac{1}{2}$ (d) $\frac{-1}{2}$

9. The minimum value of the function $f(x) = |x|$ is :
(a) 0 (b) –1 (c) +1 (d) $-\infty$

10. The range of $f(x) = |x|$ for all $x \in \mathbb{R}$ is :
(a) $(0, \infty)$ (b) $[0, \infty)$
(c) $(-\infty, \infty)$ (d) $[1, \infty)$

11. $\frac{d}{dx}\left(\frac{1}{x}\right)$ is equal to :
(a) $-\frac{1}{x^2}$ (b) $\frac{-1}{x}$ (c) $\log x$ (d) $\frac{1}{x^2}$

12. For the cost function $C = \frac{1}{25}e^{5x}$ the marginal cost is :
(a) $\frac{1}{25}$ (b) $\frac{1}{5}e^{5x}$ (c) $\frac{1}{125}e^{5x}$ (d) $25\ e^{5x}$

13. The demand function is always :
(a) Increasing function (b) Decreasing function
(c) Non decreasing function (d) undefined function

14. The % of income on 7% stock at ₹80 is :
(a) 9% (b) 8.75% (c) 8% (d) 7%

15. The correlation co-efficient lies between :
(a) 0 to ∞ (b) –1 to + 1
(c) –1 to 0 (d) –1 to ∞

16. If $Q_1 = 30$ and $Q_3 = 50$, then the co-efficient of quartile deviation is ________.
(a) 20 (b) 40 (c) 10 (d) 0.25

17. Let a sample space of an experiment be $S = \{E_1, E_2, ..., E_n\}$ then $\sum_{i=1}^{n} P(E_i)$ is equal to
(a) 0 (b) 1 (c) $\frac{1}{2}$ (d) $\frac{1}{3}$

18. The correlation co-efficient is :
(a) $r(X, Y) = \frac{\sigma x \sigma y}{\text{cov}(x, y)}$ (b) $r(X, Y) = \frac{\text{cov}(x, y)}{\sigma x \sigma y}$
(c) $r(X, Y) = \frac{\text{cov}(x, y)}{\sigma y}$ (d) $r(X, Y) = \frac{\text{cov}(x, y)}{\sigma x}$

19. The correlation co-efficient from the following data $N = 25$, $\Sigma X = 125$, $\Sigma Y = 100$, $\Sigma X^2 = 650$, $\Sigma Y^2 = 436$, $\Sigma XY = 520$ is :
(a) 0.667 (b) – 0.006 (c) – 0.667 (d) 0.70

20. In a network while numbering the events which one of the following statement is false ?

(a) Event numbers should be unique

(b) Event numbering should be carried out on a sequential basis from left to right

(c) The initial event is numbered 0 or 1.

(d) The head of an arrow should always bear a number lesser than the one assigned at the tail of the arrow.

PART - II

Note : Answer **any seven** questions. Question No. **30** is **compulsory.** **7 × 2 = 14**

21. Evaluate $\begin{vmatrix} x & x+1 \\ x-1 & x \end{vmatrix}$.

22. How many five digit telephone numbers can be constructed using the digits 0 to 9 if each number starts with 67 with no digit appear more than once?

23. Find the acute angle between the lines $2x - y + 3 = 0$ and $x + y + 2 = 0$.

24. Find the value of cosec 1125°.

25. If $f(x) = \frac{x+1}{x-1}$, $x \neq 1$ then prove that $f(f(x)) = x$

26. For the given demand function $p = 40 - x$, find the output when $\eta_d = 1$.

27. Which is better investment ? 7% of ₹100 shares at ₹120 (or) 8% of ₹100 shares at ₹135.

28. From the following data calculate the correlation co-efficient $\Sigma xy = 120$, $\Sigma x^2 = 90$. $\Sigma y^2 = 640$.

29. Draw the network for the project whose activities with their relationships are given below :

Activities A, D, E can start simultaneously :

B, C > A ; G, F > D, C ; H > E, F.

30. A bag contains 5 white and 4 black balls. Two balls are drawn at random one after the other without replacement. Find the probability that both balls drawn are black.

PART - III

Note : Answer **any seven** questions. Question no. **40** is **compulsory**. **7 × 3 = 21**

31. Prove that $\begin{vmatrix} \frac{1}{a} & bc & b+c \\ \frac{1}{b} & ca & c+a \\ \frac{1}{c} & ab & a+b \end{vmatrix} = 0$.

32. If $4\,(nC_2) = (n+2)C_3$, then find n.

33. Determine whether the points P(1, 0), Q (2, 1) and R (2, 3) lie outside the circle, on the circle or inside the circle $x^2 + y^2 - 4x - 6y + 9 = 0$.

34. Prove that

$$\frac{\sin(180° - \theta)\cos(90° + \theta)\tan(270° - \theta)\tan(270° - \theta)\cot(360° - \theta)}{\sin(360° - \theta)\cos(360° + \theta)\sin(270° - \theta)\text{cosec}(-\theta)} = -1.$$

35. Find $\frac{dy}{dx}$ of the following function $x = a\,[\theta - \sin\theta]$, $y = a\,[1 - \cos\theta]$.

36. The demand and cost functions of a firm are $x = 6000 - 30p$ and $C = 72000 + 60x$ respectively. Find the level of output and price at which the profit is maximum.

37. A person buys 20 shares of par value of ₹10 of a company which pays 9% dividend at such a price that he gets 12% on his money. Find the market value of a share.

38. A die is thrown. Find the probability of getting.

(i) a prime number.

(ii) a number greater than or equal to 3.

39. Calculate the correlation co-efficient for the following data.

X	5	10	5	11	12	4	3	2	7	1
Y	1	6	2	8	5	1	4	6	5	2

40. Differentiate $x^2 \log x$ with respect to x.

PART - IV

Note : Answer **all** the questions. **7 × 5 = 35**

41. (a) Sundar bought for ₹4,500, 12% of ₹10 shares at par. He sold them when the price rose to ₹23 and invested the proceeds in ₹25 shares paying 10% per annum at ₹18. Find the change in his income.

(OR)

(b) Show that the point (7, –5) lies on the circle $x^2 + y^2 - 6x + 4y - 12 = 0$ and find the co-ordinates of the other end of the diameter through this point.

42. (a) In an economy there are two industries P_1 and P_2 and the following table gives the supply and the demand position in crores of rupees.

Production Sector	Consumption Sector		Final Demand	Gross Output
	P_1	P_2		
P_1	10	25	15	50
P_2	20	30	10	60

Determine the outputs when the final demand changes to 35 for P_1 and 42 for P_2.

(OR)

(b) Ten competitors in a beauty contest are ranked by three judges in the following order.

First Judge	1	4	6	3	2	9	7	8	10	5
Second Judge	2	6	5	4	7	10	9	3	8	1
Third Judge	3	7	4	5	10	8	9	2	6	1

Use the method of rank correlation co-efficient to determine which pair of judges has the nearest approach to common taste in beauty.

43. (a) By the principle of Mathematical Induction, prove that $1+4+7+....+(3n-2)=\frac{n(3n-1)}{2}$ for all $n\in\mathbb{N}$.

(OR)

(b) Examine the following function for continuity at indicated points $f(x)=\begin{cases}\frac{x^2-4}{x-2}, & \text{if } x\neq 2\\ 0, & \text{if } x=2\end{cases}$ at $x=2$.

44. (a) How many numbers lesser than 1000 can be formed using the digits 5, 6, 7, 8 and 9 if no digit are repeated?

(OR)

(b) If tan A – tan B = x and cot B – cot A = y prove that cot (A – B) = $\frac{1}{x}+\frac{1}{y}$.

45. (a) Show that the function $f|x|$ is not differentiable at $x=0$.

(OR)

(b) Probability of solving specific problem independently by A and B are $\frac{1}{2}$ and $\frac{1}{3}$ respectively. If both try to solve the problem independently. Find the probability that the problem is :

(i) solved.

(ii) exactly one of them solves the problem.

46. (a) The following table gives the activities of a project and their duration in days.

Activity	1 - 2	1 - 3	2 - 3	2 - 4	3 - 4	3- 5	4 - 5
Duration	5	8	6	7	5	4	8

Construct the network and calculate the Earliest Start Time (EST), Earliest Finish Time (EFT), Latest Start Time (LST) and Latest Finish Time (LFT) of each activity and determine the critical path of the project and duration to complete the project.

(OR)

(b) For the demand function $p=550-3x-6x^2$ where x is quantity demand and P is unit price. Show that $MR=p\left[1-\frac{1}{\eta d}\right]$

47. (a) The total cost function of a firm is $C(x)=\frac{x^3}{3}-5x^2+28x+10$ where x is the output. A tax at the rate of ₹ 2 per unit of output is imposed and the producer adds it to his cost. If the market demand function is given by $p=2530-5x$ where p is the price per unit of output, find the profit maximizing the output and price.

(OR)

(b) Compute the mean deviation about mean from the following data.

Class Interval	0 – 5	5 –10	10 – 15	15 – 20	20 – 25
Frequency f	3	5	12	6	4

❋ ❋ ❋ ❋ ❋

ANSWERS

PART - I

1. (c) $\frac{30}{7}\begin{bmatrix}\frac{1}{2} & \frac{5}{12}\\ \frac{2}{5} & \frac{4}{5}\end{bmatrix}$
2. (c) Prof. Wassily W. Leontief
3. (d) 24
4. (a) 18
5. (d) (1, –1)
6. (c) ± 4
7. (b) $\frac{\sqrt{3}-1}{2\sqrt{2}}$
8. (d) $\frac{-1}{2}$
9. (a) 0
10. (b) $[0,\infty)$
11. (a) $-\frac{1}{x^2}$
12. (b) $\frac{1}{5}e^{5x}$
13. (b) Decreasing function
14. (b) 8.75%
15. (b) –1 to + 1
16. (d) 0.25
17. (b) 1
18. (b) $r(X,Y)=\frac{\text{cov}(x,y)}{\sigma x\sigma y}$
19. (a) 0.667
20. (d) The head of an arrow should always bear a number lesser than the one assigned at the tail of the arrow.

PART - II

21. $\begin{vmatrix}x & x+1\\ x-1 & x\end{vmatrix}=x^2-(x+1)(x-1)$

$=x^2-(x^2-1)=x^2-x^2+1=1$

22.

6	7			

For a five digit number, the number starts with 67. So we have to fill the unit place, tens place hundreds place.

- Unit place can be filled up in 8 ways using numbers 0, 1, 2, 3, 4, 5, 8, 9
- Tens place can be filled up in 7 ways since repetition is not allowed.
- Hundreds place can be filled up in 6 ways.

∴ By fundamental principle of multiplication, the total number of 5 digits telephone numbers.