
BASED ON THE UPDATED NEW TEXTBOOK
Limited stock Only
ENGLISH மற்றும் தமிழ் மீடியம்
SURA'S
11th std
School Guides
100% SUCCESS
அனைத்து புத்தகக் கடைகளிலும் கிடைக்கிறது
100/100
Available on
2023-24 பதிப்பு
orders@surabooks.com
call @ 8124201000 | 8124301000
9600175757 / 8056294222 / 7871802000

SURA'S

Mathematics

11th Standard

VOLUME - I & II

FREE Practice Workbook

Based on the Updated New Textbook

Salient Features

- Complete solutions to Textbook Exercises.
- Exhaustive Additional Questions & Answers in all chapters.
- Govt. Model Question Paper-2018, First Mid-Term Test - 2018, Quarterly Exam - 2018 & 2019, Half yearly Exam - 2018 & 2019, Public Exam March - 2019, 2020 & May - 2022 Instant Supplementary Exam June - 2019 & August - 2022 (June - 2019 & Aug. - 2022), Govt. Supplementary Exam September 2020 & 2021, Common Revision Test (CRT - 2022) Questions are incorporated in the appropriate sections.
- Instant Supplementary Exam August - 2022 Question Paper is given with answers.

SURA PUBLICATIONS
Chennai

2023-24 *Edition*

ISBN : 978-93-5330-538-3
Code No : SG 262

Author :
Mr.H.S.Raj *M.Sc.,B.Ed.,*
Chennai.

NOTE FROM PUBLISHER

It gives me great pride and pleasure in bringing to you **Sura's Mathematics Guide Volume I and Volume II** for 11th Standard. A deep understanding of the text and exercises is rudimentary to have an insight into Mathematics. The students and teachers have to carefully understand the topics and exercises.

Sura's Mathematics 11th Standard Guide encompasses all the requirements of the students to comprehend the text and the evaluation of the textbook.

- It will be a teaching companion to teachers and a learning companion to students.
- Exhaustive additional Questions and Answers have been given to help students practice and learn effectively all the sections of the textbook.
- In order to learn effectively, I advise students to learn the subject sectionwise and practice the exercises given.

Though these salient features are available in our Sura's Mathematics Guide 11th Standard, I cannot negate the indispensable role of the teachers in assisting the student to understand the Mathematics.

I sincerely believe this guide satisfies the needs of the students and bolsters the teaching methodologies of the teachers.

I pray the almighty to bless the students for consummate success in their examinations.

Subash Raj, B.E., M.S.
- Publisher
Sura Publications

All the Best

Head Office:
Sura Publications
1620, 'J' Block, 16th Main Road,
Anna Nagar, **Chennai - 600 040.**
Phones : 044 - 4862 9977, 044 - 4862 7755.
e-mail : orders@surabooks.com
website : www.surabooks.com

CONTENTS

Volume - I

Volume - II

01 SETS, RELATIONS AND FUNCTIONS

MUST KNOW DEFINITIONS

A set is a collection of well defined objects.

Type of sets

Empty set	:	A set containing no element.
Finite set	:	The number of elements in the set is finite.
Infinite set	:	The number of elements in the set is not finite.
Singleton set	:	A set containing only one element.
Equivalent set	:	Two sets having same number of elements.
Equal sets	:	Two sets exactly having the same elements.
Subset	:	A set X is a subset of Y if every element of X is also an element of Y. $(X \subseteq Y)$
Proper subset	:	X is a proper subset of Y if $X \subset Y$ and $X \neq Y$.
Power set	:	The set of all subsets of A is the power set of A.
Universal set	:	The set contains all the elements under consideration

Algebra of sets

Union	:	The union of two sets A and B is the set of elements which are either in A or in B $(A \cup B)$
Intersection	:	The intersection of two sets A and B is the set of all elements common to both A and B $(A \cap B)$.
Complement of a set	:	The complement of a set is the set of all elements of U (Universal set) that are not elements of A. (A′) Set different(A\B) or (A – B)
Difference of two sets	:	The difference of the two sets A and B is the set of all elements belonging to A but not to B. Set different(A\B) or (A – B)
Disjoint sets	:	Two sets A and B are said to be disjoint if there is no element common to both A and B.
Open interval	:	The set $\{x: a < x < b\}$ is called an open interval and denoted by (a, b)
Closed interval	:	The set $\{x: a \leq x \leq b\}$ is called a closed interval and denoted by $[a, b]$
Neighbourhood of a point	:	Let a be any real number. Let $\in > 0$ be arbitrarily small real number. Then $(a - \in, a + \in)$ is called an "$\in$" neighbourhood of the point a and denoted by $N_{a,\in}$

Cartesian product of sets	:	The set of all ordered pairs (a, b) such that $a \in$ A and $b \in$ B is called the cartesian product of A and B and is denoted by A × B.

Types of relation

Reflexive	:	A relation R on a set A is said to be reflexive if every element of A is related to itself.
Symmetric	:	A relation R on a set A is said to be symmetric if $(a, b) \in$ R $\Rightarrow (b, a) \in$ R for all $a, b \in$ A.
Transitive	:	A relation R on a set A is said to be transitive if $(a, b) \in$ R and $(b, c) \in$ R $\Rightarrow (a, c) \in$ R for all $a, b, c \in$ A.
Equivalent	:	A relation R on a set A is said to be equivalence relation if it is reflexive, symmetric and transitive.
Function	:	A function f from a set A to a set B is a rule which assigns to each element of A, a unique element of B. If f: A → B, then A is the domain, B is the co-domain.

Types of algebraic functions

Identity function	:	A function that associates each real number to itself.
Absolute value function	:	The function $f(x)$ defined by $f(x) = \|x\| = \begin{cases} x, & x \geq 0 \\ -x, & x < 0 \end{cases}$
Constant function	:	A function $f(x)$ defined by $f(x) = k$ where k is a real number.
Greatest integer function	:	The greatest integer function $f: \mathbb{R} \to \mathbb{R}$ defined by $f(x) = \lfloor x \rfloor$ for all $x \in$ R.
Smallest integer function	:	The smallest integer function $f: \mathbb{R} \to \mathbb{R}$ defined by $f(x) = \lceil x \rceil$ for all $x \in$ R.
Signum function	:	The function f defined by $f(x) = \begin{cases} \frac{\|x\|}{x}, & x \neq 0 \\ 0, & x = 0 \end{cases}$
Polynomial function	:	The function $f: \mathbb{R} \to \mathbb{R}$ defined by $f(x) = a_0x^n + a_1x^{n-1} + a_2x^{n-2} + \ldots + a_n$ where $a_0, a_1, \ldots, a_n$ are constants.
Rational function	:	The function $f: \mathbb{R} \to \mathbb{R}$ defined by $f(x) = \frac{p(x)}{q(x)}$, $q(x) \neq 0$ and $p(x), q(x)$ are polynomial.

Algebra of functions

Addition	:	If f: $D_1 \to$ R and g: $D_2 \to$ R, then their sum $f + g$: $D_1 \cap D_2 \to$ R such that $(f + g)(x) = f(x)\ \ g(x)$ for all $x \in D_1 \cap D_2$.
Subtraction	:	If f_1: $D_1 \to$ R and g: $D_2 \to$ R, then their difference $f - g$: $D_1 \cap D_2 \to$ R such that $(f - g)(x) = f(x) - g(x)$ for all $x \in D_1 \cap D_2$.
Product	:	If f_1: $D_1 \to$ R and g: $D_2 \to$ R, then their product $f \cdot g$: $D_1 \cap D_2 \to$ R such that $(fg)(x) = f(x) \,.\, g(x)$ for all $x \in D_1 \cap D_2$.
Quotient	:	If f_1: $D_1 \to$ R and g: $D_2 \to$ R, then their quotient $\frac{f}{g}: D_1 \cap D_2 - \{x : g(x) = 0\} \to$ R such that $\left(\frac{f}{g}\right)(x) = \frac{f(x)}{g(x)}$ such that for all $x \in D_1 \cap D_2 - \{x : g(x) = 0\}$.
Composition of functions	:	If f: A → B and g: B → C then gof : A → C defined by gof$(x) = g[f(x)]$ for all $x \in$ A.

Kinds of functions

One-one	:	A function $f: A \to B$ is said to be a one-one function (injection) if different elements of A have different images in B.
Onto	:	A function $f: A \to B$ is said to be an onto (surjection) function if every element of B is the image of some element of A.
Bijection	:	A function $f: A \to B$ is a bijection if one-one as well as onto.
Inverse of a function	:	Let $f: A \to B$ be a bijection. Then $g: B \to A$ which associates each element $y \in B$ to a unique element $x \in A$ such that $g(y) = x$ is called the inverse of f, and it denoted as f'.

Formulae to remember

Demorgan's laws	:	1. $(A \cup B)' = A' \cap B'$ 2. $(A \cap B) = A' \cup B'$ 3. $A\backslash(B \cup C) = (A\backslash B) \cap (A\backslash C)$ 4. $A\backslash(B \cap C) = (A\backslash B) \cup (A\backslash C)$
Reflexive	:	aRa for all $a \in A$
Symmetric	:	$aRb \Rightarrow bRa$ for all $a, b \in A$
Transitive	:	$aRb, bRc \Rightarrow aRc$ for all $a, b, c \in A$
Antisymmetric	:	aRb and $bRa \Rightarrow a = b$ for all $a, b \in A$ $A \Delta B = (A\backslash B) \cup (B\backslash A)$
One-one function	:	If f: $A \to A$ then $f(x_1) = f(x_2) \Rightarrow x_1 = x_2$ for all $x_1, x_2 \in A$
Onto function	:	Co-domain = Range.
		If a set has n elements, then total number of subsets is 2^n.

TEXTUAL QUESTIONS

Exercise 1.1

1. Write the following in roster form.

(i) $\{x \in \mathbb{N} : x^2 < 121$ **and** x **is a prime}.**

(ii) **the set of all positive roots of the equation** $(x-1)(x+1)(x^2-1) = 0$.

(iii) $\{x \in \mathbb{N} : 4x + 9 < 52\}$.

(iv) $\left\{x : \frac{x-4}{x+2} = 3, x \in \mathbb{R} - \{-2\}\right\}$

Sol : **(i)** $\{x \in \mathbb{N} : x^2 < 121$ **and** x **is a prime}.**

Let $A = \{x \in \mathbb{N}: x^2 < 121$, and x is a prime$\}$

$A = \{2, 3, 5, 7\}$.

(ii) the set of all positive roots of the equation $(x-1)(x+1)(x^2-1) = 0$.

Let B = {the set of positive roots of the equation $(x-1)(x+1)(x^2-1) = 0$}

$$(x-1)(x+1)(x-1)(x+1) = 0$$
$$(x+1)^2 (x-1)^2 = 0$$
$$(x+1)^2 = 0 \text{ or } (x-1)^2 = 0$$
$$x + 1 = 0 \text{ or } x - 1 = 0$$
$$x = -1 \text{ or } x = 1$$
$$\Rightarrow x = 1, -1$$
$$B = \{1\}.$$

(iii) $\{x \in \mathbb{N} : 4x + 9 < 52\}$.

Let $C = \{x \in \mathbb{N}: 4x + 9 < 52\}$

$\Rightarrow C = \{x \in \mathbb{N}: 4x < 52 - 9\}$

$\Rightarrow C = \{x \in \mathbb{N}: 4x < 43\}$

$\Rightarrow C = \left\{x \in \mathbb{N}: x < \frac{43}{4}\right\}$

$\Rightarrow C = \{x \in \mathbb{N}: x < 10.75\}$

$\Rightarrow C = \{1, 2, 3, 4, 5, 6, 7, 8, 9, 10\}$.

(iv) $\left\{x : \frac{x-4}{x+2} = 3, x \in \mathbb{R} - \{-2\}\right\}$.

Let $D = \left\{x : \frac{x-4}{x+2} = 3, x \in \mathbb{R} - \{-2\}\right\}$

$\Rightarrow D = \{x: x - 4 = 3x + 6, x \in \mathbb{R}\}$

$\Rightarrow D = \{x: -4 - 6 = 3x - x, x \in \mathbb{R}\}$

$\Rightarrow D = \{x: 2x = -10, x \in \mathbb{R}\}$

$\Rightarrow D = \{x: x = -5, x \in \mathbb{R}\}$

$\Rightarrow D = \{-5\}$

2. Write the set {–1, 1} in set builder form.

Sol : Let $P = \{-1, 1\}$

$\Rightarrow P = \{x : x \text{ is a root of } x^2 - 1 = 0\}$

$\Rightarrow P = \{x : x^2 - 1 = 0, x \in \mathbb{R}\}$

3. State whether the following sets are finite or infinite.

(i) $\{x \in \mathbb{N} : x$ **is an even prime number}**

(ii) $\{x \in \mathbb{N} : x$ **is an odd prime number}**

(iii) $\{x \in \mathbb{Z} : x$ **is even and less than 10}**

(iv) $\{x \in \mathbb{R} : x$ **is a rational number}**

(v) $\{x \in \mathbb{N} : x$ **is a rational number}**

Sol : (i) $\{x \in \mathbb{N} : x$ **is an even prime number}**

Let A = $\{x \in \mathbb{N}: x$ is an even prime number}

$\Rightarrow$ A = {2} $\Rightarrow$ A is a finite set.

(ii) $\{x \in \mathbb{N} : x$ **is an odd prime number}**

Let B = $\{x \in \mathbb{N}: x$ is an odd prime number}

$\Rightarrow$ B = {1, 3, 5, 7, 11, ...}

$\Rightarrow$ B is an infinite set.

(iii) $\{x \in \mathbb{Z} : x$ **is even and less than 10}**

Let C = $\{x \in \mathbb{Z}: x$ is even and < 10}

$\Rightarrow$ C = {...–4, –2, 0, 2, 4, 6, 8}. C is a infinite set.

(iv) $\{x \in \mathbb{R} : x$ **is a rational number}**

Let D = $\{x \in \mathbb{R}: x$ is a rational number}

$\Rightarrow$ D = {set of all rational number}

$\Rightarrow$ D is an infinite set.

(v) $\{x \in \mathbb{N} : x$ **is a rational number}**

Let $\mathbb{N}$ = $\{x \in \mathbb{N}: x$ is a rational number}

$$\Rightarrow \mathbb{N} = \left\{\frac{1}{1}, \frac{2}{1}, \frac{3}{1}, \frac{4}{1}, ...\infty\right\}$$

$\Rightarrow \mathbb{N}$ is an infinite set.

4. By taking suitable sets A, B, C, verify the following results:

(i) $A \times (B \cap C) = (A \times B) \cap (A \times C)$

(ii) $A \times (B \cup C) = (A \times B) \cup (A \times C)$

(iii) $(A \times B) \cap (B \times A) = (A \cap B) \times (B \cap A)$

(iv) $C - (B - A) = (C \cap A) \cup (C \cap B')$

(v) $(B - A) \cap C = (B \cap C) - A = B \cap (C - A)$

(vi) $(B - A) \cup C = (B \cup C) - (A - C)$

Sol : (i) $A \times (B \cap C) = (A \times B) \cap (A \times C)$

Let A = {1, 2, 3}, B = (4, 5, 6, 7}

C = {3,4,5,9}

and U = {1, 2, 3, 4, 5, 6, 7, 9}

LHS = A × (B ∩ C)

= A × {4, 5} [∵ B ∩ C = {4,5}]

= {1, 2, 3} × {4, 5}

= {(1, 4) (1, 5) (2, 4) (2, 5) (3, 4) (3, 5)} ...(1)

A × B = {1, 2, 3} × {4, 5, 6, 7}

= {(1, 4) (1, 5) (1, 6) (1, 7) (2, 4) (2, 5) (2, 6) (2, 7) (3, 4) (3, 5) (3, 6) (3, 7)}

A × C = {1, 2, 3} × {3, 4, 5, 9}

= {(1, 3) (1, 4) (1, 5) (1, 9) (2, 3) (2, 4) (2, 5) (2, 9) (3, 3) (3, 4) (3, 5) (3, 9)}

RHS = (A × B) ∩ (A × C)

= {(1, 4) (1, 5) (2, 4) (2, 5) (3, 4) (3, 5)} ...(2)

From (1) and (2), LHS = RHS. Hence verified.

(ii) $A \times (B \cup C) = (A \times B) \cup (A \times C)$

(B ∪ C) = {3, 4, 5, 6, 7, 9}

Now, A × (B ∪ C) = {1, 2, 3} × {3, 4, 5, 6, 7, 9}

= {(1, 3) (1, 4) (1, 5) (1, 6) (1, 7) (1, 9) (2, 3) (2, 4) (2, 5) (2, 6) (2, 7) (2, 9) (3, 3) (3, 4) (3, 5) (3, 6) (3, 7) (3, 9)} ...(1)

Now A × B = {1, 2, 3} × {4, 5, 6, 7}

= {(1, 4) (1, 5) (1, 6) (1, 7) (2, 4) (2, 5) (2, 6) (2, 7) (3, 4) (3, 5) (3, 6) (3, 7)}

A × C = {1, 2, 3} × {3, 4, 5, 9}

= {(1, 3) (1, 4) (1, 5) (1, 9) (2, 3) (2, 4) (2, 5) (2, 9) (3, 3) (3, 4) (3, 5) (3, 9)}

RHS (A × B) ∪ (A × C)

= {(1, 3) (1, 4) (1, 5) (1, 6) (1, 7) (1, 9) (2, 3) (2, 4) (2, 5) (2, 6) (2, 7) (2, 9) (3, 3) (3, 4) (3, 5) (3, 6) (3, 7) (3,9)} ...(2)

From (1) & (2), LHS = RHS Hence verified

(iii) $(A \times B) \cap (B \times A) = (A \cap B) \times (B \cap A)$

(A × B) = {(1, 4) (1, 5) (1, 6) (1, 7) (2, 4) (2, 5) (2, 6) (2, 7) (3, 4) (3, 5) (3, 6) (3, 7)}

(B × A) = {(4, 1) (4, 2) (4, 3) (5, 1) (5, 2) (5, 3) (6, 1) (6, 2) (6, 3) (7, 1) (7, 2) (7, 3)}

LHS = (A × B) ∩ (B × A) = { } ...(1)

(A ∩ B) = { }, (B ∩ A) = { }

∴ RHS = (A ∩ B) × (B ∩ A) = { } ...(2)

From (1) and (2), LHS = RHS

(iv) $C - (B - A) = (C \cap A) \cup (C \cap B')$

B – A = {4, 5, 6, 7}

LHS = C – (B – A) = {3, 9} ...(1)

C ∩ A = {3}

B′ = {1, 2, 3, 9}

C ∩ B′ = {3, 9}

RHS = (C ∩ A) ∪ (C ∩ B′)

= {3, 9} ...(2)

From (1) and (2), LHS = RHS

(v) $(B - A) \cap C = (B \cap C) - A = B \cap (C - A)$

B – A = {4, 5, 6, 7}

(B – A) ∩ C = {4, 5} ...(1)

B ∩ C = {4, 5}

(B ∩ C) – A = {4, 5} ...(2)

$C - A = \{4, 5, 9\}$

$B \cap (C - A) = 4, 5\}$ ...(3)

From (1), (2) and (3),

$(B - A) \cap C = (B \cap C) - A = B \cap (C - A)$.

(vi) $(B - A) \cup C = (B \cup C) - (A - C)$

$B - A = \{4, 5, 6, 7\}$

$(B - A) \cup C = \{3, 4, 5, 6, 7, 9\}$ ...(1)

$B \cup C = \{3, 4, 5, 6, 7, 9\}$

$A - C = \{1, 2\}$

$(B \cup C) - (A - C) = \{3, 4, 5, 6, 7, 9\}$ ...(2)

From (1) and (2), $(B - A) \cup C = (B \cup C) - (A - C)$

Hence verified.

5. Justify the trueness of the statement "*An element of a set can never be a subset of itself*".

Sol : Let $P = \{a, b, c, d\}$.

Each and every element of the set P can be a subset of the set itself

Eg : $\{a\}, \{b\}, \{c\}, \{d\}$.

Hence, the given statement is not true.

6. If $n(p(A)) = 1024$, $n(A \cup B) = 15$ and $n(P(B)) = 32$, then find $n(A \cap B)$.

Sol : Given $n(P(A)) = 1024 = 2^{10}$ [$\because$ If $n(A) = n$, then $n(P(A)) = 2^n$]

$\Rightarrow n(A) = 10$

$n(P(B)) = 32 = 2^5$

$\Rightarrow n(B) = 5$.

We know that,

$n(A \cup B) = n(A) + n(B) - n(A \cap B)$

$\Rightarrow 15 = 10 + 5 - n(A \cap B)$

$\Rightarrow n(A \cap B) = 0$.

7. If $n(A \cap B) = 3$ and $n(A \cup B) = 10$, then find $n(P(A \Delta B))$ [Qy. - 2018; CRT - 2022]

Sol : We know that $n(A \cup B) = n(A - B) + n(B - A) + n(A \cap B)$ if A and B are not disjoint.

$\Rightarrow n(A - B) + n(B - A) = n(A \cup B) - n(A \cap B)$

$\Rightarrow n(A \Delta B) = 10 - 3$

$\Rightarrow \therefore n(A \Delta B) = 7$

$\therefore n[P(A \Delta B)] = 2^7 = 128$.

8. For a set A, A × A contains 16 elements and two of its elements are (1, 3) and (0, 2). Find the elements of A.

Sol : Since A × A contains 16 elements, then A must have 4 elements

$\Rightarrow n(A) = 4$.

The elements of A × A are (1, 3) and (0, 2)

$\therefore$ The possibilities of elements of A are $\{0, 1, 2, 3\}$

9. Let A and B be two sets such that $n(A) = 3$ and $n(B) = 2$. If $(x, 1) (y, 2) (z, 1)$ are in A × B, find A and B, where x, y, z are distinct elements. [Hy. - 2018]

Sol : Given $A \times B = \{(x, 1) (y, 2) (z, 1)\}$

Since $n(A) = 3$ and $n(B) = 2$,

A × B will have 6 elements.

The remaining elements of A × B will be $(x, 2) (y, 1) (z, 2)$

$\therefore A \times B = \{(x, 1) (y, 2) (z, 1) (x, 2)(y, 1) (z, 2)\}$

$\therefore A = \{x, y, z\}$ and $B = \{1, 2\}$

10. If A × A has 16 elements, $S = \{(a, b) \in A \times A: a < b\}$; (–1, 2) and (0, 1) are two elements of S, then find the remaining elements of S. [Qy. - 2018; June - 2019]

Sol : $n(A \times A) = 16 \Rightarrow n(A) = 4$.

Given $S = \{(a, b) \in A \times A: a < b\}$

$\therefore A = \{-1, 0, 1, 2\}$.

$A \times A = \{(-1, -1) (-1, 0) (-1, 1) (-1, 2)(0, -1),(0,0) (0,1) (0,2) (1, -1) (1, 0)(1, 1) (1, 2) (2, -1)(2, 0) (2, 1) (2, 2)\}$

Now, $S = \{(-1, 0) (-1, 1) (-1, 2) (0, 1) (0, 2) (1,2)\}$

$\therefore$ The remaining elements of S are (–1, 0) (–1, 1) (0, 2) (1,2)

EXERCISE 1.2

1. Discuss the following relations for reflexivity, Symmetric and Transitive :

(i) The relation R defined on the set of all positive integers by "mRn if m divides n".

(ii) Let P denote the set of all straight lines in a plane. The relation R defined by "lRm if l is perpendicular to m". [Qy. - 2019]

(iii) Let A be the set consisting of all the members of a family. The relation R defined by "aRb if a is not a sister of b".

(iv) Let A be the set consisting of all the female members of a family. The relation R defined by "aRb if a is not a sister of b".

(v) On the set of natural numbers the relation R defined by "xRy if $x + 2y = 1$".

Sol : (i) The relation R defined on the set of all positive integers by "mRn" if m divides n".

Given relation is "mRn if m divides n".

Reflexive : mRm since m divides m for all positive integers m.

$\therefore$ R is reflexive.

Symmetric : mR$n \neq n$Rm.
m divides $n \Rightarrow$ 4 divides 2 $\neq$ 2 divides 4.
$\therefore$ R is not symmetric

Transitive : mRn and nR$p \Rightarrow m$Rp.
m divides n and n divides p then m divides p.
[For eg : 4 divides 2, 2 divides 8, then4 divides 8]
$\therefore$ R is transitive.

$\therefore$ R is reflexive, not symmetric and transitive.

(ii) Let *P* denote the set of all straight lines in a plane. The relation *R* defined by "*l*R*m* if *l* is perpendicular to *m*".

Let $l, m, n \in p$.

Reflexive : We cannot say l is perpendicular to l itself.
$\therefore l \not{R} l \Rightarrow$ R is not reflexive.

Symmetric : lR$m \Rightarrow m$Rl
l is perpendicular to m = m is perpendicular to l
$\therefore$ R is symmetric

Transitive : lRm and mR$n \neq l$Rn.
l is perpendicular to m and m is perpendicular to n.
$\Rightarrow$ l is not perpendicular to n.
$\therefore$ R is not transitive.
$\Rightarrow$ R is only symmetric.

(iii) Let A be the set consisting of all the members of a family. The relation R defined by "*a*R*b* if *a* is not a sister of *b*".

Given relation is "aRb if a is not a sister of b". and $a, b, c \in$ A.

Reflexive : aR$a \Rightarrow a$ is not a sister of a
$\therefore$ R is reflexive.

Symmetric : aR$b \neq b$Ra
a is not a sister of b but b may be a sister of a
$\therefore$ R is not symmetric.

Transitive : aRb and bR$c \neq a$Rc
a is not a sister of b and b is not a sister of c, but a may be a sister of c.
$\therefore$ R is not transitive.
$\therefore$ R is only reflexive.

(iv) Let A be the set consisting of all the female members of a family. The relation R defined by "*a*R*b* if *a* is not a sister of *b*".

Given relation is aRb if a is not a sister of b.
Let $a, b, c \in$ A.

Reflexive : aR$a \Rightarrow a$ is not a sister of a
$\therefore$ R is reflexive.

Symmetric : aR$b \Rightarrow b$Ra
a is not a sister of $b \Rightarrow b$ is not a sister of a.
$\therefore$ R is symmetric.

Transitive : aRb and bR$c \neq a$Rc
a is not a sister of b, b is not a sister of c does not imply a is not a sister of c. [Eg :Mother is not a sister of daughter, daughter is not a sister of chithi, but mother is a sister of chithi.]
$\therefore$ R is not transitive.
$\therefore$ R is reflexive, symmetric and not transitive.

(v) On the set of natural numbers, the relation R is defined by "*x*R*y* if *x* + 2*y* = 1".

The relation R is defined by xRy if $x + 2y = 1$ for $x, y \in$ N.

Reflexive : Let $x \in \mathbb{N}$
xR$x \Rightarrow x + 2x = 1$
$\Rightarrow 3x = 1 \quad \Rightarrow x = \dfrac{1}{3} \notin \mathbb{N}$
$\therefore$ R is not reflexive.

Symmetric : R is an empty relation.
$\therefore$ R is symmetric (by definition)

Transitive : R is an empty relation.
$\therefore$ R is transitive. (by definition)

$\therefore$ R is an empty set; not reflexive; symmetric; transitive.

2. Let X = {*a*, *b*, *c*, *d*}, and R = {(*a*, *a*) (*b*, *b*) (*a*, *c*)}. Write down the minimum number of ordered pairs to be included to R to make it
(i) reflexive (ii) symmetric
(iii) transitive (iv) equivalence.

Sol : Given X = $\{a, b, c, d\}$ and R = $\{(a, a)\ (b, b)\ (a, c)\}$

(i) To make the relation R reflexive we must have (c, c) and $(d, d) \in$ R
$\therefore$ Minimum number of ordered pairs to be included to R to make it reflexive is (c, c) and (d, d)

(ii) To make R symmetric, we must have $(c, a) \in$ R
$\therefore$ Minimum number of ordered pairs to be included to R to make it symmetric is (c, a).

(iii) R is transitive.
$\therefore$ Nothing need to be included.

(iv) Minimum number of ordered pairs to be included to make R equivalence is (c, c) (d, d) (c, a).

3. Let A = {a, b, c}, and R = {(a, a) (b, b) (a, c)}. Write down the minimum number of ordered pairs to be included to R to make it

(i) reflexive (ii) symmetric
(iii) transitive (iv) equivalence.

Sol : (i) The ordered pairs (c, c) should be included to R to make it reflexive.
$\therefore$ Minimum number of ordered pair is (c, c)

(ii) The ordered pairs (c,a) should be included to R to make it symmetric.
$\therefore$ Minimum number of ordered pair is (c, a).

(iii) The relation is transitive.
$\therefore$ Nothing needs to included.

(iv) The ordered pairs (c, c) and (c, a) should be included to R to make it equivalence.
$\therefore$ Minimum number of ordered pairs are (c, c) and (c, a).

4. Let P be the set of all triangles in a plane and R be the relation defined on P as aRb if a is similar to b. Prove that R is an equivalence relation.

Sol : Let P be the set of all triangles in a plane and R is defined as aRb if a is similar to b.
Let $a, b, c \in$ P.

Reflexive : aR$a \Rightarrow a$ is similar to a for all $a \in$ P.
$\therefore$ R is reflexive

Symmetric : aR$b \Rightarrow b$Ra
a is similar to b
$\Rightarrow b$ is similar to a for all $a, b \in$ P.

Transitive : aRb, and bR$c \Rightarrow a$Rc.
a is similar to b and b is similar to c
$\Rightarrow a$ is similar to c.

Hence R is an equivalence relation.

5. On the set of natural numbers let R be the relation defined by aRb if $2a + 3b = 30$. Write down the relation by listing all the pairs. Check whether it is

(i) reflexive (ii) symmetric
(iii) transitive (iv) equivalence.

Sol : Given relation is $2a + 3b = 30$ for all $a, b \in \mathbb{N}$.

$2a + 3b = 30 \Rightarrow 2a = 30 - 3b$

$$\Rightarrow \quad a = \frac{30-3b}{2}$$

a	12	9	6	3
b	2	4	6	8

$\therefore$ The list of ordered pairs are (12, 2) (9, 4) (6, 6) (3, 8)

(i) **Reflexive** : (12, 12) $\notin$ R $\Rightarrow$ R is not reflexive.

(ii) **Symmetric** : (9, 4) $\in$ R but (4, 9) $\notin \mathbb{R}$
$\therefore$ R is not symmetric

(iii) **Transitive** : Clearly R is transitive.
[$\because$ (a, b) (b, c) $\notin \mathbb{R}$]

(iv) R is not an equivalence relation.

6. Prove that the relation "friendship" is not an equivalence relation on the set of all people in Chennai.

Sol : Let a, b, c are people in Chennai

Reflexive : "a" is a friend of "a" $\Rightarrow a \not R a$.
$\Rightarrow$ R is not reflexive.

Symmetric : a is friend of $b \Rightarrow b$ is the friend of a.
$\therefore a$R$b \Rightarrow b$R$a \Rightarrow$ R is symmetric

Transitive : a is the friend of b and b is the friend of $c \Rightarrow a$ need not be the friend of c.
$\therefore a$R$b \Rightarrow b$R$c \neq a$R$c \Rightarrow$ R is not transitive

Hence, the relation "friendship" is not equivalent.

7. On the set of natural number let R be the relation defined by aRb if $a + b \leq 6$. Write down the relation by listing all the pairs. Check whether it is

(i) reflexive (ii) symmetric
(iii) transitive (iv) equivalence.

Sol : The relation is defined by aRb if $a + b \leq 6$ for all $a, b \in \mathbb{N}$. $a + b \leq 6 \Rightarrow a \leq 6 - b$

a	1	1	1	1	1	2	2	2	2	3	3	3	4	4	5
b	1	2	3	4	5	1	2	3	4	1	2	3	1	2	1

$\therefore$ The list of ordered pairs are (1, 1) (1, 2) (1, 3) (1, 4) (1, 5), (2, 1), (2, 2), (2,3), (2,4), (3,1), (3,2), (3,3), (4,1), (4,2),(5, 1)

(i) **Reflexive** : R is not reflexive since (4, 4) $\notin \mathbb{R}$.

(ii) **Symmetric** : (1, 5) $\in \mathbb{R} \Rightarrow$ (5, 1) $\in \mathbb{R}$
(2, 4) $\in \mathbb{R} \Rightarrow$ (4, 2) $\in \mathbb{R}$
(1, 2) $\in \mathbb{R} \Rightarrow$ (2, 1) $\in \mathbb{R}$
(1, 3) $\in \mathbb{R} \Rightarrow$ (3, 1) $\in \mathbb{R}$
(1, 4) $\in \mathbb{R} \Rightarrow$ (4, 1) $\in \mathbb{R}$
(2, 3) $\in \mathbb{R} \Rightarrow$ (3, 2) $\in \mathbb{R}$
$\therefore$ R is symmetric

(iii) **Transitive** : (3, 1) $\in \mathbb{R}$ and (1, 5) $\in \mathbb{R}$
$\Rightarrow$ (3, 5) $\notin$ R
$\therefore$ R is not transitive.

(iv) R is not an equivalence relation.

8. Let A = {a, b, c}. What is the equivalence relation of smallest cardinality on A? What is the equivalence relation of largest cardinality on A?

Sol : Given A = {a, b, c}

(i) Let R = {$(a, a)\ (b, b)\ (c, c)$}

R is reflexive

R is symmetric [$\because (a, b) \notin \mathbb{R} \Rightarrow (b, a) \notin \mathbb{R}$] and

R is transitive [$\because (a, b)\ (b, c) \notin \mathbb{R} \Rightarrow (a, c) \notin \mathbb{R}$]

R is an equivalence relation.

This is the equivalence relation of smallest cardinality on A.

$\therefore n$ (R) = 3

(ii) Let R = {$(a, a)\ (a, b)\ (a, c)\ (b, a)\ (b, b) (b, c) (c, a)\ (c, b)\ (c, c)$}

R is reflexive since $(a, a)\ (b, b)$ and $(c, c) \in \mathbb{R}$

R is symmetric since $(a, b) \in$ R $\Rightarrow (b, a) \in \mathbb{R}$

$(b, c) \in \mathbb{R} \Rightarrow (c, b) \in \mathbb{R}$

$(c, a) \in \mathbb{R} \Rightarrow (a, c) \in \mathbb{R}$

R is also transitive since $(a, b)\ (b, c) \in \mathbb{R}$

$\Rightarrow (a, c) \in \mathbb{R}$

Hence R is are equivalence relation of largest cardinality on A.

$\therefore n$ (R) = 9

9. In the set Z of integers, define mRn if $m - n$ is divisible by 7. Prove that R is an equivalence relation. **[Sep. - 2020]**

Sol : As $m - m = 0$,

$m - m$ is divisible by 7 $\Rightarrow m$Rm

$\therefore$ R is reflexive.

Let mRn.

Then $m - n = 7k$ for some integer k

Thus $n - m = 7(-k)$ and hence nRm

$\therefore$ R is symmetric.

Let mRn and nRp

$\Rightarrow m - n = 7k$ and $n - p = 7l$

$\Rightarrow m = 7k + n$ and

$-p = 7l - n$ for some integers k and l

so $m - p = 7k + \cancel{n} + 7l - \cancel{n}$

$\Rightarrow m - p = 7(k + l) \Rightarrow mRp$

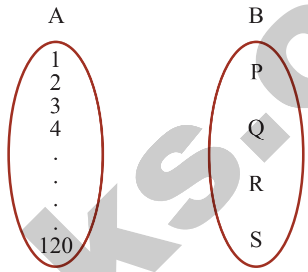
$\therefore$ R is transitive.

Thus, R is an equivalence relation.

Exercise 1.3

1. Suppose that 120 students are studying in 4 sections of eleventh standard in a school. Let A denote the set of students and B denote the set of the sections. Define a relation from A to B as "x related to y if the student x belongs to the section y". Is this relation a function? What can you say about the inverse relation? Explain your answer.

Sol : Given n(A) = 120, n(B) = 4



xRy is the student x belongs to the section y.

This relation is a function since every student of set A will be mapped on to some section in B.

$\therefore f$ is a function from A $\to$ B.

The inverse relation is f^{-1}: B $\to$ A.

The inverse relation is not a function since one section will have more than one student.

2. Write the values of f at –4, 1, –2, 7, 0 if

$$f(x) = \begin{cases} -x+4 & \text{if } -\infty < x \leq -3 \\ x+4 & \text{if } -3 < x < -2 \\ x^2 - x & \text{if } -2 \leq x < 1 \\ x - x^2 & \text{if } 1 \leq x < 7 \\ 0, & \text{otherwise} \end{cases}$$

[Aug. - 2022]

Sol : Now $f(-4) = 4 + 4 = 8$ [$\because f(x) = -x + 4$ when $x = -4$]

$f(1) = 1 - 1^2$ [$\because f(x) = x - x^2$ when $x = 1$]

$f(1) = 0$

$f(-2) = (-2)^2 - (-2) = 4 + 2 = 6$

[$\because f(x) = x^2 - x$ when $x = -2$]

$f(7) = 0$ [$\because f(x) = 0$ when $x = 7$]

$f(0) = 0^2 - 0 = 0$.[$\because f(x) = x^2 - x$ when $x = 0$]

$\therefore f(-4) = 8, f(1) = 0,$

$f(-2) = 6, f(7) = 0$ and $f(0) = 0$

3. Write the values of f at –3, 5, 2, –1, 0 if **[Hy. - 2019]**

$$f(x) = \begin{cases} x^2 + x - 5 & \text{if } x \in (-\infty, 0) \\ x^2 + 3x - 2 & \text{if } x \in (3, \infty) \\ x^2 & \text{if } x \in (0, 2) \\ x^2 - 3, & \text{Otherwise} \end{cases}$$

[First Mid - 2018; Sep. - 2021; CRT - 2022]

Sol :

$$f(-3) = (-3)^2 - 3 - 5 = 9 - 3 - 5 = 9 - 8 = 1$$
$[\because f(x) = x^2 + x - 5 \text{ when } x = -3]$

$$f(5) = 5^2 + 3(5) - 2 = 25 + 15 - 2 = 38$$
$[\because f(x) = x^2 + 3x - 2 \text{ when } x = 5]$

$$f(2) = 2^2 - 3 = 4 - 3 = 1$$
$[\because f(x) = x^2 - 3 \text{ when } x = 2]$

$$f(-1) = (-1)^2 + (-1) - 5 = \not{1} - \not{1} - 5 = -5$$
$[\because f(x) = x^2 + x - 5 \text{ when } x = -1]$

$$f(0) = 0^2 - 3 = -3$$
$[\because f(x) = x^2 - 3 \text{ when } x = 0]$

$$\therefore f(-3) = 1, f(5) = 38,$$
$$f(2) = 1, f(-1) = -5, f(0) = -3$$

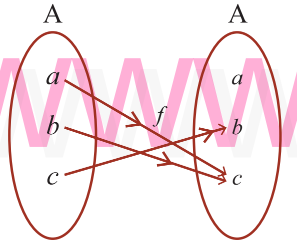
4. State whether the following relations are functions or not. If it is a function check for one-to-oneness and ontoness . If it is not a function state why?

(i) If A = $\{a, b, c\}$ and $f = \{(a, c)\ (b, c)\ (c, b)\}$: ($f : A \to A$).

(ii) If X = $\{x, y, z\}$ and $f = \{(x, y)\ (x, z)\ (z, x)\}$: ($f : X \to X$)

Sol : (i) If A = $\{a, b, c\}$ and $f = \{(a, c)\ (b, c)\ (c, b)\}$ ($f : A \to A$).

Given $f : A \to A$



This is a function. Since different elements of A does not have different images in A.

$\therefore f$ is not one-one.

Here

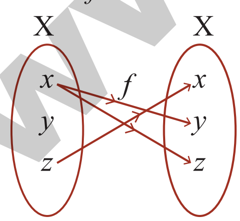
Co-domain = $\{a, b, c\}$

But Range = $\{b, c\}$

f is not onto since co-domain ≠ Range.

(ii) If X = $\{x, y, z\}$ and $f = \{(x, y)\ (x, z)\ (z, x)\}$: ($f : X \to X$)

Given $f : X \to X$



f is not a function since the element x have two images namely y and z.

5. Let A = {1, 2, 3, 4} and B = $\{a, b, c, d\}$. Give a function from A → B for each of the following :

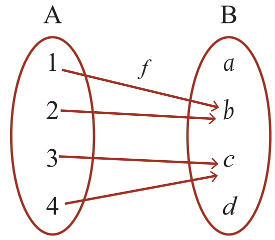
(i) neither one- to -one nor onto.

(ii) not one-to-one but onto.

(iii) one-to-one but not onto.

(iv) one-to-one and onto.

Sol : (i) neither one- to -one nor onto.



Let $f = \{(1, b)\ (2, b)\ (3, c)\ (4, c)\}$

Different elements in A does not have different images in B

$\therefore f$ is not one-one

Now, Co-domain = $\{a, b, c, d\}$,

Range = $\{b, c\}$

Co-domain ≠ range

$\therefore f$ is not onto.

Hence f is neither one-one and nor onto.

(ii) not one-to-one but onto.

Given A = {1, 2, 3, 4}, and B = $\{a, b, c, d\}$

Let f : A → B.

The function does not exist for not one-one but onto.

Since f : A → B, f is onto

$\Rightarrow f$ must be one one since $n(A) = n(B)$

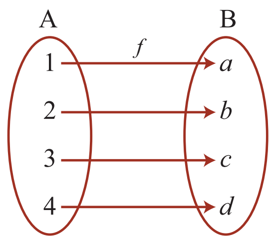
(iii) one-to-one but not onto.

The function does not exist for one-to-one but not onto.

Since f : A→ B, f is one-one $\Rightarrow f$ must be onto

$[\because n(A) = n(B)]$

(iv) one-to-one and onto.



Let f: A → B defined by

$f = \{(1, a)\ (2, b)\ (3, c)\ (4, d)\}$

Here different elements have different images

$\therefore f$ is one-to-one.

Also Co-domain = $\{a, b, c, d\}$ = Range.

$\therefore f$ is onto.

$\therefore f$ is one-to-one and onto.

6. Find the domain of $\dfrac{1}{1 - 2\sin x}$. **[Sep. 2020; Hy. - 2019]**

Sol :

$$\text{Let } f(x) = \frac{1}{1 - 2\sin x}.$$

When the denominator is 0,

$$1 - 2\sin x = 0 \Rightarrow 1 = 2\sin x$$

$$\Rightarrow \sin x = \frac{1}{2} \Rightarrow \sin x = \sin\frac{\pi}{6}$$

$$\Rightarrow x = n\pi + (-1)^n\frac{\pi}{6}, n \in \mathbb{Z}$$

$[\because \sin x = \sin\alpha \Rightarrow x = n\pi + (-1)^n\alpha, n \in \mathbb{Z}]$

Domain of $f(x)$ is $R - \left\{n\pi + (-1)^n\frac{\pi}{6}\right\}, n \in \mathbb{Z}$

7. Find the largest possible domain of the real valued function $f(x) = \frac{\sqrt{4-x^2}}{\sqrt{x^2-9}}$.

Sol : Given $f(x) = \frac{\sqrt{4-x^2}}{\sqrt{x^2-9}}$.

When $x = 2, f(x) = 0$

When $x = -2, f(x) = 0$

For all the other values, we get negative value in the square root which is not possible.

$\therefore$ Domain $= \phi$

8. Find the range of the function $\frac{1}{2\cos x - 1}$. **[Govt. MQP - 2018; June - 2019]**

Sol : Range of cosine function is $-1 \le \cos x \le 1$.

$\Rightarrow -2 \le 2\cos x \le 2$ (Multiplied by 2)

$\Rightarrow -2 - 1 \le 2\cos x - 1 \le 2 - 1$

$\Rightarrow -3 \le 2\cos x - 1 \le 1$

$$\Rightarrow \frac{-1}{3} \ge \frac{1}{2\cos x - 1} \ge \frac{1}{1} \Rightarrow \frac{-1}{3} \ge f(x) \ge 1$$

$\therefore$ Range of $f(x)$ is $\left(-\infty, -\frac{1}{3}\right] \cup [1, \infty)$

9. Show that the relation $xy = -2$ is a function for a suitable domain. Find the domain and the range of the function.

Sol : Given relation is $xy = -2$.

$$\Rightarrow x = -\frac{2}{y}$$

$$\text{Now } f(x_1) = f(x_2) \Rightarrow -\frac{2}{y_1} = -\frac{2}{y_2}$$

$$\Rightarrow \frac{1}{y_1} = \frac{1}{y_2} \Rightarrow y_1 = y_2$$

$\therefore f$ is a one-one function

The element $0 \in$ the domain will not have the image.

$\therefore$ Domain $= R - \{0\}$ and Range $= R - \{0\}$.

10. If $f, g : \mathbb{R} \to \mathbb{R}$ are defined by $f(x) = |x| + x$ and $g(x) = |x| - x$, find $g \circ f$ and $f \circ g$. **[March - 2020]**

Sol : Given $f(x) = |x| + x$

$$= \begin{cases} x + x = 2x & \text{if } x \ge 0 \\ -x + x = 0 & \text{if } x < 0 \end{cases}$$

$g(x) = |x| - x$

$$= \begin{cases} x - x = 0 & \text{if } x \ge 0 \\ -x - x = -2x & \text{if } x < 0 \end{cases}$$

Now, $fog(x) = f(g(x))$

$$= \begin{cases} f(0) & \text{if } x \ge 0 \\ f(-2x) & \text{if } x < 0 \end{cases}$$

$$\Rightarrow fog(x) = \begin{cases} 2 \times 0 = 0 & \text{if } x \ge 0 \\ 0 & \text{if } x < 0 \end{cases}$$

$\therefore fog(x) = 0$ for all $x \in \mathbb{R}$.

$$\text{and } gof(x) = g(f(x)) = \begin{cases} g(2x) & \text{if } x \ge 0 \\ g(0) & \text{if } x < 0 \end{cases}$$

$$\Rightarrow gof(x) = \begin{cases} 0 \text{ if } x \ge 0 \\ 0 \text{ if } x < 0 \end{cases}$$

$\Rightarrow gof(x) = 0$ for all $x \in \mathbb{R}$.

11. If f, g, h are real valued functions defined on $\mathbb{R}$, then prove that $(f + g)oh = foh + goh$. What can you say about $fo(g + h)$? Justify your answer.

Sol : (i) Since f, g, h are functions from $\mathbb{R} \to \mathbb{R}$, $(f + g)oh: \mathbb{R} \to \mathbb{R}$ and $foh + goh: \mathbb{R} \to \mathbb{R}$.

For any $x \in \mathbb{R}$,

$$[(f+g)oh](x) = (f+g)(h(x))$$
$$= f(h(x)) + g(h(x))$$
$$= foh(x) + goh(x)$$

$\therefore (f + g)oh = foh + goh$

(ii) Also $fo(g + h) = f[(g + h)(x)]$ for any $x \in \mathbb{R}$

$$= f[g(x) + h(x)] = f(g(x)) + f(h(x))$$
$$= fog(x) + foh(x).$$

$\therefore fo(g + h) = fog(x) + foh(x)$.

12. If $f : \mathbb{R} \to \mathbb{R}$ is defined by $f(x) = 3x - 5$, prove that f is a bijection and find its inverse. **[Govt.MQP & Qy. - 2018]**

Sol : Let $y = 3x - 5$.

$$\Rightarrow y + 5 = 3x \Rightarrow \frac{y+5}{3} = x.$$

Let $g(y) = \frac{y+5}{3}$.

$gof(x) = g(f(x)) = g(3x - 5)$

$$= \frac{3x - \cancel{5} + \cancel{5}}{3} = \frac{\cancel{3}x}{\cancel{3}} = x$$

Also $fog(y) = f(g(y)) = f\left(\frac{y+5}{3}\right)$

$$= 3\left(\frac{y+5}{3}\right) - 5 = y + 5 - 5 = y.$$

Thus $gof(x) = I_x$ and $fog(y) = I_y$.

Where I is identify function.

This implies that f and g are bijections and inverses to each other.

Hence f is a bijection and $f^{-1}(y) = \frac{y+5}{3}$.

Replacing y by x we get, $f^{-1}(x) = \frac{x+5}{3}$

13. The weight of the muscles of a man is a function of his body weight x and can be expressed as W(x) = 0.35x. Determine the domain of this function.

Sol : Given $W(x) = 0.35x$

(Note that x is positive real numbers)

$W(0) = 0, W(1) = 0.35,$

$W(2) = 7, W(\infty) = \infty$

Since x denotes the number of men, it will take only positive integers.

$\therefore W = W \to R^+$

Hence the domain is the set of whole numbers. (or) $x > 0$.

14. The distance of an object falling is a function of time t and can be expressed as $s(t) = -16t^2$. Graph the function and determine if it is one-to-one.

Sol : Given $s(t) = -16t^2$

Now, $s(t_1) = s(t_2)$

$\Rightarrow -16t_1^2 = -16t_2^2 \Rightarrow t_1^2 = t_2^2$

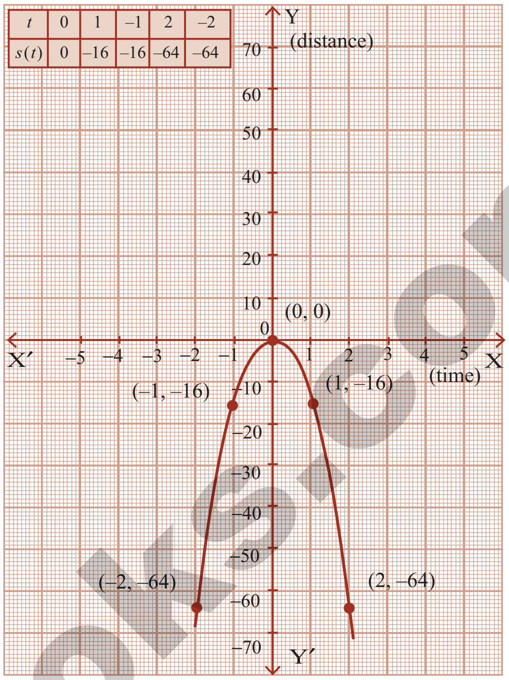
$\Rightarrow \pm t_1 = \pm t_2$

Since $s(t_1) = s(t_2) \neq t_1 = t_2$,

the function $s(t)$ is not one-one.

Graph of $s(t) = -16t^2$

Let X - axis represents the time and Y - axis represents the distance.



15. The total cost of airfare on a given route is comprised of the base cost C and the fuel surcharge S in rupee. Both C and S are functions of the mileage m; C(m) = 0.4 m + 50 and S(m) = 0.03 m. Determine a function for the total cost of a ticket in terms of the mileage and find the airfare for flying 1600 miles.

Sol : Given base cost function and fuel surcharge function are as follows:

$c(m) = 0.4\,m + 50$

and $s(m) = 0.03\,m.$

$\therefore$ Total cost of a ticket $= c(m) + s(m)$

$\therefore f(x) = 0.4\,m + 50 + 0.03\,m$

Total cost $= 0.43\,m + 50$

Given $m = 1600$ miles

Airfare for flying 1600 miles = 0.43(1600) +50

= ₹738

16. A salesperson whose annual earnings can be represented by the function A(x) = 30,000 + 0.04x, where x is the rupee value of the merchandise he sells. His son is also in sales and his earnings are represented by the function S(x) = 25,000 + 0.05x. Find (A + S) (x) and determine the total family income if they each sell ₹1,50,00,000 worth of merchandise.

Sol : Given $A(x) = 30,000 + 0.04\,x$

$S(x) = 25,000 + 0.05\,x.$

$\therefore (A + S)(x) = 30,000 + 0.04x + 25,000 + 0.05x$

$= 55,000 + 0.09x$

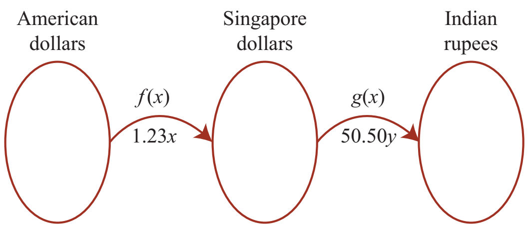
Given x = ₹1,50,00,000

Then Family income is $= 55{,}000 + 0.09\,(1{,}50{,}00{,}000)$
$= 55{,}000 + 13{,}50{,}000$
$= 14{,}05{,}000.$

Hence total family income = ₹ 14,05,000.

17. The function for exchanging American dollars for Singapore Dollar on a given day is $f(x) = 1.23x$, where x represents the number of American dollars. On the same day the function for exchanging Singapore Dollar to Indian Rupee is $g(y) = 50.50y$, where y represents the number of Singapore dollars. Write a function which will give the exchange rate of American dollars in terms of Indian rupee.

Sol : Given $f(x) = 1.23x$ where x represents the number of American dollars and $g(y) = 50.50y$ where y represents the number of Singapore dollars.



To convert American dollars to Indian rupees, we have to find out go $f(x)$

$$\therefore \text{go } f(x) = g(f(x)) = g(1.23x)$$
$$= 50.50[1.23x] = 62.115x$$

∴The function for exchange rate of American dollars in terms of Indian rupee is $gof(x) = 62.115x$.

18. The owner of a small restaurant can prepare a particular meal at a cost of Rupees 100. He estimates that if the menu price of the meal is x rupees, then the number of customers who will order that meal at that price in an evening is given by the function $D(x) = 200 - x$. Express his day revenue, total cost and profit on this meal as a function of x.

Sol : Number of customers $= 200 - x$
Cost of one meal = ₹100
Total cost $= 100\,(200 - x)$
Revenue on one meal $= x$
Total revenue $= x\,(200 - x)$
Profit = Revenue – Cost
= ₹ $x(200 - x) - 100\,(200 - x)$

19. The formula for converting from Fahrenheit to Celsius temperatures is $y = \frac{5x}{9} - \frac{160}{9}$. Find the inverse of this function and determine whether the inverse is also a function.

Sol :

$$\text{Let } f(x) = \frac{5x-160}{9}$$
$$\text{Given } y = \frac{5x}{9} - \frac{160}{9} \Rightarrow y = \frac{5x-160}{9}$$
$$\text{Then } 9y = 5x - 160$$
$$\Rightarrow 5x = 9y + 160 \Rightarrow x = \frac{9y+160}{5}$$
$$\text{Let } g(y) = \frac{9y+160}{5}.$$
$$\text{Now } gof(x) = g[f(x)] = g\left(\frac{5x-160}{9}\right)$$
$$= \frac{\cancel{9}\left(\frac{5x-160}{\cancel{9}}\right)+160}{5}$$
$$= \frac{5x - \cancel{160} + \cancel{160}}{5} = \frac{\cancel{5}x}{\cancel{5}} = x$$
$$\text{and } fog(y) = f(g(y)) = f\left(\frac{9y+160}{5}\right)$$
$$= \frac{\cancel{5}\left(\frac{9y+160}{\cancel{5}}\right)-160}{9} = \frac{9y + \cancel{160} - \cancel{160}}{9} = y$$

Thus $gof = I_x$ and $fog = I_y$.

This implies that f and g are bijections and inverses to each other.

$$\Rightarrow f^{-1}(y) = \frac{9y+160}{5}$$

Replacing y by x, we get $f^{-1}(x) = \frac{9x+160}{5} = \frac{9x}{5} + 32$

∴ The inverse function is bijection.

20. A simple cipher takes a number and codes it, using the function $f(x) = 3x - 4$. Find the inverse of this function, determine whether the inverse is also a function and verify the symmetrical property about the line $y = x$ (by drawing the lines).

Sol :

$$\text{Given } f(x) = 3x - 4$$
$$\text{Let } y = 3x - 4 \Rightarrow y + 4 = 3x$$
$$\Rightarrow x = \frac{y+4}{3}$$
$$\text{Let } g(y) = \frac{y+4}{3}.$$
$$\text{Now } gof\,(x) = g(f(x)) = g(3x-4)$$
$$= \frac{3x - \cancel{4} + \cancel{4}}{3} = \frac{\cancel{3}x}{\cancel{3}} = x$$
$$\text{and } fog\,(y) = f(g(y)) = f\left(\frac{y+4}{3}\right)$$
$$= \cancel{3}\left(\frac{y+4}{\cancel{3}}\right) - 4 = y + \cancel{4} - \cancel{4} = y$$

04 Combinatorics and Mathematical Induction

MUST KNOW DEFINITIONS

Factorial	:	The continuous product of first 'n' natural numbers is called the "n factorial" and is denoted by $n!$ or $\angle n$.
Fundamental principle of multiplication	:	If there are two jobs such that one of them can be completed in m ways, second job can be completed in n ways, then the two jobs together can be completed in mn ways.
Fundamental principle of addition	:	If there are two jobs such that they can be performed independently in m and n ways, then either of the two jobs can be performed in $(m + n)$ ways.
Permutations	:	Each of the arrangements which can be made by taking some or all of a number of things is called a permutation.
Combinations	:	Each of the different selections made by taking some or all of a number of objects, irrespective of their arrangements is called a combination.
Mathematical statements	:	Statements involving mathematical relations are known as the mathematical statements.
Mathematical Induction	:	Mathematical induction is a mathematical technique which is used to prove a statement is true for every natural number.
Formulae to remember	:	**1.** $n! = n(n-1)(n-2)\ldots 3.2.1$ **2.** $n! = n(n-1)!$ **3.** $0! = 1$ and $1! = 1$ **4.** $nPr = n(n-1)(n-2)\ldots(n-(r+1))$ **5.** $nP_r = P(n, r) = \dfrac{n!}{(n-r)!}$ **6.** The number of permutations of n distinct things, taken all at a time is $n!$

7. The number of permutations of n things, of which p_1 are alike of one kind, p_2 are alike of second kind, p_3 are alike of third kind . . . p_r are alike of r^{th} kind such that $p_1 + p_2 + \ldots p_r = n$ is $\dfrac{n!}{p_1!p_2!\ldots p_r!}$

8. Number of ways of selecting r objects from n objects is $nC_r = C(n, r) = \dfrac{n!}{(n-r)!r!}$

9. **Properties of combinations** :
(i) $nC_r = nC_{n-r}$ for $0 \le r \le n$
(ii) $nC_x = nC_y \Rightarrow x = y$ or $x + y = n$.
(iii) $nC_r + nC_{r-1} = {}^{n+1}C_r$ where n and r are non-negative.
(iv) $nC_r = \frac{n}{r} \times n-1\ C_{r-1}$, $1 \le r \le n$
(v) $n \,.\, n{-}1C_{r-1} = (n - r + 1).\ nC_{r-1}$, $1 \le r \le n$

10. **Alogarithm for mathematical induction** :
(i) Obtain $p(n)$
(ii) Prove that $p(1)$ is true
(iii) Assume that $p(m)$ is true.
(iv) Prove that $p(m + 1)$ is true.

11. **Relation between permutations and Combinations** : $nC_r = \dfrac{nP_r}{r!}$

TEXTUAL QUESTIONS

Exercise 4.1

1. (i) **A person went to a restaurant for dinner. In the menu card, the person saw 10 Indian and 7 Chinese food items. In how many ways the person can select either an Indian or a Chinese food?**

Sol : The person can select 10 Indian food in 10 ways and 7 Chinese food in 7 ways.

By fundamental principle of addition, number of ways of selecting 10 Indian or 7 Chinese food is $(10 + 7) = 17$ ways.

(ii) **There are 3 types of toy car and 2 types of toy train are available in a shop. Find the number of ways a baby can buy a toy car and a toy train?**

Sol : Number of ways of buying a toy car from 3 types of car = 3

Number of ways of buying a toy train from 2 types of train = 2.

By fundamental principle of multiplication, number of ways of buying a toy car and a toy train = $3 \times 2 = 6$ ways.

(iii) **How many two-digit numbers can be formed using 1, 2, 3, 4, 5 without repetition of digits?**

Sol :

tens	one's
4	5

The one's place can be filled up in 5 ways using 1, 2, 3, 4, 5 and tens place can be filled up in 4 ways.

Number of two digit numbers using the digits 1, 2, 3, 4, 5 is $4 \times 5 = 20$ ways.

(iv) **Three persons enter into a conference hall in which there are 10 seats. In how many ways they can take their seats?**

Sol : Number of ways of getting a seat for 1st person = 10
Number of ways of getting a seat for 2nd person = 9
Number of ways of getting a seat for 3rd person = 8
By fundamental principle of multiplication, number of ways of getting seats for 3 persons in conference hall = 10 × 9 × 8 = 720 ways.

(v) **In how many ways 5 persons can be seated in a row?**

Sol : To arrange 5 persons in a row, we need 5 place.
Number of ways of 1st person can be seated in a row = 5
Number of ways of 2nd person can be seated in a row = 4
Number of ways of 3rd person can be seated in a row = 3
Number of ways of 4th person can be seated in a row = 2
Number of ways of 5th person can be seated in a row = 1
Number of ways of 5 persons can be seated in a row = 5 × 4 × 3 × 2 × 1 = 5! = 120

2. (i) **A mobile phone has a passcode of 6 distinct digits. What is the maximum number of attempts one makes to retrieve the passcode?**

Sol : Since the pass code has 6 distinct digits, the first digit can be tried in 10 ways using 0, 1, 2, 3, 4, 5, 6, 7, 8, 9.
Similarly 2nd, 3rd, 4th, 5th and 6th digit can also be tried in 9, 8, 7, 6, 5 respectively.
Maximum number of attempts made to retrieve the pass code = 10 × 9 × 8 × 7 × 6 × 5
= 151200.

(ii) **Given four flags of different colours, how many different signals can be generated if each signal requires to use of three flags, one below the other?**

Sol : The total number of signals is equal to the number of ways of filling 3 places in succession by 4 flags of different colours.
The upper place can be filled in 4 ways, following which the next place can be filled in 3 ways and the lower place can be filled in 2 ways.
Hence, by fundamental principle of multiplication, the required number of signals = 4 × 3 × 2 = 24.

3. **Four children are running a race.**
(i) **In how many ways can the first two places be filled?**
(ii) **In how many different ways could they finish the race?**

Sol : (i) **In how many ways can the first two place be filled?**
First place can be given to any one of the 4 children and second place can be given to any one of the 3 remain children.
The first two places can be filled in 4 × 3 = 12 ways [By fundamental principle of multiplication]

(ii) **In how many different ways could they finish the race?**
The race can be finished in 4 × 3 × 2 × 1 = 4!
= 24 ways

4. **Count the number of three – digit numbers which can be formed from the digits 2, 4, 6, 8 if**
(i) **repetitions of digits is allowed?**
(ii) **repetitions of digits is not allowed?**

Sol : (i) **repetition of digits is allowed**

hundreds	tens	unit
4	4	4

The unit place can be filled in 4 ways.
Since repetition is allowed, the tens place and hundreds place can also be filled in 4 ways each.
Total number of 3 digit numbers = 4 × 4 × 4
= 64 ways

(ii) **repetition of digits is not allowed**

hundreds	tens	unit
2	3	4

The unit place can be filled in 4 ways.
Since repetition of digits is not allowed, the tens place can be filled in 3 ways.
Hundreds place can be filled in 2 ways.
Total number of 3 - digit numbers without repetition = 4 × 3 × 2 = 24 ways

5. **How many three-digit numbers are there with 3 in the unit place?**
(i) **with repetition** (ii) **without repetition**

Sol : (i) **with repetition**

hundreds	tens	unit
9	10	1

The given digits are 0, 1, 2, 3, 4, 5, 6, 7, 8, 9. The unit place can be filled in only one way using 3.

Since repetition is allowed, the tens place can be filled in 10 ways using any one of the digits from 0, 1, 2, 3, 4, 5, 6, 7, 8, 9.

The hundreds place can be filled in 9 ways using the digits 1, 2, 3, 4, 5, 6, 7, 8, 9 (excluding 0)

By fundamental principle of multiplication, total number of 3 digit numbers = 9 × 10 × 1 = 90.

(ii) without repetition

hundreds	tens	unit
8	8	1

The unit place can be filled in only one way using the digit 3.

The hundreds place can be filled in 8 ways using the digits 1, 2, 3, 4, 5, 6, 7, 8, 9 (excluding 0 and 3)

Since repetition is not allowed and tens place can be filled in 8 ways (including 0 and excluding the number used in hundred's place)

By fundamental principle of multiplication, total number of 3 digit numbers = 8 × 8 × 1 = 64

6. How many numbers are there between 100 and 500 with the digits 0, 1, 2, 3, 4, 5 ? if

(i) repetition of digits allowed

(ii) the repetition of digits is not allowed

Sol : **(i) repetition of digit is allowed**

4	6	6

Since we are going to find numbers between 100 and 500 it has 3 digits

The unit place can be filled in 6 ways using the digits 0, 1, 2, 3, 4, 5

The tens place also can be filled in 6 ways since repetition of digits is allowed.

The hundreds place can be filled in 4 ways using the digits 1, 2, 3, 4 [excluding 0 and 5]

By fundamental principle of multiplication, required number of 3-digit numbers

= 4 × 6 × 6 = 144

(ii) Repetition of digits is not allowed.

[Govt. MQP - 2018]

4	5	4

Hundreds place can be filled in 4 ways excluding 0 and 5

Tens place can be filled in 5 ways since repetition of digits is not allowed

Unit place can be filled in 4 ways.

By fundamental principle of multiplication, required number of three-digit numbers = 4 × 5 × 4 = 80.

7. How many three-digit odd numbers can be formed by using the digits 0, 1, 2, 3, 4, 5? if

(i) the repetition of digits is not allowed

(ii) the repetition of digits is allowed

Sol : **(i) repetition of digits is not allowed**

hundreds	tens	unit
4	4	3

Since we need 3 – digit odd numbers, the unit place can be filled in 3 ways using the digits 1, 3 or 5.

Hundred's place can be filled in 4 ways (excluding 0 and the number used for one's place) since repetition is not allowed.

Ten's place can be filled 4 ways including 0.

∴ By fundamental principle of multiplication, number of required 3 digit odd numbers = 4 × 4 × 3 = 48

(ii) the repetition of digits is allowed

hundreds	tens	unit
5	6	3

The unit place can be filled in 3 ways using the digits 1, 3, or 5 since we need 3 digit odd numeric. Hundreds place can be filled in 5 ways excluding 0 and repetition of digits is allowed.

Tens place can be filled in 6 ways.

∴ By fundamental principle of multiplication, required number of 3-digit odd numbers

= 5 × 6 × 3 = 30 × 3 = 90.

8. Count the numbers between 999 and 10,000 subject to the condition that there are

(i) no restriction.

(ii) no digit is repeated.

(iii) at least one of the digits is repeated.

Sol : **(i) no restriction.**

Given digits are 0, 1, 2, 3, 4, 5, 6, 7, 8, 9.

Since we need numbers between 999 and 10000, it has only 4 digits.

thous	hun	tens	ones
9	10	10	10

Thousands place can be filled in 9 ways (excluding 0) since there is no restriction, hundreds place, tens place and unit place can be filled in 10 ways each using all the digits.

∴ By fundamental principle of multiplication, required number of 4-digit numbers,

$= 9 \times 10 \times 10 \times 10 = 9000$.

(ii) no digit is repeated.

thous	hun	tens	ones
9	7	8	9

Thousands place can be filled in 9 ways (excluding 0)

Since repetition is not allowed, unit place can be filled in 9 ways, tens place can be filled in 8 ways and hundreds place can be filled in 7 ways.

∴ By fundamental principle of multiplication, required number of 4 digit numbers

$= 9 \times 7 \times 8 \times 9 = 4536$.

(iii) at least one of the digits is repeated.

Required number of 4 digit numbers = Total number of 4 digit numbers - number of 4 digit numbers when no digit is repeated

$= 9000 - 4536 = 4464$

9. How many three-digit numbers, which are divisible by 5, can be formed using the digits 0, 1, 2, 3, 4, 5 if

(i) repetition of digits are not allowed?

(ii) repetition of digits are allowed?

Sol : **(i) repetition of digits is not allowed**

When the unit place is 0, the tens and hundreds place can be filled in 5 and 4 ways respectively

∴ Number of ways $= 4 \times 5 \times 1 = 20$

When the unit place is 5, the tens and hundreds place can be filled in 4 ways each

∴ Number of ways $= 4 \times 4 \times 1 = 16$

∴ Total number of ways $= 20 + 16 = 36$.

(ii) repetition of digits are allowed?

hundreds	tens	unit
5	6	2

Unit place can be filled in 2 ways using the digit 0 and 5, since the three digit number is divisible by 5.

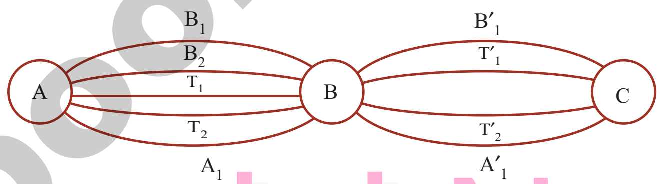
Hundreds place can be filled in 5 ways excluding 0

Tens place can be filled in 6 ways, since repetition of digits are allowed.

∴ By fundamental principle of multiplication, required number of three digit numbers $= 5 \times 6 \times 2 = 60$ ways.

10. To travel from a place A to place B, there are two different bus routes B_1, B_2, two different train routes T_1, T_2 and one air route A_1. From place B to place C there is one bus route say B'_1, two different train routes say T'_1; T'_2 and one air route A'_1. Find the number of routes of commuting from place A to place C via place B without using similar mode of transportation.

Sol : The given data can be converted as a route map diagram as follows:



The possible choices for number of routes commuting from A to place C via place B, without using similar mode of transportation are

(B_1, T'_1) (B_1, T'_2) (B_1, A'_1) (B_2, T'_1) (B_2, T'_2) (B_2, A'_1) (T_1, B'_1) (T_1, A'_1) (T_2, B'_1) (T_2, A'_1) (A_1, B'_1) (A_1, T'_1) and (A_1, T'_2).

∴ Required number of routes are 13.

11. How many numbers are there between 1 and 1000 (both inclusive) which are divisible neither by 2 nor by 5?

Sol : Given digits are 0, 1, 2, 3, 4, 5, 6, 7, 8, 9.

Numbers which are neither divisible by 2 nor 5 should have 1, 3, 7, 9 in the unit place.

∴ 1, 3, 7, 9 are the one digit numbers which are neither divisible by 2 nor by 5.

∴ Required one digit numbers = 4 ...(1)

Two-digit numbers

tens	unit
9	4

The unit place can be filled in 4 ways using the digits 1, 3, 7, 9.

Tens place can be filled in 9 ways using all the digits excluding 0.

∴ Required number of 2-digit numbers $= 9 \times 4 = 36$ ...(2)

Three digit numbers

hundreds	tens	unit
9	10	4

Unit place can be filled in 4 ways using the digits 1, 3, 7, 9.

Hundreds place can be filled in 9 ways excluding 0

Tens place can be filled in 10 ways including 0.

∴ Required number of 3 digit numbers = 9 × 10 × 4 = 360 ...(3)

There is only one 4-digit number 1000 but it is divisible by 2 and 5.

∴ Required numbers using fundamental principle of addition = 4 + 36+ 360 [From (1), (2) and (3)] = 400.

12. How many strings can be formed using the letters of the word LOTUS if the word

(i) either starts with L or ends with S. [CRT - 2022]

(ii) neither starts with L nor ends with S?

Sol : **(i) either starts with L or ends with S.**

1	4	3	2	1
L				

Since the words starts with L, the remaining 4 boxes can be filled in 4 × 3 × 2 × 1 ways by the remaining letters 'O', T, U, S.

∴ Number of words starting with L = 1 × 4 × 3 × 2 × 1 = 24.

1	2	3	4	1
				S

... (1)

Here also, the remaining 4 boxes can be filled in 4 × 3 × 2 × 1 ways = 24.

Number of words ending with S = 24 ...(2)

Number of words starting with L and end with S are 3 × 2 × 1 = 6. ...(3)

∴ By fundamental principle of addition, number of words either starts with L or ends with S = 24 + 24 – 6 = 48 – 6 = 42

(ii) Neither starts with L nor ends with S.

	3	2	1	
F				S

Total number of words formed by the letters of the word LOTUS is 5 × 4 × 3 × 2 × 1 = 120.

Now, number of words neither starts with L nor end with S.

= (Total number of words) – (Number of words either starts with L nor ends with S)

= 120 − 42 = 78.

13. (i) Count the total number of ways of answering 6 objective type questions, each question having 4 choices.

Sol : Since each question can be answered in 4 ways, the total number of ways of answering 6 questions is $4 \times 4 \times 4 \times 4 \times 4 \times 4 = 4^6$.

(ii) In how many ways 10 pigeons can be placed in 3 different pigeon holes ?

Sol : Since each pigeon can occupy any of these 3 holes, total number of ways of placing 10 pigeons = 3^{10}.

(iii) Find the number of ways of distributing 12 distinct prizes to 10 students?

Sol : Each I prize can be distributed to any one of the 10 students.

By the rule of product, the number of ways of distributing 12 distinct prizes to 10 students are $10 \times 10 \times 10 \times 10 \times 10 \times 10 \times 10 \times 10 \times 10 \times 10 \times 10 \times 10 = 10^{12}$.

14. Find the value of

(i) 6! (ii) 4! + 5! (iii) 3! – 2! (iv) 3! × 4!

(v) $\dfrac{12!}{9! \times 3!}$ **(vi)** $\dfrac{(n+3)!}{(n+1)!}$

Sol : (i) 6! $= 6 \times 5 \times 4 \times 3 \times 2 \times 1 = 720$

(ii) 4! + 5! $= (4 \times 3 \times 2 \times 1) + (5 \times 4 \times 3 \times 2 \times 1)$
$= 24 + 120 = 144$

(iii) 3! – 2! $= (3 \times 2 \times 1) - (2 \times 1) = 6 - 2 = 4$

(iv) 3! × 4! $= (3 \times 2 \times 1) \times (4 \times 3 \times 2 \times 1)$
$= 6 \times 24 = 144$

(v) $\dfrac{12!}{9! \times 3!} = \dfrac{12 \times 11 \times 10 \times \cancel{9!}}{\cancel{9!} \times 3 \times 2} = \dfrac{\overset{2}{\cancel{12}} \times 11 \times 10}{\cancel{6}}$
$= 2 \times 11 \times 10 = 220$

(vi) $\dfrac{(n+3)!}{(n+1)!} = \dfrac{(n+3)(n+2)\cancel{(n+1)!}}{\cancel{(n+1)!}}$
$= (n+3)(n+2)$

15. Evaluate $\dfrac{n!}{r!\,(n-r)!}$ **when**

(i) $n = 6, r = 2$ **(ii)** $n = 10, r = 3$.

(iii) For any n **with** $r = 2$.

Sol : (i) $n = 6, r = 2$

Given $n = 6, r = 2$

$\therefore \dfrac{n!}{r!\,(n-r)!} = \dfrac{6!}{2!\,(6-2)!} = \dfrac{6!}{2!\;4!} = \dfrac{6 \times 5 \times \cancel{4!}}{2!\;\cancel{4!}}$

$= \dfrac{\overset{3}{\cancel{6}} \times 5}{\cancel{2}} = 15$

(ii) Given $n = 10$, $r = 3$.

$$\therefore \frac{n!}{r!\,(n-r)!} = \frac{10!}{3!\,(10-3)!} = \frac{10!}{3!\;7!}$$

$$= \frac{10\times9\times8\times7!}{3\times2\times7!} = \frac{10\times9\times8}{3\times2} = 120.$$

(iii) For any n with $r = 2$.

$$\therefore \frac{n!}{r!\,(n-r)!} = \frac{n!}{2!\,(n-2)!}$$

$$= \frac{n(n-1)\,(n-2)!}{2\times1\;(n-2)!} = \frac{n^2-n}{2} = \frac{n(n-1)}{2}$$

16. Find the value of n if

(i) $(n + 1)! = 20\,(n - 1)!$ (ii) $\frac{1}{8!}+\frac{1}{9!}=\frac{n}{10!}$

Sol : (i) $(n + 1)! = 20\,(n - 1)!$

Given $(n + 1)! = 20\,(n - 1)!$

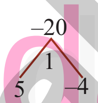
$\Rightarrow (n + 1)\,(n)\,(n-1)! = 20\,(n-1)!$

$\Rightarrow (n + 1)\,(n) = 20$

$\Rightarrow n^2 + n - 20 = 0.$

$\Rightarrow (n + 5)\,(n - 4) = 0$

$n = -5$ or $n = 4$



But $n = -5$ is not possible. $\therefore n = 4$

(ii) $\frac{1}{8!}+\frac{1}{9!}=\frac{n}{10!}$

$$\frac{1}{8!}+\frac{1}{9!} = \frac{n}{10!}$$

$$\frac{1}{8!}+\frac{1}{9\times8!} = \frac{n}{10\times9\times8!}$$

Multiplying by 8! throughout we get,

$$1+\frac{1}{9} = \frac{n}{90} \Rightarrow \frac{9+1}{9} = \frac{n}{90} \Rightarrow \frac{10}{9} = \frac{n}{90}$$

$$\Rightarrow n = \frac{10\times90}{9} = 100$$

$$\therefore n = 100.$$

Exercise 4.2

1. If $^{(n-1)}P_3 : {^nP_4} = 1 : 10$, find n.

Sol : Given $^{(n-1)}P_3 : {^nP_4} = 1 : 10$

$$\Rightarrow \frac{^{(n-1)}P_3}{^nP_4} = \frac{1}{10}$$

$$\Rightarrow 10.\,{^{(n-1)}P_3} = 1.{^nP_4} \qquad \left[\because nPr = \frac{n!}{(n-r)!}\right]$$

$$\Rightarrow 10\times\frac{(n-1)!}{(n-1-3)!} = \frac{n!}{(n-4)!}$$

$$\Rightarrow \frac{10\times(n-1)!}{(n-4)!} = \frac{n\,(n-1)!}{(n-4)!}$$

$$\Rightarrow 10 = n$$

$$\therefore n = 10.$$

2. If $^{10}P_{r-1} = 2 \times 6P_r$, find r **[Qy. - 2018]**

Sol : Given $^{10}P_{r-1} = 2 \times {^6P_r}$

$$\Rightarrow \frac{10!}{(10-r+1)!} = 2\times\frac{6!}{(6-r)!} \qquad \left[\because nP_r = \frac{n!}{(n-r)!}\right]$$

$$\Rightarrow \frac{10\times9\times8\times7\times6!}{(11-r)!} = \frac{2\times6!}{(6-r)!}$$

$$\Rightarrow \frac{10\times9\times8\times7}{(11-r)(10-r)(9-r)(8-r)(7-r)(6-r)!} = \frac{2}{(6-r)!}$$

$$\Rightarrow \frac{10\times9\times8\times7}{(11-r)(10-r)(9-r)(8-r)(7-r)} = 2$$

$\Rightarrow (11 - r)\,(10 - r)\,(9 - r)\,(8 - r)\,(7 - r)$
$= 5 \times 9 \times 8 \times 7$

$\Rightarrow (11 - r)\,(10 - r)\,(9 - r)\,(8 - r)\,(7 - r)$
$= 7 \times 6 \times 5 \times 4 \times 3$

$\Rightarrow (11 - r)\,(10 - r)\,(9 - r)\,(8 - r)\,(7 - r)$
$= (11 - 4)\,(10 - 4)\,(9 - 4)\,(8 - 4)\,(7 - 4)$

$\Rightarrow r = 4$

3. (i) Suppose 8 people enter an event in a swimming meet. In how many ways could the gold, silver and bronze prizes be awarded?

Sol : (i) Gold medal can be awarded to any one of the 8 candidates in 8 ways.

Silver medal can be awarded to any one of the remaining 7 candidates in 7 ways.

Bronze medal can be awarded to any one of the remaining 6 candidates in 6 ways.

$\therefore$ Total numbers of ways of awarding the prize

$= 8 \times 7 \times 6 = 336$

(ii) Three men have 4 coats, 5 waist coats and 6 caps. In how many ways can they wear them?

05 BINOMIAL THEOREM, SEQUENCES AND SERIES

MUST KNOW DEFINITIONS

Binomial theorem for positive integral index	:	If x and a are real numbers, then for all $n \in \mathbb{N}$, $(x+a)^n = nC_0x^na^0 + nC_1x^{n-1}a^1 + nC_2x^{n-2}a^2 + ... + nC_rx^{n-r}a^r + ... + a^n$.
Sequence	:	A sequence is a function whose domain is the set N of natural numbers.
Series	:	If $a_1, a_2, . . . a_n$ is a sequence, then the expression $a_1 + a_2 + a_3 + . . . + a_n$ is a series.
A.P	:	A sequence is called an arithmetic progression (A.P) if the difference of a term and the previous term is always same.
G.P	:	A sequence of non-Zero numbers is called a Geometric progression (G.P) if the ratio of a term and the term proceeding to it is always a constant.
H. P:	:	The reciprocals of the terms of an, A.P form a H.P.
Arithmetico – geometric progression (AGP)	:	An AGP is a progression in which each term can be represented as the product of the terms of an AP and a G.P.

FORMULAE TO REMEMBER

- $(x+a)^n = x^n + nC_1 x^{n-1} a^1 + nC_2 x^{n-2} a^2 + \ldots + nC_r x^{n-r} a^r + \ldots + a^n$.
- $(x-a)^n = \sum_{r=0}^{n} (-1)^r \; nC_r x^{n-r} a^r$
- Middle terms in binomial expansion:
- If n is even, then $\left(\frac{n}{2}+1\right)^{th}$ term
- If n is odd, then $\left(\frac{n+1}{2}\right)^{th}$ and $\left(\frac{n+3}{2}\right)^{th}$ term are the middle terms.
- **A.P**
- n^{th} term of A.P., $t_n = a + (n-1)d$
- Sum to n terms of an A.P.

$S_n = \frac{n}{2}\left[2a+(n-1)d\right]$ or $S_n = \frac{n}{2}(a+l)$ where l is the last term of an A.P.

Properties of A. P.

1. If a constant is added or subtracted from each term of an A.P., then the resulting sequence is also an A.P with the same common difference
2. If each term of an A.P is multiplied or divided by a non-zero constant K, then the resulting sequence is an A.P. with the common difference Kd or $\frac{d}{k}$
3. In a finite A.P, the sum of the terms equidistant from the beginning and the end is always same and is equal to the sum of first and last term.
4. Three numbers a, b, c are in A.P if $2b = a + c$.
5. **G. P**
   - nth term of a G. P is $a\,r^{n-1}$
   - nth term from the end $= l\left(\frac{1}{r}\right)^{n-1}$
   - Sum to n terms of a G.P

   $S_n = a\left(\frac{r^n-1}{r-1}\right)$ if $r > 1$ (OR) $S_n = a\left(\frac{1-r^n}{1-r}\right)$ if $r < 1$.

✦ $\frac{e+e^{-1}}{2}=1+\frac{1}{2!}+\frac{1}{4!}+..$ ✦ $\frac{e-e^{-1}}{2}=\frac{1}{1!}+\frac{1}{3!}+\frac{1}{5!}+..$

Logarithmic Series:

If $-1 < x \le 1$, then $\log(1+x)=x-\frac{x^2}{2}+\frac{x^3}{3}-\frac{x^4}{4}+\cdots$ and $\log(1-x)=-\left[x+\frac{x^2}{2}+\frac{x^3}{3}+\frac{x^4}{4}+\cdots\right]$

Binomial theorem for a rational index:

For any rational number n, other than positive integer $(1+x)^n=1+nx+\frac{n(n-1)}{1.2}x^2+\frac{n(n-1)(n-2)}{1.2.3}x^3+\cdots$ provided $|x|<1$.

✦ $(1-x)^n=1-nx+\frac{n(n-1)}{2!}x^2-\frac{n(n-1)(n-2)}{3!}x^3+\cdots$ ✦ $(1+x)^{-n}=1-nx+\frac{n(n+1)}{2!}x^2-\frac{n(n+1)(n+2)}{3!}x^3+\cdots$

✦ $(1-x)^{-n}=1+nx+\frac{n(n+1)}{2!}x^2+\frac{n(n+1)(n+2)}{3!}x^3+\cdots$

TEXTUAL QUESTIONS

Exercise 5.1

1. Expand

(i) $\left(2x^2-\frac{3}{x}\right)^3$

(ii) $\left(2x^2-3\sqrt{1-x^2}\right)^4+\left(2x^2+3\sqrt{1-x^2}\right)^4$

Sol : **(i)** $\left(2x^2-\frac{3}{x}\right)^3$

$(a-b)^n = nC_0\, a^n b^0 - nC_1\, a^{n-1} b^1 + nC_a\, a^0 b^n$

$=\left(2x^2\right)^3+3C_1\left(2x^2\right)^2\left(\frac{3}{x}\right)+3C_2\left(2x^2\right)^1\left(\frac{3}{x}\right)^2-\left(\frac{3}{x}\right)^3$

$=8x^6+3\left(4x^4\right)\left(-\frac{3}{x}\right)+\frac{3\times \not{2}}{\not{2}\times 1}\left(2\not{x^2}\right)\left(\frac{9}{\not{x^2}}\right)-\frac{27}{x^3}$

$=8x^6-36x^3+54-\frac{27}{x^3}$

(ii) $\left(2x^2-3\sqrt{1-x^2}\right)^4+\left(2x^2+3\sqrt{1-x^2}\right)^4$

$=\Big[\left(2x^2\right)^4-{}^4C_1\left(2x^2\right)^3\left(3\sqrt{1-x^2}\right)^1+{}^4C_2\left(2x^2\right)^2\left(3\sqrt{1-x^2}\right)^2-{}^4C_3\left(2x^2\right)^1\left(3\sqrt{1-x^2}\right)^3+\left(3\sqrt{1-x^2}\right)^4\Big]+$

$\Big[\left(2x^2\right)^4+{}^4C_1\left(2x^2\right)^3\left(3\sqrt{1-x^2}\right)^1+{}^4C_2\left(2x^2\right)^2\left(3\sqrt{1-x^2}\right)^2+{}^4C_3\left(2x^2\right)^1\left(3\sqrt{1-x^2}\right)^3+\left(3\sqrt{1-x^2}\right)^4\Big]$

$=2\left[\left(2x^2\right)^4+{}^4C_2\left(2x^2\right)^2\left(3\sqrt{1-x^2}\right)^2+\left(3\sqrt{1-x^2}\right)^4\right]$

$=2\left[\left(16x^8\right)+\frac{\not{4}^{2}\times 3}{\not{2}\times 1}\times 4x^4\times 9\left(1-x^2\right)+3^4\left(1-x^2\right)^2\right]$

$=2[16x^8+216x^4(1-x^2)+81(1-x^2)^2]$

2. Compute

(i) 102^4 **(ii)** 99^4 **(iii)** 9^7

Sol : **(i)** $102^4 = (100+2)^4$

$(a+b)^n = {}^nC_0\, a^n b^0 + {}^nC_1\, a^{n-1} b + {}^nC_2\, a^{n-2} b^2 + \ldots + {}^nC_n\, a^0 b^n, n\in\mathbb{N}$

$= 100^4 + {}^4C_1(100)^3(2)^1 + {}^4C_2(100)^2(2^2) + {}^4C_3(100)^1(2^3)+2^4$

$= 100000000+4\,(1000000)(2)+\frac{\not{4}^{2}\times(3)}{2(1)}(10000)(4)+400(8)+16$

$= 100000000+8000000+240000+3200+16$

$= 108{,}243{,}216$

(ii) 99^4

$=(100-1)^4$

$(a-b)^n = nC_0\, a^n b^0 - nC_1\, a^{n-1} b^1 + \ldots + nC_n \times a^0 b^n, n\in\mathbb{N}$

$=100^4-{}^4C_1(100)^3(1)^1+{}^4C_2(100)^2(1)^2-{}^4C_3(100)(1^3)+1^4$

$=100000000+4(1000000)+\frac{\not{4}^{2}\times 3}{\not{2}\times 1}(10000)-400+1$

$=100000000-4000000+60000-400+1$

$=96{,}059{,}601$

(iii) 9^7 [Qy. - 2018]

$= (10-1)^7$

$(a-b)^n = nC_0\, a^n b^0 - nC_1\, a^{n-1} b^1 + ... + nC_n\, a^0 b^n, n \in \mathbb{N}$

$= 10^7 - {}^7C_1\, 10^6(1) + {}^7C_2 10^5(1)^2 - {}^7C_3 10^4(1^3) + {}^7C_4(10)^3(1)^4 - {}^7C_5(10)^2(1)^5 + {}^7C_6(10)^1(1)^6 - (1)^7$

$= 10000000 - 7(1000000) + \frac{7\times 6}{2\times 1}(100000) - \frac{7\times 6\times 5}{3\times 2\times 1}10000 + \frac{7\times 6\times 5\times 4}{4\times 3\times 2\times 1}1000 - \frac{7\times 6}{2\times 1}(100) + 7(10) - 1$

$= 10000000 - 7000000 + 2100000 - 350000 + 35000 - 2100 + 70 - 1 = 4782969$

3. Using binomial theorem, indicate which of the following two number is larger $(1.01)^{1000000}$, 10000

Sol : Consider $(1.01)^{1000000} - 10{,}000$

$= (1+.01)^{1000000} - 10{,}000$

$= {}^{1000000}C_0 + {}^{1000000}C_1(.01) + {}^{1000000}C_2(.01)^2 + ... + (.01)^{1000000} - 10{,}000$

$= (1 + 1000000\times(.01) + \text{other positive terms}) - 10{,}000$

$= 1 + \text{other positive terms}$

$\therefore (1.01)^{1000000} - 10{,}000 > 0$

$\Rightarrow (1.01)^{1000000} > 10{,}000.$

$\therefore (1.01)^{1000000}$ is larger.

4. Find the co-efficient of x^{15} in $\left(x^2+\frac{1}{x^3}\right)^{10}$ [Govt. MQP - 2018]

Sol : In $\left(x^2+\frac{1}{x^3}\right)^{10}$, $n = 10$, $x = x^2$, $a = \frac{1}{x^3}$,

So the general term is $T_{r+1} = {}^nCr\, x^{n-r} a^r$

$\Rightarrow T_{r+1} = {}^{10}C_r\,(x^2)^{10-r}\left(\frac{1}{x^3}\right)^r$

$= {}^{10}C_r\, x^{20-2r}\cdot x^{-3r}$

$= {}^{10}C_r\, x^{20-5r}$ ...(1)

To find the Co-efficient of x^{15},

put $20-5r = 15$

$\Rightarrow 20-15 = 5r$

$\Rightarrow 5 = 5r$

$\Rightarrow r = 1$

Putting $r = 1$ in (1) we get

$T_2 = {}^{10}C_1 x^{20-5} = {}^{10}C_1 x^{15}$

$\therefore$ Co-efficient of x^{15} is 10.

5. Find the Co-efficient of x^2 and the co-efficient of x^6 in $\left(x^2-\frac{1}{x^3}\right)^6$

Sol : In $\left(x^2-\frac{1}{x^3}\right)^6$, $n = 6$, $x = x^2$ and $a = \frac{-1}{x^3}$.

$\therefore$ The general term is $T_{r+1} = nC_r\, x^{n-r} a^r$

$T_{r+1} = {}^6C_r(x^2)^{6-r}\cdot\left(\frac{-1}{x^3}\right)^r$

$= {}^6C_r\, x^{12-2r}(-1)^r\cdot x^{-3r}$

$= (-1)^r\, {}^6C_r\, x^{12-5r}$ ...(1)

To find the Co-efficient x^6, put

$12-5r = 6$

$\Rightarrow 12-6 = 5r$

$\Rightarrow 6 = 5r$

$r = \frac{6}{5}$ which is not possible.

$\therefore$ There wont be x^6 term.

To find the Co-efficient of x^2, put $12-5r = 2$

$12-2 = 5r$

$\Rightarrow 10 = 5r$

$\Rightarrow r = 2$

Putting $r = 2$ in (1) we get,

$T_3 = (-1)^2\, {}^6C_2\, x^{12-10} = \frac{6\times 5}{2\times 1}x^2 = 15x^2$

$\therefore$ Co-efficient of x^2 is 15.

6. Find the Co-efficient of x^4 in the expansion of $(1+x^3)^{50}\left(x^2+\frac{1}{x}\right)^5$.

Sol : Given $\left(x^2+\frac{1}{x}\right)^5(1+x^3)^{50}$.

Let us expand $\left(x^2+\frac{1}{x}\right)^5$

$= (x^2)^5 + {}^5C_1(x^2)^4\left(\frac{1}{x}\right)^1 + {}^5C_2(x^2)^3\left(\frac{1}{x}\right)^2 + {}^5C_3(x^2)^2\left(\frac{1}{x}\right)^3 + {}^5C_4(x^2)^1\left(\frac{1}{x}\right)^4 + \frac{1}{x^5}$

$$= x^{10} + 5\frac{x^8}{x} + \frac{5\times 4}{2\times 1}\cdot\frac{x^6}{x^2} + \frac{5\times 4}{2\times 1}\frac{x^4}{x^3} + 5\frac{x^2}{x^4} + \frac{1}{x^5}$$

$[\because 5C_3 = 5C_2]$

$$= x^{10} + 5x^7 + 10x^4 + 10x + \frac{5}{x^2} + \frac{1}{x^5}$$

General term in $(1 + x^3)^{50}$

$$T_{r+1} = {}^{50}C_r (1)^{50-r} (x^3)^r = {}^{50}C_r \cdot x^{3r} \quad ...(1)$$

Consider $\left(x^2 + \frac{1}{x}\right)^5 (1+x^3)^{50}$

$$= \left(x^{10} + 5x^7 + 10x^4 + 10x + \frac{5}{x^2} + \frac{1}{x^5}\right)(1+x^3)^{50}$$

$$= \left(x^{10} + 5x^7 + 10x^4 + 10x + \frac{5}{x^2} + \frac{1}{x^5}\right)({}^{50}C_0 + {}^{50}C_1 x^3 + {}^{50}C_2 x^6 + {}^{50}C_3 x^9 + ...)$$

$$= \left(x^{10} + 5x^7 + 10x^4 + 10x + \frac{5}{x^2} + \frac{1}{x^5}\right)(1 + 50x^3 + 1225x^6 + 19600x^9 + ...)$$

Now, Co-efficient of x^4

= 10[Constant term in $(1 + x^3)^{50}$] + 10[Co-efficient of x^3 in $(1 + x^3)^{50}$] + 5[Co-efficient of x^6] + 1 × [Co efficient of x^9]

= 10(1) + 10(50) + 5(1225) + 1(19600) [using (1)]

= 10 + 500 + 6125 + 19600 = 26235

∴ Co-efficient of x^4 is 26235.

7. Find the constant term of $\left(2x^3 - \frac{1}{3x^2}\right)^5$.

Sol : In $\left(2x^3 - \frac{1}{3x^2}\right)^5$, $n = 5, x = 2x^3, a = -\frac{1}{3x^2}$

∴ General term is

$$T_{r+1} = nC_r x^{n-r} a^r = {}^5C_r (2x^3)^{5-r}\left(-\frac{1}{3x^2}\right)^r$$

$$= {}^5C_r\, 2^{5-r} x^{15-3r} \frac{(-1)^r}{3^r \cdot x^{2r}} = {}^5C_r \frac{2^{5-r}}{3^r}(-1)^r x^{15-3r-2r}.$$

$$= {}^5C_r \frac{2^{5-r}}{3^r}(-1)^r x^{15-5r} \quad ...(1)$$

To get the constant term, put $15 - 5r = 0$

$\Rightarrow 15 - 5r = 0 \quad \Rightarrow 15 = 5r$

$\Rightarrow r = \frac{15}{5} = 3$

Putting $r = 3$ in (1) we get

$$T_4 = {}^5C_3 \frac{2^2}{3^3}\cdot(-1)^3 \cdot x^0$$

$$= -{}^5C_3 \frac{(4)}{27}$$

$$= -\frac{5\times 4\times 3}{3\times 2\times 1}\times\frac{4}{27} = -\frac{40}{27}$$

Hence the constant term is $-\frac{40}{27}$

8. Find the last two digits of the number 3^{600}. **[Qy. - 2018]**

Sol : Consider $3^{600} = (3^2)^{300} = 9^{300}$

$3^{600} = (10 - 1)^{300}$

Using binomial theorem,

$3^{600} = {}^{300}C_0(10)^{300} - {}^{300}C_1(10)^{299} + ... -300C_{299}(10)^1 + 1$

$= (10)^{300} - 300(10)^{299} + ... -300(10) + 1$

$3^{600} = (10)^{300} - 300\,(10)^{299} + ... -3000 + 1$

Hence, it is clear that the last two digits in 3^{600} are 01.

9. If n is a positive integer using Binomial theorem, show that, $9^{n+1} - 8n - 9$ is always divisible by 64. **[Qy. - 2019]**

Sol : We know $(1 + x)^n = {}^nC_0 + {}^nC_1 x + {}^nC_2 x^2 + ... + {}^nC_{n-1} x^{n-1} + {}^nC_n x^n$.

putting $x = 8$ we get,

$(1 + 8)^n = {}^nC_0 + {}^nC_1(8) + {}^nC_2(8)^2 + ... + {}^nC_{n-1}(8)^{n-1} + {}^nC_n \cdot (8)^n$.

$\Rightarrow 9^n = 1 + 8n + nC_2 8^2 + nC_3 8^3 + ... + nC_{n-1} 8^{n-1} + nC_n 8^n$.

$\Rightarrow 9^n - 1 - 8n = 8^2 [{}^nC_2 + 8\cdot{}^nC_3 + ... + {}^nCn\, 8^{n-2}]$.

$\Rightarrow 9^n - 1 - 8n = 64 [{}^nC_2 + 8\,{}^nC_3 + ... + {}^nCn\, 8^{n-2}]$

$\Rightarrow 9^n - 8n - 1$ is divisible by 64 for all positive integer n.

Putting $n = n + 1$, we get,

$9^{n+1} - 8(n + 1) - 1$ is divisible by 64 for all positive integer n.

$\Rightarrow (9^{n-1} - 8n - 8 - 1)$ is divisible by 64

$\Rightarrow 9^{n+1} - 8n - 9$ is always divisible by 64.

10. If n is an odd positive integer, prove that the co-efficients of the middle terms in the expansion of $(x + y)^n$ are equal.

Sol : Given $(x + y)^n$

If n is odd, the two middle terms in $(x + y)^n$ are $T_{\frac{n-1}{2}+1}$ and $T_{\frac{n+1}{2}+1}$

$T_{\frac{n-1}{2}+1} = {}^nC_{\frac{n-1}{2}} x^{\frac{n+1}{2}} y^{\frac{n-1}{2}}$ and $T_{\frac{n+1}{2}+1} = {}^nC_{\frac{n+1}{2}} x^{\frac{n-1}{2}} y^{\frac{n+1}{2}}$

The co-efficients of middle terms are ${}^nC_{\frac{n+1}{2}}$ and ${}^nC_{\frac{n-1}{2}}$

We know $nC_r = nC_{n-r}$

$$ {}^nC_{\frac{n-1}{2}} = {}^nC_{n-\frac{(n-1)}{2}} = {}^nC_{\frac{2n-n+1}{2}} = {}^nC_{\frac{n+1}{2}}$$

$\Rightarrow {}^nC_{\frac{n-1}{2}}$ and ${}^nC_{\frac{n+1}{2}}$ are same

$\Rightarrow$ Coefficient of middle terms are equal.

11. If n is a positive integer and r is a nonnegative integer, prove that the co-efficients of x^r and x^{n-r} in the expansion of $(1 + x)^n$ are equal.

Sol : In $(1 + x)^n$, $n = n$, $x = 1$, $a = x$

$\therefore$ General term is $t_{r+1} = {}^nC_r x^{n-r} a^r$

$\Rightarrow t_{r+1} = {}^nC_r (1)^{n-r} x^r$

$\Rightarrow t_{r+1} = {}^nC_r x^r$ ...(1)

$\therefore$ Co-efficient of x^r is nC_r.

Putting $r = n - r$ in (1) we get,

$T_{n-r+1} = nC_{n-r} x^{n-r}$

$\Rightarrow$ Co-efficient of x^{n-r} is nC_{n-r} ... (2)

But $nC_r = nC_{n-r}$ using the property of combination

$\therefore$ Co-efficients of x^r and co-efficients of x^{n-r} are equal.

12. If a and b are distinct integers, prove that $a - b$ is a factor of $a^n - b^n$, whenever n is a positive integer. [Hint: write $a^n = (a - b + b)^n$ and expand] [Qy. - 2018]

Sol : Let $a = a + b - b = b + (a - b)$

$\therefore a^n = [b + (a - b)]^n$

Using binomial theorem,

$$a^n = b^n + nC_1 b^{n-1} (a - b) + nC_2 b^{n-2} (a-b)^2 + ... + (a-b)^n$$

$$\Rightarrow a^n - b^n = nC_1 b^{n-1} (a - b) + nC_2 (a-b)^2 b^{n-2} + ... + (a-b)^n$$

$$\Rightarrow a^n - b^n = (a - b) \times [nC_1 b^{n-1} + nC_2 b^{n-2} (a - b) + ... + (a - b)^{n-1}]$$

$$\Rightarrow a^n - b^n = (a - b) K$$

[Where $K = nC_1 b^{n-1} + nC_2 b^{n-2} (a - b) + ... + (a - b)^{n-1}$]

Thus, $(a - b)$ is a factor of $a^n - b^n$.

13. In the binomial expansion of $(a + b)^n$, if the coefficients of the 4th and 13th terms are equal then, find n.

Sol : In $(a + b)^n$, the general term is

$T_{r+1} = {}^nC_r a^{n-r} \cdot b^r$. ...(1)

To find the co-efficient of 4th term, put $r = 3$ in (1)

$\therefore T_4 = {}^nC_3 a^{n-3} \cdot b^3$

To find the co-efficient of 13th term

put $r = 12$ in (1)

$\therefore T_{13} = {}^nC_{12} a^{n-12} b^{12}$.

Given ${}^nC_3 = {}^nC_{12} \Rightarrow 3 + 12 = n$

$[\therefore nC_x = nC_y \Rightarrow x = y$ or $x + y = n]$

$\Rightarrow n = 15$

14. If the binomial co-efficients of three consecutive terms in the expansion of $(a + x)^n$ are in the ratio 1 : 7 : 42, then find n.

Sol : Let the three consecutive terms be r^{th}, $(r + 1)^{th}$ and $(r + 2)^{th}$ terms.

General term in $(a + x)^n$ is $T_{r+1} = nC_r a^{n-r} x^r$. ... (1)

Then, co-efficients of r^{th}..., $(r + 1)^{th}$ and $(r + 2)^{th}$.. terms are ${}^nC_{r-1}$, ${}^nC_r$ and ${}^nC_{r+1}$ respectively.

Given that ${}^nC_{r-1} : {}^nC_r : {}^nC_{r+1} = 1 : 7 : 42$

Consider $\frac{{}^nC_{r-1}}{{}^nC_r} = \frac{1}{7}$

$$\frac{\frac{n!}{(r-1)!(n-r+1)!}}{\frac{n!}{r!(n-r)!}} = \frac{1}{7} \quad \left[\because {}^nC_r = \frac{n!}{r!(n-r)!}\right]$$

$$\frac{\cancel{n!}}{\cancel{(r-1)!}(n-r+1)\cancel{(n-r)!}} \times \frac{r\cancel{(r-1)!}\cancel{(n-r)!}}{\cancel{n!}} = \frac{1}{7}$$

$$\frac{r}{n-r+1} = \frac{1}{7}$$

$\Rightarrow 7r = n - r + 1$

$\Rightarrow n - 8r + 1 = 0$ ...(2)

and $\frac{{}^nC_r}{{}^nC_{r+1}} = \frac{\cancel{7}}{\cancel{42}_6}$

$$\frac{r+1}{n-r} = \frac{1}{6}$$

$\Rightarrow 6r + 6 = n - r$

$\Rightarrow n - 7r - 6 = 0$ ...(3)

(2) – (3)

$(n - 8r + 1) - (n - 7r - 6) = 0$

06 Two dimensional Analytical Geometry

MUST KNOW DEFINITIONS

Locus	:	The curve described by a point which moves under given conditions or condition is called its locus.
Straight line	:	A straight line is a curve such that every point on the line segment joining any two points on it lies on it.
Slope or Gradient	:	The trigonometrical tangent of the angle that a line makes with the positive direction of the X – axis in the anti clock wise direction.

FORMULAE TO REMEMBER

* Slope (m)
  1. $m = \tan\theta$
  2. $m = \dfrac{y_2 - y_1}{x_2 - x_1}$
  3. $m = -\dfrac{\text{Co-efficient of } x}{\text{Co-efficient of } y}$
* **Angle between two lines:**

  $\tan\theta = \dfrac{m_1 - m_2}{1 + m_1 m_2}$ where m_1, m_2 are slopes of the lines
* Condition for parallel lines is $m_1 = m_2$
* Condition for perpendicular lines is $m_1\, m_2 = -1$
* **Equation of straight lines:**
  1. Equation of X-axis is $y = 0$ and equation of any line parallel to X-axis is $y = k$.
  2. Equation of Y-axis is $x = 0$ and equation of any line parallel to y-axis is $x = k$, where K is the distance between the line and the Y-axis
  3. **Slope point form**: $y - y_1 = m(x - x_1)$
  4. **Slope – intercept form:** $y = mx + C$.
  5. **Two points form:** $\dfrac{y - y_1}{y_2 - y_1} = \dfrac{x - x_1}{x_2 - x_1}$
  6. **Intercepts form:** $\dfrac{x}{a} + \dfrac{y}{b} = 1$
  7. **Normal form:** $x\cos\alpha + y\sin\alpha = p$
  8. **Symmetric form/parametric form:**

     $\dfrac{x - x_1}{\cos\theta} = \dfrac{y - y_1}{\sin\theta} = r$
  9. **General form:** $ax + by + c = 0$
  10. Equation of pair of straight lines passing through the origin is $ax^2 + 2hxy + by^2 = 0$
  11. The straight lines are real and distinct if $h^2 > ab$
  12. The straight lines are coincident if $h^2 = ab$
  13. The straight lines are imaginary if $h^2 < ab$
  14. If $m_1 m_2$ are slopes of the pair of straight lines then $m_1 + m_2 = -\dfrac{2h}{b}$ and $m_1 m_2 = \dfrac{a}{b}$

15. $\tan\theta = 2\dfrac{\sqrt{h^2-ab}}{a+b}$ where θ is the angle between pair of straight lines.

16. The condition for $ax^2 + 2hxy + by^2 + 2gx + 2fy + c = 0$ to represent a pair of straight lines is $abc + 2fgh - af^2 - bg^2 - ch^2 = 0$.

17. If the straight lines are parallel, then $h^2 = ab$. For pair of straight lines

18. If the straight lines are perpendicular the $a + b = 0$

19. Length of the perpendicular from $P(x_1, y_1)$ to the line $ax + by + c = 0$ is $\pm\dfrac{ax_1+by_1+c}{\sqrt{a^2+b^2}}$

20. Length of the perpendicular from the origin to $ax + by + c = 0$ is $\pm\dfrac{c}{\sqrt{a^2+b^2}}$

21. Distance between two parallel lines $ax + by + c_1 = 0$ and $ax + by + c_2 = 0$ is $\left|\dfrac{c_1-c_2}{\sqrt{a^2+b^2}}\right|$

22. Equations of the bisectors of the angles between the straight lines $ax + by + c = 0$ and $a_1x + b_1y + c_1 = 0$ are $\dfrac{ax+by+c}{\sqrt{a^2+b^2}} = \pm\dfrac{a_1x+b_1y+c_1}{\sqrt{a_1^2+b_1^2}}$

23. Equation of the bisectors of the angles between the line $ax^2 + 2hxy + by^2 = 0$ is $\dfrac{x^2-y^2}{a-b} = \dfrac{xy}{h}$

TEXTUAL QUESTIONS

EXERCISE 6.1

1. Find the locus of P, if for all values of α, the co-ordinates of a moving point P is

(i) (9 cos α, 9 sin α) (ii) (9 cos α, 6 sin α)

Sol : **(i) (9 cos α, 9 sin α)**

Let P (h, k) be any point on the required path

From the given information, we have

$h = 9\cos\alpha$ and $k = 9\sin\alpha$

$\Rightarrow \dfrac{h}{9} = \cos\alpha$ and $\dfrac{k}{9} = \sin\alpha$

To eliminate the parameter α, squaring and adding we get

$\left(\dfrac{h}{9}\right)^2 + \left(\dfrac{k}{9}\right)^2 = \cos^2\alpha + \sin^2\alpha$

$\Rightarrow \dfrac{h^2}{81} + \dfrac{k^2}{81} = 1 \quad [\because \sin^2\alpha + \cos^2\alpha = 1]$

$\Rightarrow h^2 + k^2 = 81$

∴ Locus of (h, k) is $x^2 + y^2 = 81$

(ii) (9 cos α, 6 sin α)

Let P (h, k) be any point on the required path

From the given information, we have

$h = 9\cos\alpha$ and $k = 6\sin\alpha$

$\Rightarrow \dfrac{h}{9} = \cos\alpha$ and $\dfrac{k}{6} = \sin\alpha$

To eliminate the parameter α,

Squaring and adding we get

$\left(\dfrac{h}{9}\right)^2 + \left(\dfrac{k}{6}\right)^2 = \cos^2\alpha + \sin^2\alpha$

$\Rightarrow \dfrac{h^2}{81} + \dfrac{k^2}{36} = 1 \quad [\because \cos^2\alpha + \sin^2\alpha = 1]$

∴ Locus of (h, k) is $\dfrac{x^2}{81} + \dfrac{y^2}{36} = 1$

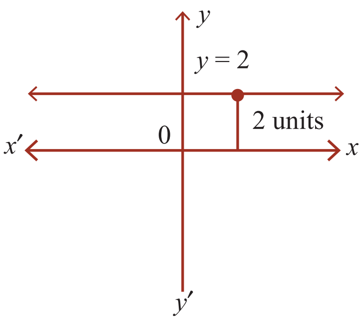
2. Find the locus of a point P that moves at a constant distant of

(i) two units from the x - axis

(ii) three units from the y - axis

Sol : **(i) two units from the x- axis**

Let P (h, k) be any point on the required path

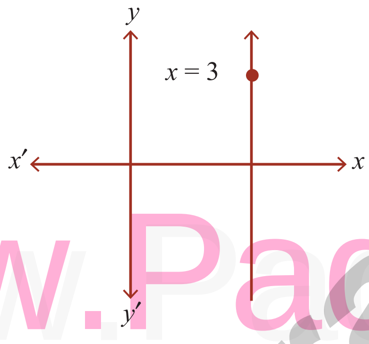


Any line parallel to x - axis will be of the form $y = \pm\, c$.

By the given condition, the distance from x - axis is 2 units $\Rightarrow c = \pm\, 2$.

$\therefore$ Locus of the point P is $y = \pm\, 2$.

(ii) Any line parallel to y - axis will be of the form $x = \pm\, c$.



By the given condition, the distance from y - axis is 3 units $\Rightarrow c = \pm\, 3$.

$\therefore$ Locus of the point P is $x = \pm\, 3$.

3. If θ is a parameter, find the equation of the locus of a moving point, whose coordinates are $x = a\cos^3\theta, y = a\sin^3\theta$.

Sol : Given $x = a\cos^3\theta$, $y = a\sin^3\theta$

$$\Rightarrow \quad \frac{x}{a} = \cos^3\theta \text{ and } \frac{y}{a} = \sin^3\theta$$

Taking power $\left(\frac{2}{3}\right)$ for both the equations, we get

$$\left(\frac{x}{a}\right)^{\frac{2}{3}} = (\cos^3\theta)^{\frac{2}{3}} \text{ and}$$

$$\left(\frac{y}{a}\right)^{\frac{2}{3}} = (\sin^3\theta)^{\frac{2}{3}}$$

$$\left(\frac{x}{a}\right)^{\frac{2}{3}} = \cos^2\theta \text{ and } \left(\frac{y}{a}\right)^{\frac{2}{3}} = \sin^2\theta$$

We know that $\cos^2\theta + \sin^2\theta = 1$

$$\therefore \left(\frac{x}{a}\right)^{\frac{2}{3}} + \left(\frac{y}{a}\right)^{\frac{2}{3}} = 1 \Rightarrow \frac{x^{\frac{2}{3}}}{a^{\frac{2}{3}}} + \frac{y^{\frac{2}{3}}}{a^{\frac{2}{3}}} = 1$$

$$\Rightarrow \frac{x^{\frac{2}{3}} + y^{\frac{2}{3}}}{a^{\frac{2}{3}}} = 1 \Rightarrow x^{\frac{2}{3}} + y^{\frac{2}{3}} = a^{\frac{2}{3}}$$

Which is the required locus.

4. Find the value of k and b, if the points P(–3,1) and Q(2,b) lie on the locus of $x^2 - 5x + ky = 0$.

Sol : Given that P (–3,1) lie on the locus of $x^2 - 5x + ky = 0$

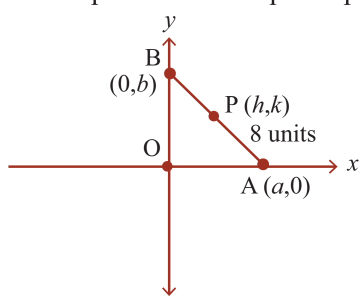
$$\Rightarrow \quad (-3)^2 - 5(-3) + k(1) = 0$$
$$\Rightarrow \quad 9 + 15 + k = 0$$
$$\Rightarrow \quad k = -24$$

Also, it is given that (2, b) lie on the locus of $x^2 - 5x + ky = 0$

$$\Rightarrow \quad 2^2 - 5(2) + kb = 0$$
$$\Rightarrow \quad 4 - 10 - 24(b) = 0$$
$$\Rightarrow \quad -6 - 24b = 0 \Rightarrow -24b = 6$$
$$\Rightarrow \quad b = \frac{-6}{24} = \frac{-1}{4}$$
$$\Rightarrow \quad \boxed{b = -\frac{1}{4}}$$

5. A straight rod of length 8 units slides with its ends A and B always on the x and y axes respectively. Find the locus of the mid point of the line segment AB.

Sol : Let (h, k) be the mid-point on the required path.



Let the co-ordinates of A and B be A (a, 0) and B (0,b).

As the rod slides, the values of a and b change.

So a and b are two variables.

$$\text{Then } h = \frac{a+0}{2}, \text{ and } k = \frac{0+b}{2}$$

$$\Rightarrow \quad h = \frac{a}{2} \text{ and } k = \frac{b}{2}$$

$$\Rightarrow \quad a = 2h \text{ and } b = 2k$$

From ΔOAB we have

$AB^2 = OA^2 + OB^2$

$\Rightarrow a^2 + b^2 = 8^2$ [$\because$ AB = 8 units]

$\Rightarrow (2h)^2 + (2k)^2 = 64$

$\Rightarrow 4h^2 + 4k^2 = 64$

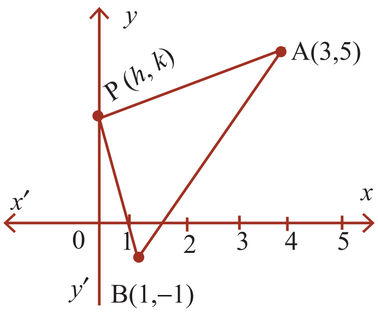
$\Rightarrow h^2 + k^2 = 16$ [Dividing by 4]

$\therefore$ Locus of (h,k) is $x^2 + y^2 = 16$

6. Find the equation of the locus of a point such that the sum of the squares of the distance from the points (3, 5), (1,–1) is equal to 20. **[CRT - 2022]**

Sol : Let P(h, k) be the point on the locus and A(3,5), B(1,–1) be the given points.

By the given condition, $PA^2 + PB^2 = 20$.



Given $PA^2 + PB^2 = 20$

$\Rightarrow (h-3)^2 + (k-5)^2 + (h-1)^2 + (k+1)^2 = 20$

$h^2 - 6h + 9 + k^2 - 10k + 25 + h^2 - 2h + 1 + k^2 + 2k + 1 = 20$

$\Rightarrow 2h^2 + 2k^2 - 8h - 8k + 36 = 20$

$\Rightarrow 2h^2 + 2k^2 - 8h - 8k + 36 - 20 = 0$

$\Rightarrow 2h^2 + 2k^2 - 8h - 8k + 16 = 0$

Dividing by 2 we get,

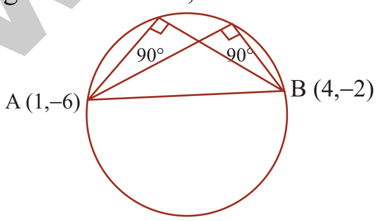
$h^2 + k^2 - 4h - 4k + 8 = 0$

$\therefore$ Locus of (h,k) is $x^2 + y^2 - 4x - 4y + 8 = 0$

7. Find the equation of the locus of the point P such that the line segment AB, joining the points A(1, –6) and B(4,–2), subtends a right angle at P.

Sol : Let P(h, k) be the point on the locus and A(1,–6), B(4,–2) be the given points.

By the given condition, ∠APB = 90º.



$\therefore$ΔAPB is a right angled triangle

$\Rightarrow AB^2 = PA^2 + PB^2$

$\Rightarrow (1-4)^2 + (-6+2)^2 = (h-1)^2 + (k+6)^2 + (h-4)^2 + (k+2)^2$

$\Rightarrow 9 + 16 = h^2 - 2h + 1 + k^2 + 36 + 12k + h^2 - 8h + 16 + k^2 + 4k + 4$

$\Rightarrow 25 = 2h^2 + 2k^2 - 10h + 16k + 57$

$\Rightarrow 2h^2 + 2k^2 - 10h + 16k + 57 - 25 = 0$

$\Rightarrow 2h^2 + 2k^2 - 10h + 16k + 32 = 0$

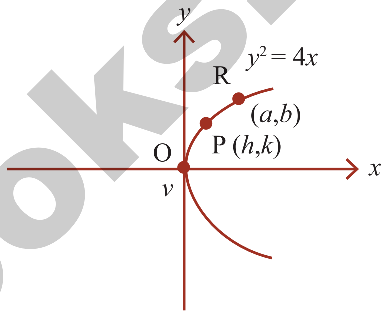
Dividing by 2, we get,

$\Rightarrow h^2 + k^2 - 5h + 8k + 16 = 0$

$\therefore$ Locus of (h, k) is $x^2 + y^2 - 5x + 8y + 16 = 0$

8. If O is origin and R is a variable point on $y^2 = 4x$, then find the equation of the locus of the mid-point of the line segment OR.

Sol : Given (0, 0) is the vertex of $y^2 = 4x$ and P (h, k) be the mid-point on the segment OR and the co-ordinates of R be (a, b)



Now P is the mid-point of OR

$\therefore (h, k) = \left(\frac{0+a}{2}, \frac{0+b}{2}\right) = \left(\frac{a}{2}, \frac{b}{2}\right)$

$\Rightarrow h = \frac{a}{2}$ and $k = \frac{b}{2}$

$\Rightarrow a = 2h$ and $b = 2k$ ... (1)

Here a and b are two variable which are to be eliminated.

Since (a, b) lies on $y^2 = 4x \Rightarrow b^2 = 4a$

$\Rightarrow (2k)^2 = 4(2h)$

$\Rightarrow 4k^2 = 8h$

$\Rightarrow k^2 = 2h$

$\therefore$ Locus of (h, k) is $y^2 = 2x$

9. The coordinates of a moving point P are $\left(\frac{a}{2}(\text{cosec}\,\theta + \sin\theta), \frac{b}{2}(\text{cosec}\,\theta - \sin\theta)\right)$, where θ is a variable parameter. Show that the equation of the locus P is $b^2x^2 - a^2y^2 = a^2b^2$.

Sol : Let P (h,k) be any point on the locus

By the given condition,

$h = \frac{a}{2}(\text{cosec}\,\theta + \sin\theta)$,

$k = \frac{b}{2}(\text{cosec}\,\theta - \sin\theta)$

$\Rightarrow \quad \frac{2h}{a} = \text{cosec}\,\theta + \sin\theta$ and $\frac{2k}{b} = \text{cosec}\,\theta - \sin\theta$

Squaring and subtracting we get,

$$\Rightarrow \left(\frac{2h}{a}\right)^2 - \left(\frac{2k}{b}\right)^2 = (\text{cosec}\,\theta + \sin\theta)^2 - (\text{cosec}\,\theta - \sin\theta)^2$$

$$\Rightarrow \frac{4h^2}{a^2} - \frac{4k^2}{b^2} = \text{cosec}^2\theta + \sin^2\theta + 2\,\text{cosec}\,\theta\,\sin\theta - (\text{cosec}^2\theta + \sin^2\theta - 2\,\text{cosec}\,\theta\,\sin\theta)$$

$$\Rightarrow \frac{4h^2}{a^2} - \frac{4k^2}{b^2} = \text{cosec}^2\theta + \sin^2\theta + 2\,\text{cosec}\,\theta\,\sin\theta - \text{cosec}^2\theta - \sin^2\theta + 2\,\text{cosec}\,\sin\theta$$

$$\Rightarrow \frac{4h^2}{a^2} - \frac{4k^2}{b^2} = 4\,\text{cosec}\,\theta\,\sin\theta$$

$$\Rightarrow \frac{4h^2}{a^2} - \frac{4k^2}{b^2} = 4 \quad \left[\because \text{cosec}\,\theta \sin\theta = \frac{1}{\sin\theta}.\sin\theta = 1\right]$$

Dividing by 4 we get,

$$\Rightarrow \frac{h^2}{a^2} - \frac{k^2}{b^2} = 1 \Rightarrow \frac{b^2h^2 - a^2k^2}{a^2b^2} = 1$$

$$\Rightarrow b^2h^2 - a^2k^2 = a^2b^2$$

$\therefore$ Locus of (h, k) is $b^2x^2 - a^2y^2 = a^2b^2$

10. If P(2,–7) is a given point and Q is a point on $(2x^2 + 9y^2 = 18)$, then find the equations of the locus of the mid-point of PQ.

Sol : Let R(h, k) be the locus of the mid-point of PQ where, P is (2, –7) and Q is a point on $(2x^2 + 9y^2 = 18)$

Given equation is $2x^2 + 9y^2 = 18$

Dividing by 18 we get,

$$\Rightarrow \frac{x^2}{9} + \frac{y^2}{2} = 1 \Rightarrow \frac{x^2}{3^2} + \frac{y^2}{(\sqrt{2})^2} = 1$$

$$\Rightarrow a = 3 \text{ and } b = \sqrt{2}$$

Any point on the ellipse is $(a\cos\theta, b\sin\theta)$

$\therefore$ Q is $(3\cos\theta, \sqrt{2}\sin\theta)$

Since R is the mid-point of PQ, we get,

$$(h,k) = \left(\frac{2+3\cos\theta}{2}, \frac{-7+\sqrt{2}\sin\theta}{2}\right)$$

$$\Rightarrow h = \frac{2+3\cos\theta}{2} \qquad k = \frac{-7+\sqrt{2}\sin\theta}{2}$$

$$\Rightarrow 2h = 2 + 3\cos\theta \qquad \Rightarrow 2k = -7 + \sqrt{2}\sin\theta$$

$$\Rightarrow 2h - 2 = 3\cos\theta \qquad \Rightarrow 2k + 7 = \sqrt{2}\sin\theta$$

$$\Rightarrow \frac{2h-2}{3} = \cos\theta \qquad \Rightarrow \frac{2k+7}{\sqrt{2}} = \sin\theta$$

Squaring and adding we get, $\left(\frac{2h-2}{3}\right)^2 + \left(\frac{2k+7}{\sqrt{2}}\right)^2 = \cos^2\theta + \sin^2\theta$

$$\Rightarrow \frac{4h^2 + 4 - 8h}{9} + \frac{4k^2 + 49 + 28k}{2} = 1 \quad [\because \cos^2\theta + \sin^2\theta = 1]$$

$$\Rightarrow 2(4h^2 + 4 - 8h) + 9(4k^2 + 49 + 28k) = 18$$

$$\Rightarrow 8h^2 + 8 - 16h + 36k^2 + 441 + 252k - 18 = 0$$

$$\Rightarrow 8h^2 + 36k^2 - 16h + 252k + 431 = 0$$

$\therefore$ Locus of (h, k) is $8x^2 + 36y^2 - 16x + 252y + 431 = 0$

11. If R is any point on the x - axis and Q is any point on the y- axis and P is a variable point on RQ with RP = b, PQ = a, then find the equation of locus of P.

Sol : Let P(h, k) be the point on RQ such that $\angle$ORQ = θ where θ is a variable.

From triangle PLR, we get $\sin\theta = \frac{k}{b}$ ... (1)

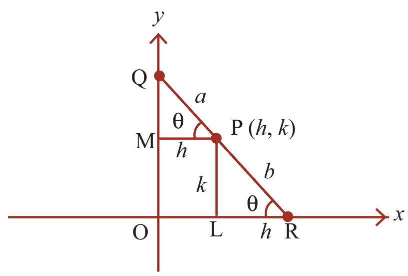
$$\left[\because \sin\theta = \frac{\text{opp}}{\text{hyp}} \text{ and PR} = b\right]$$

From triangle PMQ, we get $\cos\theta = \frac{h}{b}$ ... (2)

$$\left[\because \cos\theta = \frac{\text{adj}}{\text{hyp}} = \frac{h}{a} \text{ and PQ} = a\right]$$

Squaring and adding (1) and (2), we get,

$$\sin^2\theta + \cos^2\theta = \left(\frac{k}{b}\right)^2 + \left(\frac{h}{a}\right)^2$$



$$\Rightarrow 1 = \frac{k^2}{b^2} + \frac{h^2}{a^2} \quad [\because \sin^2\theta + \cos^2\theta = 1]$$

$$\Rightarrow \frac{h^2}{a^2} + \frac{k^2}{b^2} = 1$$

The locus of $p(h, k)$ is obtained by replacing h by x and k by y.

$\therefore$ Locus of (h, k) is $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$

07 MATRICES AND DETERMINANTS

MUST KNOW DEFINITIONS

Matrix	:	A matrix is a rectangular array or arrangement of entries or elements displayed in rows and columns put within a square bracket [ ].
Order of Matrix	:	If a matrix A has m rows and n columns then the order or size of the matrix A is defined to be $m \times n$.
Column Matrix	:	A matrix having only one column is called a column matrix.
Row matrix	:	A matrix having only one row is called a row matrix.
Square matrix	:	A matrix in which number of rows is equal to the number of columns, is called a square matrix.
Diagonal matrix	:	A square matrix A = $[a_{ij}]_{m\times n}$ is called a diagonal matrix. If $a_{ij} = 0$ whenever $i \neq j$
Scalar matrix	:	A diagonal matrix whose entries along the principal diagonal are equal is called a scalar matrix.
Unit matrix	:	A square matrix in which all the diagonal entries are 1 and the rest are all zero is called a unit matrix.
Triangular matrix	:	A square matrix which is either upper triangular or lower triangular is called a triangular matrix.
Singular and Non - Singular Matrix	:	A square matrix A is said to be singular if $\|A\| = 0$. A square matrix A is said to be non-singular if $\|A\| \neq 0$.

Properties of Determinants :

1. The value of the determinant remains unchanged if its rows and columns are interchanged.
2. If any two rows (or columns) of a determinant are interchanged, then sign of determinant changes.
3. If any two rows (or columns) of a determinant are identical, then the value of the determinant is zero.
4. If each element of a row (or column) of a determinant is multiplied by a constant k, then its value gets multiplied by k.
5. If some or all elements of a row or column of a determinant are expressed as sum of two (or more) terms, then the determinant can be expressed as sum of two (or more) determinants.
6. The value of the determinant remain same if we apply the operation. $R_i \to R_i + kR_j$ or $C_i \to C_i + kC_j$

Minor of an element

- The concept of determinant can be extended to the case of square matrix or order n, $n \geq 4$. Let $A = [a_{ij}]_{m\times n}$, $n \geq 4$.
- If we delete the i^{th} row and j^{th} column from the matrix of $A = [a_{ij}]_{n\times m}$, we obtain a determinant of order $(n-1)$, which is called the minor of the element a_{ij}.

Adjoint

- Adjoint of a square matrix $A = [a_{ij}]_{n\times n}$ is defined as the transpose of the matrix $[A_{ij}]_{n\times n}$ where A_{ij} is the co-factor of the element a_{ij}.

Solving linear equations by Gaussian Elimination method

- Gaussian elimination is a method of solving a linear system by bringing the augmented matrix to an upper triangular form.

FORMULAE TO REMEMBER

- $kA = [a_{ij}]_{m\times n}$ $[ka_{ij}]_{m\times n}$ where k is a scalar.
- $-A = (-1)A$, $A - B = A + (-1)B$
- $A + B = B + A$, (Commutative property for addition)
- $(A + B) + C = A + (B + C)$, (Associative property for addition)
- $k(A + B) = kA + kB$ where A, B are of same order, k is a constant.
- $(k + l)A = kA + lA$ where k and l are constants.
- $A(BC) = (AB)C$, $A(B + C) = AB + AC$, $(A + B)C = AC + BC$. (Distributive law)
- If $A = (a_{ij})_{m\times n}$, then $A^T = (a_{ji})_{n\times m}$
- Elementary operations of a matrix are as follows
  (i) $R_i \leftrightarrow R_j$ or $C_i \leftrightarrow C_j$ (ii) $R_i \rightarrow kR_i$ or $C_i \rightarrow kC_j$ (iii) $R_i \rightarrow R_i + kR_j$ or $C_i \rightarrow C_i + kC_j$
- Evaluation of determinant $A = [a_{11}]_{1\times 1} = |A| = a_{11}$
- Evaluation of determinant $A = \begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix} = a_{11}a_{22} - a_{21}a_{12}$
- Evaluation of determinant $A = \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}$ is $|A| = a_1 \begin{vmatrix} b_2 & c_3 \\ b_3 & c_3 \end{vmatrix} - b_1 \begin{vmatrix} a_2 & c_2 \\ a_3 & c_3 \end{vmatrix} + c_1 \begin{vmatrix} a_2 & b_2 \\ a_3 & b_3 \end{vmatrix}$
- If $A = [a_{ij}]_{3\times 3}$, then $|k \cdot A| = k^3 |A|$.
- $A(adj\ A) = (adj\ A)A = |A|$. I where A is a square matrix of order n.
- A square matrix A is said to be singular or non-singular according as $|A| = 0$ or $|A| \neq 0$.
- **Transpose of a matrix:** $(A^T)^T = A$, $(kA)^T = kA^T$. $(A + B)^T = A^T + B^T$, $(AB)^T = B^T A^T$.
- Co-factor of a_{ij} of $A_{ij} = (-1)^{i+j} m_{ij}$ where m_{ij} is the minor of a_{ij}.
- $|AB| = |A| \cdot |B|$ where A and B are square matrices of same order.

TEXTUAL QUESTIONS

Exercise 7.1

1. Construct an $m \times n$ matrix $A = [a_{ij}]$, where a_{ij} is given by

(i) $a_{ij} = \frac{(i-2j)^2}{2}$ with $m = 2, n = 3$ [Sep. - 2021]

(ii) $a_{ij} = \frac{|3i-4j|}{4}$ with $m = 3, n = 4$

Sol : (i) Given $a_{ij} = \frac{(i-2j)^2}{2}$ with $m = 2, n = 3$

we need to construct a 2×3 matrix.

$$\therefore a_{11} = \frac{(1-2(1))^2}{2} = \frac{(-1)^2}{2} = \frac{1}{2}$$

$$a_{12} = \frac{(1-2(2))^2}{2} = \frac{(-3)^2}{2} = \frac{9}{2}$$

$$a_{13} = \frac{(1-2(3))^2}{2} = \frac{(-5)^2}{2} = \frac{25}{2}$$

$$a_{21} = \frac{(2-2(1))^2}{2} = \frac{0}{2} = 0$$

$$a_{22} = \frac{(2-2(2))^2}{2} = \frac{(-2)^2}{2} = \frac{4}{2}$$

$$a_{23} = \frac{(2-2(3))^2}{2} = \frac{(-4)^2}{2} = \frac{16}{2}$$

$$\therefore A = \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \end{pmatrix}$$

$$= \begin{pmatrix} \frac{1}{2} & \frac{9}{2} & \frac{25}{2} \\ 0 & \frac{4}{2} & \frac{16}{2} \end{pmatrix} = \frac{1}{2}\begin{pmatrix} 1 & 9 & 25 \\ 0 & 4 & 16 \end{pmatrix}$$

(ii) Given $a_{ij} = \frac{|3i-4j|}{4}$ with $m = 3, n = 4$.

Let B be a 3×4 matrix with entries as

$$B = \begin{pmatrix} a_{11} & a_{12} & a_{13} & a_{14} \\ a_{21} & a_{22} & a_{23} & a_{24} \\ a_{31} & a_{32} & a_{33} & a_{34} \end{pmatrix}$$

$$a_{ij} = \frac{|3i-4j|}{4}$$

$$a_{11} = \frac{|3-4|}{4} = \frac{|-1|}{4} = \frac{1}{4}$$

$$a_{12} = \frac{|3-8|}{4} = \frac{|-5|}{4} = \frac{5}{4}$$

$$a_{13} = \frac{|3-12|}{4} = \frac{|-9|}{4} = \frac{9}{4}$$

$$a_{14} = \frac{|3-16|}{4} = \frac{|-13|}{4} = \frac{13}{4}$$

$$a_{21} = \frac{|3(2)-4(1)|}{4} = \frac{|6-4|}{4} = \frac{2}{4}$$

$$a_{22} = \frac{|3(2)-4(2)|}{4} = \frac{|6-8|}{4} = \frac{2}{4}$$

$$a_{23} = \frac{|3(2)-4(3)|}{4} = \frac{|6-12|}{4} = \frac{6}{4}$$

$$a_{24} = \frac{|3(2)-4(4)|}{4} = \frac{|6-16|}{4} = \frac{10}{4}$$

$$a_{31} = \frac{|3(3)-4(1)|}{4} = \frac{|9-4|}{4} = \frac{5}{4}$$

$$a_{32} = \frac{|3(3)-4(2)|}{4} = \frac{|9-8|}{4} = \frac{1}{4}$$

$$a_{33} = \frac{|3(3)-4(3)|}{4} = \frac{|9-12|}{4} = \frac{3}{4}$$

$$a_{34} = \frac{|3(3)-4(4)|}{4} = \frac{|9-16|}{4} = \frac{7}{4}$$

$$\therefore B = \begin{bmatrix} 1/4 & 5/4 & 9/4 & 13/4 \\ 2/4 & 2/4 & 6/4 & 10/4 \\ 5/4 & 1/4 & 3/4 & 7/4 \end{bmatrix} = \frac{1}{4}\begin{bmatrix} 1 & 5 & 9 & 13 \\ 2 & 2 & 6 & 10 \\ 5 & 1 & 3 & 7 \end{bmatrix}$$

2. Find the values of p, q, r, and s if

$$\begin{bmatrix} p^2-1 & 0 & -31-q^3 \\ 7 & r+1 & 9 \\ -2 & 8 & s-1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & -4 \\ 7 & \frac{3}{2} & 9 \\ -2 & 8 & -\pi \end{bmatrix}$$

Sol : Given $\begin{bmatrix} p^2-1 & 0 & -31-q^3 \\ 7 & r+1 & 9 \\ -2 & 8 & s-1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & -4 \\ 7 & 3/2 & 9 \\ -2 & 8 & -\pi \end{bmatrix}$

Since the matrices are equal, the corresponding entries on both sides are equal.

$\therefore p^2 - 1 = 1 \Rightarrow p^2 = 2 \Rightarrow p = \pm\sqrt{2}$ [Equating a_{11}]

$-31 - q^3 = -4 \Rightarrow -q^3 = -4 + 31$ [Equating a_{13}]

(i) $\therefore A_\alpha A_\beta = \begin{bmatrix} \cos\alpha & -\sin\alpha \\ \sin\alpha & \cos\alpha \end{bmatrix}\begin{bmatrix} \cos\beta & -\sin\beta \\ \sin\beta & \cos\beta \end{bmatrix}$

$$= \begin{bmatrix} \cos\alpha\cos\beta - \sin\alpha\sin\beta & -\cos\alpha\sin\beta - \sin\alpha\cos\beta \\ \sin\alpha\cos\beta + \cos\alpha\sin\beta & -\sin\alpha\sin\beta + \cos\alpha\cos\beta \end{bmatrix}$$

$$= \begin{bmatrix} \cos(\alpha+\beta) & -\sin(\alpha+\beta) \\ \sin(\alpha+\beta) & \cos(\alpha+\beta) \end{bmatrix}$$

$$\begin{bmatrix} \text{since } \sin(\alpha+\beta) = \sin\alpha\cos\beta + \cos\alpha\sin\beta \\ \cos(\alpha+\beta) = \cos\alpha\cos\beta - \sin\alpha\sin\beta \end{bmatrix}$$

$$A_\alpha . A_\beta = A_{\alpha+\beta}$$

Hence proved.

(ii) Given $A_\alpha = \begin{bmatrix} \cos\alpha & -\sin\alpha \\ \sin\alpha & \cos\alpha \end{bmatrix}$

$$A_\alpha^T = \begin{bmatrix} \cos\alpha & \sin\alpha \\ -\sin\alpha & \cos\alpha \end{bmatrix}$$

Also, it is given that $A_\alpha + A_\alpha^T = I$

$$\Rightarrow \begin{bmatrix} \cos\alpha & -\sin\alpha \\ \sin\alpha & \cos\alpha \end{bmatrix} + \begin{bmatrix} \cos\alpha & \sin\alpha \\ -\sin\alpha & \cos\alpha \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} 2\cos\alpha & 0 \\ 0 & 2\cos\alpha \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

Equating the corresponding entries on both sides, we get

$$2\cos\alpha = 1 \Rightarrow \cos\alpha = \frac{1}{2} = \cos\frac{\pi}{3}.$$

$$\Rightarrow \alpha = 2n\pi \pm \frac{\pi}{3},\ n \in \mathbb{Z}$$

$[\because \cos\alpha = \cos\theta \Rightarrow \alpha = 2n\pi \pm \theta,\ n \in \mathbb{Z}]$

$$\therefore \alpha = 2n\pi \pm \frac{\pi}{3},\ n \in \mathbb{Z}$$

7. If $A = \begin{bmatrix} 4 & 2 \\ -1 & x \end{bmatrix}$ and such that $(A - 2I)(A - 3I) = 0$, find the value of x.

Sol : Given $A = \begin{bmatrix} 4 & 2 \\ -1 & x \end{bmatrix}$

Also, $(A - 2I)(A - 3I) = 0$

$$\therefore A - 2I = \begin{bmatrix} 4 & 2 \\ -1 & x \end{bmatrix} - 2\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 4-2 & 2-0 \\ -1-0 & x-2 \end{bmatrix} = \begin{bmatrix} 2 & 2 \\ -1 & x-2 \end{bmatrix}$$

$$A - 3I = \begin{bmatrix} 4 & 2 \\ -1 & x \end{bmatrix} - 3\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 4-3 & 2-0 \\ -1-0 & x-3 \end{bmatrix} = \begin{bmatrix} 1 & 2 \\ -1 & x-3 \end{bmatrix}$$

$$\therefore (A - 2I)(A - 3I) = \begin{bmatrix} 2 & 2 \\ -1 & x-2 \end{bmatrix}\begin{bmatrix} 1 & 2 \\ -1 & x-3 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} 2-2 & 4+2(x-3) \\ -1-1(x-2) & -2+(x-2)(x-3) \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} 0 & 4+2x-6 \\ -1-x+2 & -2+x^2-5x+6 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} 0 & 2x-2 \\ -x+1 & x^2-5x+4 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$$

Equating the corresponding entries we get,

$$x^2 - 5x + 4 = 0$$

$$\Rightarrow (x-4)(x-1) = 0$$

$$\Rightarrow x = 1,\ x = 4$$

($x = 4$ not possible)

$$2x - 2 = 0 \Rightarrow 2x = 2 \Rightarrow x = 1$$

$$-x + 1 = 0 \Rightarrow -x = -1 \Rightarrow x = 1$$

Since $x = 1$ alone satisfies the equation $(A - 2I)(A - 3I) = 0$, we get $x = 1$.

8. If $A = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ a & b & -1 \end{bmatrix}$, show that A^2 is a unit matrix.

Sol : Given $A = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ a & b & -1 \end{bmatrix}$

$$A^2 = A.A = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ a & b & -1 \end{bmatrix}\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ a & b & -1 \end{bmatrix}$$

$$= \begin{bmatrix} 1+0+0 & 0+0+0 & 0+0+0 \\ 0+0+0 & 0+1+0 & 0+0+0 \\ a+0-a & 0+b-b & 0+0+1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$\therefore A^2$ is a unit matrix.

$$= \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$$

$$BA = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} 0+0 & 1+0 \\ 0+0 & 0+0 \end{bmatrix}$$

$$= \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$$

$\therefore$ AB = 0 and BA $\neq$ 0

11. Show that $f(x)f(y) = f(x+y)$, where

$$f(x) = \begin{bmatrix} \cos x & -\sin x & 0 \\ \sin x & \cos x & 0 \\ 0 & 0 & 1 \end{bmatrix}.$$

Sol : Given $f(x) = \begin{bmatrix} \cos x & -\sin x & 0 \\ \sin x & \cos x & 0 \\ 0 & 0 & 1 \end{bmatrix}$

Also $f(y) = \begin{bmatrix} \cos y & -\sin y & 0 \\ \sin y & \cos y & 0 \\ 0 & 0 & 1 \end{bmatrix}$

$$f(x).f(y) = \begin{bmatrix} \cos x & -\sin x & 0 \\ \sin x & \cos x & 0 \\ 0 & 0 & 1 \end{bmatrix}\begin{bmatrix} \cos y & -\sin y & 0 \\ \sin y & \cos y & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} \cos x\cos y - \sin x\sin y & -\cos x\sin y - \sin x\cos y & 0 \\ \sin x\cos y + \cos x\sin y & -\sin x\sin y + \cos x\cos y & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} \cos(x+y) & -\sin(x+y) & 0 \\ \sin(x+y) & \cos(x+y) & 0 \\ 0 & 0 & 1 \end{bmatrix} = f(x+y)$$

[$\because \cos(x+y) = \cos x\cos y - \sin x \sin y$
$\sin(x+y) = \sin x\cos y + \cos x\sin y$]

12. If A is a square matrix such that $A^2 = A$, find the value of $7A - (I + A)^3$.

Sol : Given A is a square matrix and $A^2 = A$

Consider $7A - (I+A)^3 = 7A - (I^3 + 3I^2A + 3IA^2 + A^3)$

[$\because (a+b)^3 = a^3 + 3a^2b + 3ab^2 + b^3$]

$= 7A - (I + 3A + 3A^2 + A^2.A)$

$= 7A - (I + 3A + 3A + A.A)$ [$\because A^2 = A$]

[$\because I^3 = I, I^2 = I$]

$= 7A - (I + 6A + A^2)$

$= 7A - (I + 6A + A)$

$= 7A - (I + 7A) = \cancel{7A} - I - \cancel{7A} = -I$

13. Verify the property A(B + C) = AB + AC, when the matrices A, B, and C are given by

$$A = \begin{bmatrix} 2 & 0 & -3 \\ 1 & 4 & 5 \end{bmatrix}, B = \begin{bmatrix} 3 & 1 \\ -1 & 0 \\ 4 & 2 \end{bmatrix}, \text{ and } C = \begin{bmatrix} 4 & 7 \\ 2 & 1 \\ 1 & -1 \end{bmatrix}.$$

Sol : Given $A = \begin{bmatrix} 2 & 0 & -3 \\ 1 & 4 & 5 \end{bmatrix}, B = \begin{bmatrix} 3 & 1 \\ -1 & 0 \\ 4 & 2 \end{bmatrix}$

and $C = \begin{bmatrix} 4 & 7 \\ 2 & 1 \\ 1 & -1 \end{bmatrix}$

$$B + C = \begin{bmatrix} 3 & 1 \\ -1 & 0 \\ 4 & 2 \end{bmatrix} + \begin{bmatrix} 4 & 7 \\ 2 & 1 \\ 1 & -1 \end{bmatrix}$$

$$= \begin{bmatrix} 3+4 & 1+7 \\ -1+2 & 0+1 \\ 4+1 & 2-1 \end{bmatrix} = \begin{bmatrix} 7 & 8 \\ 1 & 1 \\ 5 & 1 \end{bmatrix}$$

$$\therefore A(B+C) = \begin{bmatrix} 2 & 0 & -3 \\ 1 & 4 & 5 \end{bmatrix}\begin{bmatrix} 7 & 8 \\ 1 & 1 \\ 5 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 14+0-15 & 16+0-3 \\ 7+4+25 & 8+4+5 \end{bmatrix}$$

$$\text{LHS} = A(B+C) = \begin{bmatrix} -1 & 13 \\ 36 & 17 \end{bmatrix} \quad \dots (1)$$

$$AB = \begin{bmatrix} 2 & 0 & -3 \\ 1 & 4 & 5 \end{bmatrix}\begin{bmatrix} 3 & 1 \\ -1 & 0 \\ 4 & 2 \end{bmatrix}$$

$$= \begin{bmatrix} 6+0-12 & 2+0-6 \\ 3-4+20 & 1+0+10 \end{bmatrix}$$

$$= \begin{bmatrix} -6 & -4 \\ 19 & 11 \end{bmatrix}$$

$$AC = \begin{bmatrix} 2 & 0 & -3 \\ 1 & 4 & 5 \end{bmatrix}\begin{bmatrix} 4 & 7 \\ 2 & 1 \\ 1 & -1 \end{bmatrix}$$

$$= \begin{bmatrix} 8+0-3 & 14+0+3 \\ 4+8+5 & 7+4-5 \end{bmatrix} = \begin{bmatrix} 5 & 17 \\ 17 & 6 \end{bmatrix}$$

RHS = AB + AC

$$= \begin{bmatrix} -6 & -4 \\ 19 & 11 \end{bmatrix} + \begin{bmatrix} 5 & 17 \\ 17 & 6 \end{bmatrix}$$

$$= \begin{bmatrix} -6+5 & -4+17 \\ 19+17 & 11+6 \end{bmatrix} = \begin{bmatrix} -1 & 13 \\ 36 & 17 \end{bmatrix} \quad \dots (2)$$

From (1) and (2), A(B + C) = AB + AC.

14. Find the matrix A which satisfies the matrix relation $A\begin{bmatrix}1 & 2 & 3\\4 & 5 & 6\end{bmatrix} = \begin{bmatrix}-7 & -8 & -9\\2 & 4 & 6\end{bmatrix}$.

Sol : Given $A\begin{bmatrix}1 & 2 & 3\\4 & 5 & 6\end{bmatrix} = \begin{bmatrix}-7 & -8 & -9\\2 & 4 & 6\end{bmatrix}$

The order of the matrix $\begin{bmatrix}1 & 2 & 3\\4 & 5 & 6\end{bmatrix}$ is 2 × 3 and the order of the matrix $\begin{bmatrix}-7 & -8 & -9\\2 & 4 & 6\end{bmatrix}$ is also 2 × 3.

∴ A must be of order 2 × 2.

$$\text{Let } A = \begin{bmatrix}a & b\\c & d\end{bmatrix}$$

$$\therefore \begin{bmatrix}a & b\\c & d\end{bmatrix}\begin{bmatrix}1 & 2 & 3\\4 & 5 & 6\end{bmatrix} = \begin{bmatrix}-7 & -8 & -9\\2 & 4 & 6\end{bmatrix}$$

$$\Rightarrow \begin{bmatrix}a+4b & 2a+5b & 3a+6b\\c+4d & 2c+5d & 3c+6d\end{bmatrix} = \begin{bmatrix}-7 & -8 & -9\\2 & 4 & 6\end{bmatrix}$$

Equating the corresponding entries on both sides, we get

$$a + 4b = -7 \quad \text{... (1)}$$
$$2a + 5b = -8 \quad \text{... (2)}$$
$$c + 4d = 2 \quad \text{... (3)}$$
$$2c + 5d = 4 \quad \text{... (4)}$$

(1) × 2 ⇒ $2a + 8b = -14$

(2) ⇒ $2a + 5b = -8$ [(–) (–) (+)]

$3b = -6 \quad \Rightarrow b = -2$

Substituting $b = -2$ in (1) we get,

$a - 8 = -7 \Rightarrow a = -7 + 8 \Rightarrow a = 1$

(3) × 2 ⇒ $2c + 8d = 4$

(4) ⇒ $2c + 5d = 4$ [(–) (–) (–)]

$3d = 0 \quad \Rightarrow d = 0$

Substituting $d = 0$ in (3) we get,

$c = 2$

$$\therefore A = \begin{bmatrix}1 & -2\\2 & 0\end{bmatrix}$$

15. If $A^T = \begin{bmatrix}4 & 5\\-1 & 0\\2 & 3\end{bmatrix}$ **and** $B = \begin{bmatrix}2 & -1 & 1\\7 & 5 & -2\end{bmatrix}$, **verify the following**

(i) $(A + B)^T = A^T + B^T = B^T + A^T$

(ii) $(A - B)^T = A^T - B^T$ **(iii)** $(B^T)^T = B$.

Sol : Given $A^T = \begin{bmatrix}4 & 5\\-1 & 0\\2 & 3\end{bmatrix}$ and $B = \begin{bmatrix}2 & -1 & 1\\7 & 5 & -2\end{bmatrix}$

(i) Verify $(A + B)^T = A^T + B^T = B^T + A^T$

$$(A^T)^T = \begin{bmatrix}4 & 5\\-1 & 0\\2 & 3\end{bmatrix}^T \Rightarrow A = \begin{bmatrix}4 & -1 & 2\\5 & 0 & 3\end{bmatrix}$$

$$\text{Now, } A + B = \begin{bmatrix}4 & -1 & 2\\5 & 0 & 3\end{bmatrix} + \begin{bmatrix}2 & -1 & 1\\7 & 5 & -2\end{bmatrix} = \begin{bmatrix}6 & -2 & 3\\12 & 5 & 1\end{bmatrix}$$

$$\therefore (A+B)^T = \begin{bmatrix}6 & 12\\-2 & 5\\3 & 1\end{bmatrix} \quad \text{... (1)}$$

$$B^T = \begin{bmatrix}2 & 7\\-1 & 5\\1 & -2\end{bmatrix}$$

$$\therefore A^T + B^T = \begin{bmatrix}4 & 5\\-1 & 0\\2 & 3\end{bmatrix} + \begin{bmatrix}2 & 7\\-1 & 5\\1 & -2\end{bmatrix} = \begin{bmatrix}6 & 12\\-2 & 5\\3 & 1\end{bmatrix} \quad \text{... (2)}$$

$$B^T + A^T = \begin{bmatrix}2 & 7\\-1 & 5\\1 & -2\end{bmatrix} + \begin{bmatrix}4 & 5\\-1 & 0\\2 & 3\end{bmatrix} = \begin{bmatrix}6 & 12\\-2 & 5\\3 & 1\end{bmatrix} \quad \text{... (3)}$$

From (1), (2) and (3), $(A + B)^T = A^T + B^T = B^T + A^T$

(ii) Verify $(A - B)^T = A^T - B^T$

$$A - B = \begin{bmatrix}4 & -1 & 2\\5 & 0 & 3\end{bmatrix} - \begin{bmatrix}2 & -1 & 1\\7 & 5 & -2\end{bmatrix} = \begin{bmatrix}2 & 0 & 1\\-2 & -5 & 5\end{bmatrix}$$

$$(A - B)^T = \begin{bmatrix}2 & -2\\0 & -5\\1 & 5\end{bmatrix} \quad \text{... (4)}$$

$$A^T - B^T = \begin{bmatrix} 4 & 5 \\ -1 & 0 \\ 2 & 3 \end{bmatrix} - \begin{bmatrix} 2 & 7 \\ -1 & 5 \\ 1 & -2 \end{bmatrix} = \begin{bmatrix} 2 & -2 \\ 0 & -5 \\ 1 & 5 \end{bmatrix} \quad \dots (5)$$

From (4) and (5), $(A - B)^T = A^T - B^T$

(iii) Verify $(B^T)^T = B$

$$\text{Given } B = \begin{bmatrix} 2 & -1 & 1 \\ 7 & 5 & -2 \end{bmatrix} \quad \dots (6)$$

$$\therefore B^T = \begin{bmatrix} 2 & 7 \\ -1 & 5 \\ 1 & -2 \end{bmatrix}$$

$$\text{Also, } (B^T)^T = \begin{bmatrix} 2 & -1 & 1 \\ 7 & 5 & -2 \end{bmatrix} \quad \dots (7)$$

From (6) and (7), $(B^T)^T = B$

16. If A is a 3 × 4 matrix and B is a matrix such that both A^TB and BA^T are defined, what is the order of the matrix B?

Sol : Given A is a 3 × 4 matrix.

A^T is a 4 × 3 matrix.

A^TB and BA^T are defined.

A^TB	BA^T
$4 \times 3 \quad m \times n$	$m \times n \quad 4 \times 3$
$\Rightarrow m = 3$	$\Rightarrow n = 4$

$\therefore$ B matrix is of the order (3, 4) (i.e) 3 × 4

17. Express the following matrices as the sum of a symmetric matrix and a skew-symmetric matrix:

(i) $\begin{bmatrix} 4 & -2 \\ 3 & -5 \end{bmatrix}$ **and (ii)** $\begin{bmatrix} 3 & 3 & -1 \\ -2 & -2 & 1 \\ -4 & -5 & 2 \end{bmatrix}$.

Sol : (i) Let $A = \begin{bmatrix} 4 & -2 \\ 3 & -5 \end{bmatrix} \Rightarrow A^T = \begin{bmatrix} 4 & 3 \\ -2 & -5 \end{bmatrix}$

Let $P = \frac{1}{2}(A + A^T)$

$$= \frac{1}{2}\left\{\begin{bmatrix} 4 & -2 \\ 3 & -5 \end{bmatrix} + \begin{bmatrix} 4 & 3 \\ -2 & -5 \end{bmatrix}\right\} = \frac{1}{2}\begin{bmatrix} 8 & 1 \\ 1 & -10 \end{bmatrix}$$

$$\Rightarrow P^T = \frac{1}{2}\begin{bmatrix} 8 & 1 \\ 1 & -10 \end{bmatrix} = P$$

$\therefore$ P is a symmetric matrix.

$$\text{Let } Q = \frac{1}{2}\left[A - A^T\right]$$

$$= \frac{1}{2}\left\{\begin{bmatrix} 4 & -2 \\ 3 & -5 \end{bmatrix} - \begin{bmatrix} 4 & 3 \\ -2 & -5 \end{bmatrix}\right\} = \frac{1}{2}\begin{bmatrix} 0 & -5 \\ 5 & 0 \end{bmatrix}$$

$$Q^T = \frac{1}{2}\begin{bmatrix} 0 & 5 \\ -5 & 0 \end{bmatrix} = -Q$$

$\therefore$ Q is a skew-symmetric matrix.

$$\text{Now } A = P + Q = \frac{1}{2}\begin{bmatrix} 8 & 1 \\ 1 & -10 \end{bmatrix} + \frac{1}{2}\begin{bmatrix} 0 & -5 \\ 5 & 0 \end{bmatrix}$$

Thus A is expressed as the sum of symmetric and skew-symmetric matrices.

$$\text{(ii) Let } B = \begin{bmatrix} 3 & 3 & -1 \\ -2 & -2 & 1 \\ -4 & -5 & 2 \end{bmatrix} \Rightarrow B^T = \begin{bmatrix} 3 & -2 & -4 \\ 3 & -2 & -5 \\ -1 & 1 & 2 \end{bmatrix}$$

$$\text{Let } P = \frac{1}{2}\left[B + B^T\right] = \frac{1}{2}\left\{\begin{bmatrix} 3 & 3 & -1 \\ -2 & -2 & 1 \\ -4 & -5 & 2 \end{bmatrix} + \begin{bmatrix} 3 & -2 & -4 \\ 3 & -2 & -5 \\ -1 & 1 & 2 \end{bmatrix}\right\}$$

$$= \frac{1}{2}\begin{bmatrix} 6 & 1 & -5 \\ 1 & -4 & -4 \\ -5 & -4 & 4 \end{bmatrix}$$

$$\Rightarrow P^T = \frac{1}{2}\begin{bmatrix} 6 & 1 & -5 \\ 1 & -4 & -4 \\ -5 & -4 & 4 \end{bmatrix} = P$$

$\therefore$ P is a symmetric matrix.

$$\text{Let } Q = \frac{1}{2}\left[B - B^T\right] = \frac{1}{2}\left\{\begin{bmatrix} 3 & 3 & -1 \\ -2 & -2 & 1 \\ -4 & -5 & 2 \end{bmatrix} - \begin{bmatrix} 3 & -2 & -4 \\ 3 & -2 & -5 \\ -1 & 1 & 2 \end{bmatrix}\right\}$$

$$= \frac{1}{2}\begin{bmatrix} 0 & 5 & 3 \\ -5 & 0 & 6 \\ -3 & -6 & 0 \end{bmatrix}$$

$$\Rightarrow Q^T = \frac{1}{2}\begin{bmatrix} 0 & -5 & -3 \\ 5 & 0 & -6 \\ 3 & 6 & 0 \end{bmatrix} = -Q$$

$\therefore$ Q is a skew-symmetric matrix.

Now B = P + Q

$$= \frac{1}{2}\begin{bmatrix} 6 & 1 & -5 \\ 1 & -4 & -4 \\ -5 & -4 & 4 \end{bmatrix} + \frac{1}{2}\begin{bmatrix} 0 & 5 & 3 \\ -5 & 0 & 6 \\ -3 & -6 & 0 \end{bmatrix}$$

Thus, B is expressed as the sum of a symmetric and a skew-symmetric matrix.

18. Find the matrix A such that $\begin{bmatrix} 2 & -1 \\ 1 & 0 \\ -3 & 4 \end{bmatrix} A^T = \begin{bmatrix} -1 & -8 & -10 \\ 1 & 2 & -5 \\ 9 & 22 & 15 \end{bmatrix}$.

Sol : Given $\begin{bmatrix} 2 & -1 \\ 1 & 0 \\ -3 & 4 \end{bmatrix}_{3\times2} A^T = \begin{bmatrix} -1 & -8 & -10 \\ 1 & 2 & -5 \\ 9 & 22 & 15 \end{bmatrix}_{3\times3}$

A^T is a matrix of order 2×3.

Let $A^T = \begin{bmatrix} a & b & c \\ d & e & f \end{bmatrix}$

$$\therefore \begin{bmatrix} 2 & -1 \\ 1 & 0 \\ -3 & 4 \end{bmatrix}\begin{bmatrix} a & b & c \\ d & e & f \end{bmatrix} = \begin{bmatrix} -1 & -8 & -10 \\ 1 & 2 & -5 \\ 9 & 22 & 15 \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} 2a-d & 2b-e & 2c-f \\ a & b & c \\ -3a+4d & -3b+4e & -3c+4f \end{bmatrix} = \begin{bmatrix} -1 & -8 & -10 \\ 1 & 2 & -5 \\ 9 & 22 & 15 \end{bmatrix}$$

Equating the corresponding entries on both sides, we get,

$a = 1, b = 2, c = -5$ and $2a - d = -1$

$\Rightarrow \quad 2 - d = -1$

$\Rightarrow \quad 2 + 1 = d$

$\Rightarrow \quad d = 3$

$2b - e = -8$

$\Rightarrow \quad 4 - e = -8$

$\Rightarrow \quad 4 + 8 = e$

$\Rightarrow \quad e = 12$

$2c - f = -10$

$\Rightarrow \quad -10 - f = -10$

$\Rightarrow \quad f = 0$

$$\therefore A^T = \begin{bmatrix} 1 & 2 & -5 \\ 3 & 12 & 0 \end{bmatrix}$$

$$\Rightarrow (A^T)^T = A = \begin{bmatrix} 1 & 3 \\ 2 & 12 \\ -5 & 0 \end{bmatrix}$$

19. If $A = \begin{bmatrix} 1 & 2 & 2 \\ 2 & 1 & -2 \\ x & 2 & y \end{bmatrix}$ is a matrix such that $AA^T = 9I$, find the values of x and y.

Sol : Given $A = \begin{bmatrix} 1 & 2 & 2 \\ 2 & 1 & -2 \\ x & 2 & y \end{bmatrix} \Rightarrow A^T = \begin{bmatrix} 1 & 2 & x \\ 2 & 1 & 2 \\ 2 & -2 & y \end{bmatrix}$

Also, $AA^T = 9I$

$$\Rightarrow \begin{bmatrix} 1 & 2 & 2 \\ 2 & 1 & -2 \\ x & 2 & y \end{bmatrix}\begin{bmatrix} 1 & 2 & x \\ 2 & 1 & 2 \\ 2 & -2 & y \end{bmatrix} = 9\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} 1+4+4 & 2+2-4 & x+4+2y \\ 2+2-4 & 4+1+4 & 2x+2-2y \\ x+4+2y & 2x+2-2y & x^2+4+y^2 \end{bmatrix} = \begin{bmatrix} 9 & 0 & 0 \\ 0 & 9 & 0 \\ 0 & 0 & 9 \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} 9 & 0 & x+2y+4 \\ 0 & 9 & 2x-2y+2 \\ x+2y+4 & 2x-2y+2 & x^2+y^2+4 \end{bmatrix} = \begin{bmatrix} 9 & 0 & 0 \\ 0 & 9 & 0 \\ 0 & 0 & 9 \end{bmatrix}$$

Equating the corresponding entries on both sides, we get

$x + 2y + 4 = 0$

$2x - 2y + 2 = 0$

$\Rightarrow \quad x + 2y = -4 \quad$ ...(1)

$\Rightarrow \quad 2x - 2y = -2 \quad$ .. (2)

$3x = -6 \Rightarrow x = -2$

Substituting $x = -2$ in (1) we get,

$-2 + 2y = -4$

$\Rightarrow \quad 2y = -4 + 2 = -2$

$\Rightarrow \quad y = -1$

Hence, $x = -2, y = -1$

20. (i) For what value of x, the matrix $A = \begin{bmatrix} 0 & 1 & -2 \\ -1 & 0 & x^3 \\ 2 & -3 & 0 \end{bmatrix}$ is skew-symmetric. **[Hy. - 2018]**

(ii) If $\begin{bmatrix} 0 & p & 3 \\ 2 & q^2 & -1 \\ r & 1 & 0 \end{bmatrix}$ is skew-symmetric, find the values of p, q, and r.

$= BA + AB$ [using (1)]

$= AB + BA$

$\Rightarrow (AB + BA)^T = AB + BA$

$\therefore$ (AB + BA) is a symmetric matrix.

(ii) Given A and B are symmetric matrices

$\Rightarrow A^T = A$ and $B^T = B$ ... (2)

To prove that (AB – BA) is a skew-symmetric matrix.

Consider $(AB - BA)^T = (AB)^T - (BA)^T$

$= B^TA^T - A^TB^T$ $[\because (AB)^T = B^TA^T]$

$= BA - AB$ [using (2)]

$= -(AB - BA)$

$\Rightarrow (AB - BA)^T = -(AB - BA)$

$\therefore$ (AB – BA) is a skew-symmetric matrix.

24. A shopkeeper in a Nuts and Spices shop makes gift packs of cashew nuts, raisins and almonds. Pack I contains 100 gm of cashew nuts, 100 gm of raisins and 50 gm of almonds. Pack-II contains 200 gm of cashew nuts, 100 gm of raisins and 100 gm of almonds. Pack-III contains 250 gm of cashew nuts, 250 gm of raisins and 150 gm of almonds. The cost of 50 gm of cashew nuts is ₹ 50/-, 50 gm of raisins is ₹10/-, and 50 gm of almonds is ₹ 60/-. What is the cost of each gift pack?

Sol : Gift pack matrix is as follows:

	I	II	III
Weight of Cashew nuts	100	200	250
Weight of Raisins	100	100	250
Weight of Almonds	50	100	150

Let us consider 50 gm of cashew nuts as one packet, 50 gm of raisins as one packet and 50 gm of almonds as one packet, we get the matrix as

	I	II	III
No. of packets of cashewnuts	2	4	5
No. of packets of raisins	2	2	5
No. of packets of almonds	1	2	3

$= A$

Given cost matrix is $\begin{bmatrix} 50 & 10 & 60 \end{bmatrix} = B$

$$\therefore \text{Cost of gift pack} = AB = \begin{bmatrix} 50 & 10 & 60 \end{bmatrix}\begin{bmatrix} 2 & 4 & 5 \\ 2 & 2 & 5 \\ 1 & 2 & 3 \end{bmatrix}$$

$$= \begin{bmatrix} 100+20+60 \\ 200+20+120 \\ 250+50+180 \end{bmatrix} = \begin{bmatrix} 180 \\ 340 \\ 480 \end{bmatrix}$$

$\therefore$ Cost of I gift pack = ₹ 180

Cost of II gift pack = ₹ 340 and cost of III gift pack = ₹ 480

EXERCISE 7.2

1. Without expanding the determinant, prove that $\begin{vmatrix} s & a^2 & b^2+c^2 \\ s & b^2 & c^2+a^2 \\ s & c^2 & a^2+b^2 \end{vmatrix} = 0$

Sol : Let $A = \begin{vmatrix} s & a^2 & b^2+c^2 \\ s & b^2 & c^2+a^2 \\ s & c^2 & a^2+b^2 \end{vmatrix}$

Applying $C_2 \to C_2 + C_3$ we get,

$$A = \begin{vmatrix} s & a^2+b^2+c^2 & b^2+c^2 \\ s & a^2+b^2+c^2 & c^2+a^2 \\ s & a^2+b^2+c^2 & a^2+b^2 \end{vmatrix}$$

Taking 's' common from C_1 and $(a^2 + b^2 + c^2)$ common from C_2 we get

$$A = s(a^2+b^2+c^2)\begin{vmatrix} 1 & 1 & b^2+c^2 \\ 1 & 1 & c^2+a^2 \\ 1 & 1 & a^2+b^2 \end{vmatrix}$$

$$= s(a^2+b^2+c^2)\,(0) = 0 \quad [\because C_1 \equiv C_2]$$

Hence, $\begin{vmatrix} s & a^2 & b^2+c^2 \\ s & b^2 & c^2+a^2 \\ s & c^2 & a^2+b^2 \end{vmatrix} = 0$

2. Show that $\begin{vmatrix} b+c & bc & b^2c^2 \\ c+a & ca & c^2a^2 \\ a+b & ab & a^2b^2 \end{vmatrix} = 0.$

Sol : Applying $R_1 \to aR_1$, $R_2 \to bR_2$ and $R_3 \to cR_3$ we get,

$$A = \begin{vmatrix} ab+ac & abc & ab^2c^2 \\ bc+ab & abc & a^2bc^2 \\ ac+bc & abc & a^2b^2c \end{vmatrix}$$

Taking out (abc) common from C_2 and C_3 we get,

10 Differential Calculus - Differentiability and Methods of Differentiation

MUST KNOW DEFINITIONS

- **Tangent line with slope m :** Let f be defined on an open interval containing x_0 and if the limit $\lim_{\Delta x \to 0} \frac{\Delta y}{\Delta x} = \lim_{\Delta x \to 0} \frac{f(x_0 + \Delta x) - f(x)}{\Delta x} = m_{\tan}$ exists, then the line passing through $(x_0, f(x_0))$ with slope m is the tangent line to the graph of f at the point $(x_0, f(x_0))$

- **Position functions :** Suppose an object moves along a straight line according to an equation of motion $s = f(t)$ where s is the displacement (directed distance) of the object from the origin at time t. The function f that describes the motion is called the position function, of the object.

- **Differentiation :** The process of finding the derivative of a function is called differentiation.

- **Leibnitz symbol :** The notation $\frac{dy}{dx}$ is read as "derivative of y with respect to x" or simply "dy-dx", or we should rather read it as "Dee y Dee x" or "Dee Dee x of y". But it is cautioned that we should not regard $\frac{dy}{dx}$ as the quotient $dy \div dx$ and should not refer it as "dy by dx". The symbol $\frac{dy}{dx}$ is know as Leibnitz symbol.

- **Derivatives from first Principle :** The process of finding the derivative of a function using the conditions stated in the definition of derivatives is known as derivatives from first principle.

- **Intermediate Argument :** Thus, to differentiate a function $y = f(g(x))$, we have to take the derivative of the outer function f regarding the argument $g(x) = u$, and multiply the derivative of the inner function $g(x)$ with respect to the independent variable x. The variable u is known as intermediate argument.

- **Logarithmic differentiation:** The operation consists of first taking the logarithm of the function $f(x)$ (to base e) then differentiating is called logarithmic differentiation.

- **Parameter :** If two variables x and y are defined separately as a function of an intermediating (auxiliary) variable t, then the specification of a functional relationship between x and y is described as parametric and the auxiliary variable is known as parameter.

FORMULAE TO REMEMBER

- $(u \pm v)' = u' \pm v'$ where u and v are functions of x.
- $(uv)' = u'v + uv'$ (Product Rule)
- $\left(\frac{u}{v}\right)' = \frac{u'v - uv'}{v^2}, v \neq 0$ (Quotient Rule)
- Let $y = f(u)$ be function of u and let $u = g(x)$ be a function of x so that $y = f(g(x)) = f \circ g(x)$

  Then $\frac{d}{dx}(f(g(x)) = f'(g(x))g'(x)$ [Chain Rule/composite function rule]
- The first derivative of y with respect to 'x' is $\frac{dy}{dx}$
- The second derivative of y with respect to 'x' is $\frac{d^2y}{dx^2}$
- The third derivative of y with respect to 'x' is $\frac{d^3y}{dx^3}$

Following are some of the standard derivatives:

- $\frac{d}{dx}(x^n) = nx^{n-1}$
- $\frac{d}{dx}(e^x) = e^x \Rightarrow \frac{d}{dx}(e^{ax+b}) = a \cdot e^{ax+b}$
- $\frac{d}{dx}(\log x) = \frac{1}{x} \Rightarrow \frac{d}{dx}(\log(x+a)) = \frac{1}{x+a}$
- $\frac{d}{dx}(\sin x) = \cos x$
- $\frac{d}{dx}(\cos x) = -\sin x$
- $\frac{d}{dx}(\tan x) = \sec^2 x$
- $\frac{d}{dx}\left(\frac{1}{x}\right) = -\frac{1}{x^2}$
- $\frac{d}{dx}(\sin^{-1} x) = \frac{1}{\sqrt{1-x^2}}$
- $\frac{d}{dx}(\cos^{-1} x) = \frac{-1}{\sqrt{1-x^2}}$
- $\frac{d}{dx}(\tan^{-1} x) = \frac{1}{1+x^2}$
- $\frac{d}{dx}(\cot^{-1} x) = \frac{-1}{1+x^2}$
- $\frac{d}{dx}(\sec^{-1} x) = \frac{1}{x\sqrt{x^2-1}}$
- $\frac{d}{dx}(\text{cosec}^{-1} x) = \frac{-1}{x\sqrt{x^2-1}}$
- $\frac{d}{dx}(c) = 0$ where c is a constant.
- $\frac{d}{dx}(a^x) = a^x \cdot \log a$.

TEXTUAL QUESTIONS

Exercise 10.1

1. Find the derivatives of the following functions using first principle.

(i) $f(x) = 6$ (ii) $f(x) = -4x + 7$

(iii) $f(x) = -x^2 + 2$

Sol : (i) Given $f(x) = 6$ ... (1)

$$f'(x) = \lim_{\Delta x \to 0} \frac{f(x+\Delta x) - f(x)}{\Delta x}$$

Given $f(x) = 6$

$\Rightarrow$ $f(x + \Delta x) = 6$ ... (2)

$$\therefore f'(x) = \lim_{\Delta x \to 0} \frac{6-6}{\Delta x} = \lim_{\Delta x \to 0} \frac{0}{\Delta x} = 0$$

[Using (1) and (2)]

$\therefore f'(x) = 0$

(ii) Given $f(x) = -4x + 7$ ... (1)

$\therefore f(x + \Delta x) = -4(x + \Delta x) + 7$

$= -4x - 4\Delta x + 7$ ... (2)

$$\therefore \lim_{\Delta x \to 0} \frac{f(x+\Delta x) - f(x)}{\Delta x} = \frac{-4x - 4\Delta x + 7 - (-4x+7)}{\Delta x}$$

[Using (1) and (2)]

$$= \frac{-4x - 4\Delta x + 7 + 4x - 7}{\Delta x} = \frac{-4\Delta x}{\Delta x} = -4$$

$\therefore f'(x) = -4$

(iii) Given $f(x) = -x^2 + 2$ ... (1)

$f(x + \Delta x) = -(x + \Delta x)^2 + 2$

$= -(x^2 + 2x\,\Delta x + (\Delta x)^2) + 2$

$f(x + \Delta x) = -x^2 - 2x\,\Delta x - (\Delta x)^2 + 2$ ... (2)

$(2)-(1) \Rightarrow f(x+\Delta x) - f(x) = -x^2 - 2x\Delta x - (\Delta x)^2 + 2 + x^2 - 2$

$= -2x\,\Delta x - (\Delta x)^2 = \Delta x(-2x - \Delta x)$

$$\therefore \lim_{\Delta x \to 0} \frac{f(x+\Delta x) - f(x)}{\Delta x} = \lim_{\Delta x \to 0} \frac{\Delta x(-2x - \Delta x)}{\Delta x}$$

$$= \lim_{\Delta x \to 0} -2x - \Delta x = -2x - (0) = -2x$$

$\therefore f'(x) = -2x$

2. Find the derivatives from the left and from the right at $x = 1$ (if they exist) of the following functions. Are the functions differentiable at $x = 1$?

(i) $f(x) = |x - 1|$ (ii) $f(x) = \sqrt{1-x^2}$

(iii) $f(x) = \begin{cases} x, & x \le 1 \\ x^2, & x > 1 \end{cases}$

Sol :

(i) Given $f(x) = |x - 1|$

$$f'(1^-) = \lim_{x \to 1^-} \frac{f(x) - f(1)}{x-1} = \lim_{x \to 1^-} \frac{|x-1| - 0}{x-1}$$

$$= \lim_{x \to 1^-} \frac{|x-1|}{x-1} = \lim_{x \to 1^-} -\frac{(x-1)}{x-1} = -1$$

$\therefore f'(1^-) = -1$

$$\therefore f'(1^+) = \lim_{x \to 1^+} \frac{f(x) - f(1)}{x-1}$$

$$= \lim_{x \to 1^+} \frac{|x-1| - 0}{x-1} = \lim_{x \to 1^+} \frac{|x-1|}{x-1}$$

$$= \lim_{x \to 1^+} \frac{x-1}{x-1} = 1$$

$\therefore f'(1^+) = 1$

Since the one sided derivatives $f'(1^-)$ and $f'(1^+)$ are not equal, $f'(1)$ does not exist.

$\therefore f$ is not differentiable at $x = 1$.

(ii) Given $f(x) = \sqrt{1-x^2}$

$f(1) = \sqrt{1-1} = 0$

$$\therefore f'(1^-) = \lim_{x \to 1^-} \frac{\sqrt{1-x^2} - 0}{x-1} \qquad [\because f(1) = 0]$$

$$= \lim_{x \to 1^-} \frac{\sqrt{1-x^2}}{x-1} = \lim_{x \to 1^-} \frac{\sqrt{1+x}.\sqrt{1-x}}{-(1-x)}$$

$$= \lim_{x \to 1^-} \frac{\sqrt{1+x}.\sqrt{1-x}}{-(\sqrt{1-x}).\sqrt{1-x}} = \lim_{x \to 1^-} -\frac{\sqrt{1+x}}{\sqrt{1-x}}$$

$$= \lim_{x \to 1^-} -\sqrt{\frac{1+x}{1-x}} = -\sqrt{\frac{2}{0}} = -\infty \qquad ... (1)$$

Since $f'(1^-) = -\infty$, we can say that f is not differentiable at $x = 1$.

(iii) $f(x) = \begin{cases} x, & x \le 1 \\ x^2, & x > 1 \end{cases}$

$$\therefore f'(1^-) = \lim_{x\to1^-} \frac{f(x)-f(1)}{x-1} = \lim_{x\to1^-} \frac{x-1}{x-1}$$

$$= \lim_{x\to1^-} 1 = 1 \quad \text{... (1)}$$

$$\therefore f'(1^+) = \lim_{x\to1^+} \frac{f(x)-f(1)}{x-1}$$

$$= \lim_{x\to1^+} \frac{x^2-1}{x-1} = \lim_{x\to1^+} \frac{(x+1)(x-1)}{(x-1)}$$

$$= \lim_{x\to1^+} x+1 = 1+1 = 2$$

Since $f'(1^-) \ne f'(1^+)$, $f(x)$ is not differentiable at $x = 1$.

3. Determine whether the following function is differentiable at the indicated values.

(i) $f(x) = x\,|\,x\,|$ at $x = 0$

(ii) $f(x) = |\,x^2 - 1|$ at $x = 1$

(iii) $f(x) = |\,x\,| + |x - 1|$ at $x = 0, 1$

(iv) $f(x) = \sin|\,x\,|$ at $x = 0$

Sol : (i) Given $f(x) = x\,|\,x\,| = \begin{cases} x^2, & x \ge 0 \\ -x^2, & x < 0 \end{cases}$

$$f'(0^-) = \lim_{x\to0^-} \frac{f(x)-f(0)}{x-0} = \lim_{x\to0^-} \frac{-x^2-0}{x-0}$$

$$= \lim_{x\to0^-} \frac{-x^2}{x} = \lim_{x\to0^-} (-x) = 0 \quad \text{... (1)}$$

$$\therefore f'(0^+) = \lim_{x\to0^+} \frac{f(x)-f(0)}{x-0} = \lim_{x\to0^+} \frac{x^2-0}{x-0}$$

$$= \lim_{x\to0^+} \frac{x^2}{x} = \lim_{x\to0^+} x = 0 \quad \text{... (2)}$$

From (1) and (2), $f'(0^-) = f'(0^+)$

Hence, $f(x) = x\,|\,x\,|$ is differentiable at $x = 0$.

(ii) Given $f(x) = |\,x^2 - 1|$

$$f(x) = \begin{cases} -(x^2-1), & x < 1 \\ +(x^2-1), & x \ge 1 \end{cases}$$

$$\therefore f'(1^-) = \lim_{x\to1^-} \frac{f(x)-f(1)}{x-1} = \lim_{x\to1^-} \frac{-(x^2-1)-0}{x-1}$$

$$= \lim_{x\to1^-} \frac{1-x^2}{-(1-x)} = \lim_{x\to1^-} \frac{(1+x)(1-x)}{-(1-x)}$$

$$= \lim_{x\to1^-} -(1+x) = -2 \quad \text{... (1)}$$

$$\therefore f'(1^+) = \lim_{x\to1^+} \frac{f(x)-f(1)}{x-1} = \lim_{x\to1^+} \frac{x^2-1-0}{x-1}$$

$$= \lim_{x\to1^+} \frac{x^2-1}{x-1} = \lim_{x\to1^+} \frac{(x+1)(x-1)}{(x-1)}$$

$$= \lim_{x\to1^+} x+1 = 2 \quad \text{... (2)}$$

From (1) and (2), $f'(1^-) \ne f'(1^+)$

Hence $f(x)$ is not differentiable at $x = 1$.

(iii) Given $f(x) = |\,x\,| + |\,x - 1\,|$

$$f(x) = \begin{cases} -x-(x-1) = -2x+1 & \text{if } x<0 \\ x-(x-1) = 1 & \text{if } 0 \le x < 1 \\ x+x-1 = 2x-1 & \text{if } x \ge 1 \end{cases}$$

$$\therefore f'(0^-) = \lim_{x\to0^-} \frac{f(x)-f(0)}{x-0} = \lim_{x\to0^-} \frac{-2x+1-(1)}{x-0}$$

$$= \lim_{x\to0^-} \frac{-2x}{x} = -2$$

$$\therefore f'(0^+) = \lim_{x\to0^+} \frac{f(x)-f(0)}{x-0} = \lim_{x\to0^+} \frac{1-1}{x} = 0$$

$\therefore$ $f(x)$ is not differentiable at $x = 0$

$$f'(1^-) = \lim_{x\to1^-} \frac{f(x)-f(1)}{x-1}$$

$$= \lim_{x\to1^-} \frac{0}{x-1} = 0$$

$$f'(1^+) = \lim_{x\to1^+} \frac{f(x)-f(1)}{x-1} = \lim_{x\to1^+} \frac{(2x-1)-1}{x-1}$$

$$= \lim_{x\to1^+} \frac{2x-2}{x-1} = \lim_{x\to1^+} \frac{2(x-1)}{x-1} = 2$$

$\therefore$ $f(x)$ is not differentiable at $x = 1$

(iv) Given $f(x) = \sin|\,x\,| = \begin{cases} \sin x \text{ if} & x \ge 0 \\ -\sin x \text{ if} & x < 0 \end{cases}$

$$\therefore f'(0^-) = \lim_{x\to0^-} \frac{f(x)-f(0)}{x-0} = \lim_{x\to0^-} \frac{-\sin x - \sin 0}{x-0}$$

$$= \lim_{x\to0^-} \frac{-\sin x}{x} = -1 \quad \left[\because \lim_{x\to0} \frac{\sin x}{x} = 1\right]$$

$$\therefore f'(0^+) = \lim_{x\to0^+} \frac{f(x)-f(0)}{x-0}$$

$$= \lim_{x\to0^+} \frac{\sin x - \sin 0}{x-0} = \lim_{x\to0^+} \frac{\sin x}{x} = 1$$

$$f'(0^-) \ne f'(0^+)$$

$\therefore$ $f(x)$ is not differentiable at $x = 0$.

4. Show that the following functions are not differentiable at the indicated value of x.

(i) $f(x)=\begin{cases}-x+2, & x\le 2\\ 2x-4, & x>2\end{cases};\ x=2$

(ii) $f(x)=\begin{cases}3x, & x<0\\ -4x, & x\ge 0\end{cases};\ x=0$

Sol : (i) $f(x) = \begin{cases}-x+2 & x\le 2\\ 2x-4 & x>2\end{cases};\ x=2$

$$f'(2^-) = \lim_{x\to 2^-}\frac{f(x)-f(2)}{x-2}$$

$$= \lim_{x\to 2^-}\frac{-x+2-(0)}{x-2} \quad \left[\begin{array}{l}\because f(x)=-x+2\\ \Rightarrow f(2)=-2+2=0\end{array}\right]$$

$$= \lim_{x\to 2^-}\frac{-x+2}{x-2} = \lim_{x\to 2^-}\frac{-(x-2)}{x-2} = -1 \quad \ldots (1)$$

$$\therefore f'(2^+) = \lim_{x\to 2^+}\frac{f(x)-f(2)}{x-2}$$

$$= \lim_{x\to 2^+}\frac{(2x-4)-0}{x-2} \quad \left[\begin{array}{l}\because f(x)=2x-4\\ f(2)=4-4=0\end{array}\right]$$

$$= \lim_{x\to 2^+}\frac{2(x-2)}{x-2} = 2 \quad \ldots (2)$$

From (1) and (2), $f'(2^-)\neq f'(2^+)$

Hence, $f(x)$ is not differentiable at $x = 2$.

(ii) $f(x) = \begin{cases}3x, & x<0\\ -4x, & x\ge 0\end{cases};\ x=0$

$$\therefore f'(0^-) = \lim_{x\to 0^-}\frac{f(x)-f(0)}{x-0} = \lim_{x\to 0^-}\frac{3x-0}{x}$$

$$= \lim_{x\to 0^-}\frac{3x}{x} = 3 \quad [f(x) = 3x] \quad \ldots (1)$$

$$\therefore f'(0^+) = \lim_{x\to 0^+}\frac{f(x)-f(0)}{x-0} = \lim_{x\to 0^+}\frac{-4x-0}{x}$$

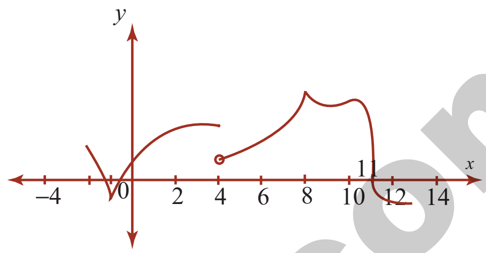
$$\left[\begin{array}{l}\because f(x)=-4x\\ f(0)=4(0)=0\end{array}\right]$$

$$= \lim_{x\to 0^+}\frac{-4x}{x} = -4 \quad \ldots (2)$$

From (1) and (2), $f'(0^-)\neq f'(0^+)$

Hence $f(x)$ is not differentiable at $x = 0$.

5. The graph of f is shown below. State with reasons that x values (the numbers), at which f is not differentiable.



Sol : (i) From the graph it is clear that at $x = -1$, the graph has a sharp edge.

$\therefore$ It is not differentiable at $x = -1$.

(ii) At $x = 4$, it is discontinuous.

$\therefore f$ is not differentiable at $x = 4$.

(iii) At $x = 8$, it has a sharp peak.

$\therefore f$ is not differentiable at $x = 8$.

(iv) At $x = 11$, the tangent is perpendicular

$\Rightarrow$ At $x = 11$, it has a vertical tangent

$\therefore f$ is also not differentiable at $x = 11$.

6. If $f(x) = |x + 100| + x^2$, test whether $f'(-100)$ exists.

Sol : Given $f(x) = |x + 100| + x^2$

$$\therefore f'(-100^-) = \lim_{x\to -100^-}\frac{f(x)-f(-100)}{x-(-100)}$$

$$= \lim_{x\to -100^-}\frac{|x+100|+x^2-(100)^2}{x+100}$$

$$\left[\begin{array}{l}\because f(x)=|x+100|+x^2\\ f(-100)=|-100+100|+(-100)^2=100^2\end{array}\right]$$

$$= \lim_{x\to -100^-}\frac{|x+100|+x^2-100^2}{x+100}$$

$$= \lim_{x\to -100^-}\frac{-(x+100)+x^2-100^2}{x+100} = \lim_{x\to -100^-}\frac{-x-100+x^2-100^2}{x+100}$$

$$= \lim_{x\to -100^-}\frac{-(x+100)+(x+100)(x-100)}{x+100} = \lim_{x\to -100^-}\frac{(x+100)(-1+x-100)}{x+100}$$

$$= \lim_{x\to -100^-}(x-101) = -201 \quad \ldots (1)$$

$$\therefore f'(-100^+) = \lim_{x\to -100^+}\frac{f(x)-f(-100)}{x-(-100)}$$

$$= \lim_{x\to -100^+}\frac{|x+100|+x^2-(100)^2}{x+100}$$

$$= \lim_{x \to -100^+} \frac{(x+100)+x^2-100^2}{x+100}$$

$$= \lim_{x \to -100^+} \frac{(x+100)+(x-100)(x+100)}{x+100}$$

$$= \lim_{x \to -100^+} \frac{\cancel{(x+100)}\,[1+x-100]}{\cancel{x+100}}$$

$$= \lim_{x \to -100^+} [1+x-100] = -199 \qquad \ldots (2)$$

From (1) and (2), $f(x)$ is not differentiable at $x = -100$

$\Rightarrow f'(-100)$ does not exist.

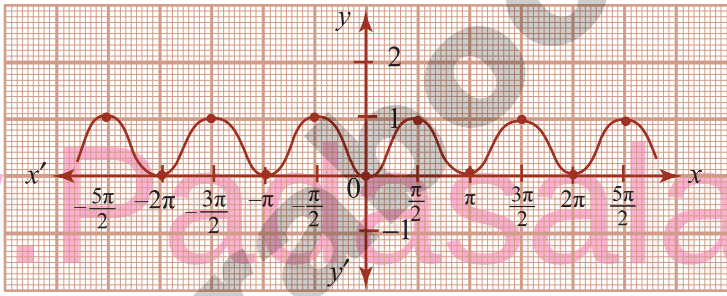
7. Examine the differentiability of functions in $\mathbb{R}$ by drawing the diagrams.

(i) $|\sin x|$ (ii) $|\cos x|$.

Sol :

(i) Let $f(x) = |\sin x|$ when $x = 0$, $f(x) = |\sin 0| = 0$

x	0	$\frac{\pi}{2}$	$-\frac{\pi}{2}$	π	$-\pi$	$\frac{3\pi}{2}$	$-\frac{3\pi}{2}$	$\frac{5\pi}{2}$	$-\frac{5\pi}{2}$	2π	-2π
$f(x)$	0	1	1	0	0	1	1	1	1	0	0

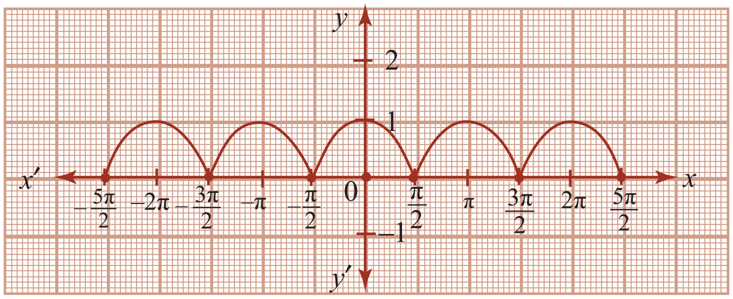


The curve $f(x) = |\sin x|$ has got vertical tangents at $x = \pi, -\pi, 2\pi, -2\pi$ etc.

$\therefore f(x) = |\sin x|$ is not differentiable at $x = n\pi$, $n \in \mathbb{Z}$.

(ii) Let $f(x) = |\cos x|$ when $x = 0$, $f(0) = |\cos 0| = |1| = 1$

x	0	$\frac{\pi}{2}$	$-\frac{\pi}{2}$	π	$-\pi$	$\frac{3\pi}{2}$	$-\frac{3\pi}{2}$	$\frac{5\pi}{2}$	$-\frac{5\pi}{2}$	2π
$f(x)$	1	0	0	1	1	0	0	0	0	1



$f(x) = |\cos x|$ has got vertical tangents at $x = \frac{\pi}{2}, -\frac{\pi}{2}, \frac{3\pi}{2}, -\frac{3\pi}{2}, \frac{5\pi}{2}, -\frac{5\pi}{2}$ etc.

$\therefore f(x) = |\cos x|$ is not differentiable at $x = (2n+1)\frac{\pi}{2}$, $n \in \mathbb{Z}$.

11 INTEGRAL CALCULUS

MUST KNOW DEFINITIONS

Anti-derivative function	:	A function F(x) is called an anti derivative (Newton - Leibniz integral or primitive) of a function $f(x)$ on an interval I if $F'(x) = f(x)$, for every value of x in I.
Integration	:	$\int f(x)dx = F(x) + c$ where c is called on arbitrary constant. The function $f(x)$ is called Integrand. The variable x in dx is called variable of integration or integrator. The process of finding the integral is called integration or anti differentiation.

FORMULAE TO REMEMBER

- $\int x^n dx = \frac{x^{n+1}}{n+1} + c$
- $\int \frac{1}{x} dx = \log |x| + c$
- $\int \sin x \, dx = -\cos x + c$
- $\int \cos x \, dx = \sin x + c$
- $\int \tan x \, dx = \log |\sec| + c$
- $\int \cot x \, dx = \log |\sin x| + c$
- $\int \sec x \, dx = \log |\sec x + \tan x| + c$
- $\int \text{cosec}\, x \, dx = \log |\text{cosec}\, x - \cot x| + c$
- $\int \sec^2 x \, dx = \tan x + c$
- $\int \text{cosec}^2 x \, dx = -\cot x + c$
- $\int \sec x \tan x \, dx = \sec x + c$
- $\int \text{cosec}\, x \cot x dx = -\text{cosec}\, x + c$
- $\int \frac{1}{\sqrt{1-x^2}} dx = \sin^{-1} x + c$
- $\int \frac{1}{1+x^2} dx = \tan^{-1} x + c$

- $\int a^x dx = \frac{a^x}{\log a} + c$
- $\int e^x dx = e^x + c$
- $\int \frac{dx}{a^2+x^2} = \frac{1}{a}\tan^{-1}\left(\frac{x}{a}\right) + c$
- $\int e^x (f(x) + f'(x))\, dx = e^x f(x) + c$
- $\int \frac{dx}{a^2-x^2} = \frac{1}{2a}\log\left|\frac{a+x}{a-x}\right| + c$
- $\int \frac{dx}{x^2-a^2} = \frac{1}{2a}\log\left|\frac{x-a}{x+a}\right| + c$
- $\int \frac{dx}{\sqrt{a^2-x^2}} = \sin^{-1}\left(\frac{x}{a}\right) + c$
- $\int \frac{dx}{\sqrt{x^2+a^2}} = \log\left|x+\sqrt{x^2+a^2}\right| + c$
- $\int \sqrt{a^2-x^2}\,dx = \frac{1}{2}x\sqrt{a^2-x^2} + \frac{a^2}{2}\sin^{-1}\left(\frac{x}{a}\right) + c$
- $\int \frac{dx}{\sqrt{x^2-a^2}} = \log\left|x+\sqrt{x^2-a^2}\right| + c$
- $\int \sqrt{x^2+a^2}\,dx = \frac{x}{2}\sqrt{x^2+a^2} + \frac{a^2}{2}\log\left|x+\sqrt{x^2+a^2}\right| + c$
- $\int \frac{1}{x\sqrt{x^2-a^2}}dx = \frac{1}{a}\sec^{-1}\left(\frac{x}{a}\right) + c$
- $\int \sqrt{x^2-a^2}\,dx = \frac{x}{2}\sqrt{x^2-a^2} - \frac{a^2}{2}\log\left|x+\sqrt{x^2-a^2}\right| + c$

TEXTUAL QUESTIONS

Exercise 11.1

Integrate the following with respect to x:

1. (i) x^{11} (ii) $\frac{1}{x^7}$ [Sep.-2021] (iii) $\sqrt[3]{x^4}$ (iv) $\left(x^5\right)^{\frac{1}{8}}$

Sol : (i) We know that $\int x^n dx = \frac{x^{n+1}}{n+1} + c$

$$\therefore \int x^{11} dx = \frac{x^{11+1}}{11+1} + c = \frac{x^{12}}{12} + c$$

(ii) $\frac{1}{x^7}$

$$\int \frac{1}{x^7} dx = \int x^{-7} dx = \frac{x^{-7+1}}{-7+1} = \frac{x^{-6}}{-6} + c$$
$$= -\frac{1}{6x^6} + c$$

(iii) $\sqrt[3]{x^4}$

$$\int \sqrt[3]{x^4}\,dx = \int \left(x^4\right)^{1/3} dx = \int x^{4/3} dx$$
$$= \frac{x^{\frac{4}{3}+1}}{\frac{4}{3}+1} + c = \frac{x^{\frac{7}{3}}}{\frac{7}{3}} + c = \frac{3}{7}x^{\frac{7}{3}} + c$$

(iv) $\left(x^5\right)^{\frac{1}{8}}$

$$\int \left(x^5\right)^{1/8} dx = \int x^{5/8} dx = \frac{x^{\frac{5}{8}+1}}{\frac{5}{8}+1} + c = \frac{x^{\frac{13}{8}}}{\frac{13}{8}} + c$$
$$= \frac{8}{13}x^{\frac{13}{8}} + c$$

2. (i) $\frac{1}{\sin^2 x}$ (ii) $\frac{\tan x}{\cos x}$ (iii) $\frac{\cos x}{\sin^2 x}$ (iv) $\frac{1}{\cos^2 x}$

Sol : (i) $\frac{1}{\sin^2 x}$

$$\int \frac{1}{\sin^2 x} dx = \int \text{cosec}^2 x\, dx = -\cot x + c$$

(ii) $\frac{\tan x}{\cos x}$

$$\int \frac{\tan x}{\cos x} dx = \int \tan x \times \frac{1}{\cos x} dx$$
$$= \int \tan x \sec x\, dx = \sec x + c$$

(iii) $\dfrac{\cos x}{\sin^2 x}$

$$\int \frac{\cos x}{\sin^2 x} dx = \int \frac{\cos x}{\sin x} \times \frac{1}{\sin x} dx$$
$$= \int \cot x \operatorname{cosec} x \, dx = -\operatorname{cosec} x + c$$

(iv) $\dfrac{1}{\cos^2 x}$

$$\int \frac{1}{\cos^2 x} dx = \int \sec^2 x \, dx = \tan x + c$$

3. (i) 12^3 (ii) $\dfrac{x^{24}}{x^{25}}$ (iii) e^x

Sol : (i) $12^3 = \int 12^3 dx = 12^3 \int dx = 12^3 x + c$

(ii) $\dfrac{x^{24}}{x^{25}} = \int \dfrac{x^{24}}{x^{25}} dx = \int \dfrac{1}{x} dx = \log|x| + c$

(iii) $e^x = \int e^x dx = e^x + c$

4. (i) $\left(1+x^2\right)^{-1}$ [March - 2020] (ii) $\left(1-x^2\right)^{-\frac{1}{2}}$

Sol : (i) $\int \left(1+x^2\right)^{-1} dx = \int \dfrac{1}{1+x^2} dx = \tan^{-1} x + c$

(ii) $\int \left(1-x^2\right)^{-1/2} dx = \int \dfrac{1}{\left(1-x^2\right)^{1/2}} dx$

$$= \int \frac{dx}{\sqrt{1-x^2}} + c = \sin^{-1} x + c$$

Exercise 11.2

Integrate the following functions with respect to x:

1. (i) $(x+5)^6$ (ii) $\dfrac{1}{(2-3x)^4}$ (iii) $\sqrt{3x+2}$

Sol : (i) $(x+5)^6$

$$\int (x+5)^6 dx = \frac{(x+5)^{6+1}}{6+1} + c = \frac{(x+5)^7}{7} + c$$

(ii) $\dfrac{1}{(2-3x)^4}$

$$\int \frac{1}{(2-3x)^4} dx = \int (2-3x)^{-4} dx$$

$$= \frac{(2-3x)^{-4+1}}{(-4+1)(-3)} = \frac{(2-3x)^{-3}}{9} = \frac{1}{9(2-3x)^3} + c$$

(iii) $\sqrt{3x+2}$

$$\int \sqrt{3x+2}\, dx = \int (3x+2)^{\frac{1}{2}} dx = \frac{(3x+2)^{\frac{1}{2}+1}}{\left(\frac{1}{2}+1\right)(3)} + c$$

$$= \frac{(3x+2)^{\frac{3}{2}}}{\frac{3}{2} \times 3} + c = \frac{2}{9}(3x+2)^{\frac{3}{2}} + c$$

2. (i) $\sin 3x$ (ii) $\cos(5-11x)$ (iii) $\operatorname{cosec}^2(5x-7)$

Sol : (i) $\int \sin 3x \, dx = -\dfrac{1}{3}\cos 3x + c \Rightarrow -\dfrac{\cos 3x}{3} + c$

(ii) $\int \cos(5-11x) dx = \sin\dfrac{(5-11x)}{-11} + c$

$$= -\frac{\sin(5-11x)}{11} + c$$

(iii) $\int \operatorname{cosec}^2(5x-7) dx = -\dfrac{\cot(5x-7)}{5} + c$

3. (i) e^{3x-6} (ii) e^{8-7x} (iii) $\dfrac{1}{6-4x}$

Sol : (i) $\int e^{3x-6} dx = \dfrac{e^{3x-6}}{3} + c = \dfrac{1}{3} e^{3x-6} + c$

(ii) $\int e^{8-7x} dx = \dfrac{e^{8-7x}}{(-7)} + c = -\dfrac{e^{8-7x}}{7} + c$

(iii) $\int \dfrac{1}{6-4x} dx = \dfrac{\log|6-4x|}{-4} + c$

$$= -\frac{1}{4}\log|6-4x| + c$$

4. (i) $\sec^2 \dfrac{x}{5}$ (ii) $\operatorname{cosec}(5x+3)\cot(5x+3)$

(iii) $\sec(2-15x)\tan(2-15x)$

Sol : (i) $\int \sec^2\left(\dfrac{x}{5}\right) dx = \dfrac{\tan\left(\dfrac{x}{5}\right)}{\dfrac{1}{5}} + c = 5\tan\left(\dfrac{x}{5}\right) + c$

(ii) $\int \operatorname{cosec}(5x+3)\cot(5x+3) dx$

$$= -\operatorname{cosec}\frac{(5x+3)}{5} + c = -\frac{1}{5}\operatorname{cosec}(5x+3) + c$$

(iii) $\int \sec(2-15x)\tan(2-15x) dx$

$$= \frac{\sec(2-15x)}{-15} + c = -\frac{1}{15}\sec(2-15x) + c$$

5. (i) $\dfrac{1}{\sqrt{1-(4x)^2}}$ (ii) $\dfrac{1}{\sqrt{1-81x^2}}$ (iii) $\dfrac{1}{1+36x^2}$

Sol : (i) We know that $\int \dfrac{dx}{\sqrt{1-x^2}} = \sin^{-1}(x)+c$

$$= \frac{\sin^{-1}(4x)}{4}+c = \frac{1}{4}\sin^{-1}(4x)+c$$

(ii) $\int \dfrac{dx}{\sqrt{1-81x^2}} = \int \dfrac{dx}{\sqrt{1-(9x)^2}} = \dfrac{\sin^{-1}(9x)}{9}+c$

$$= \frac{1}{9}\sin^{-1}(9x)+c$$

(iii) $\int \dfrac{1}{1+36x^2}dx = \int \dfrac{1}{1+(6x)^2}dx = \dfrac{\tan^{-1}(6x)}{6}$

$$= \frac{1}{6}\tan^{-1}(6x)+c \quad \left[\because \int \frac{dx}{1+x^2} = \tan^{-1}x+c\right]$$

Exercise 11.3

Integrate the following with respect to x:

1. $(x+4)^5 + \dfrac{5}{(2-5x)^4} - \text{cosec}^2(3x-1)$

Sol : $\int (x+4)^5 + \dfrac{5}{(2-5x)^4} - \text{cosec}^2(3x-1)dx$

$$= \int (x+4)^5\,dx + 5\int \frac{1}{(2-5x)^4}dx - \int \text{cosec}^2(3x-1)dx$$

$$= \frac{(x+4)^6}{6} + 5\int (2-5x)^{-4}dx + \frac{\cot(3x-1)}{3}+c$$

$$= \frac{(x+4)^6}{6} + \not{5}\frac{(2-5x)^{-4+1}}{(-4+1)(-\not{5})} + \frac{1}{3}\cot(3x-1)+c$$

$$= \frac{(x+4)^6}{6} + \frac{(2-5x)^{-3}}{-(-3)} + \frac{1}{3}\cot(3x-1)+c$$

$$= \frac{(x+4)^6}{6} + \frac{1}{3(2-5x)^3} + \frac{1}{3}\cot(3x-1)+c$$

2. $4\cos(5-2x)+9e^{3x-6}+\dfrac{24}{6-4x}$

Sol : $\int 4\cos(5-2x) + 9e^{3x-6} + \dfrac{24}{6-4x}dx$

$$= 4\int \cos(5-2x)dx + 9\int e^{3x-6}dx + 24\int \frac{dx}{6-4x}$$

$$= \overset{2}{\not{4}}.\frac{\sin(5-2x)}{-\not{2}} + \overset{3}{\not{9}}.\frac{e^{3x-6}}{\not{3}} + \overset{6}{\not{24}}.\frac{\log|6-4x|}{-\not{4}}+c$$

$$= -2\sin(5-2x) + 3e^{3x-6} - 6\log|6-4x|+c$$

3. $\sec^2\dfrac{x}{5}+18\cos 2x+10\sec(5x+3)\tan(5x+3)$

Sol :

$$\int \sec^2(x/5)+18\cos 2x+10\sec(5x+3)\tan(5x+3)\,dx$$

$$= \int \sec^2\left(\frac{x}{5}\right)dx + 18\int\cos 2x\,dx + 10\int\sec(5x+3)\tan(5x+3)\,dx$$

$$= \frac{\tan\left(\frac{x}{5}\right)}{\frac{1}{5}} + \overset{9}{\not{18}}\frac{\sin 2x}{\not{2}} + \frac{\overset{2}{\not{10}}\sec(5x+3)}{\not{5}}+c$$

$$= 5\tan\left(\frac{x}{5}\right) + 9\sin 2x + 2\sec(5x+3)+c$$

4. $\dfrac{8}{\sqrt{1-(4x)^2}}+\dfrac{27}{\sqrt{1-9x^2}}-\dfrac{15}{1+25x^2}$

Sol : $\int \dfrac{8}{\sqrt{1-(4x)^2}} + \dfrac{27}{\sqrt{1-9x^2}} - \dfrac{15}{1+25x^2}dx$

$$= 8\int \frac{1}{\sqrt{1-(4x)^2}}dx + 27\int \frac{dx}{\sqrt{1-(3x)^2}} - 15\int \frac{dx}{1+25x^2}$$

$$= 8.\frac{\sin^{-1}(4x)}{4} + 27.\frac{\sin^{-1}(3x)}{3} - 15\int \frac{dx}{1+(5x)^2}+c$$

$$= 2\sin^{-1}(4x) + 9\sin^{-1}(3x) - \overset{3}{\not{15}}\frac{\tan^{-1}(5x)}{\not{5}}+c$$

$$= 2\sin^{-1}(4x) + 9\sin^{-1}(3x) - 3\tan^{-1}(5x)+c$$

5. $\dfrac{6}{1+(3x+2)^2}-\dfrac{12}{\sqrt{1-(3-4x)^2}}$

Sol : $\int \dfrac{6}{1+(3x+2)^2} - \dfrac{12}{\sqrt{1-(3-4x)^2}}dx$

$$= 6\int \frac{dx}{1+(3x+2)^2} - 12\int \frac{dx}{\sqrt{1-(3-4x)^2}}$$

$$= \overset{2}{\not{6}}.\frac{\tan^{-1}(3x+2)}{\not{3}} - \overset{3}{\not{12}}.\frac{\sin^{-1}(3-4x)}{-\not{4}}+c$$

$$= 2\tan^{-1}(3x+2) + 3\sin^{-1}(3-4x)+c$$

6. $\dfrac{1}{3}\cos\left(\dfrac{x}{3}-4\right)+\dfrac{7}{7x+9}+e^{\frac{x}{5}+3}$

Sol : $\int \dfrac{1}{3}\cos\left(\dfrac{x}{3}-4\right) + \dfrac{7}{7x+9} + e^{\frac{x}{5}+3}dx$

11th STD

Reg. No.

INSTANT SUPP. EXAM AUGUST - 2022

Part - III

Mathematics (with answers)

Time Allowed : 3.00 Hours] [Maximum Marks : 90

Instructions :

(1) Check the question paper for fairness of printing. If there is any lack of fairness, inform the Hall Supervisor immediately.

(2) Use **Blue** or **Black** ink to write and underline and pencil to draw diagrams

PART - I

Note : (i) Answer **all** the questions. **[20 × 1 = 20]**

(ii) Choose the most appropriate answer from the given four alternatives and write the option code and the corresponding answer.

1. If $A = \{(x, y) : y = \sin x, x \in \mathbb{R}\}$ and $B = \{(x, y) : y = \cos x, x \in \mathbb{R}\}$ then $A \cap B$ contains

(a) no element
(b) infinitely many elements
(c) only one element
(d) cannot be determined

2. If $|x + 2| \leq 9$, then x belong to :

(a) $(-\infty, -7)$ (b) $[-11, 7]$
(c) $(-\infty, -7) \cup [11, \infty)$ (d) $(-11, 7)$

3. The value of sin 18° is :

(a) $\frac{\sqrt{5}-1}{4}$ (b) $\frac{\sqrt{5}+1}{4}$
(c) $\frac{1-\sqrt{5}}{4}$ (d) $\frac{\sqrt{5}-1}{2}$

4. The value of cos 150° is :

(a) $\frac{1}{2}$ (b) $\frac{-1}{2}$ (c) $\frac{\sqrt{3}}{2}$ (d) $\frac{-\sqrt{3}}{2}$

5. Number of sides of a polygon having 44 diagonals is :

(a) 4 (b) 4! (c) 11 (d) 22

6. The n^{th} term of the sequence 1, 2, 4, 7, 11, ... is :

(a) $n^3 + 3n^2 + 2n$ (b) $n^3 - 3n^2 + 3n$
(c) $\frac{n(n+1)(n+2)}{3}$ (d) $\frac{n^2 - n + 2}{2}$

7. The equation of the line with slope 2 and the length of the perpendicular from the origin equal to $\sqrt{5}$ is :

(a) $x - 2y = \sqrt{5}$ (b) $2x - y = \sqrt{5}$
(c) $2x - y = 5$ (d) $x - 2y - 5 = 0$

8. If $a_{ij} = \frac{1}{2}(3i - 2j)$ and $A = [a_{ij}]_{2\times 2}$ then A is :

(a) $\begin{bmatrix} \frac{1}{2} & 2 \\ \frac{-1}{2} & 1 \end{bmatrix}$ (b) $\begin{bmatrix} \frac{1}{2} & \frac{-1}{2} \\ 2 & 1 \end{bmatrix}$
(c) $\begin{bmatrix} 2 & 2 \\ \frac{1}{2} & \frac{-1}{2} \end{bmatrix}$ (d) $\begin{bmatrix} \frac{-1}{2} & \frac{1}{2} \\ 1 & 2 \end{bmatrix}$

9. The value of the determinant of $A = \begin{bmatrix} 0 & a & -b \\ -a & 0 & c \\ b & -c & 0 \end{bmatrix}$ is :

(a) $-2abc$ (b) abc
(c) 0 (d) $a^2 + b^2 + c^2$

10. The vectors $\vec{a}-\vec{b}, \vec{b}-\vec{c}, \vec{c}-\vec{a}$ are :

(a) parallel to each other
(b) unit vectors
(c) mutually perpendicular vectors
(d) coplanar vectors

11. If $|\vec{a}+\vec{b}| = 60$, $|\vec{a}-\vec{b}| = 40$ and $|\vec{b}| = 46$ then, $|\vec{a}|$ is :

(a) 42 (b) 12 (c) 22 (d) 32

12. If $\vec{a} = \hat{i} + 2\hat{j} + 2\hat{k}$, $|\vec{b}| = 5$ and the angle between $\vec{a}$ and $\vec{b}$ is $\frac{\pi}{6}$ then, the area of the triangle formed by these two vectors as two sides, is :

(a) $\frac{7}{4}$ (b) $\frac{15}{4}$ (c) $\frac{3}{4}$ (d) $\frac{17}{4}$

13. $\lim_{x\to 0} \frac{\sqrt{1-\cos 2x}}{x}$ is :

(a) 0 (b) 1
(c) $\sqrt{2}$ (d) None of the above

14. $\lim_{x\to 0} \dfrac{e^{\tan x} - e^x}{\tan x - x}$ is :

(a) 1 (b) e (c) $\frac{1}{2}$ (d) 0

15. $\frac{d}{dx}[\log 5x]$ is :

(a) $\frac{1}{5x}$ (b) $\frac{1}{x}$ (c) $5x$ (d) $\log 5x$

16. If $f(x) = \begin{cases} x+1, & \text{when } x<2 \\ 2x-1, & \text{when } x \geq 2 \end{cases}$, then, $f'(2)$ is :

(a) 0 (b) 1
(c) 2 (d) does not exist

17. $\int \frac{\sqrt{\tan x}}{\sin 2x}dx$ is :

(a) $\sqrt{\tan x} + C$ (b) $2\sqrt{\tan x} + C$
(c) $\frac{1}{2}\sqrt{\tan x} + C$ (d) $\frac{1}{4}\sqrt{\tan x} + C$

18. $\int x^2 \cos x\, dx$ is :

(a) $x^2 \sin x + 2x\cos x - 2\sin x + C$
(b) $x^2 \sin x - 2x\cos x - 2\sin x + C$
(c) $-x^2 \sin x + 2x\cos x + 2\sin x + C$
(d) $-x^2 \sin x - 2x\cos x + 2\sin x + C$

19. If A and B are two events such that $A \subset B$ and $P(B) \neq 0$, then which of the following is correct :

(a) $P(A/B) = \frac{P(A)}{P(B)}$ (b) $P(A/B) < P(A)$
(c) $P(A/B) \geq P(A)$ (d) $P(A/B) > P(B)$

20. It is given that the events A and B are such that $P(A) = \frac{1}{4}$, $P(A/B) = \frac{1}{2}$ and $P(B/A) = \frac{2}{3}$. Then P(B) is :

(a) $\frac{1}{6}$ (b) $\frac{1}{3}$ (c) $\frac{2}{3}$ (d) $\frac{1}{2}$

PART - II

Note : Answer **any seven** questions.
Question number **30** is **compulsory.** **7 × 2 = 14**

21. Solve $|x - 9| < 2$ for x.

22. Express $\cos 6\theta + \cos 2\theta$ as a product.

23. Find the value of sin 150°.

24. Find the value of $\frac{8!}{5! \times 2!}$.

25. Write the first 4 terms of the sequences whose n^{th} term a_n is given as, $a_n = \begin{cases} 1 & \text{if } n=1 \\ 2 & \text{if } n=2 \\ a_{n-1}+a_{n-2} & \text{if } n>2 \end{cases}$

26. Find the equation of the lines passing through the points (1, 1) with slope 3.

27. Find |A| if $A = \begin{bmatrix} 3 & 4 & 1 \\ 0 & -1 & 2 \\ 5 & -2 & 6 \end{bmatrix}$

28. Find $\vec{a}\cdot\vec{b}$ when $\vec{a} = \hat{i} - 2\hat{j} + \hat{k}$ and $\vec{b} = 3\hat{i} - 4\hat{j} - 2\hat{k}$.

29. Differentiate $y = e^x + \sin x + 2$ with respect to x.

30. Integrate $\cos 3x$ with respect to x.

PART - III

Note : Answer **any seven** questions.
Question number **40** is **compulsory.** **7 × 3 = 21**

31. If $n(A) = 10$ and $n(A \cap B) = 3$ find $n[(A \cap B)' \cap A]$.

32. Resolve into partial fractions : $\frac{3x+1}{(x-2)(x+1)}$

33. Expand $(1 + x)^{2/3}$ up to four terms for $|x| < 1$.

34. Find the distance between two parallel lines $3x + 4y - 12 = 0$ and $6x + 8y + 1 = 0$.

35. Show that $\begin{vmatrix} 0 & c & b \\ c & 0 & a \\ b & a & 0 \end{vmatrix}^2 = \begin{vmatrix} b^2+c^2 & ab & ac \\ ab & c^2+a^2 & bc \\ ac & bc & a^2+b^2 \end{vmatrix}$.

36. Show that the points whose position vectors are $2\hat{i} + 3\hat{j} - 5\hat{k}$, $3\hat{i} + \hat{j} - 2\hat{k}$ and $6\hat{i} - 5\hat{j} + 7\hat{k}$ are collinear.

37. Differentiate $y = e^{\sin x}$ with respect to x.

38. Integrate $(2x - 5)(36 + 4x)$ with respect to x.

39. If $P(A) = 0.5$, $P(B) = 0.8$ and $P(B/A) = 0.8$, find $P(A/B)$.

40. Find the distinct permutation of the letters of the word MATHEMATICS.

PART - IV

Note : Answer **all** the questions. **7 × 5 = 35**

41. (a) Write the values of f at –4, 1, –2, 7, 0 if : $f(x) = \begin{cases} -x+4 & \text{if } -\infty < x \le -3 \\ x+4 & \text{if } -3 < x < -2 \\ x^2 - x & \text{if } -2 \le x < 1 \\ x - x^2 & \text{if } 1 \le x < 7 \\ 0 & \text{otherwise} \end{cases}$

(OR)

(b) Compute : $\lim\limits_{x \to 3} \dfrac{(x^2 - 6x + 5)}{x^3 - 8x + 7}$

42. (a) If $\sin x = \dfrac{15}{17}$ and $\cos y = \dfrac{12}{13}$, $0 < x < \dfrac{\pi}{2}$, $0 < y < \dfrac{\pi}{2}$, find the value of $\cos(x - y)$.

(OR)

(b) If $y = e^{\tan^{-1}x}$, show that $(1 + x^2)y'' + (2x - 1)y' = 0$

43. (a) Evaluate the following :

(i) ${}^4P_4$ (ii) ${}^6P_5$

(iii) ${}^{10}C_3$ (iv) ${}^{100}C_{99}$ (v) ${}^{50}C_{50}$

(OR)

(b) Find all values of x that satisfies the inequality $\dfrac{2x-3}{(x-2)(x-4)} < 0$.

44. (a) Prove that $\sqrt[3]{x^3+6} - \sqrt[3]{x^3+3}$ is approximately equal to $\dfrac{1}{x^2}$ when x is sufficiently large.

(OR)

(b) Differentiate : $\dfrac{2x-5}{(x^2 - 2x + 5)}$ with respect to x.

45. (a) Prove that $\begin{vmatrix} 1 & 1 & 1 \\ x & y & z \\ x^2 & y^2 & z^2 \end{vmatrix} = (x - y)(y - z)(z - x)$ using factor theorem.

(OR)

(b) Integrate $x \log x$ with respect to x.

46. (a) If ABCD is a quadrilateral and E and F are the midpoints of AC and BD respectively, then prove that $\overrightarrow{AB} + \overrightarrow{AD} + \overrightarrow{CB} + \overrightarrow{CD} = 4\overrightarrow{EF}$

(OR)

(b) Evaluate : $\displaystyle\int \frac{1}{\sin^2 x \cos^2 x}\,dx$

47. (a) A factory has two machines I and II. Machine I produces 40% of items of the output and Machine II produces 60% of the items. Further 4% of items produced by Machine I are defective and 5% produced by Machine II are defective. An item is drawn at random. If the drawn item is defective, find the probability that it was produced by Machine II.

(OR)

(b) Rewrite $\sqrt{3}x + y + 4 = 0$ into normal form.

ANSWERS

PART - I

1. (b) infinitely many elements

2. (b) [–11, 7]

3. (a) $\dfrac{\sqrt{5}-1}{4}$

4. (d) $\dfrac{-\sqrt{3}}{2}$

5. (c) 11

6. (d) $\dfrac{n^2 - n + 2}{2}$

7. (c) $2x - y = 5$

8. (b) $\begin{bmatrix} \frac{1}{2} & \frac{-1}{2} \\ 2 & 1 \end{bmatrix}$

9. (c) 0

10. (d) coplanar vectors

11. (c) 22

12. (b) $\dfrac{15}{4}$

13. (d) None of the above

14. (a) 1

15. (a) $\dfrac{1}{5x}$

16. (d) does not exist

17. (a) $\sqrt{\tan x} + C$

18. (a) $x^2 \sin x + 2x\cos x - 2\sin x + C$

19. (c) $P(A/B) \ge P(A)$

20. (b) $\dfrac{1}{3}$