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HIGHER SECONDARY SECOND YEAR – MATHEMATICS

Chapter: I Applications of Matrices & Determinants
Theorems & Properties

Theorem: 1
If a square matrix has an inverse then it is unique
Proof
Let A be a square matrix of order n, such that A^{-1} exists.
Let B and C be two inverses of A
By definition

$$AB = BA = I_n$$
$$AC = CA = I_n$$

Let $B = BI_n = B(AC) = (BA)C = I_nC = C$
$\Rightarrow B = C$
Hence an inverse of a square matrix is unique.

Theorem: 2
Let A be a square matrix of order n. then A^{-1} exists if and only iff A is non – singular
Proof
Let A be a square matrix of order n
Suppose that A^{-1} exists

$$\Rightarrow AA^{-1} = A^{-1}A = I_n$$
$$|AA^{-1}| = |A||A^{-1}|$$
$$= |A^{-1}||A|$$
$$= |I_n|$$
$$= 1$$
$$\Rightarrow |A| \neq 0$$

Hence A is non singular.
Conversely,
Suppose that A is non singular

$$\Rightarrow |A| \neq 0$$
$$\text{Let } A(\text{adj}A) = (\text{adj}A)A = |A|I_n$$
$$\div \text{by} |A| \Rightarrow A\frac{\text{adj}A}{|A|} = \frac{\text{adj}A}{|A|}A = I_n$$
$$A\left(\frac{1}{|A|}\text{adj}A\right) = \left(\frac{1}{|A|}\text{adj}A\right)A = I_n, \text{ Hence } A^{-1} = \frac{1}{|A|}\text{adj}A$$

Theorem: 3

If A is non singular then $|A^{-1}| = \frac{1}{|A|}$

Proof

Let A be a non singular

$\Rightarrow |A| \neq 0$

Hence A^{-1} exists

By definition

$AA^{-1} = A^{-1}A = I_n$

$|AA^{-1}| = |A^{-1}A| = |I_n|$

Let $|AA^{-1}| = |I_n|$

$\Rightarrow |A||A^{-1}| = 1$

Hence $|A^{-1}| = \frac{1}{|A|}$

Theorem: 4

If A is non singular then $(A^T)^{-1} = (A^{-1})^T$

Proof

Let A be a non singular

$\Rightarrow |A| \neq 0$

Hence A^{-1} exists

By definition

$AA^{-1} = A^{-1}A = I_n$

$(AA^{-1})^T = (A^{-1}A)^T = (I_n)^T$

$(A^{-1})^T A^T = A^T (A^{-1})^T = I_n$

$\Rightarrow (A^T)^{-1} = (A^{-1})^T$

Theorem: 5

If A is non singular then $(\lambda A)^{-1} = \frac{1}{\lambda} A^{-1}$

where λ **is a non zero scalar**

Proof

Let A be a non singular

$\Rightarrow |A| \neq 0$

Hence A^{-1} exists

By definition

$AA^{-1} = A^{-1}A = I_n$

Since λ is a non zero scalar

$\Rightarrow (\lambda A)\left(\frac{1}{\lambda} A^{-1}\right) = \left(\frac{1}{\lambda} A^{-1}\right)(\lambda A) = I_n$

Hence $(\lambda A)^{-1} = \frac{1}{\lambda} A^{-1}$

Theorem: 6 (Left cancellation law)

Let A, B and C be square matrices of order n. If A is non singular and $AB = AC$ **then** $B = C$

Proof

Since A is non singular, A^{-1} exists

$$\Rightarrow AA^{-1} = A^{-1}A = I_n$$
$$\text{Let } AB = AC$$
$$A^{-1}(AB) = A^{-1}(AC)$$
$$(A^{-1}A)B = (A^{-1}A)C$$
$$I_nB = I_nC$$
$$\text{Hence } B = C$$

Theorem: 7 (Right cancellation law)

Let A, B and C be square matrices of order n. If A is non singular and $BA = CA$ **then** $B = C$

Proof

Since A is non singular, A^{-1} exists

$$\Rightarrow AA^{-1} = A^{-1}A = I_n$$
$$\text{Let } BA = CA$$
$$(BA)A^{-1} = (CA)A^{-1}$$
$$B(AA^{-1}) = C(AA^{-1})$$
$$BI_n = CI_n$$
$$\text{Hence } B = C$$

Theorem: 8 (Reversal Law for Inverses)

If A and B are non – singular matrices of the same order then the product AB is also non – singular and $(AB)^{-1} = B^{-1}A^{-1}$

Proof

Assume that A and B are non – singular matrices of same order n

$\Rightarrow |A| \neq 0, \ |B| \neq 0$

$\Rightarrow A^{-1}$ and B^{-1} exists

Let $|AB| = |A||B| \neq 0$

$\Rightarrow |AB| \neq 0$

$\therefore (AB)^{-1}$ exists

Let $(AB)(B^{-1}A^{-1}) = A(BB^{-1})A^{-1} = A(I_n)A^{-1}$
$= AA^{-1}$
$= I_n$ -------------- (1)

Let $(B^{-1}A^{-1})(AB) = B^{-1}(A^{-1}A)B = B^{-1}(I_n)B$
$= B^{-1}B$
$= I_n$ -------------- (2)

From (1) & (2) $\Rightarrow (AB)(B^{-1}A^{-1}) = (B^{-1}A^{-1})(AB) = I_n$

$B^{-1}A^{-1}$ is an inverse of AB

$\therefore (AB)^{-1} = B^{-1}A^{-1}$

Theorem: 9 (Law of Double Inverse)

If A is non – singular, then A^{-1} is also non – singular and $(A^{-1})^{-1} = A$

Proof

Assume that A is non – singular

$\Rightarrow |A| \neq 0$

$\therefore A^{-1}$ exists

Let $|A^{-1}| = \frac{1}{|A|} \neq 0$

$\Rightarrow A^{-1}$ is also non – singular

Let $AA^{-1} = A^{-1}A = I_n$

$AA^{-1} = I \Rightarrow (AA^{-1})^{-1} = I$

$\Rightarrow (A^{-1})^{-1} A^{-1} = I$

Pre multiply by A on both sides

$\Rightarrow (A^{-1})^{-1} A^{-1}A = IA$

$\therefore (A^{-1})^{-1} = A$

Theorem: 10

If A is a non – singular square matrix of order n, then $(\text{adj}A)^{-1} = \text{adj}(A^{-1}) = \frac{1}{|A|}A$

Proof

Since A is a non – singular square matrix

$\Rightarrow |A| \neq 0$

$\therefore A^{-1}$ exists

Let $A^{-1} = \frac{1}{|A|}\text{adj}A$

$\Rightarrow \text{adj}A = |A|A^{-1}$

$(\text{adj}A)^{-1} = (|A|A^{-1})^{-1}$

$= \frac{1}{|A|}A$ -------------- (1)

Replacing A by A^{-1} we get

$\text{adj}(A^{-1}) = |A^{-1}|(A^{-1})^{-1}$

$= \frac{1}{|A|}A$ -------------- (2)

From (1) and (2)

$\Rightarrow (\text{adj}A)^{-1} = \text{adj}(A^{-1}) = \frac{1}{|A|}A$

Theorem: 11

If A is a non – singular square matrix of order n, then $|\text{adj}A| = |A|^{n-1}$

Proof

Since A is non – singular square matrix of order n

Let $A(adjA)=(adjA)A=|A|I_n$

$$|A(adjA)|=|(adjA)A|=|(|A|I_n)|$$

$$\Rightarrow |A||adjA|=|adjA||A|=|A|^n$$

$$\Rightarrow |A||adjA|=|A|^n$$

$$\therefore |adjA|=|A|^{n-1}$$

Theorem: 12

If A is a non – singular square matrix of order n, then $adj(adjA)=|A|^{n-2}A$

Proof

Let B be a non – singular matrix of order n

$\Rightarrow B(adjB)=(adjB)(B)=|B|I_n$

Put $B=adjA$

$\Rightarrow (adjA)(adj(adjA))=(adj(adjA))(adjA)=|adjA|I_n$

$\Rightarrow (adjA)(adj(adjA))=|adjA|I_n$

Since $|adjA|=|A|^{n-1}$

$\Rightarrow (adjA)(adj(adjA))=|A|^{n-1}I_n$

Pre – multiplying both sides by A

$\Rightarrow A[(adjA)(adj(adjA))]=A(|A|^{n-1}I_n)$

$\Rightarrow [A(adjA)]adj(adjA)=A(|A|^{n-1}I_n)$

$\Rightarrow (|A|I_n)adj(adjA)=|A|^{n-1}A$

$\therefore adj(adjA)=|A|^{n-2}A$

Theorem: 13

If A is a non – singular square matrix of order n, then $adj(\lambda A)=\lambda^{n-1}adj(A)$, λ **is a non – zero scalar**

Proof

Let A be a non – singular square matrix of order n

$adj(A)=|A|A^{-1}$

Replacing A by λA

$adj(\lambda A)=|\lambda A|(\lambda A)^{-1}$

$$=\lambda^n|A|\frac{1}{\lambda}A^{-1}$$

$$=\lambda^{n-1}|A|A^{-1}$$

$$=\lambda^{n-1}adj(A)$$

Theorem: 14

If A is a non – singular square matrix of order n, then $|adj(adjA)|=|A|^{(n-1)^2}$

Proof

Let A be a non – singular square matrix of order n

Let $adj(adjA)=|A|^{n-2}A$

$$|\operatorname{adj}(\operatorname{adj}A)| = \left||A|^{n-2}A\right|$$
$$= \left(|A|^{n-2}\right)^n |A|$$
$$= |A|^{n^2-2n+1}$$
$$= |A|^{(n-1)^2}$$

Theorem: 15

If A is a non – singular square matrix of order n then $(\operatorname{adj}A)^T = \operatorname{adj}(A^T)$

Proof

Let A be a non – singular square matrix of order n

Let $A^{-1} = \frac{1}{|A|}\operatorname{adj}A$

Replacing A by A^T

$$(A^T)^{-1} = \frac{1}{|A^T|}\operatorname{adj}(A^T)$$
$$\Rightarrow \operatorname{adj}(A^T) = |A^T|(A^T)^{-1}$$
$$= |A^T|(A^{-1})^T$$
$$= (|A|A^{-1})^T$$
$$= \left(|A|\frac{1}{|A|}\operatorname{adj}A\right)^T$$
$$= (\operatorname{adj}A)^T$$

Theorem: 16

If A and B are any two non – singular square matrices of order n then $\operatorname{adj}(AB) = (\operatorname{adj}B)(\operatorname{adj}A)$

Proof

Let A and B are any two non – singular square matrices of order n

Let $\operatorname{adj}(A) = |A|A^{-1}$

Replacing A by AB

$$\operatorname{adj}(AB) = |AB|(AB)^{-1}$$
$$= (|A||B|)(B^{-1}A^{-1})$$
$$= |A|(|B|B^{-1})A^{-1}$$
$$= (|B|B^{-1})(|A|A^{-1})$$
$$= (\operatorname{adj}B)(\operatorname{adj}A)$$
$$\therefore \operatorname{adj}(AB) = (\operatorname{adj}B)(\operatorname{adj}A)$$