HIGHER SECONDARY SECOND YEAR
MATHEMATICS QUESTION BANK 2023-24

S.No	Topic Name	Page Number	No of Theorems	No of Examples	No of Exercise Problems	No of One mark questions
1	Applications of Matrices and Determinants	2	10	40	38	25
2	Complex Numbers	6	10	36	45	25
3	Theory of Equations	10	3	31	38	10
4	Inverse Trigonometric Functions	12	10	29	31	20
5	Two Dimensional Analytical Geometry II	15	6	41	39	25
6	Application of Vector Algebra	20	23	56	73	25
7	Applications of Differential Calculus	27	12	72	65	20
8	Differentials and Partial Derivatives	33	2	22	55	15
9	Applications of Integration	36	10	69	28	20
10	Ordinary Differential Equations	40	-	30	56	25
11	Probability Distributions	43	3	22	35	20
12	Discrete Mathematics	47	2	19	25	20
Total			91	467	528	250

	No of Theorems	No of examples	No of exercise problems	No of One mark questions
Volume I	62	233	264	130
Volume II	29	234	264	120
Total	91	467	528	250

1.APPLICATION OF MATRICES AND DETERMINANTS

Theorem 1.1: For every square matrix A of order n, $A(adjA) = (adjA)A = |A|I_n$.

Theorem 1.2: If a square matrix has an inverse, then it is unique.

Theorem 1.3: Let A be square matrix of order n. Then A^{-1} exists if and only if A is non-singular.

Theorem 1.4: If A is non-singular, then i)$|A^{-1}| = \frac{1}{|A|}$ ii)$(A^T)^{-1} = (A^{-1})^T$ iii)$(\lambda A)^{-1} = \frac{1}{\lambda}A^{-1}$, where λ is a non-zero scalar.

Theorem 1.5(Left Cancellation Law): Let A, B and C be square matrices of order n. If A is non-singular and $AB = AC$, then $B = C$.

Theorem1.6 (Right Cancellation Law): Let A, B and C be square matrices of order n. If A is non-singular and $BA = CA$, then $B = C$.

Theorem 1.7(Reversal Law for Inverses): If A and B are non-singular matrices of the same order, then the product AB is also non-singular and $(AB)^{-1} = B^{-1}A^{-1}$.

Theorem 1.8(Law of Double Inverse): If A is non-singular, then A^{-1} is also non-singular and $(A^{-1})^{-1} = A$.

Theorem 1.9: If A is a non-singular square matrix of order n, then

(i) $(adj\ A)^{-1} = adj\ (A^{-1}) = \frac{1}{|A|}A$ (ii) $|adj\ A| = |A|^{n-1}$

(iii) $adj(adj\ A) = |A|^{n-2}A$ (iv) $(adj\lambda A) = \lambda^{n-1}\ adj\ (A)$ where λ is a nonzero scalar

(v) $|adj(adj\ A)| = |A|^{(n-1)^2}$ (vi) $(adjA)^T = adj\ (A^T)$

Theorem 1.10: If A and B are any two non-singular square matrices of order n, then $adj(AB) = (adj\ B)(adj\ A)$.

Ex:1.1 If $A = \begin{bmatrix} 8 & -6 & 2 \\ -6 & 7 & -4 \\ 2 & -4 & 3 \end{bmatrix}$, verify that $A(adjA) = (adjA)A = |A|I_3$.

Ex:1.2 If $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$is non-singular, find A^{-1}

Ex:1.3 Find the inverse of the matrix $\begin{bmatrix} 2 & -1 & 3 \\ -5 & 3 & 1 \\ -3 & 2 & 3 \end{bmatrix}$.

Ex:1.4 If A is a non-singular matrix of odd order, prove that $|adjA|$ is positive.

Ex:1.5 Find a matrix A if $adj(A) = \begin{bmatrix} 7 & 7 & -7 \\ -1 & 11 & 7 \\ 11 & 5 & 7 \end{bmatrix}$.

Ex:1.6 If $adjA = \begin{bmatrix} -1 & 2 & 2 \\ 1 & 1 & 2 \\ 2 & 2 & 1 \end{bmatrix}$, find A^{-1}.

Ex:1.7 If A is symmetric, prove that $adj\ A$ is also symmetric.

Ex:1.8 Verify the property $(A^T)^{-1} = (A^{-1})^T$ with $A = \begin{bmatrix} 2 & 9 \\ 1 & 7 \end{bmatrix}$. **Mar 2020- 3M**

Ex:1.9 Verify $(AB)^{-1} = B^{-1}A^{-1}$ with $A = \begin{bmatrix} 0 & -3 \\ 1 & 4 \end{bmatrix}, B = \begin{bmatrix} -2 & -3 \\ 0 & -1 \end{bmatrix}$.**Sep 2020 3M & July-2022 3M**

Ex:1.10 If $A = \begin{bmatrix} 4 & 3 \\ 2 & 5 \end{bmatrix}$, find x and y such that $A^2 + xA + yI_2 = O_2$. Hence find A^{-1}.

Ex:1.11 Prove that $\begin{bmatrix} cos\theta & -sin\theta \\ sin\theta & cos\theta \end{bmatrix}$ is orthogonal. **Mar 2023- 2M**

Ex:1.12 If $A = \frac{1}{7}\begin{bmatrix} 6 & -3 & a \\ b & -2 & 6 \\ 2 & c & 3 \end{bmatrix}$ is orthogonal, find a, b and c, and hence A^{-1}.

Exercise 1.1

1. Find the adjoint of the following (i) $\begin{bmatrix} -3 & 4 \\ 6 & 2 \end{bmatrix}$ (ii) $\begin{bmatrix} 2 & 3 & 1 \\ 3 & 4 & 1 \\ 3 & 7 & 2 \end{bmatrix}$ (iii)$\frac{1}{3}\begin{bmatrix} 2 & 2 & 1 \\ -2 & 1 & 2 \\ 1 & -2 & 2 \end{bmatrix}$

2. Find the inverse (if it exists) of the following (i) $\begin{bmatrix} -2 & 4 \\ 1 & -3 \end{bmatrix}$ (ii) $\begin{bmatrix} 5 & 1 & 1 \\ 1 & 5 & 1 \\ 1 & 1 & 5 \end{bmatrix}$ (iii) $\begin{bmatrix} 2 & 3 & 1 \\ 3 & 4 & 1 \\ 3 & 7 & 2 \end{bmatrix}$

3. If $F(\alpha) = \begin{bmatrix} cos\alpha & 0 & sin\alpha \\ 0 & 1 & 0 \\ -sin\alpha & 0 & cos\alpha \end{bmatrix}$, show that $[F(\alpha)]^{-1} = F(-\alpha)$. **Mar 2023- 3M**

4. If $A = \begin{bmatrix} 5 & 3 \\ -1 & -2 \end{bmatrix}$, show that $A^2 - 3A - 7I_2 = O_2$. Hence find A^{-1}.

5. If $A = \frac{1}{9}\begin{bmatrix} -8 & 1 & 4 \\ 4 & 4 & 7 \\ 1 & -8 & 4 \end{bmatrix}$, prove that $A^{-1} = A^T$.

6. If $A = \begin{bmatrix} 8 & -4 \\ -5 & 3 \end{bmatrix}$, verify that $A(adjA) = (adjA)A = |A|I_2$. **Sep 2020 – 3M & Aug 2021- 3M**

7. If $A = \begin{bmatrix} 3 & 2 \\ 7 & 5 \end{bmatrix}$ and $B = \begin{bmatrix} -1 & -3 \\ 5 & 2 \end{bmatrix}$, verify $(AB)^{-1} = B^{-1}A^{-1}$.

8. If $adj(A) = \begin{bmatrix} 2 & -4 & 2 \\ -3 & 12 & -7 \\ -2 & 0 & 2 \end{bmatrix}$, find A.

9. If $adj(A) = \begin{bmatrix} 0 & -2 & 0 \\ 6 & 2 & -6 \\ -3 & 0 & 6 \end{bmatrix}$, find A^{-1}.

10. Find $adj(adj(A))$ if $adjA = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 2 & 0 \\ -1 & 0 & 1 \end{bmatrix}$.

11. $A = \begin{bmatrix} 1 & tanx \\ -tanx & 1 \end{bmatrix}$, show that $A^T A^{-1} = \begin{bmatrix} cos2x & -sin2x \\ sin2x & cos2x \end{bmatrix}$.

12. Find the matrix A for which $A\begin{bmatrix} 5 & 3 \\ -1 & -2 \end{bmatrix} = \begin{bmatrix} 14 & 7 \\ 7 & 7 \end{bmatrix}$.

13. Given $A = \begin{bmatrix} 1 & -1 \\ 2 & 0 \end{bmatrix}, B = \begin{bmatrix} 3 & -2 \\ 1 & 1 \end{bmatrix}$ and $C = \begin{bmatrix} 1 & 1 \\ 2 & 2 \end{bmatrix}$, find a matrix X such that $AXB = C$.

14. If $A = \begin{bmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{bmatrix}$, show that $A^{-1} = \frac{1}{2}(A^2 - 3I)$.

15. Decrypt the received encoded message $[2 \quad -3][20 \quad 4]$ with the encryption matrix $\begin{bmatrix} -1 & -1 \\ 2 & 1 \end{bmatrix}$ and the decryption matrix as its inverse, where the system of codes are described by the numbers $1 - 26$ to the letters $A - Z$ respectively, and the number 0 to a blank space.

Ex:1.13 Reduce the matrix $\begin{bmatrix} 3 & -1 & 2 \\ -6 & 2 & 4 \\ -3 & 1 & 2 \end{bmatrix}$ to a row-echelon form.

Ex:1.14 Reduce the matrix $\begin{bmatrix} 0 & 3 & 1 & 6 \\ -1 & 0 & 2 & 5 \\ 4 & 2 & 0 & 0 \end{bmatrix}$ to a row-echelon form.

Ex:1.15 Find the rank of each of the following matrices: (i) $\begin{bmatrix} 3 & 2 & 5 \\ 1 & 1 & 2 \\ 3 & 3 & 6 \end{bmatrix}$ (ii) $\begin{bmatrix} 4 & 3 & 1 & -2 \\ -3 & -1 & -2 & 4 \\ 6 & 7 & -1 & 2 \end{bmatrix}$

Ex:1.16 Find the rank of the following matrices which are in row-echelon form :

(i) $\begin{bmatrix} 2 & 0 & -7 \\ 0 & 3 & 1 \\ 0 & 0 & 1 \end{bmatrix}$ (ii) $\begin{bmatrix} -2 & 2 & -1 \\ 0 & 5 & 1 \\ 0 & 0 & 0 \end{bmatrix}$ (iii) $\begin{bmatrix} 6 & 0 & -9 \\ 0 & 2 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$

Ex:1.17 Find the rank of the matrix $\begin{bmatrix} 1 & 2 & 3 \\ 2 & 1 & 4 \\ 3 & 0 & 5 \end{bmatrix}$ by reducing it to an row-echelon form.

Ex:1.18 Find the rank of the matrix $\begin{bmatrix} 2 & -2 & 4 & 3 \\ -3 & 4 & -2 & -1 \\ 6 & 2 & -1 & 7 \end{bmatrix}$ by reducing it to an row-echelon form.

Ex:1.19 Show that the matrix $\begin{bmatrix} 3 & 1 & 4 \\ 2 & 0 & -1 \\ 5 & 2 & 1 \end{bmatrix}$ is non-singular and reduce it to the identity matrix by elementary row transformations.

Ex:1.20 Find the inverse of the non-singular matrix $A = \begin{bmatrix} 0 & 5 \\ -1 & 6 \end{bmatrix}$, by Gauss-Jordan method.

Ex:1.21 Find the inverse of $A = \begin{bmatrix} 2 & 1 & 1 \\ 3 & 2 & 1 \\ 2 & 1 & 2 \end{bmatrix}$ by Gauss-Jordan method.

Exercise *1.2*

1.Find the rank of the following matrices by minor method:

(i) $\begin{bmatrix} 2 & -4 \\ -1 & 2 \end{bmatrix}$ (ii) $\begin{bmatrix} -1 & 3 \\ 4 & -7 \\ 3 & -4 \end{bmatrix}$ (iii) $\begin{bmatrix} 1 & -2 & -1 & 0 \\ 3 & -6 & -3 & 1 \end{bmatrix}$ (iv) $\begin{bmatrix} 1 & -2 & 3 \\ 2 & 4 & -6 \\ 5 & 1 & -1 \end{bmatrix}$ **July-2022 3M**

(v) $\begin{bmatrix} 0 & 1 & 2 & 1 \\ 0 & 2 & 4 & 3 \\ 8 & 1 & 0 & 2 \end{bmatrix}$ **Aug 2021- 3M**

2. Find the rank of the following matrices by row reduction method:

(i) $\begin{bmatrix} 1 & 1 & 1 & 3 \\ 2 & -1 & 3 & 4 \\ 5 & -1 & 7 & 11 \end{bmatrix}$ (ii) $\begin{bmatrix} 1 & 2 & -1 \\ 3 & -1 & 2 \\ 1 & -2 & 3 \\ 1 & -1 & 1 \end{bmatrix}$ **May 2022 - 3M** (iii) $\begin{bmatrix} 3 & -8 & 5 & 2 \\ 2 & -5 & 1 & 4 \\ -1 & 2 & 3 & -2 \end{bmatrix}$

3. Find the inverse of each of the following by Gauss-Jordan method:

(i) $\begin{bmatrix} 2 & -1 \\ 5 & -2 \end{bmatrix}$ (ii) $\begin{bmatrix} 1 & -1 & 0 \\ 1 & 0 & -1 \\ 6 & -2 & -3 \end{bmatrix}$ (iii) $\begin{bmatrix} 1 & 2 & 3 \\ 2 & 5 & 3 \\ 1 & 0 & 8 \end{bmatrix}$

Ex:1.22 Solve the following system of linear equations, using matrix inversion method:

$5x + 2y = 3,\ 3x + 2y = 5.$ **May 2022 - 3M**

Ex:1.23 Solve the following system of equations, using matrix inversion method:

$2x_1 + 3x_2 + 3x_3 = 5,\ \ x_1 - 2x_2 + x_3 = -4,\ \ 3x_1 - x_2 - 2x_3 = 3.$

Ex:1.24 If $A = \begin{bmatrix} -4 & 4 & 4 \\ -7 & 1 & 3 \\ 5 & -3 & -1 \end{bmatrix}$ and $B = \begin{bmatrix} 1 & -1 & 1 \\ 1 & -2 & -2 \\ 2 & 1 & 3 \end{bmatrix}$, find the products AB and BA and hence solve the system of equations $x - y + z = 4,\ x - 2y - 2z = 9,\ 2x + y + 3z = 1.$

Exercise *1.3*

1. Solve the following system of linear equations by matrix inversion method:

(i) $2x + 5y = -2,\ \ x + 2y = -3$ (ii) $2x - y = 8,\ \ 3x + 2y = -2$

(iii) $2x + 3y - z = 9,\ \ x + y + z = 9,\ \ 3x - y - z = -1$

(iv) $x + y + z - 2 = 0,\ \ 6x - 4y + 5z - 31 = 0,\ \ 5x + 2y + 2z = 13$

2. If $A = \begin{bmatrix} -5 & 1 & 3 \\ 7 & 1 & -5 \\ 1 & -1 & 1 \end{bmatrix}$ and $B = \begin{bmatrix} 1 & 1 & 2 \\ 3 & 2 & 1 \\ 2 & 1 & 3 \end{bmatrix}$, find the products AB and BA and hence solve the system of equations $x + y + 2z = 1,\ \ 3x + 2y + z = 7,\ \ 2x + y + 3z = 2.$

3. A man is appointed in a job with a monthly salary of certain amount and a fixed amount of annual increment. If his salary was Rs.19,800 per month at the end of the first month after 3 years of service and Rs.23,400 per month at the end of the first month after 9 years of service, find his starting salary and his annual increment. (Use matrix inversion method to solve the problem.)

4. Four men and 4 women can finish a piece of work jointly in 3 days while 2 men and 5 women can finish the same work jointly in 4 days. Find the time taken by one man alone and that of one woman alone to finish the same work by using matrix inversion method.

5. The prices of three commodities A, B and C are Rs. x, y and z per units respectively. A person P purchases 4 units of B and sells *two* units of A and 5 units of C. Person Q purchases 2 units of C and sells 3 units of A and *one* unit of B. Person R purchases *one* unit of A and sells 3 unit of B and

one unit of C. In the process, P, Q and R earn Rs.15,000, Rs.1,000 and Rs.4,000 respectively. Find the prices per unit of A, B and C. (Use matrix inversion method to solve the problem.)

Ex:1.25 Solve, by Cramer's rule, the system of equations

$$x_1 - x_2 = 3, \quad 2x_1 + 3x_2 + 4x_3 = 17, \quad x_2 + 2x_3 = 7.$$

Ex:1.26 In a T20 match, a team needed just 6 runs to win with 1 ball left to go in the last over. The last ball was bowled and the batsman at the crease hit it high up. The ball traversed along a path in a vertical plane and the equation of the path is $y = ax^2 + bx + c$ with respect to a xy-coordinate system in the vertical plane and the ball traversed through the points$(10,8), (20,16), (40,22)$, can you conclude that Chennai Super Kings won the match? Justify your answer. (All distances are measured in meters and the meeting point of the plane of the path with the farthest boundary line is $(70,0)$.)

Exercise *1.4*

1.Solve the following systems of linear equations by Cramer's rule:

(i) $5x - 2y + 16 = 0,\ x + 3y - 7 = 0$ (ii) $\frac{3}{x} + 2y = 12,\ \frac{2}{x} + 3y = 13$

(iii)$3x + 3y - z = 11, 2x - y + 2z = 9, 4x + 3y + 2z = 25$ **July-2022 5M**

(iv) $\frac{3}{x} - \frac{4}{y} - \frac{2}{z} - 1 = 0,\ \frac{1}{x} + \frac{2}{y} + \frac{1}{z} - 2 = 0,\ \frac{2}{x} - \frac{5}{y} - \frac{4}{z} + 1 = 0$

2.In a competitive examination, one mark is awarded for every correct answer while $\frac{1}{4}$ mark is deducted for every wrong answer. A student answered 100 questions and got 80 marks. How many questions did he answer correctly? (Use Cramer's rule to solve the problem).

3.A chemist has one solution which is 50% acid and another solution which is 25% acid. How much each should be mixed to make 10 litres of a 40% acid solution? (Use Cramer's rule to solve the problem).

4.A fish tank can be filled in 10 minutes using both pumps A and B simultaneously. However, pump B can pump water in or out at the same rate. If pump B is inadvertently run in reverse, then the tank will be filled in 30 minutes. How long would it take each pump to fill the tank by itself? (Use Cramer's rule to solve the problem).

5. A family of 3 people went out for dinner in a restaurant. The cost of two dosai, three idlies and two vadais is $Rs.150$. The cost of the two dosai, two idlies and four vadais is $Rs.200$. The cost of five dosai, four idlies and two vadais is $Rs.250$. The family has $Rs.350$ in hand and they ate 3 dosai and six idlies and six vadais. Will they be able to manage to pay the bill within the amount they had?

Ex:1.27 Solve the following system of linear equations, by Gaussian elimination method :

$$4x + 3y + 6z = 25, \quad x + 5y + 7z = 13, \quad 2x + 9y + z = 1$$

Ex:1.28 The upward speed $v(t)$of a rocket at time t is approximated by $v(t) = at^2 + bt + c, 0 \le t \le 100$ where $a, b,$ and c are constants. It has been found that the speed at times $t = 3, t = 6$ and $t = 9$ seconds are respectively, $64, 133$ and 208 miles per second respectively. Find the speed at time $t = 15$ seconds. (Use Gaussian elimination method.)

Exercise *1.5*

1. Solve the following systems of linear equations by Gaussian elimination method:
   (i) $2x - 2y + 3z = 2, \quad x + 2y - z = 3, \quad 3x - y + 2z = 1$
   (ii)$2x + 4y + 6z = 22, \quad 3x + 8y + 5z = 27, \quad -x + y + 2z = 2$
2. If $ax^2 + bx + c$ is divided by $x + 3, x - 5$ and $x - 1$, the remainders are $21, 61$ and 9 respectively. Find a, b and c (Use Gaussian elimination method).
3. An amount of Rs.65,000 is invested in three bonds at the rates of 6 %, 8% and 9% per annum respectively.The total annual income is Rs.4,800. The income from the third bond is Rs.600 more than that from the second bond. Determine the price of each bond. (Use Gaussian elimination method.)

(OR)

3. An amount of Rs.65,000 is invested in three bonds at the rates of 6 %, 8% and 10% per annum respectively.The total annual income is Rs.5,000. The income from the third bond is Rs.800 more

than that from the second bond. Determine the price of each bond. (Use Gaussian elimination method.)

4. A boy is walking along the path $y = ax^2 + bx + c$ through the points $(-6,8), (-2,-12)$ and $(3,8)$. He wants to meet his friend at $P(7,60)$. Will he meet his friend? (Use Gaussian elimination method) **Mar 2023- 5M**

Ex:1.29 Test for consistency of the following system of linear equations and if possible solve:

$$x + 2y - z = 3, \quad 3x - y + 2z = 1, \quad x - 2y + 3z = 3, \quad x - y + z + 1 = 0$$

Ex:1.30 Test for consistency of the following system of linear equations and if possible solve:

$$4x - 2y + 6z = 8, \quad x + y - 3z = -1, \quad 15x - 3y + 9z = 21$$

Ex:1.31 Test for consistency of the following system of linear equations and if possible solve:

$$x - y + z = -9, \quad 2x - 2y + 2z = -18, \quad 3x - 3y + 3z + 27 = 0$$

Ex:1.32 Test for consistency of the following system of linear equations

$$x - y + z = -9, \quad 2x - y + z = 4, \quad 3x - y + z = 6, \quad 4x - y + 2z = 7$$ **Mar 2020-5M**

Ex:1.33 Find the condition on a, b and c so that the following system of linear equations has one parameter family of solutions: $x + y + z = a, \quad x + 2y + 3z = b, \quad 3x + 5y + 7z = c$.

Ex:1.34 Investigate for what values of λ and μ the system of linear equations

$x + 2y + z = 7, \quad x + y + \lambda z = \mu, \quad x + 3y - 5z = 5$ has (i) no solution (ii) a unique solution (iii) an infinite number of solutions.

Exercise *1.6*

1. Test for consistency and if possible, solve the following systems of equations by rank method.
   (i) $x - y + 2z = 2, \quad 2x + y + 4z = 7, \quad 4x - y + z = 4$
   (ii) $3x + y + z = 2, \quad x - 3y + 2z = 1, \quad 7x - y + 4z = 5$
   (iii) $2x + 2y + z = 5, \quad x - y + z = 1, \quad 3x + y + 2z = 4$ **Sep 2020 – 5M**
   (iv) $2x - y + z = 2, \quad 6x - 3y + 3z = 6, \quad 4x - 2y + 2z = 4$
2. Find the value of k for which the equations $kx - 2y + z = 1, \quad x - 2ky + z = -2,$ $x - 2y + kz = 1$ have (i) no solution (ii) unique solution (iii) infinitely many solution
3. Investigate the values of λ and μ the system of linear equations
   $2x + 3y + 5z = 9, \quad 7x + 3y - 5z = 8, \quad 2x + 3y + \lambda z = \mu$ have
   (i) no solution (ii) a unique solution (iii) an infinite number of solutions.

Ex:1.35 Solve the following system: $x + 2y + 3z = 0, \quad 3x + 4y + 4z = 0, \quad 7x + 10y + 12z = 0$

Ex:1.36 Solve the system: $x + 3y - 2z = 0, \quad 2x - y + 4z = 0, \quad x - 11y + 14z = 0$

Ex:1.37 Solve the system:

$$x + y - 2z = 0, \quad 2x - 3y + z = 0, \quad 3x - 7y + 10z = 0, \quad 6x - 9y + 10z = 0.$$

Ex:1.38 Determine the values of λ for which the following system of equations $(3\lambda - 8)x + 3y + 3z = 0, \quad 3x + (3\lambda - 8)y + 3z = 0, \quad 3x + 3y + (3\lambda - 8)z = 0$ has a non-trivial solution.

Ex:1.39 By using Gaussian elimination method, balance the chemical reaction equation:

$C_5H_8 + O_2 \rightarrow CO_2 + H_2O.$

Ex:1.40 If the system of equations $px + by + cz = 0; ax + qy + cz = 0; ax + by + rz = 0$ has a non-trivial solution and $p \neq a, \ q \neq b, \ r \neq c$, then prove that $\frac{p}{p-a} + \frac{q}{q-b} + \frac{r}{r-c} = 2$.

Exercise *1.7*

1. Solve the following system of homogenous equations.
   (i) $3x + 2y + 7z = 0, \quad 4x - 3y - 2z = 0, \quad 5x + 9y + 23z = 0$
   (ii) $2x + 3y - z = 0, \quad x - y - 2z = 0, \quad 3x + y + 3z = 0$
2. Determine the values of λ for which the following system of equations $x + y + 3z = 0,$ $4x + 3y + \lambda z = 0, \ 2x + y + 2z = 0$ has (i) a unique solution (ii) a non-trivial solution.
3. By using Gaussian elimination method, balance the chemical reaction equation:
   $C_2H_6 + O_2 \rightarrow H_2O + CO_2.$

2. COMPLEX NUMBERS

Property 1: (The commutative property under addition)

For any two complex numbers z_1 and z_2, we have $z_1 + z_2 = z_2 + z_1$.

Property 2: (Inverse property under multiplication) Prove that the multiplicative inverse of a nonzero complex number $z = x + iy$ is $\frac{x}{x^2+y^2} - i\frac{y}{x^2+y^2}$.

Property 3: For any two complex numbers z_1 and z_2, prove that $\overline{z_1 + z_2} = \bar{z}_1 + \bar{z}_2$

Property 4: For any two complex numbers z_1 and z_2, prove that $\overline{z_1 . z_2} = \bar{z}_1 . \bar{z}_2$

Property 5: A complex number z is purely imaginary if and only if $z = -\bar{z}$.

Property6:(Triangle inequality)

For any two complex number z_1 and z_2, prove that $|z_1 + z_2| \le |z_1| + |z_2|$.

Property 7: For any two complex number z_1 and z_2, prove that $|z_1 z_2| = |z_1||z_2|$

Property 8: If $z = r(cos\theta + isin\theta)$, then $z^{-1} = \frac{1}{r}(cos\theta - isin\theta)$

Property 9: If $z_1 = r_1(cos\theta_1 + isin\theta_1)$ and $z_2 = r_2(cos\theta_2 + isin\theta_2)$ then

$z_1 z_2 = r_1 r_2(\cos(\theta_1 + \theta_2) + isin(\theta_1 + \theta_2))$

Property 10: If $z_1 = r_1(cos\theta_1 + isin\theta_1)$ and $z_2 = r_2(cos\theta_2 + isin\theta_2)$ then

$\frac{z_1}{z_2} = \frac{r_1}{r_2}\ (\cos(\theta_1 - \theta_2)\ + isin\theta(\theta_1 - \theta_2)$

Ex:2.1 Simplify: (i) i^7 (ii) i^{1729} (iii) $i^{-1924} + i^{2018}$ (iv) $\sum_{n=1}^{102} i^n$ (v) $i\, i^2 i^3 \cdots i^{40}$

Exercise 2.1

1. Simplify (i) $i^{1947} + i^{1950}$ (ii) $i^{1948} - i^{-1869}$ (iii) $\sum_{n=1}^{12} i^n$
(iv) $i^{59} + \frac{1}{i^{59}}$ (v) $i\, i^2 i^3 \cdots i^{2000}$ (vi) $\sum_{n=1}^{10} i^{n+50}$

Ex:2.2 Find the value of the real numbers x and y, if the complex number $(2 + i)x + (1 - i)y + 2i - 3$ and $x + (-1 + 2i)y + 1 + i$ are equal.

Exercise 2.2

1. Evaluate the following if $z = 5 - 2i$ and $w = -1 + 3i$
(i) $z + w$ (ii) $z - iw$ (iii) $2z + 3w$ (iv) zw (v) $z^2 + 2zw + w^2$ (vi) $(z + w)^2$
2. Given the complex number $z = 2 + 3i$, represent the complex numbers in Argand diagram
(i) z, iz and $z + iz$ (ii) $z, -iz$, and $z - iz$.
3. Find the value of the real numbers x and y, if the complex numbers $(3 - i)x - (2 - i)y + 2i + 5$ and $2x + (-1 + 2i)y + 3 + 2i$ are equal.

Exercise 2.3

1. If $z_1 = 1 - 3i, z_2 = -4i$ and $z_3 = 5$, show that (i) $(z_1 + z_2) + z_3 = z_1 + (z_2 + z_3)$
(ii) $(z_1 . z_2). z_3 = z_1 . (z_2 . z_3)$
2. If $z_1 = 3, z_2 = -7i$ and $z_3 = 5 + 4i$, show that (i) $z_1 . (z_2 + z_3) = z_1 . z_2 + z_1 . z_3$ **July-2022 2M**
(ii) $(z_1 + z_2). z_3 = z_1 . z_3 + z_2 . z_3$
3. If $z_1 = 2 + 5i,\ z_2 = -3 - 4i$ and $z_3 = 1 + i$, find the additive and multiplicative inverse of z_1, z_2 and z_3.

Ex:2.3 Write $\frac{3+4i}{5-12i}$ in the $x + iy$ form, hence find its real and imaginary parts.

Ex:2.4 Simplify $\left(\frac{1+i}{1-i}\right)^3 - \left(\frac{1-i}{1+i}\right)^3$ into rectangular form. **Mar-2020 – 2M**

Ex:2.5 If $\frac{z+3}{z-5i} = \frac{1+4i}{2}$, find the complex number z in the rectangular form.

Ex:26 If $z_1 = 3 - 2i$ and $z_2 = 6 + 4i$ find $\frac{z_1}{z_2}$ in the rectangular form.

Ex:2.7 Find z^{-1}, if $z = (3 + 2i)(1 - i)$

Ex:2.8 Show that (i) $\left(2 + i\sqrt{3}\right)^{10} + \left(2 - i\sqrt{3}\right)^{10}$ is real (ii) $\left(\frac{19+9i}{5-3i}\right)^{15} - \left(\frac{8+i}{1+2i}\right)^{15}$ is purely imaginary

Exercise 2.4

1. Write the following in the rectangular form: (i) $\overline{(5+9i)+(2-4i)}$ (ii) $\frac{10-5i}{6+2i}$ (iii) $\bar{3i}+\frac{1}{2-i}$

2. If $z = x + iy$, find the following in the rectangular form (i) $Re\left(\frac{1}{z}\right)$ (ii) $Re(i\bar{z})$ (iii) $Im(3z + 4\bar{z} - 4i)$

3. If $z_1 = 2 - i$ and $z_2 = -4 + 3i$, find the inverse of z_1z_2 and $\frac{z_1}{z_2}$

4. The complex numbers u, v and w are related by $\frac{1}{u} = \frac{1}{v} + \frac{1}{w}$. If $v = 3 - 4i$ and $w = 4 + 3i$, find u in rectangular form.

5. Prove the following properties: (i) z is real if and only if $z = \bar{z}$

(ii) $Re(z) = \frac{z+\bar{z}}{2}$ and $Im(z) = \frac{z-\bar{z}}{2i}$ **May 2022 - 2M**

6. Find the least value of the positive integer n for which $\left(\sqrt{3}+i\right)^n$ (i) real (ii) purely imaginary

7. Show that (i) $\left(2+i\sqrt{3}\right)^{10} - \left(2-i\sqrt{3}\right)^{10}$ is purely imaginary **July-2022 5M**

(ii) $\left(\frac{19-7i}{9+i}\right)^{12} + \left(\frac{20-5i}{7-6i}\right)^{12}$ is real

Ex:2.9 If $z_1 = 3 + 4i, z_2 = 5 - 12i$ and $z_3 = 6 + 8i$, find $|z_1|, |z_2|, |z_3|, |z_1 + z_2|, |z_2 - z_3|, |z_1 + z_3|$

Ex:2.10 Find the following (i) $\left|\frac{2+i}{-1+2i}\right|$ (ii) $\left|\overline{(1+i)}(2+3i)(4i-3)\right|$ (iii) $\left|\frac{i(2+i)^3}{(1+i)^2}\right|$

Ex:2.11 Which one of the points $i, -2 + i$ and 3 is farthest from the origin.

Ex:2.12 If z_1, z_2 and z_3 are complex numbers such that $|z_1| = |z_2| = |z_3| = |z_1 + z_2 + z_3| = 1$, find the value of $\left|\frac{1}{z_1} + \frac{1}{z_2} + \frac{1}{z_3}\right|$.

Ex:2.13 If $|z| = 2$, then show that $3 \le |z + 3 + 4i| \le 7$ **Mar 2023- 2M**

Ex:2.14 Show that the points $1, -\frac{1}{2} + \frac{i\sqrt{3}}{2}$ and $-\frac{1}{2} - \frac{i\sqrt{3}}{2}$ are the vertices of an equilateral triangle.

Aug 2021- 3M

Ex:2.15 Let z_1, z_2 and z_3 be complex numbers such that $|z_1| = |z_2| = |z_3| = r > 0$ and $z_1 + z_2 + z_3 \neq 0$. Prove that $\left|\frac{z_1z_2+z_2z_3+z_3z_1}{z_1+z_2+z_3}\right| = r$.

Ex:2.16 Show that the equation $z^2 = \bar{z}$ has 4 solutions.

Ex:2.17 Find the square root of $6 - 8i$. **July-2022 3M**

Exercise 2.5

1. Find the modulus of the following complex numbers

(i) $\frac{2i}{3+4i}$ (ii) $\frac{2-i}{1+i} + \frac{1-2i}{1-i}$ (iii) $(1-i)^{10}$ (iv) $2i(3-4i)(4-3i)$

2. For any two complex numbers z_1 and z_2 such that $|z_1| = |z_2| = 1$ and $z_1z_2 \neq -1$ then show that $\frac{z_1+z_2}{1+z_1z_2}$ is a real number.

3. Which one of the points $10 - 8i, 11 + 6i$ is closest to $1 + i$. **May 2022 - 3M**

4. If $|z| = 3$, show that $7 \le |z + 6 - 8i| \le 13$.

5. If $|z| = 1$, show that $2 \le |z^2 - 3| \le 4$.

6. If $|z| = 2$, show that $8 \le |z + 6 + 8i| \le 12$.

7. Let z_1, z_2 and z_3 are three complex numbers such that $|z_1| = 1, |z_2| = 2, |z_3| = 3$ and $|z_1 + z_2 + z_3| = 1$. show that $|9z_1z_2 + 4z_1z_3 + z_2z_3| = 6$.

8. If the area of the triangle formed by the vertices $z, iz,$, and $z + iz$ is 50 square units, find the value of $|z|$.

9. Show that the equation $z^3 + 2\bar{z} = 0$ has five solutions.

10. Find the square roots of (i) $4 + 3i$ (ii) $-6 + 8i$ (iii) $-5 - 12i$.

Ex:2.18 Given the complex number $z = 3 + 2i$, represent the complex numbers $z, iz,$, and $z + iz$ in one Argand plane. Show that these complex numbers form the vertices of an isosceles right triangle.

Ex:2.19 Show that $|3z - 5 + i| = 4$ represents a circle, and, find its centre and radius.

Ex:2.20 Show that $|z + 2 - i| < 2$ represents interior points of a circle. Find its centre and radius.

Ex:2.21 Obtain the Cartesian form of the locus of z in each of the following cases.

(i) $|z| = |z-i|$ (ii) $|2z-3-i| = 3$

Exercise 2.6

1. If $z = x+iy$ is a complex number such that $\left|\frac{z-4i}{z+4i}\right| = 1$, show that the locus of z is real axis. **Aug 2021- 5M**
2. If $z = x+iy$ is a complex number such that $Im\left(\frac{2z+1}{iz+1}\right) = 0$. show that the locus of z is $2x^2+2y^2+x-2y = 0$.
3. Obtain the Cartesian form of the locus of $z = x+iy$ in each of the following cases.
   (i) $[Re(iz)]^2 = 3$ (ii) $Im[(1-i)z+1] = 0$
   (iii) $|z+i| = |z-1|$ **Sep 2020 – 2M & May 2022 - 5M** (iv) $\bar{z} = z^{-1}$
4. Show that the following equations represent a circle, and, find its centre and radius.
   (i) $|z-2-i| = 3$ (ii) $|2z+2-4i| = 2$ (iii) $|3z-6+12i| = 8$
5. Obtain the Cartesian equation for the locus of $z = x+iy$ in each of the following cases:
   (i) $|z-4| = 16$ (ii) $|z-4|^2 - |z-1|^2 = 16$

Ex:2.22 Find the modulus and principal argument of the following complex numbers.
(i) $\sqrt{3}+i$ (ii) $-\sqrt{3}+i$ (iii) $-\sqrt{3}-i$ (iv) $\sqrt{3}-i$

Ex:2.23 Represent the complex number (i) $-1-i$ (ii) $1+i\sqrt{3}$ in polar form

Ex:2.24 Find the principal argument $Arg\ z$, when $z = -\frac{2}{1+i\sqrt{3}}$.

Ex:2.25 Find the product of $\frac{3}{2}(\cos\frac{\pi}{3}+isin\frac{\pi}{3}).6(\cos\frac{5\pi}{6}+isin\frac{5\pi}{6})$ in rectangular form.

Ex:2.26 Find the quotient $\frac{2\left(\cos\frac{9\pi}{4}+isin\frac{9\pi}{4}\right)}{4\left(\cos\left(-\frac{3\pi}{2}\right)+isin\left(-\frac{3\pi}{2}\right)\right)}$ in rectangular form.

Ex:2.27 If $z = x+iy$ and $arg\left(\frac{z-1}{z+1}\right) = \frac{\pi}{2}$, then show that $x^2+y^2 = 1$.

Exercise 2.7

1. Write in polar form of the following complex numbers
   (i) $2+i2\sqrt{3}$ (ii) $3-i\sqrt{3}$ (iii) $-2-i2$ (iv) $\frac{i-1}{\cos\frac{\pi}{3}+isin\frac{\pi}{3}}$
2. Find the rectangular form of the complex numbers
   (i)$(\cos\frac{\pi}{6}+isin\frac{\pi}{6}).\left(\cos\frac{\pi}{12}+isin\frac{\pi}{12}\right)$ (ii) $\frac{\cos\frac{\pi}{6}-isin\frac{\pi}{6}}{2\left(\cos\frac{\pi}{3}+isin\frac{\pi}{3}\right)}$
3. If $(x_1+iy_1)(x_2+iy_2)\cdots(x_n+iy_n) = a+ib$ show that
   (i) $(x_1^2+y_1^2)(x_2^2+y_2^2)\cdots(x_n^2+y_n^2) = a^2+b^2$ (ii) $\sum_{r=1}^{n}\tan^{-1}\left(\frac{y_r}{x_r}\right) = \tan^{-1}\left(\frac{b}{a}\right)+2k\pi, k\in\mathbb{Z}$.
4. If $\frac{1+z}{1-z} = cos2\theta+isin2\theta$, show that $z = itan\theta$.
5. If $cos\alpha+cos\beta+cos\gamma = sin\alpha+sin\beta+sin\gamma = 0$, show that
   $(i)cos3\alpha+cos3\beta+cos3\gamma = 3\cos(\alpha+\beta+\gamma)$ (ii)$sin3\alpha+sin3\beta+sin3\gamma = 3\sin(\alpha+\beta+\gamma)$.
6. If $z = x+iy$ and $arg\left(\frac{z-i}{z+2}\right) = \frac{\pi}{4}$ then show that $x^2+y^2+3x-3y+2 = 0$.

Ex:2.28 If $z = (cos\theta+isin\theta)$, show that $z^n+\frac{1}{z^n} = 2\cos n\theta$ and $z^n-\frac{1}{z^n} = 2i\ sin\ n\theta$

Ex:2.29 Simplify $\left(\sin\frac{\pi}{6}+icos\frac{\pi}{6}\right)^{18}$ **Ex:2.30** Simplify $\left(\frac{1+cos2\theta+isin2\theta}{1+cos2\theta-isin2\theta}\right)^{30}$.

Ex:2.31 Simplify (i) $(1+i)^{18}$ (ii) $\left(-\sqrt{3}+3i\right)^{31}$ **Ex:2.32** Find the cube roots of unity.

Ex:2.33 Find the fourth roots of unity.

Ex:2.34 Solve the equation $z^3+8i = 0$,where $z\in\mathbb{C}$. **Mar 2023- 5M**

Ex:2.35 Find the all cube roots of $\sqrt{3}+i$.

Ex:2.36 Suppose z_1, z_2 and z_3 are the vertices of an equilateral triangle inscribed in the circle $|z| = 2$. If $z_1 = 1+i\sqrt{3}$, then find z_2 and z_3.

Exercise 2.8

1. If $\omega \neq 1$ is a cube root of unity, then show that $\frac{a+b\omega+c\omega^2}{b+c\omega+a\omega^2}+\frac{a+b\omega+c\omega^2}{c+a\omega+b\omega^2}=-1$.

2. Show that $\left(\frac{\sqrt{3}}{2}+\frac{i}{2}\right)^5+\left(\frac{\sqrt{3}}{2}-\frac{i}{2}\right)^5=-\sqrt{3}$.

3. Find the value of $\left(\frac{1+\sin\frac{\pi}{10}+i\cos\frac{\pi}{10}}{1+\sin\frac{\pi}{10}-i\cos\frac{\pi}{10}}\right)^{10}$

4. If $2\cos\alpha = x+\frac{1}{x}$ and $2\cos\beta = y+\frac{1}{y}$, show that

(i) $\frac{x}{y}+\frac{y}{x}=2\cos(\alpha-\beta)$ (ii) $xy-\frac{1}{xy}=2i\sin(\alpha+\beta)$

(iii) $\frac{x^m}{y^n}-\frac{y^n}{x^m}=2i\sin(m\alpha-n\beta)$ (iv) $x^m y^n+\frac{1}{x^m y^n}=2\cos(m\alpha+n\beta)$ **Mar 2020 – 5M**

5. Solve the equation $z^3+27=0$.

6. If $\omega \neq 1$ is a cube root of unity, then show that the roots of the equation $(z-1)^3+8=0$ are -1, $1-2\omega, 1-2\omega^2$.

7. Find the value of $\sum_{k=1}^{8}\left(\cos\frac{2k\pi}{9}+i\sin\frac{2k\pi}{9}\right)$.

8. If $\omega \neq 1$ is a cube root of unity, show that (i) $(1-\omega+\omega^2)^6+(1+\omega-\omega^2)^6=128$

(ii) $(1+\omega)(1+\omega^2)(1+\omega^4)(1+\omega^8)\cdots\left(1+\omega^{2^{11}}\right)=1$.

9. If $z=2-2i$ find the rotation of z by θ radians in the counter clockwise direction about the origin

(i) $\theta=\frac{\pi}{3}$ (ii) $\theta=\frac{2\pi}{3}$ (iii) $\theta=\frac{3\pi}{2}$

10. Prove that the values of $\sqrt[4]{-1}$ are $\pm\frac{1}{\sqrt{2}}(1\pm i)$.

3. THEORY OF EQUATIONS

Theorem1: (The Fundamental theorem of Algebra) Every polynomial equation of degree $n \geq 1$ has atleast one root in $\mathbb{C}$.

Theorem2:(Complex conjugate root theorem) If a complex number z_0 is a root of a polynomial equation with real coefficients, then its complex conjugate $\overline{z_0}$ is also a root.

Theorem3:(Rational root theorem) Let $a_n x^n+\cdots\cdots\cdots+a_1 x+a_0$ *with* $a_n \neq 0$ *and* $a_0 \neq 0$ be a polynomial with integer coefficients. If $\frac{p}{q}$ with $(p,q)=1$, is a root of the polynomial, then p is a factor of a_0 and q is a factor of a_n.

Ex:3.1 If α and β are the roots of the quadratic equation $17x^2+43x-73=0$, construct a quadratic equation whose roots are $\alpha+2$ and $\beta+2$.

Ex:3.2 If α and β are the roots of the quadratic equation $2x^2-7x+13=0$, construct a quadratic equation whose roots are $\alpha^2$and β^2.

Ex:3.3 If α,β and γ are the roots of the equation $x^3+px^2+qx+r=0$, find the value of $\sum\frac{1}{\beta\gamma}$ in terms of the coefficients.

Ex:3.4 Find the sum of the squares of the roots of $ax^4+bx^3+cx^2+dx+e=0, a\neq 0$.

Ex:3.5 Find the condition that the roots of cubic equation $x^3+ax^2+bx+c=0$ are in the ratio $p:q:r$.

Ex:3.6 Form the equation whose roots are the squares of the roots of the cubic equation $x^3+ax^2+bx+c=0$.

Ex:3.7 If p is real, discuss the nature of the roots of the equation $4x^2+4px+p+2=0$, in terms of p. **Mar 2020 – 3M**

Exercise 3.1

1. If the sides of a cubic box are increased by $1,2,3$ units respectively to form a cuboid, then the volume is increased by 52 cubic units. Find the volume of the cuboid. **Aug 2021- 3M**

2. Construct a cubic equation with roots (i) 1,2 and 3 (ii) 1,1 and -2 (iii) $2,\frac{1}{2}$ and 1.

3. If α , β and γ are the roots of the cubic equation $x^3+2x^2+3x+4=0$, form a cubic equation whose roots are (i) $2\alpha, 2\beta, 2\gamma$ (ii) $\frac{1}{\alpha},\frac{1}{\beta},\frac{1}{\gamma}$ (iii) $-\alpha,-\beta,-\gamma$

4. Solve the equation $3x^3 - 16x^2 + 23x - 6 = 0$ if the product of two roots is 1.
5. Find the sum of squares of roots of the equation $2x^4 - 8x^3 + 6x^2 - 3 = 0$.
6. Solve the equation $x^3 - 9x^2 + 14x + 24 = 0$ if it is given that two of its roots are in the ratio $3 : 2$.
7. If α, β and γ are the roots of the polynomial equation $ax^3 + bx^2 + cx + d = 0$, find the value of $\sum \frac{\alpha}{\beta\gamma}$ in terms of the coefficients.
8. If α, β, γ and δ are the roots of the polynomial equation $2x^4 + 5x^3 - 7x^2 + 8 = 0$, find a quadratic equation with integer coefficients whose roots are $\alpha + \beta + \gamma + \delta$ and $\alpha\beta\gamma\delta$. **Sep 2020 – 2M**
9. If p and q are the roots of the equation $lx^2 + nx + n = 0$, show that $\sqrt{\frac{p}{q}} + \sqrt{\frac{q}{p}} + \sqrt{\frac{n}{l}} = 0$.

Mar 2023- 2M

10. If the equations $x^2 + px + q = 0$ and $x^2 + p'x + q' = 0$ have a common root, show that it must be equal to $\frac{pq' - p'q}{q - q'}$ or $\frac{q - q'}{p' - p}$.
11. A 12 metre tall tree was broken into two parts. It was found that the height of the part which was left standing was the cube root of the length of the part that was cut away. Formulate this into a mathematical problem to find the height of the part which was left standing.

Ex:3.8 Find the monic polynomial equation of minimum degree with real coefficients having $2 - \sqrt{3}i$ as a root.

Ex:3.9 Find a polynomial equation of minimum degree with rational coefficients, having $2 - \sqrt{3}$ as a root. **May 2022 - 2M**

Ex:3.10 Form a polynomial equation with integer coefficients with $\sqrt{\frac{\sqrt{2}}{\sqrt{3}}}$ as a root.

Ex:3.11 Show that the equation $2x^2 - 6x + 7 = 0$ cannot be satisfied by any real values of x.

Ex:3.12 If $x^2 + 2(k + 2)x + 9k = 0$ has equal roots, find k.

Ex:3.13 Show that, if p, q, r are rational, the roots of the equation $x^2 - 2px + p^2 - q^2 + 2qr - r^2 = 0$ are rational.

Ex:3.14 Prove that a line cannot intersect a circle at more than two points.

Exercise 3.2

1. If k is real, discuss the nature of the roots of the polynomial equation $2x^2 + kx + k = 0$, in terms of k.
2. Find a polynomial equation of minimum degree with rational coefficients, having $2 + \sqrt{3}i$ as a root.
3. Find a polynomial equation of minimum degree with rational coefficients, having $2i + 3$ as a root.
4. Find a polynomial equation of minimum degree with rational coefficients, having $\sqrt{5} - \sqrt{3}$ as a root.
5. Prove that the straight line and parabola cannot intersect at more than two points.

Ex:3.15 If $2 + i$ and $3 - \sqrt{2}$ are roots of the equation $x^6 - 13x^5 + 62x^4 - 126x^3 + 65x^2 + 127x - 140 = 0$. Find all roots.

Ex:3.16 Solve the equation $x^4 - 9x^2 + 20 = 0$.

Ex:3.17 Solve the equation $x^3 - 3x^2 - 33x + 35 = 0$.

Ex:3.18 Solve the equation $2x^3 + 11x^2 - 9x - 18 = 0$.

Ex:3.19 Obtain the condition that the roots of $x^3 + px^2 + qx + r = 0$ are in A.P. **Sep 2020 – 3M**

Ex:3.20 Find the condition that the roots of $ax^3 + bx^2 + cx + d = 0$ are in G.P. assume $a, b, c, d \neq 0$

Ex:3.21 If the roots of $x^3 + px^2 + qx + r = 0$ are in H.P. prove that $9pqr = 2q^3 + 27r^2$.

Ex:3.22 It is known that the roots of the equation $x^3 - 6x^2 - 4x + 24 = 0$ are in A.P. Find its roots.

Exercise 3.3

1. Solve the cubic equation $2x^3 - x^2 - 18x + 9 = 0$ if sum of two of its roots vanishes.
2. Solve the equation $9x^3 - 36x^2 + 44x - 16 = 0$ if the roots form an arithmetic progression.
3. Solve the equation $3x^3 - 26x^2 + 52x - 24 = 0$ if its roots form a geometric progression.
4. Determine k and solve the equation $2x^3 - 6x^2 + 3x + k = 0$ if one of its roots is twice the sum of

the other two roots.

5. Find all zeros of the polynomial $x^6 - 3x^5 - 5x^4 + 22x^3 - 39x^2 - 39x + 135$, if it is known that $1 + 2i$ and $\sqrt{3}$ are two of its zeros.

6. Solve the cubic equations: (i) $2x^3 - 9x^2 + 10x = 3$ **May 2022 - 3M** (ii) $8x^3 - 2x^2 - 7x + 3 = 0$

7. Solve the equation : $x^4 - 14x^2 + 45 = 0$.

Ex:3.23 Solve the equation $(x-2)(x-7)(x-3)(x+2) + 19 = 0$.

Ex:3.24 Solve the equation $(2x-3)(6x-1)(3x-2)(x-2) - 5 = 0$.

Exercise 3.4

1. Solve: (i) $(x-5)(x-7)(x+6)(x+4) = 504$ (ii) $(x-4)(x-7)(x-2)(x+1) = 16$

2. Solve: $(2x-1)(x+3)(x-2)(2x+3) + 20 = 0$.

Ex:3.25 Solve the equation $x^3 - 5x^2 - 4x + 20 = 0$.

Ex:3.26 Find the roots of $2x^3 + 3x^2 + 2x + 3 = 0$.

Ex:3.27 Solve the equation $7x^3 - 43x^2 = 43x - 7$.

Ex:3.28 Solve the equation $x^4 - 10x^3 + 26x^2 - 10x + 1 = 0$.

Ex:3.29 Find solution, if any, of the equation $2\cos^2 x - 9\cos x + 4 = 0$.

Exercise 3.5

1. Solve the following equations (i) $\sin^2 x - 5\sin x + 4 = 0$ (ii) $12x^3 + 8x = 29x^2 - 4$

2. Examine for the rational roots of (i) $2x^3 - x^2 - 1 = 0$ (ii) $x^8 - 3x + 1 = 0$

3. Solve: $8x^{\frac{3}{2n}} - 8x^{\frac{-3}{2n}} = 63$

4. Solve: $2\sqrt{\frac{x}{a}} + 3\sqrt{\frac{a}{x}} = \frac{b}{a} + \frac{6a}{b}$

5. Solve the equations: (i) $6x^4 - 35x^3 + 62x^2 - 35x + 6 = 0$ (ii) $x^4 + 3x^3 - 3x - 1 = 0$

6. Find all real numbers satisfying $4^x - 3(2^{x+2}) + 2^5 = 0$.

7. Solve the equation $6x^4 - 5x^3 - 38x^2 - 5x + 6 = 0$ if it is known that $\frac{1}{3}$ is a solution **Mar 2023-5M**

Ex:3.30 Show that the polynomial $9x^9 + 2x^5 - x^4 - 7x^2 + 2$ has at least six imaginary roots

Ex:3.31 Discuss the nature of the roots of the following polynomials

$(i) x^{2018} + 1947x^{1950} + 15x^8 + 26x^6 + 2019$ $(ii) x^5 - 19x^4 + 2x^3 + 5x^2 + 11$

Exercise 3.6

1.Discuss the maximum possible number of positive and negative roots of the polynomial equation $9x^9 - 4x^8 + 4x^7 - 3x^6 + 2x^5 + x^3 + 7x^2 + 7x + 2 = 0$.

2. Discuss the maximum possible number of positive and negative zeros of the polynomials $x^2 - 5x + 6$ and $x^2 - 5x + 16$. Also draw rough sketch of the graphs.

3. Show that the equation $x^9 - 5x^5 + 4x^4 + 2x^2 + 1 = 0$ has atleast 6 imaginary solutions.

4. Determine the number of positive and negative roots of the equation $x^9 - 5x^8 - 14x^7 = 0$.

5. Find the exact number of real zeros and imaginary of the equation $x^9 + 9x^7 + 7x^5 + 5x^3 + 3x$.

4.INVERSE TRIGONOMETRIC FUNCTIONS

Property 1:

(i) $\sin^{-1}(\sin\theta) = \theta$,if $\theta \in \left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$ (ii) $\cos^{-1}(\cos\theta) = \theta$,if $\theta \in [0, \pi]$

(iii) $\tan^{-1}(\tan\theta) = \theta$,if $\theta \in \left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$ (iv) $\operatorname{cosec}^{-1}(\operatorname{cosec}\theta) = \theta$,if $\theta \in \left[-\frac{\pi}{2}, \frac{\pi}{2}\right] \backslash \{0\}$

(v) $\sec^{-1}(\sec\theta) = \theta$,if $\theta \in [0, \pi] \backslash \left\{\frac{\pi}{2}\right\}$ (vi) $\cot^{-1}(\cot\theta) = \theta$,if $\theta \in (0, \pi)$

Property 2:

(i) $\sin(\sin^{-1} x) = x$, if $x \in [-1, 1]$ (ii) $\cos(\cos^{-1} x) = x$, if $x \in [-1, 1]$

(iii) $\tan(\tan^{-1} x) = x$, if $x \in \mathbb{R}$ (iv) $\operatorname{cosec}(\operatorname{cosec}^{-1} x) = x$, if $x \in \mathbb{R} \backslash (-1, 1)$

(v) $\sec(\sec^{-1} x) = x$, if $x \in \mathbb{R} \backslash (-1, 1)$ (vi) $\cot(\cot^{-1} x) = x$, if $x \in \mathbb{R}$

Property 3: (Reciprocal inverse identities)

(i) $\sin^{-1}\left(\frac{1}{x}\right) = \operatorname{cosec}^{-1} x$, if $x \in \mathbb{R} \backslash (-1, 1)$ (ii) $\cos^{-1}\left(\frac{1}{x}\right) = \sec^{-1} x$, if $x \in \mathbb{R} \backslash (-1, 1)$

(iii) $\tan^{-1}\left(\frac{1}{x}\right) = \begin{cases} \cot^{-1} x & , if\ x > 0 \\ -\pi + \cot^{-1} x & , if\ x < 0 \end{cases}$

Property 4: (Reflection Identities)

(i) $\sin^{-1}(-x) = -\sin^{-1} x$, if $x \in [-1, 1]$ (ii) $\tan^{-1}(-x) = -\tan^{-1} x$, if $x \in \mathbb{R}$

(iii) $\text{cosec}^{-1}(-x) = -\text{cosec}^{-1}x$, if $|x| \geq 1$ or $x \in \mathbb{R}\backslash(-1, 1)$

(iv) $\cos^{-1}(-x) = \pi - \cos^{-1} x$, if $x \in [-1, 1]$ (v) $\cot^{-1}(-x) = \pi - \cot^{-1} x$, if $x \in \mathbb{R}$

(vi)$\sec^{-1}(-x) = \pi - \sec^{-1}x$, if $|x| \geq 1$ or $x \in \mathbb{R}\backslash(-1, 1)$

Property 5: (Cofunction Inverse Identities)

(i) $\sin^{-1} x + \cos^{-1} x = \frac{\pi}{2}$, $x \in [-1, 1]$ (ii) $\tan^{-1} x + \cot^{-1} x = \frac{\pi}{2}$, if $x \in \mathbb{R}$

(iii) $\text{cosec}^{-1}x + \sec^{-1} x = \frac{\pi}{2}$, if $|x| \geq 1$ or $x \in \mathbb{R}\backslash(-1, 1)$

Property 6:

(i) $\sin^{-1} x + \sin^{-1} y = \sin^{-1}\left(x\sqrt{1-y^2} + y\sqrt{1-x^2}\right)$, where either $x^2 + y^2 \leq 1$ or $xy < 0$

(ii) $\sin^{-1} x - \sin^{-1} y = \sin^{-1}\left(x\sqrt{1-y^2} - y\sqrt{1-x^2}\right)$, where either $x^2 + y^2 \leq 1$ or $xy > 0$

(iii) $\cos^{-1} x + \cos^{-1} y = \cos^{-1}\left(xy - \sqrt{1-x^2}\sqrt{1-y^2}\right)$, if $x + y \geq 0$

(iv) $\cos^{-1} x - \cos^{-1} y = \cos^{-1}\left(xy + \sqrt{1-x^2}\sqrt{1-y^2}\right)$, if $x \leq y$

(v) $\tan^{-1} x + \tan^{-1} y = \tan^{-1}\left(\frac{x+y}{1-xy}\right)$, if $xy < 1$

(vi)$\tan^{-1} x - \tan^{-1} y = \tan^{-1}\left(\frac{x-y}{1+xy}\right)$, if $xy > -1$

Property 7:

(i) $2\tan^{-1} x = \tan^{-1}\left(\frac{2x}{1-x^2}\right)$, $|x| < 1$ (ii) $2\tan^{-1} x = \cos^{-1}\left(\frac{1-x^2}{1+x^2}\right)$, $x \geq 0$

(iii) $2\tan^{-1} x = \sin^{-1}\left(\frac{2x}{1+x^2}\right)$, $|x| \leq 1$

Property 8:

(i) $\sin^{-1}\left(2x\sqrt{1-x^2}\right) = 2\sin^{-1} x$ if $|x| \leq \frac{1}{\sqrt{2}}$ or $-\frac{1}{\sqrt{2}} \leq x \leq \frac{1}{\sqrt{2}}$

(ii) $\sin^{-1}\left(2x\sqrt{1-x^2}\right) = 2\cos^{-1} x$ if $\frac{1}{\sqrt{2}} \leq x \leq 1$

Property 9:

(i) $\sin^{-1} x = \cos^{-1}\sqrt{1-x^2}$, if $0 \leq x \leq 1$ (ii) $\sin^{-1} x = -\cos^{-1}\sqrt{1-x^2}$,if $-1 \leq x < 0$

(iii) $\sin^{-1} x = \tan^{-1}\left(\frac{x}{\sqrt{1-x^2}}\right)$, if $-1 < x < 1$ (iv) $\cos^{-1} x = \sin^{-1}\sqrt{1-x^2}$, if $0 \leq x \leq 1$

(v) $\cos^{-1} x = \pi - \sin^{-1}\sqrt{1-x^2}$, if $-1 \leq x < 0$

(vi) $\tan^{-1} x = \sin^{-1}\left(\frac{x}{\sqrt{1+x^2}}\right) = \cos^{-1}\left(\frac{1}{\sqrt{1+x^2}}\right)$,if $x > 0$

Property 10:

(i) $3\sin^{-1} x = \sin^{-1}(3x - 4x^3)$, $x \in [-\frac{1}{2}, \frac{1}{2}]$ (ii) $3\cos^{-1} x = \cos^{-1}(4x^3 - 3x)$, $x \in [\frac{1}{2}, 1]$

Ex:4.1 Find the principal value of $\sin^{-1}\left(-\frac{1}{2}\right)$ (in radians and degree).

Ex:4.2 Find the principal value of $\sin^{-1}(2)$, if it exists.

Ex:4.3 Find the principal value of (i) $\sin^{-1}\left(\frac{1}{\sqrt{2}}\right)$ (ii) $\sin^{-1}\left(sin\left(-\frac{\pi}{3}\right)\right)$ (iii) $\sin^{-1}\left(sin\left(\frac{5\pi}{6}\right)\right)$

Ex:4.4 Find the domain of $\sin^{-1}(2 - 3x^2)$.

Exercise 4.1

1. Find all the values of x such that (i) $-10\pi \leq x \leq 10\pi$ and $\sin x = 0$ (ii) $-3\pi \leq x \leq 3\pi$ and $\sin x = -1$
2. Find the period and amplitude of (i) $y = \sin 7x$ (ii) $y = -\sin\left(\frac{1}{3}x\right)$ (iii) $y = 4\sin(-2x)$
3. Sketch the graph of $y = \sin\left(\frac{1}{3}x\right)$ for $0 \leq x < 6\pi$

4. Find the value of (i) $\sin^{-1}\left(sin\left(\frac{2\pi}{3}\right)\right)$ (ii) $\sin^{-1}\left(sin\left(\frac{5\pi}{4}\right)\right)$ **Mar-2020 2M**

5. For what value of x does $\sin x = \sin^{-1} x$? **Aug 2021- 2M**

6. Find the domain of the following (i) $f(x) = \sin^{-1}\left(\frac{x^2+1}{2x}\right)$ (ii) $g(x) = 2\sin^{-1}(2x-1) - \frac{\pi}{4}$

7. Find the value of $\sin^{-1}\left(\sin\frac{5\pi}{9}\cos\frac{\pi}{9} + \cos\frac{5\pi}{9}\sin\frac{\pi}{9}\right)$ **July-2022 5M**

Ex:4.5 Find the principal value of $\cos^{-1}\left(\frac{\sqrt{3}}{2}\right)$ **Ex:4.7** Find the domain of $\cos^{-1}\left(\frac{2+\sin x}{3}\right)$

Ex:4.6 Find (i) $\cos^{-1}\left(-\frac{1}{\sqrt{2}}\right)$ (ii) $\cos^{-1}\left(\cos\left(-\frac{\pi}{3}\right)\right)$ (iii) $\cos^{-1}\left(\cos\left(\frac{7\pi}{6}\right)\right)$

Exercise 4.2

1. Find all the values of x such that (i) $-6\pi \le x \le 6\pi$ and $\cos x = 0$ (ii) $-5\pi \le x \le 5\pi$ and $\cos x = 1$.

2. State the reason for $\cos^{-1}\left[\cos\left(-\frac{\pi}{6}\right)\right] \ne -\frac{\pi}{6}$.

3. Is $\cos^{-1}(-x) = \pi - \cos^{-1}(x)$ true? Justify your answer.

4. Find the principal value of $\cos^{-1}\left(\frac{1}{2}\right)$

5. Find the value of (i) $2\cos^{-1}\left(\frac{1}{2}\right) + \sin^{-1}\left(\frac{1}{2}\right)$ (ii) $\cos^{-1}\left(\frac{1}{2}\right) + \sin^{-1}(-1)$
(iii) $\cos^{-1}\left(\cos\frac{\pi}{7}\cos\frac{\pi}{17} - \sin\frac{\pi}{7}\sin\frac{\pi}{17}\right)$

6. Find the domain of (i) $f(x) = sin^{-1}\left(\frac{|x|-2}{3}\right) + cos^{-1}\left(\frac{1-|x|}{4}\right)$ (ii) $g(x) = sin^{-1}x + cos^{-1}x$

7. For what value of x, the inequality $\frac{\pi}{2} < cos^{-1}(3x-1) < \pi$ holds? **Mar 2023- 3M**

8. Find the value of (i) $\cos\left(\cos^{-1}\left(\frac{4}{5}\right) + sin^{-1}\left(\frac{4}{5}\right)\right)$
(ii) $cos^{-1}\left(cos\left(\frac{4\pi}{3}\right)\right) + cos^{-1}\left(cos\left(\frac{5\pi}{4}\right)\right)$ **Aug 2021- 5M**

Ex:4.8 Find the principal value of $\tan^{-1}(\sqrt{3})$ **Sep 2020 – 2M & May 2022 - 2M**

Ex:4.9 Find (i) $\tan^{-1}(-\sqrt{3})$ (ii) $\tan^{-1}\left(\tan\frac{3\pi}{5}\right)$ (iii) $tan(\tan^{-1}(2019))$

Ex:4.10 Find the value of $\tan^{-1}(-1) + \cos^{-1}\left(\frac{1}{2}\right) + \sin^{-1}\left(-\frac{1}{2}\right)$

Ex:4.11 Prove that $\tan(\sin^{-1} x) = \frac{x}{\sqrt{1-x^2}}, -1 < x < 1$.

Exercise 4.3

1. Find the domain of the following functions: (i) $\tan^{-1}(\sqrt{9-x^2})$ (ii) $\frac{1}{2}\tan^{-1}(1-x^2) - \frac{\pi}{4}$

2. Find the value of (i) $\tan^{-1}\left(\tan\frac{5\pi}{4}\right)$ (ii) $\tan^{-1}\left(\tan\left(-\frac{\pi}{6}\right)\right)$

3. Find the value of (i) $\tan\left(\tan^{-1}\left(\frac{7\pi}{4}\right)\right)$ (ii) $\tan(\tan^{-1}(1947))$ (iii) $\tan(\tan^{-1}(-0.2021))$

4. Find the value of (i) $\tan\left(\cos^{-1}\left(\frac{1}{2}\right) - \sin^{-1}\left(-\frac{1}{2}\right)\right)$ (ii) $\sin\left(\tan^{-1}\left(\frac{1}{2}\right) - \cos^{-1}\left(\frac{4}{5}\right)\right)$
(iii) $\cos\left(\sin^{-1}\left(\frac{4}{5}\right) - \tan^{-1}\left(\frac{3}{4}\right)\right)$

Ex:4.12 Find the principal value of (i) $\text{cosec}^{-1}(-1)$ (ii) $\sec^{-1}(-2)$

Ex:4.13 Find the value of $\sec^{-1}\left(-\frac{2\sqrt{3}}{3}\right)$. **Ex:4.14** If $\cot^{-1}\left(\frac{1}{7}\right) = \theta$, then find the value of $\cos\theta$.

Ex:4.15 Show that $\cot^{-1}\left(\frac{1}{\sqrt{x^2-1}}\right) = \sec^{-1} x, |x| > 1$.

Exercise 4.4

1. Find the principal value of (i) $\sec^{-1}\left(\frac{2}{\sqrt{3}}\right)$ (ii) $\cot^{-1}(\sqrt{3})$ (iii) $\text{cosec}^{-1}(-\sqrt{2})$

2. Find the value of (i) $\tan^{-1}(\sqrt{3}) - \sec^{-1}(-2)$ (ii) $\sin^{-1}(-1) + \cos^{-1}\left(\frac{1}{2}\right) + \cot^{-1}(2)$
(iii) $\cot^{-1}(1) + \sin^{-1}\left(-\frac{\sqrt{3}}{2}\right) - \sec^{-1}(-\sqrt{2})$ **May 2022 - 5M**

Ex:4.16 Prove that $\frac{\pi}{2} \le \sin^{-1} x + 2\cos^{-1} x \le \frac{3\pi}{2}$

Ex:4.17 Simplify (i) $\cos^{-1}\left(\cos\left(\frac{13\pi}{3}\right)\right)$ (ii) $\tan^{-1}\left(\tan\left(\frac{3\pi}{4}\right)\right)$ (iii)$\sec^{-1}\left(\sec\left(\frac{5\pi}{3}\right)\right)$ (iv) $\sin^{-1}(\sin 10)$

Ex:4.18 Find the value of (i) $\sin\left[\frac{\pi}{3} - \sin^{-1}\left(-\frac{1}{2}\right)\right]$ (ii) $\cos\left[\frac{1}{2}\, cos^{-1}\left(\frac{1}{8}\right)\right]$ (iii) $\tan\left[\frac{1}{2}\sin^{-1}\left(\frac{2a}{1+a^2}\right) + \frac{1}{2}\cos^{-1}\left(\frac{1-a^2}{1+a^2}\right)\right]$

Ex:4.19 Prove that $tan(\sin^{-1} x) = \frac{x}{\sqrt{1-x^2}}$ for $|x| < 1$. **Ex:4.20** Evaluate $\sin\left(\sin^{-1}\left(\frac{3}{5}\right) + \sec^{-1}\left(\frac{5}{4}\right)\right)$

Ex:4.21 Prove that (i) $\tan^{-1}\left(\frac{1}{2}\right) + \tan^{-1}\left(\frac{1}{3}\right) = \frac{\pi}{4}$ (ii) $2\tan^{-1}\left(\frac{1}{2}\right) + \tan^{-1}\left(\frac{1}{7}\right) = \tan^{-1}\left(\frac{31}{17}\right)$

Ex:4.22 If $\cos^{-1} x + \cos^{-1} y + \cos^{-1} z = \pi$ and $0 < x, y, z < 1$, show that $x^2 + y^2 + z^2 + 2xyz = 1$. **Mar 2023- 5M**

Ex:4.23 If $a_1, a_2, a_3, \cdots, a_n$ is an arithmetic progression with common difference d. Prove that $\tan\left[\tan^{-1}\left(\frac{d}{1+a_1a_2}\right) + \tan^{-1}\left(\frac{d}{1+a_2a_3}\right) + \cdots + \tan^{-1}\left(\frac{d}{1+a_{n-1}a_n}\right)\right] = \frac{a_n - a_1}{1+a_1a_n}$

Ex:4.24 Solve $\tan^{-1}\left(\frac{1-x}{1+x}\right) = \frac{1}{2}\tan^{-1} x$ for $x > 0$ **Ex:4.25** Solve $\sin^{-1} x > \cos^{-1} x$.

Ex:4.26 Show that $\cot(\sin^{-1} x) = \frac{\sqrt{1-x^2}}{x}, -1 \le x \le 1$ and $x \ne 0$.

Ex:4.27 Solve $\tan^{-1} 2x + \tan^{-1} 3x = \frac{\pi}{4}$, if $6x^2 < 1$. **Ex:4.28** Solve $\tan^{-1}\left(\frac{x-1}{x-2}\right) + \tan^{-1}\left(\frac{x+1}{x+2}\right) = \frac{\pi}{4}$

Ex:4.29 Solve $\cos\left(\sin^{-1}\left(\frac{x}{\sqrt{1+x^2}}\right)\right) = \sin\left\{\cot^{-1}\left(\frac{3}{4}\right)\right\}$.

Exercise 4.5

1. Find the value, if it exists. If not, give the reason for non-existence.
   (i) $\sin^{-1}(\cos\pi)$ (ii) $\tan^{-1}\left(\sin\left(-\frac{5\pi}{2}\right)\right)$ (iii) $\sin^{-1}(\sin 5)$
2. Find the value of the expression in terms of x, with the help of a reference triangle.
   (i) $\sin(\cos^{-1}(1-x))$ (ii) $\cos(\tan^{-1}(3x-1))$ (iii) $\tan\left(\sin^{-1}\left(x+\frac{1}{2}\right)\right)$
3. Find the value of (i) $\sin^{-1}\left(\cos\left(\sin^{-1}\left(\frac{\sqrt{3}}{2}\right)\right)\right)$ (ii) $\cot\left(\sin^{-1}\frac{3}{5} + \sin^{-1}\frac{4}{5}\right)$ (iii)$\tan\left(\sin^{-1}\frac{3}{5} + \cot^{-1}\frac{3}{2}\right)$
4. Prove that (i) $\tan^{-1}\frac{2}{11} + \tan^{-1}\frac{7}{24} = \tan^{-1}\frac{1}{2}$ (ii) $\sin^{-1}\frac{3}{5} - \cos^{-1}\frac{12}{13} = \sin^{-1}\frac{16}{65}$
5. Prove that $\tan^{-1} x + \tan^{-1} y + \tan^{-1} z = \tan^{-1}\left[\frac{x+y+z-xyz}{1-xy-yz-zx}\right]$.
6. If $\tan^{-1} x + \tan^{-1} y + \tan^{-1} z = \pi$, show that $x + y + z = xyz$.
7. Prove that $\tan^{-1} x + \tan^{-1}\frac{2x}{1-x^2} = \tan^{-1}\frac{3x-x^3}{1-3x^2}$, $|x| < \frac{1}{\sqrt{3}}$. 8. Simplify: $\tan^{-1}\frac{x}{y} - \tan^{-1}\frac{x-y}{x+y}$.
9. Solve: (i) $\sin^{-1}\frac{5}{x} + \sin^{-1}\frac{12}{x} = \frac{\pi}{2}$ (ii) $2\tan^{-1} x = \cos^{-1}\frac{1-a^2}{1+a^2} - \cos^{-1}\frac{1-b^2}{1+b^2}$, $a > 0, b > 0$
   (iii) $2\tan^{-1}(\cos x) = \tan^{-1}(2\ \operatorname{cosec} x)$ (iv) $\cot^{-1} x - \cot^{-1}(x+2) = \frac{\pi}{12}$, $x > 0$
10. Find the number of solution of the equation $\tan^{-1}(x-1) + \tan^{-1} x + \tan^{-1}(x+1) = \tan^{-1}(3x)$

5.TWO DIMENSIONAL ANALYTICAL GEOMETRY

Theorem 1: The circle passing through the points of intersection (real or imaginary) of the line $lx + my + n = 0$ and the circle $x^2 + y^2 + 2gx + 2fy + c = 0$ is the circle of the form $x^2 + y^2 + 2gx + 2fy + c + \lambda(lx + my + n) = 0, \lambda \in \mathbb{R}^1$.

Theorem 2: The equation of a circle with (x_1, y_1) and (x_2, y_2) as extremities of one of the diameters of the circle is $(x - x_1)(x - x_2) + (y - y_1)(y - y_2) = 0$

Theorem 3: The position of a point $P(x_1, y_1)$ with respect to a given circle $x^2 + y^2 + 2gx + 2fy + c = 0$ in the plane containing the circle is outside or on or inside the circle according as

$$x_1^2 + y_1^2 + 2gx_1 + 2fy_1 + c \text{ is } \begin{cases} > 0, & or \\ = 0, & or \\ < 0, \end{cases}$$

Theorem 4: From any point outside the circle $x^2 + y^2 = a^2$ two tangents can be drawn.

Theorem 5: The sum of the focal distances of any point on the ellipse is equal to length of the major axis.

Theorem 6: Three normals can be drawn to a parabola $y^2 = 4ax$ from a given point, one of which is always real.

Ex:5.1 Find the general equation of a circle with centre $(-3, -4)$ and radius 3 units.

Ex:5.2 Find the equation of the circle described on the chord $3x + y + 5 = 0$ of the circle $x^2 + y^2 = 16$ as diameter.

Ex:5.3 Determine whether $x + y - 1 = 0$ is the equation of a diameter of the circle $x^2 + y^2 - 6x + 4y + c = 0$ for all possible values of c.

Ex:5.4 Find the general equation of the circle whose diameter is the line segment joining the points $(-4, -2)$ and $(1, 1)$.

Ex:5.5 Examine the position of the point $(2, 3)$ with respect to the circle $x^2 + y^2 - 6x - 8y + 12 = 0$

Ex:5.6 The line $3x + 4y - 12 = 0$ meets the coordinate axes at A and B. Find the equation of the circle drawn on AB as diameter.

Ex:5.7 A line $3x + 4y + 10 = 0$ cuts a chord of length 6 units on a circle with centre of the circle $(2, 1)$. Find the equation of the circle in general form.

Ex:5.8 A circle of radius 3 units touches both the axes. Find the equations of all possible circles formed in the general form.

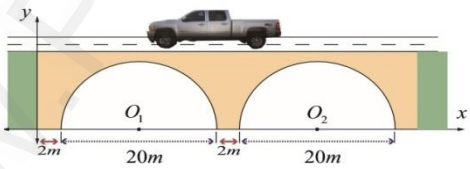
Ex:5.9 Find the centre and radius of the circle $3x^2 + (a + 1)y^2 + 6x - 9y + a + 4 = 0$.

Ex:5.10 Find the equation of the circle passing through the points $(1, 1), (2, -1)$ and $(3, 2)$. **Mar2020-5M**

Ex:5.11 Find the equations of the tangent and normal to the circle $x^2 + y^2 = 25$ at $P(-3, 4)$.

Ex:5.12 If $y = 4x + c$ is a tangent to the circle $x^2 + y^2 = 9$, find c **Mar 2023- 2M**

Ex:5.13 A road bridge over an irrigation canal has two semi circular vents each with a span of $20m$ and the supporting pillars of width $2m$. Use Fig to write the equations that represent the semi-vertical vents.



Exercise 5.1

1. Obtain the equation of the circles with radius $5cm$ and touching x −axis at the origin in general form.
2. Find the equation of the circle with centre $(2, -1)$ and passing through the point $(3, 6)$ in standard form.
3. Find the equation of circles that touch both the axes and pass through $(-4, -2)$ in general form.
4. Find the equation of the circle with centre $(2, 3)$ and passing through the intersection of the lines $3x - 2y - 1 = 0$ and $4x + y - 27 = 0$.
5. Obtain the equation of the circle for which $(3, 4)$ and $(2, -7)$ are the ends of a diameter.
6. Find the equation of the circle through the points $(1, 0), (-1, 0)$ and $(0, 1)$.
7. A circle of area 9π square units has two of its diameters along the lines $x + y = 5$ and $x - y = 1$. Find the equation of the circle. **Sep 2020 – 3M**
8. If $y = 2\sqrt{2}x + c$ is a tangent to the circle $x^2 + y^2 = 16$, find the value of c.
9. Find the equation of the tangent and normal to the circle $x^2 + y^2 - 6x + 6y - 8 = 0$ at $(2, 2)$.
10. Determine whether the points $(-2, 1), (0, 0)$ and $(-4, -3)$ lie outside, on or inside the circle

$x^2 + y^2 - 5x + 2y - 5 = 0.$

11. Find centre and radius of the following circles.

(i) $x^2 + (y+2)^2 = 0$ (ii) $x^2 + y^2 + 6x - 4y + 4 = 0$ **July-2022 3M**

(iii) $x^2 + y^2 - x + 2y - 3 = 0$ (iv) $2x^2 + 2y^2 - 6x + 4y + 2 = 0$

12. If the equation $3x^2 + (3-p)xy + qy^2 - 2px = 8pq$ represents a circle, find p and q. Also determine the centre and radius of the circle.

Ex:5.14 Find the length of Latus rectum of the parabola $y^2 = 4ax$.

Ex:5.15 Find the length of Latus rectum of the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$.

Ex:5.16 Find the equation of the parabola with focus $(-\sqrt{2}, 0)$ and directrix $x = \sqrt{2}$ **May 2022 - 5M**

Ex:5.17 Find the equation of the parabola whose vertex is $(5, -2)$ and focus $(2, -2)$.

Ex:5.18 Find the equation of the parabola with vertex $(-1, -2)$, axis parallel to y-axis and passing through $(3, 6)$. **Mar 2023- 3M**

Ex:5.19 Find the vertex, focus, directrix, and length of the latusrectum of the parabola $x^2 - 4x - 5y - 1 = 0$.

Ex:5.20 Find the equation of the ellipse with foci $(\pm 2, 0)$, vertices $(\pm 3, 0)$.

Ex:5.21 Find the equation of the ellipse whose eccentricity is $\frac{1}{2}$, one of the foci is $(2, 3)$and a directrix is $x = 7$.Also find the length of the major and minor axes of the ellipse.

Ex:5.22 Find the foci, vertices and length of major and minor axis of the conic $4x^2 + 36y^2 + 40x - 288y + 532 = 0$.

Ex:5.23 For the ellipse $4x^2 + y^2 + 24x - 2y + 21 = 0$, find the centre, vertices, and the foci. Also prove that the length of latus rectum is 2.

Ex:5.24 Find the equation of the hyperbola with vertices $(0, \pm 4)$ and foci $(0, \pm 6)$

Ex:5.25 Find the vertices, foci for the hyperbola $9x^2 - 16y^2 = 144$.

Ex:5.26 Find the centre, foci, and eccentricity of the hyperbola $11x^2 - 25y^2 - 44x + 50y - 256 = 0$ **Sep 2020 – 5M**

Ex:5.27 The orbit of Halley's Comet is an ellipse 36.18 astronomical units long and by 9.12 astronomical units wide. Find its eccentricity.

Exercise 5.2

1. Find the equation of the parabola in each of the cases given below:

(i) focus $(4, 0)$and directrix $x = -4$. **Aug 2021- 3M**

(ii) passes through $(2, -3)$ and symmetric about y −axis.

(iii)vertex $(1, -2)$ and focus $(4, -2)$ (iv) end points of latus rectum $(4, -8)$ and $(4, 8)$

2. Find the equation of the ellipse in each of the cases given below:

(i) foci $(\pm 3, 0), e = \frac{1}{2}$ (ii) foci $(0, \pm 4)$ and end points of major axis are $(0, \pm 5)$

(iii) length of latus rectum 8, eccentricity $= \frac{3}{5}$, centre $(0, 0)$ and major axis on x −axis.

(iv) length of latus rectum 4, distance between foci $4\sqrt{2}$, centre $(0, 0)$ and major axis as y − axis.

3. Find the equation of the hyperbola in each of the cases given below:

(i) foci $(\pm 2, 0)$ eccentricity $= \frac{3}{2}$

(ii) Centre $(2, 1)$, one of the foci $(8, 1)$ and corresponding directrix $x = 4$

(iii) passing through $(5, -2)$ and length of the transverse axis along x −axis and of length 8 units.

4. Find the vertex, focus, equation of directrix and length of the latus rectum of the following:

(i) $y^2 = 16x$ (ii) $x^2 = 24y$ (iii) $y^2 = -8x$

(iv)$x^2 - 2x + 8y + 17 = 0$ (v) $y^2 - 4y - 8x + 12 = 0$

5. Identify the type of conic and find centre, foci, vertices, and directrices of each of the following:

(i) $\frac{x^2}{25} + \frac{y^2}{9} = 1$ **July-2022 5M** (ii) $\frac{x^2}{3} + \frac{y^2}{10} = 1$ (iii) $\frac{x^2}{25} - \frac{y^2}{144} = 1$ (iv) $\frac{y^2}{16} - \frac{x^2}{9} = 1$

6. Prove that the length of the latus rectum of the hyperbola $\frac{x^2}{a^2}-\frac{y^2}{b^2}=1$ is $\frac{2b^2}{a}$.
7. Show that the absolute value of difference of the focal distances of any point P on the hyperbola is the length of its transverse axis.
8. Identify the type of conic and find centre, foci, vertices, and directrices of each of the following:
   (i) $\frac{(x-3)^2}{225}+\frac{(y-4)^2}{289}=1$ (ii) $\frac{(x+1)^2}{100}+\frac{(y-2)^2}{64}=1$
   (iii) $\frac{(x+3)^2}{225}-\frac{(y-4)^2}{64}=1$ (iv) $\frac{(y-2)^2}{25}-\frac{(x+1)^2}{16}=1$
   (v) $18x^2+12y^2-144x+48y+120=0$ **Mar 2023- 5M**
   (vi) $9x^2-y^2-36x-6y+18=0$

Ex:5.28 Identify the type of the conic for the following equations: (i) $16y^2=-4x^2+64$
(ii) $x^2+y^2=-4x-y+4$ (iii) $x^2-2y=x+3$ (iv) $4x^2-9y^2-16x+18y-29=0$

Exercise 5.3

1. Identify the type of the conic section for the following equations: (i) $2x^2-y^2=7$
(ii) $3x^2+3y^2-4x+3y+10=0$ (iii) $3x^2+2y^2=14$ (iv) $x^2+y^2+x-y=0$
(v) $11x^2-25y^2-44x+50y-256=0$ (vi) $y^2+4x+3y+4=0$

Ex:5.29 Find the equations of tangent and normal to the parabola $x^2+6x+4y+5=0$ at $(1,-3)$.

Ex:5.30 Find the equations of tangent and normal to the ellipse $x^2+4y^2=32$ when $\theta=\frac{\pi}{4}$.

Exercise 5.4

1. Find the equations of the two tangents that can be drawn from $(5,2)$to the ellipse $2x^2+7y^2=14$.
2. Find the equations of tangents to the hyperbola $\frac{x^2}{16}-\frac{y^2}{64}=1$ which are parallel to $10x-3y+9=0$
3. Show that the line $x-y+4=0$ is a tangent to the ellipse $x^2+3y^2=12$. Also find the coordinates of the point of contact.
4. Find the equation of the tangent to the parabola $y^2=16x$ perpendicular to $2x+2y+3=0$
5. Find the equation of the tangent at $t=2$ to the parabola $y^2=8x$. (Hint: use parametric form)
6. Find the equations of the tangent and normal to hyperbola $12x^2-9y^2=108$ at $\theta=\frac{\pi}{3}$. (Use parametric form).
7. Prove that the point of intersection of the tangents at 't_1' and 't_2'on the parabola $y^2=4ax$ is $[at_1t_2,\ a(t_1+t_2)]$.
8. If the normal at the point 't_1' on the parabola $y^2=4ax$ meets the parabola again at the point 't_2', then prove that $t_2=-\left(t_1+\frac{2}{t_1}\right)$.

Ex:5.31 A semielliptical archway over a one-way road has a height of 3m and a width of $12m$. The truck has a width of $3m$ and a height of $2.7m$. Will the truck clear the opening of the archway?

Ex:5.32 The maximum and minimum distances of the Earth from the Sun respectively are $152\times10^6\ km$ and $94.5\times10^6\ km$. The Sun is at one focus of the elliptical orbit. Find the distance from the Sun to the other focus. **May 2022 - 5M & Mar 2023- 3M**

Ex:5.33 A concrete bridge is designed as a parabolic arch. The road over bridge is $40m$ long and the maximum height of the arch is $15m$. Write the equation of the parabolic arch. **Mar2020-3M**

Ex:5.34 The parabolic communication antenna has a focus at $2m$ distance from the vertex of the antenna. Find the width of the antenna $3m$ from the vertex. **July-2022 5M**

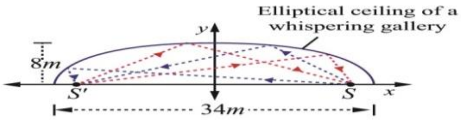
Ex:5.35 The equation $y=\frac{1}{32}x^2$ models cross sections of parabolic mirrors that are used for solar energy. There is a heating tube located at the focus of each parabola; how high is this tube located above the vertex of the parabola?

Ex:5.36 A search light has a parabolic reflector (has a cross section that forms a 'bowl'). The parabolic bowl is $40cm$ wide from rim to rim and $30cm$ deep. The bulb is located at the focus. (i)

What is the equation of the parabola used for reflector? (ii) How far from the vertex is the bulb to be placed so that the maximum distance covered?

Ex:5.37 An equation of the elliptical part of an optical lens system is $\frac{x^2}{16}+\frac{y^2}{9}=1$. The parabolic part of the system has a focus in common with the right focus of the ellipse .The vertex of the parabola is at the origin and the parabola opens to the right. Determine the equation of the parabola.

Ex:5.38 A room $34m$ long is constructed to be a whispering gallery. The room has an elliptical ceiling, as shown in Fig. If the maximum height of the ceiling is $8m$, determine where the foci are located.



Ex:5.39 If the equation of the ellipse is $\frac{(x-11)^2}{484}+\frac{y^2}{64}=1$ (x and y are measured in centimeters) where to the nearest centimeter, should the patient's kidney stone be placed so that the reflected sound hits the kidney stone?

Ex:5.40 Two coast guard stations are located $600km$ apart at points $A(0,0)$ and $B(0,\ 600)$. A distress signal from a ship at P is received at slightly different times by two stations. It is determined that the ship is $200km$ farther from station A than it is from station B. Determine the equation of hyperbola that passes through the location of the ship.

Ex:5.41 Certain telescopes contain both parabolic mirror and a hyperbolic mirror. In the telescope, the parabola and hyperbola share focus F_1 which is $14m$ above the vertex of the parabola. The hyperbola's second focus F_2 is $2m$ above the parabola's vertex. The vertex of the hyperbolic mirror is $1m$ below F_1. Position a coordinate system with the origin at the centre of the hyperbola and with the foci on the y-axis. Then find the equation of the hyperbola.

Exercise 5.5

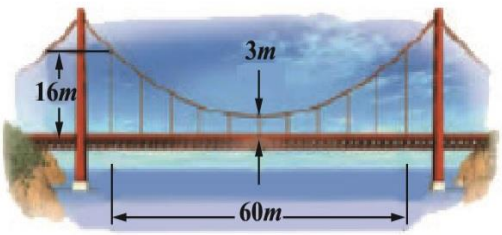
1. A bridge has a parabolic arch that is 10m high in the centre and $30m$ wide at the bottom. Find the height of the arch $6m$ from the centre, on either sides.

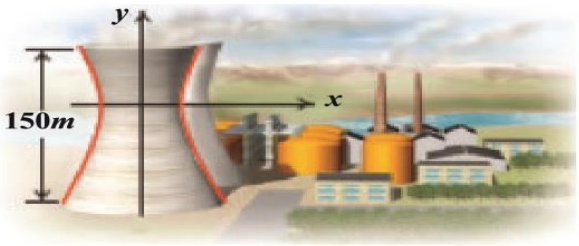
2. A tunnel through a mountain for a four lane highway is to have a elliptical opening. The total width of the highway (not the opening) is to be $16m$, and the height at the edge of the road must be sufficient for a truck $4m$ high to clear if the highest point of the opening is to be $5m$ approximately. How wide must the opening be?

3. At a water fountain, water attains a maximum height of $4m$ at horizontal distance of $0.5m$ from its origin. If the path of water is a parabola, find the height of water at a horizontal distance of $0.75m$ from the point of origin.

4. An engineer designs a satellite dish with a parabolic cross section. The dish is $5m$ wide at the opening, and the focus is placed $1.2m$ from the vertex (a) Position a coordinate system with the origin at the vertex and the x −axis on the parabola's axis of symmetry and find an equation of the parabola. (b) Find the depth of the satellite dish at the vertex.

5. Parabolic cable of a $60m$ portion of the roadbed of a suspension bridge are positioned as shown below. Vertical Cables are to be spaced every 6m along this portion of the roadbed. Calculate the lengths of first two of these vertical cables from the vertex.





6. Cross section of a Nuclear cooling tower is in the shape of a hyperbola with equation $\frac{x^2}{30^2} - \frac{y^2}{44^2} = 1$. The tower is $150m$ tall and the distance from the top of the tower to the centre of the hyperbola is half the distance from the base of the tower to the centre of the hyperbola. Find the diameter of the top and base of the tower

7. A rod of length $1.2m$ moves with its ends always touching the coordinate axes. The locus of a point P on the rod, which is $0.3m$ from the end in contact with x −axis is an ellipse. Find the eccentricity. **Sep 2020 – 5M**

8. Assume that water issuing from the end of a horizontal pipe, $7.5m$ above the ground, describes a parabolic path. The vertex of the parabolic path is at the end of the pipe. At a position $2.5m$ below the line of the pipe, the flow of water has curved outward $3m$ beyond the vertical line through the end of the pipe. How far beyond this vertical line will the water strike the ground? **Mar 2020-5M**

9. On lighting a rocket cracker it gets projected in a parabolic path and reaches a maximum height of $4m$ when it is $6m$ away from the point of projection. Finally it reaches the ground $12m$ away from the starting point. Find the angle of projection. **Aug 2021- 5M**

10. Points A and B are $10km$ apart and it is determined from the sound of an explosion heard at those points at different times that the location of the explosion is 6 km closer to A than B. Show that the location of the explosion is restricted to a particular curve and find an equation of it.

6. APPLICATION OF VECTOR ALGEBRA

Theorem1: If $\vec{a} = a_1\hat{\imath} + a_2\hat{\jmath} + a_3\hat{k},\ \vec{b} = b_1\hat{\imath} + b_2\hat{\jmath} + b_3\hat{k}\ ,\ and\ \vec{c} = c_1\hat{\imath} + c_2\hat{\jmath} + c_3\hat{k},$ then

$$(\vec{a} \times \vec{b}).\vec{c} = \begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix}$$

Theorem 2: For any three vectors $\vec{a}, \vec{b}$ and $\vec{c}$, $(\vec{a} \times \vec{b}).\vec{c} = \vec{a}.(\vec{b} \times \vec{c})$

Theorem 3: The scalar triple product preserves addition and scalar multiplication

(i) $[(\vec{a} + \vec{b}),\ \vec{c},\ \vec{d}] = [\vec{a},\ \vec{c},\ \vec{d}] + [b,\ \vec{c},\ \vec{d}]$ (ii) $[\lambda\vec{a},\ \vec{b},\ \vec{c}] = \lambda[\vec{a},\ \vec{b},\ \vec{c}], \forall\, \lambda \in \mathbb{R}$

(iii) $[\vec{a}, (\vec{b} + \vec{c}), \vec{d}] = [\vec{a},\ \vec{b},\ \vec{d}] + [\vec{a},\ \vec{c},\ \vec{d}]$ (iv) $[\vec{a},\ \lambda\vec{b},\ \vec{c}] = \lambda[\vec{a},\ \vec{b},\ \vec{c}], \forall\, \lambda \in \mathbb{R}$

(v) $[\vec{a}, \vec{b}, (\vec{c} + \vec{d})] = [\vec{a},\ \vec{b},\ \vec{c}] + [\vec{a},\ \vec{b},\ \vec{d}]$ (vi) $[\vec{a},\ \vec{b},\ \lambda\vec{c}] = \lambda[\vec{a},\ \vec{b},\ \vec{c}], \forall\, \lambda \in \mathbb{R}$

Theorem 4: The scalar triple product of three non-zero vectors is zero if, and only if, the three vectors are coplanar.

Theorem 5: Three vectors $\vec{a}, \vec{b}, \vec{c}$ are coplanar if, and only if, there exist scalars $r, s, t\ \in \mathbb{R}$ such that at least one of them is non-zero and $r\vec{a} + s\vec{b} + t\vec{c} = 0$.

Theorem 6: If $\vec{a}, \vec{b}, \vec{c}$ and $\vec{p}, \vec{q}, \vec{r}$ are any two system of three vectors, and if $\vec{p} = x_1\vec{a} + y_1\vec{b} + z_1\vec{c}$, $\vec{q} = x_2\vec{a} + y_2\vec{b} + z_2\vec{c}$ and $\vec{r} = x_3\vec{a} + y_3\vec{b} + z_3\vec{c}$, then $[\vec{p},\ \vec{q},\ \vec{r}] = \begin{vmatrix} x_1 & y_1 & z_1 \\ x_2 & y_2 & z_2 \\ x_3 & y_3 & z_3 \end{vmatrix} [\vec{a},\ \vec{b},\ \vec{c}]$

Theorem 7: The vector triple product satisfies the following properties:

$$(\overrightarrow{a_1} + \overrightarrow{a_2}) \times (\vec{b} \times \vec{c}) = \overrightarrow{a_1} \times (\vec{b} \times \vec{c}) + \overrightarrow{a_2} \times (\vec{b} \times \vec{c}),\quad (\lambda\vec{a}) \times (\vec{b} \times \vec{c}) = \lambda\left(\vec{a} \times (\vec{b} \times \vec{c})\right), \lambda \in \mathbb{R}$$

$$\vec{a} \times \left((\overrightarrow{b_1} + \overrightarrow{b_2}) \times \vec{c}\right) = \vec{a} \times (\overrightarrow{b_1} \times \vec{c}) + \vec{a} \times (\overrightarrow{b_2} \times \vec{c}),\quad \vec{a} \times \left((\lambda\vec{b}) \times \vec{c}\right) = \lambda\left(\vec{a} \times (\vec{b} \times \vec{c})\right),\ \lambda \in \mathbb{R}$$

$$\vec{a} \times \left(\vec{b} \times (\overrightarrow{c_1} + \overrightarrow{c_2})\right) = \vec{a} \times (\vec{b} \times \overrightarrow{c_1}) + \vec{a} \times (\vec{b} \times \overrightarrow{c_2}),\quad \vec{a} \times \left(\vec{b} \times (\lambda\vec{c})\right) = \lambda\left(\vec{a} \times (\vec{b} \times \vec{c})\right),\ \lambda \in \mathbb{R}$$

Theorem 8: For any three vectors $\vec{a}, \vec{b}, \vec{c}$ we have $\vec{a} \times (\vec{b} \times \vec{c}) = (\vec{a} \cdot \vec{c})\vec{b} - (\vec{a} \cdot \vec{b})\vec{c}$

Theorem 9: (Jacobi's Identity) For any three vectors $\vec{a}, \vec{b}, \vec{c}$ we have $\vec{a} \times (\vec{b} \times \vec{c}) + \vec{b} \times (\vec{c} \times \vec{a}) + \vec{c} \times (\vec{a} \times \vec{b}) = \vec{0}$.

Theorem 10:(Lagrange's Identity) For any four vectors $\vec{a}, \vec{b}, \vec{c}, \vec{d}$ we have

$$(\vec{a} \times \vec{b}) \cdot (\vec{c} \times \vec{d}) = \begin{vmatrix} \vec{a} \cdot \vec{c} & \vec{a} \cdot \vec{d} \\ \vec{b} \cdot \vec{c} & \vec{b} \cdot \vec{d} \end{vmatrix}$$

Theorem 11: The vector equation of a straight line passing through a fixed point with position vector $\vec{a}$ and parallel to a given vector $\vec{b}$ is $\vec{r} = \vec{a} + t\vec{b}$, where $t \in \mathbb{R}$.

Theorem 12: The parametric form of vector equation of a line passing through two given points whose position vectors are $\vec{a}$ and $\vec{b}$ respectively is $\vec{r} = \vec{a} + t(\vec{b} - \vec{a})$, where $t \in \mathbb{R}$.

Theorem 13: The shortest distance between the two parallel lines $\vec{r} = \vec{a} + s\vec{b}$ and $\vec{r} = \vec{c} + t\vec{b}$ is given by $d = \frac{|(c-\vec{a})\times\vec{b}|}{|\vec{b}|}$ where $|\vec{b}| \neq 0$.

Theorem 14: The shortest distance between the two skew lines $\vec{r} = \vec{a} + s\vec{b}$ and $\vec{r} = \vec{c} + t\vec{d}$ is given by $\delta = \frac{|(c-\vec{a})\times(\vec{b}\times\vec{d})|}{|\vec{b}\times\vec{d}|}$ where $|\vec{b} \times \vec{d}| \neq 0$.

Theorem 15: The equation of the plane at a distance p from the origin and perpendicular to the unit normal vector $\hat{d}$ is $\vec{r} \cdot \hat{d} = p$.

Theorem 16: The general equation $ax + by + cz + d = 0$ of first degree in x, y, z represents a plane.

Theorem 17: If three non-collinear points with position vectors $\vec{a}, \vec{b}, \vec{c}$ are given, then the vector equation of the plane passing through the given points in parametric form is $\vec{r} = \vec{a} + s(\vec{b} - \vec{a}) + t(\vec{c} - \vec{a})$, where $\vec{b} \neq \vec{0}, \vec{c} \neq \vec{0}$ and $s, t \in \mathbb{R}$.

Theorem 18: The acute angle θ between the two planes $\vec{r} \cdot \vec{n_1} = p_1$ and $\vec{r} \cdot \vec{n_2} = p_2$ is given by

$$\theta = \cos^{-1}\left(\frac{|\vec{n_1}\cdot\vec{n_2}|}{|\vec{n_1}||\vec{n_2}|}\right)$$

Theorem 19: The acute angle θ between the two planes $a_1x + b_1y + c_1z + d_1 = 0$ and $a_2x + b_2y + c_2z + d_2 = 0$ is given by $\theta = \cos^{-1}\left(\frac{|a_1a_2+b_1b_2+c_1c_2|}{\sqrt{a_1^2+b_1^2+c_1^2}\sqrt{a_2^2+b_2^2+c_2^2}}\right)$.

Theorem 20: The perpendicular distance from a point with position vector $\vec{u}$ to the plane $\vec{r} \cdot \vec{n} = p$ is given by $\delta = \frac{|\vec{u}\cdot\vec{n}-p|}{|\vec{n}|}$

Theorem 21: The distance between two parallel planes $ax + by + cz + d_1 = 0$ and $ax + by + cz + d_2 = 0$ is given by $\frac{|d_1-d_2|}{\sqrt{a^2+b^2+c^2}}$.

Theorem 22: The vector equation of a plane which passes through the line of intersection of the planes $\vec{r} \cdot \vec{n_1} = d_1$ and $\vec{r} \cdot \vec{n_2} = d_2$ is given by $(\vec{r} \cdot \vec{n_1} - d_1) + \lambda(\vec{r} \cdot \vec{n_2} - d_2) = 0$, where $\lambda \in \mathbb{R}$.

Theorem 23: The position vector of the point of intersection of the straight line $\vec{r} = \vec{a} + t\vec{b}$ and the plane $\vec{r} \cdot \vec{n} = p$ is $\vec{a} + \left(\frac{p-(\vec{a}\cdot\vec{n})}{\vec{b}\cdot\vec{n}}\right)\vec{b}$, provided $\vec{b} \cdot \vec{n} \neq 0$

Ex:6.1 (Cosine formulae) With usual notations, in any triangle ABC, prove the following by vector method. (i) $a^2 = b^2 + c^2 - 2bc\cos A$ (ii) $b^2 = c^2 + a^2 - 2ca\cos B$ (iii) $c^2 = a^2 + b^2 - 2ab\cos C$

Ex:6.2 With usual notations, in any triangle ABC, prove the following by vector method.
(i) $a = b\cos C + c\cos B$ (ii) $b = c\cos A + a\cos C$ (iii) $c = a\cos B + b\cos A$

Ex:6.3 By vector method, prove that $\cos(\alpha + \beta) = \cos\alpha\cos\beta - \sin\alpha\sin\beta$. **Mar2020-5M**

Ex:6.4 With usual notations, in any triangle ABC, prove by vector method that $\frac{a}{\sin A} = \frac{b}{\sin B} = \frac{c}{\sin C}$.
Sep 2020 – 3M

Ex:6.5 Prove by vector method that $\sin(\alpha - \beta) = \sin\alpha\cos\beta - \cos\alpha\sin\beta$.

Ex:6.6 (Apollonius's theorem): If D is the midpoint of the side BC of a triangle ABC, show by vector method that $|\overrightarrow{AB}|^2 + |\overrightarrow{AC}|^2 = 2\left(|\overrightarrow{AD}|^2 + |\overrightarrow{BD}|^2\right)$.

Ex:6.7 Prove by vector method that the perpendiculars (attitudes) from the vertices to the opposite sides of a triangle are concurrent.

Ex:6.8 In triangle, ABC, the points D, E, F are the midpoints of the sides BC, CA and AB respectively. Using vector method, show that the area of ΔDEF is equal to $\frac{1}{4}(area\ of\ \Delta ABC)$.

Ex:6.9 A particle acted upon by constant forces $2\hat{\imath}+5\hat{\jmath}+6\hat{k}$ and $-\hat{\imath}-2\hat{\jmath}-\hat{k}$ is displaced from the point $(4,-3,-2)$ to the point $(6,1,-3)$. Find the total work done by the forces.

Ex:6.10 A particle is acted upon by the forces $3\hat{\imath}-2\hat{\jmath}+2\hat{k}$ and $2\hat{\imath}+\hat{\jmath}-\hat{k}$ is displaced from the point $(1,3,-1)$ to the point $(4,-1,\lambda)$ .If the work done by the forces is 16 units, find the value of λ.

Ex:6.11 Find the magnitude and the direction cosines of the torque about the point $(2,0,-1)$ of a force $2\hat{\imath}+\hat{\jmath}-\hat{k}$ whose line of action passes through the origin. **Mar2020 2M & May 2022 - 3M**

Exercise 6.1

1. Prove by vector method that if a line is drawn from the centre of a circle to the midpoint of a chord, then the line is perpendicular to the chord.
2. Prove by vector method that the median to the base of an isosceles triangle is perpendicular to the base.
3. Prove by vector method that an angle in a semi-circle is a right angle.
4. Prove by vector method that the diagonals of a rhombus bisect each other at right angles.
5. Using vector method, prove that if the diagonals of a parallelogram are equal, then it is a rectangle.
6. Prove by vector method that the area of the quadrilateral $ABCD$ having diagonals AC and BD is $\frac{1}{2}\left|\overrightarrow{AC}\times\overrightarrow{BD}\right|$.
7. Prove by vector method that the parallelograms on the same base and between the same parallels are equal in area.
8. If G is the centroid of a ΔABC, prove that
$(area\ of\ \Delta\ GAB) = (area\ of\ \Delta GBC) = (area\ of\ \Delta GCA) = \frac{1}{3}\ (area\ of\ \Delta ABC)$.
9. Using vector method, prove that $\cos(\alpha-\beta)=\cos\alpha\cos\beta+\sin\alpha\sin\beta$. **Sep2020 5M, July2022 5M, Aug 2021- 5M Mar 2023- 5M**
10. Prove by vector method that $\sin(\alpha+\beta)=\sin\alpha\cos\beta+\cos\alpha\sin\beta$. **May 2022 5M**
11. A particle acted on by constant forces $8\hat{\imath}+2\hat{\jmath}-6\hat{k}$ and $6\hat{\imath}+2\hat{\jmath}-2\hat{k}$ is displaced from the point $(1,2,3)$ to the point $(5,4,1)$. Find the total work done by the forces. **July-2022 3M**
12. Forces of magnitudes $5\sqrt{2}$ and $10\sqrt{2}$ units acting in the directions $3\hat{\imath}+4\hat{\jmath}+5\hat{k}$ and $10\hat{\imath}+6\hat{\jmath}-8\hat{k}$, respectively, act on a particle which is displaced from the point with position vector $4\hat{\imath}-3\hat{\jmath}-2\hat{k}$ to the point with position vector $6\hat{\imath}+\hat{\jmath}-3\hat{k}$.Find the work done by the forces.
13. Find the magnitude and direction cosines of the torque of a force represented by $3\hat{\imath}+4\hat{\jmath}-5\hat{k}$ about the point with position vector $2\hat{\imath}-3\hat{\jmath}+4\hat{k}$ acting through a point whose position vector is $4\hat{\imath}+2\hat{\jmath}-3\hat{k}$.
14. Find the torque of the resultant of the three forces represented by $-3\hat{\imath}+6\hat{\jmath}-3\hat{k}$, $4\hat{\imath}-10\hat{\jmath}+12\hat{k}$ and $4\hat{\imath}+7\hat{\jmath}$ acting at the point with position vector $8\hat{\imath}-6\hat{\jmath}-4\hat{k}$, about the point with position vector $18\hat{\imath}+3\hat{\jmath}-9\hat{k}$.

Ex:6.12 If $\vec{a}=-3\hat{\imath}-\hat{\jmath}+5\hat{k},\ \vec{b}=\hat{\imath}-2\hat{\jmath}+\hat{k},\vec{c}=4\hat{\jmath}-5\hat{k}$, find $\vec{a}\cdot(\vec{b}\times\vec{c})$

Ex:6.13 Find the volume of the parallelepiped whose coterminous edges are given by the vectors $2\hat{\imath}-3\hat{\jmath}+4\hat{k},\ \hat{\imath}+2\hat{\jmath}-\hat{k}$ and $3\hat{\imath}-\hat{\jmath}+2\hat{k}$.

Ex:6.14 Show that the vectors $\hat{\imath}+2\hat{\jmath}-3\hat{k},\ 2\hat{\imath}-\hat{\jmath}+2\hat{k}$ and $3\hat{\imath}+\hat{\jmath}-\hat{k}$ are coplanar.

Ex:6.15 If $2\hat{\imath}-\hat{\jmath}+3\hat{k},\ 3\hat{\imath}+2\hat{\jmath}+\hat{k},\ \hat{\imath}+m\hat{\jmath}+4\hat{k}$ are coplanar, find the value of m.

Ex:6.16 Show that the four points $(6,-7,0),(16,-19,-4),(0,3,-6),(2,-5,10)$ lie on a same plane.

Ex:6.17 If the vectors $\vec{a},\vec{b},\vec{c}$ are coplanar, then prove that the vectors $\vec{a}+\vec{b},\ \vec{b}+\vec{c},\ \vec{c}+\vec{a}$ are also coplanar.

Ex:6.18 If $\vec{a},\vec{b},\vec{c}$ are three vectors, prove that $[\vec{a}+\vec{c},\vec{a}+\vec{b},\vec{a}+\vec{b}+\vec{c}]=[\vec{a},\vec{b},\vec{c}]$.

Exercise 6.2

1. If $\vec{a} = \hat{\imath} - 2\hat{\jmath} + 3\hat{k},\ \vec{b} = 2\hat{\imath} + \hat{\jmath} - 2\hat{k}, \vec{c} = 3\hat{\imath} + 2\hat{\jmath} + \hat{k}$, find $\vec{a} \cdot (\vec{b} \times \vec{c})$.

2. Find the volume of the parallelepiped whose coterminous edges are represented by the vectors $-6\hat{\imath} + 14\hat{\jmath} + 10\hat{k},\ 14\hat{\imath} - 10\hat{\jmath} - 6\hat{k}$ and $2\hat{\imath} + 4\hat{\jmath} - 2\hat{k}$.

3. The volume of the parallelepiped whose coterminous edges are $7\hat{\imath} + \lambda\hat{\jmath} - 3\hat{k},\ \hat{\imath} + 2\hat{\jmath} - \hat{k}$, $-3\hat{\imath} + 7\hat{\jmath} + 5\hat{k}$ is 90 cubic units. Find the value of λ.

4. If $\vec{a}, \vec{b}, \vec{c}$ are three non-coplanar vectors represented by concurrent edges of a parallelepiped of volume 4 cubic units, find the value of $(\vec{a} + \vec{b}) \cdot (\vec{b} \times \vec{c}) + (\vec{b} + \vec{c}) \cdot (\vec{c} \times \vec{a}) + (\vec{c} + \vec{a}) \cdot (\vec{a} \times \vec{b})$.

5. Find the altitude of a parallelepiped determined by the vectors $\vec{a} = -2\hat{\imath} + 5\hat{\jmath} + 3\hat{k},\ \vec{b} = \hat{\imath} + 3\hat{\jmath} - 2\hat{k}$ and $\vec{c} = -3\hat{\imath} + \hat{\jmath} + 4\hat{k}$ if the base is taken as the parallelogram determined by $\vec{b}$ and $\vec{c}$.

6. Determine whether the three vectors $2\hat{\imath} + 3\hat{\jmath} + \hat{k}, \hat{\imath} - 2\hat{\jmath} + 2\hat{k}$ and $3\hat{\imath} + \hat{\jmath} + 3\hat{k}$ are coplanar.
Aug 2021- 2M

7. Let $\vec{a} = \hat{\imath} + \hat{\jmath} + \hat{k},\ \vec{b} = \hat{\imath}$ and $\vec{c} = c_1\hat{\imath} + c_2\hat{\jmath} + c_3\hat{k}$. If $c_1 = 1$ and $c_2 = 2$,find c_3 such that, $\vec{a}, \vec{b}$ and $\vec{c}$ are coplanar.

8. If $\vec{a} = \hat{\imath} - \hat{k},\ \vec{b} = x\hat{\imath} + \hat{\jmath} + (1 - x)\hat{k},\ \vec{c} = y\hat{\imath} + x\hat{\jmath} + (1 + x - y)\hat{k}$, show that $[\vec{a}, \vec{b}, \vec{c}]$ depends on neither x nor y.

9. If the vectors $a\hat{\imath} + a\hat{\jmath} + c\hat{k}, \hat{\imath} + \hat{k}$ and $c\hat{\imath} + c\hat{\jmath} + b\hat{k}$ are coplanar, prove that c is the geometric mean of a and b.

10. Let $\vec{a}, \vec{b}, \vec{c}$ be three non-zero vectors such that $\vec{c}$ is a unit vector perpendicular to both $\vec{a}$ and $\vec{b}$. If the angle between $\vec{a}$ and $\vec{b}$ is $\frac{\pi}{6}$, show that $[\vec{a}, \vec{b}, \vec{c}]^2 = \frac{1}{4}|\vec{a}|^2|\vec{b}|^2$.

Ex:6.19 Prove that $[\vec{a} \times \vec{b},\ \vec{b} \times \vec{c},\ \vec{c} \times \vec{a}] = [\vec{a}, \vec{b}, \vec{c}]^2$.

Ex:6.20 Prove that $\left(\vec{a} \cdot (\vec{b} \times \vec{c})\right)\vec{a} = (\vec{a} \times \vec{b}) \times (\vec{a} \times \vec{c})$.

Ex:6.21 For any four vectors $\vec{a}, \vec{b}, \vec{c}, \vec{d}$ we have

$$(\vec{a} \times \vec{b}) \times (\vec{c} \times \vec{d}) = [\vec{a}, \vec{b}, \vec{d}]\vec{c} - [\vec{a}, \vec{b}, \vec{c}]\vec{d} = [\vec{a}, \vec{c}, \vec{d}]\vec{b} - [\vec{b}, \vec{c}, \vec{d}]\vec{a}$$

Ex:6.22 If $\vec{a} = -2\hat{\imath} + 3\hat{\jmath} - 2\hat{k},\ \vec{b} = 3\hat{\imath} - \hat{\jmath} + 3\hat{k},\ \vec{c} = 2\hat{\imath} - 5\hat{\jmath} + \hat{k}$, find $(\vec{a} \times \vec{b}) \times \vec{c}$ and $\vec{a} \times (\vec{b} \times \vec{c})$. State whether they are equal.

Ex:6.23 If $\vec{a} = \hat{\imath} - \hat{\jmath},\ \vec{b} = \hat{\imath} - \hat{\jmath} - 4\hat{k},\ \vec{c} = 3\hat{\jmath} - \hat{k}$ and $\vec{d} = 2\hat{\imath} + 5\hat{\jmath} + \hat{k}$, verify that

(i)$(\vec{a} \times \vec{b}) \times (\vec{c} \times \vec{d}) = [\vec{a}, \vec{b}, \vec{d}]\vec{c} - [\vec{a}, \vec{b}, \vec{c}]\vec{d}$ (ii)$(\vec{a} \times \vec{b}) \times (\vec{c} \times \vec{d}) = [\vec{a}, \vec{c}, \vec{d}]\vec{b} - [\vec{b}, \vec{c}, \vec{d}]\vec{a}$

Exercise 6.3

1. If $\vec{a} = \hat{\imath} - 2\hat{\jmath} + 3\hat{k},\ \vec{b} = 2\hat{\imath} + \hat{\jmath} - 2\hat{k},\ \vec{c} = 3\hat{\imath} + 2\hat{\jmath} + \hat{k}$, find (i) $(\vec{a} \times \vec{b}) \times \vec{c}$ (ii) $\vec{a} \times (\vec{b} \times \vec{c})$

2. For any vector $\vec{a}$, prove that $\vec{\imath} \times (\vec{a} \times \vec{\imath}) + \vec{\jmath} \times (\vec{a} \times \vec{\jmath}) + \vec{k} \times (\vec{a} \times \vec{k}) = 2\vec{a}$.

3. Prove that $[\vec{a} - \vec{b},\ \vec{b} - \vec{c},\ \vec{c} - \vec{a}] = 0$.

4. If $\vec{a} = 2\hat{\imath} + 3\hat{\jmath} - \hat{k},\ \vec{b} = 3\hat{\imath} + 5\hat{\jmath} + 2\hat{k},\ \vec{c} = -\hat{\imath} - 2\hat{\jmath} + 3\hat{k}$, verify that
(i) $(\vec{a} \times \vec{b}) \times \vec{c} = (\vec{a} \cdot \vec{c})\vec{b} - (\vec{b} \cdot \vec{c})\vec{a}$ (ii) $\vec{a} \times (\vec{b} \times \vec{c}) = (\vec{a} \cdot \vec{c})\vec{b} - (\vec{a} \cdot \vec{b})\vec{c}$

5. $\vec{a} = 2\hat{\imath} + 3\hat{\jmath} - \hat{k},\ \vec{b} = -\hat{\imath} + 2\hat{\jmath} - 4\hat{k},\ \vec{c} = \hat{\imath} + \hat{\jmath} + \hat{k}$ then find the value of $(\vec{a} \times \vec{b}) \cdot (\vec{a} \times \vec{c})$

6. If $\vec{a}, \vec{b}, \vec{c}, \vec{d}$ are coplanar vectors, then show that $(\vec{a} \times \vec{b}) \times (\vec{c} \times \vec{d}) = \vec{0}$

7. If $\vec{a} = \hat{\imath} + 2\hat{\jmath} + 3\hat{k},\ \vec{b} = 2\hat{\imath} - \hat{\jmath} + \hat{k},\ \vec{c} = 3\hat{\imath} + 2\hat{\jmath} + \hat{k}$ and $\vec{a} \times (\vec{b} \times \vec{c}) = l\vec{a} + m\vec{b} + n\vec{c}$, find the values of l, m, n.

8. If $\hat{a}, \hat{b}, \hat{c}$ are three unit vectors such that $\hat{b}$ and $\hat{c}$ are non-parallel and $\hat{a} \times (\hat{b} \times \hat{c}) = \frac{1}{2}\hat{b}$, find the angle between $\hat{a}$ and $\hat{c}$. **Sep 2020 – 2M**

Ex:6.24 A straight line passes through the point $(1, 2, -3)$ and parallel to $4\hat{\imath} + 5\hat{\jmath} - 7\hat{k}$. Find (i) vector equation in parametric form (ii) vector equation in non-parametric form (iii) Cartesian equations of the straight line.

Ex:6.25 The vector equation in parametric form of a line is $\vec{r} = (3\hat{\imath} - 2\hat{\jmath} + 6\hat{k}) + t(2\hat{\imath} - \hat{\jmath} + 3\hat{k})$. Find (i) the direction cosines of the straight line (ii) vector equation in non-parametric form of the line (iii) Cartesian equations of the line.

Ex:6.26 Find the vector equation in parametric form and Cartesian equations of the line passing through $(-4, 2, -3)$ and is parallel to the line $\frac{-x-2}{4} = \frac{y+3}{-2} = \frac{2z-6}{3}$

Ex:6.27 Find the vector equation in parametric form and Cartesian equations of a straight passing through the points $(-5, 7, -4)$ and $(13, -5, 2)$. Find the point where the straight line crosses the xy-plane. **Mar2020-3M**

Ex:6.28 Find the angle between the straight line $\frac{x+3}{2} = \frac{y-1}{2} = -z$ with coordinate axes. **Mar2023- 3M**

Ex:6.29 Find the acute angle between the lines $\vec{r} = (\hat{\imath} + 2\hat{\jmath} + 4\hat{k}) + t(2\hat{\imath} + 2\hat{\jmath} + \hat{k})$ and the straight line passing through the points (5,1,4) and (9,2,12).

Ex:6.30 Find the acute angle between the straight lines $\frac{x-4}{2} = \frac{y}{1} = \frac{z+1}{-2}$ and $\frac{x-1}{4} = \frac{y+1}{-4} = \frac{z-2}{2}$ and state whether they are parallel or perpendicular. **July-2022 2M**

Ex:6.31 Show that the straight line passing through the points $A(6, 7, 5)$ and $B(8, 10, 6)$ is perpendicular to the straight line passing through the points $C(10, 2, -5)$ and $D(8, 3, -4)$.

Ex:6.32 Show that the lines $\frac{x-1}{4} = \frac{2-y}{6} = \frac{z-4}{12}$ and $\frac{x-3}{-2} = \frac{y-3}{3} = \frac{5-z}{6}$ are parallel.

Exercise 6.4

1. Find the non-parametric form of vector equation and Cartesian equations of the straight line passing through the point with position vector $4\hat{\imath} + 3\hat{\jmath} - 7\hat{k}$ and parallel to the vector $2\hat{\imath} - 6\hat{\jmath} + 7\hat{k}$.
2. Find the parametric form of vector equation and Cartesian equations of the straight line passing through the point $(-2,3,4)$ and parallel to the straight line $\frac{x-1}{-4} = \frac{y+3}{5} = \frac{8-z}{6}$.
3. Find the points where the straight line passes through $(6, 7, 4)$ and $(8, 4, 9)$ cuts the xz and yz planes.
4. Find the direction cosines of the straight line passing through the points $(5, 6, 7)$ and $(7, 9, 13)$. Also, find the parametric form of vector equation and Cartesian equations of the straight line passing through two given points.
5. Find the acute angle between the following lines.
   (i) $\vec{r} = (4\hat{\imath} - \hat{\jmath}) + t(\hat{\imath} + 2\hat{\jmath} - 2\hat{k})$, $\vec{r} = (\hat{\imath} - 2\hat{\jmath} + 4\hat{k}) + s(-\hat{\imath} - 2\hat{\jmath} + 2\hat{k})$
   (ii) $\frac{x+4}{3} = \frac{y-7}{4} = \frac{z+5}{5}$, $\vec{r} = 4\hat{k} + t(2\hat{\imath} + \hat{\jmath} + \hat{k})$ (iii) $2x = 3y = -z$ and $6x = -y = -4z$
6. The vertices of ΔABC are $A(7, 2, 1)$, $B(6, 0, 3)$, and $C(4, 2, 4)$. Find $\angle ABC$
7. If the straight line joining the points $(2, 1, 4)$ and $(a - 1, 4, -1)$ is parallel to the line joining the points $(0, 2, b - 1)$ and $(5, 3, -2)$, find the values of a and b.
8. If the straight lines $\frac{x-5}{5m+2} = \frac{2-y}{5} = \frac{1-z}{-1}$ and $x = \frac{2y+1}{4m} = \frac{1-z}{-3}$ are perpendicular to each other, find the value of m.
9. Show that the points $(2, 3, 4), (-1, 4, 5)$ and $(8, 1, 2)$ are collinear.

Ex:6.33 Find the point of intersection of the lines $\frac{x-1}{2} = \frac{y-2}{3} = \frac{z-3}{4}$ and $\frac{x-4}{5} = \frac{y-1}{2} = z$.

Ex:6.34 Find the parametric form of a vector equation of a straight line passing through the point of intersection of the straight lines $\vec{r} = (\hat{\imath} + 3\hat{\jmath} - \hat{k}) + t(2\hat{\imath} + 3\hat{\jmath} + 2\hat{k})$ and $\frac{x-2}{1} = \frac{y-4}{2} = \frac{z+3}{4}$, and perpendicular to both straight lines.

Ex:6.35 Determine whether the pair of straight lines $\vec{r} = (2\hat{\imath} + 6\hat{\jmath} + 3\hat{k}) + t(2\hat{\imath} + 3\hat{\jmath} + 4\hat{k})$, $\vec{r} = (2\hat{\jmath} - 3\hat{k}) + s(\hat{\imath} + 2\hat{\jmath} + 3\hat{k})$ are parallel. Find the shortest distance between them.

Ex:6.36 Find the shortest distance between the two given straight lines

$\vec{r} = (2\hat{\imath} + 3\hat{\jmath} + 4\hat{k}) + t(-2\hat{\imath} + \hat{\jmath} - 2\hat{k})$ and $\frac{x-3}{2} = \frac{y}{-1} = \frac{z+2}{2}$.

Ex:6.37 Find the coordinates of the foot of the perpendicular drawn from the point $(-1, 2, 3)$ to the straight line $\vec{r} = (\hat{\imath} - 4\hat{\jmath} + 3\hat{k}) + t(2\hat{\imath} + 3\hat{\jmath} + \hat{k})$. Also, find the shortest distance from the point to the straight line.

Exercise 6.5

1. Find the parametric form of vector equation and Cartesian equations of a straight line passing through$(5, 2, 8)$ and is perpendicular to the straight lines $\vec{r} = (\hat{\imath} + \hat{\jmath} - \hat{k}) + s(2\hat{\imath} - 2\hat{\jmath} + \hat{k})$ and $\vec{r} = (2\hat{\imath} - \hat{\jmath} - 3\hat{k}) + t(\hat{\imath} + 2\hat{\jmath} + 2\hat{k})$.

2. Show that the lines $\vec{r} = (6\hat{\imath} + \hat{\jmath} + 2\hat{k}) + s(\hat{\imath} + 2\hat{\jmath} - 3\hat{k})$ and $\vec{r} = (3\hat{\imath} + 2\hat{\jmath} - 2\hat{k}) + t(2\hat{\imath} + 4\hat{\jmath} - 5\hat{k})$ are skew lines and hence find the shortest distance between them.

3. If the two lines $\frac{x-1}{2} = \frac{y+1}{3} = \frac{z-1}{4}$ and $\frac{x-3}{1} = \frac{y-m}{2} = z$ intersect at a point, find the value of m.

4. Show that the lines $\frac{x-3}{3} = \frac{y-3}{-1}$, $z - 1 = 0$ and $\frac{x-6}{2} = \frac{z-1}{3}, y - 2 = 0$, intersect. Also find the point of intersection.

5. Show that the straight lines $x + 1 = 2y = -12z$ and $x = y + 2 = 6z - 6$ are skew and hence find the shortest distance between them.

6. Find the parametric form of vector equation of the straight line passing through $(-1, 2, 1)$ and parallel to the straight line $\vec{r} = (2\hat{\imath} + 3\hat{\jmath} - \hat{k}) + t(\hat{\imath} - 2\hat{\jmath} + \hat{k})$ and hence find the shortest distance between the lines.

7. Find the foot of the perpendicular drawn from the point $(5, 4, 2)$ to the line $\frac{x+1}{2} = \frac{y-3}{3} = \frac{z-1}{-1}$. Also, find the equation of the perpendicular.

Ex:6.38 Find the vector and Cartesian form of the equations of a plane which is at a distance of 12 units from the origin and perpendicular to $6\hat{\imath} + 2\hat{\jmath} - 3\hat{k}$.

Ex:6.39 If the Cartesian equation of a plane is $3x - 4y + 3z = -8$, find the vector equation of the plane in the standard form.

Ex:6.40 Find the direction cosines of the normal to the plane and length of the perpendicular from the origin to the plane $\vec{r} \cdot (3\hat{\imath} - 4\hat{\jmath} + 12\hat{k}) = 5$.

Ex:6.41 Find the vector and Cartesian equations of the plane passing through the point with position vector $4\hat{\imath} + 2\hat{\jmath} - 3\hat{k}$ and normal to vector $2\hat{\imath} - \hat{\jmath} + \hat{k}$

Ex:6.42 A variable plane moves in such a way that the sum of the reciprocals of its intercepts on the coordinate axes is a constant. Show that the plane passes through a fixed point.

Exercise 6.6

1. Find the vector equation of a plane which is at a distance of 7 units from the origin having $3, -4, 5$ as direction ratios of a normal to it. **Mar 2023- 2M**

2. Find the direction cosines of the normal to the plane $12x + 3y - 4z = 65$. Also,find the non parametric form of vector equation of a plane and the length of the perpendicular to the plane from the origin.

3. Find the vector and Cartesian equations of the plane passing through the point with position vector $2\hat{\imath} + 6\hat{\jmath} + 3\hat{k}$ and normal to the vector $\hat{\imath} + 3\hat{\jmath} + 5\hat{k}$.

4. A plane passes through the point $(-1, 1, 2)$ and the normal to the plane of magnitude $3\sqrt{3}$ makes equal acute angles with the coordinate axes. Find the equation of the plane.

5. Find the intercepts cut off by the plane $\vec{r} \cdot (6\hat{\imath} + 4\hat{\jmath} - 3\hat{k}) = 12$ on the coordinate axes.

6. If a plane meets the coordinate axes at A, B, C such that the centriod of the triangle ABC is the point (u, v, w), find the equation of the plane.

Ex:6.43 Find the non-parametric form of vector equation, and Cartesian equation of the plane passing through the point $(0, 1, -5)$ and parallel to the straight lines $\vec{r} = (\hat{\imath} + 2\hat{\jmath} - 4\hat{k}) + s(2\hat{\imath} + 3\hat{\jmath} + 6\hat{k})$

and $\vec{r} = (\hat{\imath} - 3\hat{\jmath} + 5\hat{k}) + t(\hat{\imath} + \hat{\jmath} - \hat{k})$. **Mar2020-5M**

Ex:6.44 Find the vector parametric, vector non-parametric and Cartesian form of the equation of the plane passing through the points $(-1, 2, 0), (2, 2, -1)$ and parallel to the straight line $\frac{x-1}{1} = \frac{2y+1}{2} = \frac{z+1}{-1}$.

Exercise 6.7

1. Find the non-parametric form of vector equation, and Cartesian equation of the plane passing through the point $(2, 3, 6)$ and parallel to the straight lines $\frac{x-1}{2} = \frac{y+1}{3} = \frac{z-3}{1}$ and $\frac{x+3}{2} = \frac{y-3}{-5} = \frac{z+1}{-3}$.

2. Find the parametric form of vector equation, and Cartesian equations of the plane passing through the points $(2, 2, 1), (9, 3, 6)$and perpendicular to the plane $2x + 6y + 6z = 9$. **May 2022 - 5M**

3. Find parametric form of vector equation and Cartesian equations of the plane passing through the points$(2, 2, 1), (1, -2, 3)$ and parallel to the straight line passing through the points $(2, 1, -3)$ and $(-1, 5, -8)$.

4. Find the non-parametric form of vector equation and Cartesian equation of the plane passing through the point $(1, -2, 4)$ and perpendicular to the plane $x + 2y - 3z = 11$ and parallel to the line $\frac{x+7}{3} = \frac{y+3}{-1} = \frac{z}{1}$. **Sep 2020 – 5M**

5. Find the parametric form of vector equation, and Cartesian equations of the plane containing the line $\vec{r} = (\hat{\imath} - \hat{\jmath} + 3\hat{k}) + t(2\hat{\imath} - \hat{\jmath} + 4\hat{k})$ and perpendicular to plane $\vec{r} \cdot (\hat{\imath} + 2\hat{\jmath} + \hat{k}) = 8$.
Mar 2023- 5M

6. Find the parametric vector, non-parametric vector and Cartesian form of the equations of the plane passing through the three non-collinear points $(3, 6, -2), (-1, -2, 6)$, and $(6, 4, -2)$.

7. Find the non-parametric form of vector equation, and Cartesian equations of the plane
$\vec{r} = (6\hat{\imath} - \hat{\jmath} + \hat{k}) + s(-\hat{\imath} + 2\hat{\jmath} + \hat{k}) + t(-5\hat{\imath} - 4\hat{\jmath} - 5\hat{k})$.

Ex:6.45 Verify whether the line $\frac{x-3}{-4} = \frac{y-4}{-7} = \frac{z+3}{12}$ lies in the plane $5x - y + z = 8$.

Ex:6.46 Show that the lines $\vec{r} = (-\hat{\imath} - 3\hat{\jmath} - 5\hat{k}) + s(3\hat{\imath} + 5\hat{\jmath} + 7\hat{k})$ and $\vec{r} = (2\hat{\imath} + 4\hat{\jmath} + 6\hat{k}) + t(\hat{\imath} + 4\hat{\jmath} + 7\hat{k})$ are coplanar. Also, find the non-parametric form of vector equation of the plane containing these lines.

Exercise 6.8

1. Show that the straight lines $\vec{r} = (5\hat{\imath} + 7\hat{\jmath} - 3\hat{k}) + s(4\hat{\imath} + 4\hat{\jmath} - 5\hat{k})$ and $\vec{r} = (8\hat{\imath} + 4\hat{\jmath} + 5\hat{k}) + t(7\hat{\imath} + \hat{\jmath} + 3\hat{k})$ are coplanar. Find the vector equation of the plane in which they lie.

2. Show that the lines $\frac{x-2}{1} = \frac{y-3}{1} = \frac{z-4}{3}$ and $\frac{x-1}{-3} = \frac{y-4}{2} = \frac{z-5}{1}$are coplanar. Also, find the plane containing these lines.

3. If the straight lines $\frac{x-1}{1} = \frac{y-2}{2} = \frac{z-3}{m^2}$ and $\frac{x-3}{1} = \frac{y-2}{m^2} = \frac{z-1}{2}$ are coplanar, find the distinct real values of m.

4. If the straight lines $\frac{x-1}{2} = \frac{y+1}{\lambda} = \frac{z}{2}$ and $\frac{x+1}{5} = \frac{y+1}{2} = \frac{z}{\lambda}$ are coplanar, find λ and equations of the planes containing these two lines.

Ex:6.47 Find the acute angle between the planes $\vec{r} \cdot (2\hat{\imath} + 2\hat{\jmath} + 2\hat{k}) = 11$ and $4x - 2y + 2z = 15$.

Ex:6.48 Find the angle between the straight line $\vec{r} = (2\hat{\imath} + 3\hat{\jmath} + \hat{k}) + t(\hat{\imath} - \hat{\jmath} + \hat{k})$ and the plane $2x - y + z = 5$.

Ex:6.49 Find the distance of a point $(2, 5, -3)$ from the plane $\vec{r} \cdot (6\hat{\imath} - 3\hat{\jmath} + 2\hat{k}) = 5$.

Ex:6.50 Find the distance of the point $(5, -5, -10)$ from the point of intersection of a straight line passing through the points $A(4, 1, 2)$ and $B(7, 5, 4)$ with the plane $x - y + z = 5$.

Ex:6.51 Find the distance between the parallel planes $x + 2y - 2z + 1 = 0$ and $2x + 4y - 4z + 5 = 0$

Ex:6.52 Find the distance between the planes $\vec{r} \cdot (2\hat{\imath} - \hat{\jmath} - 2\hat{k}) = 6$ and $\vec{r} \cdot (6\hat{\imath} - 3\hat{\jmath} - 6\hat{k}) = 27$.

Ex:6.53 Find the equation of the plane passing through the intersection of the planes

$\vec{r}\cdot(\hat{\imath}+\hat{\jmath}+\hat{k})+1=0$ and $\vec{r}\cdot(2\hat{\imath}-3\hat{\jmath}+5\hat{k})=2$ and the point $(-1,2,1)$.

Ex:6.54 Find the equation of the plane passing through the intersection of the planes $2x+3y-z+7=0$ and $x+y-2z+5=0$ and is perpendicular to the plane $x+y-3z-5=0$.

Ex:6.55 Find the image of the point whose position vector is $\hat{\imath}+2\hat{\jmath}+3\hat{k}$ in the plane $\vec{r}\cdot(\hat{\imath}+2\hat{\jmath}+4\hat{k})=38$.

Ex:6.56 Find the coordinates of the point where the straight line $\vec{r}=(2\hat{\imath}-\hat{\jmath}+2\hat{k})+t(3\hat{\imath}+4\hat{\jmath}+2\hat{k})$ intersects the plane $x-y+z-5=0$.

Exercise 6.9

1. Find the equation of the plane passing through the line of intersection of the planes $\vec{r}\cdot(2\hat{\imath}-7\hat{\jmath}+4\hat{k})=3$ and $3x-5y+4z+11=0$, and the point $(-2,1,3)$.
2. Find the equation of the plane passing through the line of intersection of the planes $x+2y+3z=2$ and $x-y+z+11=3$, and at a distance $\frac{2}{\sqrt{3}}$ from the point $(3,1,-1)$.
3. Find the angle between the line $\vec{r}=(2\hat{\imath}-\hat{\jmath}+\hat{k})+t(\hat{\imath}+2\hat{\jmath}-2\hat{k})$ and the plane $\vec{r}\cdot(6\hat{\imath}+3\hat{\jmath}+2\hat{k})=8$.
4. Find the angle between the planes $\vec{r}\cdot(\hat{\imath}+\hat{\jmath}-2\hat{k})=3$ and $2x-2y+z=2$.
5. Find the equation of the plane which passes through the point $(3,4,-1)$ and is parallel to the plane $2x-3y+5z+7=0$. Also, find the distance between the two planes.
6. Find the length of the perpendicular from the point $(1,-2,3)$ to the plane $x-y+z=5$
7. Find the point of intersection of the line $x-1=\frac{y}{2}=z+1$ with the plane $2x-y+2z=2$. Also, find the angle between the line and the plane.
8. Find the coordinates of the foot of the perpendicular and length of the perpendicular from the point $(4,3,2)$ to the plane $x+2y+3z=2$.

7. APPLICATIONS OF DIFFERENTIAL CALCULUS

Theorem 1: Intermediate Value Theorem. **Theorem 2:** Rolle's Theorem

Theorem 3: Lagrange's Mean Value Theorem (or) Rotated Rolle's Theorem.

Theorem 4: If $f(x)$ is continuous in $[a,b]$ and differentiable in (a,b) and if $f'(x)>0, \forall\, x\in(a,b)$, then for $x_1,x_2\in[a,b]$, such that $x_1<x_2$ we have, $f(x_1)<f(x_2)$.

Theorem 5: Taylor's Series and Maclaurin's Series

Theorem 6: Limit of a composite function theorem.

Theorem 7: Extreme Value Theorem. **Theorem 8:** Fermat Theorem.

Theorem 9: First Derivative test. **Theorem 10:** Test of Concavity.

Theorem 11: Test for Points of Inflection. **Theorem 12:** The Second derivative test.

Ex:7.1 For the function $f(x)=x^2, x\in[0,2]$ compute the average rate of changes in the subintervals [0, 0.5], [0.5, 1], [1, 1.5], [1.5, 2] and the instantaneous rate of changes at the points x =0.5, 1, 1.5, 2.

Ex:7.2 The temperature T in Celsius in a long rod of length 10m, insulated at both ends, is a function of length x given by $T=x(10-x)$. Prove that the rate of change of temperature at the midpoint of the rod is zero.

Ex:7.3 A person learnt 100 words for an English test. The number of words the person remembers in t days after learning is given by $W(t)=100\times(1-0.1t)^2, 0\le t\le 10$. What is the rate at which the person forgets the words 2 days after learning?

Ex:7.4 A particle moves so that the distance moved is according to the law $s(t)=\frac{t^3}{3}-t^2+3$. At what time the velocity and acceleration are zero respectively?

Ex:7.5 A particle is fired straight up from the ground to reach a height s feet in t seconds, where $s(t)=128t-16t^2$. (i) Compute the maximum height of the particle reached. (ii) What is the velocity when the particle hits the ground? **July-2022 5M**

Ex:7.6 A particle moves along a horizontal line such that its position at any time $t\ge 0$ is given by

$s(t) = t^3 - 6t^2 + 9t + 1$, where s is measured in meters and t in seconds. (i) At what time the particle is at rest? (ii) At what time the particle changes its direction? (iii) Find the total distance travelled by the particle in the first 2 seconds?

Ex:7.7 If we blow air into a balloon of spherical shape at a rate of $1000cm^3$ per second, at what rate the radius of the balloon changes when the radius is 7cm? Also compute the rate at which the surface area changes.

Ex:7.8 The price of a product is related to the number of units available (supply) by the equation $Px + 3P - 16x = 234$, where P is the price of the product per unit in Rupees(Rs) and x is the number of units. Find the rate at which the price is changing with respect to time when 90 units are available and the supply is increasing at a rate of 15units/week.

Ex:7.9 Salt is poured from a conveyer belt at a rate of 30 cubic meter per minute forming a conical pile with a circular base whose height and diameter of base are always equal. How fast is the height of the pile increasing when the pile is 10 metre high?

Ex:7.10 A road running north to south crosses a road going east to west at the point P. Car A is driving north along the first road, and car B is driving east along the second road. At a particular time car A is 10 kilometers to the north of P and travelling at 80km/hr, while car B is 15 kilometers to the east of P and travelling at 100km/hr. How fast is the distance between the two cars changing?

Exercise 7.1

1. A particle moves along a straight line in such a way that after t seconds its distance from the origin is $s = 2t^2 + 3t$ meters. (i) Find the average velocity between $t = 3$ and $t = 6$ seconds. (ii) Find the instantaneous velocities at $t = 3$ and $t = 6$ seconds.

2. A camera is accidentally knocked off an edge of a cliff 400 ft high. The camera falls a distance of $s = 16t^2$ in t seconds. (i) How long does the camera fall before it hits the ground? (ii) What is the average velocity with which the camera falls during the last 2 seconds? (iii) What is the instantaneous velocity of the camera when it hits the ground? **Aug 2021- 5M**

3. A particle moves along a line according to the law $s(t) = 2t^3 - 9t^2 + 12t - 4$, where $t \geq 0$. (i)At what times the particle changes direction? (ii)Find the total distance travelled by the particle in the first 4 seconds? (iii)Find the particle's acceleration each time the velocity is zero.

4. If the volume of a cube of side length x is $v = x^3$. Find the rate of change of the volume with respect to x when $x = 5$ units.

5. If the mass $m(x)$ (in kilograms) of a thin rod of length x (in meters) is given by, $m(x) = \sqrt{3x}$ then what is the rate of change of mass with respect to the length when it is $x = 3$ and $x = 27$ meters.

6. A stone is dropped into a pond causing ripples in the form of concentric circles. The radius r of the outer ripple is increasing at a constant rate at 2cm per second. When the radius is 5cm find the rate of changing of the total area of the disturbed water?

7. A beacon makes one revolution every 10 seconds. It is located on a ship which is anchored 5km from a straight shore line. How fast is the beam moving along the shore line when it makes an angle of $45^o$with the shore?

8. A conical water tank with vertex down of 12 meters height has a radius of 5 meters at the top. If water flows into the tank at a rate 10 cubic m/min, how fast is the depth of the water increases when the water is 8 meters deep?

9. A ladder 17 meter long is against the wall. The base of the ladder is pulled away from the wall at a rate of 5m/s. When the base of the ladder is 8 meters from the wall, (i) how fast is the top of the ladder moving down the wall? (ii) at what rate, the area of the triangle formed by the ladder, wall, and the floor, is changing?

10. A police jeep, approaching an orthogonal intersection from the northern direction, is chasing a speeding car that has turned and moving straight east. When the jeep is 0.6 km north of the intersection and the car is 0.8 km to the east. The police determine with a radar that the distance

between them and the car is increasing at 20 km/hr. If the jeep is moving at 60km/hr at the instant of measurement, what is the speed of the car? **Mar2020-5M**

Ex:7.11 Find the equation of the tangent and normal to the curve $y = x^2 + 3x - 2$ at the point (1,2). **Mar 2023- 2M**

Ex:7.12 Find the points on the curve $y = x^3 - 3x^2 + x - 2$ at which the tangent is parallel to the line $y = x$. **May 2022 - 2M**

Ex:7.13 Find the equation of the tangent and normal at any point to the Lissajous curve given by $x = 2cos3t$ and $y = 3sin2t$, $t \in \mathbb{R}$.

Ex:7.14 Find the angle between $y = x^2$ and $y = (x-3)^2$.

Ex:7.15 Find the angle between the curves $y = x^2$ and $x = y^2$ at their points of intersection (0,0) and (1,1). **May 2022 - 5M**

Ex:7.16 Find the angle of intersection of the curve $y = sinx$ with the positive x-axis.

Ex:7.17 If the curves $ax^2 + by^2 = 1$ and $cx^2 + dy^2 = 1$ intersect each other orthogonally then, show that $\frac{1}{a} - \frac{1}{b} = \frac{1}{c} - \frac{1}{d}$.

Ex:7.18 Prove that the ellipse $x^2 + 4y^2 = 8$ and the hyperbola $x^2 - 2y^2 = 4$ intersect orthogonally. **Mar 2023- 5M**

Exercise 7.2

1. Find the slope of the tangent to the following curves at the respective given points.
   (i) $y = x^4 + 2x^2 - x$ at $x = 1$ (ii) $x = acos^3t,\ y = bsin^3t$ at $t = \frac{\pi}{2}$.
2. Find the point on the curve $y = x^2 - 5x + 4$ at which the tangent is parallel to the line $3x + y = 7$. **Aug 2021- 3M**
3. Find the points on the curve $y = x^3 - 6x^2 + x + 3$ where the normal is parallel to the line $x + y = 1729$.
4. Find the points on the curve $y^2 - 4xy = x^2 + 5$ for which the tangent is horizontal.
5. Find the equation of the tangent and normal to the curves at the given points on the curve
   (i) $y = x^2 - x^4$ at (1,0) **July-2022 2M** (ii) $y = x^4 + 2e^x$ at (0,2)
   (iii) $y = xsinx$ at $(\frac{\pi}{2}, \frac{\pi}{2})$ (iv) $x = cost, y = 2sin^2t$ at $t = \frac{\pi}{3}$.
6. Find the equations of the tangents to the curve $y = 1 + x^3$ for which the tangent is orthogonal with the line $x + 12y = 12$.
7. Find the equations of the tangents to the curve $y = \frac{x+1}{x-1}$ which are parallel to the line $x + 2y = 6$.
8. Find the equation of the tangent and normal to the curve given by $x = 7cost,\ y = 2sint, t \in \mathbb{R}$ at any point on the curve.
9. Find the angle between the rectangular hyperbola $xy = 2$ and the parabola $x^2 + 4y = 0$.
10. Show that the two curves $x^2 - y^2 = r^2$ and $xy = c^2$ where c, r are constants, cut orthogonally.

Ex:7.19 Compute the value of 'c' satisfied by the Rolle's theorem for the function $f(x) = x^2(1-x)^2, x \in [0,1]$.

Ex:7.20 Find the value in the interval $\left(\frac{1}{2}, 2\right)$ satisfied by the Rolle's theorem for the function $f(x) = x + \frac{1}{x},\ x \in \left[\frac{1}{2}, 2\right]$. **Mar2020-2M**

Ex:7.21 Compute the value of 'c' satisfied by the Rolle's theorem for the function $f(x) = log\left(\frac{x^2+6}{5x}\right)$ in the interval [2,3].

Ex:7.22 Without actually solving show that the equation $x^4 + 2x^3 - 2 = 0$ has only one real root in the interval (0,1).

Ex:7.23 Prove using Rolle's theorem that between any two distinct real zeros of the polynomial $a_nx^n + a_{n-1}x^{n-1} + \cdots + a_1x + a_0$ there is a zero of the polynomial $na_nx^{n-1} + (n-1)a_{n-1}x^{n-2} + \cdots + a_1$.

Ex:7.24 Prove that there is a zero of the polynomial, $2x^3 - 9x^2 - 11x + 12$ in the interval (2,7) given that 2 and 7 are the zeros of the polynomial $x^4 - 6x^3 - 11x^2 + 24x + 28$.

Ex:7.25 Find the values in the interval (1,2) of the mean value theorem satisfied by the function $f(x) = x - x^2$ for $1 \le x \le 2$.

Ex:7.26 A truck travels on a toll road with a speed limit of 80 km/hr. The truck completes a 164 km journey in 2 hours. At the end of the toll road the trucker is issued with a speed violation notice. Justify this using the Mean Value Theorem.

Ex:7.27 Suppose $f(x)$ is a differentiable function for all x with $f'(x) \le 29$ and $f(2) = 17$. What is the maximum value of $f(7)$?

Ex:7.28 Prove, using mean value theorem, that $|sin\alpha - sin\beta| \le |\alpha - \beta|, \ \ \alpha, \beta \in \mathbb{R}$.

Ex:7.29 A thermometer was taken from a freezer and placed in a boiling water. It took 22 seconds for the thermometer to raise from -10^oC to 100^oC. Show that the rate of change of temperature at some time t is 5^oC per second.

Exercise 7.3

1. Explain why Rolle's theorem is not applicable to the following functions in the respective intervals.
(i) $f(x) = \left|\frac{1}{x}\right|, x \in [-1, 1]$ (ii) $f(x) = tanx, x \in [0, \pi]$ (iii) $f(x) = x - 2logx, x \in [2,7]$..

2. Using the Rolle's theorem, determine the values of x at which the tangent is parallel to the x-axis for the following functions:
(i) $f(x) = x^2 - x, x \in [0, 1]$ (ii) $f(x) = \frac{x^2-2x}{x+2}, x \in [-1, 6]$ (iii) $f(x) = \sqrt{x} - \frac{x}{3}, x \in [0, 9]$

3. Explain why Lagrange's mean value theorem is not applicable to the following functions in the respective intervals: (i) $f(x) = \frac{x+1}{x}, x \in [-1, 2]$ (ii) $f(x) = |3x + 1|, \ x \in [-1,3]$

4. Using Lagrange's mean value theorem determine the values of x at which the tangent is parallel to the secant line at the end points of the given interval
(i) $f(x) = x^3 - 3x + 2, \ x \in [-2,2]$ (ii) $f(x) = (x - 2)(x - 7), \ x \in [3,11]$

5. Show that the value in the conclusion of the mean value theorem for (i) $f(x) = \frac{1}{x}$ on the closed interval of the positive numbers $[a, b]$ is $\sqrt{ab}$. (ii) $f(x) = Ax^2 + Bx + C$ on any interval $[a, b]$ is $\frac{a+b}{2}$.

6. A race car driver is racing at 20th km. If his speed never exceeds 150 km/hr, what is the maximum distance he can cover in the next two hours.

7. Suppose that for a function $f(x), f'(x) \le 1$ for all $1 \le x \le 4$. Show that $f(4) - f(1) \le 3$.

8. Does there exist a differentiable function $f(x)$ such that $f(0) = -1, f(2) = 4$ and $f'(x) \le 2$ for all x. Justify your answer.

9. Show that there lies a point on the curve $f(x) = x(x + 3)e^{\frac{-\pi}{2}}, -3 \le x \le 0$ where tangent drawn is parallel to the x-axis.

10. Using mean value theorem prove that for , $a > 0, b > 0, |e^{-a} - e^{-b}| < |a - b|$.

Ex:7.30 Expand $\log(1 + x)$ as a Maclaurin's series upto 4 non-zero terms for $-1 < x \le 1$.

Ex:7.31 Expand $tanx$ in ascending powers of x upto 5th power for $\frac{-\pi}{2} < x < \frac{\pi}{2}$.

Ex:7.32 Write the Taylor series expansion of $\frac{1}{x}$ about $x = 2$ by finding the first three non-zero terms.

Exercise 7.4

1. Write the Maclaurin series expansion of the following functions:
(i) e^x (ii) $sinx$ (iii) $cosx$ (iv) $\log(1 - x)$; $-1 \le x < 1$ (v) $tan^{-1}x$; $-1 \le x \le 1$ (vi) cos^2x

2. Write down the Taylor series expansion, of the function $logx$ about $x = 1$ upto three non-zero terms for $x > 0$.

3. Expand $sinx$ in ascending powers $x - \frac{\pi}{4}$ upto three non-zero terms.

4. Expand the polynomial $f(x) = x^2 - 3x + 2$ in powers of $x - 1$.

Ex:7.33 Evaluate $\lim_{x\to 1}\left(\frac{x^2-3x+2}{x^2-4x+3}\right)$ **Ex:7.34** Compute the limit $\lim_{x\to a}\left(\frac{x^n-a^n}{x-a}\right)$

Ex:7.35 Evaluate the limit $\lim_{x\to 0}\left(\frac{\sin mx}{x}\right)$ **Sep 2020 – 2M** **Ex:7.36** Evaluate the lmit $\lim_{x\to 0}\left(\frac{\sin x}{x^2}\right)$

Ex:7.37 If $\lim_{\theta\to 0}\left(\frac{1-\cos m\theta}{1-\cos n\theta}\right)=1$, then prove that $m=\pm n$. **Ex:7.38** Evaluate $\lim_{x\to 1^-}\left(\frac{\log(1-x)}{\cot(\pi x)}\right)$.

Ex:7.39 Evaluate $\lim_{x\to 0^+}\left(\frac{1}{x}-\frac{1}{e^x-1}\right)$. **Ex:7.40** Evaluate $\lim_{x\to 0^+} x\log x$. **July-2022 3M**

Ex:7.41 Evaluate $\lim_{x\to\infty}\left(\frac{x^2+17x+29}{x^4}\right)$. **Ex:7.42** Evaluate $\lim_{x\to\infty}\left(\frac{e^x}{x^m}\right), m\in N$. **Aug 2021- 2M**

Ex:7.43 Using the L'Hopital rule, prove that $\lim_{x\to 0^+}(1+x)^{\frac{1}{x}}=e$.

Ex:7.44 Evaluate $\lim_{x\to\infty}(1+2x)^{\frac{1}{2\log x}}$. **Ex:7.45** Evaluate $\lim_{x\to 1} x^{\frac{1}{1-x}}$.

Exercise 7.5

Evaluate the following limits, if necessary use L'Hospital rule:

1. $\lim_{x\to 0}\frac{1-\cos x}{x^2}$ 2. $\lim_{x\to\infty}\frac{2x^2-3}{x^2-5x+3}$ **May 2022 - 3M** 3. $\lim_{x\to\infty}\frac{x}{\log x}$ 4. $\lim_{x\to\frac{\pi}{2}^-}\frac{\sec x}{\tan x}$ 5. $\lim_{x\to\infty} e^{-x}\sqrt{x}$

6. $\lim_{x\to 0}\left(\frac{1}{\sin x}-\frac{1}{x}\right)$ 7. $\lim_{x\to 1^+}\left(\frac{2}{x^2-1}-\frac{x}{x-1}\right)$ 8. $\lim_{x\to 0^+} x^x$ 9. $\lim_{x\to\infty}\left(1+\frac{1}{x}\right)^x$ 10. $\lim_{x\to\frac{\pi}{2}}(\sin x)^{\tan x}$

11. $\lim_{x\to 0^+}(\cos x)^{\frac{1}{x^2}}$

12. If an initial amount A_0 of money is invested at an interest rate r compounded n times a year, the value of the investment after t years is $A=A_0(1+\frac{r}{n})^{nt}$. If the interest is compounded continuously,(that is as $n\to\infty$), show that the amount after t years is $A=A_0e^{rt}$.

Ex:7.46 Prove that the function $f(x)=x^2+2$ is strictly increasing in the interval (2,7) and strictly decreasing in the interval $(-2,0)$.

Ex:7.47 Prove that the function $f(x)=x^2-2x-3$ is strictly increasing in $(2,\infty)$.

Ex:7.48 Find the absolute maximum and absolute minimum values of the function $f(x)=2x^3+3x^2-12x$ on $[-3,2]$.

Ex:7.49 Find the absolute extrema of the function $f(x)=3\cos x$ on the closed interval $[0,2\pi]$.

Ex:7.50 Find the intervals of monotonicity and hence find the local extrema for the function $f(x)=x^2-4x+4$.

Ex:7.51 Find the intervals of monotonicity and hence find the local extrema for the function $f(x)=x^{\frac{2}{3}}$.

Ex:7.52 Prove that the function $f(x)=x-\sin x$ is increasing on the real line. Also discuss for the existence of local extrema.

Ex:7.53 Discuss the monotonicity and local extrema of the function $f(x)=\log(1+x)-\frac{x}{1+x}, x>-1$ and hence find the domain where, $\log(1+x)>\frac{x}{1+x}$.

Ex:7.54 Find the intervals of monotonicity and local extrema of the function $f(x)=x\log x+3x$.

Ex:7.55 Find the intervals of monotonicity and local extrema of the function $f(x)=\frac{1}{1+x^2}$.

Ex:7.56 Find the intervals of monotonicity and local extrema of the function $f(x)=\frac{x}{1+x^2}$.

Exercise 7.6

1. Find the absolute extrema of the following functions on the given closed interval.

(i) $f(x)=x^2-12x+10;\ [1,2]$ **Sep 2020 – 3M** (ii) $f(x)=3x^4-4x^3;\ [-1,2]$

(iii) $f(x)=6x^{\frac{4}{3}}-3x^{\frac{1}{3}};\ [-1,1]$ (iv) $f(x)=2\cos x+\sin 2x;\ \left[0,\frac{\pi}{2}\right]$

2. Find the intervals of monotonicities and hence find the local extremum for the following functions:

(i) $f(x)=2x^3+3x^2-12x$ (ii) $f(x)=\frac{x}{x-5}$ (iii) $f(x)=\frac{e^x}{1-e^x}$

(iv) $f(x) = \frac{x^3}{3} - logx$ (v) $f(x) = sinxcosx + 5, x \in (0,2\pi)$

Ex:7.57 Determine the intervals of concavity of the curve $f(x) = (x-1)^3(x-5)$, $x \in \mathbb{R}$ and, points of inflection if any.

Ex:7.58 Determine the intervals of concavity of the curve $f(x) = 3 + sinx$.

Ex:7.59 Find the local extremum of the function $f(x) = x^4 + 32x$.

Ex:7.60 Find the local extrema of the function $f(x) = 4x^6 - 6x^4$. **May 2022 - 5M**

Ex:7.61 Find the local maximum and minimum of the function x^2y^2 on the line $x + y = 10$.

Exercise 7.7

1. Find intervals of concavity and points of inflexion for the following fuctions:

(i) $f(x) = x(x-4)^3$ (ii) $f(x) = sinx + cosx, 0 < x < 2\pi$ (iii) $f(x) = \frac{1}{2}(e^x - e^{-x})$

2. Find the local extrema for the following fuctions using second derivative test:

(i) $f(x) = -3x^5 + 5x^3$ (ii) $f(x) = xlogx$ (iii) $f(x) = x^2e^{-2x}$

3. For the function $f(x) = 4x^3 + 3x^2 - 6x + 1$ find the intervals of monotonicity, local extrema, intervals of concavity and points of inflexion.

Ex:7.62 We have a 12 square unit piece of thin material and want to make an open box by cutting small squares from the corners of our material and folding the sides up. The question is, which cut produces the box of maximum value?

Ex:7.63 Find the points on the unit circle $x^2 + y^2 = 1$ nearest and farthest from (1,1).

Ex:7.64 A steel plant is capable of producing x tonnes per day of a low-grade steel and y tonnes per day of a high-grade steel, where $y = \frac{40-5x}{10-x}$. If the fixed market price of low-grade steel is half that of high-grade steel, then what should be optimal productions in low-grade steel and high-grade steel in order to have maximum receipts.

Ex:7.65 Prove that among all the rectangles of the given area square has the least perimeter.

Exercise 7.8

1. Find two positive numbers whose sum is 12 and their product is maximum.
2. Find two positive numbers whose product is 20 and their sum is minimum. **Aug 2021- 5M**
3. Find the smallest positive value of $x^2 + y^2$ given that $x + y = 10$.
4. A garden is to be laid out in a rectangular area and protected by wire fence. What is the largest possible area of the fenced garden with 40 meters of wire.
5. A rectangular page is to contain 24 cm^2 of print. The margins at the top and bottom of the page are 1.5 cm and the margins at other sides of the page is 1 cm. What should be the dimensions of the page so that the area of the paper used is minimum.
6. A farmer plans to fence a rectangular pasture adjacent to a river. The pasture must contain 1,80,000 sq.mtrs in order to provide enough grass for herds. No fencing is needed along the river. What is the length of the minimum needed fencing material?
7. Find the dimensions of the rectangle with maximum area that can be inscribed in a circle of radius 10 cm.
8. Prove that among all the rectangles of the given perimeter, the square has the maximum area.
   Sep 2020 – 5M & July-2022 5M
9. Find the dimensions of the largest rectangle that can be inscribed in a semi circle of radius r cm.
10. A manufacturer wants to design an open box having a square base and a surface area of 108 sq.cm. Determine the dimensions of the box for the maximum volume.
11. The volume of a cylinder is given by the formula $V = \pi r^2 h$. Find the greatest and least values of V if $r + h = 6$.
12. A hollow cone with base radius a cm and height b cm is placed on a table. Show that the volume of the largest cylinder that can be hidden underneath is $\frac{4}{9}$ times volume of the cone.

Ex:7.66 Find the asymptotes of the function $f(x) = \frac{1}{x}$.

Ex:7.67 Find the slant (oblique) asymptote for the function $f(x) = \frac{x^2-6x+7}{x+5}$.

Ex:7.68 Find the asymptotes of the curve $f(x) = \frac{2x^2-8}{x^2-16}$.

Ex:7.69 Sketch the curve $y = f(x) = x^2 - x - 6$.

Ex:7.70 Sketch the curve $y = f(x) = x^3 - 6x - 9$.

Ex:7.71 Sketch the curve $y = \frac{x^2-3x}{x-1}$ **Ex:7.72** Sketch the graph of the function $y = \frac{3x}{x^2-1}$

Exercise 7.9

1. Find the asymptotes of the following curves:

(i) $f(x) = \frac{x^2}{x^2-1}$ (ii) $f(x) = \frac{x^2}{x+1}$ (iii) $f(x) = \frac{3x}{\sqrt{x^2+2}}$ (iv) $f(x) = \frac{x^2-6x-1}{x+3}$ (v) $f(x) = \frac{x^2+6x-4}{3x-6}$

2. Sketch the graphs of the following functions:

(i) $y = \frac{-1}{3}(x^3 - 3x + 2)$ (ii) $y = x\sqrt{4-x}$ (iii) $y = \frac{x^2+1}{x^2-4}$ (iv) $y = \frac{1}{1+e^{-x}}$ (v) $y = \frac{x^3}{24} - logx$

8.DIFFERENTIALS AND PARTIAL DERIVATIVES

Theorem 8.1: Clairaut's Theorem. **Theorem 8.2:** Euler's Theorem.

Ex:8.1 Find the linear approximation for $f(x) = \sqrt{1+x}, x \geq -1$, at x_0 =3. Use the linear approximation to estimate $f(3.2)$.

Ex:8.2 Use linear approximation to find an approximate value of $\sqrt{9.2}$ without using a calculator.

Ex:8.3 Let us assume that the shape of a soap bubble is a sphere. Use linear approximation to approximate the increase in the surface area of a soap bubble as its radius increases from 5cm to 5.2cm. Also, calculate the percentage error.

Ex:8.4 A right circular cylinder has radius $r = 10cm$ and height $h = 20cm$. Suppose that the radius of the cylinder is increased from $10cm$ to $10.1cm$ and the height does not change. Estimate the change in the volume of the cylinder. Also, calculate the relative error and percentage error.

Exercise 8.1

1. Let $f(x) = \sqrt[3]{x}$ Find the linear approximation at $x = 27$. Use the linear approximation to approximate $\sqrt[3]{27.2}$

2. Use the linear approximation to find the approximate values of (i)$(123)^{\frac{2}{3}}$ **Mar 2023- 3M**
   (ii) $\sqrt[4]{15}$ (iii) $\sqrt[3]{26}$

3. Find the linear approximation for the following functions at the indicated points.
   (i) $f(x) = x^3 - 5x + 12, x_0 = 2$ (ii) $g(x) = \sqrt{x^2+9}, x_0 = -4$ (iii) $h(x) = \frac{x}{x+1}$, $x_0 = 1$

4. The radius of a circular plate is measured as 12.65cm instead of the actual length 12.5cm. Find the following in calculating the area of the circular plate: (i) Absolute error (ii) Relative error (iii) Percentage error.

5. A sphere is made of ice having radius 10cm. Its radius decreases from 10cm to 9.8cm. Find approximations for the following: (i) change in the volume (ii) change in the surface area

6. The time T, taken for a complete oscillation of a single pendulum with length l, is given by the equation $T = 2\pi\sqrt{\frac{l}{g}}$, where g is a constant. Find the approximate percentage error in the calculated value of T corresponding to an error of 2 percent in the value of l.

7. Show that the percentage error in the nth root of a number is approximately $\frac{1}{n}$ times the percentage error in the number.

Ex:8.5 Let $f, g: (a, b) \to \mathbb{R}$ be differentiable functions. Show that $d(fg) = fdg + gdf$.

Ex:8.6 Let $g(x) = x^2 + \sin x$. Calculate the differential dg. **Aug 2021- 2M**

Ex:8.7 If the radius of a sphere, with radius $10cm$, has to decrease by $0.1cm$, approximately how much will its volume decrease?

Exercise 8.2

1. Find differential dy for each of the following functions:

(i) $y = \frac{(1-2x)^3}{3-4x}$ (ii) $y = (3 + sin2x)^{\frac{2}{3}}$ (iii) $y = e^{x^2-5x+7}\cos(x^2 - 1)$

2. Find df for $f(x) = x^2 + 3x$ and evaluate it for

(i) $x = 2$ and $dx = 0.1$ **Mar2020 2M** & **May 2022 - 2M** (ii) $x = 3$ and $dx = 0.02$ **July-2022 2M**

3. Find Δf and df for the function f for the indicated values of x, Δx and compare

(i) $f(x) = x^3 - 2x^2$; $x = 2$, $\Delta x = dx = 0.5$ (ii) $f(x) = x^2 + 2x + 3$; $x = -0.5$, $\Delta x = dx = 0.1$

4. Assuming $log_{10}e = 0.4343$, find an approximate value of $log_{10}1003$.

5. The trunk of a tree has diameter 30cm. During the following year, the circumference grew 6cm. (i) Approximately, how much did the tree's diameter grow? (ii) What is the percentage increase in area of the tree's cross-section?

6. An egg of a particular bird is very nearly spherical. If the radius to the inside of the shell is 5mm and radius to the outside of the shell is 5.3mm, find the volume of the shell approximately.

7. Assume that the cross section of the artery of human is circular. A drug is given to a patient to dilate his arteries. If the radius of an artery is increased from 2mm to 2.1mm, how much is cross-sectional area increased approximately? **May 2022 - 3M**

8. In a newly developed city, it is estimated that the voting population (in thousands) will increase according to $V(t) = 30 + 12t^2 - t^3$, $0 \le t \le 8$ where t is the time in years. Find the approximate change in voters for the time change from 4 to $4\frac{1}{6}$ year.

9. The relation between the number of words y a person learns in x hours is given by $y = 52\sqrt{x}$, $0 \le x \le 9$. What is the approximate number of words learned when x changes from (i) 1 to 1.1 hour? (ii) 4 to 4.1 hour?

10. A circular plate expands uniformly under the influence of heat. If it's radius increases from 10.5cm to 10.75cm, then find an approximate change in the area and the approximate percentage change in the area. **July-2022 3M**

11. A coat of paint of thickness 0.2cm is applied to the faces of a cube whose edge is 10cm. Use the differentials to find approximately how many cubic centimetres of paint is used to paint this cube. Also calculate the exact amount of paint used to paint this cube.

Ex:8.8 Let $f(x, y) = \frac{3x-5y+8}{x^2+y^2+1}$ for all $(x, y) \in \mathbb{R}^2$. Show that f is continuous on $\mathbb{R}^2$.

Ex:8.9 Consider $f(x, y) = \frac{xy}{x^2+y^2}$ if $(x, y) \neq (0,0)$ and $f(0,0) = 0$. Show that f is not continuous at $(0,0)$ and continuous at all other points of $\mathbb{R}^2$.

Ex:8.10 Consider $g(x, y) = \frac{2x^2y}{x^2+y^2}$ if $(x, y) \neq (0,0)$ and $g(0,0) = 0$. Show that g is continuous on $\mathbb{R}^2$.

Exercise 8.3

1. Evaluate $\lim_{(x,y)\to(1,2)} g(x, y)$, if the limit exists, where $g(x, y) = \frac{3x^2-xy}{x^2+y^2+3}$.

2. Evaluate $\lim_{(x,y)\to(0,0)} cos\left(\frac{x^3+y^2}{x+y+2}\right)$ if the limit exists.

3. Let $f(x, y) = \frac{y^2-xy}{\sqrt{x}-\sqrt{y}}$ for $(x, y) \neq (0,0)$. Show that $\lim_{(x,y)\to(0,0)} f(x, y) = 0$.

4. Evaluate $\lim_{(x,y)\to(0,0)} cos\left(\frac{e^x siny}{y}\right)$, if the limit exists.

5. Let $g(x, y) = \frac{x^2y}{x^4+y^2}$ for $(x, y) \neq (0,0)$ and $g(0,0) = 0$. (i)Show that $\lim_{(x,y)\to(0,0)} g(x, y) = 0$ along every line $y = mx, m \in \mathbb{R}$. (ii) Show that $\lim_{(x,y)\to(0,0)} g(x, y) = \frac{k}{1+k^2}$ along every parabola $y = kx^2, k \in \mathbb{R}\backslash\{0\}$.

6. Show that $f(x, y) = \frac{x^2-y^2}{y^2+1}$ is continuous at every $(x, y) \in \mathbb{R}^2$.

7. Let $g(x,y) = \frac{e^y sinx}{x}$, for $x \neq 0$ and $g(0,0) = 1$. Show that g is continuous at (0,0).

Ex:8.11 Let $f(x,y) = 0$ if $xy \neq 0$ and $f(x,y) = 1$ if $xy = 0$. (i) Calculate: $\frac{\partial f}{\partial x}(0,0)$, $\frac{\partial f}{\partial y}(0,0)$. (ii) Show that f is not continuous at (0,0).

Ex:8.12 Let $F(x,y) = x^3y + y^2x + 7$ for all $(x,y) \in \mathbb{R}^2$. Calculate $\frac{\partial F}{\partial x}(-1,3)$ and $\frac{\partial F}{\partial y}(-2,1)$.

Ex:8.13 Let $f(x,y) = \sin(xy^2) + e^{x^3+5y}$ for all $(x,y) \in \mathbb{R}^2$. Calculate $\frac{\partial f}{\partial x}$, $\frac{\partial f}{\partial y}$, $\frac{\partial^2 f}{\partial y \partial x}$ and $\frac{\partial^2 f}{\partial x \partial y}$.

Ex:8.14 Let $w(x,y) = xy + \frac{e^y}{y^2+1}$ for all $(x,y) \in \mathbb{R}^2$. Calculate $\frac{\partial^2 w}{\partial y \partial x}$ and $\frac{\partial^2 w}{\partial x \partial y}$.

Ex:8.15 Let $u(x,y) = e^{-2y}\cos(2x)$ for all $(x,y) \in \mathbb{R}^2$. Prove that u is a harmonic function in $\mathbb{R}^2$.

Exercise 8.4

1. Find the partial derivatives of the following functions at the indicated points.
(i) $f(x,y) = 3x^2 - 2xy + y^2 + 5x + 2$, $(2,-5)$ (ii) $g(x,y) = 3x^2 + y^2 + 5x + 2$, $(1,-2)$
(iii) $h(x,y,z) = xsin(xy) + z^2x$, $(2,\frac{\pi}{4},1)$ (iv) $G(x,y) = e^{x+3y}\log(x^2+y^2)$, $(-1,1)$

2. For each of the following functions find the f_x, f_y and show that $f_{xy} = f_{yx}$.
(i) $f(x,y) = \frac{3x}{y+sinx}$ (ii) $f(x,y) = tan^{-1}\left(\frac{x}{y}\right)$ (iii) $f(x,y) = \cos(x^2 - 3xy)$

3. If $U(x,y,z) = \frac{x^2+y^2}{xy} + 3z^2y$, find $\frac{\partial U}{\partial x}$, $\frac{\partial U}{\partial y}$ *and* $\frac{\partial U}{\partial z}$.

4. If $U(x,y,z) = \log(x^3+y^3+z^3)$, find $\frac{\partial U}{\partial x} + \frac{\partial U}{\partial y} + \frac{\partial U}{\partial z}$. **Mar 2020-3M**

5. For each of the following functions find the g_{xy}, g_{xx}, g_{yy} and g_{yx}.
(i) $g(x,y) = xe^y + 3x^2y$ (ii) $g(x,y) = \log(5x+3y)$ (iii) $g(x,y) = x^2 + 3xy - 7y + \cos(5x)$

6. Let $w(x,y,z) = \frac{1}{\sqrt{x^2+y^2+z^2}}$, $(x,y,z) \neq (0,0,0)$. Show that $\frac{\partial^2 w}{\partial x^2} + \frac{\partial^2 w}{\partial y^2} + \frac{\partial^2 w}{\partial z^2} = 0$.

7. If $V(x,y) = e^x(xcosy - ysiny)$, then prove that $\frac{\partial^2 V}{\partial x^2} + \frac{\partial^2 V}{\partial y^2} = 0$.

8. If $w(x,y) = xy + \sin(xy)$, then prove that $\frac{\partial^2 w}{\partial y \partial x} = \frac{\partial^2 w}{\partial x \partial y}$.

9. If $v(x,y,z) = x^3 + y^3 + z^3 + 3xyz$, show that $\frac{\partial^2 v}{\partial y \partial z} = \frac{\partial^2 v}{\partial z \partial y}$.

10. A firm produces two types of calculators each week, x number of type A and y number of type B. The weekly revenue and cost functions (in rupees) are $R(x,y) = 80x + 90y + 0.04xy - 0.05x^2 - 0.05y^2$ and $C(x,y) = 8x + 6y + 2000$ respectively. (i) Find the profit function $P(x,y)$ (ii) Find $\frac{\partial P}{\partial x}(1200,1800)$ and $\frac{\partial P}{\partial y}(1200,1800)$ and interpret these results.

Ex:8.16 If $w(x,y,z) = x^2y + y^2z + z^2x, x,y,z \in \mathbb{R}$, find the differential dw.

Ex:8.17 Let $U(x,y,z) = x^2 - xy + 3sinz$, $x,y,z \in \mathbb{R}$. Find the linear approximation for U at $(2,-1,0)$.

Exercise 8.5

1. If $w(x,y) = x^3 - 3xy + 2y^2, x,y \in \mathbb{R}$, Find the linear approximation for w at $(1,-1)$.
2. Let $z(x,y) = x^2y + 3xy^4, x,y \in \mathbb{R}$, Find the linear approximation for z at $(2,-1)$.
3. If $v(x,y) = x^2 - xy + \frac{1}{4}y^2 + 7$, $x,y \in \mathbb{R}$, find the differential dv.
4. Let $W(x,y,z) = x^2 - xy + 3sinz, x,y,z \in \mathbb{R}$. Find the linear approximation at $(2,-1,0)$.
5. Let $V(x,y,z) = xy + yz + zx, x,y,z \in \mathbb{R}$. Find the differential dV.

Ex:8.18 Verify $\frac{\partial F}{\partial x}\frac{dx}{dt} + \frac{\partial F}{\partial y}\frac{dy}{dt} = \frac{dF}{dt}$ for the function $F(x,y) = x^2 - 2y^2 + 2xy$ and $x(t) = cost$, $y(t) = sint, t \in [0,2\pi]$.

Ex:8.19 Let $(x,y) = x^2 - yx + \sin(x+y), x(t) = e^{3t}, y(t) = t^2, t \in \mathbb{R}$. Find $\frac{dg}{dt}$.

Ex:8.20 Let $g(x,y) = 2y + x^2, x = 2r - s, y = r^2 + 2s, r,s \in \mathbb{R}$. Find $\frac{\partial g}{\partial r}, \frac{\partial g}{\partial s}$.

Exercise 8.6

1. If $u(x,y) = x^2y + 3xy^4$, $x = e^t$ and $y = sint$, find $\frac{du}{dt}$ and evaluate it at $t = 0$.

2. If $u(x,y,z) = xy^2z^3$, $x = sint$, $y = cost$, $z = 1 + e^{2t}$, find $\frac{du}{dt}$.

3. If $w(x,y,z) = x^2 + y^2 + z^2$, $x = e^t$, $y = e^t sint$, $z = e^t cost$, find $\frac{dw}{dt}$.

4. If $U(x,y,z) = xyz$, $x = e^{-t}$, $y = e^{-t}cost$, $z = sint$, $t\epsilon\mathbb{R}$. Find $\frac{dU}{dt}$.

5. If $w(x,y) = 6x^3 - 3xy + 2y^2$, $x = e^s$, $y = cos\mathbf{s}$, $s \in \mathbb{R}$, find $\frac{dw}{ds}$, and evaluate at $s = 0$.

6. If $z(x,y) = xtan^{-1}(xy)$, $x = t^2$, $y = se^t$, $s,t \in \mathbb{R}$. Find $\frac{\partial z}{\partial s}$ and $\frac{\partial z}{\partial t}$ at $s = t = 1$.

7. Let $U(x,y) = e^x siny$, where $x = st^2$, $y = s^2t$, $s,t \in \mathbb{R}$. Find $\frac{\partial U}{\partial s}$, $\frac{\partial U}{\partial t}$ and evaluate them at $s = t = 1$.

8. Let $z(x,y) = x^3 - 3x^2y^3$, where $x = se^t$, $y = se^{-t}$, $s,t \in \mathbb{R}$. Find $\frac{\partial z}{\partial s}$ and $\frac{\partial z}{\partial t}$.

9. $W(x,y,z) = xy + yz + zx$, $x = u - v$, $y = uv$, $z = u + v$, $u,v \in \mathbb{R}$. Find $\frac{\partial W}{\partial u}$, $\frac{\partial W}{\partial v}$, and evaluate them at $\left(\frac{1}{2}, 1\right)$.

Ex:8.21 Show that $F(x,y) = \frac{x^2+5xy-10y^2}{3x+7y}$ is a homogeneous function of degree 1.

Ex:8.22 If $u = sin^{-1}\left(\frac{x+y}{\sqrt{x}+\sqrt{y}}\right)$, Show that $x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} = \frac{1}{2}tanu$.

Exercise 8.7

1. In each of the following cases, determine whether the following function is homogeneous or not. If it is so, find the degree.

(i) $f(x,y) = x^2y + 6x^3 + 7$ (ii) $h(x,y) = \frac{6x^2y^3-\pi y^5+9x^4y}{2020x^2+2019y^2}$

(iii) $g(x,y,z) = \frac{\sqrt{3x^2+5y^2+z^2}}{4x+7y}$ (iv) $U(x,y,z) = xy + sin\left(\frac{y^2-2z^2}{xy}\right)$.

2. Prove that $f(x,y) = x^3 - 2x^2y + 3xy^2 + y^3$ is homogeneous; what is the degree? Verify Euler's Theorem for f.

3. Prove that $g(x,y) = xlog\left(\frac{y}{x}\right)$ is homogeneous; what is the degree? Verify Euler's Theorem for g.

4. If $u(x,y) = \frac{x^2+y^2}{\sqrt{x+y}}$, prove that $x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} = \frac{3}{2}u$.

5. If $v(x,y) = log\left(\frac{x^2+y^2}{x+y}\right)$, prove that $x\frac{\partial v}{\partial x} + y\frac{\partial v}{\partial y} = 1$.

6. If $w(x,y,z) = log\left(\frac{5x^3y^4+7y^2xz^4-75y^3z^4}{x^2+y^2}\right)$, find $x\frac{\partial w}{\partial x} + y\frac{\partial w}{\partial y} + z\frac{\partial w}{\partial z}$.

9.APPLICATIONS OF INTEGRATION

Theorem 9.1: First Fundamental Theorem of Integral Calculus.

Theorem 9.2: Second Fundamental Theorem of Integral Calculus.

Property6: $\int_a^b f(x)dx = \int_a^b f(a+b-x)\,dx$ **Property7:** $\int_0^{2a} f(x)dx = \int_0^a [f(x) + f(2a-x)]dx$.

Property8: If $f(x)$ is an even function, then $\int_{-a}^a f(x)dx = 2\int_0^a f(x)dx$.

Property9: If $f(x)$ is an odd function, then $\int_{-a}^a f(x)dx = 0$.

Property10: If $f(2a-x) = f(x)$, then $\int_0^{2a} f(x)dx = 2\int_0^a f(x)dx$.

Property11: If $f(2a-x) = -f(x)$, then $\int_0^{2a} f(x)dx = 0$.

Property12: $\int_0^a xf(x)dx = \frac{a}{2}\int_0^a f(x)dx$. if $f(a-x) = f(x)$.

Define: Gamma Integral Prove that $\Gamma(n+1) = n\Gamma n$.

Ex:9.1 Estimate the value of $\int_0^{0.5} x^2\,dx$ using the Riemann sums corresponding to 5 subintervals of equal width and applying (i) left-end rule (ii) right-end rule (iii) the mid-point rule.

Exercise 9.1

1. Find an approximate value of $\int_1^{1.5} x dx$ by applying the left-end rule with the partition $\{1.1, 1.2, 1.3, 1.4, 1.5\}$.

2. Find an approximate value of $\int_1^{1.5} x^2 dx$ by applying the right-end rule with the partition $\{1.1, 1.2, 1.3, 1.4, 1.5\}$.

3. Find an approximate value of $\int_1^{1.5} (2-x) dx$ by applying the mid-point rule with the partition $\{1.1, 1.2, 1.3, 1.4, 1.5\}$.

Ex:9.2 Evaluate $\int_0^1 x dx$, as the limit of a sum. **Ex:9.3** Evaluate $\int_0^1 x^3 dx$, as the limit of a sum.

Ex:9.4 Evaluate $\int_1^4 (2x^2+3) dx$, as the limit of a sum.

Exercise 9.2

1. Evaluate the following integrals as the limit of sums: (i) $\int_0^1 (5x+4) dx$ (ii) $\int_1^2 (4x^2-1) dx$

Ex:9.5 Evaluate $\int_0^3 (3x^2-4x+5) dx$. **Ex:9.6** Evaluate $\int_0^1 \frac{2x+7}{5x^2+9} dx$.

Ex:9.7 Evaluate $\int_0^1 [2x] dx$ where $[\cdot]$ is the greatest integer function.

Ex:9.8 Evaluate $\int_0^{\frac{\pi}{3}} \frac{secx\, tanx}{1+sec^2x} dx$. **May 2022 - 3M** **Ex:9.9** Evaluate $\int_0^9 \frac{1}{x+\sqrt{x}} dx$.

Ex:9.10 Evaluate $\int_1^2 \frac{x}{(x+1)(x+2)} dx$. **Ex:9.11** Evaluate: $\int_0^{\frac{\pi}{2}} \frac{cos\theta}{(1+sin\theta)(2+sin\theta)} d\theta$.

Ex:9.12 Evaluate: $\int_0^{\frac{1}{\sqrt{2}}} \frac{sin^{-1}x}{(1-x^2)^{\frac{3}{2}}} dx$. **Ex:9.13** Evaluate: $\int_0^{\frac{\pi}{2}} (\sqrt{tanx} + \sqrt{cotx})\, dx$.

Ex:9.14 Evaluate: $\int_0^{1.5} [x^2]\, dx$, where $[x]$ is the greatest integer function.

Ex:9.15 Evaluate: $\int_{-4}^4 |x+3|\, dx$. **Ex:9.16** Show that $\int_0^{\frac{\pi}{2}} \frac{dx}{4+5sinx} = \frac{1}{3}\log_e 2$.

Ex:9.17 Prove that $\int_0^{\frac{\pi}{4}} \frac{sin2x\, dx}{sin^4x+cos^4x} = \frac{\pi}{4}$. **Ex:9.19** Evaluate: $\int_0^{\frac{\pi}{4}} \frac{1}{sinx+cosx} dx$.

Ex:9.18 Prove that $\int_0^{\frac{\pi}{4}} \frac{dx}{a^2sin^2x+b^2cos^2x} = \frac{1}{ab} tan^{-1}\left(\frac{a}{b}\right)$, where $a, b > 0$.

Ex:9.20 Show that $\int_0^{\pi} g(sinx) dx = 2\int_0^{\frac{\pi}{2}} g(sinx) dx$, where $g(sinx)$ is a function of $sinx$.

Ex:9.21 Evaluate $\int_0^{\pi} \frac{x}{1+sinx} dx$.

Ex:9.22 Show that $\int_0^{2\pi} g(cosx) dx = 2\int_0^{\pi} g(osx) dx$, where $g(cosx)$ is a function of $cosx$.

Ex:9.23 If $f(x) = f(a+x)$, then prove that $\int_0^{2a} f(x)\, dx = 2\int_0^a f(x)\, dx$.

Ex:9.24 Evaluate: $\int_{\frac{-\pi}{2}}^{\frac{\pi}{2}} x\, cosx\, dx$. **Ex:9.25** Evaluate: $\int_{-log2}^{log2} e^{-|x|}\, dx$.

Ex:9.26 Evaluate: $\int_0^a \frac{f(x)}{f(x)+f(a-x)} dx$. **May 2022 - 5M**

Ex 9.27: Prove that $\int_0^{\pi/4} \log(1+tanx)\, dx = \frac{\pi}{8} log2$.

Ex 9.28: Show that $\int_0^1 (tan^{-1}x + tan^{-1}(1-x))\, dx = \frac{\pi}{2} - log_e 2$. **Sep 2020 – 5M**

Ex 9.29: Evaluate $\int_2^3 \frac{\sqrt{x}}{\sqrt{5-x}+\sqrt{x}} dx$. **Ex 9.30:** Evaluate $\int_{-\pi}^{\pi} \frac{cos^2x}{1+a^x} dx$. **Mar 2020-5M**

Exercise 9.3

1. Evaluate the following definite integrals:

(i) $\int_3^4 \frac{dx}{x^2-4}$ **Sep 2020 – 2M** (ii) $\int_{-1}^1 \frac{dx}{x^2+2x+5}$ (iii) $\int_0^1 \sqrt{\frac{1-x}{1+x}} dx$ **Aug 2021- 5M**

(iv) $\int_0^{\pi/2} e^x \left(\frac{1+sinx}{1+cosx}\right) dx$ (v) $\int_0^{\pi/2} \sqrt{cos\theta}\, sin^3\theta d\theta$ (vi) $\int_0^1 \frac{1-x^2}{(1+x^2)^2} dx$

2. Evaluate the following integrals using properties of integration:

(i) $\int_{-5}^{5} x\cos x\left(\frac{e^x-1}{e^x+1}\right)dx$ (ii) $\int_{-\pi/2}^{\pi/2}(x^5+x\cos x+\tan^3 x+1)\,dx$ (iii) $\int_{-\pi/4}^{\pi/4}\sin^2 x dx$

(iv) $\int_0^{2\pi} x\log\left(\frac{3+\cos x}{3-\cos x}\right)dx$ (v) $\int_0^{2\pi}\sin^4 x\cos^3 x\,dx$ (vi) $\int_0^1|5x-3|dx$

(vii) $\int_0^{\sin^2 x}\sin^{-1}\sqrt{t}\,dt+\int_0^{\cos^2 x}\cos^{-1}\sqrt{t}\,dt$ (viii) $\int_0^1\frac{\log(1+x)}{1+x^2}dx$ (ix) $\int_0^{\pi}\frac{x\sin x}{1+\sin x}dx$

(x) $\int_{\pi/8}^{3\pi/8}\frac{1}{1+\sqrt{\tan x}}dx$ (xi) $\int_0^{\pi} x[\sin^2(\sin x)+\cos^2(\cos x)]\,dx$

Ex:9.31 Evaluate $\int_0^{\pi} x^2\cos nx\,dx$, where n is a positive integer.

Ex:9.32 Evaluate: $\int_0^1 e^{-2x}(1+x-2x^3)\,dx$.

Ex:9.33 Evaluate: $\int_0^{2\pi} x^2\sin nx\,dx$, where n is a positive integer.

Ex:9.34 Evaluate: $\int_{-1}^{1} e^{-\lambda x}(1-x^2)\,dx$.

Exercise 9.4

Evaluate the following:

1. $\int_0^1 x^3e^{-2x}\,dx$ 2. $\int_0^1\frac{\sin(3\tan^{-1}x)\tan^{-1}x}{1+x^2}dx$ 3. $\int_0^{\frac{1}{\sqrt{2}}}\frac{e^{\sin^{-1}x}\sin^{-1}x}{\sqrt{1-x^2}}dx$ 4. $\int_0^{\pi/2} x^2\cos 2x\,dx$

Ex:9.35 Evaluate $\int_b^{\infty}\frac{1}{a^2+x^2}dx, a>0, b\in\mathbb{R}$. **Mar 2023- 2M**

Ex:9.36 Evaluate $\int_0^{\pi/2}\frac{dx}{4\sin^2 x+5\cos^2 x}$.

Exercise 9.5

1. Evaluate the following: (i) $\int_0^{\pi/2}\frac{dx}{1+5\cos^2 x}$ (ii) $\int_0^{\pi/2}\frac{dx}{5+4\sin^2 x}$

Ex:9.37 Evaluate $\int_0^{\pi/2}(\sin^2 x+\cos^4 x)\,dx$. **Ex:9.38** Evaluate $\int_0^{\pi/2}\begin{vmatrix}\cos^4 x & 7\\ \sin^5 x & 3\end{vmatrix}dx$.

Ex:9.39 Find the values of the following: (i) $\int_0^{\pi/2}\sin^5 x\cos^4 x\,dx$ (ii) $\int_0^{\pi/2}\sin^4 x\cos^6 x\,dx$

Ex:9.40 Evaluate $\int_0^{2a} x^2\sqrt{2ax-x^2}\,dx$. **Ex:9.41** Evaluate $\int_0^1 x^5(1-x^2)^5\,dx$.

Ex:9.42 Evaluate $\int_0^1 x^3(1-x)^4\,dx$.

Exercise 9.6

1. Evaluate the following: (i) $\int_0^{\pi/2}\sin^{10}x\,dx$ (ii) $\int_0^{\pi/2}\cos^7 x\,dx$ (iii) $\int_0^{\pi/4}\sin^6 2x\,dx$

(iv) $\int_0^{\pi/6}\sin^5 3x\,dx$ (v) $\int_0^{\pi/2}\sin^2 x\cos^4 x\,dx$ (vi) $\int_0^{2\pi}\sin^7\frac{x}{4}dx$ (vii) $\int_0^{\pi/2}\sin^3\theta\cos^5\theta\,d\theta$

(viii) $\int_0^1 x^2(1-x)^3\,dx$

Ex:9.43 Prove that $\int_0^{\infty} e^{-x}x^n\,dx=n!$, where n is a positive integer.

Ex:9.44 Evaluate $\int_0^{\infty} e^{-ax}x^n\,dx$, where $a>0$. **Ex:9.45** Show that $\Gamma(n)=2\int_0^{\infty} e^{-x^2}x^{2n-1}\,dx$.

Ex:9.46 Evaluate $\int_0^{\infty}\frac{x^n}{n^x}dx$, where n is a positive integer ≥ 2.

Exercise 9.7

1. Evaluate the following: (i) $\int_0^{\infty} x^5e^{-3x}dx$ (ii) $\int_0^{\pi/2}\frac{e^{-\tan x}}{\cos^6 x}dx$

2. If $\int_0^{\infty} e^{-\alpha x^2}x^3\,dx=32, \alpha>0$, find α.

Ex:9.47 Find the area of the region bounded by the line $6x+5y=30$, x -axis and the lines $x=-1$ and $x=3$.

Ex:9.48 Find the area of the region bounded by the line $7x-5y=35$, x -axis and the lines $x=-2$ and $x=3$.

Ex:9.49 Find the area of the region bounded by the ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$. **July-2022 5M**

Ex:9.50 Find the area of the region bounded between the parabola $y^2=4ax$ and its latus rectum.

Ex:9.51 Find the area of the region bounded by the y- axis and the parabola $x=5-4y-y^2$.

Ex:9.52 Find the area of the region bounded by x- axis, the sine curve $y = sinx$, the lines $x = 0$ and $x = 2\pi$.

Ex:9.53 Find the area of the region bounded by x-axis, the curve $y = |cosx|$, the lines $x = 0$ and $x = \pi$. **Mar 2020-5M**

Ex:9.54 Find the area of the region bounded between the parabolas $y^2 = 4x$ and $x^2 = 4y$.

Ex:9.55 Find the area of the region bounded between the parabola $x^2 = y$ and the curve $y = |x|$.

Ex:9.56 Find the area of the region bounded by $y = cosx, y = sinx$, the lines $x = \frac{\pi}{4}$ and $x = \frac{5\pi}{4}$.

Ex:9.57 The region enclosed by the circle $x^2 + y^2 = a^2$ is divided into two segments by the line $x = h$. Find the area of the smaller segment.

Ex:9.58 Find the area of the region in the first quadrant bounded by the parabola $y^2 = 4x$, the line $x + y = 3$ and y-axis.

Ex:9.59 Find, by integration, the area of the region bounded by the lines $5x - 2y = 15, x + y + 4 = 0$ and the x-axis.

Ex:9.60 Using integration, find the area of the region bounded by triangle ABC, whose vertices A,B and C are $(-1,1), (3,2)$ and $(0,5)$ respectively.

Ex:9.61 Using integration, find the area of the region which is bounded by x-axis, the tangent and normal to the circle $x^2 + y^2 = 4$ drawn at $(1, \sqrt{3})$.

Exercise 9.8

1. Find the area of the region bounded by $3x - 2y + 6 = 0$, $x = -3$, $x = 1$ and x –axis. **Aug 2021-5M & May2022-5M**
2. Find the area of the region bounded by $2x - y + 1 = 0$, $y = -1$, $y = 3$ and y –axis.
3. Find the area of the region bounded by the curve $2 + x - x^2 + y = 0$, x-axis, $x = -3$, and $x = 3$.
4. Find the area of the region bounded by the line $y = 2x + 5$ and the parabola $y = x^2 - 2x$.
5. Find the area of the region bounded between the curves $y = sinx$ and $y = cosx$ and the lines $x = 0$, and $x = \pi$.
6. Find the area of the region bounded by $y = tanx, y = cotx$ and the lines $x = 0$, $x = \frac{\pi}{2}$, $y = 0$.
7. Find the area of the region bounded by the parabola $y^2 = x$ and the line $y = x - 2$.
8. Father of a family wishes to divide his square field bounded by $x = 0$, $x = 4$, $y = 4$ and $y = 0$ along the curve $y^2 = 4x$ and $x^2 = 4y$ into three equal parts for his wife, daughter and son. Is it possible to divide? If so, find the area to be divided among them.
9. The curve $y = (x - 2)^2 + 1$ has a minimum point at P. A point Q on the curve is such that the slope of PQ is 2. Find the area bounded by the curve and the chord PQ.
10. Find the area of the region common to the circle $x^2 + y^2 = 16$ and the parabola $y^2 = 6x$.

Ex:9.62 Find the volume of a sphere of radius a.

Ex:9.63 Find the volume of a right-circular cone of base radius r and height h.

Ex:9.64 Find the volume of the spherical cap of height h cut of from a sphere of radius r.

Ex:9.65 Find the volume of the solid formed by revolving the region bounded by the parabola $y = x^2$, x-axis, ordinates $x = 0$ and $x = 1$ about x-axis.

Ex:9.66 Find the volume of the solid formed by revolving the region bounded by the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$, $a > b$ about the major axis.

Ex:9.67 Find, by integration, the volume of the solid generated by revolving about y-axis the region bounded between the parabola $x = y^2 + 1$, the y-axis, and the lines $y = 1$ and $y = -1$.

Ex:9.68 Find, by integration, the volume of the solid generated by revolving about y-axis the region bounded between the curve $y = \frac{3}{4}\sqrt{x^2 - 16}$, $x \geq 4$, the y-axis, and the lines $y = 1$ and $y = 6$.

Ex:9.69 Find, by integration, the volume of the solid generated by revolving about y-axis the region bounded by the curves $y = logx, y = 0, x = 0$ and $y = 2$.

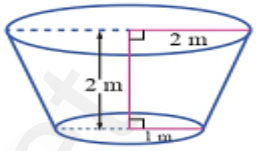
Exercise 9.9

1. Find, by integration, the volume of the solid generated by revolving about the x-axis, the region enclosed by $y = 2x^2, y = 0$ and $x = 1$.

2. Find, by integration, the volume of the solid generated by revolving about the x-axis, the region enclosed by $y = e^{-2x}, y = 0, x = 0$ and $x = 1$.

3. Find, by integration, the volume of the solid generated by revolving about y-axis the region enclosed by $x^2 = 1 + y$ and $y = 3$.

4. The region enclosed between the graphs of $y = x$ and $y = x^2$ is denoted by R, Find the volume generated when R is rotated through 360° about x-axis.



5. Find, by integration, the volume of the container which is in the shape of a right circular conical frustum as shown in the above figure.

6. A watermelon has an ellipsoid shape which can be obtained by revolving an ellipse with major-axis 20cm and minor-axis 10cm about its major-axis. Find its volume using integration.

10. ORDINARY DIFFERENTIAL EQUATIONS

Ex:10.1 Determine the order and degree (if exists) of the following differential equations:

(i) $\frac{dy}{dx} = x + y + 5$ (ii) $\left(\frac{d^4y}{dx^4}\right)^3 + 4\left(\frac{dy}{dx}\right)^7 + 6y = 5cos3x$ (iii) $\frac{d^2y}{dx^2} + 3\left(\frac{dy}{dx}\right)^2 = x^2 log\left(\frac{d^2y}{dx^2}\right)$

(iv) $3\frac{d^2y}{dx^2} = \left[4 + \left(\frac{dy}{dx}\right)^2\right]^{\frac{3}{2}}$ (v) $dy + (xy - cosx)dx = 0$

Exercise 10.1

1. For each of the following differential equations, determine its order, degree (if exists)

(i) $\frac{dy}{dx} + xy = cotx$ (ii) $\left(\frac{d^3y}{dx^3}\right)^{\frac{2}{3}} - 3\frac{d^2y}{dx^2} + 5\frac{dy}{dx} + 4 = 0$ (iii) $\left(\frac{d^2y}{dx^2}\right)^2 + \left(\frac{dy}{dx}\right)^2 = xsin\left(\frac{d^2y}{dx^2}\right)$

(iv) $\sqrt{\frac{dy}{dx}} - 4\frac{dy}{dx} - 7x = 0$ (v) $y\left(\frac{dy}{dx}\right) = \frac{x}{\left(\frac{dy}{dx}\right) + \left(\frac{dy}{dx}\right)^3}$ (vi) $x^2\frac{d^2y}{dx^2} + \left[1 + \left(\frac{dy}{dx}\right)^2\right]^{\frac{1}{2}}$ (vii) $\left(\frac{d^2y}{dx^2}\right)^3 = \sqrt{1 + \frac{dy}{dx}}$

(viii) $\frac{d^2y}{dx^2} = xy + cos\left(\frac{dy}{dx}\right)$ (ix) $\frac{d^2y}{dx^2} + 5\frac{dy}{dx} + \int ydx = x^3$ (x) $x = e^{xy\left(\frac{dy}{dx}\right)}$

Exercise 10.2

1. Express each of the following physical statements in the form of differential equation.

(i) Radium decays at a rate proportional to the amount Q present.

(ii)The population P of a city increases at a rate proportional to the product of population and to the difference between 5,00,000 and the population.

(iii) For a certain substance, the rate of change of vapour pressure P with respect to temperature T is proportional to the vapour pressure and inversely proportional to the square of the temperature.

(iv) A saving amount pays 8% interest per year, compounded continuously. In addition, the income from another investment is credited to the amount continuously at the rate of Rs400 per year.

2. Assume that a spherical rain drop evaporates at a rate proportional to its surface area. Form a differential equation involving the rate of change of the radius of the rain drop.

Ex:10.2 Find the differential equation for the family of all straight lines passing through the origin.

Ex:10.3 Form the differential equation by eliminating the arbitrary constants A and B from $y = Acosx + Bsinx$.

Ex:10.4 Find the differential equation of the family of circles passing through the points $(a, 0)$ and $(-a, 0)$.

Ex:10.5 Find the differential equation of the family of parabolas $y^2 = 4ax$, where a is an arbitrary constant. **Mar 2020-2M**

Ex:10.6 Find the differential equation of the family of all ellipses having foci on the x-axis and centre at the origin.

Exercise 10.3

1. Find the differential equation of the family of (i) all non-vertical lines in a plane (ii) all non-horizontal lines in a plane.
2. Find the differential equation of all straight lines touching the circle $x^2 + y^2 = r^2$.
3. Find the differential equation of the family of circles passing through the origin and having their centres on the x-axis.
4. Find the differential equation of the family of all the parabolas with latus rectm $4a$ and whose axes are parallel to the x-axis.
5. Find the differential equation of the family of parabolas with vertex at $(0, -1)$ and having axis along the y-axis.
6. Find the differential equations of the family of all the ellipses having foci on the y-axis and centre at the origin.
7. Find the differential equation corresponding to the family of curves represented by the equation $y = Ae^{8x} + Be^{-8x}$, where A and B are arbitrary constants.
8. Find the differential equation of the curve represented by $xy = ae^x + be^{-x} + x^2$.

Ex:10.7 Show that $x^2 + y^2 = r^2$, where r is a constant, is a solution of the differential equation $\frac{dy}{dx} = -\frac{x}{y}$.

Ex:10.8 Show that $y = mx + \frac{7}{m}, m \neq 0$ is a solution of the differential equation $xy' + 7\frac{1}{y'} - y = 0$.

Ex:10.9 Show that $y = 2(x^2 - 1) + Ce^{-x^2}$ is a solution of the differential equation $\frac{dy}{dx} + 2xy - 4x^3 = 0$.

Ex:10.10 Show that $y = \text{acos}(logx) + bsin(logx), x > 0$ is a solution of the differential equation $x^2y'' + xy' + y = 0$.

Exercise 10.4

1. Show that each of the following expressions is a solution of the corresponding given differential equation. (i) $y = 2x^2$; $xy' = 2y$ (ii) $y = ae^x + be^{-x}$; $y'' - y = 0$ **July-2022 2M**
2. Find the value of m so that the function $y = e^{mx}$ is a solution of the given differential equation. (i) $y' + 2y = 0$ (ii) $y'' - 5y' + 6y = 0$
3. The slope of the tangent to the curve at any point is the reciprocal of four times the ordinate at that point. The curve passes through (2,5). Find the equation of the curve.
4. Show that $y = e^{-x} + mx + n$ is a solution of the differential equation $e^x\left(\frac{d^2y}{dx^2}\right) - 1 = 0$.
5. Show that $y = ax + \frac{b}{x}, x \neq 0$ is a solution of the differential equation $x^2y'' + xy' - y = 0$.
6. Show that $y = ae^{-3x} + b$, where a and b are arbitrary constants, is a solution of the differential equation $\frac{d^2y}{dx^2} + 3\frac{dy}{dx} = 0$.
7. Show that the differential equation representing the family of curves $y^2 = 2a\left(x + a^{\frac{2}{3}}\right)$, where a is a positive parameter, is $\left(y^2 - 2xy\frac{dy}{dx}\right)^3 = 8\left(y\frac{dy}{dx}\right)^5$.
8. Show that $y = acosbx$ is a solution of the differential equation $\frac{d^2y}{dx^2} + b^2y = 0$.

Ex:10.11 Solve: $(1 + x^2)\frac{dy}{dx} = 1 + y^2$. **May 2022 - 5M**

Ex:10.12 Find the particular solution of $(1 + x^3)dy - x^2ydx = 0$ satisfying the condition $y(1) = 2$.

Ex:10.13 Solve: $y' = sin^2(x - y + 1)$.

Ex:10.14 Solve: $\frac{dy}{dx} = \sqrt{4x + 2y - 1}$.

Ex:10.15 Solve: $\frac{dy}{dx} = \frac{x-y+5}{2(x-y)+7}$.

Ex:10.16 Solve: $\frac{dy}{dx} = (3x + y + 4)^2$.

Exercise 10.5

1. If F is the constant force generated by the motor of an automobile of mass M, its velocity V is given by $M\frac{dV}{dt} = F - kV$, where k is a constant. Express V in terms of t given that $V = 0$ when $t = 0$. **Mar2020- 5M**

2. The velocity v, of a parachute falling vertically satisfies the equation $v\frac{dv}{dx} = g\left(1 - \frac{v^2}{k^2}\right)$, where g and k are constants. If v and x are both initially zero, find v in terms of x.

3. Find the equation of the curve whose slope is $\frac{y-1}{x^2+x}$ and which passes through the point (1,0).

4. Solve the following differential equations:

(i) $\frac{dy}{dx} = \sqrt{\frac{1-y^2}{1-x^2}}$ **Aug 2021-2M & May 2022 - 2M** (ii) $ydx + (1+x^2)tan^{-1}xdy = 0$

(iii) $sin\frac{dy}{dx} = a, y(0) = 1$ (iv) $\frac{dy}{dx} = e^{x+y} + x^3e^y$ **Sep 2020 – 5M**

(v) $(e^y + 1)cosxdx + e^y sinxdy = 0$ **Aug 2021- 5M**

(vi) $(ydx - xdy)cot\left(\frac{x}{y}\right) = ny^2dx$ (vii) $\frac{dy}{dx} - x\sqrt{25 - x^2} = 0$

(viii) $xcosydy = e^x(xlogx + 1)dx$ **Mar 2023 -3M**

(ix) $tany\frac{dy}{dx} = \cos(x+y) + cos(x-y)$ (x) $\frac{dy}{dx} = tan^2(x+y)$

Ex:10.17 Solve $(x^2 - 3y^2)dx + 2xydy = 0$.

Ex:10.18 Solve $\left(y + \sqrt{x^2+y^2}\right)dx - xdy = 0, y(1) = 0$.

Ex:10.19 Solve: $(2x + 3y)dx + (y - x)dy = 0$. **Ex:10.20** Solve: $y^2 + x^2\frac{dy}{dx} = xy\frac{dy}{dx}$.

Ex:10.21 Solve: $\left(1 + 2e^{\frac{x}{y}}\right)dx + 2e^{\frac{x}{y}}\left(1 - \frac{x}{y}\right)dy = 0$.

Exercise 10.6

Solve the following differential equations:

1. $\left[x + ycos\left(\frac{y}{x}\right)\right]dx = xcos\left(\frac{y}{x}\right)dx$ 2. $(x^3 + y^3)dy - x^2ydx = 0$

3. $ye^{\frac{x}{y}}dx = \left(xe^{\frac{x}{y}} + y\right)dy$ 4. $2xydx + (x^2 + 2y^2)dy = 0$

5. $(y^2 - 2xy)dx = (x^2 - 2xy)dy$ 6. $x\frac{dy}{dx} = y - xcos^2\left(\frac{y}{x}\right)$

7. $\left(1 + 3e^{\frac{y}{x}}\right)dy + 3e^{\frac{y}{x}}\left(1 - \frac{y}{x}\right)dx = 0$, given that $y = 0$ when$x = 1$.

8. $(x^2 + y^2)dy = xydx$. It is given that $y(1) = 1$ and $y(x_0) = e$. Find the value of x_0.

Ex:10.22 Solve: $\frac{dy}{dx} + 2y = e^{-x}$. **Ex:10.23** Solve: $[y(1 - xtanx) + x^2cosx]dx - xdy = 0$.

Ex:10.24 Solve: $\frac{dy}{dx} + 2ycotx = 3x^2cosec^2x$. **Ex:10.25** Solve: $(1 + x^3)\frac{dy}{dx} + 6x^2y = 1 + x^2$.

Ex:10.26 Solve: $ye^ydx = (y^3 + 2xe^y)dy$.

Exercise 10.7

Solve the following differential equations:

1. $cosx\frac{dy}{dx} + ysinx = 1$ 2. $(1 - x^2)\frac{dy}{dx} - xy = 1$

3. $\frac{dy}{dx} + \frac{y}{x} = sinx$ **July-2022 5M** 4. $(x^2 + 1)\frac{dy}{dx} + 2xy = \sqrt{x^2 + 4}$

5. $(2x - 10y^3)dy + ydx = 0$ 6. $xsinx\frac{dy}{dx} + (xcosx + inx)y = sinx$

7. $\left(y - e^{sin^{-1}x}\right)\frac{dx}{dy} + \sqrt{1 - x^2} = 0$ 8. $\frac{dy}{dx} + \frac{y}{(1-x)\sqrt{x}} = 1 - \sqrt{x}$

9. $(1 + x + xy^2)\frac{dy}{dx} + (y + y^3) = 0$ **Mar 2023 - 5M** 10. $\frac{dy}{dx} + \frac{y}{xlogx} = \frac{sin2x}{logx}$

11. $(x + a)\frac{dy}{dx} - 2y = (x + a)^4$ 12. $\frac{dy}{dx} = \frac{sin^2x}{1+x^3} - \frac{3x^2}{1+x^3}y$

13. $x\frac{dy}{dx} + y = xlogx$ 14. $x\frac{dy}{dx} + 2y - x^2logx = 0$

15. $\frac{dy}{dx} + \frac{3y}{x} = \frac{1}{x^2}$, given that $y = 2$ when $x = 1$.

Ex:10.27 The growth of a population is proportional to the number present. If the population of a colony doubles in 50 years, in how many years will the population become triple?

Ex:10.28 A radioactive isotope has an initial mass $200mg$, which two years later is $50mg$. Find the expression for the amount of the isotope remaining at any time. What is its half-life? (half-life means the time taken for the radioactivity of a specified isotope to fall to half its original value).

Ex:10.29 In a murder investigation, a corpse was found by a defective at exactly $8\,p.m.$ Being alert, the detective also measured the body temperature and found it to be 70°F. Two hours later, the detective measured the body temperature again and found it to be 60°F. If the room temperature is 50°F, and assuming that the body temperature of the person before death was 98.6°F, at what time did the murder occur? $[log(2.43) = 0.88789; \log(0.5) = -0.69315]$ **Mar 2020-5M**

Ex10.30 A tank contains 1000 litres of water in which 100 grams of salt is dissolved. Brine (*Brine is a high-concentration solution of salt(usually sodium chloride) in water*) runs in a rate of 10 litres per minute, and each litre contains 5 grams of dissolved salt. The mixure of the tank is kept uniform by stirring. Brine runs out at 10 litres per minute. Find the amount of salt at any time t.

Exercise 10.8

1. The rate of increase in the number of bacteria in a certain bacteria culture is proportional to the number present. Given that the number triples in 5 hours, find how many bacteria will be present after 10 hours?

2. Find the population of a city at any time t, given that the rate of increase of population is proportional to the population at that instant and that in a period of 40 years the population increased from 3,00,000 to 4,00,000.

3. The equation of electromotive force for an electric circuit containing resistance and self-inductance is $E = Ri + L\frac{di}{dt}$, where E is the electromotive force is given to the circuit, R the resistance and L, the coefficient of induction. Find the current i at time t when $E = 0$.

4. The engine of a motor boat moving at $10\,m/s$ is shut off. Given that the retardation at any subsequent time (after shutting off the engine) equal to the velocity at that time. Find the velocity after 2 seconds of switching off the engine.

5. Suppose a person deposits 10,000 Indian rupees in a bank account at the rate of 5% per annum compounded continuously. How much money will be in his bank account 18 months later?

Mar 2023 - 5M

6. Assume that the rate at which radioactive nuclei decay is proportional to the number of such nuclei that are present in a given sample. In a certain sample 10% of the original number of radioactive nuclei have undergone disintegration in a period of 100 years. What percentage of the original radioactive nuclei will remain after 1000 years?

7. Water at temperature 100°C cools in 10 minutes to 80°C in a room temperature of 25°C. Find (i) The temperature of water after 20 minutes (ii) The time when the temperature is 40°C.

$\left[log_e\frac{11}{15} = -0.3101;\ \ log_e 5 = 1.6094\right]$

8. At 10.00 A.M. a woman took a cup of hot instant coffee from her microwave oven and placed it on a nearby Kitchen counter to cool. At this instant the temperature of the coffee was 180°F, and 10 minutes later it was 160°F. Assume that constant temperature of the kitchen was 70°F. (i) What was the temperature of the coffee at 10.15 A.M.? (ii) The woman like to drink coffee when its temperature is between 130°F and 140°F, what times should she have drunk the coffee?

9. A pot of boiling water at 100°C is removed from a stove at time $t = 0$ and left to cool in the kitchen. After 5 minutes, the water temperature has decreased to 80°C, and another 5 minutes later it has dropped to 65°C. Determine the temperature of the kitchen. **Sep 2020 – 5M**

10. A tank initially contains 50 litres of pure water. Starting at time $t = 0$ a brine containing with 2

grams of dissolved salt per litre flows into the tank at the rate of 3 litres per minute. The mixture is kept uniform by stirring and the well-stirred mixture simultaneously flows out of the tank at the same rate. Find the amount of salt present in the tank at any time $t > 0$.

11. PROBABILITY DISTRIBUTIONS

Property1: $E(aX + b) = aE(X) + b$, where a and b are constants.

Property2: $Var(X) = E(X^2) - [E(X)]^2$

Property3: $Var(aX + b) = a^2 Var(X)$, where a and b are constants.

Ex:11.1 Suppose two coins are tossed once. If X denotes the number of tails, (i) write down the sample space (ii) find the inverse image of 1 (iii) the values of the random variable and number of elements in its inverse images.

Ex:11.2 Suppose a pair of unbiased dice is rolled once. If X denotes the total score of two dice, write down (i) the sample space (ii) the values taken by the random variable X (iii) the inverse image of 10, and (iv) the number of elements in inverse image of X.

Ex:11.3 An urn contains 2 white balls and 3 red balls. A sample of 3 balls are chosen at random from the urn. If X denotes the number of red balls chosen, find the values taken by the random variable X and its number of inverse images.

Ex:11.4 Two balls are chosen randomly from an urn containing 6 white and 4 black balls. Suppose that we win Rs30 for each black ball selected and we lose Rs20 for each white ball selected. If X denotes the winning amount, find the values of X and number of points in its inverse images.

Exercise 11.1

1. Suppose X is the number of tails occurred when three fair coins are tossed simultaneously. Find the values of the random variable X and number of points in its inverse images. **May 2022 - 2M**

2. In a pack of 52 playing cards, two cards are drawn at random simultaneously. If the number of black cards drawn is a random variable, find the values of the random variable and number of points in its inverse images.

3. An urn contains 5 mangoes and 4 apples. Three fruits are taken at random. If the number of apple taken is a random variable , then find the values of the random variable and number of points in its inverse images.

4. Two balls are chosen randomly from an urn containing 6 red and 8 black balls. Suppose that we win Rs15 for each red ball selected and we lose Rs10 for each black ball selected. If X denotes the winning amount, find the values of X and number of points in its inverse images.

5. A six sided die is marked '2' on one face, '3' on two of its faces, and '4' on remaining three faces. The die is thrown twice. If X denotes that total score in two throws, find the values of the random variable and number of points in its inverse images.

Ex:11.5 Two fair coins are tossed simultaneously(equivalent to a fair coin is tossed twice). Find the probability mass function for number of heads occurred.

Ex:11.6 A pair of fair dice is rolled once. Find the probability mass function to get the number of fours. **Sep 2020 – 3M**

Ex:11.7 If the probability mass function $f(x)$ of a random variable X is

x	1	2	3	4
$f(x)$	$\frac{1}{12}$	$\frac{5}{12}$	$\frac{5}{12}$	$\frac{1}{12}$

Find (i) its cumulative distribution function, hence find (ii) $P(X \leq 3)$ and (iii) $P(X \geq 2)$.

Ex:11.8 A six sided die is marked '1' on one face, '2' on two of its faces, and '3' on remaining three faces. The die is rolled twice. If X denotes the total score in two throws. (i) Find the probability mass function (ii) Find the cumulative distribution function (iii) Find $P(3 \leq X < 6)$ $(iv) Find$ $P(X \geq 4)$.

Ex:11.9 Find the probability mass function $f(x)$ of the discrete random varable X whose cumulative distribution function $F(x)$ is given by $F(x) = \begin{cases} 0 & -\infty < x < -2 \\ 0.25 & -2 \le x < -1 \\ 0.60 & -1 \le x < 0 \\ 0.90 & 0 \le x < 1 \\ 1 & 1 \le x < \infty \end{cases}$

Also find (i) $P(X < 0)$ and (ii) $P(X \ge -1)$.

Ex:11.10 A random variable X has the following probability mass function.

x	1	2	3	4	5	6
$f(x)$	k	$2k$	$6k$	$5k$	$6k$	$10k$

Find k **May 2022 - 2M** (i) $P(2 < X < 6)$ **-Mar2020-3M** (ii) $P(2 \le X < 5)$ (iii) $P(X \le 4)$ (iv) $P(3 < X)$

Exercise 11.2

1. Three fair coins are tossed simultaneously. Find the probability mass function for number of heads occurred.

2. A six sided die is marked '1' on one face, '3' on two of its faces, and '5' on remaining three faces. The die is thrown twice. If X denotes the total score in two throws, find (i) the probability mass function (ii) the cumulative distribution function (iii) $P(4 \le X < 10)$ (iv) $P(X \ge 6)$.

3. Find the probability mass function and cumulative distribution function of number of girl child in families with 4 children, assuming equal probabilities for boys and girls.

4. Suppose a discrete random variable can only take the values 0,1, and 2. The probability mass function is defined by $f(x) = \begin{cases} \frac{x^2+1}{k}, & for\ x = 0,1,2 \\ 0 & otherwise \end{cases}$

Find (i) the value of k (ii) cumulative distribution function (iii) $P(X \ge 1)$.

5. The cumulative distribution function of a discrete random variable is given by $F(x) = \begin{cases} 0 & -\infty < x < -1 \\ 0.15 & -1 \le x < 0 \\ 0.35 & 0 \le x < 1 \\ 0.60 & 1 \le x < 2 \\ 0.85 & 2 \le x < 3 \\ 1 & 3 \le x < \infty \end{cases}$ Find (i) the probability mass function (ii) $P(X < 1)$ (iii) $P(X \ge 2)$.

6. A random variable X has the following probability mass function

x	1	2	3	4	5
$f(x)$	k^2	$2k^2$	$3k^2$	$2k$	$3k$

Find (i) the value of k **July-2022 2M** (ii) $P(2 \le X < 5)$ (iii) $P(3 < X)$. **Aug 2021 - 5M**

7. The cumulative distribution function of a discrete random variable is given by

$$F(x) = \begin{cases} 0 & for -\infty < x < 0 \\ \frac{1}{2} & for\ 0 \le x < 1 \\ \frac{3}{5} & for\ 1 \le x < 2 \\ \frac{4}{5} & for\ 2 \le x < 3 \\ \frac{9}{10} & for\ 3 \le x < 4 \\ 1 & for\ 4 \le x < \infty \end{cases}$$

Find (i) the probability mass function (ii) $P(X < 3)$ (iii)$P(X \ge 2)$ **July-2022 5M**

Ex:11.11 Find the constant C such that the function $f(x) = \begin{cases} Cx^2 & 1 < x < 4 \\ 0 & Othewise \end{cases}$ is a density function, and compute (i) $P(1.5 < X < 3.5)$ (ii) $P(X \le 2)$ (iii) $P(3 < X)$.

Ex:11.12 If X is a random variable with probability density function $f(x)$ is given by, $f(x) = \begin{cases} x-1, & 1 \le x < 2 \\ -x+3, & 2 \le x < 3 \\ 0 & Otherwise \end{cases}$ Find (i) the distribution function F(x) (ii) $P(1.5 \le X \le 2.5)$.

Ex:11.13 If X is the random variable with distribution function F(x) given by, $F(x) = \begin{cases} 0, & x < 0 \\ x, & 0 \le x < 1 \\ 1, & 1 \le x \end{cases}$ then find (i) the probability density function $f(x)$ (ii) $P(0.2 \le X \le 0.7)$.

Ex:11.14 The probability density function of a random variable X is given by $f(x) = \begin{cases} k & 1 \le x \le 5 \\ 0 & Otherwise \end{cases}$ Find(i)Distribution function (ii)$P(X < 3)$ (iii) $P(2 < X < 4)$ (iv) $P(3 \le X)$.

Ex:11.15 Let X be a random variable denoting the life time of an electrical equipment having probability density function $f(x) = \begin{cases} ke^{-2x}, & x > 0 \\ 0, & x \le 0 \end{cases}$ Find (i) the value of k (ii) Distribution function (iii) $P(X < 2)$ (iv) calculate the probability that X is at least for four unit of time (v) $P(X = 3)$

Exercise 11.3

1. The probability density function of X is given by $f(x) = \begin{cases} kxe^{-2x}, & x > 0 \\ 0, & x \le 0 \end{cases}$ Find the value of k. **Aug 2021 - 2M**

2. The probability density function of X is $f(x) = \begin{cases} x, & 0 < x < 1 \\ 2-x, & 1 \le x < 2 \\ 0, & Otherwise \end{cases}$

Find (i) $P(0.2 \le X < 0.6)$ (ii) $P(1.2 \le X < 1.8)$ (iii) $P(0.5 \le X < 1.5)$

3. Suppose the amount of milk sold daily at a milk booth is distributed with a minimum of 200 liters and a maximum of 600 literswith probability density function $f(x) = \begin{cases} k & 200 \le x \le 600 \\ 0 & Otherwise \end{cases}$

Find (i) the value of k **July-2022 2M** (ii) the distribution function (iii) the probability that daily sales will fall between 300 liters and 500 liters? **Mar 2023 - 5M**

4. The probability density function of X is given by $f(x) = \begin{cases} ke^{-\frac{x}{3}}, & x > 0 \\ 0, & x \le 0 \end{cases}$ Find (i) the value of k (ii) the distribution function (iii) $P(X < 3)$ (iv) $P(5 \le X)$ (v) $P(X \le 4)$.

5. If X is the random variable with probability density function $f(x)$ given by, $f(x) = \begin{cases} x+1, & -1 \le x < 0 \\ -x+1, & 0 \le x < 1 \\ 0 & otherwise \end{cases}$ then find (i) the distribution function F(x) (ii) $P(-0.5 \le X \le 0.5)$

6. If X is a random variable with distribution function F(x) given by, $F(x) = \begin{cases} 0, & -\infty < x < 0 \\ \frac{1}{2}(x^2+x), & 0 \le x < 1 \\ 1, & 1 \le x < \infty \end{cases}$

then find (i) the probability density function $f(x)$ **Aug 2021 - 2M** (ii) $P(0.3 \le X \le 0.6)$

Ex:11.16 Suppose that f(x) given below represents a probability mass function,

x	1	2	3	4	5	6
$f(x)$	c^2	$2c^2$	$3c^2$	$4c^2$	c	$2c$

Find (i) the value of c (ii) Mean and Variance.

Ex:11.17 Two balls are chosen randomly from an urn containing 8 white and 4 black balls. Suppose that we win Rs.20 for each black ball selected and we lose Rs.10 for each white ball selected. Find the expected winning amount and variance..

Ex11.18 Find the mean and variance of a random variable X, whose probability density function is

$$f(x) = \begin{cases} \lambda e^{-\lambda x}, & for x \ge 0 \\ 0, & otherwise \end{cases}$$

Exercise 11.4

1. For the random variable X with the given probability mass function as below, find the mean and variance. (i) $f(x) = \begin{cases} \frac{1}{10}, & x = 2,5 \\ \frac{1}{5}, & x = 0,1,3,4 \end{cases}$ (ii) $f(x) = \begin{cases} \frac{4-x}{6}, & x = 1,2,3 \end{cases}$

(iii) $f(x) = \begin{cases} 2(x-1), & 1 < x < 2 \\ 0, & Otherwise \end{cases}$ (iv) $f(x) = \begin{cases} \frac{1}{2}e^{-\frac{x}{2}}, & x > 0 \\ 0, & Otherwise \end{cases}$

2. Two balls are drawn in succession without replacement from an urn containing 4 red balls and 3 black balls. Let X be the possible outcomes drawing red balls. Find the probability mass function and mean for X.

3. If μ and σ^2 are the mean and variance of the discrete random variable X, $E(X+3) = 10$ and $E(X+3)^2 = 116$, find μ and σ^2.

4. Four fair coins are tossed once. Find the probability mass function, mean and variance for number of heads occurred.

5. A commuter train arrives punctually at a station every half hour. Each morning , a student leaves his house to the train station. Let X denote the amount of time, in minutes, that the student waits for the train from the time he reaches the train station. It is known that the pdf of X is

$f(x) = \begin{cases} \frac{1}{30}, & 0 < x < 30 \\ 0, & elsewhere \end{cases}$ Obtain and interpret the expected value of the random variable X.

6. The time to failure in thousands of hours of an electronic equipment used in a manufactured computer has the density function $f(x) = \begin{cases} 3e^{-3x}, & for x > 0 \\ 0, & otherwise \end{cases}$

Find the expected life of this electronic equipment.

7. The probability density function of the random variable X is given by $f(x) = \begin{cases} 16xe^{-4x}, & x > 0 \\ 0, & x \le 0 \end{cases}$ find the mean and variance of X.

8. A lottery with 600 tickets gives one prize of Rs.200, four prizes of Rs.100, and six prizes of Rs.50. If the ticket costs is Rs.2, find the expected winning amount of a ticket.

Ex:11.19 Find the binomial distribution for each of the following. (i) Five fair coins are tossed once and X denotes the number of heads. (ii)A fair die is rolled 10 times and X denotes the number of times 4 appeared.

Ex:11.20 A multiple choice examination has ten questions, each question has four distracters with exactly one correct answer. Suppose a student answers by guessing and if X denotes the number of correct answers, find (i) binomial distribution (ii) probability that the student will get seven correct answers (iii) the probability of getting at least one correct answer.

Ex:11.21 The mean and variance of a binomial variate X are respectively 2 and 1.5

Find (i) $P(X = 0)$ **Sep 2020 – 3M** (ii) $P(X = 1)$ (iii) $P(X \ge 1)$

Ex:11.22 On the average,20% of the products manufactured by ABC company are found to be defective. If we select 6 of these products at random and X denotes the number of defective products, find the probability that (i) two products are defective (ii) at most one product is defective (iii) at least two products are defective.

Exercise 11.5

1. Compute $P(X = k)$ for the binomial distribution, B(n, p) where $(i) n = 6, p = \frac{1}{3}, k = 3$ $(ii)\ n = 10, p = \frac{1}{5}, k = 4$ $(iii)\ n = 9, p = \frac{1}{2}, k = 7$.

2. The probability that Mr.Q hits a target at any trail is $\frac{1}{4}$. Suppose he tries at the target 10 times. Find the probability that he hits the target (i) exactly 4 times (ii) at least one time.

3. Using binomial distribution find the mean and variance of X for the following experiments. (i) A

fair coin is tossed 100 times, and X denote the number of heads (ii) A fair die is rolled 240 times, and X denote the number of times that four appeared.

4. The probability that a certain kind of component will survive a electrical test is $\frac{3}{4}$. Find the probability that exactly 3 of the 5 components tested survive.

5. A retailer purchases a certain kind of electronic device from a manufacturer. The manufacturer indicates that the defective rate of the device is 5%. The inspector of the retailer randomly picks 10 items from a shipment. What is the probability that there will be (i) at least one defective item (ii) exactly two defective items?

6. If the probability that a fluorescent light has a useful life of at least 600 hours is 0.9, find the probabilities that among 12 such lights (i) exactly 10 will have a useful life of at least 600 hours (ii) at least 11 will have a useful life of at least 600 hours (iii) at least 2 will not have a useful life of at least 600 hours.

7. The mean and standard deviation of a binomial variate X are respectively 6 and 2. Find (i) the probability mass function (ii) $P(X = 3)$ (iii) $P(X \geq 2)$.

8. If $X \sim B(n, p)$ such that $4P(X = 4) = P(X = 2)$ and n=6. Find the distribution mean and standard deviation of X. **Sep 2020 – 5M**

9. In a binomial distribution consisting of 5 independent trails, the probability of 1 and 2 successes are 0.4096 and 0.2048 respectively. Find the mean and variance of the random variable.

12. DISCRETE MATHEMATICS

Theorem1: (Uniqueness of Identity) In an algebraic structure the identity element(if exists) must be unique. **Mar 2020- 2M**

Theorem2: (Uniqueness of Inverse) In an algebraic structure the inverse of an element(if exists) must be unique.

Ex:12.1 Examine the binary operation (closure property) of the following operations on the respective sets(if it is not, make it binary). (i) $a * b = a + 3ab - 5b^2,\ \forall a, b \in \mathbb{Z}$ (ii) $a * b = \left(\frac{a-1}{b-1}\right),\ \forall\, a, b \in \mathbb{Q}$. **Sep 2020 – 2M**

Ex:12.2 Verify the (i) closure property (ii) commutative property (iii) associative property (iv) existence of identity and (v) existence of inverse for the arithmetic operation + on $\mathbb{Z}$.

Ex:12.3 Verify the (i) closure property (ii) commutative property (iii) associative property (iv) existence of identity and (v) existence of inverse for the arithmetic operation $-$ on $\mathbb{Z}$.

Ex:12.4 Verify the (i) closure property (ii) commutative property (iii) associative property (iv) existence of identity and (v) existence of inverse for the arithmetic operation + on $\mathbb{Z}_e$ = the set of all even integers.

Ex:12.5 Verify the (i) closure property (ii) commutative property (iii) associative property (iv) existence of identity and (v) existence of inverse for the arithmetic operation + on $\mathbb{Z}_o$ = the set of all odd integers.

Ex:12.6 Verify (i) closure property (ii) commutative property (iii) associative property of the following operation on the given set. $a * b = a^b,\ \forall a, b \in \mathbb{N}$. **July-2022 3M**

Ex:12.7 Verify (i) closure property (ii) commutative property (iii) associative property (iv) existence of identity and (v) existence of inverse for the following operation on the given set $m * n = m + n - mn, \forall\, m, n \in \mathbb{Z}$.

Ex:12.8 Let $A = \begin{pmatrix} 0 & 1 \\ 1 & 1 \end{pmatrix}$ and $B = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$ be any two Boolean matrices of the same type. Find $A \vee B$ and $A \wedge B$. **Mar 2023 - 2M**

Ex:12.9 Verify (i) closure property (ii) commutative property (iii) associative property (iv) existence of identity and (v) existence of inverse for the operation $+_5$ on $\mathbb{Z}_5$ using table corresponding to addition modulo 5.

Ex:12.10 Verify (i) closure property (ii) commutative property (iii) associative property (iv) existence of identity and (v) existence of inverse for the operation $\times_{11}$ on a subset $A = \{1,3,4,5,9\}$ of the set of remainders $\{0,1,2,3,4,5,6,7,8,9,10\}$.

Exercise 12.1

1. Determine whether $*$ is a binary operation on the sets given below. (i) $a * b = a.|b|$ on $\mathbb{R}$.
(ii) $a * b = \min(a, b)$ on $A = \{1,2,3,4,5\}$ (iii) $a * b = a\sqrt{b}$ is binary on $\mathbb{R}$.

2. On $\mathbb{Z}$, define $\otimes$ by $m \otimes n = m^n + n^m$, $\forall\, m, n \in \mathbb{Z}$. Is $\otimes$ binary on $\mathbb{Z}$?

3. Let $*$ be defined on $\mathbb{R}$ by $a * b = a + b + ab - 7$. Is $*$ binary on $\mathbb{R}$? If so, find $3 * \left(\frac{-7}{15}\right)$.
May 2022 - 3M

4. Let $A = \{a + \sqrt{5}b : a, b \in \mathbb{Z}\}$. Check whether the usual multiplication is a binary operation on A.

5. Define on operation $*$ on $\mathbb{Q}$ as follows: $a * b = \left(\frac{a+b}{2}\right); a, b \in \mathbb{Q}$. (i) Examine the closure, commutative and associative properties satisfied by $*$ on $\mathbb{Q}$.
Aug 2021 - 3M
(ii) Examine the existence of identity and the existence of inverse for the operation $*$ on $\mathbb{Q}$.

$*$	a	b	c
a	b		
b	c	b	a
c	a		c

6.Fill in the following table so that the binary operation $*$ on A={a,b,c} is commutative.

7. Consider the binary operation $*$ defined on the set A={a, b, c, d} by the following table. Is it commutative and associative?

$*$	a	b	c	d
a	a	c	b	d
b	d	a	b	c
c	c	d	a	a
d	d	b	a	C

8. Let $A = \begin{pmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ 1 & 0 & 0 & 1 \end{pmatrix}$, $B = \begin{pmatrix} 0 & 1 & 0 & 1 \\ 1 & 0 & 1 & 0 \\ 1 & 0 & 0 & 1 \end{pmatrix}$, $C = \begin{pmatrix} 1 & 1 & 0 & 1 \\ 0 & 1 & 1 & 0 \\ 1 & 1 & 1 & 1 \end{pmatrix}$ be any three Boolean matrices of the same type. Find (i) $A \vee B$ (ii) $A \wedge B$ (iii) $(A \vee B) \wedge C$ (iv) $(A \wedge B) \vee C$.

9.Let $M = \left\{\begin{pmatrix} x & x \\ x & x \end{pmatrix} : x \in \mathbb{R} - \{0\}\right\}$ and let $*$ be the matrix multiplication. Determine whether M is closed under $*$. If so, (i) examine the commutative and associative properties satisfied by $*$ on M. (ii) examine the existence of identity and existence of inverse properties for the operation $*$ on M.

10. Let A be $\mathbb{Q}\backslash\{1\}$. Define $*$ on A by $x * y = x + y - xy$. Is $*$ binary on A? If so,(i) examine the commutative and associative properties satisfied by $*$ on A. (ii) examine the existence of identity and existence of inverse properties for the operation $*$ on A.

Ex:12.11 Identify the valid statements from the following sentences.
1) Mount Everest is the highest mountain of the world. 2) 3+4=8. 3) 7+5>10 4) Give me that book. 5) $10-x=7$ 6) How beautiful this flower is! 7) Where are you going? 8) Wish you all success. 9) This is the beginning of the end.

Ex:12.12 Write the statements in words corresponding to $\neg p$, $p \wedge q$, $p \vee q$ and $q \vee \neg p$, where p is 'It is cold' and q is 'It is raining'.

Ex:12.13 How many rows are needed for following statement formulae?
(i) $p \vee \neg t \wedge (p \vee \neg s)$ (ii) $((p \wedge q) \vee (\neg r \vee \neg s)) \wedge (\neg t \wedge v)$

Ex:12.14 Consider $p \rightarrow q$: If today is Monday, then 4+4=8. Give the truth value of $p \rightarrow q$.

Ex:12.15 Write down the (i) conditional statement (ii) converse statement (iii) inverse statement and (iv) contra positive statement for the two statements p and q given below. p: The number of primes is infinite. q: Ooty is in kerala.

Ex:12.16 Construct the truth table for $(p \veebar q) \wedge (p \veebar \neg q)$.

Ex:12.17 Establish the equivalence property: $p \rightarrow q \equiv \neg p \vee q$. **Mar 2020-3M**

Laws of equivalence: 1.Idempotent laws: (i) $p \vee p \equiv p$ (ii) $p \wedge p \equiv p$
2. Commutative laws: (i) $p \vee q \equiv q \vee p$ (ii) $p \wedge q \equiv q \wedge p$
3. Associative laws: (i) $p \vee (q \vee r) \equiv (p \vee q) \vee r$ (ii) $p \wedge (q \wedge r) \equiv (p \wedge q) \wedge r$
4. Distributive laws: (i) $p \vee (q \wedge r) \equiv (p \vee q) \wedge (p \vee r)$ (ii) $p \wedge (q \vee r) \equiv (p \wedge q) \vee (p \wedge r)$

5. Identity laws: (i) $p \vee \mathbb{T} \equiv \mathbb{T}$ and $p \vee \mathbb{F} \equiv p$ (ii) $p \wedge \mathbb{T} \equiv p$ and $p \wedge \mathbb{F} \equiv \mathbb{F}$
6. Complement laws: (i) $p \vee \neg p \equiv \mathbb{T}$ and $p \wedge \neg p \equiv \mathbb{F}$ (ii) $\neg \mathbb{T} \equiv \mathbb{F}$ and $\neg \mathbb{F} \equiv \mathbb{T}$
7. Involution law or Double negation law: $\neg(\neg p) \equiv p$
8. Demorgan's laws: (i) $\neg(p \wedge q) \equiv \neg p \vee \neg q$ (ii) $\neg(p \vee q) \equiv \neg p \wedge \neg q$
9. Absorption laws: (i) $p \vee (p \wedge q) \equiv p$ (ii) $p \wedge (p \vee q) \equiv p$

Ex:12.18 Establish the equivalence property connecting the bi-conditional with conditional:
$p \leftrightarrow q \equiv (p \rightarrow q) \wedge (q \rightarrow p)$

Ex:12.19 Using the equivalence property, show that $p \leftrightarrow q \equiv (p \wedge q) \vee (\neg p \wedge \neg q)$.

Exercise 12.2

1.Let p: Jupiter is a planet and q: India is an island be any two simple statements. Give verbal sentence describing each of the following statements. (i) $\neg p$ (ii) $p \wedge \neg q$ (iii) $\neg p \vee q$ (iv) $p \rightarrow \neg q$ (v) $p \leftrightarrow q$

2.Write each of the following sentences in symbolic form using statement variables p and q. (i) 19 is not a prime number and all the angles of a triangle are equal.(ii) 19 is a prime number or all the angles of a triangle are not equal. (iii) 19 is a prime number and all the angles of a triangle are equal. (iv)19 is not a prime number.

3. Determine the truth value of each of the following statements. (i) If 6+2=5, then the milk is white. (ii) China is in Europe or $\sqrt{3}$ is an integer. (iii) It is not true that 5+5=9 or Earth is a planet. (iv) 11 is a prime number and all the sides of a rectangle are equal.

4.Which one of the following sentences is a proposition? (i)4+7=12 (ii)What are you doing? (iii) $3^n \leq 81,\ n \in \mathbb{N}$ (iv)Peacock is our national bird (v)How tall this mountain is!

5. Write the converse, inverse and contra positive of each of the following implication: (i) If x and y are numbers such that $x = y$, then $x^2 = y^2$ (ii) If a quadrilateral is a square then it is a rectangle.

6. Construct the truth table for the following statements. (i) $\neg p \wedge \neg q$ (ii) $\neg(p \wedge \neg q)$
(iii) $(p \vee q) \vee \neg q$ (iv) $(\neg p \rightarrow r) \wedge (p \leftrightarrow q)$

7. Verify whether the following compound propositions are tautologies or contradictions or contingency
(i) $(p \wedge q) \wedge \neg(p \vee q)$ (ii) $((p \vee q) \wedge \neg p) \rightarrow q$
(iii) $(p \rightarrow q) \leftrightarrow (\neg p \rightarrow q)$ **July-2022 5M** (iv) $((p \rightarrow q) \wedge (q \rightarrow r)) \rightarrow (p \rightarrow r)$

8.Show that (i) $\neg(p \wedge q) \equiv \neg p \vee \neg q$ (ii) $\neg(p \rightarrow q) \equiv p \wedge \neg q$ **Aug 2021 - 5M**

9. Prove that $q \rightarrow p \equiv \neg p \rightarrow \neg q$.

10. Show that $p \rightarrow q$ and $q \rightarrow p$ are not equivalent. **Mar 2023 - 3M**

11. Show that $\neg(p \leftrightarrow q) \equiv p \leftrightarrow \neg q$

12. Check whether the statements $p \rightarrow (q \rightarrow p)$ is a tautology or a contradiction without using the truth table.

13. Using the truth table check whether the statements $\neg(p \wedge q) \vee (\neg p \wedge q)$ and $\neg p$ are logically equivalent.

14. Prove $p \rightarrow (q \rightarrow r) \equiv (p \wedge q) \rightarrow r$ without using truth table. **May 2022 - 5M**

15. Prove that $p \rightarrow (\neg q \vee r) \equiv \neg p \vee (\neg q \vee r)$ using truth table. **Mar 2023 - 5M**

1. APPLICATIONS OF MATRICES AND DETERMINANTS

1. If $|adj(adj\ A)| = |A|^9$, then the order of the square matrix A is
(1) 3 (2) 4 (3) 2 (4) 5

2. If A is a 3×3 non-singular matrix such that $AA^T = A^T A$ and $B = A^{-1}A^T$, then $BB^T =$
(1) A (2) B (3) I_3 (4) B^T

3. If $A = \begin{bmatrix} 3 & 5 \\ 1 & 2 \end{bmatrix}$, $B = adjA$ and $C = 3A$, then $\frac{|adj\ B|}{|C|} =$
(1) $\frac{1}{3}$ (2) $\frac{1}{9}$ (3) $\frac{1}{4}$ (4) 1

4. If $A\begin{bmatrix}1 & -2\\1 & 4\end{bmatrix}=\begin{bmatrix}6 & 0\\0 & 6\end{bmatrix}$, then $A=$

(1) $\begin{bmatrix}1 & -2\\1 & 4\end{bmatrix}$ (2) $\begin{bmatrix}1 & 2\\-1 & 4\end{bmatrix}$ (3) $\begin{bmatrix}4 & 2\\-1 & 1\end{bmatrix}$ (4) $\begin{bmatrix}4 & -1\\2 & 1\end{bmatrix}$

5. If $A=\begin{bmatrix}7 & 3\\4 & 2\end{bmatrix}$, then $9I_2-A=$

(1) A^{-1} (2) $\frac{A^{-1}}{2}$ (3) $3A^{-1}$ (4) $2A^{-1}$

6. If $A=\begin{bmatrix}2 & 0\\1 & 5\end{bmatrix}$ and $B=\begin{bmatrix}1 & 4\\2 & 0\end{bmatrix}$, then $|adj(AB)|=$

(1) -40 (2) -80 (3) -60 (4) -20

7. If $P=\begin{bmatrix}1 & x & 0\\1 & 3 & 0\\2 & 4 & -2\end{bmatrix}$ is the adjoint of 3×3 matrix A and $|A|=4$, then x is

(1) 15 (2) 12 (3) 14 (4) 11

8. If $A=\begin{bmatrix}3 & 1 & -1\\2 & -2 & 0\\1 & 2 & -1\end{bmatrix}$ and $A^{-1}=\begin{bmatrix}a_{11} & a_{12} & a_{13}\\a_{21} & a_{22} & a_{23}\\a_{31} & a_{32} & a_{33}\end{bmatrix}$, then the value of a_{23} is

(1) 0 (2) -2 (3) -3 (4) -1

9. If A, B and C are invertible matrices of same order, then which one of the following is not true?

(1) $adj\ A=|A|A^{-1}$ (2) $adj(AB)=(adjA)(adjB)$

(3) $detA^{-1}=(detA)^{-1}$ (4) $(ABC)^{-1}=C^{-1}B^{-1}A^{-1}$

10. If $(AB)^{-1}=\begin{bmatrix}12 & -17\\-19 & 27\end{bmatrix}$ and $A^{-1}=\begin{bmatrix}1 & -1\\-2 & 3\end{bmatrix}$, then $B^{-1}=$

(1) $\begin{bmatrix}2 & -5\\-3 & 8\end{bmatrix}$ (2) $\begin{bmatrix}8 & 5\\3 & 2\end{bmatrix}$ (3) $\begin{bmatrix}3 & 1\\2 & 1\end{bmatrix}$ (4) $\begin{bmatrix}8 & -5\\-3 & 2\end{bmatrix}$

11. If A^TA^{-1} is symmetric, then $A^2=$

(1) A^{-1} (2) $(A^T)^2$ (3) A^T (4) $(A^{-1})^2$

12. If A is non-singular matrix such that $A^{-1}=\begin{bmatrix}5 & 3\\-2 & -1\end{bmatrix}$, then $(A^T)^{-1}=$

(1) $\begin{bmatrix}-5 & 3\\2 & 1\end{bmatrix}$ (2) $\begin{bmatrix}5 & 3\\-2 & -1\end{bmatrix}$ (3) $\begin{bmatrix}-1 & -3\\2 & 5\end{bmatrix}$ (4) $\begin{bmatrix}5 & -2\\3 & -1\end{bmatrix}$

13. If $A=\begin{bmatrix}\frac{3}{5} & \frac{4}{5}\\x & \frac{3}{5}\end{bmatrix}$ and $A^T=A^{-1}$, then the value of x is

(1) $-\frac{4}{5}$ (2) $-\frac{3}{5}$ (3) $\frac{3}{5}$ (4) $\frac{4}{5}$

14. If $A=\begin{bmatrix}1 & \tan\frac{\theta}{2}\\-\tan\frac{\theta}{2} & 1\end{bmatrix}$ and $AB=I_2$, then $B=$

(1) $(\cos^2\frac{\theta}{2})A$ (2) $(\cos^2\frac{\theta}{2})A^T$ (3) $(\cos^2\theta)\,I$ (4) $(\sin^2\frac{\theta}{2})A$

15. If $A=\begin{bmatrix}cos\theta & sin\theta\\-sin\theta & cos\theta\end{bmatrix}$ and $A(adjA)=\begin{bmatrix}k & 0\\0 & k\end{bmatrix}$, then $k=$

(1) 0 (2) $sin\theta$ (3) $cos\theta$ (4) 1

16. If $A=\begin{bmatrix}2 & 3\\5 & -2\end{bmatrix}$ be such that $\lambda A^{-1}=A$, then λ is

(1) 17 (2) 14 (3) 19 (4) 21

17. If $adjA=\begin{bmatrix}2 & 3\\4 & -1\end{bmatrix}$ and $adjB=\begin{bmatrix}1 & -2\\-3 & 1\end{bmatrix}$ then $adj(AB)$ is

(1) $\begin{bmatrix}-7 & -1\\7 & -9\end{bmatrix}$ (2) $\begin{bmatrix}-6 & 5\\-2 & -10\end{bmatrix}$ (3) $\begin{bmatrix}-7 & 7\\-1 & -9\end{bmatrix}$ (4) $\begin{bmatrix}-6 & -2\\5 & -10\end{bmatrix}$

18. The rank of the matrix $\begin{bmatrix} 1 & 2 & 3 & 4 \\ 2 & 4 & 6 & 8 \\ -1 & -2 & -3 & -4 \end{bmatrix}$ is

(1) 1 (2) 2 (3) 4 (4) 3

19. If $x^a y^b = e^m, x^c y^d = e^n$, $\Delta_1 = \begin{vmatrix} m & b \\ n & d \end{vmatrix}$, $\Delta_2 = \begin{vmatrix} a & m \\ c & n \end{vmatrix}$, $\Delta_3 = \begin{vmatrix} a & b \\ c & d \end{vmatrix}$, then the value of x and y are respectively

(1) $e^{\frac{\Delta_2}{\Delta_1}}, e^{\frac{\Delta_3}{\Delta_1}}$ (2) $\log\left(\frac{\Delta_1}{\Delta_3}\right), \log\left(\frac{\Delta_2}{\Delta_3}\right)$ (3) $\log\left(\frac{\Delta_2}{\Delta_1}\right), \log\left(\frac{\Delta_3}{\Delta_1}\right)$ (4) $e^{\frac{\Delta_1}{\Delta_3}}, e^{\frac{\Delta_2}{\Delta_3}}$

20. Which of the following is/are correct?

(i) Adjoint of a symmetric matrix is also a symmetric matrix.

(ii) Adjoint of a diagonal matrix is also a diagonal matrix.

(iii) If A is a square matrix of order n and λ is a scalar, then $adj(\lambda A) = \lambda^n \, adjA$.

(iv) $A(adjA) = (adjA)A = |A|I_n$

(1) only (i) (2) (ii) and (iii) (3) (iii) and (iv) (4) (i), (ii) and (iv)

21. If $\rho\,(A) = \rho\,([A\,|B])$, then the system $AX = B$ of linear equations is

(1) consistent and has a unique solution (2) consistent

(3)consistent and has infinitely many solution (4) inconsistent

22. If $0 \le \theta \le \pi$ and the system of equations $x + (sin\theta)y - (cos\theta)z = 0$, $(cos\theta)x - y + z = 0$, $(sin\theta)x + y - z = 0$ has a non-trivial solution then θ is

(1) $\frac{2\pi}{3}$ (2) $\frac{3\pi}{4}$ (3) $\frac{5\pi}{6}$ (4) $\frac{\pi}{4}$

23. The augmented matrix of the system of linear equations is $\begin{bmatrix} 1 & 2 & 7 & 3 \\ 0 & 1 & 4 & 6 \\ 0 & 0 & \lambda - 7 & \mu + 5 \end{bmatrix}$. The system has infinitely many solutions if

(1) $\lambda = 7, \mu \neq 5$ (2) $\lambda = -7, \mu = 5$ (3) $\lambda \neq 7, \mu \neq -5$ (4) $\lambda = 7, \mu = -5$

24. Let $A = \begin{bmatrix} 2 & -1 & 1 \\ -1 & 2 & -1 \\ 1 & -1 & 2 \end{bmatrix}$ and $4B = \begin{bmatrix} 3 & 1 & -1 \\ 1 & 3 & x \\ -1 & 1 & 3 \end{bmatrix}$. If B is the inverse of A, then the value of x is

(1) 2 (2) 4 (3) 3 (4) 1

25. If $A = \begin{bmatrix} 3 & -3 & 4 \\ 2 & -3 & 4 \\ 0 & -1 & 1 \end{bmatrix}$, then $adj(adjA)$ is

(1) $\begin{bmatrix} 3 & -3 & 4 \\ 2 & -3 & 4 \\ 0 & -1 & 1 \end{bmatrix}$ (2) $\begin{bmatrix} 6 & -6 & 8 \\ 4 & -6 & 8 \\ 0 & -2 & 2 \end{bmatrix}$ (3) $\begin{bmatrix} -3 & 3 & -4 \\ -2 & 3 & -4 \\ 0 & 1 & -1 \end{bmatrix}$ (4) $\begin{bmatrix} 3 & -3 & 4 \\ 0 & -1 & 1 \\ 2 & -3 & 4 \end{bmatrix}$

2.COMPLEX NUMBERS

1. $i^n + i^{n+1} + i^{n+2} + i^{n+3}$ is

(1) 0 (2) 1 (3) -1 (4) i

2. The value of $\sum_{i=1}^{13}(i^n + i^{n-1})$ is

(1) $1 + i$ (2) i (3) 1 (4) 0

3. The area of the triangle formed by the complex numbers z, iz and $z + iz$ in the Argand's diagram is

(1) $\frac{1}{2}|z|^2$ (2) $|z|^2$ (3) $\frac{3}{2}|z|^2$ (4) $2|z|^2$

4. The conjugate of the complex number is $\frac{1}{i-2}$. Then the complex number is

(1) $\frac{1}{i+2}$ (2) $\frac{-1}{i+2}$ (3) $\frac{-1}{i-2}$ (4) $\frac{1}{i-2}$

5. If $z = \frac{(\sqrt{3}+i)^3(3i+4)^2}{(8+6i)^2}$ then $|z|$ is equal to

(1) 0 (2) 1 (3) 2 (4) 3

6. If z is a non-zero complex number, such that $2iz^2 = \bar{z}$ then $|z|$is

(1) $\frac{1}{2}$ (2) 1 (3) 2 (4) 3

7. If $|z-2+i| \le 2$ then the greatest value of $|z|$ is

(1) $\sqrt{3}-2$ (2) $\sqrt{3}+2$ (3) $\sqrt{5}-2$ (4) $\sqrt{5}+2$

8. If $\left|z-\frac{3}{z}\right| = 2$ then the least value of $|z|$is

(1) 1 (2) 2 (3) 3 (4) 5

9. If $|z| = 1$, then the value of $\frac{1+z}{1+\bar{z}}$ is

(1) z (2) $\bar{z}$ (3) $\frac{1}{z}$ (4) 1

10. The solution of the equation $|z| - z = 1 + 2i$ is

(1) $\frac{3}{2} - 2i$ (2) $-\frac{3}{2} + 2i$ (3) $2 - \frac{3i}{2}$ (4) $2 + \frac{3i}{2}$

11. If $|z_1| = 1, |z_2| = 2, |z_3| = 3$and $|9z_1z_2 + 4z_1z_3 + z_2z_3| = 12$, then the value of $|z_1 + z_2 + z_3|$ is

(1) 1 (2) 2 (3) 3 (4) 4

12. If z is a complex number such that $z \in \mathbb{C}\backslash\mathbb{R}$ and $z + \frac{1}{z} \in \mathbb{R}$, then $|z|$ is

(1) 0 (2) 1 (3) 2 (4) 3

13. If z_1, z_2 and z_3 be complex numbers such that $z_1 + z_2 + z_3 = 0$ and $|z_1| = |z_2| = |z_3| = 1$ then

$z_1^2 + z_2^2 + z_3^2$ is

(1) 3 (2) 2 (3) 1 (4) 0

14. If $\frac{z-1}{z+1}$ is purely imaginary then $|z|$ is

(1) $\frac{1}{2}$ (2) 1 (3) 2 (4) 3

15. If $z = x + iy$ is a complex number such that $|z+2| = |z-2|$ then the locus of z is

(1) real axis (2) imaginary axis (3) ellipse (4) circle

16. The principal argument of $\frac{3}{-1+i}$ is

(1) $-\frac{5\pi}{6}$ (2) $-\frac{2\pi}{3}$ (3) $-\frac{3\pi}{4}$ (4) $-\frac{\pi}{2}$

17. The principal argument of $(\sin 40° + i\cos 40°)^5$ is

(1) $-110°$ (2) $-70°$ (3) $70°$ (4) $110°$

18. If $(1+i)(1+2i)(1+3i)\cdots(1+ni) = x + iy$ then $2.5.10\cdots(1+n^2)$ is

(1) 1 (2) i (3) $x^2 + y^2$ (4) $1 + n^2$

19. If $\omega \neq 1$ is a cube root of unity and $(1+\omega)^7 = A + B\omega$, then (A, B) equals

(1) $(1, 0)$ (2) $(-1, 1)$ (3) $(0, 1)$ (4) $(1, 1)$

20. The principal argument of the complex number $\frac{(1+i\sqrt{3})^2}{4i(1-i\sqrt{3})}$ is

(1) $\frac{2\pi}{3}$ (2) $\frac{\pi}{6}$ (3) $\frac{5\pi}{6}$ (4) $\frac{\pi}{2}$

21. If α and β are the roots of $x^2 + x + 1 = 0$, then $\alpha^{2020} + \beta^{2020}$ is

(1) -2 (2) -1 (3) 1 (4) 2

22. The product of all four values of $\left(\cos\frac{\pi}{3} + i\sin\frac{\pi}{3}\right)^{\frac{3}{4}}$ is

(1) -2 (2) -1 (3) 1 (4) 2

23. If $\omega \neq 1$ is a cube root of unity and $\begin{vmatrix} 1 & 1 & 1 \\ 1 & -\omega^2-1 & \omega^2 \\ 1 & \omega^2 & \omega^7 \end{vmatrix} = 3k$ then k is equal to

(1) 1 (2) -1 (3) $\sqrt{3}i$ (4) $-\sqrt{3}i$

24. The value of $\left(\frac{1+\sqrt{3}i}{1-\sqrt{3}i}\right)^{10}$ is

(1) $cis\frac{2\pi}{3}$ (2) $cis\frac{4\pi}{3}$ (3) $-cis\frac{2\pi}{3}$ (4) $-cis\frac{4\pi}{3}$

25. If $\omega = cis\frac{2\pi}{3}$, then the number of distinct roots of $\begin{vmatrix} z+1 & \omega & \omega^2 \\ \omega & z+\omega^2 & 1 \\ \omega^2 & 1 & z+\omega \end{vmatrix} = 0$

(1) 1 (2) 2 (3) 3 (4) 4

3.THEORY OF EQUATIONS

1. A zero of $x^3 + 64$ is

(1) 0 (2) 4 (3) $4i$ (4) -4

2. If f and g are polynomials of degrees m and n respectively, and if $h(x) = (f \circ g)(x)$, then the degree of h is

(1)m (2) $m+n$ (3) m^n (4)n^m

3. A polynomial equation in x of degree n always has

(1) n distinct roots (2) n real roots (3) n complex roots (4) at most one root.

4. If α, β, and γ are the zeros of $x^3 + px^2 + qx + r$, then $\sum \frac{1}{\alpha}$ is

(1) $-\frac{q}{r}$ (2) $-\frac{p}{r}$ (3) $\frac{q}{r}$ (4) $-\frac{q}{p}$

5. According to the rational root theorem, which number is not possible rational zero of $4x^7 + 2x^4 - 10x^3 - 5$?

(1) -1 (2) $\frac{5}{4}$ (3) $\frac{4}{5}$ (4) 5

6. The polynomial $x^3 - kx^2 + 9x$ has three real zero s if and only if, k satisfies

(1) $|k| \le 6$ (2) $k = 0$ (3) $|k| > 6$ (4) $|k| \ge 6$

7. The number of real numbers in $[0, 2\pi]$ satisfying $\sin^4 x - 2\sin^2 x + 1$ is

(1) 2 (2) 4 (3) 1 (4) ∞

8. If $x^3 + 12x^2 + 10ax + 1999$ definitely has a positive zero, if and only if

(1) $a \ge 0$ (2) $a > 0$ (3) $a < 0$ (4) $a \le 0$

9. The polynomial $x^3 + 2x + 3$ has

(1) one negative and two imaginary zeros (2) one positive and two imaginary zeros
(3) three real zeros (4) no zeros

10. The number of positive zeros of the polynomial $\sum_{j=0}^{n} nCr\,(-1)^r x^r$ is

(1) 0 (2) n (3) $< n$ (4) r

4.INVERSE TRIGONOMETRIC FUNCTIONS

1. The value of $\sin^{-1}(\cos x)$, $0 \le x \le \pi$ is

(1) $\pi - x$ (2) $x - \frac{\pi}{2}$ (3) $\frac{\pi}{2} - x$ (4) $\pi - x$

2. If $\sin^{-1} x + \sin^{-1} y = \frac{2\pi}{3}$; then $\cos^{-1} x + \cos^{-1} y$ is equal to

(1) $\frac{2\pi}{3}$ (2) $\frac{\pi}{3}$ (3) $\frac{\pi}{6}$ (4) π

3. $\sin^{-1}\frac{3}{5} - \cos^{-1}\frac{12}{13} + \sec^{-1}\frac{5}{3} - \operatorname{cosec}^{-1}\frac{13}{12}$ is equal to

(1) 2π (2) π (3) 0 (4) $\tan^{-1}\frac{12}{65}$

4. If $\sin^{-1} x = 2\sin^{-1}\alpha$ has a solution, then

(1) $|\alpha| \le \frac{1}{\sqrt{2}}$ (2) $|\alpha| \ge \frac{1}{\sqrt{2}}$ (3) $|\alpha| < \frac{1}{\sqrt{2}}$ (4) $|\alpha| > \frac{1}{\sqrt{2}}$

5. $\sin^{-1}(\cos x) = \frac{\pi}{2} - x$ is valid for

(1) $-\pi \le x \le 0$ (2) $0 \le x \le \pi$ (3) $-\frac{\pi}{2} \le x \le \frac{\pi}{2}$ (4) $-\frac{\pi}{4} \le x \le \frac{3\pi}{4}$

6. If $\sin^{-1} x + \sin^{-1} y + \sin^{-1} z = \frac{3\pi}{2}$, the value of $x^{2017} + y^{2018} + z^{2019} - \frac{9}{x^{101} + y^{101} + z^{101}}$ is

(1) 0 (2) 1 (3) 2 (4) 3

7. If $\cot^{-1} x = \frac{2\pi}{5}$, for some $x \in$ R, the value of $\tan^{-1} x$ is

(1) $-\frac{\pi}{10}$ (2) $\frac{\pi}{5}$ (3) $\frac{\pi}{10}$ (4) $-\frac{\pi}{5}$

8. The domain of the function defined by $f(x) = sin^{-1}\sqrt{x-1}$ is

(1) [1, 2] (2) [−1, 1] (3) [0, 1] (4) [−1, 0]

9. If $x = \frac{1}{5}$, the value of $cos(\cos^{-1} x + 2\sin^{-1} x)$ is

(1) $-\sqrt{\frac{24}{25}}$ (2) $\sqrt{\frac{24}{25}}$ (3) $\frac{1}{5}$ (4) $-\frac{1}{5}$

10. $\tan^{-1}\left(\frac{1}{4}\right) + \tan^{-1}\left(\frac{2}{9}\right)$ is equal to

(1) $\frac{1}{2}\cos^{-1}\left(\frac{3}{5}\right)$ (2) $\frac{1}{2}\sin^{-1}\left(\frac{3}{5}\right)$ (3) $\frac{1}{2}\tan^{-1}\left(\frac{3}{5}\right)$ (4) $\tan^{-1}\left(\frac{1}{2}\right)$

11. If the function $f(x) = sin^{-1}(x^2 - 3)$, then x belongs to

(1) $[-1, 1]$ (2) $[\sqrt{2}, 2]$ (3) $[-2, -\sqrt{2}] \cup [\sqrt{2}, 2]$ (4) $[-2, -\sqrt{2}]$

12. If $\cot^{-1} 2$ and $\cot^{-1} 3$ are two angles of a triangle, then the third angle is

(1) $\frac{\pi}{4}$ (2) $\frac{3\pi}{4}$ (3) $\frac{\pi}{6}$ (4) $\frac{\pi}{3}$

13. $\sin^{-1}\left(\tan\frac{\pi}{4}\right) - \sin^{-1}\left(\sqrt{\frac{3}{x}}\right) = \frac{\pi}{6}$. Then x is a root of the equation

(1) $x^2 - x - 6 = 0$ (2) $x^2 - x - 12 = 0$ (3) $x^2 + x - 12 = 0$ (4) $x^2 + x - 6 = 0$

14. $\sin^{-1}(2cos^2 x - 1) + \cos^{-1}(1 - 2sin^2 x) =$

(1) $\frac{\pi}{2}$ (2) $\frac{\pi}{3}$ (3) $\frac{\pi}{4}$ (4) $\frac{\pi}{6}$

15. If $\cot^{-1}\left(\sqrt{\sin\alpha}\right) + \tan^{-1}\left(\sqrt{\sin\alpha}\right) = u$, then $cos2u$ is equal to

(1) $tan^2\alpha$ (2) 0 (3) -1 (4) $tan2\,\alpha$

16. If $|x| \le 1$, then $2\tan^{-1} x - \sin^{-1}\frac{2x}{1+x^2}$ is equal to

(1) $\tan^{-1} x$ (2) $\sin^{-1} x$ (3) 0 (4) π

17. The equation $\tan^{-1} x - \cot^{-1} x = \tan^{-1}\left(\frac{1}{\sqrt{3}}\right)$ has

(1) no solution (2) unique solution (3) two solutions (4) infinite number of solutions

18. If $\sin^{-1} x + \cot^{-1}\left(\frac{1}{2}\right) = \frac{\pi}{2}$, then x is equal to

(1) $\frac{1}{2}$ (2) $\frac{1}{\sqrt{5}}$ (3) $\frac{2}{\sqrt{5}}$ (4) $\frac{\sqrt{3}}{2}$

19. If $\sin^{-1}\frac{x}{5} + \text{cosec}^{-1}\frac{5}{4} = \frac{\pi}{2}$, then the value of x is

(1) 4 (2) 5 (3) 2 (4) 3

20. $\sin(\tan^{-1} x)$, $|x| < 1$ is equal to

(1) $\frac{x}{\sqrt{1-x^2}}$ (2) $\frac{1}{\sqrt{1-x^2}}$ (3) $\frac{1}{\sqrt{1+x^2}}$ (4) $\frac{x}{\sqrt{1+x^2}}$

5.TWO DIMENSIONAL ANALYTIAL GEOMENTRY II

1. The equation of the circle passing through (1, 5) and (4, 1) and touching y-axis is $x^2 + y^2 - 5x - 6y + 9 + \lambda(4x + 3y - 19) = 0$ where λ is equal to

(1) $0, -\frac{40}{9}$ (2) 0 (3) $\frac{40}{9}$ (4) $-\frac{40}{9}$

2. The eccentricity of the hyperbola whose latus rectum is 8 and conjugate axis is equal to half the distance between the foci is

(1) $\frac{4}{3}$ (2) $\frac{4}{\sqrt{3}}$ (3) $\frac{2}{\sqrt{3}}$ (4) $\frac{3}{2}$

3. The circle $x^2 + y^2 = 4x + 8y + 5$ intersects the line $3x - 4y = m$ at two distinct points if

(1) $15 < m < 65$ (2) $35 < m < 85$ (3) $-85 < m < -35$ (4) $-35 < m < 15$

4. The length of the diameter of the circle which touches the x-axis at the point (1,0) and

passes through the point (2, 3)

(1) $\frac{6}{5}$ (2) $\frac{5}{3}$ (3) $\frac{10}{3}$ (4) $\frac{3}{5}$

5. The radius of the circle $x^2 + by^2 + 4bx - 6by + b^2 = 0$ is

(1) 1 (2) 3 (3) $\sqrt{10}$ (4) $\sqrt{11}$

6. The centre of the circle inscribed in a square formed by the lines $x^2 - 8x - 12 = 0$ and $y^2 - 14y + 45 = 0$ is

(1) (4, 7) (2) (7, 4) (3) (9, 4) (4) (4, 9)

7. The equation of the normal to the circle $x^2 + y^2 - 2x - 2y + 1 = 0$ which is parallel to the line $2x + 4y = 3$ is

(1) $x + 2y = 3$ (2) $x + 2y + 3 = 0$ (3) $2x + 4y + 3 = 0$ (4) $x - 2y + 3 = 0$

8. If $P(x, y)$ be any point on $16x^2 + 25y^2 = 400$ with foci $F_1(3, 0)$ and $F_2(-3, 0)$ then $PF_1 + PF_2$ is

(1) 8 (2) 6 (3) 10 (4) 12

9. The radius of the circle passing through the point (6, 2) two of whose diameter are $x + y = 6$ and $x + 2y = 4$ is

(1) 10 (2) $2\sqrt{5}$ (3) 6 (4) 4

10. The area of quadrilateral formed with foci of the hyperbolas $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$ and $\frac{x^2}{a^2} - \frac{y^2}{b^2} = -1$ is

(1) $4(a^2 + b^2)$ (2) $2(a^2 + b^2)$ (3) $a^2 + b^2$ (4) $\frac{1}{2}(a^2 + b^2)$

11. If the normals of the parabola $y^2 = 4x$ drawn at the end points of its latus rectum are tangents to the circle $(x - 3)^2 + (y + 2)^2 = r^2$, then the value of r^2 is

(1) 2 (2) 3 (3) 1 (4) 4

12. If $x + y = k$ is a normal to the parabola $y^2 = 12x$, then the value of k is

(1) 3 (2) -1 (3) 1 (4) 9

13. The ellipse $E_1: \frac{x^2}{9} + \frac{y^2}{4} = 1$ is inscribed in a rectangle R whose sides are parallel to the coordinate axes. Another ellipse E_2 passing through the point (0, 4) circumscribes the rectangle R. The eccentricity of the ellipse is

(1) $\frac{\sqrt{2}}{2}$ (2) $\frac{\sqrt{3}}{2}$ (3) $\frac{1}{2}$ (4) $\frac{3}{4}$

14. Tangents are drawn to the hyperbola $\frac{x^2}{9} - \frac{y^2}{4} = 1$parallel to the straight line $2x - y = 1$. One of the points of contact of tangents on the hyperbola is

(1) $\left(\frac{9}{2\sqrt{2}}, \frac{-1}{\sqrt{2}}\right)$ (2) $\left(\frac{-9}{2\sqrt{2}}, \frac{1}{\sqrt{2}}\right)$ (3) $\left(\frac{9}{2\sqrt{2}}, \frac{1}{\sqrt{2}}\right)$ (4) $\left(3\sqrt{3}, -2\sqrt{2}\right)$

15. The equation of the circle passing through the foci of the ellipse $\frac{x^2}{16} + \frac{y^2}{9} = 1$ having centre at $(0, 3)$ is

(1) $x^2 + y^2 - 6y - 7 = 0$ (2) $x^2 + y^2 - 6y + 7 = 0$

(3) $x^2 + y^2 - 6y - 5 = 0$ (4) $x^2 + y^2 - 6y + 5 = 0$

16. Let C be the circle with centre at (1, 1) and radius =1. If T is the circle centered at (0, y) passing through the origin and touching the circle C externally, then the radius of T is equal to

(1) $\frac{\sqrt{3}}{\sqrt{2}}$ (2) $\frac{\sqrt{3}}{2}$ (3) $\frac{1}{2}$ (4) $\frac{1}{4}$

17. Consider an ellipse whose centre is of the origin and its major axis is along x −axis. If its eccentrcity is $\frac{3}{5}$ and the distance between its foci is 6, then the area of the quadrilateral inscribed in the ellipse with diagonals as major and minor axis of the ellipse is

(1) 8 (2) 32 (3) 80 (4) 40

18. Area of the greatest rectangle inscribed in the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ is

(1) $2ab$ (2) ab (3) $\sqrt{ab}$ (4) $\frac{a}{b}$

19. An ellipse has OB as semi minor axes, F and F' its foci and the angle FBF' is a right angle. Then the eccentricity of the ellipse is

(1) $\frac{1}{\sqrt{2}}$ (2) $\frac{1}{2}$ (3) $\frac{1}{4}$ (4) $\frac{1}{\sqrt{3}}$

20. The eccentricity of the ellipse $(x-3)^2+(y-4)^2 = \frac{y^2}{9}$ is

(1) $\frac{\sqrt{3}}{2}$ (2) $\frac{1}{3}$ (3) $\frac{1}{3\sqrt{2}}$ (4) $\frac{1}{\sqrt{3}}$

21. If the two tangents drawn from a point P to the parabola $y^2 = 4x$ are at right angles then the locus of P is

(1) $2x + 1 = 0$ (2) $x = -1$ (3) $2x - 1 = 0$ (4) $x = 1$

22. The circle passing through (1, –2) and touching the axis of x at (3, 0) passing through the point

(1) (–5, 2) (2) (2, –5) (3) (5, –2) (4) (– 2, 5)

23. The locus of a point whose distance from $(-2, 0)$ is $\frac{2}{3}$ times its distance from the line $x = -\frac{9}{2}$ is

(1) a parabola (2) a hyperbola (3) an ellipse (4) a circle

24. The values of m for which the line $y = mx + 2\sqrt{5}$ touches the hyperbola $16x^2 - 9y^2 = 144$ are the roots of $x^2 - (a+b)x - 4 = 0$, then the value of $(a+b)$is

(1) 2 (2) 4 (3) 0 (4) – 2

25. If the coordinates at one end of a diameter of the circle $x^2 + y^2 - 8x - 4y + c = 0$ are $(11, 2)$, the coordinates of the other end are

(1) $(-5, 2)$ (2) $(-3, 2)$ (3) $(5, -2)$ (4) $(-2, 5)$

6.APPLICATIONS OF VECTOR ALGEBRA

1. If $\vec{a}$ and $\vec{b}$ are parallel vectors, then $[\vec{a}, \vec{c}, \vec{b}]$ is equal to

(1) 2 (2) −1 (3) 1 (4) 0

2. If a vector $\vec{\alpha}$ lies in the plane of $\vec{\beta}$ and $\vec{\gamma}$, then

(1) $[\vec{\alpha}, \vec{\beta}, \vec{\gamma}]=1$ (2) $[\vec{\alpha}, \vec{\beta}, \vec{\gamma}]=-1$ (3) $[\vec{\alpha}, \vec{\beta}, \vec{\gamma}]=0$ (4) $[\vec{\alpha}, \vec{\beta}, \vec{\gamma}]=2$

3. If $\vec{a}\cdot\vec{b} = \vec{b}\cdot\vec{c} = \vec{c}\cdot\vec{a} = 0$, then the value of $[\vec{a}, \vec{b}, \vec{c}]$ is

(1) $|\vec{a}||\vec{b}||\vec{c}|$ (2) $\frac{1}{3}|\vec{a}||\vec{b}||\vec{c}|$ (3) 1 (4) −1

4. If $\vec{a}, \vec{b}, \vec{c}$ are three unit vectors such that $\vec{a}$ is perpendicular to $\vec{b}$, and is parallel to $\vec{c}$ then $\vec{a}\times(\vec{b}\times\vec{c})$ is equal to

(1) $\vec{a}$ (2) $\vec{b}$ (3) $\vec{c}$ (4) $\vec{0}$

5. If $[\vec{a}, \vec{b}, \vec{c}]=1$ then the value of $\frac{\vec{a}.(\vec{b}\times\vec{c})}{(\vec{c}\times\vec{a}).\vec{b}} + \frac{\vec{b}.(\vec{c}\times\vec{a})}{(\vec{a}\times\vec{b}).\vec{c}} + \frac{\vec{c}.(\vec{a}\times\vec{b})}{(\vec{c}\times\vec{b}).\vec{a}}$ is

(1) 1 (2) −1 (3) 2 (4) 3

6. The volume of the parallelepiped with its edges represented by the vectors $\hat{\imath}+\hat{\jmath}, \hat{\imath}+2\hat{\jmath}, \hat{\imath}+\hat{\jmath}+\pi\hat{k}$ is

(1) $\frac{\pi}{2}$ (2) $\frac{\pi}{3}$ (3) π (4) $\frac{\pi}{4}$

7. If $\vec{a}$ and $\vec{b}$ are unit vectors such that $[\vec{a}, \vec{b}, \vec{a}\times\vec{b}]=\frac{1}{4}$, then the angle between $\vec{a}$ and $\vec{b}$ is

(1) $\frac{\pi}{6}$ (2) $\frac{\pi}{4}$ (3) $\frac{\pi}{3}$ (4) $\frac{\pi}{2}$

8. If $\vec{a}=\hat{\imath}+\hat{\jmath}+\hat{k}$, $\vec{b}=\hat{\imath}+\hat{\jmath}$, $\vec{c}=\hat{\imath}$ and $(\vec{a}\times\vec{b})\times\vec{c} = \lambda\vec{a}+\mu\vec{b}$ then the value of $\lambda+\mu$ is

(1) 0 (2) 1 (3) 6 (4) 3

9. If $\vec{a}, \vec{b}, \vec{c}$ are non-coplanar, non-zero vectors such that $[\vec{a}, \vec{b}, \vec{c}]= 3$, then $\{[\vec{a}\times\vec{b}, \vec{b}\times\vec{c}, \vec{c}\times\vec{a}]\}^2$ is equal to

(1) 81 (2) 9 (3) 27 (4)18

10. If $\vec{a}, \vec{b}, \vec{c}$ are three non-coplanar vectors such that $\vec{a}\times(\vec{b}\times\vec{c})=\frac{(\vec{b}+\vec{c})}{\sqrt{2}}$, then the angle between

$\vec{a}$ and $\vec{b}$ is

(1) $\frac{\pi}{2}$ (2) $\frac{3\pi}{4}$ (3) $\frac{\pi}{4}$ (4) π

11. If the volume of the parallelepiped with $\vec{a}\times\vec{b}$, $\vec{b}\times\vec{c}$, $\vec{c}\times\vec{a}$ as coterminous edges is 8 cubic units, then the volume of the parallelepiped with $(\vec{a}\times\vec{b})\times(\vec{b}\times\vec{c})$, $(\vec{b}\times\vec{c})\times(\vec{c}\times\vec{a})$ and $(\vec{c}\times\vec{a})\times(\vec{a}\times\vec{b})$ as coterminous edges is,

(1) 8 cubic units (2) 512 cubic units (3) 64 cubic units (4) 24 cubic units

12. Consider the vectors $\vec{a}, \vec{b}, \vec{c}, \vec{d}$ such that $(\vec{a}\times\vec{b})\times(\vec{c}\times\vec{d}) = \vec{0}$. Let P_1 and P_2 be the planes determined by the pairs of vectors $\vec{a}, \vec{b}$ and $\vec{c}, \vec{d}$ respectively. Then the angle between P_1 and P_2 is

(1) 0° (2) 45° (3) 60° (4) 90°

13. If $\vec{a}\times(\vec{b}\times\vec{c}) = (\vec{a}\times\vec{b})\times\vec{c}$, where $\vec{a}, \vec{b}, \vec{c}$ are any three vectors such that $\vec{b}\cdot\vec{c} \neq 0$ and $\vec{a}\cdot\vec{b} \neq 0$, then $\vec{a}$ and $\vec{c}$ are

(1) perpendicular (2) parallel

(3) inclined at an angle $\frac{\pi}{3}$ (4) inclined at an angle $\frac{\pi}{6}$

14. If $\vec{a} = 2\hat{\imath} + 3\hat{\jmath} - \hat{k}$, $\vec{b} = \hat{\imath} + 2\hat{\jmath} - 5\hat{k}$, $\vec{c} = 3\hat{\imath} + 5\hat{\jmath} - \hat{k}$ then a vector perpendicular to $\vec{a}$ and lies in the plane containing $\vec{b}$ and $\vec{c}$ is

(1) $-17\hat{\imath} + 21\hat{\jmath} - 97\hat{k}$ (2) $17\hat{\imath} + 21\hat{\jmath} - 123\hat{k}$ (3) $-17\hat{\imath} - 21\hat{\jmath} + 97\hat{k}$ (4) $-17\hat{\imath} - 21\hat{\jmath} - 97\hat{k}$

15. The angle between the lines $\frac{x-2}{3} = \frac{y+1}{-2}$, z =2 and $\frac{x-1}{1} = \frac{2y+3}{3} = \frac{z+5}{2}$ is

(1) $\frac{\pi}{6}$ (2) $\frac{\pi}{4}$ (3) $\frac{\pi}{3}$ (4) $\frac{\pi}{2}$

16. If the line $\frac{x-2}{3} = \frac{y-1}{-5} = \frac{z+2}{2}$ lies in the plane $x + 3y - \alpha z + \beta = 0$ then (α, β) is

(1) (−5, 5) (2) (−6, 7) (3) (5, −5) (4) (6, −7)

17. The angle between the line $\vec{r} = (\hat{\imath} + 2\hat{\jmath} - 3\hat{k}) + t(2\hat{\imath} + \hat{\jmath} - 2\hat{k})$ and the plane $\vec{r}\cdot(\hat{\imath} + \hat{\jmath}) + 4 = 0$ is

(1) 0° (2) 30° (3) 45° (4) 90°

18. The coordinates of the point where the line $\vec{r} = (6\hat{\imath} - \hat{\jmath} - 3\hat{k}) + t(-\hat{\imath} + 4\hat{k})$ meets the plane $\vec{r}\cdot(\hat{\imath} + \hat{\jmath} - \hat{k}) = 3$ are

(1) (2, 1, 0) (2) (7, −1, −7) (3) (1, 2, −6) (4) (5, −1, 1)

19. Distance from the origin to the plane $3x - 6y + 2z + 7 = 0$ is

(1) 0 (2) 1 (3) 2 (4) 3

20. The distance between the planes $x + 2y + 3z + 7 = 0$ and $2x + 4y + 6z + 7 = 0$ is

(1) $\frac{\sqrt{7}}{2\sqrt{2}}$ (2) $\frac{7}{2}$ (3) $\frac{\sqrt{7}}{2}$ (4) $\frac{7}{2\sqrt{2}}$

21. If the direction cosines of a line are $\frac{1}{c}, \frac{1}{c}, \frac{1}{c}$ then

(1) c =±3 (2) c=$\pm\sqrt{3}$ (3) c >0 (4) 0<c<1

22. The vector equation $\vec{r} = (\hat{\imath} - 2\hat{\jmath} - \hat{k}) + t(6\hat{\jmath} - \hat{k})$ represents a straight line passing through the points

(1) (0, 6, −1) and (1, −2, −1) (2) (0, 6, −1) and (−1, −4, −2)

(3) (1, −2, −1) and (1, 4, −2) (4) (1, −2, −1) and (0, −6, 1)

23. If the distance of the point (1, 1, 1) from the origin is half of its distance from the plane x + y + z + k=0, then the values of k are

(1) ±3 (2) ±6 (3) − 3, 9 (4) 3, −9

24. If the planes $\vec{r}\cdot(2\hat{\imath} - \lambda\hat{\jmath} + \hat{k}) = 3$ and $\vec{r}\cdot(4\hat{\imath} + \hat{\jmath} - \mu\hat{k}) = 5$ are parallel, then the value of λ and μ are

(1) $\frac{1}{2}, -2$ (2) $-\frac{1}{2}, 2$ (3) $-\frac{1}{2}, -2$ (4) $\frac{1}{2}, 2$

25. If the length of the perpendicular from the origin to the plane $2x + 3y + \lambda z = 1$, $\lambda > 0$ is $\frac{1}{5}$, then the value of λ is

(1) $2\sqrt{3}$ (2) $3\sqrt{2}$ (3) 0 (4) 1

7. APPLICATIONS OF DIFFERENTIAL CALCULUS

1. The volume of a sphere is increasing in volume at the rate of $3\pi cm^3/sec$. The rate of change of its radius when radius is $\frac{1}{2}cm$

(1) 3cm/s (2) 2cm/s (3) 1cm/s (4) $\frac{1}{2}cm/s$

2. A balloon rises straight up at 10 m/s. An observer is 40m away from the spot where the balloon left the ground. Find the rate of change of the balloon's angle of elevation in radian per second when the balloon is 30 meters above the ground.

(1) $\frac{3}{25}radians/sec$ (2) $\frac{4}{25}radians/sec$ (3) $\frac{1}{5}radians/sec$ (4) $\frac{1}{3}radians/sec$

3.The position of a particle moving along a horizontal line of any time t is given by $s(t) = 3t^2 - 2t - 8$. The time at which the particle is at rest is

(1) $t = 0$ (2) $t = \frac{1}{3}$ (3) $t = 1$ (4) $t = 3$

4. A stone is thrown up vertically. The height it reaches at time t seconds is given by $x = 80t - 16t^2$. The stone reaches the maximum height in time t seconds is given by

(1) 2 (2) 2.5 (3) 3 (4) 3.5

5. Find the point on the curve $6y = x^3 + 2$ at which y-coordinate changes 8 times as fast as x-coordinate is

(1) (4,11) (2) $(4, -11)$ (3) $(-4,11)$ (4) $(-4, -11)$

6. The abscissa of the point on the curve $f(x) = \sqrt{8 - 2x}$ at which the slope of the tangent is -0.25?

(1) -8 (2) -4 (3) -2 (4) 0

7. The slope of the line normal to the curve $f(x) = 2cos4x$ at $x = \frac{\pi}{12}$ is

(1) $-4\sqrt{3}$ (2) -4 (3) $\frac{\sqrt{3}}{12}$ (4) $4\sqrt{3}$

8. The tangent to the curve $y^2 - xy + 9 = 0$ is vertical when

(1) $y = 0$ (2) $y = \pm\sqrt{3}$ (3) $y = \frac{1}{2}$ (4) $y = \pm 3$

9. Angle between $y^2 = x$ and $x^2 = y$ at the origin is

(1) $tan^{-1}\frac{3}{4}$ (2) $tan^{-1}\frac{4}{3}$ (3) $\frac{\pi}{2}$ (4) $\frac{\pi}{4}$

10. The value of the limit $\lim_{x\to 0}\left(cotx - \frac{1}{x}\right)$ is

(1) 0 (2) 1 (3) 2 (4) ∞

11. The function $sin^4x + cos^4x$ is increasing in the interval

(1) $\left[\frac{5\pi}{8}, \frac{3\pi}{4}\right]$ (2) $\left[\frac{\pi}{2}, \frac{5\pi}{8}\right]$ (3) $\left[\frac{\pi}{4}, \frac{\pi}{2}\right]$ (4) $\left[0, \frac{\pi}{4}\right]$

12. The number given by the Rolle's theorem for the function $x^3 - 3x^2, x \in [0,3]$ is

(1) 1 (2) $\sqrt{2}$ (3) $\frac{3}{2}$ (4) 2

13. The number given by the Mean value theorem for the function $\frac{1}{x}, x \in [1,9]$ is

(1) 2 (2) 2.5 (3) 3 (4) 3.5

14. The minimum value of the function $|3 - x| + 9$ is

(1) 0 (2) 3 (3) 6 (4) 9

15. The maximum slope of the tangent to the curve $y = e^x sinx, x \in [0, 2\pi]$ is at

(1) $x = \frac{\pi}{4}$ (2) $x = \frac{\pi}{2}$ (3) $x = \pi$ (4) $x = \frac{3\pi}{2}$

16. The maximum value of the function x^2e^{-2x}, $x > 0$ is

(1) $\frac{1}{e}$ (2) $\frac{1}{2e}$ (3) $\frac{1}{e^2}$ (4) $\frac{4}{e^4}$

17. One of the closest points on the curve $x^2 - y^2 = 4$ to the point (6,0) is

(1) (2,0) (2) $(\sqrt{5}, 1)$ (3) $(3, \sqrt{5})$ (4) $(\sqrt{13}, -\sqrt{3})$

18. The maximum value of the product of two positive numbers, when their sum of the squares is 200, is

(1) 100 (2) $25\sqrt{7}$ (3) 28 (4) $24\sqrt{14}$

19. The curve $y = ax^4 + bx^2$ with $ab > 0$

(1) has no horizontal tangent (2) is concave up (3) is concave down (4) has no points of inflection

20. The point of inflection of the curve $y = (x-1)^3$ is

(1) (0,0) (2) (0,1) (3) (1,0) (4) (1,1)

8.DIFFERENTIALS AND PARTIAL DERIVATIVES

1. A circular template has a radius of 10cm. The measurement of radius has an approximate error of 0.02cm. Then the percentage error in calculating area of this template is

(1) 0.2% (2) 0.4% (3) 0.04% (4) 0.08%

2. The percentage error of fifth root of 31 is approximately how many times the percentage error in 31?

(1) $\frac{1}{31}$ (2) $\frac{1}{5}$ (3) 5 (4) 31

3. If $u(x,y) = e^{x^2+y^2}$, then $\frac{\partial u}{\partial x}$ is equal to

(1)$e^{x^2+y^2}$ (2) $2xu$ (3) x^2u (4) y^2u

4. If $v(x,y) = \log(e^x + e^y)$, then $\frac{\partial v}{\partial x} + \frac{\partial v}{\partial y}$ is equal to

(1) $e^x + e^y$ (2) $\frac{1}{e^x+e^y}$ (3) 2 (4) 1

5. If $w(x,y) = x^y, x > 0$, then $\frac{\partial w}{\partial x}$ is equal to

(1) $x^y logx$ (2) $ylogx$ (3) yx^{y-1} (4) $xlogy$

6. If $f(x,y)) = e^{xy}$, then $\frac{\partial^2 f}{\partial x \partial y}$ is equal to

(1) xye^{xy} (2) $(1+xy)e^{xy}$ (3) $(1+y)e^{xy}$ (4) $(1+x)e^{xy}$

7. If we measure the side of a cube to be 4cm with an error of 0.1cm, then the error in our calculation of the volume is

(1) 0.4 $cu.cm$ (2) 0.45 $cu.cm$ (3) 2 $cu.cm$ (4) 4.8 $cu.cm$

8. The change in the surface area $S = 6x^2$ of a cube when the edge length varies from x_0 to $x_0 + dx$ *is*

(1) $12x_0 + dx$ (2) $12x_0dx$ (3) $6x_0dx$ (4) $6x_0 + dx$

9. The approximate change in the volume V of a cube of side x meters caused by increasing the side by 1% is

(1) $0.3xdx\ m^3$ (2) $0.03x\ m^3$ (3) $0.03x^2\ m^3$ (4) $0.03x^3\ m^3$

10. If $g(x,y) = 3x^2 - 5y + 2y^2, x(t) = e^t$ and $y(t) = cost$, then $\frac{dg}{dt}$ is equal to

(1) $6e^{2t} + 5sint - 4costsint$ (2) $6e^{2t} - 5sint + 4costsint$

(3) $3e^{2t} + 5sint + 4costsint$ (4) $3e^{2t} - 5sint + 4costsint$

11. If $f(x) = \frac{x}{x+1}$, then its differential is given by

(1) $\frac{-1}{(x+1)^2}dx$ (2) $\frac{1}{(x+1)^2}dx$ (3) $\frac{1}{x+1}dx$ (4) $\frac{-1}{x+1}dx$

12. If $u(x,y) = x^2 + 3xy + y - 2019$, then $\left.\frac{\partial u}{\partial x}\right|_{(4,-5)}$ is equal to

(1) -4 (2) -3 (3) -7 (4) 13

13. Linear approximation for $g(x) = cosx$ at $x = \frac{\pi}{2}$ is

(1) $x + \frac{\pi}{2}$ (2) $-x + \frac{\pi}{2}$ (3) $x - \frac{\pi}{2}$ (4) $-x - \frac{\pi}{2}$

14. If $w(x,y,z) = x^2(y-z) + y^2(z-x) + z^2(x-y)$, then $\frac{\partial w}{\partial x} + \frac{\partial w}{\partial y} + \frac{\partial w}{\partial z}$ is

(1) $xy + yz + zx$ (2) $x(y+z)$ (3) $y(z+x)$ (4) 0

15. If $f(x,y,z) = xy + yz + zx$, then $f_x - f_z$ is equal to

(1) $z - x$ (2) $y - z$ (3) $x - z$ (4) $y - x$

9.APPLICATIONS OF INTEGRATION

1. The value of $\int_0^{\frac{2}{3}} \frac{dx}{\sqrt{4-9x^2}}$

(1) $\frac{\pi}{6}$ (2) $\frac{\pi}{2}$ (3) $\frac{\pi}{4}$ (4) π

2. The value of $\int_{-1}^{2} |x|\, dx$

(1) $\frac{1}{2}$ (2) $\frac{3}{2}$ (3) $\frac{5}{2}$ (4) $\frac{7}{2}$

3. For any value of $n \in \mathbb{Z}$, $\int_0^{\pi} e^{cos^2 x} cos^3[(2n+1)x]dx$ is

(1) $\frac{\pi}{2}$ (2) π (3) 0 (4) 2

4. The value of $\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} sin^2 x cosx\, dx$ is

(1) $\frac{3}{2}$ (2) $\frac{1}{2}$ (3) 0 (4) $\frac{2}{3}$

5. The value of $\int_{-4}^{4} \left[tan^{-1}\left(\frac{x^2}{x^4+1}\right) + tan^{-1}\left(\frac{x^4+1}{x^2}\right)\right] dx$ is

(1) π (2) 2π (3) 3π (4) 4π

6. The value of $\int_{\frac{-\pi}{4}}^{\frac{\pi}{4}} \left(\frac{2x^7-3x^5+7x^3-x+1}{cos^2 x}\right) dx$ is

(1) 4 (2) 3 (3) 2 (4) 0

7. If $f(x) = \int_0^x tcost\, dt$, then $\frac{df}{dx}$=

(1) $cosx - xsinx$ (2) $sinx + xcosx$ (3) $xcosx$ (4) $xsinx$

8. The area between $y^2 = 4x$ and its latus rectum is

(1) $\frac{2}{3}$ (2) $\frac{4}{3}$ (3) $\frac{8}{3}$ (4) $\frac{5}{3}$

9. The value of $\int_0^1 x(1-x)^{99}\, dx$ is

(1) $\frac{1}{11000}$ (2) $\frac{1}{10100}$ (3) $\frac{1}{10010}$ (4) $\frac{1}{10001}$

10. The value of $\int_0^{\pi} \frac{dx}{1+5^{cosx}}$ is

(1) $\frac{\pi}{2}$ (2) π (3) $\frac{3\pi}{2}$ (4) 2π

11. If $\frac{\Gamma(n+2)}{\Gamma(n)} = 90$ then n is

(1) 10 (2) 5 (3) 8 (4) 9

12. The value of $\int_0^{\frac{\pi}{6}} cos^3 3x\, dx$ is

(1) $\frac{2}{3}$ (2) $\frac{2}{9}$ (3) $\frac{1}{9}$ (4) $\frac{1}{3}$

13. The value of $\int_0^{\pi} sin^4 x\, dx$ is

(1) $\frac{3\pi}{10}$ (2) $\frac{3\pi}{8}$ (3) $\frac{3\pi}{4}$ (4) $\frac{3\pi}{2}$

14. The value of $\int_0^{\infty} e^{-3x} x^2\, dx$ is

(1) $\frac{7}{27}$ (2) $\frac{5}{27}$ (3) $\frac{4}{27}$ (4) $\frac{2}{27}$

15. If $\int_0^a \frac{1}{4+x^2} dx = \frac{\pi}{8}$ then a is

(1) 4 (2) 1 (3) 3 (4) 2

16. The volume of solid of revolution of the region bounded by $y^2 = x(a-x)$ about x −axis is

(1) πa^3 (2) $\frac{\pi a^3}{4}$ (3) $\frac{\pi a^3}{5}$ (4) $\frac{\pi a^3}{6}$

17. If $f(x) = \int_1^x \frac{e^{sinu}}{u} du, x > 1$ and $\int_1^3 \frac{e^{sinx^2}}{x} dx = \frac{1}{2}[f(a) - f(1)]$, then one of the possible value of

a is

(1) 3 (2) 6 (3) 9 (4) 5

18. The value of $\int_0^1 (sin^{-1}x)^2\, dx$ is

(1) $\frac{\pi^2}{4} - 1$ (2) $\frac{\pi^2}{4} + 2$ (3) $\frac{\pi^2}{4} + 1$ (4) $\frac{\pi^2}{4} - 2$

19. The value of $\int_0^a \left(\sqrt{a^2 - x^2}\right)^3 dx$ is

(1) $\frac{\pi a^3}{16}$ (2) $\frac{3\pi a^4}{16}$ (3) $\frac{3\pi a^2}{8}$ (4) $\frac{3\pi a^4}{8}$

20. If $\int_0^x f(t)\, dt = x + \int_x^1 tf(t)\, dt$, then the value of $f(1)$ is

(1) $\frac{1}{2}$ (2) 2 (3) 1 (4) $\frac{3}{4}$

10. ORDINARY DIFFERENTIAL EQUATIONS

1. The order and degree of the differential equation $\frac{d^2y}{dx^2} + \left(\frac{dy}{dx}\right)^{\frac{1}{3}} + x^{1/4} = 0$ are respectively

(1) 2,3 (2) 3,3 (3) 2,6 (4) 2,4

2. The differential equation representing the family of curves $y = Acos(x + B)$, where A and B are parameters, is

(1) $\frac{d^2y}{dx^2} - y = 0$ (2) $\frac{d^2y}{dx^2} + y = 0$ (3) $\frac{d^2y}{dx^2} = 0$ (4) $\frac{d^2x}{dy^2} = 0$

3. The order and degree of the differential equation $\sqrt{sinx}\,(dx + dy) = \sqrt{cosx}\,(dx - dy)$ is

(1) 1,2 (2) 2,2 (3) 1,1 (4) 2,1

4. The order of the differential equation of all circles with centre at (h, k) and radius 'a' is

(1) 2 (2) 3 (3) 4 (4) 1

5. The differential equation of the family of curves $y = Ae^x + Be^{-x}$, where A and B are arbitrary constants is

(1) $\frac{d^2y}{dx^2} + y = 0$ (2) $\frac{d^2y}{dx^2} - y = 0$ (3) $\frac{dy}{dx} + y = 0$ (4) $\frac{dy}{dx} - y = 0$

6. The general solution of the differential equation $\frac{dy}{dx} = \frac{y}{x}$ is

(1) $xy = k$ (2) $y = klogx$ (3) $y = kx$ (4) $logy = kx$

7. The solution of the differential equation $2x\frac{dy}{dx} - y = 3$ represents

(1) straight lines (2) circles (3) parabola (4) ellipse

8. The solution of $\frac{dy}{dx} + p(x)y = 0$ is

(1) $y = ce^{\int pdx}$ (2) $y = ce^{-\int pdx}$ (3) $x = ce^{-\int pdy}$ (4) $x = ce^{\int pdy}$

9. The integrating factor of the differential equation $\frac{dy}{dx} + y = \frac{1+y}{\lambda}$ is

(1) $\frac{x}{e^{\lambda}}$ (2) $\frac{e^{\lambda}}{x}$ (3) λe^x (4) e^x

10. The integrating factor of the differential equation $\frac{dy}{dx} + P(x)y = Q(x)$ is x, then $P(x)$

(1) x (2) $\frac{x^2}{2}$ (3) $\frac{1}{x}$ (4) $\frac{1}{x^2}$

11. The degree of the differential equation $y(x) = 1 + \frac{dy}{dx} + \frac{1}{1.2}\left(\frac{dy}{dx}\right)^2 + \frac{1}{1.2.3}\left(\frac{dy}{dx}\right)^3 + \cdots\cdots$ is

(1) 2 (2) 3 (3) 1 (4) 4

12. If p and q are the order and degree of the differential equation $y\frac{dy}{dx} + x^3\left(\frac{d^2y}{dx^2}\right) + xy = cosx$, when

(1) $p < q$ (2) $p = q$ (3) $p > q$ (4) p exists` and q does not exists

13. The solution of the differential equation $\frac{dy}{dx} + \frac{1}{\sqrt{1-x^2}} = 0$ is

(1) $y + sin^{-1}x = c$ (2) $x + sin^{-1}y = 0$ (3) $y^2 + 2sin^{-1}x = c$ (4) $x^2 + 2sin^{-1}y = 0$

14. The solution of the differential equation $\frac{dy}{dx} = 2xy$

(1) $y = Ce^{x^2}$ (2) $y = 2x^2$+C (3) $y = Ce^{-x^2}$+C (4) $y = x^2 + C$

15. The general solution of the differential equation $log\left(\frac{dy}{dx}\right) = x + y$ is

(1) $e^x + e^y = C$ (2) $e^x + e^{-y} = C$ (3) $e^{-x} + e^y = C$ (4) $e^{-x} + e^{-y} = C$

16. The solution of $\frac{dy}{dx} = 2^{y-x}$ is

(1) $2^x + 2^y = C$ (2) $2^x - 2^y = C$ (3) $\frac{1}{2^x} - \frac{1}{2^y} = C$ (4) $x + y = C$

17. The solution of the differential equation $\frac{dy}{dx} = \frac{y}{x} + \frac{\phi\left(\frac{y}{x}\right)}{\phi'\left(\frac{y}{x}\right)}$ is

(1) $x\phi\left(\frac{y}{x}\right) = k$ (2) $\phi\left(\frac{y}{x}\right) = kx$ (3) $y\phi\left(\frac{y}{x}\right) = k$ (4) $\phi\left(\frac{y}{x}\right) = ky$

18. If $sinx$ is the integrating factor of the linear differential equation $\frac{dy}{dx} + Py = Q$, then P is

(1) $\log sinx$ (2) $cosx$ (3) $tanx$ (4) $cotx$

19. The number of arbitrary constants in the general solutions of order n and $n + 1$ are respectively

(1) $n - 1, n$ (2) $n, n + 1$ (3) $n + 1, n + 2$ (4) $n + 1, n$

20. The number of arbitrary constants in the particular solution of a differential equation of third order is

(1) 3 (2) 2 (3) 1 (4) 0

21. Integrating factor of the differential equation $\frac{dy}{dx} = \frac{x+y+1}{x+1}$ is

(1) $\frac{1}{x+1}$ (2) $x + 1$ (3) $\frac{1}{\sqrt{x+1}}$ (4) $\sqrt{x + 1}$

22. The population P in any year t is such that the rate of increase in the population is proportional to the population. Then

(1) $P = Ce^{kt}$ (2) $P = Ce^{-kt}$ (3) $P = Ckt$ (4) $P = C$

23. P is the amount of certain substance left in after time t. If the rate of evaporation of the substance is proportional to the amount remaining, then

(1) $P = Ce^{kt}$ (2) $P = Ce^{-kt}$ (3) $P = Ckt$ (4) $Pt = C$

24. If the solution of the differential equation $\frac{dy}{dx} = \frac{ax+3}{2y+f}$ represents a circle, then the value of a is

(1) 2 (2) -2 (3) 1 (4) -1

25. The slope at any point of the curve $y = f(x)$ is given by $\frac{dy}{dx} = 3x^2$ and it passes through $(-1,1)$.Then the equation of the curve is

(1) $y = x^3 + 2$ (2) $y = 3x^2 + 4$ (3) $y = 3x^3 + 4$ (4) $y = x^3 + 5$

11. PROBABILITY DISTRIBUTIONS

1. Let X be random variable with probability density function $f(x) = \begin{cases} \frac{2}{x^3} & x \geq 1 \\ 0 & x < 1 \end{cases}$

Which of the following statement is correct?

(1) both mean and variance exist (2) mean exists but variance does not exist

(3) both mean and variance do not exist (4) variance exists but Mean does not exist.

2. A rod of length $2l$ is broken into two pieces at random. The probability density function of the shorter of the two pieces is $f(x) = \begin{cases} \frac{1}{l} & 0 < x < l \\ 0 & l \leq x < 2l \end{cases}$ The mean and variance of the shorter of the two pieces are respectively

(1) $\frac{l}{2}, \frac{l^2}{3}$ (2) $\frac{l}{2}, \frac{l^2}{6}$ (3) $l, \frac{l^2}{12}$ (4) $\frac{l}{2}, \frac{l^2}{12}$

3. Consider a game where the player tosses a six-sided fair die. If the face that comes up is 6, the player wins Rs.36, otherwise he loses Rs.k^2, where k is the face that comes up $k = \{1,2,3,4,5\}$. The expected amount to win at this game in Rs.is

(1) $\frac{19}{6}$ (2) $\frac{-19}{6}$ (3) $\frac{3}{2}$ (4) $\frac{-3}{2}$

4. A pair of dice numbered 1,2,3,4,5,6 of six-sided die and 1,2,3,4 of a four-sided die is rolled and the sum is determined. Let the random variable X denote this sum. Then the number of elements in the inverse image of 7 is

(1) 1 (2) 2 (3) 3 (4) 4

5. A random variable X has binomial distribution with $n = 25$ and $p = 0.8$ then standard deviation of X is

(1) 6 (2) 4 (3) 3 (4) 2

6. Let X represent the difference between the number of heads and the number of tails obtained when a coin is tossed n times. Then the possible values of X are

(1) $i + 2n, i = 0,1,2, \dots \dots n$ (2) $2i - n, i = 0,1,2, \dots \dots n$ (3) $n - i, i = 0,1,2, \dots \dots n$
(4) $2i + 2n, i = 0,1,2, \dots \dots n$

7. If the function $f(x) = \frac{1}{12}$ for $a < x < b$, represents a probability density function of a continuous random variable X, then which of the following cannot be the value of a and b?

(1) 0 and 12 (2) 5 and 17 (3) 7 and 19 (4)16 and 24

8. Four buses carrying 160 students from the same school arrive at a football stadium. The buses carry, respectively, 42, 36, 34 and 48 students. One of the students is randomly selected. Let X denote the number of students that were on the bus carrying the randomly selected student. One of the 4 bus drivers is also randomly selected. Let Y denote the number of students on that bus. Then $E[X]$ and $E[Y]$ respectively are

(1) 50,40 (2) 40,50 (3) 40.75,40 (4) 41,41

9. Two coins are to be flipped. The first coin will land on heads with probability 0.6, the second with probability 0.5. Assume that the results of the flips are independent, and let X equal the total number of heads that result. The value of $E[X]$ is

(1) 0.11 (2) 1.1 (3) 11 (4) 1

10. On a multiple-choice exam with 3 possible destructives for each of the 5 questions, the probability that a student will get 4 or more correct answers just by guessing is

(1) $\frac{11}{243}$ (2) $\frac{3}{8}$ (3) $\frac{1}{243}$ (4) $\frac{5}{243}$

11. If $P(X = 0) = 1 - P(X = 1)$. If $E[X] = 3Var(X)$, then $P(X = 0)$ is

(1) $\frac{2}{3}$ (2) $\frac{2}{5}$ (3) $\frac{1}{5}$ (4) $\frac{1}{3}$

12. If X is a binomial random variable with expected value 6 and variance 2.4, then $P(X = 5)$ is

(1) $\binom{10}{5}\left(\frac{3}{5}\right)^6\left(\frac{2}{5}\right)^4$ (2) $\binom{10}{5}\left(\frac{3}{5}\right)^{10}$ (3) $\binom{10}{5}\left(\frac{3}{5}\right)^4\left(\frac{2}{5}\right)^6$ (4) $\binom{10}{5}\left(\frac{3}{5}\right)^5\left(\frac{2}{5}\right)^5$

13. The random variable X has the probability density function $f(x) = \begin{cases} ax + b & 0 < x < 1 \\ 0 & otherwise \end{cases}$ and $E(X) = \frac{7}{12}$, then a and b are respectively

(1)1 and $\frac{1}{2}$ (2) $\frac{1}{2}$ and 1 (3) 2 and 1 (4) 1 and 2

14. Suppose that X takes on one of the values 0,1 and 2. If for some constant k, $P(X = i) = kP(X = i - 1)$ for $i = 1,2$ and $P(X = 0) = \frac{1}{7}$, then the value of k is

(1) 1 (2) 2 (3) 3 (4) 4

15. Which of the following is a discrete random variable?

I. The number of cars crossing a particular signal in a day.
II. The number of customers in a queue to buy train tickets at a moment.

III. The time taken to complete a telephone call.

(1) I and II (2) II only (3)III only (4) II and III

16. If $f(x) = \begin{cases} 2x & 0 \le x \le a \\ 0 & otherwise \end{cases}$ is a probability density function of a random variable, then the value of a is

(1) 1 (2) 2 (3) 3 (4) 4

17. The probability mass function of random variable is defined as:

x	-2	-1	0	1	2
$f(x)$	k	2k	3k	4k	5k

Then $E(X)$ is equal to

(1) $\frac{1}{15}$ (2) $\frac{1}{10}$ (3) $\frac{1}{3}$ (4) $\frac{2}{3}$

18. Let X have a Bernoulli distribution with mean 0.4, then the variance of $(2X-3)$ is

(1) 0.24 (2) 0.48 (3) 0.6 (4) 0.96

19. If in 6 trails, X is a binomial variable which follows the relation $9P(X=4) = P(X=2)$, then the probability of success is

(1) 0.125 (2) 0.25 (3) 0.375 (4) 0.75

20. A computer salesperson knows from his past experience that he sells computers to one in every twenty customers who enter the showroom. What is the probability that he will sell a computer to exactly two of the next three customers?

(1) $\frac{57}{20^3}$ (2) $\frac{57}{20^2}$ (3) $\frac{19^3}{20^3}$ (4) $\frac{57}{20}$

12. DISCRETE MATHEMATICS

1. A binary operation on a set S is a function from

(1) $S \to S$ (2) $(S \times S) \to S$ (3) $S \to (S \times S)$ (4) $(S \times S) \to (S \times S)$

2. Subtraction is not a binary operation in

(1) $\mathbb{R}$ (2) $\mathbb{Z}$ (3) $\mathbb{N}$ (4) $\mathbb{Q}$

3. Which of the following is a binary operation on $\mathbb{N}$?

(1) Subtraction (2) Multiplication (3) Division (4) All the above

4. In the set $\mathbb{R}$ of real numbers '$*$' is defined as follows. Which one of the following is not a binary operation on $\mathbb{R}$?

(1) $a * b = \min(a, b)$ (2) $a * b = \max(a, b)$ (3) $a * b = a$ (4) $a * b = a^b$

5. The operation $*$ defined by $a * b = \frac{ab}{7}$ is not binary operation on

(1) $\mathbb{Q}^+$ (2) $\mathbb{Z}$ (3) $\mathbb{R}$ (4) $\mathbb{C}$

6. In the set $\mathbb{Q}$ define $a \odot b = a + b + ab$. For what value of y, $3 \odot (y \odot 5) = 7$?

(1) $y = \frac{2}{3}$ (2) $y = \frac{-2}{3}$ (3) $y = \frac{-3}{2}$ (4) $y = 4$

7. If $a * b = \sqrt{a^2 + b^2}$ on the real numbers then $*$ is

(1) commutative but not associative (2) associative but not commutative

(3) both commutative and associative (4) neither commutative nor associative

8. Which of the following statements has the truth value T?

(1) $sinx$ is an even function (2) Every square matrix is non-singular (3) The product of complex number and its conjugate is purely imaginary (4) $\sqrt{5}$ is an irrational numbber

9. Which of the following statements has the truth value F?

(1) Chennai is in India or $\sqrt{2}$ is an integer (2) Chennai is in India or $\sqrt{2}$ is an irrational number

(3) Chennai is in China or $\sqrt{2}$ is an integer (4) Chennai is in China or $\sqrt{2}$ is an irrational number

10. If a compound statement involves 3 simple statements, then the number of rows in the truth table is

(1) 9 (2) 8 (3) 6 (4) 3

11. Which one is the inverse of the statement $(p \vee q) \to (p \wedge q)$?

(1) $(p \wedge q) \rightarrow (p \vee q)$ (2) $\neg(p \vee q) \rightarrow (p \wedge q)$
(3) $(\neg p \vee \neg q) \rightarrow (\neg p \wedge \neg q)$ (4) $(\neg p \wedge \neg q) \rightarrow (\neg p \vee \neg q)$

12. Which one is the contra positive of the statement $(p \vee q) \rightarrow r$?
(1) $\neg r \rightarrow (\neg p \wedge \neg q)$ (2) $\neg r \rightarrow (p \vee q)$ (3) $r \rightarrow (p \wedge q)$ (4) $p \rightarrow (q \vee r)$

13. The truth table for $(p \wedge q) \vee (\neg q)$ is given below

p	q	$(p \wedge q) \vee (\neg q)$
T	T	(a)
T	F	(b)
F	T	(c)
F	F	(d)

Which one of the following is true?

	(a)	(b)	(c)	(d)
(1)	T	T	T	T
(2)	T	F	T	T
(3)	T	T	F	T
(4)	T	F	F	F

14. In the last column of the truth table for $(\neg(p \vee \neg q))$ the number of final outcomes of the truth value 'F' are
(1) 1 (2) 2 (3) 3 (4) 4

15. Which one of the following is incorrect? For any two propositions p and q, we have
(1) $\neg(p \vee q) \equiv \neg p \wedge \neg q$ (2) $\neg(p \wedge q) \equiv \neg p \vee \neg q$ (3) $\neg(p \wedge q) \equiv \neg p \vee \neg q$
(4) $\neg(\neg p) \equiv p$

16.

p	q	$(p \wedge q) \vee (\neg p)$
T	T	(a)
T	F	(b)
F	T	(c)
F	F	(d)

Which one of the following is correct for the truth value of $(p \wedge q) \vee (\neg p)$?

	(a)	(b)	(c)	(d)
(1)	T	T	T	T
(2)	F	T	T	T
(3)	F	F	T	T
(4)	T	T	T	F

17. The dual of $\neg(p \vee q) \vee [p \vee (p \wedge \neg r)]$ is
(1) $(\neg(p \wedge q) \wedge [p \vee (p \wedge \neg r)]$ (2) $(p \wedge q) \wedge [p \wedge (p \vee \neg r)]$
(3) $\neg(p \wedge q) \wedge [p \wedge (p \wedge r)]$ (4) $\neg(p \wedge q) \wedge [p \wedge (p \vee \neg r)]$

18. The proposition is $p \wedge (\neg p \vee q)$ is
(1) a tautology (2) a contradiction (3) logically equivalent to $p \wedge q$
(4) logically equivalent to $p \vee q$

19. Determine the truth value of each of the following statements:
(a) $4 + 2 = 5$ and $6 + 3 = 9$ (b) $3 + 2 = 5$ and $6 + 1 = 7$
(c) $4 + 5 = 9$ and $1 + 2 = 4$ (d) $3 + 2 = 5$ and $4 + 7 = 11$

	(a)	(b)	(c)	(d) is
(1)	F	T	F	T
(2)	T	F	T	F
(3)	T	T	F	F
(4)	F	F	T	T

20. Which one of the following is not true?
(1) Negation of a negation of a statement is the statement itself.
(2) If the last column of the truth table contains only T then it is tautology.
(3) If the last column of the truth table contains only F then it is a contradiction.
(4) If p and q are any two statements then $p \leftrightarrow q$ is a tautology.

Previous year creative questions:

1. Prove that $\int_0^{\frac{\pi}{2}} \frac{f(sinx)}{f(six)+f(cosx)}\, dx = \frac{\pi}{4}$. **March 2020 – 2M**

2. Find the equation of the parabola if the curve is open leftward, vertex is $(2,1)$ and passing through the point $(1,3)$. **March 2020 – 2M**

3. Find the critical numbers(only x values) of the function $f(x) = x^{\frac{4}{5}}(x-4)^2$. **March 2020 – 3M**

4. Let X be a continuous random variable and $f(x)$ is defined as: $f(x) = \begin{cases} kx(1-x)^{10}, & 0 < x < 1 \\ 0 & otherwise \end{cases}$
Find the value k. **March 2020 – 3M**

5.If the lines $\frac{x-x_1}{l_1} = \frac{y-y_1}{m_1} = \frac{z-z_1}{n_1}$ and $\frac{x-x_2}{l_2} = \frac{y-y_2}{m_2} = \frac{z-z_2}{n_2}$ lie on the same plane, then write the number of ways to find the Cartesian equation of the above plane and explain in detail. **March 2020 – 3M**

6.A square shaped thin material with area 196 sq.units to make into an open box by cutting small equal squares from the four corners and folding the sides upward. Prove that the length of the side of a removed square is $\frac{7}{3}$ when the volume of the box is maximum. **March 2020 – 5M**

7. Three fair coins are tossed once. Find the probability mass function, mean and variance for number of heads occurred. Verify the results by binomial distribution. **March 2020 – 5M**

8. Find the least positive integer n such that $\left(\frac{1+i}{1-i}\right)^n = 1$. **Sep 2020 – 2M**

9. Find the differential equation of the family of $y = ax^2 + bx + c$ where a, b are parameters and c is a constant. **Sep 2020 – 2M**

10. Show that, if $x = rcos\theta$, $y = rsin\theta$, then $\frac{\partial r}{\partial x}$ is equal to $cos\theta$. **Sep 2020 – 2M**

11. Suppose that $z = ye^{x^2}$, where $x = 2t$ and $y = 1 - t$ then find $\frac{dz}{dt}$. **Sep 2020 – 3M**

12. Show that $((\neg q)\wedge p)\wedge q$ is a contradiction. **Sep 2020 – 3M**

13. Prove that $\arg(z_1 z_2) = \arg(z_1) + \arg(z_2)$. **Sep 2020 – 5M**

14. Draw the graph of $tanx$ in $\left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$ and $tan^{-1}x$ in $(-\infty, \infty)$. **Sep 2020 – 5M**

15. A Car A is travelling from west at 50 km/hr and Car B is travelling towards north at 60 km/hr. Both are headed for the intersection of the two roads. At what rate are the Cars approaching each other when Car A is 0.3 kilometers and Car B is 0.4 kilometers from the intersection? **Sep 2020 – 5M**

16. Find the area of the region bounded by the ellipse $\frac{x^2}{25} + \frac{y^2}{16} = 1$ and its latus rectums. **Sep 2020- 5M**

17. If $z = (2+3i)(1-i)$, then prove that $z^{-1} = \frac{5}{6} - i\frac{1}{26}$. **Aug 2021 – 2M**

18. If α and β are the roots of $x^2 - 5x + 6 = 0$ then prove that $\alpha^2 - \beta^2 = \pm 5$. **Aug 2021 – 2M**

19. Show that the differential equation corresponding to $y = Asinx$, where A is an arbitrary constant is, $y = y'tanx$. **Aug 2021 – 2M**

20. A force $13\hat{\imath} + 10\hat{\jmath} - 3\hat{k}$ acts on a particle which is displaced from the point with position vector $4\hat{\imath} - 3\hat{\jmath} - 2\hat{k}$ to the point with position vector $6\hat{\imath} + \hat{\jmath} - 3\hat{k}$. Show that the work done by the force is 69 units. **Aug 2021 – 3M**

21. An egg of a particular bird is spherical shape. If the radius to the inside of the shell is 4 mm and radius to the outside of the shell is 4.2 mm, prove that the approximate volume of the shell is $12.8\pi\ mm^3$ **Aug 2021 – 3M**

22. Show that $\int_0^1 \frac{\sqrt{x}}{\sqrt{1-x}+\sqrt{x}}dx = \frac{1}{2}$. **Aug 2021 – 3M**

23. Solve the system of equations $x - y + 2z = 2, 2x + y + 4z = 7, 4x - y + z = 4$ by cramer's rule. **Aug 2021 – 5M**

24. Find the eccentricity, centre, vertices and foci of the ellipse $\frac{x^2}{16} + \frac{y^2}{9} = 1$ and also draw the rough diagram. **Aug 2021 – 5M**

25. Show that the Cartesian equation of the plane passing through the points $(1, 2, 3)$ ad $(2, 3, 1)$ and also perpendicular to the plane $3x - 2y + 4z + 5 = 0$ is $2y + z - 7 = 0$. **Aug 2021 – 5M**

26. Show that the differential equation of the family of curves $y = Ae^x + Be^{-x}$, where A and B are arbitrary constants, is $\frac{d^2y}{dx^2} - y = 0$. **May 2022 – 2M**

27. Show that the distance from the origin to the plane $3x + 6y + 2z + 7 = 0$ is 1. **May 2022 – 2M**

28. Prove that the general equation of the circle whose diameter is the line segment joining the points $(-4, -2)$ and $(-1, -1)$, is $x^2 + y^2 + 5x + 3y + 6 = 0$. **May 2022 – 3M**

29. Cramer's rule is not applicable to solve the system $3x + y + z = 2, x - 3y + 2z = 1, 7x - y + 4z = 5$. Why? **May 2022 – 5M**

30. The distribution function of a continuous random variable X is:

$F(x) = \begin{cases} 0, & x < 1 \\ \frac{x-1}{4}, & 1 \le x \le 5 \\ 1, & x > 5 \end{cases}$ Find (i) $P(X < 3)$ (ii) $P(2 < X < 4)$ (iii) $P(3 \le X)$ **May 2022 – 5M**

31. If α and β are the roots of $x^2 + 5x + 6 = 0$ then show that $\alpha^2 + \beta^2 = 12$ **July 2022 – 2M**

32. Find the value of $sin^{-1}(1) + cos^{-1}(1)$ **July 2022 – 2M**

33. Form the differential equation of the curve $y = ax^2 + bx + c$, where a, b, c are arbitrary constants. **July 2022 – 2M**

34. Prove that the roots of the equation $x^4 - 3x^2 - 4 = 0$ are $\pm 2, \pm i$ **July 2022 – 3M**

35. Prove that $\int_0^1 xe^x dx = 1$. **July 2022 – 3M**

36. Show that the area between the parabola $y^2 = 16x$ and its latusrectum (using integration) is $\frac{128}{3}$ **July 2022 – 3M**

37.Show that the Cartesian equation of the plane passing through the points $(a, 0, 0), (0, b, 0)$, $(0, 0, c)$ is $\frac{x}{a} + \frac{y}{b} + \frac{z}{c} = 1$ **July 2022 – 5M**

38. If the radius of a sphere with radius $10cm$, has to decrease by $0.1cm$, approximately how much will its volume decrease? **March 2023 – 2M**

39. Express $e^{cos\theta + isin\theta}$ in $a + ib$ form. **March 2023 – 2M**

40. If $z = (2 + 3i)(1 - i)$, then find z^{-1}. **March 2023 – 3M**

41. If $a + b + c = 0$ and a, b, c are rational numbers then, prove that the roots of the equation $(b + c - a)x^2 + (c + a - b)x + (a + b - c) = 0$ are rational numbers. **March 2023 – 3M**

42. Find the maximum value of $\frac{\log x}{x}$ **March 2023 – 5M**

43. Find the area of the smallest region to the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ and the straight line $\frac{x}{a} + \frac{y}{b} = 1$. **March 2023 – 5M**