CHAPTER - 7

EXERCISE 7.1

① Soln:-

Let $s(t) = 2t^2+3t$

put $t=3$, $s(3) = 2(3^2)+3(3) = 18+9$

$s(3) = 27$

Put $t=6$, $s(6) = 2(6^2)+3(6) = 72+18$

$s(6) = 90$

(i) Average velocity $= \frac{s(6)-s(3)}{6-3}$

$= \frac{90-27}{3}$

$= 21$ m/s

(ii) Instantaneous velocity

$s'(t) = 4t+3$

Put $t=3$, $s'(3) = 4(3)+3 = 12+3$

$s'(3) = 15$ m/s

put $t=6$, $s'(6) = 4(6)+3 = 24+3$

$s'(6) = 27$ m/s

② Soln:-

Given $s(t) = 400$ ft

(i) $s(t) = 16t^2$

$400 = 16t^2$

$25 = t^2$

$t = 5$ sec

It taken 5 sec to hit the ground

(ii) Average velocity $= \frac{s(5)-s(3)}{5-3}$

$= \frac{16(25)-16(9)}{2}$

$= \frac{16(25-9)}{2}$

$= 8\times16$

$= 128$ ft/sec

(iii) Instantaneous velocity at $t=5$ is

$s'(t) = 32t$

$s'(5) = 32(5) = 160$ ft/sec

③ Soln:-

Let $s(t) = 2t^3-9t^2+12t-4$

Velocity $s'(t) = 6t^2-18t+12$

$v = 6(t^2-3t+2)$

(i) At $v=0$

$6(t^2-3t+2)=0$

$t^2-3t+2=0$

$(t-1)(t-2)=0$

$t=1, t=2$

partical change direction at $t=1$ and $t=2$

(ii) Distance travelled first 4 sec is

$|s(0)-s(1)|+|s(1)-s(2)|+|s(2)-s(4)|$

$=|-4-1|+|1-0|+|0-28|$

$=5+1+28$

$=34$ m

(iii) Acceleration $(a) = v'(t) = 6(2t-3)$

$a = 6(2t-3)$

At, $t=1$

$a = 6(2-3) = -6$ units/sec

At, $t=2$

$a = 6(4-3) = 6$ units/sec

④ Soln:-

$V = x^3$

$\frac{dV}{dx} = 3x^2$

At $x=5$

$\frac{dV}{dx} = 3(25) = 75$ units

⑤ Soln:-

Let $m = \sqrt{3x} = \sqrt{3}\,x^{\frac{1}{2}}$

$\frac{dm}{dx} = \sqrt{3}\times\frac{1}{2}x^{-\frac{1}{2}}$

$\frac{dm}{dx} = \frac{\sqrt{3}}{2\sqrt{x}}$

At $x=3$

$\frac{dm}{dx} = \frac{\sqrt{3}}{2\sqrt{3}} = \frac{1}{2}$ kg/m

At $x=27$

$\frac{dm}{dx} = \frac{\sqrt{3}}{2\sqrt{27}} = \frac{\sqrt{3}}{2\times3\sqrt{3}} = \frac{1}{6}$ kg/m

⑥ Soln:-

Given $r = 5$cm and $\frac{dr}{dt} = 2$cm/sec

Area $(A) = \pi r^2$

$\frac{dA}{dt} = \pi(2r)\frac{dr}{dt}$

$= \pi(2)(5)(2)$

$= 20\pi$ cm^2/sec

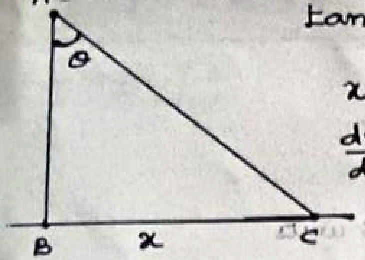
⑦ **Soln:-**

10 sec per one revolution is 2π

1 sec revolution $= \frac{2\pi}{10}$

$$\frac{d\theta}{dt} = \frac{\pi}{5} \text{ rad/sec}$$

$\theta = 45°$



$\tan\theta = \frac{x}{5}$

$x = 5\tan\theta$

$$\frac{dx}{dt} = 5\sec^2\theta \frac{d\theta}{dt}$$

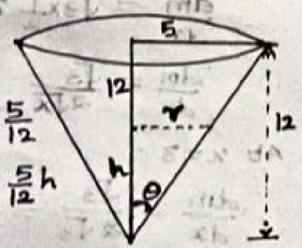
$$= 5(\sec 45°)^2 \frac{\pi}{5}$$

$$= 5(\sqrt{2})^2 \frac{\pi}{5}$$

$$\frac{dx}{dt} = 2\pi \text{ km/sec}$$

⑧ **Soln:-**

Let $\frac{dV}{dt} = 10$



$\frac{r}{h} = \frac{5}{12}$

$r = \frac{5}{12}h$

Volume$(V) = \frac{1}{3}\pi r^2 h$

$$V = \frac{1}{3}\pi\left(\frac{5}{12}h\right)^2 h$$

$$V = \frac{\pi(25)h^3}{3\times144}$$

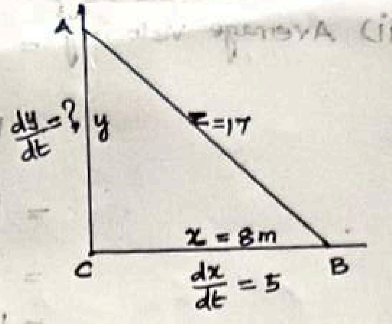
$$\frac{dV}{dt} = \frac{25\pi}{3\times144} 3h^2 \frac{dh}{dt}$$

$$10 = \frac{25\pi}{144}(8)^2 \frac{dh}{dt}$$

$$2 = \frac{5\pi\times64}{144} \frac{dh}{dt}$$

$$\frac{dh}{dt} = \frac{2\times144}{5\pi\times64} = \frac{9}{10\pi} \text{ m/min.}$$

⑨ **Soln:-**



In ΔABC

$x^2 + y^2 = 17^2$

$64 + y^2 = 289$

$y^2 = 225$

$\boxed{y = 15}$

(i)

$x^2 + y^2 = 17^2$

Diff. w. r. to 't'

$$2x\frac{dx}{dt} + 2y\frac{dy}{dt} = 0$$

$$x\frac{dx}{dt} + y\frac{dy}{dt} = 0$$

$$8(5) + (15)\frac{dy}{dt} = 0$$

$$\frac{dy}{dt} = \frac{-40}{15}$$

$$\frac{dy}{dt} = \frac{-8}{3} \text{ m/sec}$$

(ii)

Area$(A) = \frac{1}{2}xy$

Diff. w. r. to 't'

$$\frac{dA}{dt} = \frac{1}{2}\left[x\frac{dy}{dt} + y\frac{dx}{dt}\right]$$

$$= \frac{1}{2}\left[8\left(\frac{-8}{3}\right) + 15(5)\right]$$

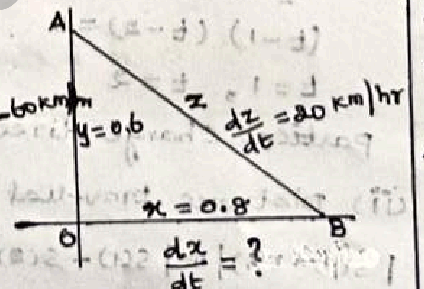
$$= \frac{1}{2}\left[\frac{-64}{3} + 75\right]$$

$$= \frac{1}{2}\left[\frac{-64+225}{3}\right]$$

$$= \frac{161}{6}$$

$$\frac{dA}{dt} = 26.83 \text{ m}^2\text{/sec}$$

⑩ **Soln:-**



In ΔOAB

$z^2 = x^2 + y^2$

$z^2 = (0.8)^2 + (0.6)^2$

$z^2 = 0.64 + 0.36$

$z^2 = 1$

$\boxed{z = 1}$

$z^2 = x^2 + y^2$

Diff. w. r. to 't'

$$2z\left(\frac{dz}{dt}\right) = 2x\left(\frac{dx}{dt}\right) + 2y\left(\frac{dy}{dt}\right)$$

$$(1)(20) = (0.8)\left(\frac{dx}{dt}\right) + (0.6)(-60)$$

$$20 = (0.8)\left(\frac{dx}{dt}\right) + (-36)$$

$$56 = (0.8)\left(\frac{dx}{dt}\right)$$

$$\frac{dx}{dt} = \frac{56}{0.8}$$

$$\frac{dx}{dt} = 70 \text{ km/hr}$$

Example 7.1

Soln:-

Let $f(x) = x^2$

At $[0, 0.5]$

$$\text{Average velocity} = \frac{f(0.5)-f(0)}{0.5-0} = \frac{0.25}{0.5} = 0.5$$

At $[0.5, 1]$

$$\text{Average Velocity} = \frac{f(1)-f(0.5)}{1-0.5} = \frac{1-0.25}{0.5} = \frac{0.75}{0.5} = 1.5$$

At $[1, 1.5]$

$$\text{Avg velocity} = \frac{f(1.5)-f(1)}{1.5-1} = \frac{2.25-1}{0.5} = \frac{1.25}{0.5} = 2.5$$

At $[1.5, 2]$

$$\text{Avg Velocity} = \frac{f(2)-f(1.5)}{2-1.5} = \frac{4-2.25}{0.5} = \frac{1.75}{0.5} = 3.5$$

Intantaneous Velocity:

$f'(x) = 2x$

At $x = 0.5$

$f'(0.5) = 2(0.5) = 1$

At $x=1$,

$f'(1) = 2(1) = 2$

At $x=1.5$

$f'(1.5) = 2(1.5) = 3$

At $x=2$

$f'(2) = 2(2) = 4$

Example 7.2

Soln:-

Let $T = x(10-x)$

$T = 10x - x^2$

$\frac{dT}{dx} = 10 - 2x$

At $x=5$

$\frac{dT}{dx} = 10 - 2(5) = 0$

Example 7.3

Soln:-

Let $W(t) = 100(1-0.1t)^2$

$W'(t) = 100 \times 2(1-0.1t)(-0.1)$

$W'(t) = -20(1-0.1t)$

At $t=2$

$W'(t) = -20(1-0.2)$

$= -20(0.8)$

$= -16$

$\therefore$ The person forget at the rate of 16 words after 2 days.

Example 7.4

Soln:-

$S(t) = \frac{t^3}{3} - t^2 + 3$

$V = S'(t) = \frac{3t^2}{3} - 2t + 0$

$\boxed{V = t^2 - 2t}$

$a = V' = 2t - 2$

$\boxed{a = 2t-2}$

At $V=0$

$t^2 - 2t = 0$

$t(t-2) = 0$

$\boxed{t=0}, \boxed{t=2}$

At $t=2$, Velocity is 0

At $a=0$

$2t - 2 = 0$

$2(t-1) = 0$

$t - 1 = 0$

$\boxed{t=1}$

At $t=1$, acceleration is 0.

Example 7.5

Soln:-

(i) $S(t) = 128t - 16t^2 \rightarrow ①$

$V = S'(t) = 128 - 32t \rightarrow ②$

At $V=0$

$128 - 32t = 0$

$32t = 128$

$\boxed{t=4}$

At $t=4$

Max hight $= 128(4) - 16(4)^2$

$= 512 - 256$

$= 256$ ft

When the partical hits the ground then $S=0$

$① \Rightarrow$

$128t - 16t^2 = 0$

$t(128 - 16t) = 0$

$\boxed{t=0}$ $\quad 128 = 16t$

$\boxed{t=8}$

$\therefore$ The particle hits the ground at $t=8$ sec

$\therefore$ The Velocity at $t=8$

$② \Rightarrow V(8) = 128 - 32(8)$

$= 128 - 256$

$V = -128$ ft/s

Example 7.6

Soln:-

$S(t) = t^3 - 6t^2 + 9t + 1$

$V = S'(t) = 3t^2 - 12t + 9$

$a = V'(t) = 6t - 12$

(i) What time the particle is at rest

$V=0$

$3t^2 - 12t + 9 = 0$

$\div 3 \quad t^2 - 4t + 3 = 0$

$(t-3)(t-1) = 0$

$\boxed{t=1}$ $\boxed{t=3}$ $\therefore$ Particle rest at $t=1,3$

(ii) Particle change direction at $t=1$ and $t=3$

(iii) Total distance is 2 sec $=$

$|S(1) - S(0)| + |S(2) - S(1)| \longrightarrow ①$

$s(0) = 1$

$s(1) = 1-6+9+1 = 5$

$s(2) = 8-24+18+1 = 3$

① → Total distance $= |5-1| + |3-5|$

$= 4+2$

$= 6\,m$

Example 7.7

Soln:- Given $\frac{dV}{dt} = 1000\ cm^3/sec$ | $\frac{dr}{dt} = ?$

$r = 7\ cm$ | $\frac{ds}{dt} = ?$

Volume of baloon (V) $= \frac{4}{3}\pi r^3$

$$\frac{dV}{dt} = \frac{4\pi}{3} \times 3r^2\left(\frac{dr}{dt}\right)$$

$$1000 = 4\pi \times 49 \left(\frac{dr}{dt}\right)$$

$$\frac{250}{49\pi} = \frac{dr}{dt}$$

Surface area (S) $= 4\pi r^2$

$$\frac{ds}{dt} = 4\pi(2r)\frac{dr}{dt}$$

$$= 8\pi(7) \times \frac{250}{49\pi}$$

$$\frac{ds}{dt} = \frac{2000}{7}\ cm^2/sec$$

Rate of change of radius is $\frac{250}{49\pi}$

Rate of change of surface area is $\Big\} = \frac{2000}{7}\ cm^2/s$

Example 7.8

Soln:-

$$Px + 3P - 16x = 234$$

$$P(x+3) = 16x + 234$$

$$P = \frac{16x+234}{x+3}$$

$$\frac{dP}{dt} = \frac{(x+3)(16)\frac{dx}{dt} - (16x+234)\frac{dx}{dt}}{(x+3)^2}$$

$$\frac{dP}{dt} = \frac{\frac{dx}{dt}(16x+48-16x-234)}{(x+3)^2}$$

$$\frac{dP}{dt} = \frac{dx}{dt}\left(\frac{-186}{(x+3)^2}\right)$$

At $x = 90$, $\frac{dx}{dt} = 15$

$$\frac{dP}{dt} = \frac{15 \times -186}{(93)^2} = \frac{-930}{8649} = -0.32$$

∴ Price is decreasing at a rate of 0.32 rupees/week

Example 7.9

Soln:- Given $\frac{dV}{dt} = 30\ m^3/min$

$D = h \Rightarrow 2r = h \Rightarrow \boxed{r = \frac{h}{2}}$

$h = 10\,m$

Volume $V = \frac{1}{3}\pi r^2 h$

$$= \frac{1}{3}\pi\left(\frac{h}{2}\right)^2 h$$

$$V = \frac{\pi}{12}h^3$$

$$\frac{dV}{dt} = \frac{\pi}{12}(3h^2)\frac{dh}{dt}$$

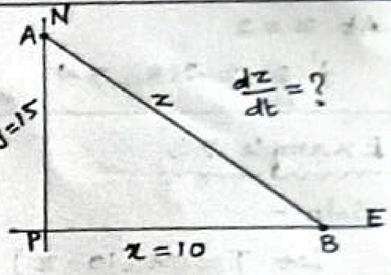
$$30 = \frac{\pi}{12} \times 3 \times 100 \left(\frac{dh}{dt}\right)$$

$$\frac{6}{5\pi} = \frac{dh}{dt}$$

$$\therefore \frac{dh}{dt} = \frac{6}{5\pi}\ m/min$$

Example 7.10

Soln:-



$$z^2 = x^2 + y^2$$

$$z^2 = 100 + 225$$

$$z = \sqrt{325}$$

$$z = 5\sqrt{13}$$

$\frac{dx}{dt} = 80\ km/hr$

$\frac{dy}{dt} = 100\ km/hr$

$$z^2 = x^2 + y^2$$

$$2z\left(\frac{dz}{dt}\right) = 2x\left(\frac{dx}{dt}\right) + 2y\left(\frac{dy}{dt}\right)$$

$$5\sqrt{13}\left(\frac{dz}{dt}\right) = 10(80) + 15(100)$$

$$\frac{dz}{dt} = \frac{800}{5\sqrt{13}} + \frac{1500}{5\sqrt{13}} = \frac{2300}{5\sqrt{13}}$$

$$\frac{dz}{dt} = \frac{460}{\sqrt{13}} \times \frac{\sqrt{13}}{\sqrt{13}}$$

$$= \frac{460 \times 3.61}{13}$$

$$\frac{dz}{dt} = 127.7\ km/hr$$

Distance between two cars increasing at a ratio of 127.7 km/hr.

Example : 7.11

Soln:-

$y = x^2 + 3x - 2$

$\frac{dy}{dx} = 2x + 3$

At (1,2)

$\frac{dy}{dx} = 2 + 3 = 5$

Equation of tangent

$y - y_1 = m(x - x_1)$

$y - 2 = 5(x - 1)$

$y - 2 = 5x - 5$

$5x - y - 3 = 0$

Equation of Normal

$x + 5y + k = 0 \rightarrow ①$

At (1,2)

$1 + 10 + k = 0$

$k = -11$

$① \Rightarrow \quad x + 5y - 11 = 0$

Example 7.12

Soln:-

$y = x^3 - 3x^2 + x - 2 \rightarrow ①$

$\frac{dy}{dx} = 3x^2 - 6x + 1$

Given $y - x = 0$

$\frac{dy}{dx} - 1 = 0 \Rightarrow \frac{dy}{dx} = 1$

Since slopes are parallel

$3x^2 - 6x + 1 = 1$

$3x(x - 2) = 0$

$3x = 0 \qquad x - 2 = 0$

$\boxed{x = 0} \qquad \boxed{x = 2}$

$① \Rightarrow$

$y = -2 \qquad \Big| \quad y = 8 - 3(4) + 2 - 2$

$\therefore y = -4$

∴ points are

$(0, -2)$ and $(2, -4)$

Example 7.13

Soln:-

$x = 2\cos 3t, \qquad y = 3\sin 2t$

$\frac{dx}{dt} = -2\sin 3t(3) \qquad \frac{dy}{dt} = 3\cos 2t(2)$

$= -6\sin 3t \qquad = 6\cos 2t$

$\frac{dy/dt}{dx/dt} = \frac{6\cos 2t}{-6\sin 3t}$

$\frac{dy}{dx} = -\frac{\cos 2t}{\sin 3t}$

Equation of tangent is

$y - y_1 = m(x - x_1)$

$y - 3\sin 2t = -\frac{\cos 2t}{\sin 3t}(x - 2\cos 3t)$

$y\sin 3t - 3\sin 2t\sin 3t = -x\cos 2t + 2\cos 2t\cos 3t$

$x\cos 2t + y\sin 3t = 3\sin 2t\sin 3t + 2\cos 2t\cos 3t$

Equation of Normal is

$x\sin 3t - y\cos 2t = k \rightarrow ①$

At $(2\cos 3t, 3\sin 2t)$

$2\cos 3t\sin 3t - 3\sin 2t\cos 2t = k$

$\sin 2(3t) - 3\frac{\sin 2(2t)}{2} = k$

$\sin 6t - \frac{3}{2}\sin 4t = k$

$① \Rightarrow$

$x\sin 3t - y\cos 2t = \sin 6t - \frac{3}{2}\sin 4t.$

Example 7.14

Soln:-

$y = x^2 \rightarrow ①$

$y = (x - 3)^2 \rightarrow ②$

Sub ① in ②

$x^2 = (x - 3)^2$

$x^2 = x^2 - 6x + 9$

$6x = 9$

$\boxed{x = 3/2}$

$① \Rightarrow y = \frac{9}{4}$

∴ point of intersection

$(x, y) = \left(\frac{3}{2}, \frac{9}{4}\right)$

Given $y = x^2$

$\frac{dy}{dx} = 2x$

$m_1 = \frac{dy}{dx} = 2\left(\frac{3}{2}\right)$

$\boxed{m_1 = 3}$

$y = (x - 3)^2$

$\frac{dy}{dx} = 2(x - 3)$

$m_2 = \frac{dy}{dx} = 2\left(\frac{3}{2} - 3\right)$

$= 3 - 6$

$\boxed{m_2 = -3}$

$$\therefore \theta = \tan^{-1}\left|\frac{m_1-m_2}{1+m_1m_2}\right|$$

$$\therefore \theta = \tan^{-1}\left|\frac{3+3}{1-9}\right|$$

$$= \tan^{-1}\left|\frac{6}{-8}\right|$$

$$\theta = \tan^{-1}\left(\frac{3}{4}\right)$$

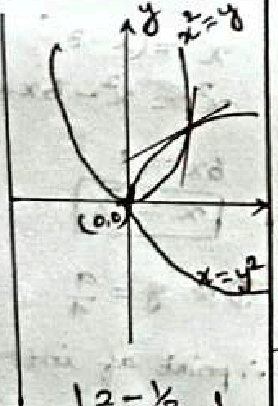
Example 7.15

Soln:-

At $(0,0)$

Angle between two curve $=$ { Angle between their tangents x and y axis

$$= \frac{\pi}{2}$$



Given,

$y = x^2$

$\frac{dy}{dx} = 2x$

$m_1 = \left(\frac{dy}{dx}\right)_{(1,1)} = 2(1)$

$\boxed{m_1 = 2}$

$x = y^2$

$1 = 2y\left(\frac{dy}{dx}\right)$

$\frac{1}{2y} = \frac{dy}{dx}$

$\boxed{m_2 = \frac{1}{2}}$

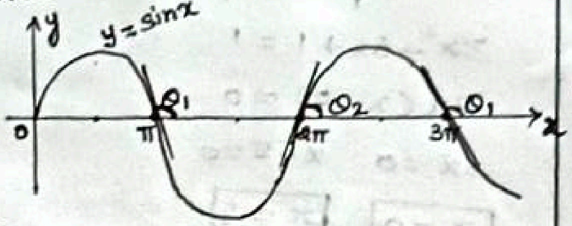
$$\theta = \tan^{-1}\left|\frac{m_1-m_2}{1+m_1m_2}\right| = \tan^{-1}\left|\frac{2-\frac{1}{2}}{1+2(\frac{1}{2})}\right|$$

$$= \tan^{-1}\left|\frac{3/2}{2}\right|$$

$$\theta = \tan^{-1}\left(\frac{3}{4}\right)$$

Example 7.16

Soln:-



$y = \sin x$ | $y = 0$

$\frac{dy}{dx} = \cos x$ | $\frac{dy}{dx} = 0$

(i) At $x = \pi, 3\pi, 5\pi, 7\pi \ldots$

$\frac{dy}{dx} = \cos(\pi) = -1$ | $\frac{dy}{dx} = 0$

$\boxed{m_1 = -1}$ | $\boxed{m_2 = 0}$

$$\theta_1 = \tan^{-1}\left(\frac{m_1-m_2}{1+m_1m_2}\right)$$

$$= \tan^{-1}\left(\frac{-1-0}{1+(-1)0}\right)$$

$$= \tan^{-1}(-1)$$

$$= \pi - \frac{\pi}{4}$$

$$\boxed{\theta_1 = \frac{3\pi}{4}}$$

(ii) At $x = 2\pi, 4\pi \ldots$

$m_1 = 1,\ m_2 = 0$

$$\theta_2 = \tan^{-1}\left(\frac{1-0}{1+0}\right)$$

$$\theta_2 = \tan^{-1}(1)$$

$$\boxed{\theta_2 = \frac{\pi}{4}}$$

Example 7.17

Soln:-

Let point of intersection be (x_1, y_1)

$$ax^2 + by^2 = 1$$

Diff. w.r.to 'x'

$$a(2x) + b(2y)\frac{dy}{dx} = 0$$

$$by\left(\frac{dy}{dx}\right) = -ax$$

$$\frac{dy}{dx} = \frac{-ax}{by}$$

$$m_1 = \left(\frac{dy}{dx}\right)_{(x_1,y_1)} = \frac{-ax_1}{by_1}$$

similarly;

$$m_2 = \frac{-cx_1}{dy_1}$$

Since, orthogonal

$$m_1 \cdot m_2 = -1$$

$$\frac{-ax_1}{by_1} \cdot \frac{-cx_1}{dy_1} = -1$$

$$acx_1^2 = -bdy_1^2$$

$$acx_1^2 + bdy_1^2 = 0 \to (1)$$

Given at (x_1, y_1)

$$ax_1^2 + by_1^2 = 1$$

$$cx_1^2 + dy_1^2 = 1$$

$$(-) \quad (-) \quad (-)$$

$$(a-c)x_1^2 + (b-d)y_1^2 = 0 \to (2)$$

From (1) & (2)

$$\frac{a-c}{ac} = \frac{b-d}{bd}$$

$$\frac{a}{ac} - \frac{c}{ac} = \frac{b}{bd} - \frac{d}{bd}$$

$$\frac{1}{c} - \frac{1}{a} = \frac{1}{d} - \frac{1}{b}$$

$$\frac{1}{c} - \frac{1}{d} = \frac{1}{a} - \frac{1}{b}$$

Example 7.18

Soln:-

Let the point of intersection be (x_1, y_1)

$$x^2+4y^2=8$$

Diff w.r.to 'x'

$$2x+8y\left(\frac{dy}{dx}\right)=0$$

$$8y\left(\frac{dy}{dx}\right)=-2x$$

$$m_1=\frac{dy}{dx}=\frac{-x}{4y}$$

$$m_1=\frac{-x_1}{4y_1}$$

$$x^2-2y^2=4$$

Diff w.r.to 'x'

$$2x-4y\left(\frac{dy}{dx}\right)=0$$

$$-4y\left(\frac{dy}{dx}\right)=-2x$$

$$m_2=\frac{dy}{dx}=\frac{x}{2y}$$

$$m_2=\frac{x_1}{2y_1}$$

Given at (x_1,y_1)

$$x_1^2+4y_1^2-8=0$$

$$x_1^2-2y_1^2-4=0$$

$$\begin{matrix} & x_1^2 & & y_1^2 & & 1 & \\ 4 & & -8 & & 1 & & 4 \\ -2 & & -4 & & 1 & & -2 \end{matrix}$$

$$\frac{x_1^2}{-16-16}=\frac{y_1^2}{-8+4}$$

$$\frac{x_1^2}{y_1^2}=\frac{-32}{-4}$$

$$\frac{x_1^2}{y_1^2}=8 \rightarrow ①$$

$$\therefore m_1 . m_2=\frac{-x_1}{4y_1}\cdot\frac{x_1}{2y_1}$$

$$=\frac{-x_1^2}{8y_1^2}$$

$$=\frac{-8}{8} \quad \text{by } ①$$

$$m_1 . m_2=-1$$

Hence given curves cut orthogonally.

EXERCISE 7.2

① (i) Soln:-

$$y=x^4+2x^2-x$$

$$\frac{dy}{dx}=4x^3+4x-1$$

At $x=1$

$$\frac{dy}{dx}=4+4-1$$

$$\frac{dy}{dx}=7$$

(ii) Soln:-

$$x=a[\cos t]^3$$

$$\frac{dx}{dt}=a\,3(\cos t)^2(-\sin t)$$

$$\frac{dx}{dt}=-3a\cos^2 t\sin t$$

$$y=b[\sin t]^3$$

$$\frac{dy}{dt}=b\,3\sin^2 t(\cos t)$$

$$\frac{dy}{dt}=3b\sin^2 t\cos t$$

$$\frac{dy}{dx}=-\frac{3b\sin^2 t\cos t}{3a\cos^2 t\sin t}=\frac{-b}{a}\tan t$$

At $t=\frac{\pi}{2}$

$$\frac{dy}{dx}=\frac{-b}{a}\tan\left(\frac{\pi}{2}\right)$$

$$\frac{dy}{dx}=\infty$$

slope of the tangent at $t=\frac{\pi}{2}$ is ∞

② Soln:-

$$y=x^2-5x+4 \rightarrow ①$$

$$\frac{dy}{dx}=2x-5$$

Given $3x+y-7=0$

$$y=-3x+7$$

$$\frac{dy}{dx}=-3$$

Since slope parallel

$$2x-5=-3$$

$$2x=2$$

$$\boxed{x=1}$$

$① \Rightarrow y=1-5+4$

$$\boxed{y=0}$$

$\therefore$ The point is $(x,y)=(1,0)$

③ Soln:-

$$y=x^3-6x^2+x+3 \rightarrow ①$$

$$\frac{dy}{dx}=3x^2-12x+1$$

slope of the normal

$$-\frac{1}{m}=\frac{-1}{3x^2-12x+1}$$

Given $x+y-1729=0$

$$\text{slope}=-\frac{a}{b}=-1$$

Since, parallel

$$\frac{-1}{3x^2-12x+1}=-1$$

$$3x^2-12x+1=1$$

$$3x^2-12x=0$$

$$3x(x-4)=0$$

$$3x=0$$

$$x=0$$

$① \Rightarrow y=3$

$$x-4=0$$

$$x=4$$

$$y=64-6(16)+4$$

$$y=-25$$

$\therefore$ The points are $(0,3)$ and $(4,-25)$

④ Soln:-

$y^2 - 4xy = x^2 + 5 \rightarrow ①$

Diff w.r.to 'x'

$2y\left(\frac{dy}{dx}\right) - 4\left(x\frac{dy}{dx} + y\right) = 2x$

$2y\left(\frac{dy}{dx}\right) - 4x\left(\frac{dy}{dx}\right) - 4y = 2x$

$2\left(\frac{dy}{dx}\right)[y - 2x] = 2x + 4y$

$\frac{dy}{dx} = \frac{2(x+2y)}{2(y-2x)}$

$\frac{dy}{dx} = \frac{x+2y}{y-2x}$

since, horizontal

$\frac{dy}{dx} = 0,$

$\frac{x+2y}{y-2x} = 0$

$x + 2y = 0$

$x = -2y \rightarrow ②$

Sub ② in ①

$y^2 - 4(-2y)y = (-2y)^2 + 5$

$y^2 + 8y^2 = 4y^2 + 5$

$5y^2 = 5$

$y^2 = 1$

$y = \pm 1$

At $y = 1$

② ⇒ $x = -2$

At $y = -1$

② ⇒ $x = 2$

∴ The points are

$(-2, 1)$ and $(2, -1)$

⑤ Soln:-

(i)

$y = x^2 - x^4$

$\frac{dy}{dx} = 2x - 4x^3$

At $(1, 0)$

$\frac{dy}{dx} = 2 - 4 = -2$

Equation of tangent

$y - y_1 = m(x - x_1)$

$y - 0 = -2(x - 1)$

$y = -2x + 2$

$2x + y - 2 = 0$

Equation of Normal

$x - 2y + k = 0 \rightarrow ①$

At $(1, 0)$

$1 - 0 + k = 0$

$\boxed{k = -1}$

① ⇒ $x - 2y - 1 = 0$

(ii)

$y = x^4 + 2e^x$

$\frac{dy}{dx} = 4x^3 + 2e^x$

At $(0, 2)$

$\frac{dy}{dx} = 0 + 2(1) = 2$

Equation of tangent

$y - y_1 = m(x - x_1)$

$y - 2 = 2(x - 0)$

$y - 2 = 2x$

$2x - y + 2 = 0$

Equation of Normal

$x + 2y + k = 0 \rightarrow ①$

At $(0, 2)$

$0 + 4 + k = 0$

$k = -4$

① ⇒

$x + 2y - 4 = 0.$

(iii)

$y = x \sin x \rightarrow ①$

$\frac{dy}{dx} = x\cos x + \sin x$

At $x = \frac{\pi}{2}$

$\frac{dy}{dx} = \frac{\pi}{2}(0) + 1$

$\frac{dy}{dx} = 1 \quad \Big| \; -\frac{1}{\left(\frac{dy}{dx}\right)} = -1$

Given at $x = \pi/2$

① ⇒ $y = \frac{\pi}{2}(1)$

$y = \frac{\pi}{2}$

∴ $(x, y) = \left(\frac{\pi}{2}, \frac{\pi}{2}\right)$

Equation of tangent

$y - y_1 = m(x - x_1)$

$y - \frac{\pi}{2} = 1\left(x - \frac{\pi}{2}\right)$

$x - y = 0$

Equation of Normal

$x + y + k = 0 \rightarrow ②$

At $\left(\frac{\pi}{2}, \frac{\pi}{2}\right)$

$\frac{\pi}{2} + \frac{\pi}{2} + k = 0$

$k = -\pi$

② ⇒

$x + y - \pi = 0.$

(ix)

$x = \cos t$

$\frac{dx}{dt} = -\sin t$

$y = 2\sin^2 t$

$\frac{dy}{dt} = 2(2\sin t \cos t)$

$= 4\sin t \cos t$

$\frac{dy}{dx} = -\frac{4\sin t \cos t}{\sin t} = -4\cos t$

At $t = \frac{\pi}{3}$

$\frac{dy}{dx} = -4\cos(\frac{\pi}{3}) = -4(\frac{1}{2}) = -2$

$m = -2$

$(x, y) = (\cos t, 2\sin^2 t)$

At $t = \frac{\pi}{3}$

$(x, y) = \left[\cos\frac{\pi}{3}, 2(\sin\frac{\pi}{3})^2\right] = (\frac{1}{2}, \frac{3}{2})$

Equation of tangent

$y - y_1 = m(x - x_1)$

$y - \frac{3}{2} = -2(x - \frac{1}{2})$

$\frac{2y-3}{2} = -2x + 1$

$2y - 3 = -4x + 2$

$4x + 2y - 5 = 0$

Equation of Normal

$2x - 4y + k = 0 \rightarrow (1)$

At $(\frac{1}{2}, \frac{3}{2})$

$2(\frac{1}{2}) - 4(\frac{3}{2}) + k = 0$

$1 - 6 + k = 0$

$k = 5$

$(1) \Rightarrow 2x - 4y + 5 = 0.$

(6) Soln:-

$y = 1 + x^3 \rightarrow (1)$

$\frac{dy}{dx} = 3x^2$

$x + 12y - 12 = 0$

slope $= -\frac{a}{b} = -\frac{1}{12}$

Since, perpendicular, $m_1 \cdot m_2 = -1$

$3x^2 \cdot (-\frac{1}{12}) = -1$

$x^2 = 4$

$x = \pm 2$

At $x = 2$

$(1) \Rightarrow y = 1 + 8$

$y = 9$

point $(2, 9)$

slope $= 12$

At $x = -2$

$y = 1 - 8$

$y = -7$

point $(-2, -7)$

slope $= 12$

Equation of tangent is

$y - y_1 = m(x - x_1)$

$y - 9 = 12(x - 2)$

$y - 9 = 12x - 24$

$12x - y - 15 = 0$

$y + 7 = 12(x + 2)$

$y + 7 = 12x + 24$

$12x - y + 17 = 0$

(7) Soln:-

$y = \frac{x+1}{x-1} \rightarrow (1)$

$\frac{dy}{dx} = \frac{(x-1)(1) - (x+1)(1)}{(x-1)^2}$

$= \frac{x - 1 - x - 1}{(x-1)^2}$

$\frac{dy}{dx} = \frac{-2}{(x-1)^2}$

$x + 2y - 6 = 0$

slope $= -\frac{a}{b} = -\frac{1}{2}$

Since, parallel, $m_1 = m_2$

$\frac{-2}{(x-1)^2} = -\frac{1}{2}$

$(x-1)^2 = 4$

$x - 1 = \pm 2$

$x = 1 \pm 2$

$x = 3, -1$

At $x = 3$

$(1) \Rightarrow y = \frac{4}{2}$

$y = 2$

$(x, y) = (3, 2)$

At $x = -1$

$y = 0$

$(x, y) = (-1, 0)$

Equation of tangent is

$y - y_1 = m(x - x_1)$

$y - 2 = -\frac{1}{2}(x - 3)$

$2y - 4 = -x + 3$

$x + 2y - 7 = 0$

$y - 0 = -\frac{1}{2}(x + 1)$

$2y = -x - 1$

$x + 2y + 1 = 0$

(8) Soln:-

$x = 7\cos t$

$\frac{dx}{dt} = -7\sin t$

$y = 2\sin t$

$\frac{dy}{dt} = 2\cos t$

$\frac{dy}{dx} = -\frac{2\cos t}{7\sin t}$

Equation of tangent

$y - y_1 = m(x - x_1)$

$y - 2\sin t = -\frac{2\cos t}{7\sin t}(x - 7\cos t)$

$7y\sin t - 14\sin^2 t = -2x\cos t + 14\cos^2 t$

$2x\cos t + 7y\sin t = 14\sin^2 t + 14\cos^2 t$

$= 14(\sin^2 t + \cos^2 t)$

$2x\cos t + 7y\sin t = 14$

Equation of Normal is

$7x\sin t - 2y\cos t + k = 0 \longrightarrow (1)$

At $(7\cos t, 2\sin t)$

$7\sin t(7\cos t) - 2\sin t(2\cos t) = k$

$49\sin t\cos t - 4\sin t\cos t = k$

$45\sin t\cos t = k$

(1) $\Rightarrow$

$7x\sin t - 2y\cos t + 45\sin t\cos t = 0$

(9) soln:-

$xy = 2$

$y = \frac{2}{x} \rightarrow (1)$

$x^2 + 4y = 0 \rightarrow (2)$

sub (1) in (2)

$x^2 + 4\left(\frac{2}{x}\right) = 0$

$x^3 + 8 = 0$

$x^3 = -8$

$\boxed{x = -2}$

(1) $\Rightarrow$ $y = \frac{2}{-2}$

$\boxed{y = -1}$

$\therefore$ point of intersection $(x, y) = (-2, -1)$

$xy = 2$

Diff w. r. to x

$x\left(\frac{dy}{dx}\right) + y = 0$

$\frac{dy}{dx} = \frac{-y}{x}$

At $(-2, -1)$

$m_1 = -\frac{1}{2}$

$x^2 + 4y = 0$

$2x + 4\frac{dy}{dx} = 0$

$4\frac{dy}{dx} = -2x$

$\frac{dy}{dx} = -\frac{x}{2}$

At $(-2, -1)$

$m_2 = \frac{2}{2} = 1$

$\theta = \tan^{-1}\left[\frac{m_1 - m_2}{1 + m_1 m_2}\right]$

$= \tan^{-1}\left|\frac{-\frac{1}{2} - 1}{1 + (-\frac{1}{2})(1)}\right|$

$= \tan^{-1}\left|\frac{-3/2}{1/2}\right|$

$\theta = \tan^{-1}(3)$

(10) soln:-

Let the point of intersection be (x_1, y_1)

$x^2 - y^2 = r^2$

Diff w. r. to 'x'

$2x - 2y\left(\frac{dy}{dx}\right) = 0$

$-2y\left(\frac{dy}{dx}\right) = -2x$

$\frac{dy}{dx} = \frac{x}{y}$

$m_1 = \left(\frac{dy}{dx}\right)_{(x_1, y_1)} = \frac{x_1}{y_1}$

$xy = c^2$

Diff w. r. to 'x'

$x\left(\frac{dy}{dx}\right) + y = 0$

$\frac{dy}{dx} = \frac{-y}{x}$

$m_2 = \left(\frac{dy}{dx}\right)_{(x_1, y_1)} = \frac{-y_1}{x_1}$

$m_1 \cdot m_2 = \frac{x_1}{y_1} \cdot \frac{-y_1}{x_1}$

$m_1 m_2 = -1$

Hence two curves cut orthogonally.

EXERCISE 7.3

① (i) $f(x) = \left|\frac{1}{x}\right|, \ x \in [-1,1]$

$x = -1, 0, 1$

$f(0) = \infty$

$\therefore f(x)$ is not continuous at $x = 0$

(ii) $f(x) = \tan x, \ x \in [0, \pi]$

$x = 0, \frac{\pi}{2}, \pi$

$f(\frac{\pi}{2}) = \tan(\frac{\pi}{2}) = \infty$

$\therefore f(x)$ is not continuous at $x = \frac{\pi}{2}$

(iii) $f(x) = x - 2\log x, \ x \in [2,7]$

$f(2) = 2 - 2\log 2$

$f(7) = 7 - 2\log 7$

$f(2) \neq f(7)$

② Soln:-

(i) $f(x) = x^2 - x, \ x \in [0,1]$

$f(x)$ is continuous in $[0,1]$

$f(x)$ is differentiable in $(0,1)$

$f(0) = 0, \quad f(1) = 1 - 1 = 0$

$\therefore f(0) = f(1) = 0$

All the three conditions are satisfied,

$f'(x) = 2x - 1 = 0$

$2x = 1$

$x = \frac{1}{2} \in (0,1)$

$\therefore c = \frac{1}{2}$

(ii) $f(x) = \frac{x^2 - 2x}{x+2}, \ x \in [-1,6]$

$f(x)$ is continuous in $[-1,6]$

$f(x)$ is differentiable in $(-1,6)$

$f(-1) = \frac{1+2}{-1+2} = 3$

$f(6) = \frac{36-12}{8} = \frac{24}{8} = 3$

$f(-1) = f(6)$

All three conditions are satisfied

$$f'(x) = \frac{(x+2)(2x-2) - (x^2-2x)(1)}{(x+2)^2}$$

$$= \frac{2x^2 + 4x - 2x - 4 - x^2 + 2x}{(x+2)^2}$$

$$f'(x) = \frac{x^2 + 4x - 4}{(x+2)^2}$$

$f'(x) = 0, \quad x^2 + 4x - 4 = 0$

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

$$x = \frac{-4 \pm \sqrt{16 + 4(-4)}}{2}$$

$$= \frac{-4 \pm \sqrt{32}}{2}$$

$$= \frac{-4 \pm 4\sqrt{2}}{2}$$

$x = -2 \pm 2\sqrt{2}$, (But $-2-2\sqrt{2}$ not possible)

$-2+2\sqrt{2} \in (-1,6)$ but $-2-2\sqrt{2} \notin (-1,6)$

$f(x) = \sqrt{x} - \frac{x}{3}, \ x \in [0,9]$

$f(x)$ is continuous in $[0,9]$

$f(x)$ is differentiable in $(0,9)$

$f(0) = 0, \quad f(9) = \sqrt{9} - \frac{9}{3} = 3 - 3 = 0$

$\therefore f(0) = f(9)$

All the three conditions are satisfied.

$f'(x) = \frac{1}{2\sqrt{x}} - \frac{1}{3}$

$f'(x) = 0, \quad \frac{1}{2\sqrt{x}} - \frac{1}{3} = 0$

$\frac{1}{2\sqrt{x}} = \frac{1}{3}$

$\frac{1}{\sqrt{x}} = \frac{2}{3}$

$\sqrt{x} = \frac{3}{2}$

$x = \frac{9}{4} \in (0,9)$

$\therefore c = \frac{9}{4}$

Example: 7.19

$f(x) = x^2(1-x)^2, \ x \in [0,1]$

$f(0) = 0, \quad f(1) = 0$

$f(x)$ is continuous in $[0,1]$

$f(x)$ is differentiable in $(0,1)$

$f(0) = f(1)$

All three conditions are satisfied

$$f(x) = x^2(1+x^2-2x)$$
$$= x^2+x^4-2x^3$$
$$f'(x) = 2x+4x^3-6x^2$$
$$f'(x)=0, \quad 2x+4x^3-6x^2=0$$
$$2x(1+2x^2-3x)=0$$
$$2x=0, \quad 2x^2-3x+1=0$$
$$x=0 \quad (x-1)(2x-1)=0$$
$$x=1, \; x=\tfrac{1}{2}$$
$$\therefore \; c=0, 1, \text{ and } \tfrac{1}{2}$$
$$\rightarrow c=\tfrac{1}{2} \in (0,1)$$

Example :- 7.20

$$f(x) = x+\frac{1}{x}, \; x\in[\tfrac{1}{2},2]$$

$f(x)$ is continuous in $[\frac{1}{2},2]$

$f(x)$ is differentiable in $(\frac{1}{2},2)$

$$f(\tfrac{1}{2}) = \tfrac{1}{2}+2 = \tfrac{5}{2}$$
$$f(2) = 2+\tfrac{1}{2} = \tfrac{5}{2}$$
$$\therefore f(\tfrac{1}{2}) = f(2)$$

All three conditions are satisfied.

$$f'(x) = 1-\frac{1}{x^2}$$
$$\therefore f'(x)=0, \quad 1-\frac{1}{x^2}=0$$
$$\frac{1}{x^2}=1$$
$$x^2=1$$
$$x=\pm 1, \quad c=1,-1$$

But $\therefore$ $c=1 \in (\frac{1}{2},2)$

Example : 7.21

$$f(x) = \log\left(\frac{x^2+6}{5x}\right), \quad x\in[2,3]$$

$f(x)$ is continuous in $[2,3]$

$f(x)$ is differentiable in $(2,3)$

$$f(2) = \log\left(\frac{4+6}{10}\right) = \log(1) = 0$$
$$f(3) = \log\left(\frac{9+6}{15}\right) = \log(1) = 0$$
$$\therefore f(2) = f(3)$$

All three conditions are satisfied.

$$f'(x) = \frac{1}{\left(\frac{x^2+6}{5x}\right)}\left[\frac{5x(2x)-(x^2+6)5}{(5x)^2}\right]$$
$$= \left(\frac{5x}{x^2+6}\right)\left[\frac{10x^2-5x^2-30}{(5x)^2}\right]$$
$$= \frac{5(x^2-6)}{(x^2+6)\,5x}$$
$$f'(x) = \frac{x^2-6}{x(x^2+6)}$$
$$\therefore f'(x)=0, \quad \frac{x^2-6}{x(x^2+6)}=0$$
$$x^2-6=0$$
$$x^2=6$$
$$x=\pm\sqrt{6}$$
$$\therefore c=\sqrt{6}, -\sqrt{6} \quad \text{But}$$
$$c=\sqrt{6}\in(2,3)$$

Example :- 7.22

Let $f(x) = x^4+2x^3-2 \qquad x\in(0,1)$

$f(x)$ is continuous in $[0,1]$

$f(x)$ is differentiable in $(0,1)$

$$f'(x) = 4x^3+6x^2$$
$$\therefore f'(x)=0, \quad 4x^3+6x^2=0$$
$$x^2(4x+6)=0$$
$$x=0, \quad 4x+6=0$$
$$4x=-6$$
$$x=-\tfrac{3}{2}$$

$\therefore x=0$, and $-\frac{3}{2}$ But

$0, -\frac{3}{2} \notin (0,1)$ Roll's theorem doesnot exist

$\rightarrow f'(x)>0, \; \forall x\in(0,1)$

But $f(0) = -2 < 0$

$f(1) = 1 > 0$

$\therefore$ The curve $y=f(x)$ crosses the x-axis between 0 and 1.

$\therefore$ The $f(x)$ has only one real root in the interval $(0,1)$

Lagrange's Mean Value Theorem:

Let $f(x)$ be continuous in $[a,b]$ and differentiable in (a,b) then there exist atleast one point $c\in(a,b)$

$$f'(c) = \frac{f(b)-f(a)}{b-a}$$

③ (i) Soln:-

$f(x) = \frac{x+1}{x}, \quad x \in [-1, 2]$

$x = -1, 0, 1, 2$

$f(0) = \frac{1}{0} = \infty$

$\therefore f(x)$ is not continuous at $x = 0$

(ii) $f(x) = |3x+1|, \; x \in [-1, 3]$

$f(x)$ is continuous in $[-1, 3]$

$f(x)$ is not differentiable at $x = -\frac{1}{3}$

④ (i) $f(x) = x^3 - 3x + 2, \quad x \in [-2, 2]$

$f(x)$ is continuous on $[-2, 2]$

$f(x)$ is differentiable on $(-2, 2)$

By Lagrange's mean value theorem,

$$f'(c) = \frac{f(b) - f(a)}{b-a}$$

$f'(x) = 3x^2 - 3,$

$f(2) = 8 - 6 + 2 = 4$

$f(-2) = -8 + 6 + 2 = 0$

$3x^2 - 3 = \frac{4-0}{4} = 1$

$3x^2 - 3 = 1 \rightarrow 3x^2 = 4$

$x^2 = \frac{4}{3} =$

$x = \pm 2/\sqrt{3}$

(ii) $f(x) = (x-2)(x-7), \quad x \in [3, 11]$

$f(x)$ is continuous on $[3, 11]$

$f(x)$ is differentiable on $(3, 11)$

By mean value theorem,

$$f'(x) = \frac{f(b) - f(a)}{b-a}$$

$f(x) = x^2 - 9x + 14$

$f'(x) = 2x - 9$

$f'(x) = 2x - 9$

$f(3) = 9 - 27 + 14 = -4$

$f(11) = 121 - 99 + 14 = 36$

$f'(x) = \frac{36+4}{11-3} = \frac{40}{8} = 5$

$2x - 9 = 5$

$2x = 5 + 9$

$2x = 14$

$x = 7 \in (3, 11)$

$\therefore c = 7$

⑤ (i) $f(x) = \frac{1}{x}, \quad x \in [a, b]$

$f'(x) = \frac{-1}{x^2}$

By mean value theorem,

$$f'(x) = \frac{f(b) - f(a)}{b-a}$$

$-\frac{1}{x^2} = \frac{\frac{1}{b} - \frac{1}{a}}{b-a}$

$-\frac{1}{x^2} = \frac{a-b}{ab} \times \frac{1}{b-a}$

$-\frac{1}{x^2} = \frac{-(b-a)}{ab} \times \frac{1}{(b-a)}$

$\frac{1}{x^2} = \frac{1}{ab}$

$x^2 = ab$

$x = \pm\sqrt{ab}, \quad x \in \sqrt{ab} \in (a, b)$

(ii) $f(x) = Ax^2 + Bx + C, \; x \in [a, b]$

$f'(x) = 2Ax + B$

$f(a) = Aa^2 + Ba + C$

$f(b) = Ab^2 + Bb + C$

By mean value theorem,

$$f'(x) = \frac{f(b) - f(a)}{b-a}$$

$2Ax + B = \frac{Ab^2 + Bb + C - Aa^2 - Ba - C}{b-a}$

$2xA + B = \frac{A(b^2 - a^2) + B(b-a)}{b-a}$

$2xA + B = \frac{[A(b+a) + B](b-a)}{(b-a)}$

$2A = b + a,$

$A = \frac{b+a}{2} \in (a, b)$

(6) Let $f(a) = 20$, $f(a+2) = ?$

$f'(x) \leq 150$

$\therefore f'(x) = \dfrac{f(a+2) - f(a)}{a+2-a}$

$150 = \dfrac{f(a+2) - 20}{2}$

$300+20 = f(a+2)$

$320 = f(a+2)$

The initial point is 20^{th} km

$\therefore$ The maximum distance that he can cover in next two hours is 300 km

(7) Let $f'(x) \leq 1 \quad \forall\ 1 \leq x \leq 4$

By Mean value theorem,

$f'(x) = \dfrac{f(b) - f(a)}{b-a}$

$1 = \dfrac{f(4) - f(1)}{4-1}$

$3 \geq f(4) - f(1)$

$\therefore f(4) - f(1) \leq 3 \quad \forall\ x \in (1,4)$

(8) Let $f(0) = -1$, $f(2) = 4$

$f'(x) \leq 2 \quad \forall\ x$

By Mean value theorem,

$f'(x) = \dfrac{f(b) - f(a)}{b-a}$

$\therefore 2 \geq \dfrac{f(2) - f(0)}{2-0}$

$2 \geq \dfrac{4+1}{2}$ (or) $f'(x) = \frac{5}{2}$

$f'(x) = 2.5$

$\therefore 4 \geq 5$

Does not exist to $x \notin (0,2)$

(9) $f(x) = x(x+3)e^{-x/2}$, $-3 \leq x \leq 0$

$f(-3) = -3(-3+3)e^{-x/2} = 0$

$f(0) = 0$

$f(x)$ is continuous on $[-3,0]$

$f(x)$ is differentiable on $(-3,0)$

By Mean value theorem,

$f'(x) = \dfrac{f(b) - f(a)}{b-a}$

$= \dfrac{0-(0)}{3} = 0$

Hence, there should be a point on the curve where tangent is parallel to x-axis

(10) Let $f(x) = e^{-x}$, $a>0$, $b>0$,

$f(x)$ is continuous on $[a,b]$

$f(x)$ is differentiable on (a,b)

By Mean value theorem, $\quad \mid f'(x) = -e^{-x}$

$f'(x) = \dfrac{f(b) - f(a)}{b-a}$

$-e^{-x} = \dfrac{e^{-b} - e^{-a}}{b-a}$

$|e^{-x}| = \left|\dfrac{e^{-b} - e^{-a}}{b-a}\right|$, $|e^{-x}| < 1$

$1 > \dfrac{|e^{-b} - e^{-a}|}{|b-a|}$

$|b-a| > |e^{-b} - e^{-a}|$

$\Rightarrow |e^{-a} - e^{-b}| < |a-b|$

Example : 7.23

Let $p(x) = a_n x^n + a_{n-1}x^{n-1} + \dots + a_1 x + a_0$

Let α, β be the real zeros of $p(x)$ $(\alpha < \beta)$

$\therefore p(\alpha) = 0$

$p(\beta) = 0$

$p'(x) = n a_n x^{n-1} + (n-1) a_{n-1} x^{n-2} + \dots + a_1$

$p(x)$ is continuous on $[\alpha, \beta]$

$p(x)$ is differentiable on (α, β)

and $p(\alpha) = p(\beta)$

By Roll's theorem, there exist $c \in (\alpha, \beta)$

$\ni$, $p'(c) = 0$

c is a zero of $p'(x)$

$p'(x) = n a_n x^{n-1} + (n-1) a_{n-1} x^{n-2} + \dots + a_1$

Example : 7.24

Let $p(x) = x^4 - 6x^3 - 11x^2 + 24x + 28.$

$p'(x) = 4x^3 - 18x^2 - 22x + 24$

$p'(x) = 2[2x^3 - 9x^2 - 11x + 12]$

$p'(x) = 2\, Q(x)$, where

$Q(x) = 2x^3 - 9x^2 - 11x + 12$

$p(x)$ is continuous on $[2, 7]$

$p(x)$ is differentiable on $(2, 7)$

$p(2) = 0, \quad p(7) = 0$

$\therefore p(2) = p(7)$

By Roll's theorem,

$\exists\, c \in (2, 7) \ni$

$p'(c) = 0$

$2(Q(c)) = 0$

$Q(c) = 0$

c is a zero of $Q(x)$

c is a zero of $2x^3 - 9x^2 - 11x + 12$

Hence proved.

Example 7.25

$f(x) = x - x^2$

$f'(x) = 1 - 2x$

$f(x)$ is continuous on $[1, 2]$

$f(x)$ is differentiable on $(1, 2)$

$\exists\, c \in (1, 2) \ni$

$f'(x) = \dfrac{f(2) - f(1)}{2-1}$

$1 - 2x = \dfrac{-2 - (0)}{1}$

$1 - 2x = -2$

$-2x = -3$

$x = 3/2 \in [1, 2)$

$\therefore c = 3/2 \in (1, 2)$

Example: 7.26

Let $f(t)$ be the distance travelled

$f(0) = 0, \quad f(2) = 164$

$f(x)$ is continuous on $[0, 2]$

$f(x)$ is differentiable on $(0, 2)$

$\exists\, c \in (0, 2) \ni$

$f'(x) = \dfrac{f(2) - f(0)}{2 - 0}$

$= \dfrac{164}{2}$

$f'(x) = 82 > 80$

$f'(c) = 82 > 80$ which justifies the issuance of speed violation notice.

Example 7.27

Let $f'(x) \le 29$

$f(2) = 17$

$f(x)$ is continuous on $[2, 7]$

$f(x)$ is differentiable on $(2, 7)$

By mean value theorem,

$\exists\, c \in (2, 7) \ni$

$f'(x) = \dfrac{f(7) - f(2)}{7 - 2}$

$f'(x) = \dfrac{f(7) - 17}{5}$

$5f'(x) = f(7) - 17$

$f(7) \leq 5f'(x) + 17 \leq 5(29) + 17$

$f(7) \leq 162$

Hence max value of $f(7)$ is 162.

Example: 7.28

$f(x) = \sin x$, $[\alpha, \beta]$

$f'(x) = \cos x$

$f(x)$ is continuous on $[\alpha, \beta]$

$f(x)$ is differentiable on (α, β)

By Mean value theorem,

$\exists\ c \in (\alpha, \beta) \ni$

$$f'(x) = \frac{f(\beta) - f(\alpha)}{\beta - \alpha}$$

$$|\cos x| = \left|\frac{\sin\beta - \sin\alpha}{\beta - \alpha}\right|$$

$$|\cos x| = \frac{|\sin\alpha - \sin\beta|}{|\alpha - \beta|} \quad |\cos x| \leq 1$$

$$\frac{|\sin\alpha - \sin\beta|}{|\alpha - \beta|} \leq 1$$

$$|\sin\alpha - \sin\beta| \leq |\alpha - \beta|$$

Hence proved.

Example 7.29

Let $f(t)$ be the temperature at time 't'

$f(0) = -10°C$

$f(22) = 100°C$

To show: $f'(t) = 5°C/sec$

$f(x)$ is continuous on $[0, 22]$

$f(x)$ is differentiable on $(0, 22)$

By mean value theorem,

$$f'(x) = \frac{f(b) - f(a)}{b - a}$$

$$= \frac{100 - (-10)}{22 - 0}$$

$$= \frac{100 + 10}{22}$$

$$= \frac{110}{22}$$

$f'(c) = 5°C/sec$

The rate of change of temperature at some time t is $5°C/sec$.

Try

Example: 7.30

$f(x) = \log(1+x)$

function	derivatives	value at $x=0$
$f(x)$	$\log(1+x)$	0
$f'(x)$	$\frac{1}{1+x}$	1
$f''(x)$	$\frac{-1}{(1+x)^2}$	-1
$f'''(x)$	$\frac{2}{(1+x)^3}$	2
$f^{IV}(x)$	$\frac{-6}{(1+x)^4}$	-6

By Maclaurin series,

$$f(x) = f(0) + \frac{f'(0)}{1!}x + \frac{f''(x)}{2!}x^2 + \dots\dots$$

$$\log x = 0 + \frac{1}{1!}x - \frac{1}{2!}x^2 + \frac{2}{3!}x^3 - \frac{6x^4}{4!} + \dots$$

Example: 7.31

$f(x) = \tan x$

function	Derivatives	Value at $x=0$
$f(x)$	$\tan x$	0
$f'(x)$	$\sec^2 x$	1
$f''(x)$	$2\sec x \sec x \tan x$ $2\sec^2 x \tan x$	0
$f'''(x)$	$4\sec x \sec x \tan x + 2\sec^2 x \sec^2 x$ $4\sec^2 x \tan x + 2\sec^4 x$	2
$f^{IV}(x)$	$16\sec^4 x \tan x + 8\sec^2 x \tan^3 x$	0
$f^{V}(x)$	$16\sec^6 x + 88\sec^4 x \tan^2 x + 16\sec^2 x \tan^4 x$	16

$$f(x) = \sum_{n=0}^{\infty} \frac{f^n(0)}{n!}x^n$$

$$f(x) = f(0) + \frac{f'(0)}{1!}x + \dots\dots + \frac{f^n(0)}{n!}x^n$$

$$\tan x = 0 + \frac{x}{1!} + 0 + 2\frac{x^3}{3!} + 0 + 16\frac{x^5}{5!} + \dots$$

$$\tan x = x + \frac{x^3}{3} + \frac{2x^5}{15} + \dots\dots$$

Example 7.32

$f(x) = \frac{1}{x}$

function	Derivatives	value at $x=2$
$f(x)$	$\frac{1}{x}$	$\frac{1}{2}$
$f'(x)$	$\frac{-1}{x^2}$	$-\frac{1}{4}$
$f''(x)$	$\frac{2}{x^3}$	$\frac{1}{4}$
$f^{IV}(x)$	$\frac{-6}{x^4}$	$-\frac{3}{8}$

By Taylor series,

$$f(x) = f(a) + \frac{f'(a)}{1!}(x-a) + \dots + \frac{f^n(a)}{n!}(x-a)^n + \dots$$

$$\frac{1}{x} = \frac{1}{2} - \frac{1}{4}\frac{(x-2)}{1!} + \frac{1}{4}\frac{(x-2)^2}{2!} - \frac{3}{8}\frac{(x-2)^3}{3!} + \dots$$

$$\frac{1}{x} = \frac{1}{2} - \frac{1}{4}(x-2) + \frac{(x-2)^2}{8} - \frac{(x-2)^3}{16} + \dots$$

EXERCISE 7.4

1(i) $f(x) = e^x$

function	Derivative	value at $x=0$
$f(x)$	e^x	1
$f'(x)$	e^x	1
$f''(x)$	e^x	1
$f'''(x)$	e^x	1

$$\therefore e^x = f(0) + x f'(0) + \frac{x^2}{2!}f''(0) + \dots$$

$$e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \dots\dots$$

(ii) $f(x) = \sin x$

function	Derivatives	Value at $x=0$
$f(x)$	$\sin x$	0
$f'(x)$	$\cos x$	1
$f''(x)$	$-\sin x$	0
$f'''(x)$	$-\cos x$	-1
$f^{IV}(x)$	$\sin x$	0
$f^{V}(x)$	$\cos x$	1

$$\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} + \cdots$$

(iii) $f(x) = \cos^2 x$

function	Derivative	value at $x=0$
$f(x)$	$\cos x$	1
$f'(x)$	$-\sin x$	0
$f''(x)$	$-\cos x$	-1
$f'''(x)$	$\sin x$	0
$f^{IV}(x)$	$\cos x$	1

$$\cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} + \cdots$$

(iv) $f(x) = \log(1-x)$

function	Derivatives	value at $x=0$
$f(x)$	$\log(1-x)$	0
$f'(x)$	$\frac{-1}{(1-x)}$	-1
$f''(x)$	$\frac{-1}{(1-x)^2}$	-1
$f'''(x)$	$\frac{-2}{(1-x)^3}$	-2

$$\log(1-x) = \frac{x}{1!}(-1) + \frac{x^2}{2!}(-1) + \frac{x^3}{3!}(-2) + \cdots$$

$$= -x - \frac{x^2}{2!} - \frac{2x^3}{3!} + \cdots$$

$$= -\left(x + \frac{x^2}{2} + \frac{x^3}{3} + \cdots\right)$$

(v) $f(x) = \tan^{-1} x$

function	Derivatives	Value at $x=0$
$f(x)$	$\tan^{-1} x$	0
$f'(x)$	$\frac{1}{1+x^2}$	1
$f''(x)$	$-2x(1+x^2)$	0
$f'''(x)$	$8x^2(1+x^2)^{-3} - 2(1+x^2)^{-2}$	-2
$f^{IV}(x)$	$12x^3(1+x^2)^{-4} + 16x(1+x^2)^{-3} - 8x(1+x^2)^{-3}$	0

$$\tan^{-1} x = 0 + \frac{x}{1}(1) + \frac{x^2}{2!}(0) + \frac{x^3}{3!}(-2) + \cdots$$

$$\tan^{-1} x = x - \frac{x^3}{3} + \cdots$$

(vi) $f(x) = \cos^2 x$

function	Derivatives	value at $x=0$
$f(x)$	$\cos^2 x$	1
$f'(x)$	$-2\cos x \sin x$ $= -\sin 2x$	0
$f''(x)$	$-2\cos 2x$	-2
$f'''(x)$	$4\sin 2x$	0
$f^{IV}(x)$	$8\cos 2x$	8

$$\cos^2 x = 1 + \frac{x}{1!}(0) + \frac{x^2}{2!}(-2) + \frac{x^3}{3!}(0) + \frac{x^3}{4!}(8)$$

$$= 1 - \frac{2x^2}{2} + \frac{8x^3}{24} - \cdots$$

$$= 1 - x^2 + \frac{1}{3}x^3 - \cdots$$

② Let $f(x) = \log x \quad x = 1$

function	Derivative	value at $x=1$
$f(x)$	$\log x$	0
$f'(x)$	$\frac{1}{x}$	1
$f''(x)$	$\frac{-1}{x^2}$	-1
$f'''(x)$	$\frac{2}{x^3}$	2
$f^{iv}(x)$	$\frac{-6}{x^4}$	-6

By Taylor series

$$\log x = 0 + (x-1) + \frac{(x-1)^2}{2!}(-1) + \frac{(x-1)^3}{3!}(2) + \cdots$$

$$\log x = (x-1) - \frac{(x-1)^2}{2} + \frac{(x-1)^3}{3} + \cdots$$

③ Let $f(x) = \sin x$

function	derivatives	value at $x = \frac{\pi}{4}$
$f(x)$	$\sin x$	$\frac{1}{\sqrt{2}}$
$f'(x)$	$\cos x$	$\frac{1}{\sqrt{2}}$
$f''(x)$	$-\sin x$	$-\frac{1}{\sqrt{2}}$
$f^{iv}(x)$	$-\cos x$	$-\frac{1}{\sqrt{2}}$

$$\sin x = \frac{1}{\sqrt{2}} + \left(x-\frac{\pi}{4}\right)\frac{1}{\sqrt{2}} + \frac{\left(x-\frac{\pi}{4}\right)^2}{2!}\left(-\frac{1}{\sqrt{2}}\right) + \cdots$$

$$\sin x = \frac{1}{\sqrt{2}}\left[1 + \left(x-\frac{\pi}{4}\right) - \frac{\left(x-\frac{\pi}{4}\right)^2}{2!} - \frac{\left(x-\frac{\pi}{4}\right)^3}{3!} - \cdots\right]$$

④

$f(x) = x^2 - 3x + 2 \qquad f(1) = 0$

$f'(x) = 2x - 3 \qquad f'(1) = -1$

$f''(x) = 2 \qquad f''(1) = 2$

$f'''(x) = 0 \qquad f'''(1) = 0$

$$x^2 - 3x + 2 = 0 + \frac{(x-1)}{1!}(-1) + \frac{(x-1)^2}{2!}2 + 0$$

$$= -(x-1) + (x-1)^2$$

(11) $\lim_{x\to 0^+} (\cos x)^{\frac{1}{x^2}}$ (1^∞ form)

Let $f(x) = (\cos x)^{1/x^2}$

$\log f(x) = \frac{1}{x^2} \log \cos x$

$\lim_{x\to 0^+} \log f(x) = \lim_{x\to 0^+} \frac{\log \cos x}{x^2} = \left(\frac{0}{0}\right)$

By rule,

$= \lim_{x\to 0^+} \frac{\frac{1}{\cos x}(-\sin x)}{2x}$

$= \lim_{x\to 0^+} \frac{-\tan x}{2x}$ ($\frac{0}{0}$ form)

Again by rule, $= \lim_{x\to 0^+} \frac{-\sec^2 x}{2} = -\frac{1}{2}$

$\Rightarrow \log \lim_{x\to 0^+} f(x) = -\frac{1}{2}$

$\lim_{x\to 0^+} f(x) = e^{-1/2} = \frac{1}{e^{1/2}}$

$\therefore \lim_{x\to 0^+} f(x) = \frac{1}{\sqrt{e}}$

(12) $A = A_0 \left(1+\frac{r}{n}\right)^{nt} \longrightarrow (1)$

Let $\lim_{n\to\infty} y = \lim_{n\to\infty} \left(1+\frac{r}{n}\right)^{nt}$

$\lim_{n\to\infty} \log y = \lim_{n\to\infty} \log \left(1+\frac{r}{n}\right)^{nt}$

$= \lim_{n\to\infty} nt \log\left(1+\frac{r}{n}\right)$

$\log \lim_{n\to\infty} y = \lim_{n\to\infty} \left[\frac{\log\left(1+\frac{r}{n}\right)}{\left(\frac{1}{nt}\right)}\right] = \left(\frac{0}{0}\right)$

By rule,

$\log \lim_{n\to\infty} y = \lim_{n\to\infty} \left[\frac{\frac{1\left(-\frac{r}{n^2}\right)}{\left(1+\frac{r}{n}\right)}}{\left(-\frac{1}{tn^2}\right)}\right]$

$= \lim_{n\to\infty} \left(\frac{rt}{1+\frac{r}{n}}\right)$

$\log \lim_{n\to\infty} y = rt$

$\lim_{n\to\infty} y = e^{rt}$

$(1) \Rightarrow$

$A = A_0 e^{rt}$

Example 7.33

$\lim_{x\to 1} \left(\frac{x^2-3x+2}{x^2-4x+3}\right) = \left(\frac{0}{0}\text{ form}\right)$

$\lim_{x\to 1} \left(\frac{2x-3}{2x-4}\right) = \frac{2-3}{2-4} = \frac{-1}{-2} = \frac{1}{2}$

Example : 7.34

$\lim_{x\to a} \left(\frac{x^n-a^n}{x-a}\right) = \lim_{x\to a} \left(\frac{nx^{n-1}}{1}\right)$

$= na^{n-1}$

Example 7.35

$\lim_{x\to 0} \left(\frac{\sin mx}{x}\right) = \left(\frac{0}{0}\text{ form}\right)$

By rule,

$\lim_{x\to 0} \left(\frac{\cos mx\,(m)}{1}\right) = \frac{m(\cos 0)}{1} = m$

Example 7.36

$\lim_{x\to 0} \left(\frac{\sin x}{x^2}\right) = \left(\frac{0}{0}\text{ form}\right)$

By rule,

$\lim_{x\to 0} \left(\frac{\cos x}{2x}\right) = \frac{1}{0} = (\infty\text{ form})$

Again by rule,

$\lim_{x\to 0} \left(\frac{\sin x}{2}\right) = \frac{0}{2} = 0.$

Example 7.37

$\lim_{\theta\to 0} \left(\frac{1-\cos m\theta}{1-\cos n\theta}\right) = \left(\frac{1-1}{1-1}\right) = \left(\frac{0}{0}\text{ form}\right)$

By rule,

$\lim_{\theta\to 0} \left(\frac{m\sin m\theta}{n\sin n\theta}\right) = \lim_{\theta\to 0} \frac{m}{n}\left[\frac{\frac{\sin m\theta}{m\theta}}{\frac{\sin n\theta}{n\theta}}\right]$

$\Rightarrow \frac{m}{n}\left(\frac{m}{n}\right) = \frac{m^2}{n^2}$

$\therefore m^2 = n^2 \Rightarrow m = \pm n$

Example 7.38

$$\lim_{x\to 1^-}\left(\frac{\log(1-x)}{\cot(\pi x)}\right) = \left(\frac{\infty}{\infty}\text{ form}\right)$$

By rule,

$$\lim_{x\to 1^-}\left[\frac{-1/(1-x)}{-\pi\,\mathrm{cosec}^2(\pi x)}\right] = \left(\frac{\infty}{\infty}\text{ form}\right)$$

$$\lim_{x\to 1^-}\left(\frac{\sin^2(\pi x)}{\pi(1-x)}\right) = \left(\frac{0}{0}\text{ form}\right)$$

Again by rule

$$\lim_{x\to 1^-}\left(\frac{2\sin(\pi x)\cdot\cos(\pi x)\,\pi}{-\pi}\right)$$

$$= \lim_{x\to 1^-}\left(-2\sin(\pi x)\cos(\pi x)\right)$$

$$= \lim_{x\to 1^-}\left(-\sin(2\pi x)\right)$$

$$= 0$$

Example 7.39

$$\lim_{x\to 0^+}\left(\frac{1}{x}-\frac{1}{e^x-1}\right) = (\infty-\infty)\text{ form}$$

$$\lim_{x\to 0^+}\left(\frac{e^x-1-x}{x(e^x-1)}\right) = \left(\frac{0}{0}\text{ form}\right)$$

By rule,

$$\lim_{x\to 0^+}\left(\frac{e^x-1}{x(e^x)+(e^x-1)}\right) = \left(\frac{1-1}{0-0}\right) = \frac{0}{0}\text{ form}$$

Again by rule,

$$\lim_{x\to 0^+}\left(\frac{e^x}{xe^x+e^x+e^x}\right) = \frac{1}{1+1} = \frac{1}{2}$$

Example 7.40

$$\lim_{x\to 0^+} x\log x = (0\times\infty)\text{ form}$$

$$\lim_{x\to 0^+}\left(\frac{\log x}{(1/x)}\right) = \left(\frac{\infty}{\infty}\text{ form}\right)$$

By rule,

$$\lim_{x\to 0^+}\left(\frac{1}{x}\times -\frac{x^2}{1}\right) = \lim_{x\to 0^+}(-x) = 0$$

Example 7.41

$$\lim_{x\to\infty}\left(\frac{x^2+17x+29}{x^4}\right) = \left(\frac{\infty}{\infty}\text{ form}\right)$$

By rule,

$$\lim_{x\to\infty}\left(\frac{2x+17}{4x^3}\right) = \left(\frac{\infty}{\infty}\text{ form}\right)$$

Again by rule,

$$\lim_{x\to\infty}\left(\frac{2}{12x^2}\right) = \frac{2}{\infty} = 0$$

Example : 7.42

$$\lim_{x\to\infty}\left(\frac{e^x}{x^m}\right) m\in N, = \left(\frac{\infty}{\infty}\text{ form}\right)$$

By rule

$$\lim_{x\to\infty}\left(\frac{e^x}{m!}\right) = \infty$$

Example 7.43:

$$\lim_{x\to 0^+}(1+x)^{\frac{1}{x}} = 1^\infty\text{ form.}$$

Let $f(x) = (1+x)^{\frac{1}{x}}$

$$\log f(x) = \frac{1}{x}\log(1+x)$$

$$\lim_{x\to 0^+}\log f(x) = \lim_{x\to 0^+}\frac{\log(1+x)}{x} = \left(\frac{0}{0}\right)$$

By rule,

$$\lim_{x\to 0^+}\log f(x) = \lim_{x\to 0^+}\frac{\left(\frac{1}{1+x}\right)}{1} = 1$$

$$\log\lim_{x\to 0^+} f(x) = 1$$

$$\lim_{x\to 0^+} f(x) = e$$

Example 7.44

$$\lim_{x\to\infty}(1+2x)^{\frac{1}{2\log x}} = (\infty^0\text{ form})$$

Let $f(x) = (1+2x)^{\frac{1}{2\log x}}$

$$\log f(x) = \frac{1}{2\log x}\log(1+2x)$$

$$\lim_{x\to\infty}\log f(x) = \lim_{x\to\infty}\left(\frac{\log(1+2x)}{2\log x}\right) = \frac{\infty}{\infty}$$

By rule,

$$\log\lim_{x\to\infty} f(x) = \lim_{x\to\infty}\left[\frac{\left(\frac{2}{1+2x}\right)}{(2/x)}\right]$$

$$= \lim_{x\to\infty}\left(\frac{x}{1+2x}\right) = \left(\frac{\infty}{\infty}\text{ form}\right)$$

By rule,

$$\log \lim_{x\to\infty} f(x) = \frac{1}{2}$$

$$\lim_{x\to\infty} f(x) = e^{1/2} = \sqrt{e}$$

Example 7.45

$$\lim_{x\to 1} x^{\left(\frac{1}{1-x}\right)} = (1^{\infty} \text{ form})$$

Let $f(x) = x^{\left(\frac{1}{1-x}\right)}$

$$\log f(x) = \left(\frac{1}{1-x}\right)\log x$$

$$\lim_{x\to 1} \log f(x) = \lim_{x\to 1}\left(\frac{\log x}{1-x}\right) = \left(\frac{0}{0} \text{ form}\right)$$

By rule,

$$\log \lim_{x\to 1} f(x) = \lim_{x\to 1}\frac{\left(\frac{1}{x}\right)}{1} = \frac{1}{1}$$

$$\log \lim_{x\to 1} f(x) = 1$$

$$\lim_{x\to 1} f(x) = e$$

EXERCISE 7.6

1) Sol:-

(i) $f(x) = x^2 - 12x + 10$, $[1,7]$

$f'(x) = 2x - 12$

$f'(x) = 0 \Rightarrow 2x - 12 = 0$

$2x = 12$

$x = 6$ is critical number

$x = 1, 6, 7$

$f(1) = 1 - 12 + 10 = -1$

$f(6) = 36 - 72 + 10 = -26$

$f(7) = 49 - 84 + 10 = -25$

$\therefore$ The absolute maximum $= -1$

The absolute minimum $= -26$

(ii) $f(x) = 3x^4 - 4x^3$, $[-1,2]$

$f'(x) = 12x^3 - 12x^2$

$= 12x^2(x-1)$

$f'(x) = 0, \Rightarrow 12x^2(x-1) = 0$

$12x^2 = 0, \quad x - 1 = 0$

$x = 0, \quad x = 1$

$x = -1, 0, 1, 2$ critical Numbers

$f(-1) = 3 + 4 = 7$

$f(0) = 0 - 0 = 0$

$f(1) = 3 - 4 = -1$

$f(2) = 3(16) - 4(8) = 48 - 32 = 16$

$\therefore$ The absolute maximum $= 16$

The absolute minimum $= -1$

(iii) $f(x) = 6x^{4/3} - 3x^{1/3}$, $[-1,1]$

$f'(x) = \frac{4}{3}6x^{\frac{4}{3}-1} - 3\left(\frac{1}{3}\right)x^{\frac{1}{3}-1}$

$f'(x) = 8x^{\frac{1}{3}} - x^{-2/3}$

$f'(x) = 0, \Rightarrow 8x^{\frac{1}{3}} - x^{-2/3} = 0$

$8x^{\frac{1}{3}} = x^{-2/3}$

$8x^{\frac{1}{3}} = \frac{1}{x^{2/3}}$

$x^{\frac{1}{3}} \cdot x^{\frac{2}{3}} = \frac{1}{8}$

$x = \frac{1}{8}$ critical point

$x = -1, \frac{1}{8}, 1$

$f(-1) = 6(-1)^{4/3} - 3(-1)^{1/3} = 6 + 3 = 9$

$f(8) = 6(8)^{4/3} - 3(8)^{1/3}$

$= 6\left(\frac{1}{2^3}\right)^{4/3} - 3\left(\frac{1}{2^3}\right)^{1/3}$

$= 6\left(\frac{1}{2}\right)^4 - 3\left(\frac{1}{2}\right)$

$= \frac{6}{16} - \frac{3}{2} \Rightarrow \frac{3}{8} - \frac{3}{2} = \frac{6-12}{8} = \frac{-9}{8}$

$\therefore$ The absolute maximum $= 9$

The absolute minimum $= -9/8$

(iv) $f(x) = 2\cos x + \sin 2x$, $\left[0, \frac{\pi}{2}\right]$

$f'(x) = -2\sin x + 2\cos 2x$

$f'(x) = 0, \quad -2\sin x + 2\cos 2x = 0$

$2\cos 2x = 2\sin x$

$\cos 2x = \sin x$

When $x = 30^\circ$ $\quad \sin 30^\circ = \cos 2(30^\circ)$

$\sin 30^\circ = \cos 60^\circ = \frac{1}{2}$

$\therefore x = \frac{\pi}{6}$

$x = 0, \frac{\pi}{6}, \frac{\pi}{2}$

$f(0) = 2\cos(0) + \sin(0) = 2(1) = 2$

$f(\frac{\pi}{6}) = 2\cos(\frac{\pi}{6}) + \sin(2\frac{\pi}{6})$

$= 2(\frac{\sqrt{3}}{2}) + \frac{\sqrt{3}}{2} = \frac{3\sqrt{3}}{2}$

$f(\frac{\pi}{2}) = 2\cos(\frac{\pi}{2}) + \sin(2\frac{\pi}{2})$

$= 0 + 0 = 0$

$\therefore$ The absolute maximum $= \frac{3\sqrt{3}}{2}$

The absolute minimum $= 0$

(2) soln:-

(i) $f(x) = 2x^3 + 3x^2 - 12x$

$f'(x) = 6x^2 + 6x - 12$

$f'(x) = 0, \quad 6x^2 + 6x - 12 = 0$

$\div 6 \quad x^2 + x - 2 = 0$

$(x+2)(x-1) = 0$

$\boxed{x = -2}$ $\boxed{x = 1}$

$-\infty \quad -2 \quad 1 \quad \infty$

Intervals	$f'(x)$	Monotoncity
$(-\infty, -2)$	+ve	st. Increasing
$(-2, 1)$	−ve	st. decreasing
$(1, \infty)$	+ve	st. Increasing

$f''(x) = 12x + 6$

When $x = -2$

$f''(-2) = -24 + 6 = -18 < 0$

$\therefore f(x)$ attains local maximum.

Local maximum value

$f(-2) = 2(-2)^3 + 3(-2)^2 - 12(-2)$

$= -16 + 12 + 24 \Rightarrow 20$

When $x = 1$

$f''(1) = 12 + 6 = 18 > 0$

$\therefore f(x)$ attains local minimum

Local minimum value

$f(1) = 2 + 3 - 12 = -7$

(ii) $f(x) = \frac{x}{x-5}$

$f'(x) = \frac{(x-5)(1) - x(1)}{(x-5)^2} = \frac{-5}{(x-5)^2}$

$f'(x) = 0, \quad \frac{-5}{(x-5)^2} = 0$ but $-5 \neq 0$

$\therefore f'(x) \neq 0$

$x = 5$ is does not exist.

$-\infty \quad 5 \quad \infty$

Intervals	$f'(x)$	Monotoncity
$(-\infty, 5)$	−ve	st. dec
$(5, \infty)$	−ve	st. dec

$f'(x)$ does not changes the sign

$\therefore$ There is no local extremum.

(iii) $f(x) = \frac{e^x}{1-e^x}$

$f'(x) = \frac{(1-e^x)e^x - e^x(-e^x)}{(1-e^x)^2}$

$f'(x) = \frac{e^x - e^x e^x + e^x e^x}{(1-e^x)^2} = \frac{e^x}{(1-e^x)^2}$

$f'(x) = 0,$

$f'(x)$ does not exist at $x = 0$

$-\infty \quad 0 \quad \infty$

Intervals	$f'(x)$	Monotoncity
$(-\infty, 0)$	+ve	st. Inc
$(0, \infty)$	+ve	st. Inc

$f'(x)$ does not changes the sign

$\therefore$ There is no local extremum.

(iv) $f(x) = \frac{x^3}{3} - \log x, \quad x > 0$

$f'(x) = \frac{3x^2}{3} - \frac{1}{x} = x^2 - \frac{1}{x}$

$f'(x) = 0, \quad x^2 - \frac{1}{x} = 0$

$x^2 = \frac{1}{x}$

$x^3 = 1$

$x = 1$

$f'(x)$ does not exist at $x=0$, But $x>0$

$0 \quad 1 \quad \infty$

Interval	$f'(x)$	Monotoncitiy
$(0,1)$	$-ve$	st. decreasing
$(1,\infty)$	$+ve$	st. increasing

$$f''(x) = 2x + \frac{1}{x^2}$$

When $x=1$

$$f''(1) = 2+1 = 3 > 0$$

$\therefore$ $f(x)$ attains local minimum.

Local minimum value

$$f(1) = \frac{1}{3} - \log(1) \quad \Big| \log(1) = 0.$$

$$= \frac{1}{3}$$

(v) $f(x) = \sin x \cos x + 5, \quad [0, \frac{\pi}{2}]$

$$f(x) = \frac{2}{2}\sin x \cos x + 5$$

$$f(x) = \frac{1}{2}\sin 2x + 5$$

$$f'(x) = \frac{\cos 2x\,(2)}{2}$$

$f'(x)=0, \quad \cos 2x = 0$

$$\cos 2x = \cos\frac{\pi}{2}$$

$$2x = \frac{\pi}{2}$$

$$x = \pi/4$$

$0 \quad \pi/4 \quad \pi/2$

Interval	$f'(x)$	Monotoncitiy
$[0, \frac{\pi}{4})$	$+ve$	st. increasing
$(\frac{\pi}{4}, \frac{\pi}{2}]$	$-ve$	st. decreasing

$$f''(x) = -2\sin 2x$$

When $x = \frac{\pi}{4}$

$$f''(\tfrac{\pi}{4}) = -2\sin(2\times\tfrac{\pi}{4})$$

$$= -2\sin(\tfrac{\pi}{2})$$

$$f''(\pi/4) = -2 < 0$$

$\therefore$ $f(x)$ attains local maximum.

Local maximum value

$$f(\tfrac{\pi}{4}) = \sin(\tfrac{\pi}{4})\cos(\tfrac{\pi}{4}) + 5$$

$$= (\tfrac{1}{\sqrt{2}})(\tfrac{1}{\sqrt{2}}) + 5$$

$$= \frac{1}{2} + 5$$

$$= \frac{11}{2}$$

Example 7.46

$$f(x) = x^2 + 2$$

$$f'(x) = 2x$$

Interval	$f'(x)$	Monotoncity
$(-2,0)$	$-ve$	st. decreasing
$(2,7)$	$+ve$	st. increasing

Example 7.47

$f(x) = x^2 - 2x - 3 \qquad (2,\infty)$

$$f'(x) = 2x - 2$$

$f'(x) \neq 0$, $+ve$ on $(2,\infty)$

$\therefore$ $f(x)$ is strictly increasing.

Example 7.50

$$f(x) = x^2 - 4x + 4$$

$$f'(x) = 2x - 4$$

$f'(x) = 0, \quad 2x - 4 = 0$

$$2x = 4$$

$\boxed{x=2}$ critical point

$-\infty \quad 2 \quad \infty$

Interval	$f'(x)$	Monotoncitiy
$(-\infty, 2)$	$-ve$	st decreasing
$(2, \infty)$	$+ve$	st. increasing

$$f''(x) = 2$$

When $x=2$

$$f''(2) = 2 > 0$$

$\therefore$ $f(x)$ attains local minimum.

Local minimum value.

$$f(2) = 4 - 8 + 4$$

$$f(2) = 0$$

Example 7.48

$f(x) = 2x^3+3x^2-12x \,,\quad [-3,2]$

$f'(x) = 6x^2+6x-12$

$f'(x)=0, \quad 6x^2+6x-12=0$

$\div 6 \quad x^2+x-2=0$

$(x+2)(x-1)=0$

$x=-2, \; x=1$

$\therefore$ critical numbers are $x=-2, 1$

$f(-3) = 2(-27)+3(9)-12(-3)$

$= -54+27+36 = -1$

$f(-2) = 2(-8)+3(4)-12(-2)$

$= -16+12+24$

$= 20$

$f(1) = 2+3-12 = -7$

$f(2) = 16+12-24 = 4$

$\therefore$ The absolute maximum is 20 at $x=-2$

The absolute minimum is -7 at $x=1$

Example 7.49

$f(x) = 3\cos x \quad [0, 2\pi]$

$f'(x) = -3\sin x$

$f'(x)=0, \quad -3\sin x = 0$

$\sin x = 0$

$x = \pi, \in (0, 2\pi)$

$x = 0, \pi, 2\pi$

$f(0) = 3\cos(0) = 3$

$f(\pi) = 3\cos\pi = -3$

$f(2\pi) = 3\cos(2\pi) = 3$

$\therefore$ The absolute maximum is 3 at $x=0, 2\pi$

The absolute minimum is -3 at $x=\pi$

Example 7.51

$f(x) = x^{2/3}$

$f'(x) = \frac{2}{3}x^{2/3-1} = \frac{2}{3}x^{-1/3} = \frac{2}{3x^{1/3}}$

$f'(x)=0, \quad \frac{2}{3x^{1/3}} \neq 0$ then $f'(x)\neq 0$

$\therefore$ The critical point at $x=0$

$-\infty \longleftarrow 0 \longrightarrow \infty$

Interval	$f'(x)$	Monotonicity
$(-\infty, 0)$	$-ve$	st. decreasing
$(0, \infty)$	$+ve$	st. increasing

$f'(x)$ changes from $-ve$ to $+ve$ at $x=0$

$\therefore f(x)$ is attains local minimum at $x=0$.

Local minimum value.

$f(0) = 0.$

Example 7.52

$f(x) = x - \sin x$

$f'(x) = 1-\cos x$

$f'(x)=0, \quad 1-\cos x = 0$

$\cos x = 1$

$x = 0, 2\pi, 4\pi, \dots$

$x = 2n\pi \quad n\in Z$

$\therefore$ There is no sign change in $f'(x)$

$\therefore$ There is no local extrema.

Example 7.53

$f(x) = \log(1+x) - \frac{x}{(1+x)}, \; x>1$

$f'(x) = \frac{1}{1+x} - \left\{\frac{x(1)-[(1+x)(1)]}{(1+x)^2}\right\}$

$f'(x) = \frac{1}{(1+x)} - \frac{1}{(1+x)^2}$

$= \frac{1+x-1}{(1+x)^2}$

$f'(x) = \frac{x}{(1+x)^2}, \quad f'(x)=0$ at $x=-1$

$f'(x) = \begin{cases} <0, & -1<x<0 \\ =0, & x=0 \\ >0, & x>0 \end{cases}$

$\therefore f(x)$ is strictly increasing for $x>0$ and $f(x)$ is strictly decreasing for $x<0$.

$f'(x)$ changes from $-ve$ to $+ve$ when passing through $x=0$,

$\therefore f$ is attains local minimum at $x=0$.

$f(0)=0$.

$\therefore \quad x>0, \quad f(x) > f(0) = 0$

$\Rightarrow \log(1+x) - \dfrac{x}{1+x} > 0$

$\log(1+x) > \dfrac{x}{1+x}$ on $(0,\infty)$

Example 7.54

$f(x) = x\log x + 3x, \quad x\in(0,\infty)$

$f'(x) = \log x + x\left(\frac{1}{x}\right)+3$

$= \log x + 1 + 3$

$f'(x) = \log x + 4$

$f'(x)=0, \Rightarrow \log x + 4 = 0$

$\log x = -4$

$x = e^{-4}$ (or) $\dfrac{1}{e^4}$

Intervals	$f'(x)$	Monotonicity
$(0, e^{-4})$	$-ve$	st. decresing
(e^{-4}, ∞)	$+ve$	st increasing

$f'(x)$ changes from $-ve$ to $+ve$ when passing through $x=e^{-4}$

$\therefore f(x)$ is attains local minimum at $x=e^{-4}$

Local minimum Value

$f(e^{-4}) = e^{-4}\log e^{-4} + 3e^{-4}$ $\quad | \log e = 1$

$= -4e^{-4} + 3e^{-4}$

$f(e^{-4}) = -e^{-4}$

Example 7.55

$f(x) = \dfrac{1}{1+x^2}$

$f'(x) = \dfrac{-2x}{(1+x^2)^2}$

$f'(x)=0, \Rightarrow \dfrac{-2x}{(1+x^2)^2}=0$

$-2x=0$

$\boxed{x=0}$

Intervals	$f'(x)$	Monotoneity
$(-\infty,0)$	$+ve$	St. increasing
$(0,\infty)$	$-ve$	St. decreases

$f'(x)$ changes from $+ve$ to $-ve$ at $x=0$

$\therefore f(x)$ attains local maximum at $x=0$

Local maximum value

$f(0) = \dfrac{1}{1+0} = 1$

Example 7.56

$f(x) = \dfrac{x}{1+x^2}$

$f'(x) = \dfrac{(1+x^2)(1) - x(2x)}{(1+x^2)^2}$

$= \dfrac{1+x^2-2x^2}{(1+x^2)^2}$

$f'(x) = \dfrac{1-x^2}{(1+x^2)^2}$

$f'(x)=0, \quad \dfrac{1-x^2}{(1+x^2)^2}=0, \Rightarrow 1-x^2=0$

$x^2=1$

$\boxed{x=\pm 1}$

$-\infty \quad -1 \quad 1 \quad \infty$

Intervals	$f'(x)$	Monotoncity.
$(-\infty,-1)$	$-ve$	st. decreasing
$(-1,1)$	$+ve$	st. increasing
$(1,\infty)$	$-ve$	st. decreasing

$f'(x)$ changes from $-ve$ to $+ve$ at $x=-1$

$\therefore f$ is attains local minimum at $x=-1$

Local minimum value

$f(-1) = \dfrac{-1}{1+1} = \dfrac{-1}{2}$

$f'(x)$ changes from $+ve$ to $-ve$ at $x=1$

$\therefore f$ is attains local maximum at $x=1$

Local maximum value

$f(1) = \dfrac{1}{1+1} = \dfrac{1}{2}$

Concavity :-

(i) If $f''(x) > 0$ on (a, b) then $f(x)$ is Concave upword

(ii) If $f''(x) < 0$ on (a, b) then $f(x)$ is Concave down word.

Example 7.57

$f(x) = (x-1)^3 (x-5), \quad x \in R$

$f'(x) = 3(x-1)^2(x-5) + (x-1)^3(1)$

$= (x-1)^2 [3x - 15 + x - 1]$

$= (x-1)^2 [4x - 16]$

$f'(x) = 4(x-1)^2 (x-4)$

$f''(x) = 4[(x-1)^2(1) + (x-4)\, 2(x-1)]$

$= 4[x^2 - 2x + 1 + 2x^2 - 10x + 8]$

$= 4[3x^2 - 12x + 9]$

$= 12[x^2 - 4x + 3]$

$f''(x) = 12(x-1)(x-3)$

$f''(x) = 0, \Rightarrow x = 1, x = 3$

$-\infty \quad 1 \quad 3 \quad \infty$

Intervals	$f''(x)$	Concavity
$(-\infty, 1)$	+ve	cancave up word
$(1, 3)$	−ve	cancave down word
$(3, \infty)$	+ve	concave upword

point of inflection $(x, f(x))$

At $x = 1$,

$f(1) = 0 \qquad (1, 0)$

At $x = 3$

$f(3) = (2^3)(-2) = -16$

$(3, -16)$

Example 7.58

$y = 3 + \sin x$

$y' = \cos x$

$y'' = -\sin x$

$y'' = 0, \quad -\sin x = 0$

$\sin x = 0,$

$x = n\pi$

Let us consider the $x \in (-\pi, \pi)$

$-\pi \quad 0 \quad \pi$

Intervals	$f''(x)$	Concavity
$(-\pi, 0)$	+ve	concave upword
$(0, \pi)$	−ve	cancave down word

$\therefore$ The point of inflection $(x, f(x))$

At $x = 0$

$f(0) = 3 + 0 = 3$

$(x, f(x)) = (0, 3)$

Example 7.59

(i) $f(x) = x^4 + 32x$

$f'(x) = 4x^3 + 32$

$f''(x) = 12x^2$

put $f'(x) = 0, \quad 4x^3 + 32 = 0$

$4x^3 = -32$

$x^3 = -8$

$\boxed{x = -2}$

At $x = -2$

$f''(-2) = 12(4) = 32 > 0$

$\therefore f$ attains local minimum at $x = -2$

Local minimum value

$f(-2) = (-2)^4 + 32(-2)$

$= +16 - 64$

$f(-2) = -48$

(ii) $f(x) = x^{\frac{4}{5}} (x-4)^2$

$f'(x) = x^{\frac{4}{5}} [2(x-4)] + (x-4)^2 \frac{4}{5} x^{-\frac{1}{5}}$

$= 2x^{\frac{4}{5}} (x-4) + \frac{4}{5} x^{-\frac{1}{5}} (x-4)^2$

$= (x-4)\left[2x^{\frac{4}{5}} + \frac{4}{5} x^{-\frac{1}{5}} (x-4)\right]$

$= (x-4)\left[2x^{4/5} + \frac{4}{5} x^{\frac{4}{5}} - \frac{16}{5} x^{-1/5}\right]$

$= (x-4)\left[\frac{14}{5} x^{\frac{4}{5}} - \frac{16}{5} x^{-\frac{1}{5}}\right]$

$f'(x) = \left(\frac{x-4}{5}\right)\left[14x^{4/5} - \frac{16}{x^{1/5}}\right]$

$f'(x) = 0,$

$\left(\frac{x-4}{5}\right) \cdot \left(14x^{4/5} - \frac{16}{x^{1/5}}\right) = 0$

$\frac{x-4}{5}=0$

$x-4=0$

$\boxed{x=4}$

$14x^{\frac{4}{5}}-\frac{16}{x^{1/5}}=0$

$14x^{\frac{4}{5}}=\frac{16}{x^{1/5}}$

$x^{\frac{4}{5}}.x^{\frac{1}{5}}=\frac{16}{14}$

$\boxed{x=\frac{8}{7}}$

When $x=0$, $f'(x)$ does not exist

$\therefore$ The critical numbers are $0, 4, \frac{8}{7}$

Example 7.60

$f(x)=4x^6-6x^4$

$f'(x)=24x^5-24x^3$

$f''(x)=120x^4-72x^2$

$f'(x)=0,\quad 24x^3(x^2-1)=0$

$24x^3=0,\quad x^2=1$

$\boxed{x=0}\quad \boxed{x=\pm 1}$

At $x=-1$ $\quad x=0,1,-1$

$f''(-1)=120-72>0$

f is attains local minimum at $x=-1$

At $x=1$

$f''(1)=120-72>0$

f is attains local minimum at $x=1$

At $x=0$

$f''(0)=0$ there is no information at $x=0$.

Intervals	$f'(x)$	Monotoncity
$(-\infty,-1)$	$-ve$	st. decreasing
$(-1,0)$	$+ve$	st. increasing
$(0,1)$	$-ve$	st. decreasing
$(1,\infty)$	$+ve$	st. increasing

Local minimum value at $x=-1$

$f(-1)=4-6=-2$

Local maximum value at $x=0$

$f(0)=0$

Example. 7.61

Given $x+y=10$

$y=10-x$

$f(x)=x^2y^2$

$f(x)=x^2(10-x)^2=x^2(100-20x+x^2)$

$f(x)=100x^2-20x^3+x^4$

$f'(x)=200x-60x^2+4x^3$

$f''(x)=200-120x+12x^2$

put $f'(x)=0$,

$4x^3-60x^2+200x=0$

$4x(x^2-15x+50)=0$

$4x(x-5)(x-10)=0$

$4x=0\quad x-5=0,\quad x-10=0$

$\boxed{x=0}\quad \boxed{x=5}\quad \boxed{x=10}$

At $x=0$

$f''(0)=200>0$

$\therefore f$ attains local minimum at $x=0$

Local minimum value is

$f(0)=0.$

At $x=5$

$f''(5)=200-600+300=-100<0$

$\therefore f$ attains local maximum at $x=5$

Local maximum value is

$f(5)=(5)^2(5)^2=25\times 25=625$

At $x=10$

$f''(10)=200>0$

f attains local minimum at $x=10$

local minimum value is

$f(10)=0.$

EXERCISE - 7.7

(1) Soln:-

(i) $f(x)=x(x-4)^3$

$f'(x)=x\,3(x-4)^2+(x-4)^3$

$f'(x)=(x-4)^2[3x+x-4]$

$f'(x)=(x-4)^2[4x-4]$

$f'(x)=4(x-4)^2(x-1)$

$f'(x)=0,\quad 4(x-4)^2(x-1)=0$

$(x-4)^2=0\quad x-1=0$

$\boxed{x=4}$ $\boxed{x=1}$

$f''(x) = 4[2(x-4)(x-1)+(x-4)^2]$

$= 4(x-4)[2x-2+x-4]$

$= 4(x-4)(3x-6)$

$f''(x) = 12(x-4)(x-2)$

$f''(x) = 0$

$f''(1) \Rightarrow 12(x-4)(x-2) = 0$

$x-4=0,\ x-2=0$

$\boxed{x=4}$ $\boxed{x=2}$

$-\infty \quad 2 \quad 4 \quad \infty$

Intervals	$f''(x)$	Convavity
$(-\infty, 2)$	+ve	Concave upward
$(2, 4)$	-ve	Concave down word
$(4, \infty)$	+ve	concave upword

point of inflection:-

At $x=2$

$f(2) = 2(-2)^3 = 2(-8) = -16$

$(x, f(x)) = (2, -16)$

At $x=4$

$f(4) = 4(0) = 0$

$(x, f(x)) = (4, 0)$

(ii) $f(x) = \sin x + \cos x \quad 0 < x < 2\pi$

$f'(x) = \cos x - \sin x$

$f''(x) = -\sin x - \cos x$

$f''(x) = 0, \quad -\sin x - \cos x = 0$

$-\sin x = \cos x$

$\frac{-\sin x}{\cos x} = 1$

$-\tan x = 1$

$\tan x = -1$

$x = \tan^{-1}(1)$

$x = \frac{3\pi}{4}, \frac{7\pi}{4}$

$0 \quad \frac{3\pi}{4} \quad \frac{7\pi}{4} \quad 2\pi$

Intervals	$f''(x)$	Concavity
$(0, \frac{3\pi}{4})$	-ve	down word
$(\frac{3\pi}{4}, \frac{7\pi}{4})$	+ve	up word
$(\frac{7\pi}{4}, 2\pi)$	-ve	down word

point of Inflection:- $(x, f(x))$

$x = \frac{3\pi}{4}$

$f(\frac{3\pi}{4}) = \sin\frac{3\pi}{4} + \cos\frac{3\pi}{4} = \frac{1}{\sqrt2} - \frac{1}{\sqrt2} = 0$

$(\frac{3\pi}{4}, 0)$

$f(\frac{7\pi}{4}) = \sin\frac{7\pi}{4} + \cos\frac{7\pi}{4} = -\frac{1}{\sqrt2} + \frac{1}{\sqrt2} = 0$

$(\frac{7\pi}{4}, 0)$

(iii) $f(x) = \frac{1}{2}(e^x - e^{-x})$

$f'(x) = \frac{1}{2}[e^x + e^{-x}]$

$f''(x) = \frac{1}{2}[e^x - e^{-x}]$

$f''(x) = 0, \quad \frac{1}{2}[e^x - e^{-x}] = 0$

$e^x - e^{-x} = 0$

$e^x = \frac{1}{e^x}$

$e^{2x} = 1$

$2x = \log(1)$

$\boxed{x=0}$

$-\infty \quad 0 \quad \infty$

Intervals	$f''(x)$	concavity
$(-\infty, 0)$	+ve	upword
$(0, \infty)$	-ve	down word.

point of Inflection $(x, f(x))$ is

$x = 0$

$f(0) = \frac{1}{2}(1-1) = 0$

$(x, f(x)) = (0, 0)$

(2) Soln:-

(i) $f(x) = -3x^5 + 5x^3$

$f'(x) = -15x^4 + 15x^2$

$f''(x) = -60x^3 + 30x$

$f'(x) = 0, \quad -15x^4 + 15x^2 = 0$

$15x^2(-x^2+1) = 0$

$15x = 0$ $\qquad x^2+1=0$

$\boxed{x=0}$ $\qquad x^2 = 1$

$x^2 = 1$

$\boxed{x = \pm 1}$

At $x=0$.

$f''(0) = 0$ no information.

At $x = \pm 1$

$f''(\pm 1) = 0$ No information

Then, $-\infty \quad -1 \quad 0 \quad 1 \quad \infty$

Intervals	$f'(x)$	Monotonicity
$(-\infty, -1)$	$-ve$	st. decreasing
$(-1, 0)$	$+ve$	st Increasing
$(0, 1)$	$+ve$	st Increasing
$(1, \infty)$	$-ve$	st decreasing

At $x=1$

$f''(1) = -60+30 = -30 < 0$

$\therefore f$ attains local maximum at $x=1$

Local maximum value is

$f(1) = -3+5 = 2$

At $x=-1$

$f''(-1) = 60-30 = 30 > 0$

f attains local minimum at $x=-1$

Local minimum value is

$f(-1) = -3+5 = -2$

(ii) $f(x) = x\log x$

$f'(x) = x\left(\frac{1}{x}\right) + \log x$

$f'(x) = 1+\log x$

$f''(x) = \frac{1}{x}$

put $f'(x) = 0$, $\quad 1+\log x = 0$

$\log x = -1$

$x = e^{-1}$

$\boxed{x = \frac{1}{e}}$

$0 \quad 1/e \quad \infty$

Intervals	$f'(x)$	Monotonicity
$(0, \frac{1}{e})$	$-ve$	st. decreasing
$(\frac{1}{e}, \infty)$	$+ve$	st. Increasing

At $x = \frac{1}{e}$

$f''(\frac{1}{e}) = \frac{1}{(\frac{1}{e})} = e > 0$

$\therefore f$ is attains local minimum at $x = \frac{1}{e}$

Local minimum value is

$f(\frac{1}{e}) = \frac{1}{e}(\log e^{-1}) = -\frac{1}{e}$

(iii) $f(x) = x^2 e^{-2x}$

$f'(x) = x^2(e^{-2x})(-2) + e^{-2x}(2x)$

$f'(x) = e^{-2x}[-2x^2+2x]$

$f''(x) = e^{-2x}[-4x+2] + (-2x^2+2x)e^{-2x}(-2)$

$f''(x) = e^{-2x}[2-4x-4x+4x^2]$

$= e^{-2x}[4x^2-8x+2]$

$f''(x) = 2e^{-2x}(2x^2-4x+1)$

$f'(x) = 0$, $\quad e^{-2x}[-2x^2+2x]$

$e^{-2x} = 0 \quad -2x^2+2x = 0$

$\boxed{x = 0, 1}$ $\quad 2x^2 = 2x$

$\boxed{x = 1}$

At $x = 0$

$f''(0) = 2(1) > 0$

$\therefore f$ attains local minimum at $x = 0$

Local minimum value is

$f(0) = 0$

At $x = 1$

$f''(1) = \frac{2}{e^2}(2-4+1) = \frac{-2}{e^2} < 0$

$\therefore f$ attains local maximum at $x=1$

Local maximum value is

$f(1) = e^{-2} = \frac{1}{e^2}$

(3) $f(x) = 4x^3+3x^2-6x+1$

$f'(x) = 12x^2+6x-6$ $\quad | \; f'(x) = 0$

$12x^2+6x-6=0$

$\boxed{x = -1, 1/2}$

$f''(x) = 24x+6$

$f''(x) = 0 \quad 24x+6=0$

$\boxed{x = -\frac{1}{4}}$

The critical points are $x = \frac{1}{2}, -1$

$-\infty \quad -1 \quad \frac{1}{2} \quad \infty$

Intervals	$f'(x)$	Monotoncity
$(-\infty, -1)$	+ve	st. Increasing
$(-1, \frac{1}{2})$	−ve	st. decreasing
$(\frac{1}{2}, \infty)$	+ve	st. Increasing

At $x = -1$

$f(-1) = -4 + 3 + 6 + 1 = 6$ (max)

At $x = \frac{1}{2}$

$f(\frac{1}{2}) = \frac{4}{8} + \frac{3}{4} - 3 + 1 = -\frac{3}{4}$ (min)

Local maximum 6 at $x = -1$

Local minimum $-\frac{3}{4}$ at $x = \frac{1}{2}$

$-\infty \quad -\frac{1}{4} \quad \infty$

Intervals	$f''(x)$	Concavity
$(-\infty, -\frac{1}{4})$	−ve	down ward
$(-\frac{1}{4}, \infty)$	+ve	up ward

point of Inflection $(x, f(x))$ is

$x = -\frac{1}{4}$

$f(-\frac{1}{4}) = -\frac{4}{64} + \frac{3}{16} + \frac{6}{4} + 1 = \frac{-4 + 12 + 96 + 64}{64}$

$= \frac{21}{8}$

$\therefore (x, f(x)) = (-\frac{1}{4}, \frac{21}{8})$

EXERCISE 7.8

(1) Soln:-

Let two positive numbers be x, y

$x+y=12$

$y=12-x \to (1)$

$P = xy$

$= x(12-x)$

$P = 12x - x^2$

$P' = 12-2x$

$P'' = -2$

put $P'=0$

$12-2x=0$

$-2x = 12$

$\boxed{x=6}$

At $x=6$

$P'' = -2 < 0$

P attains maximum at $x=6$.

$(1) \Rightarrow y = 12-6$

$\boxed{y=6}$

Hence 2 positive number are 6, 6

(2) Soln:-

Let two +ve numbers be x, y

$xy = 20$

$y = \frac{20}{x} \to (1)$

$S = x+y$

$S = x + \frac{20}{x}$

$S = x + 20x^{-1}$

$S' = 1 + 20x^{-2}(-1)$

$S'' = 20(-1)(-2)x^{-3}$

$S'' = \frac{40}{x^3}$

put $S'=0$

$1 - \frac{20}{x^2} = 0$

$1 = \frac{20}{x^2}$

$x^2 = 20$

$x = \pm\sqrt{20}$

$x = \pm 2\sqrt{5}$

Since $x > 0$, $x = 2\sqrt{5}$

At $x = 2\sqrt{5}$

$S'' = \frac{40}{(2\sqrt{5})^3} > 0$

S attains minimum at $x = 2\sqrt{5}$

At $x = 2\sqrt{5}$

$(1) \Rightarrow y = \frac{20}{2\sqrt{5}} = \frac{10}{\sqrt{5}} = \frac{2\times 5}{\sqrt{5}} = 2\sqrt{5}$

Hence 2 positive numbers are $2\sqrt{5}$, $2\sqrt{5}$.

(3) Soln:-

$x+y=10$

$y = 10-x \to (1)$

$f(x) = x^2+y^2$

$f(x) = x^2+(10-x)^2 \to (2)$

$f'(x) = 2x + 2(10-x)(-1)$

$f'(x) = 2x - 2(10-x)$

$f''(x) = 2 - 2(-1)$

$f''(x) = 4$

put $f'(x) = 0$

$2x - 20 + 2x = 0$.

$4x = 20$

$\boxed{x=5}$

At $x=5$

$f''(5) = 4 > 0$

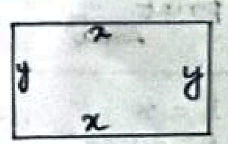
f attains minimum at $x=5$

At $x=5$

$(2) \Rightarrow f(5) = 25 + 25 = 50$

$\therefore$ Smallest value is 50.

(4) Soln:-



$x+y+x+y=40$

$2x+2y=40$

$\div 2 \quad x+y=20$

$y = 20-x \to (1)$

④ $A = xy$

$A = x(20-x)$

$A = 20x - x^2 \rightarrow$ ②

$A' = 20 - 2x$

$A'' = -20$

put $A' = 0$

$20 - 2x = 0$

$20 = 2x$

$\Rightarrow \boxed{x = 10}$

At $x = 10$

$A'' = -20 < 0$

A attains Largest at $x = 10$

At $x = 10$

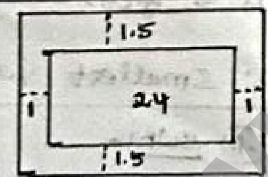
② $\Rightarrow A = 20(10) - 10^2$

$A = 200 - 100$

$A = 100\ m^2$

Largest area is $100\ m^2$

⑤ Soln:-



Print Area	Page.
$l = x$	$l = x+2$
$b = y$	$b = y+3$

$A = 24$

$xy = 24$

$y = 24/x \rightarrow$ ①

Area of page $= l \times b$

$= (x+2)(y+3)$

$= (x+2)\left(\frac{24}{x} + 3\right)$

$= 24 + 3x + \frac{48}{x} + 6$

$A = 30 + 3x + \frac{48}{x}$

$\Rightarrow A = 30 + 3x + 48x^{-1}$

$A' = 3 - 48x^{-2}$

$A'' = 96x^{-3} = \frac{96}{x^3}$

put $A' = 0$

$3 - \frac{48}{x^2} = 0$

$3 = \frac{48}{x^2}$

$3x^2 = 48$

$x^2 = 16$

$x = \pm 4$

Since $x > 0$, $\boxed{x = 4}$

At $x = 4$

$A'' = \frac{96}{4^3} > 0$

A attains local minimum at $x = 4$

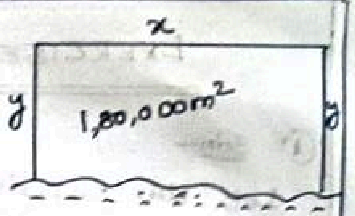
① $\Rightarrow y = \frac{24}{4} = 6$

For page,

$l = x + 2 = 4 + 2 = 6cm$

$b = y + 3 = 6 + 3 = 9cm$

⑥ Soln:-



Given, $A = 1,80,000$

$A = xy$

$180000 = xy$

$\Rightarrow y = \frac{180000}{x} \rightarrow$ ①

Perimeter $= x + 2y$

$= x + 2\left(\frac{180000}{x}\right)$

$P = x + 360000x^{-1}$

$P' = 1 - 360000x^{-2}$

$P'' = 720000x^{-3} = \frac{720000}{x^3}$

put $P' = 0$

$1 - \frac{360000}{x^2} = 0$

$1 = \frac{360000}{x^2}$

$x^2 = 360000$

$x = 600$

At $x = 600$

$P'' = \frac{720000}{(600)^3} > 0$

P attains minimum at $x = 600$

At $x = 600$

① $\Rightarrow y = \frac{180000}{600} = 300$

$\therefore P = x + 2y$

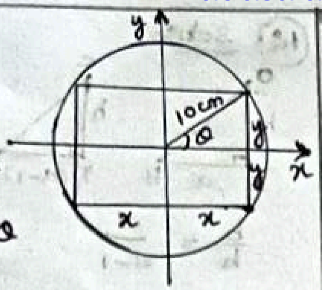
$= 600 + 2(300)$

$P = 1200\ m$

Length of minimum needed material 1200m

⑦ Soln:-

Radius = 10 cm



$\sin\theta = \frac{y}{10}$ | $\cos\theta = \frac{x}{10}$

$y = 10\sin\theta$ | $x = 10\cos\theta$

$l = 2x = 20\cos\theta$

$b = 2y = 20\sin\theta$

$A = lb$

$A = (20\cos\theta)(20\sin\theta)$

$A = 400\sin\theta\cos\theta$

$A = 200(2\sin\theta\cos\theta)$

$A = 200\sin 2\theta$

$A' = 200\cos 2\theta(2) = 400\cos 2\theta$

$A'' = 400(-\sin 2\theta)(2)$

$A'' = -800\sin 2\theta$

put $A' = 0$

$400\cos 2\theta = 0$

$\cos 2\theta = 0$

$\cos 2\theta = \cos\frac{\pi}{2}$

$2\theta = \frac{\pi}{2}$

$\boxed{\theta = \frac{\pi}{4}}$

At $\theta = \frac{\pi}{4}$

$A'' = -800\sin 2(\frac{\pi}{4})$

$= -800\sin(\frac{\pi}{2})$

$A'' = -800 < 0$

A attains maximum at $\theta = \pi/4$

$l = 20\cos\theta$ | $b = 20\sin\theta$

$l = 20\cos(\frac{\pi}{4})$ | $b = 20\sin(\pi/4)$

$l = 20(\frac{1}{\sqrt{2}})$ | $b = 20(\frac{1}{\sqrt{2}})$

$l = \frac{10\times 2}{\sqrt{2}}$ | $b = 10\sqrt{2}$ cm

$l = 10\sqrt{2}$ cm

Length is $10\sqrt{2}$ cm and breadth is $10\sqrt{2}$ cm

⑧ Soln:-

Perimeter $P = 2x + 2y$

$\Rightarrow 2y = P - 2x$

$y = \frac{P}{2} - x \rightarrow ①$

Area (A) $= xy$

$A = x(\frac{P}{2} - x)$

$A = \frac{Px}{2} - x^2$

$A' = \frac{P}{2} - 2x$

$A'' = -2$

put $A' = 0$

$\frac{P}{2} - 2x = 0$

$\frac{P}{2} = 2x$

$P = 4x$

$\Rightarrow \boxed{x = \frac{P}{4}}$

At $x = P/4$

$A'' = -2 < 0$

$\therefore$ Area is maximum

$① \Rightarrow$

$y = \frac{P}{2} - \frac{P}{4}$

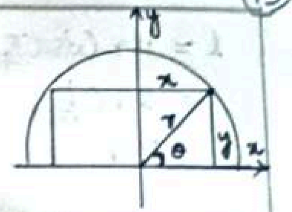
$\boxed{y = \frac{P}{4}}$

$\therefore x = y$

Thus it is a square

⑨ Soln:-

Radius $= r$ cm



$\sin\theta = \frac{y}{r}$ | $\cos\theta = \frac{x}{r}$

$y = r\sin\theta$ | $x = r\cos\theta$

$l = 2x = 2r\cos\theta$

$b = y = r\sin\theta$

$A = lb$

$A = (2r\cos\theta)(r\sin\theta)$

$A = 2r^2\sin\theta\cos\theta$

$A = r^2\sin 2\theta$

$A' = 2r^2\cos 2\theta$

$A'' = 2r^2(-\sin 2\theta)(2)$

$A'' = -4r^2\sin 2\theta$

put $A' = 0$

$2r^2\cos 2\theta = 0$

$\cos 2\theta = 0$

$2\theta = \frac{\pi}{2}$

$\boxed{\theta = \frac{\pi}{4}}$

At $\theta = \frac{\pi}{4}$

$A'' = -4r^2\sin 2(\frac{\pi}{4})$

$= -4r^2\sin(\frac{\pi}{2})$

$A'' = -4r^2 < 0$

A attains maximum at $\theta = \frac{\pi}{4}$

$l = 2r\cos\left(\frac{\pi}{4}\right)$ | $b = r\sin\left(\frac{\pi}{4}\right)$

$= 2r\left(\frac{1}{\sqrt{2}}\right)$ | $b = r\left(\frac{1}{\sqrt{2}}\right)$

$l = \sqrt{2}\,r$ | $b = \frac{r}{\sqrt{2}}$

Length $= \sqrt{2}\,r$

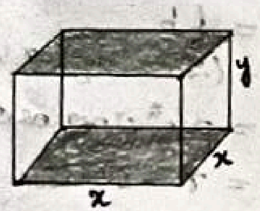
Breadth $= \frac{r}{\sqrt{2}}$

(10) Soln:-

Surface area $= 108$

$l = x$ and $b = x$

$h = y$



$S.A = x^2 + 4xy$

$\Rightarrow x^2 + 4xy = 108$

$4xy = 108 - x^2$

$\div 4x \quad y = \frac{108}{4x} - \frac{x^2}{4x}$

$y = \frac{27}{x} - \frac{x}{4} \rightarrow ①$

Volume $= lbh$

$= xxy$

$V = x^2 y$

$V = x^2\left(\frac{27}{x} - \frac{x}{4}\right)$

$V = 27x - \frac{x^3}{4}$

$V' = 27 - \frac{3}{4}x^2$

$V'' = -\frac{6}{4}x = \frac{3}{2}x$

put $V' = 0$

$27 - \frac{3x^2}{4} = 0$

$\frac{3x^2}{4} = 27$

$x^2 = \frac{27 \times 4}{3}$

$x^2 = 36$

$\boxed{x = 6}$

At $x = 6$

$V'' = -\frac{6}{4}(6) < 0$

V attains maximum at $x = 6$

At $x = 6$

$① \Rightarrow y = \frac{27}{6} - \frac{6}{4} = \frac{9}{2} - \frac{3}{2} = \frac{6}{2} = 3$

$\boxed{y = 3}$

$l = 6$ cm, $b = 6$ cm, $h = 3$ cm

(11) Soln:-

$r + h = 6$

$h = 6 - r$

$V = \pi r^2 h$

$V = \pi r^2(6 - r)$

$V = \pi(6r^2 - r^3)$

$V' = \pi(12r - 3r^2)$

$V'' = \pi(12 - 6r)$

put $V' = 0$,

$\pi(12r - 3r^2) = 0$

$12r - 3r^2 = 0$

$3r(4 - r) = 0$

$3r = 0, \quad 4 - r = 0$

$\boxed{r = 0}$ $\boxed{r = 4}$

At $r = 0$

$V'' = 12\pi > 0$

V attains minimum at $r = 0$

At $r = 4$

$V'' = \pi(12 - 24)$

$V'' = -12\pi < 0$

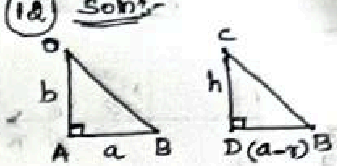
V attains max at $r = 4$

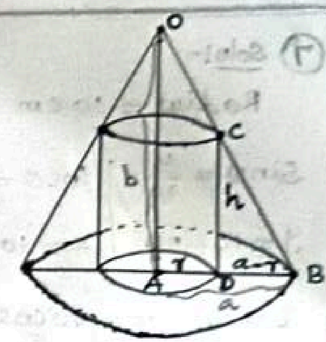
$V = \pi(16)(2)$

$V = 32\pi$

$\therefore$ Max volume is 32π

(12) Soln:-





$\frac{b}{h} = \frac{a}{a-r}$

$b(a - r) = ah$

$h = \frac{b}{a}(a - r)$

Volume of cylinder

$V = \pi r^2 h$

$= \pi r^2\left[\frac{b}{a}(a - r)\right]$

$V = \pi\frac{b}{a}(ar^2 - r^3) \rightarrow ①$

$V' = \frac{\pi b}{a}(2ar - 3r^2)$

$V'' = \frac{\pi b}{a}(2a - 6r)$

put $V' = 0$

$\frac{\pi b}{a}(2ar - 3r^2) = 0$

$2ar - 3r^2 = 0$

$r(2a - 3r) = 0$

$r = 0,$ | $2a - 3r = 0$

$r \neq 0$ | $2a = 3r$

$\Rightarrow \boxed{r = \frac{2a}{3}}$

At $r = \frac{2a}{3}$

$V'' = \frac{\pi b}{a}\left(2a - 6\left(\frac{2a}{3}\right)\right)$

$= \frac{\pi b}{a}(2a - 4a) = -\frac{\pi b}{a}(2a) < 0$

V attains maximum at $r=\frac{2a}{3}$

① ⇒ $V=\frac{\pi b}{a}\left(a\frac{4a^2}{9}-\frac{8a^3}{27}\right)$

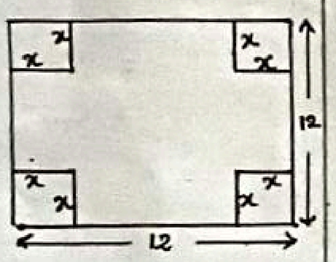
$=\frac{\pi b}{a}\left(\frac{3(4a^3)}{9\times 3}-\frac{8a^3}{27}\right)$

$=\frac{\pi b}{a}\left(\frac{4a^3}{27}\right)$

$=\frac{4}{9}\left(\frac{\pi}{3}a^2 b\right)$

Volume of cylinder $=\frac{4}{9}$ (Volume of cone)

Example : 7.62



$l=12-2x$

$b=12-2x$

$h=x$

$V=lbh$

$=(12-2x)(12-2x)x$

$V=(12-2x)^2x$

$V'=(12-2x)^2+2x(12-2x)(-2)$

$=(12-2x)[12-2x-4x]$

$V'=(12-2x)(12-6x)$

$V''=(12-2x)(-6)+(12-6x)(-2)$

put $V'=0$

$(12-2x)(12-6x)=0$

$12-2x=0, \quad 12-6x=0,$

$2x=12$, $x=6$ not possible | $12=6x$, $x=2$

At $x=2$

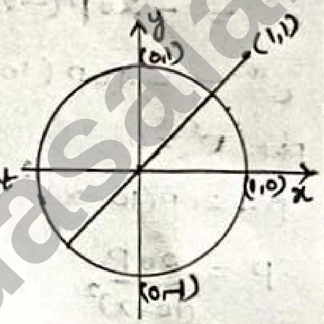
$V''=(12-4)(-6)+(12-12)(-2)$

$V''=-48<0$

V attains maximum at $x=2$

Hence maximum can be only 2 units.

Example : 7.63



Given (1,1)

Let $P(x,y)$ be the point on unit circle

$d=\sqrt{(x-1)^2+(y-1)^2}$

$d^2=(x-1)^2+(y-1)^2$

$D=(x-1)^2+(y-1)^2$

Diff w. r. to 'x'

$D'=2(x-1)+2(y-1)\frac{dy}{dx} \rightarrow$ ①

Given, $x^2+y^2=1 \rightarrow$ ②

Diff w. r. to 'x'

$2x+2y\frac{dy}{dx}=0$

$x+y\left(\frac{dy}{dx}\right)=0$

$\frac{dy}{dx}=\frac{-x}{y} \rightarrow$ ③

Sub ③ in ①

$D'=2(x-1)+2(y-1)\left(-\frac{x}{y}\right)$

$=\frac{2y(x-1)-2x(y-1)}{y}$

$=\frac{2yx-2y-2xy+2x}{y}$

$D'=\frac{2(x-y)}{y}$

$D''=2\left[\frac{y\left(1-\frac{dy}{dx}\right)-(x-y)\frac{dy}{dx}}{y^2}\right]$

$=2\left[y\left(1+\frac{x}{y}\right)+(x-y)\frac{x}{y}\right]\cdot\frac{1}{y^2}$

$=\frac{2}{y^2}\left[y\left(\frac{y+x}{y}\right)+\frac{(x-y)x}{y}\right]$

$=\frac{2}{y^2}\left[\frac{y^2+xy+x^2-xy}{y}\right]$

$=\frac{2}{y^3}\left[y^2+x^2\right]$ | $x^2+y^2=1$

$D''=\frac{2}{y^3}$

put $D'=0$

$\frac{2(x-y)}{y}=0$

$x-y=0$

$\boxed{x=y} \rightarrow$ ④

sub ④ in ②

$y^2+y^2=1$

$2y^2=1$

$y^2=\frac{1}{2}$

$y=\pm\frac{1}{\sqrt{2}}$

At $y=\frac{1}{\sqrt{2}}$

$D''=\frac{2}{(\frac{1}{\sqrt{2}})^3}>0$

D attains minimum at $y=\frac{1}{\sqrt{2}}$

④ $\Rightarrow$ $x=\frac{1}{\sqrt{2}}$

Nearest point from (1,1) is $(\frac{1}{\sqrt{2}},\frac{1}{\sqrt{2}})$

At $y=-\frac{1}{\sqrt{2}}$

$D''=\frac{2}{(-\frac{1}{\sqrt{2}})}<0$

D attains maximum at $y=-\frac{1}{\sqrt{2}}$

④ $\Rightarrow$ $x=-\frac{1}{\sqrt{2}}$

Farthest point from (1,1) is $(-\frac{1}{\sqrt{2}},-\frac{1}{\sqrt{2}})$

Example. 7.64

Low grade steel = P

high grade steel = 2P

Total cost (C) = $Px+2Py$

$y=\frac{40-5x}{10-x}$

$C=Px+2P\left(\frac{40-5x}{10-x}\right)$

Diff w.r.to 'x'

$C'=P+2P\left[\frac{(10-x)(-5)-(40-5x)(-1)}{(10-x)^2}\right]$

$C'=P+2P\left[\frac{-50+5x+40-5x}{(10-x)^2}\right]$

$C'=P+2P\left[\frac{-10}{(10-x)^2}\right]$

$C'=P-20P(10-x)^{-2}$

$C''=-20P(-2)(10-x)^{-3}(-1)$

$C''=-40P(10-x)^{-3}$

put $C'=0$

$P-20P(10-x)^{-2}=0$

$P=\frac{20P}{(10-x)^2}$

$(10-x)^2=20$

$10-x=\pm\sqrt{4\times5}$

$10-x=\pm2\sqrt{5}$

$\Rightarrow x=10\pm2\sqrt{5}$

At $x=10-2\sqrt{5}$

$C''=-40P(10-10+2\sqrt{5})^{-3}$

$C''=\frac{-40P}{(2\sqrt{5})^3}<0$

C attains maximum at $x=10-2\sqrt{5}$

$y=\frac{40-5(10-2\sqrt{5})}{...}$

$y=\frac{40-50+10\sqrt{5}}{2\sqrt{5}}$

$=\frac{10(-1+\sqrt{5})}{2\sqrt{5}}$

$=\frac{5(\sqrt{5}-1)}{\sqrt{5}}$

$=\sqrt{5}(\sqrt{5}-1)$

$y=5-\sqrt{5}$

$\therefore$ The steel produce low grade and high grad steels is $10-2\sqrt{5}$ and $5-\sqrt{5}$ tonnes per day.