SRIMAAN COACHING CENTRE-TRICHY-UG-TRB-MATHEMATICS

GRADUATE TEACHERS / BLOCK RESOURCE TEACHER EDUCATOR (BRTE)

UNIT-7- STUDY MATERIAL-TO CONTACT:8072230063.

2023-24

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UG-TRB

MATHEMATICS

UNIT-7- COMPLEX ANALYSIS

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UG-TRB: MATHEMATICS

(BT / BRTE-2023-24)

UNIT-VII-COMPLEX ANALYSIS

Algebra of Complex Number:

Definition -**Complex Number:**

A complex number z is of the form x+iy where x and y are real numbers and i is an imaginary unit with the property that $i^2=1$, x and y are called the real and imaginary part of z and we write x=Re z and y=Im z.

If x=0, the complex number z is called purely imaginary. If y=0 then z is real.

Two complex numbers are said to be equal iff they have the same real parts and the same imaginary parts.

Let C denote the set of all complex numbers.

Thus C is $\{x+iy/x, y\in R\}$

Definition

We define addition and multiplication in C as follows

Let $z_1=x_1+iy_1$ and $z_2=x_2+iy_2$

$$z_1+z_2 =(x_1 + x_2) +i(y_1 + y_2)$$

$$z_1z_2 = (x_1x_2 - y_1 y_2) +i (x_1y_2 + x_2y_1)$$

Remark 1

If $z_1=x_1+iy_1$, and $z_2=x_2+iy_2 \neq 0$ then $\frac{z_1}{z_2} = \frac{x_1x_2+y_1y_2}{x_2^2+y_2^2} + \frac{i\ y_1x_2-x_1y_2}{x_2^2+y_2^2}$

Remark 2

It is important to note that there is no order structure in the complex number system so that we cannot compare two complex numbers.

Remark 3

The complex number a+ib can also be represented by the ordered pair of real numbers (a, b).

Conjugation and modulus

Let z = x + iy be a complex number. Then the complex number x-iy is called the conjugate of z and it is denoted by $\bar{z}$.

The mapping $f : C\to C$ defined by $f(z) = \bar{z}$ is called the complex conjugation.

Note 1. z is real iff $z = \bar{z}$

2. $\bar{\bar{z}} = z$

3. $z + \bar{z} = 2 \text{ Re } z$ so that $x = \frac{z+\bar{z}}{2}$

4. $z - \bar{z} = 2i \text{ Im } z$ so that $y = \frac{z-\bar{z}}{2i}$

5. $\overline{z_1 + z_2} = \bar{z_1} + \bar{z_2}$

6. $\overline{\left(\frac{z_1}{z_2}\right)} = \frac{\bar{z_1}}{\bar{z_2}}$

Theorem

If α is a root of the polynomial equation $f(z) = a_0z^n+a_1z^{n-1}+\ldots+a_{n-1}z+a_n=0$ where $a_0, a_1, \ldots, a_n \in R$ and $a_0 \neq 0$ then $\bar{\alpha}$ is also a root of $f(z)=0$

(ie.) The non-real roots of a polynomial equation with real co-efficients occur in conjugate pairs.

Proof

Since α is a root of $f(z)=0$, we have $f(\alpha)=0$

Hence $a_0\alpha^n+a_1\alpha^{n-1}+\ldots+a_{n-1}\alpha+a_n=0$

$\Longrightarrow \overline{a_0\alpha^n + a_1\alpha^{n-1} + \cdots + a_{n-1}\alpha + a_n} = \bar{0}$

$\Longrightarrow \bar{a}_0\, \overline{\alpha^n}+\bar{a}_1\, \overline{\alpha^{n-1}}+\ldots+\bar{a}_{n-1}\, \bar{\alpha}+\bar{a}_n=0$

$\Longrightarrow a_0\bar{\alpha}^n+a_1\bar{\alpha}^{n-1}+\ldots+a_{n-1}\, \bar{\alpha}+\bar{a}_n=0$

$\Longrightarrow a_0(\bar{\alpha})^n+a_1(\bar{\alpha})^{n-1}+\ldots+a_{n-1}(\bar{\alpha})+a_n=0$

$\Longrightarrow f(\bar{\alpha})=0$ so that $\bar{\alpha}$ is also a root of $f(z)=0$.

Definition

Let $z = x+iy$ be a complex number. The modulus or absolute value of z denoted by $|z|$ is defined by $|z|=\sqrt{x^2 + y^2}$.

Remark

$|z|$ represents the distance between $z=(x, y)$ and the origin $O=(0, 0)$.

Theorem

i. $|z| \geq 0$ and $|z|=0$ iff $z=0$

ii. $z\bar{z} = |z|^2$

iii. $|z_1z_2| = |z_1|\,|z_2|$

iv. $\left|\frac{z_1}{z_2}\right| = \frac{|z_1|}{|z_2|}$ provided $z_2 \neq 0$

v. $|z_1+z_2|^2 = |z_1|^2 +|z_2|^2 + 2\text{Re}\,(z_1\, \bar{z}_2|$

vi. $|z_1-z_2|^2 = |z_1|^2 + |z_2|^2 - 2\text{Re}\,(z_1\,\bar{z}_2|$

vii. $|z_1+z_2|^2 + |z_1-z_2|^2 = 2(|z_1|^2 + |z_2|^2)$

Solved Problems

Problem 1

Find the absolute value of $\frac{(1+3i)\,(1-2i)}{3+4i}$

Solution

$$\left|\frac{(1+3i)\,(1-2i)}{3+4i}\right| = \frac{|1+3i|\,|(1-2i)|}{3+4i}$$

$$= \frac{\sqrt{10}\sqrt{5}}{5}$$

$$= \frac{\sqrt{2x5}\sqrt{5}}{5}$$

$$= \frac{\sqrt{2}x5}{5} = \sqrt{2}$$

Problem 2

Find the condition under which the equation $az+b\bar{z}+c=0$ in one complex unknown has exactly one solution and compute that solution.

Solution

$az+b\bar{z}+c=0$ (1)

Taking conjugate we have,

$\overline{az + b\bar{z} + c} = \bar{0}$

$\Longrightarrow \bar{a}\bar{z}+\bar{b}z+\bar{c} = 0$ (2)

(1) x $\bar{a} \Longrightarrow \bar{a}a\,z + \bar{a}b\,\bar{z} + \bar{a}c = 0$ (3)

(2) x b $\Longrightarrow b\bar{b}\,z + b\bar{a}\,\bar{z} + b\bar{c} = 0$ (4)

(3) – (4) $\Longrightarrow z(a\bar{a} - b\bar{b}) + \bar{a}c - b\bar{c} = 0$

$\Longrightarrow z(|a|^2 - |b|^2) = b\bar{c} - \bar{a}c$

Hence if $|a| \neq |b|$, the given equation has unique solution and the solution is given by $z = \frac{b\bar{c}-\bar{a}c}{|a|^2-|b|^2}$

Problem 3

If z_1 and z_2 are two complex numbers prove that $\left|\frac{z_1-z_2}{1-z_1\bar{z}_1}\right|=1$ if either $|z_1|=1$ or $|z_2|=1$. What exception must be made if $|z_1|=1$ and $|z_2|=1$.

Solution

Suppose $|z_1|=1$. Hence $|\bar{z}_1|=1$ and $z_1\,\bar{z}_1 = |z_1|^2 = 1$.

Now $\left|\frac{z_1-z_2}{1-\bar{z}_1 z_2}\right| = \left|\frac{z_1-z_2}{z_1\bar{z}_1-\bar{z}_1 z_2}\right|$

$= \left|\frac{z_1-z_2}{\bar{z}_1(z_1 z_2)}\right|$

$= \left|\frac{1}{\bar{z}_1}\right| = 1$

Similarly if $|\bar{z}_2|=1$, we have $\left|\frac{z_1-z_2}{1-\bar{z}_1 z_2}\right|=1$. If $|z_1|=1$ and $|z_2|=1$, then the result is true provided $1-\bar{z}_1 z_2 \neq 0$

ie. if $z_1 - z_1\,\bar{z}_1 z_2 \neq 0$

ie. if $z_1 \neq |z_1|^2 z_2$

ie. if $z_1 \neq z_2$

Inequalities

Theorem

For any three complex numbers z_1, z_2 and z_3.

i. $-|z| \leq \text{Re } z \leq |z|$
ii. $-|z| \leq \text{Im } z \leq |z|$
iii. $|z_1+z_2| \leq |z_1|+|z_2|$. (Triangle inequality)
iv. $|z_1-z_2| \geq ||z_1|-|z_2||$

Proof

Let z = +iy

Hence $|z| = \sqrt{x^2+y^2}$

Now $-\sqrt{x^2+y^2} \leq x \leq \sqrt{x^2+y^2}$

and $-\sqrt{x^2+y^2} \leq y \leq \sqrt{x^2+y^2}$

$\therefore -|z| \leq \text{Re } z \leq |z|$ and $-|z| \leq \text{Im } z \leq |z|$

Hence (i) and (ii) are proved.

iii) Triangle inequality

$|z_1 + z_2| \leq |z_1| + |z_2|$

$|z_1 + z_2|^2 = (z_1+z_2)\,(\overline{z_1+z_2})$ $\qquad \because |z|^2 = z\bar{z}$

$= (z_1+z_2)\,(\overline{z_1+z_2})$

$= z_1\bar{z}_1 + z_1\bar{z}_2 + z_2\bar{z}_1 + z_2\bar{z}_2$

$= |z_1|^2 + (z_1\bar{z}_2+\bar{z}_1z_2)+|z_2|^2$

$= |z_1|^2 + (z_1\bar{z}_2+ \overline{z_1\bar{z}_2})+|z_2|^2$

$= |z_1|^2 + 2\,\mathrm{Re}(z_1\bar{z}_2) +|z_2|^2$

$\leq |z_1|^2 + 2\,|(z_1\bar{z}_2)|+|z_2|^2$

$= |z_1|^2 + 2\,|z_1|\,|\bar{z}_2|+|z_2|^2$

$= |z_1|^2 + 2\,|z_1|\,|z_2|+|z_2|^2$ $\quad [\because |z_2| = |\bar{z}_2|]$

$= (|z_1| + |z_2|)^2$

Thus $|z_1 + z_2|^2 \leq (|z_1| + |z_2|)^2$

$\therefore |z_1 + z_2| \leq |z_1| +|z_2|$

iv) $|z_1 - z_2| \geq ||z_1| -|z_2||$

$z_1 = (z_1-z_2) + z_2$

$|z_1| = |(z_1-z_2) + z_2| \leq |z_1 - z_2| + |z_2|$

$\Rightarrow |z_1| - |z_2| \leq |z_1 - z_2|$ (1)

$z_2 = (z_2 - z_1) + z_1$

$|z_2| = |(z_2-z_1) + z_1| \leq |z_2 - z_1| + |z_1|$

$\Rightarrow |z_2|-|z_1| \leq |z_2-z_1|$

$\Rightarrow -(|z_1|-|z_2|) \leq |z_2-z_1|$

$\Rightarrow |z_1|-|z_2| - |z_2-z_1|$ (2)

From (1) and (2)

$- |z_2-z_1| \leq |z_1|-|z_2| \leq |z_1-z_2|$

ie. $- |z_1-z_2| \leq |z_1|-|z_2| \leq |z_1-z_2|$

$\Rightarrow -||z_1|-|z_2|| \leq |z_1-z_2|$

ie. $- |z_1-z_2| \geq ||z_1|-|z_2||$

Note

For any complex numbers $z_1, z_2, \ldots, z_n$ we have $|z_1+z_2+\ldots+z_n| \leq |z_1|+|z_2| +\ldots+|z_n|$

Polar form of a complex number

Consider any non zero complex number z=x+iy.

Let (r, θ) denote the polar co-ordinates of the point (x, y)

Hence $x = r\cos\theta$ and $y = r\sin\theta$

$\therefore z = r(\cos\theta + \sin\theta)$

We notice that r = |z|=$\sqrt{x^2 + y^2}$ which is the magnitude of the complex number and θ is called the amplitude or argument of z and is denoted by arg z or amp z.

We note that the value of arg z not unique. If θ = arg z then θ + 2n π where n is any integer is also a value of arg z. The value of arg z lying in the range (-π, π) is called the principal value of arg z.

Theorem

If z_1 and z_2 are any two non zero complex numbers then

i. $-\arg z_1 = \arg \bar{z}_1$

ii. $\arg z_1z_2 = \arg z_1 + \arg z_2$

ii. $\arg [\frac{z_1}{z_2}] = \arg z_1 - \arg z_2$

Proof

Let $z_1 = r_1(\cos \theta_1 + i \sin \theta_1)$

$\therefore \bar{z}_1 = r_1(\cos \theta_1 - i \sin \theta_1)$

$= r_1(\cos (-\theta_1) + i \sin (-\theta_1))$

Hence $\arg \bar{z}_1 = -\theta_1$

$= -\arg z_1$.

$\therefore \arg \bar{z}_1 = -\arg z_1$

ii) Let $z_1 = r_1(\cos \theta_1 + i \sin \theta_1)$ and

$z_2 = r_2(\cos \theta_2 + i \sin \theta_2)$

$\Longrightarrow \arg z_1 = \theta_1$ and $\arg z_2 = \theta_2$

Now $z_1 z_2 = r_1 r_2(\cos \theta, + i \sin \theta)$ $(\cos \theta_2 + i \sin \theta_2) (\cos \theta_1 + i \sin \theta_1)$

$= r_1r_2[(\cos (\theta_1+\theta_2)+ \sin\theta_1 \sin\theta_2) +i (\sin\theta_1 \cos\theta_2 + \cos\theta_1 \sin\theta_2)]$

$= r_1r_2[(\cos\theta_1 \theta_2) + i \sin(\theta_1 + \theta_2)]$

$\therefore \arg z_1z_2 = \theta_1 + \theta_2$

$= \arg z_1 + \arg z_2$

$\therefore \arg z_1z_2 = \arg z_1 + \arg z_2$

iii) $\arg (\frac{z_1}{z_2}) = \arg z_1 - \arg z_2$

$\frac{z_1}{z_2} = \frac{r_1(\cos \theta_1 + i \sin \theta_1)}{r_2(\cos \theta_2 + i \sin \theta_2)}$

$$= [\frac{r_1}{r_2}][\frac{(\cos\theta_1 + i\sin\theta_1)}{(\cos\theta_2 + \sin\theta_2)} \text{ x } \frac{\cos\theta_2 + i\sin\theta_2}{\cos\theta_2 + i\sin\theta_2}]$$

$$= (\frac{r_1}{r_2})[\frac{(\cos\theta_1\cos\theta_2 + \sin\theta_1\sin\theta_2 + i(\sin\theta_1\cos\theta_2 - \cos\theta_1\sin\theta_2)}{(\cos^2\theta_2 + \sin^2\theta_2)}]$$

$$= (\frac{r_1}{r_2})(\frac{(\cos\theta_1 - \theta_2) + i\sin(\theta_1 - \theta_2)}{1})$$

$$= (\frac{r_1}{r_2})[\cos(\theta_1 - \theta_2) + i\sin(\theta_1 - \theta_2)$$

$\arg[\frac{z_1}{z_2}] = \theta_1 - \theta_2$

$= \arg z_1 - \arg z_2$

$\therefore \arg[\frac{z_1}{z_2}] = \arg z_1 - \arg z_2$

Theorem 4.1.6

Let $z = r(\cos\theta + i\sin\theta)$ be any non zero complex number and n be any integer. Then $z^n = r^n(\cos n\theta + i\sin n\theta)$.

Proof

We first prove this result for positive integers by induction on n.

When n=1

$z^1 = r^1(\cos\theta + i\sin\theta)$

ie. $z = r(\cos\theta + i\sin\theta)$ which is true.

Hence the theorem is true when n=1.

Suppose the result is true for n=m.

Hence $z^m = r^m(\cos m\theta + i\sin m\theta)$

To prove the result is true when n=m+1

Now $z^{m+1} = z^m z$

$= r^m(\cos m\theta + i\sin m\theta)\, r(\cos\theta + i\sin\theta)$

$= r^{m+1}[(\cos m\theta \cos\theta - \sin\theta\sin m\theta) + i(\cos m\theta\sin\theta + \sin m\theta\cos\theta)]$

$= r^{m+1}[\cos(m+1)\theta + i\sin(m+1)\theta]$

Hence the result is true for n=m+1

Hence $z^n = r^n[\cos n\theta + i\sin n\theta]$ for all positive integers n.

The result is obviously true if n=0

Now $z^{-1} = \frac{1}{z}$

$= \frac{1}{r(\cos\theta + i\sin\theta)}$

$= \frac{1}{r} \times \frac{\cos\theta - i\sin\theta}{(\cos\theta + i\sin\theta)(\cos\theta - i\sin\theta)}$

$= r^{-1}\left[\frac{\cos(-\theta) + i\sin(-\theta)}{\cos^2\theta + \sin^2\theta}\right]$

$= r^{-1}[\cos(-\theta) + i\sin(-\theta)]$

∴ The result is true for n=-1. Hence it follows that the result is true for all negative integers.

Hence $z^n = r^n(\cos n\theta + i\sin n\theta)$ for all $n \in Z$.

Corollary: (De-Movire's theorem)

$(\cos\theta + i\sin\theta)^n = \cos n\theta + i\sin n\theta$

Solved Problem

Problem 1

For any three distinct complex numbers z, a, b the principal value of arg $[\frac{z-a}{z-b}]$ represents the angle between the line segment joining z and a and the line segment joining z and b taken in the appropriate sense.

Solution

Let A, B, P be the points in the complex plane representing the complex numbers a, b, z respectively.

Then $\overrightarrow{AP} = \overrightarrow{OP} - \overrightarrow{OA}$

$= z - a$

$\overrightarrow{BP} = \overrightarrow{OP} - \overrightarrow{OB}$

$= z - b$

∴ The complex numbers z-a, z-b are represented by the vectors $\overrightarrow{AP}$ and $\overrightarrow{BP}$ respectively.

Hence the principal value of arg $[\frac{z-a}{z-b}]$ gives the angle between the line segment AP and BP taken in the appropriate sense.

Circles and Straight lines

Equation of circles and straight lines in the complex plane can be expressed interms of z and $\bar{z}$.

General equation of circles

Equation of the circle with centre a and radius r is given by $|z-a|=r$

ie. $|z-a|^2 = r^2$

$\Rightarrow (z-a)\,(\overline{z-a}) = r^2$ $[\because |z|^2 = z\,\bar{z}]$

$\Rightarrow (z-a)\,(\bar{z} - \bar{a}) = r^2$

$\Rightarrow z\,\bar{z} - a\bar{z} - \bar{a}z + a\,\bar{a} - r^2 = 0$

This equation is of the form

$z\,\bar{z} + \bar{\alpha}\,z + \alpha\,\bar{z} + \beta = 0$ where β is a real number. Further any equation of the above form can be written as $|z+\alpha|^2 = \alpha\bar{\alpha} - \beta$ and hence represents a circle provided $\alpha\bar{\alpha} - \beta > 0$. It represents a circle with centre $-\alpha$ and radius $\sqrt{\alpha\bar{\alpha} - \beta}$.

Thus the general equation of a circle is given by $z\,\bar{z} + \bar{\alpha}\,z + \alpha\,\bar{z} + \beta = 0$ where β is real and $\alpha\bar{\alpha} - \beta > 0$.

General equation of straight lines

To find the general equation of the straight line passing through a and b, we note that arg $[\frac{z-a}{z-b}]$ represents the angle between the lines joining a to z and b to z where z is any point on the line joining a and b.

$\therefore$ If z, a, b are collinear then arg $[\frac{z-a}{z-b}] = 0$ or π

$\therefore \frac{z-a}{z-b}$ is real. Hence $\frac{z-a}{z-b} = [\overline{\frac{z-a}{z-b}}]$

$\therefore \frac{z-a}{z-b} = [\frac{\bar{z}-\bar{a}}{\bar{z}-\bar{b}}]$

$\Rightarrow (z-a)\,(\bar{z} - \bar{b}) = (z-b)\,(\bar{z} - \bar{a})$

$\Rightarrow z\,\bar{z} - a\,\bar{z} - \bar{b}\,z + a\,\bar{b} = z\,\bar{z} - \bar{a}\,z - b\,\bar{z} + \bar{a}\,b$

$\Rightarrow \bar{a}\,z - \bar{b}\,z - a\,\bar{z} + b\,\bar{z} + a\,\bar{b} - \bar{a}\,b = 0$

$\Rightarrow (\bar{a} - \bar{b})\,z - (a - b)\,\bar{z} + (a\,\bar{b} - \overline{a\,\bar{b}}\,) = 0$

$\Rightarrow (\bar{a} - \bar{b})\,z - (a - b)\,\bar{z} + 2i\,\mathrm{Im}\,(a\,\bar{b}) = 0$

$\lceil \because \mathrm{Im}\,z = \frac{z-\bar{z}}{2\,i} \rceil \Rightarrow z - \bar{z} = 2i\,\mathrm{Im}\,z]$

$\therefore i(\bar{a} - \bar{b})z - i(a-b)\,\bar{z} - 2\mathrm{Im}\,(a\,\bar{b}) = 0$. This equation is of the form $\bar{\alpha}\,z + \alpha\,\bar{z} + \beta = 0$ where $\alpha \neq 0$ and β is real.

Further any equation of the above form represents a straight line. This can be easily seen by changing the above equation into Cartesian form.

∴ The general equation of a straight line is given by $\overline{\alpha}\, z + \alpha\, \overline{z} + \beta = 0$. Where $\alpha \neq 0$ and β is real.

Theorem

Equation of the line joining a and b is $(\bar{a} - \bar{b})\, z + (b-a)\, \bar{z} + (a\, \bar{b} - \bar{a}\, b) = 0$

Theorem

If a and b are two distinct complex numbers where $b \neq 0$, then the equation $z = a + t\, b$ where t is a real parameter represents a straight line passing through the point a and parallel to b.

Proof

Let z be any point on the line passing through a and parallel to b. The vectors represented by $z - a$ and b are parallel.

Hence $z - a = tb$ for some real number t. Hence $z = a + t\, b$, which is the equation of the required straight line.

Definition

Two points P and Q are called reflection points for a given straight line ℓ iff ℓ is the perpendicular bisector of the segment PQ.

Theorem

Two points z_1 and z_2 are reflection points for the line $\overline{\alpha} z + \alpha \overline{z} + \beta = 0$ iff $\overline{\alpha}\, z_1 + \alpha\, \overline{z}_2 + \beta = 0$.

Proof

Let z_1 and z_2 be reflection points for the straight line $\overline{\alpha}\, z + \alpha\, \overline{z} + \beta = 0$ (1)

To prove that $\overline{\alpha}\, z_1 + \alpha\, \overline{z}_2 + \beta = 0$

For any point z on the line we have

$|z - z_1| = |z - z_2|$ [∵ z_1, z_2 are reflection points]

$\Rightarrow |z - z_1|^2 = |z - z_2|^2$

$\Rightarrow (z - z_1) \,.\, (\overline{z - z_1}) = (z - z_2)\, (\overline{z - z_2})$ [∵ $|z_1|^2 = z\, \bar{z}$]

$\Rightarrow (z - z_1)\ (\bar{z} - \bar{z}_1) = (z - z_2)\, (\bar{z} - \bar{z}_2)$

$\Rightarrow z\, \bar{z} - z\, \bar{z}_1 - z_1\, \bar{z} + z_1\, \bar{z}_1 = z\, \bar{z} - z\, \bar{z}_2 - z_2\, \bar{z} + z_2\, \bar{z}_2$

$\Rightarrow z\, \bar{z}_2 - z\, \bar{z}_1 + z_2\, \bar{z} - z_1\, \bar{z} + z_1\, \bar{z}_1 - z_2\, \bar{z}_2 = 0$

$\Rightarrow z\, (\bar{z}_2 - \bar{z}_1) + \bar{z}\, (z_2 - z_1) + z_1\, \bar{z}_1 - z_2\, \bar{z}_2 = 0$ (2)

Since the equation is true for any point z on the given line it may be regarded as the equation of the given line.

∴ From (1) and (2) we get

$$\frac{\bar{\alpha}}{\bar{z}_2-\bar{z}_1} = \frac{\alpha}{z_2-z_1} = \frac{\beta}{z_1\bar{z}_1 - z_2\bar{z}_2} = k \text{ (say)}$$

∴ $\alpha = k(z_2 - z_1)$; $\bar{\alpha} = k(\bar{z}_2 - \bar{z}_1)$ and $\beta = k(z_1\bar{z}_1 - z_2\bar{z}_2)$

∴ $\bar{\alpha}z_1 + \alpha\bar{z}_2 + \beta = k[z_1(\bar{z}_2 - \bar{z}_1) + \bar{z}_2(z_2 - z_1 + z_1\bar{z}_1 - z_2\bar{z}_2]$

$= 0$

∴ $\bar{\alpha}z_1 + \bar{\alpha}z_2 + \beta = 0$

Conversely, suppose $\bar{\alpha}z_1 + \alpha\bar{z}_2 + \beta = 0$ (3)

Subtracting (3) from (1) we get

$\bar{\alpha}(z - z_1) + -\alpha(\bar{z} - \bar{z}_2) = 0$

$\Longrightarrow \bar{\alpha}(z - z_1) = -\alpha(\bar{z} - \bar{z}_2)$

Taking modulus on both sides

$\Longrightarrow |\bar{\alpha}||z - z_1| = |\alpha||\bar{z} - \bar{z}_2|$

$\Longrightarrow |z - z_1| = |\bar{z} - \bar{z}_2| = |\overline{z - z_2}|$ $[\because |\alpha| = |\bar{\alpha}|]$

$\Longrightarrow |z - z_1| = |z - z_2|$

∴ z_1 and z_2 are reflection points for the line $\bar{\alpha}z + \alpha\bar{z} + \beta = 0$

Definition

Two points P and Q are said to be inverse points with respect to a circle with centre 0 and radius r if Q lies on the ray OP and OP. OQ = r^2.

Theorem

z_1 and z_2 are inverse points with respect to a circle $z\bar{z} + \bar{\alpha}z + \alpha\bar{z} + \beta = 0$

iff $z_1 . \bar{z}_2 + \bar{\alpha}z_1 + \bar{\alpha}z_2 + \beta = 0$

Proof

Suppose z_1 and z_2 are inverse points with respect to the circle $z\bar{z} + \bar{\alpha}z + \alpha\bar{z} + \beta = 0$ (1)

(1) can be rewritten as

$|z + \alpha|^2 = \alpha\bar{\alpha} - \beta$

∴ The centre of the circle is $-\alpha$ and radius is $\sqrt{\alpha\bar{\alpha} - \beta}$

Since

z_1 and z_2 are inverse points w.r. to (1)

we have, $\arg(z_1+\alpha) = \arg(z_2+\alpha)$ (2)

and $|z_1+\alpha|\,|\overline{z_2+\alpha}| = \alpha\bar{\alpha} - \beta$ (3)

$\therefore \arg(z_1+\alpha)\overline{z_2+\alpha} = \arg(z_1+\alpha) + \arg\overline{(z_2+\alpha)}$

$= \arg(z_1+\alpha) - \arg\overline{(z_2+\alpha)}$

$= 0$ [∵ by (2)]

$\therefore (z_1+\alpha)\overline{(z_2+\alpha)}$ is a +ve real number.

Hence using (3) we get $(z_1+\alpha)\overline{z_2+\alpha} = \alpha\bar{\alpha} - \beta$

$\Longrightarrow (z_1+\alpha)(\bar{z}_2+\bar{\alpha}) = \alpha\bar{\alpha}-\beta$

$\therefore z_1\bar{z}_2 + \bar{\alpha} z_1 + \alpha\bar{z}_2 + \beta = 0$

Converse can be similarly proved.

Note 1:

Let z_1, z_2, z_3 and z_4 be four distinct points which are either con-cyclic or collinear. Then $\arg\left[\frac{(z_1-z_3)(z_2-z_4)}{(z_1-z_4)(z_2-z_3)}\right]$ is either 0 or π depending on the relative positions of the points.

Hence $\frac{(z_1-z_3)(z_2-z_4)}{(z_1-z_4)(z_2-z_3)}$ is purely real.

Note 2 :

The equation $pz\bar{z} + \bar{\alpha}z + \alpha\bar{z} + \beta = 0$ (1)

Where p and β are real and $\alpha\bar{\alpha} - p\beta \geq 0$ can be taken as the joint equation of the family of circles and straight lines. When $p \neq 0$, it represents a circle. When p=0, it represents a straight line. Further z_1 and z_2 are inverse points or reflection points w.r.to (1) iff $pz_1\bar{z}_2 + \bar{\alpha}z_1 + \alpha\bar{z}_2 + \beta = 0$

Solved problems

Problem 1 : Prove that the equation $\left|\frac{z-z_1}{z-z_2}\right| = \lambda$ where λ is a non negative parameter represents a family of circles such that z_1 and z_2 are inverse points for every member of the family.

Solution:

Given, $\left|\frac{z-z_1}{z-z_2}\right| = \lambda \Longrightarrow \left|\frac{z-z_1}{z-z_2}\right|^2 = \lambda^2$

$\Longrightarrow \left[\frac{z-z_1}{z-z_2}\right]\left[\overline{\frac{z-z_1}{z-z_2}}\right] = \lambda^2$

$\Rightarrow [\frac{z-z_1}{z-z_2}] \frac{\bar{z}-\bar{z}_1}{\bar{z}-\bar{z}_2} = \lambda^2$

$\Rightarrow (z - z_1)(\bar{z} - \bar{z}_1) = \lambda^2 (z - z_2)(\bar{z} - \bar{z}_2)$

$\Rightarrow z\bar{z} - \bar{z}_1 z - z\bar{z} + z_1\bar{z}_1 = \lambda^2(z\bar{z} - \bar{z}_2 z - z_2\bar{z} - + z_2\bar{z}_2)$

$\Rightarrow (1-\lambda^2) z\bar{z} + (\lambda^2\bar{z}_2 - \bar{z}_1) z + (z_2\lambda^2 - z_1)\ (z_1\bar{z}_1 - \lambda^2 z_2\bar{z}_2) = 0$ (1)

(1) represents a circle when $\lambda \neq 1$

Using Note 2, it can be verified that z_1 and z_2 are inverse points w.r.to (1). When $\lambda=1$, the given equation represents a straight line which is the perpendicular bisector of the line segment joining z_1 and z_2. Clearly z_1 and z_2 are reflection points for this line.

Problem 2

Prove that arg $[\frac{z-a}{z-b}] = \mu$ where μ is a real parameter, represents a family of circles every member of which passes through a and b.

Solution

For any fixed value μ, arg $[\frac{z-a}{z-b}] = \mu$ is the locus of a point z such that the angle between the lines joining a to z and b to z is μ.

Clearly this locus is the arc of a circle passing through a and b the remaining part of the circle is represented by the equation arg $[\frac{z-a}{z-b}] = \mu+\pi$. Hence the result follows.

Exercise

1. Show that the inverse point of any point α with respect to the unit circle $|z|=1$ is $\frac{1}{\bar{\alpha}}$.
2. Find the inverse point of $-i$ with respect to the circle $2z\bar{z}+(i-1)z-(i+1)\bar{z} = 0$.

Regions in the complex plane.

Definition

Let z_0 be any complex number. Let ε be a +ve real number. Then the set of all points z satisfying $|z-z_0| < \varepsilon$ is called a neighbourhood of z_0 and is represented by $N_\varepsilon(z_0)$ or $S(z_0, \varepsilon)$. Thus $N_\varepsilon(z_0) = \{z/1z-z_0| < \varepsilon\}$.

Note 1: $|z - z_0| < \varepsilon$ represents the interior of the circle with centre z_0 and radius ε.

Note 2: $|z - z_0| \leq \varepsilon$ represents the set of points on and inside the circle with centre z_0 and radius ε and is called the closed circular disc with centre z_0 and radius ε.

Definition

Let $S \subseteq C$. Let $z_0 \in S$. Then z_0 is said to be an interior point of S if there exists a neighbourhood $N_\varepsilon(z_0)$ such that $N_\varepsilon(z_0) \subseteq S$.

S is called an open set if every point of S is an interior point of S.

Definition

Let $S \subseteq C$. Let $z_0 \in S$. Then z_0 is called a limit point of S if every neighbourhood of z_o contains infinitely many points of S.

S is called a closed set if it contains all its limit points.

Remark

A set S is closed iff its complement C-S is open.

Definition

Let $S \subseteq C$. Let $z_0 \in C$. Then z_0 is called a boundary point of S if z_o is a limit point of both S and C-S. Thus z_o is a boundary point of S iff every neighbourhood of z_o contains infinitely many points of S and infinitely many points of C-S.

Definition

Let $S \subseteq C$. Then S is called a bounded set if there exist a real number k such that $|z| \leq k$ for all $z \in S$.

Definition

Let $S \subseteq C$ then S is called a connected set if every pair of points in S can be joined by a polygon which lies in S.

Definition

A non empty open connected subset of C is called a region in C.

Example

a) Let D = {z/Rez>1}

Let z = x+iy. Then D = {z/x>1}

∴ D is nonempty, open and connected.

∴ D is a region in C.

Here D is the half plane as shown in the figure.

Example

Let $D=\{z/|z-2+i| \leq 1\}$

i.e. D is the set of all complex number satisfying $|z-(2-i)| \leq 1$. Clearly D represents the closed disc with centre 2-i and radius 1. Also D is a connected and bounded set. But the points which lie on the circle $|z-(2-i)| = 1$ are not interior points of D. Hence D is not open. Hence D is not a region.

Example

Let $D = \{z/|\text{Im } z| > 1\}$

Let $D = x + iy$

$D = \{z/|y| > 1\}$

$= \{z/y > 1 \text{ or } y < -1\}$

$= \{z/y > 1\} \cup \{z/y < -1\}$

Clearly D is the union of two half planes and it is unbounded as shown in the figure.

Obviously if z_1 is any complex number with $\text{Im } z_1 > 1$ and z_2 is any complex number with $\text{Im} z_2 < -1$ then z_1 and z_2 cannot be joined by a polygon entirely lying in D. Hence D is not connected. Hence D is not a region.

Example

$D = \{z/0 < \arg z < \pi/4\}$ is a region in C.

Example

Let $D = \{z/0 < \arg z < \frac{\pi}{4} \text{ and } |z| > 1\}$ D is as shown in the figure. Clearly D is an unbounded region in C.

Example

Let $D = \{z/1 < |z| < 2\}$ D is the region bounded by the circles $|z|=1$ and $|z|=2$. Such a region is called an annulus or annular region.

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