Sun Tuition Center

12th

Mathematics

ALL UNITS

5- MARKS

QUESTION WITH ANSWER

Life is a Good Circle,
you Choose Best Radius

Sun Tuition Center

ALL UNIT

5- MARKS QUESTION

MATIRCES AND DETERMINANTS

1. If $A = \begin{bmatrix} 4 & 3 \\ 2 & 5 \end{bmatrix}$, find x and y such that $A^2 + xA + yI_2 = 0_2$. Hence, find A^{-1}

2. If $A = \frac{1}{7}\begin{bmatrix} 6 & -3 & a \\ b & -2 & 6 \\ 2 & c & 3 \end{bmatrix}$ is orthogonal, find a, b and c, and hence A^{-1}

3. Solve the following system of equations, using matrix inversion method:

   $2x_1 + 3x_2 + 3x_3 = 5, x_1 - 2x_2 + x_3 = -4, 3x_1 - x_2 - 2x_3 = 3$

4. if $A = \begin{bmatrix} -4 & 4 & 4 \\ -7 & 1 & 3 \\ 5 & -3 & -1 \end{bmatrix}$ and $B = \begin{bmatrix} 1 & -1 & 1 \\ 1 & -2 & -2 \\ 2 & 1 & 3 \end{bmatrix}$, Find the products AB and BA and hence solve the system of equations

   $x - y + z = 4, x - 2y - 2z = 9, 2x + y + 3z = 1$

5. Solve, by Cramer's rule, the system of equations.

   $x_1 - x_2 = 3, 2x_1 + 3x_2 + 4x_3 = 17, x_2 + 2x_3 = 7$

6. In a T20 Match, Chennai Super Kings needed just 6 runs to win with 1 ball left to go in the last over. The last ball was bowled and the batsman at the crease hit it high up. the ball traversed along a path in a vertical plane and the equation of the path is $y = ax^2 + bx + c$ with respect to a xy-cordinate system in the vertical plane and the ball traversed through the points $(10, 8), (20, 16), (40,22)$ can you conclude that Chennai Super Kings won the match? Justify your answer. (All distances are measured

in metres and the meeting point of the plane of the path with the farthest boundary line is $(70, 0)$.)

7. Solve the following system of linear equations, by Gaussian elimination method

$4x + 3y + 6z = 25, x + 5y + 7z = 13, 2x + 9y + z = 1.$

8. The upward speed $v(t)$ of a rocket at time t is approximated by $v(t) = at^2 + bt + c,\ 0 \le t \le 100$ where a, b and c are constant. It has been found that the speed at times $t = 3, t = 6$ and $t = 9$ seconds are respectively, 64, 133 and 208 miles per second respectively. Find the speed at time $t = 15$ seconds. (Use Gaussian elimination method)

9. If $F(\alpha) = \begin{bmatrix} \cos\alpha & 0 & \sin\alpha \\ 0 & 1 & 0 \\ -\sin\alpha & 0 & \cos\alpha \end{bmatrix}$, show that $[F(\alpha)]^{-1} = F(-\alpha)$

10. If $A = \begin{bmatrix} 5 & 3 \\ -1 & -2 \end{bmatrix}$, show that $A^2 - 3A - 7I_2 = O_2$. Hence find A^{-1}

11. $A = \begin{bmatrix} 1 & \tan x \\ -\tan x & 1 \end{bmatrix}$, show that $A^T A^{-1} = \begin{bmatrix} \cos 2x & -\sin 2x \\ \sin 2x & \cos 2x \end{bmatrix}$.

12. If $A = \begin{bmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{bmatrix}$, show that $A^{-1} = \frac{1}{2}(A^2 - 3I)$

13. If $A = \begin{bmatrix} -5 & 1 & 3 \\ 7 & 1 & -5 \\ 1 & -1 & 1 \end{bmatrix}$ and $B = \begin{bmatrix} 1 & 1 & 2 \\ 3 & 2 & 1 \\ 2 & 1 & 3 \end{bmatrix}$, find the products AB and BA and hence solve the system of equations

$x + y + 2z = 1, \quad 3x + 2y + z = 7, 2x + y + 3z = 2$

14. The prices of three commodities A, B and C are ` x, y and z per units respectively. A person P purchases 4 units of B and sells two units of A and 5 units of C. Person Q purchases 2 units of iC and sells 3 units of A and one unit of B. Person R purchases one

unit of A and sells 3 unit of B and one unit of C. In the process, P, Q and R earn `15,000, `1,000 and `4000 respectively. Find the prices per unit of A, B and C. (Use matrix inversion method to solve the problem)

15.Solve the systems of linear equations by Cramer's rule.

(iv) $\frac{3}{x} - \frac{4}{y} - \frac{2}{z} - 1 = 0, \frac{1}{x} + \frac{2}{y} + \frac{1}{z} - 2 = 0, \frac{2}{x} - \frac{5}{y} - \frac{4}{z} + 1 = 0$

16.A family of 3 people went out for dinner in a restaurant. The cost of two dosai, three idlies and two vadais is `150. The cost of the two dosai, two idlies and four vadais is `200. The cost of five dosai, four idelies and two vadais is `250. The family has `350 in hand and they ate 3 dosai and six idlies and six vadais. Will they be able to manage to pay the bill within the amount they had?

17.If $ax^2 + bx + c$ is divided by $x + 3, x - 5$ and $x - 1$, the remainders are 21,61 and 9 respectively. Find a, b and c. (use Gaussian elimination method.)

18.An amount of `65,000 is invested in three bonds at the rates of 6%, 8% and 10% per annum respectively. The total annual income is `5,000. The income from the third bond is `800 more than that from the second bond. Determine the price of each bond. (Use Gaussian elimination method.)

19.A boy is walking along the path $y = ax^2 + bx + c$ through the points $(-6, 8), (-2, -12)$ and $(3, 8)$. He wants to meet his friend at $P(7, 60)$. will he meet his friend? (Use Gaussian elimination method)

COMPLEX NUMBERS

1. Show that $\left(\frac{19+9i}{5-3i}\right)^{15} - \left(\frac{8+i}{1+2i}\right)^{15}$ is purely imaginary.

2. Show that the points $1, \frac{-1}{2} + i\frac{\sqrt{3}}{2}$ and $\frac{-1}{2} - i\frac{\sqrt{3}}{2}$ are the vertices of an equilateral triangle.

3. Let z_1, z_2 and z_3 be complex numbers such that $|z_1| = |z_2| = |z_3| = r > 0$ and $z_1 + z_2 + z_3 \neq 0$. Prove that $\left|\frac{z_1z_2+z_2z_3+z_3z_1}{z_1+z_2+z_3}\right| = r$

4. Suppose z_1, z_2 and z_3 are the vertices of an equilateral triangle inscribed in the circle $|z| = 2$. If $z_1 = 1 + i\sqrt{3}$, then find z_2 and z_3

5. Show that $\left(\frac{19-7i}{9+i}\right)^{12} + \left(\frac{20-5i}{7-6i}\right)^{12}$ is real.

6. If z_1, z_2 and z_3 are three complex numbers such that $|z_1| = 1, |z_2| = 2, |z_3| = 3$ and $|z_1 + z_2 + z_3| = 1$, show that $|9z_1z_2 + 4z_1z_3 + z_2z_3| = 6$.

7. If $z = x + iy$ is a complex number such that $Im\left(\frac{2z+1}{iz+1}\right) = 0$, show that the locus of z is $2x^2 + 2y^2 + x - 2y = 0$.

THEORY OF EQUATIONS

1. Form the equation whose roots are the squares of the roots of the cubic equation $x^3 + ax^2 + bx + c = 0$

2. If $2 + i$ and $3 - \sqrt{2}$ are roots of the equation. $x^6 - 13x^5 + 62x^4 - 126x^3 + 65x^2 + 127x - 140 = 0$ find all roots.

3. Solve the following equation: $x^4 - 10x^3 + 26x^2 - 10x + 1 = 0$

4. Solve the equation $3x^3 - 16x^2 + 23x - 6 - 0$ if the product of two roots is 1.

5. Find the sum of squares of roots of the equation $2x^4 - 8x^3 + 6x^2 - 3 = 0$

6. Solve the equation $x^3 - 9x^2 + 14x + 24 = 0$ if it is given that two of its roots are in the ratio 3 : 2

7. If the equations $x^2 + px + q = 0$ and $x^2 + p'x + q' = 0$ have a common root, show that it must be equal to $\frac{pq' - p'q}{q - q'}$ or $\frac{q - q'}{p' - p}$

8. Find a polynomial equation of minimum degree with rational coefficients, having $\sqrt{5} - \sqrt{3}$ as a root

9. Solve the equation $9x^3 - 36x^2 + 44x - 16 = 0$ if the roots form an arithmetic progression.

10. Solve the equation $3x^3 - 26x^2 + 52x - 24 = 0$ if its roots form a geometric progression.

11. Determine k and solve the equation $2x^3 - 6x^2 + 3x + k = 0$ if one of its roots is twice the sum of the other two roots.

12. Find all zeros of the polynomial $x^6 - 3x^5 - 5x^4 + 22x^3 - 39x^2 - 39x + 135$, if it is known that $1 + 2i$ and $\sqrt{3}$ are two of its zeros.

13. Solve the equations.

    (i) $6x^4 - 35x^3 + 62x^2 - 35x + 6 = 0$

14. Solve the equation $6x^4 - 5x^3 - 38x^2 - 5x + 6 = 0$ if it is known that $\frac{1}{3}$ is a solution.

INVERSE TRIGONOMETRY FUNCTIONS

1. Find the domain of $\sin^{-1}(2-3x^2)$
2. Find the domain of $\cos^{-1}\left(\frac{2+\sin x}{3}\right)$
3. Find the domain of the following. $f(x)=\sin^{-1}\left(\frac{x^2+1}{2x}\right)$
4. Find the value of $\sin^{-1}\left(\sin\frac{5\pi}{9}\cos\frac{\pi}{9}+\cos\frac{5\pi}{9}\sin\frac{\pi}{9}\right)$.
5. Find the value of $\cos^{-1}\left(\cos\frac{\pi}{7}\cos\frac{\pi}{17}-\sin\frac{\pi}{7}\sin\frac{\pi}{17}\right)$.
6. Find the domain of (i) $f(x)=\sin^{-1}\left(\frac{|x|-2}{3}\right)+\cos^{-1}\left(\frac{1-|x|}{4}\right)$

5. TWO DIMENSIONAL ANALYTICAL GEOMETRY- II

1. Find the equation of the circle passing through the points $(1,1)$, $(2,-1)$, and $(3,2)$.
2. Find the equation of the ellipse whose eccentricity is $\frac{1}{2}$, one of the foci is $(2,3)$ and a directrix is $x=7$. Also find the length of the major axes of the ellipse.
3. Find the vertices, foci for the hyperbola $9x^2-16y^2=144$

   an equation of the elliptical part of an optical lens system is $\frac{x^2}{16}+\frac{y^2}{9}=1$. the parabolic part of the system has a focus in common with the right focus of the ellipse. The vertex of the parabola is at the origin and the parabola opens to the right. Determine the equation of the parabola.
4. Find the equation of the circle through the points $(1,0)$, $(-1,0)$ and $(0,1)$.
5. Find the vertex, focus, equation of directrix and length of the latus rectum of the following.

   (Each 5 Mark)

(i)$x^2 - 2x + 8y + 17 = 0$ (ii)$y^2 - 4y - 8ix + 12 = 0$

6. identify the type of conic and find centre, foci, vertices and directrices of each of the following.

(i)$18x^2 + 12y^2 - 144x + 48y + 120 = 0$

(ii)$9x^2 - y^2 - 36x - 6y + 18 = 0$ **(Each 5 Mark)**

7. Show that the line $x - y + 4 = 0$ is a tangent to the ellipse $x^2 + 3y^2 = 12$. Also find the coordinates of the point of contact.

8. A bridge has a parabolic arch that is $10m$ high in the centre and $30m$ wide at the bottom. Find the height of the arch $6m$ from the centre, on either sides.

9. A tunnel through a mountain for a four lane highway is to have a elliptical opening. The total width of the highway (not the opening) is to be $16m$, and the height at the edge of the road must be sufficient for a truck $4m$ high to clear if the highest point of the opening is to be $5m$ approximately. How wide must the opening be?

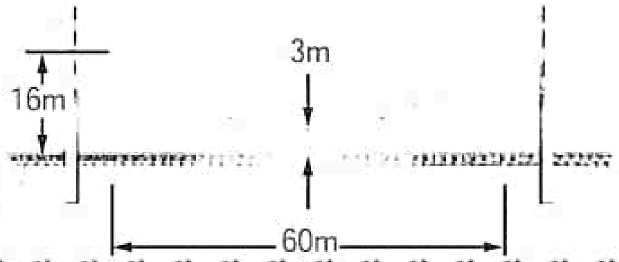
10. At a water fountain, water attains a maximum height of $4m$ at horizontal distance of $0.5\ m$ from its origin. If the path of water is a parabola, find the height of water at a horizontal distance of $0.75\ m$ from the point of origin.

11. An engineer designs a satellite dish with a parabolic cross section. the dish is $5m$ wide at the opening, and the focus is placed $1.2m$ from the vertex.

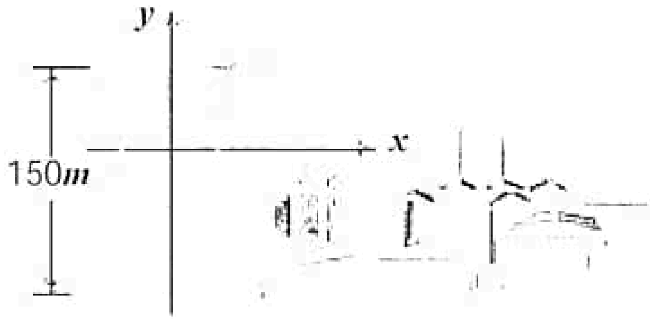
(a) Position a coordinate system with the origin at the vertex and the x-axis on the parabola's axis of symmetry and find an equation of the parabola.

(b) Find the depth of the satellite dish at the vertex.

12. Parabolic cable of a $60m$ portion of the roadbed of a suspension bridge are positioned as shown below. Vertical Cables are to be spaced every $6m$ along this portion of the roadbed. Calculate the lengths of first two of these vertical cables from the vertex.



13. Cross section of a Nuclear cooling tower is in the shape of a hyperbola with equation $\frac{x^2}{30^2} - \frac{y^2}{44^2} = 1$. The tower is $150m$ tall and the distance from the top of the tower to the centre of the hyperbola is half the distance from the base of the tower to the centre of the hyperbola. Find the diameter of the top and base of the tower.



14. A rod of length $1.2m$ moves with its ends always touching the coordinate axes. the locus of a point P on the rod, which is $0.3m$ from the end in contact with x-axis is an ellipse. find the eccentricity.
15. Assume that water issuing from the end of the horizontal pipe, $7.5m$ above the ground, describes a parabolic path. The vertex of the parabolic path is at the end of the pipe. At a position $2.5m$ below the line of the pipe, the flow of water has curved outward $3m$ beyond the vertical line through the end of the pipe. How far beyond this vertical line will the water strike the ground?
16. On lighting a rocket cracker it gets projected in a parabolic path and reaches a maximum height of $4m$ when it is $6m$ away from the point of projection. Finally it reaches the ground $12m$ away from the starting point. Find the angle of projection.

6. APPLICATIONS OF VECTOR ALGEBRA

1. By vector method, prove that $\cos(\alpha + \beta) = \cos\alpha\cos\beta - \sin\alpha\ \sin\beta$.
2. Prove by vector method that $\sin(\alpha - \beta) = \sin\alpha\cos\beta - \cos\alpha\sin\beta$
3. If D is the midpoint of the side BC of a triangle ABC, show by vector method that $|\overrightarrow{AB}|^2 + |\overrightarrow{AC}|^2 = 2\left(|\overrightarrow{AD}|^2 + |\overrightarrow{BC}|^2\right)$.

4. Prove by vector method that the perpendiculars (attitudes) from the vertices to the opposite sides of a triangle are concurrent.
5. If $\vec{a} = i - j, \vec{b} = i - j - 4k, \vec{c} = 3j - k$ and $\vec{d} = 2i + 5j + k$, verify that
   (i)$(\vec{a} \times \vec{b}) \times (\vec{c} \times \vec{d}) = [\vec{a}, \vec{b}, \vec{d}]\vec{c} - [\vec{a}, \vec{b}, \vec{c}]\vec{d}$
   (ii)$(\vec{a} \times \vec{b}) \times (\vec{c} \times \vec{d}) = [\vec{a}, \vec{c}, \vec{d}]\vec{b} - [\vec{b}, \vec{c}, \vec{d}]\vec{a}$
6. Find the vector equation in parametric form and Cartesian equations of a straight passing through the points $(-5, 7, -4)$ and $(13, -5, 2)$. Find the point where the straight line crosses the xy-plane.
7. Find the point of intersection of the lines $\frac{x-1}{2} = \frac{y-2}{3} = \frac{z-3}{4}$ and $\frac{x-4}{5} = \frac{y-1}{z} = z$
8. Find the parametric form of vector equation of a straight line passing through the point of intersection of the straight lines $\vec{r} = (\hat{\imath} + 3\hat{\jmath} - \hat{k}) + t(2\hat{\imath} + 3\hat{\jmath} + 2\hat{k})$ and $\frac{x-2}{1} = \frac{y-4}{2} = \frac{z+3}{4}$, and perpendicular to both straight lines.
9. Determine whether the pair of straight lines $\vec{r} = (2\hat{\imath} + 6\hat{\jmath} + 3\hat{k}) + t(2\hat{\imath} + 3\hat{\jmath} + 4\hat{k})$, $\vec{r} = (2\hat{\jmath} - 3\hat{k}) + s(\hat{\imath} + 2\hat{\jmath} + 3\hat{k})$ are parallel. Find the shortest distance between them.
10. Find the vector parametric, vector non-parametric and Cartesian form of the equation of the plane passing through the points $(-1, 2, 0)$, $(2, 2, -1)$ and parallel to the straight line $\frac{x-1}{1} = \frac{2y+1}{2} = \frac{z+1}{-1}$
11. Show that the lines $\vec{r} = (-\hat{\imath} - 3\hat{\jmath} - 5\hat{k}) + s(3\hat{\imath} + 5\hat{\jmath} + 7\hat{k})$ and $\vec{r} = (2\hat{\imath} + 4\hat{\jmath} + 6\hat{k}) + t(\hat{\imath} + 4\hat{\jmath} + 7\hat{k})$ are coplanar. Find the vector equation of the plane in which they lie.
12. Prove by vector method that the parallelograms on the same base and between the same parallels are equal in area.
13. If G is the centroid of a ΔABC, prove that
    (area of ΔGAB)= (area of ΔGBC)=(area of ΔGCA) = $\frac{1}{3}$ (area of ΔABC).

14.Using vector method, prove that $\cos(\alpha-\beta)=\cos\alpha\cos\beta+\sin\alpha\sin\beta$

15.Prove by vector method that $\sin(\alpha+\beta)=\sin\alpha\cos\beta+\cos\alpha\sin\beta$

16.If $\vec{a}=2\hat{\imath}+3\hat{\jmath}-\hat{k}, \vec{b}=3\hat{\imath}+5\hat{\jmath}+2\hat{k}, \vec{c}=-\hat{\imath}-2\hat{\jmath}+3\hat{k}$, verify that

(i)$(\vec{a}\times\vec{b})\times\vec{c}=(\vec{a}.\vec{c})\vec{b}-(\vec{b}.\vec{c})\vec{a}$ (ii)$\vec{a}\times(\vec{b}\times\vec{c})=(\vec{a}.\vec{c})\vec{b}-(\vec{a}.\vec{b})\vec{c}$

17.Find the points where the straight line passes through $(6,7,4)$ and $(8,4,9)$ cuts the xz and yz planes.

18. Show that the lines $\frac{x-3}{3}=\frac{y-3}{-1}, z-1=0$ and $\frac{x-6}{2}=\frac{z-1}{3}, y-2=0$ intersect. Also find the point of intersection.

19.Show that the lines $x+1=2y=-12z$ and $x=y+2=6z-6$ are skew and hence find the shortest distance between them.

20.Find the parametric form of vector equation of the straight line passing through $(-1,2,1)$ and parallel to the straight line $\vec{r}=(2\hat{\imath}+3\hat{\jmath}-\hat{k})+t(\hat{\imath}-2\hat{\jmath}+\hat{k})$ and hence find the shortest distance between the lines.

21.Find the foot of the perpendicular drawn from the point $(5,4,2)$ to the line $\frac{x+1}{2}=\frac{y-3}{3}=\frac{z-1}{-1}$. Also, find the equation of the perpendicular

22.Find the non-parametric form of vector equation, and Cartesian equation of the plane passing through the point $(2,3,6)$ and parallel to the straight lines $\frac{x-1}{2}=\frac{y+1}{3}=\frac{z-3}{1}$ and $\frac{x+3}{2}=\frac{y-3}{-5}=\frac{z+1}{-3}$

23.Find the parametric form of vector equation, and Cartesian equations of the plane passing through the points $(2,2,1),(9,3,6)$ and perpendicular to the plane $2x+6y+6z=9$.

24.Find parametric form of vector equation and Cartesian equations of the plane passing through the points $(2,2,1),(1,-2,3)$ and parallel to the straight line passing through the points $(2,1,-3)$ and $(-1,5,-8)$

25. Find the non-parametric form of vector equation and Cartesian equation of the plane passing through the point $(1,-2,4)$ and perpendicular to the plane $x+2y-3z=11$ and parallel to the line $\frac{x+7}{3}=\frac{y+3}{-1}=\frac{z}{1}$

26. Find the parametric form of vector equation, and Cartesian equations of the plane containing the line $\vec{r}=(\hat{\imath}-\hat{\jmath}+3\hat{k})+t(2\hat{\imath}-\hat{\jmath}+4\hat{k})$ and perpendicular to plane $\vec{r}.(\hat{\imath}+2\hat{\jmath}+\hat{k})=8$.

27. Find the parametric vector, non-parametric vector and Cartesian form of the equations of the plane passing through the three non-Collinear points $(3,6,-2)$ $(-1,-2,6)$ and $(6,-4,-2)$.

28. Find the non-parametric form of vector equation, and Cartesian equations of the plane $\vec{r}=(6\hat{\imath}-\hat{\jmath}+\hat{k})+s(-i+2\hat{\jmath}+\hat{k})+t(-5\hat{\imath}-4\hat{\jmath}-5\hat{k})$

29. If the straight lines $\frac{x-1}{2}=\frac{y+1}{\lambda}=\frac{z}{2}$ and $\frac{x+1}{5}=\frac{y+1}{2}=\frac{z}{\lambda}$ are coplanar, find λ and equations of the planes containing these two lines.

7.APPLICATIONS OF DIFFERENTIAL CALCULUS

1. Salt is poured from a conveyer belt at a rate of 30 cubic metre per minute forming a conical pile with a circular base whose height and diameter of base are always equal. How fast is the height of the pile increasing when the pile is 10 metre high?
2. Find the equation of the tangent and normal at any point to the Lissajous curve given by $x=2\cos 3t$ and $y=3\sin 2t$, $t\in\mathbb{R}$.
3. Find the angle between $y=x^2$ and $y=(x-3)^2$
4. Find the angle between the curves $y=x^2$ and $x=y^2$ at their points of intersection $(0,0)$ and $(1,1)$
5. It the curves $ax^2+by^2=1$ and $cx^2+dy^2=1$ intersect each other orthogonally then, show that $\frac{1}{a}-\frac{1}{b}=\frac{1}{c}-\frac{1}{d}$

6. Prove that the ellipse $x^2 + 4y^2 = 8$ and the hyperbola $x^2 - 2y^2 = 4$ intersect orthogonally.
7. Evaluate : $\lim_{x \to \infty} (1 + 2x)^{\frac{1}{2\log x}}$
8. Evaluate : $\lim_{x \to 1} x^{\frac{1}{1-x}}$
9. Find the local extrema of the function $f(x) = 4x^6 - 6x^4$
10. Find the local maximum and minimum of the function x^2y^2 on the line $x + y = 10$
11. We have a 12 square unit piece of thin material and want to make an open box by cutting small squares from the corners of our material and folding the sides up. The question is, which cut produces the box of maximum volume?
12. Find the points on the unit circle $x^2 + y^2 = 1$ nearest and farthest from $(1, 1)$.
13. A steel plant is capable of producing x tonner per day of a low-grade steel and y tonnes per day of a high-grade steel, where $y = \frac{40-5x}{10-x}$. If the fixed market price of low-grade steel is half that of high-grade steel, then what should be optimal productions in low-grade steel and high-grade steel in order to have maximum receipts.
14. Prove that among all the rectangles of the given area square has the least perimeter.
15. A stone is dropped into a pond causing ripples in the form of concentric circles. The radius *r* of the outer ripple is increasing at a constant rate at 2 cm per second. When the radius is 5 cm find the rate of changing of the total area of the disturbed water?
16. A beacon makes one revolution every 10 seconds. It is located on a ship which is anchored 5 km from a straight shore line. How fast is the beam moving along the shore line when it makes an angle of 45° with the shore?
17. A conical water tank with vertex down of 12 metres height has a radius of 5 metres at the top. If water flows into the tank at a rate 10 cubic m/min, how fast is the depth of the water increases when the water is 8 metres deep?
18. A ladder 17 metre long is leaning against the wall. The base of the ladder is pulled away from the wall at a rate of 5 m/s. When the base of the ladder is 8 metres from the wall.

(i) How fast is the top of the ladder moving down the wall?

(ii) At what rate, the area of the triangle formed by the ladder, wall, and the floor, is changing?.

19. A police jeep, approaching an orthogonal intersection from the northern direction, is chasing a speeding car that has turned and moving straight east. When the jeep is 0.6 km north of the intersection and the car is 0.8 km to the east. The police determine with a radar that the distance between them and the car is increasing at 20 km/hr. If the jeep is moving at 60 km/hr at the instant of measurement, what is the speed of the car?

20. Find the equations of the tangents to the curve $y = 1 + x^3$ for which the tangent is orthogonal with the line $x + 12y = 12$.

21. Find the equations of the tangents to the curve $y = \frac{x+1}{x-1}$ which are parallel to the line $x + 2y = 6$.

22. Find the equation of tangent and normal to the curve given by $x = 7\cos t$ and $y = 2\sin t, t \in \mathbb{R}$ at any point on the curve.

23. Find the angle between the rectangular hyperbola $xy = 2$ and the parabola $x^2 + 4y = 0$.

24. If an initial amount A_0 of money is invested at an interest rate r compounded n times a year the value of the investment after t years is $A = A_0\left(1 + \frac{r}{n}\right)^{nt}$. If the interest is compounded continuously, (that is as $n \to \infty$), show that the amount after t years is $A = A_0 e^{rt}$

25. Find the intervals of concavity and points of inflexion for the functions: $f(x) = \frac{1}{2}(e^x - e^{-x})$

26. For the function $f(x) = 4x^3 + 3x^2 - 6x + 1$ find the intervals of monotonicity, local extrema intervals of concavity and points of inflection.

27. A garden is to be laid out in a rectangular area and protected by wire fence. What is the largest possible area of the fenced garden with 40 metres of wire

28. A rectangular page is to contain 24 cm^2 of print. The margins at the top and bottom of the page are 1.5 cm and the margins at other sides of the page is 1 cm. What should be the dimensions of the page so that the area of the paper used is minimum.
29. A farmer plans to fence a rectangular pasture adjacent to a river. The pasture must contain 1,80,000 sq.mtrs in order to provide enough grass for herds. No fencing is needed along the river. What is the length of the minimum needed fencing material?
30. Find the dimensions of the rectangle with maximum area that can be inscribed in a circle of radius 10 cm.
31. Prove that among all the rectangles of the given perimeter, the square has the maximum area.
32. Find the dimensions of the largest rectangle that can be inscribed in a semi circle of radius *r* cm
33. A manufacturer wants to design an open box having a square base and a surface area of 108 sq.cm. Determine the dimensions of the box for the maximum volume.
34. The volume of a cylinder is given by the formula $V = \pi r^2 h$. Find the greatest and least values of V if $r + h = 6$.
35. A hollow cone with base radius *a* cm and height *b* cm is placed on a table. Show that the volume of the largest cylinder that can be hidden underneath is $\frac{4}{9}$ times volume of the cone.

8. DIFFERENTIALS AND PARTIAL DERIVATIVES

1. A right circular cylinder has radius *r* =10 cm. and height *h* = 20 cm. Suppose that the radius of the cylinder is increased from 10 cm to 10 1 . cm and the height does not change. Estimate the change in the volume of the cylinder. Also, calculate the relative error and percentage error.
2. The radius of a circular plate is measured as 12.65 cm instead of the actual length 12.5 cm. find the following in calculating the area of the circular plate:
   (i) Absolute error (ii) Relative error (iii) Percentage error
3. A sphere is made of ice having radius 10 cm. Its radius decreases from 10 cm to
   (i) 9 8 . cm. Find approximations for the following:
   (i) change in the volume (ii) change in the surface area

4. The time T, taken for a complete oscillation of a single pendulum with length l, is given by the equation $T = 2\pi\sqrt{\frac{l}{g}}$, where g is a constant. Find the approximate percentage error in the calculated value of T corresponding to an error of 2 percent in the value of l.
5. Show that the percentage error in the n^{th} root of a number is approximately $\frac{1}{n}$ times the percentage error in the number
6. The trunk of a tree has diameter 30 cm. During the following year, the circumference grew 6 cm.

   (i) Approximately, how much did the tree's diameter grow?

   (ii) What is the percentage increase in area of the tree's cross-section?
7. The relation between the number of words y a person learns in x hours is given by $y = 52\sqrt{x}, 0 \le x \le 9$ .What is the approximate number of words learned when x changes from

   (i) 1 to 1.1 hour? (ii) 4 to 4.1 hour?

9. APPLICATIONS OF INTEGRATION

1. Evaluate $\int_0^9 \frac{1}{x+\sqrt{x}}dx$
2. Evaluate $\int_0^{\frac{\pi}{2}} \frac{\cos\theta}{(1+\sin\theta)+(2+\sin\theta)}d\theta$.
3. Evaluate $\int_0^{\frac{1}{\sqrt{2}}} \frac{\sin^{-1}x}{(1-x^2)^{\frac{3}{2}}}dx$.
4. Evaluate $\int_0^{\frac{\pi}{2}}(\sqrt{\tan x} + \sqrt{\cot x})\, dx$.
5. Evaluate $\int_0^{1.5}[x^2]dx$, where $[x]$ is the greatest integer function.
6. Evaluate $\int_{-4}^{4}|x + 3|dx$
7. Show that $\int_0^{\frac{\pi}{2}} \frac{dx}{4+5\sin x} = \frac{1}{3}\log_{e2} 2$
8. Show that $\int_0^{\frac{\pi}{4}} \frac{\sin 2x\, dx}{\sin^4 x+\cos^4 x} = \frac{\pi}{4}$

9. Evaluate $\int_0^{\frac{\pi}{4}} \frac{1}{\sin x+\cos x} dx$

10. Evaluate $\int_0^{\pi} \frac{x}{1+\sin x}\, dx$

11. Prove that $\int_0^{\frac{\pi}{4}} \log(1+\tan x)\; dx = \frac{\pi}{8}\log 2$

12. Show that $\int_0^1 (\tan^{-1} x + \tan^{-1}(1-x))\, dx = \frac{\pi}{2} - \log_e 2$

13. Evaluate $\int_{-\pi}^{\pi} \frac{\cos^2 x}{1+a^x}\, dx$.

14. Evaluate $\int_0^{\frac{\pi}{2}} \frac{dx}{4\sin^2 x+5\cos^2 x}$

15. Find the area of the region bounded between the parabolas $y^2 = 4x$ and $x^2 = 4y$.

16. Find the area of the region bounded by $y = \cos x, y = \sin x$, the lines $x = \frac{\pi}{4}$ and $x = \frac{5\pi}{4}$.

17. Find, by integration, the area of the region bounded by the lines $5x - 2y = 15$, $x + y + 4 = 0$ and the x-axis

18. Using integration find the area of the region bounded by triangle ABC, whose vertices A, B and C are $(-1,1), (3,2)$ and $(0,5)$ respectively

19. Evaluate the definite integrals $\int_0^1 \sqrt{\frac{1-x}{1+x}}\, dx$

20. Evaluate the definite integrals $\int_0^{\frac{\pi}{2}} e^x \left(\frac{1+\sin x}{1+\cos x}\right) dx$

21. Evaluate the definite integrals $\int_0^{\frac{\pi}{2}} \sqrt{\cos\theta}\,\sin^3\theta\, d\theta$

22. Evaluate the definite integrals $\int_0^1 \frac{1-x^2}{(1+x^2)^2} dx$

23. Evaluate the integrals using properties of integration. $\int_0^{2\pi} x \log\left(\frac{3+\cos x}{3-\cos x}\right) dx$

24. Evaluate the integrals using properties of integration $\int_0^{2\pi} \sin^4 x \cos^3 x\; dx$

25. Evaluate the integrals using properties of integration $\int_0^1 |5x-3| dx$

26. Evaluate the integrals using properties of integration

$$\int_0^{\sin^2 x} \sin^{-1}\sqrt{t}\; dt + \int_0^{\cos^2 x} \cos^{-1}\sqrt{t}\, dt$$

27. Evaluate the integrals using properties of integration $\int_0^{\pi} \frac{x\sin x}{1+\sin x} dx$

28. Evaluate the integrals using properties of integration

(xi) $\int_0^{\pi} x\,[\sin^2(\sin x) + \cos^2(\cos x)]\,dx$

29. Find the area of the region bounded by the line $y = 2x + 5$ and the parabola $y = x^2 - 2x$.

30. Find the area of the region bounded between the curves $y = \sin x$ and $y = \cos x$ and the lines $x = 0$ and $x = \pi$.

31. Find the area of the region bounded by $y = \tan x, y = \cot x$ and the lines $x = 0, \ x = \frac{\pi}{2}, y = 0$.

32. Find the area of the region bounded by the parabola $y^2 = x$ and the line $y = x - 2$

33. Father of a family wishes to divided his square field bounded by $x = 0, x = 4$, $y = 4$ and $y = 0$ along the curve $y^2 = 4x$ and $x^2 = 4y$ into three equal parts for his wife, daughter and son. It is possible to divide? If so, find the area to be divided among them.

34. The curve $y = (x-2)^2 + 1$ has a minimum point at P. A point Q on the curve is such that the slope of PQ is 2. Find the area bounded by the curve and the chord PQ.

35. Find the area of the region common to the circle $x^2 + y^2 = 16$ and the parabola $y^2 = 6x$.

10.ORDINARY DIFFERENTIAL EQUATIONS.

1. Solve: $(2x + 3y)dx + (y - x)dy = 0$
2. Solve: $y^2 + x^2 \frac{dy}{dx} = xy\frac{dy}{dx}$
3. Solve: $\left(1 + 2e^{\frac{x}{y}}\right)dx + 2e^{\frac{x}{y}}\left(1 - \frac{x}{y}\right)dy = 0$
4. Solve : $[y(1 - x\tan x) + x^2\cos x]dx - xdy = 0$
5. Solve : $(1 + x^3)\frac{dy}{dx} + 6x^2 y = 1 + x^2$
6. Solve : $ye^y dx = (y^3 + 2xe^y)dy$
7. The growth of a population is proportional to the number present. If the population of a colony doubles in 50 years, in how many years will the population become triple?

8. A radioactive isotope has an initial mass 200mg, which two years later is 50mg . Find the expression for the amount of the isotope remaining at any time. What is its half-life? (half-life means the time taken for the radioactivity of a specified isotope to fall to half its original value).

9. In a murder investigation, a corpse was found by a detective at exactly 8 p.m. Being alert, the detective also measured the body temperature and found it to be 70°F. Two hours later, the detective measured the body temperature again and found it to be 60°F. If the room temperature is 50°F, and assuming that the body temperature of the person before death was 98.6°F, at what time did the murder occur?
$[\log(2.43) = 0.88789; \quad \log(0.5) = -0.69315]$

10. A tank contains 1000 litres of water in which 100 grams of salt is dissolved. Brine (*Brine is a high-concentration solution of salt (usually sodium chloride) in water*) runs in a rate of 10 litres per minute, and each litre contains 5grams of dissolved salt. The mixture of the tank is kept uniform by stirring. Brine runs out at 10 litres per minute. Find the amount of salt at any time t.

11. $\left[x + y\cos\left(\frac{y}{x}\right)\right]dx = x\cos\left(\frac{y}{x}\right)dy$

12. $(x^3 + y^3)dy - x^2ydx = 0$

13. $ye^{\frac{x}{y}}dx = \left(xe^{\frac{x}{y}} + y\right)dy$

14. $2xydx + (x^2 + 2y^2)dy = 0$

15. $(y^2 - 2xy)dx = (x^2 - 2xy)dy$

16. $x\frac{dy}{dx} = y - x\cos^2\left(\frac{y}{x}\right)$

17. $\left(1 + 3e^{\frac{y}{x}}\right)dy + 3e^{\frac{y}{x}}\left(1 - \frac{y}{x}\right)dx = 0$ given that $y = 0$ when $x = 1$

18. $(x^2 + y^2)dy = xydx$. It is given that $y(1) = 1$ and $y(x_0) = e$. Find the value of x_0

19. $\left(y - e^{\sin^{-1}x}\right)\frac{dx}{dy} + \sqrt{1 - x^2} = 0$

20. $\frac{dy}{dx} + \frac{y}{(1-x)\sqrt{x}} = 1 - \sqrt{x}$

21. $(1+x+xy^2)\frac{dy}{dx}+(y+y^3)=0$

22. $\frac{dy}{dx}+\frac{y}{x\log x}=\frac{\sin 2x}{\log x}$

23. $(x+a)\frac{dy}{dx}-2y=(x+a)^4$

24. $\frac{dy}{dx}=\frac{\sin^2 x}{1+x^3}-\frac{3x^2}{1+x^3}y$

25. $x\frac{dy}{dx}+y=x\log x$

26. $x\frac{dy}{dx}+2y-x^2\log x=0$

27. The rate of increase in the number of bacteria in a certain bacteria culture is proportional to the number present. Given that the number triples in 5 hours, find how many bacteria will be present after 10 hours?

28. Find the population of a city at any time *t*, given that the rate of increase of population is proportional to the population at that instant and that in a period of 40 years the population increased from 3,00,000 to 4,00,000.

29. The equation of electromotive force for an electric circuit containing resistance and self inductance $=Ri+L\frac{di}{dt}$, where *E* is the electromotive force is given to the circuit, *R* the resistance and *L*, the coefficient of induction. Find the current *i* at time *t* when *E* = 0.

30. The engine of a motor boat moving at 10 *m / s* is shut off. Given that the retardation at any subsequent time (after shutting off the engine) equal to the velocity at that time. Find the velocity after 2 seconds of switching off the engine.

31. Suppose a person deposits 10,000 Indian rupees in a bank account at the rate of 5% per annum compounded continuously. How much money will be in his bank account 18 months later?

32. Assume that the rate at which radioactive nuclei decay is proportional to the number of such nuclei that are present in a given sample. In a certain sample 10% of the original number of radioactive nuclei have undergone disintegration in a period of 100 years. What percentage of the original radioactive nuclei will remain after 1000 years?

33. Water at temperature 100°C cools in 10 minutes to 80°C in a room temperature of 25°C . Find

(i) The temperature of water after 20 minutes

(ii) The time when the temperature is 40°C

$$\left[\log_e \frac{11}{15} = -0.3101;\ \log_e 5 = 1.6094\right]$$

34. At 10.00 A.M. a woman took a cup of hot instant coffee from her microwave oven and placed it on a nearby Kitchen counter to cool. At this instant the temperature of the coffee was 180°F, and 10 minutes later it was 160°F . Assume that constant temperature of the kitchen was 70°F .

(i) What was the temperature of the coffee at 10.15A.M.?

(ii) The woman likes to drink coffee when its temperature is between 130°F and 140°F . between what times should she have drunk the coffee?

35. A pot of boiling water at 100°C is removed from a stove at time $t = 0$ and left to cool in the kitchen. After 5 minutes, the water temperature has decreased to 80°C , and another 5 minutes later it has dropped to 65°C . Determine the temperature of the kitchen.

36. A tank initially contains 50 litres of pure water. Starting at time $t = 0$ a brine containing with 2 grams of dissolved salt per litre flows into the tank at the rate of 3 litres per minute. The mixture is kept uniform by stirring and the well-stirred mixture simultaneously flows out of the tank at the same rate. Find the amount of salt present in the tank at any time $t > 0$.

An Equation Means Nothing to Us Unless it Express a Thought of Good God

11.PROBABILITY DISTRIBUTIONS

1. Suppose a pair of unbiased dice is rolled once. If X denotes the total score of two dice, write down (i) the sample space (ii) the values taken by the random variable X, (iii) the inverse image of 10, and (iv) the number of elements in inverse image of X.
2. A pair of fair dice is rolled once. Find the probability mass function to get the number of fours.
3. If the probability mass function $f(x)$ of a random variable X is

x	1	2	3	4
$f(x)$	$\frac{1}{12}$	$\frac{5}{12}$	$\frac{5}{12}$	$\frac{1}{12}$

find (i) its cumulative distribution function, hence find (ii) $P(X$ _ $3)$ and, (iii) $P(X \geq 2)$

4. A six sided die is marked '1' on one face, '2' on two of its faces, and '3' on remaining three faces. The die is rolled twice. If X denotes the total score in two throws.

   (i) Find the probability mass function.

   (ii) Find the cumulative distribution function.

   (iii) Find $P(3 \leq X \leq 6)$ (iv) Find $(X \geq 4)$.

5. Find the probability mass function $f(x)$ of the discrete random variable X whose cumulative distribution function $F(x)$ is given by

$$F(x) = \begin{cases} 0 & -\infty < x < -2 \\ 0.25 & -2 \leq x < -1 \\ 0.60 & -1 \leq x < 0 \\ 0.90 & 0 \leq x < 1 \\ 1 & 1 \leq x < \infty \end{cases}$$

Also find (i) $P(X < 0)$ and (ii) $P(X \leq -1)$.

6. A random variable X has the following probability mass function.

x	1	2	3	4	5	6
$f(x)$	k	$2k$	$6k$	$5k$	$6k$	$10k$

Find (i)$P(2 < X < 6)$ (ii)$P(2 \leq X < 5)$ (iii)$P(X \leq 4)$ (iv) $P(3 < X)$

7. Find the constant C such that the function $f(x) = \begin{cases} Cx^2 & 1 < x < 4 \\ 0 & \text{Otherwise} \end{cases}$ is a density function, and compute (i)$P(1.5 < X < 3.5)$ (ii)$(X \leq 2)$ (iii)$P(3 < X)$

8. If X is the random variable with probability density function $f(x)$ given by,

$$f(x) = \begin{cases} x-1, & 1 \leq x < 2 \\ -x+3, & 2 \leq x < 3 \\ 0 & \text{Other wise} \end{cases}$$

Find (i)the distribution function $F(x)$ (ii)$P(1.5 \leq X \leq 2.5)$

9. The probability density function of random variable X is given by

$$f(x) = \begin{cases} k & 1 \leq x \leq 5 \\ 0 & \text{Other wise} \end{cases}$$

Find (i)Distribution function (ii)$P(X < 3)$ (iii)$P(2 < X < 4)$ (iv)$P(3 \leq X)$

10. Let X be a random variable denoting the life time of an electrical equipment having probability density function

$$f(x) = \begin{cases} ke^{-2}x & for\ x > 0 \\ 0 & for\ x \leq 0 \end{cases}$$

Find (i)the value of k (ii)Distribution function (iii)$P(X < 2)$

(iv)calculate the probability that X is at least for four unit of time (v) $(P(X = 3)$

11. An urn contains 5 mangoes and 4 apples. Three fruits are taken at random. If the number of apples taken is a random variable, then find the values of the random variable and number of points in its inverse images.

12. Two balls are chosen randomly from an urn containing 6 red and 8 black balls. Suppose that we win ` 15 for each red ball selected and we lose ` 10 for each black ball selected. If X denotes the winning amount, then find the values of X and number of points in its inverse images

13. A six sided die is marked '1' on one face, '3' on two of its faces, and '5' on remaining three faces. The die is thrown twice. If X denotes the total score in two throws, find

(i) the probability mass function

(ii) the cumulative distribution function

(iii) $P(4 \le X < 10)$ (iv) $P(X \ge 6)$

14. Suppose a discrete random variable can only take the values 0, 1, and 2. The probability mass function is defined by

$$f(x) = \begin{cases} \dfrac{x^2+1}{k} & for\ x = 0,1,2 \\ 0 & other\ wise \end{cases}$$

Find (i) the value of k (ii) cumulative distribution function (iii) $(X \ge 1)$.

15. $$F(x) = \begin{cases} 0 & -\infty < x < -1 \\ 0.15 & -1 \le x < 0 \\ 0.35 & 0 \le x < 1 \\ 0.60 & 1 \le x < 2 \\ 0.85 & 2 \le x < 3 \\ 1 & x \le x < \infty \end{cases}$$

Find (*i*) the probability mass function (*ii*) $P(X < 1)$ and (*iii*) $P(X \ge 2)$.

16. A random variable X has the following probability mass function.

x	1	2	3	4	5
$f(x)$	k^2	$2k^2$	$3k^2$	$2k$	$3k$

Find (i) the value of k (ii) $P(2 \le X < 5)$ (iii) $P(3 < X)$

17. The cumulative distribution function of a discrete random variable is given by

$$F(x) = \begin{cases} 0 & for -\infty < x < -1 \\ \frac{1}{2} & for \quad 0 \le x < 1 \\ \frac{3}{5} & for \quad 1 \le x < 2 \\ \frac{4}{5} & for \quad 2 \le x < 3 \\ \frac{9}{10} & for \quad 3 \le x < 4 \\ 1 & for \quad 4 \le x < \infty \end{cases}$$

Find (i) the probability mass function (ii) $P(X < 3)$ and (iii) $P(X \ge 2)$.

18. The probability density function of X is given by $f(x) = \begin{cases} ke^{-\frac{x}{3}} & for\ x > 0 \\ 0 & for\ x \le 0 \end{cases}$

Find (i)the value of k (ii)the distribution function

(iii)$P(X < 3)$ (iv)$P(5 \le X)$ (v)$P(X \le 4)$

19. If X is the random variable with probability density function $f(x)$ given by,

$$f(x) = \begin{cases} x+1, & -x \le x < 0 \\ -x+1, & 0 \le x < 1 \\ 0 & \text{Otherwise} \end{cases}$$

then find (i) the distribution function $F(x)$ (ii)$P(-0.5 \le X \le 0.5)$

20. If X is the random variable with distribution function $F(x)$ given by,

$$F(x) = \begin{cases} 0, & -\infty < x < 0 \\ \frac{1}{2}(x^2 + x) & 0 \le x < 1 \\ 1, & 1 \le x < \infty \end{cases}$$

Then find (i)the probability density function $f(x)$ (ii)$P(0.3 \le X \le 0.6)$

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12. DISCRETE MATHEMATICS

1. Verify the (i) closure property, (ii) commutative property, (iii) associative property (iv) existence of identity and (v) existence of inverse for the arithmetic operation $+$ on $\mathbb{Z}$.
2. Verify the (i) closure property, (ii) commutative property, (iii) associative property (iv) existence of identity and (v) existence of inverse for the arithmetic operation - on $\mathbb{Z}$.
3. Verify the (i) closure property, (ii) commutative property, (iii) associative property (iv) existence of identity and (v) existence of inverse for the arithmetic operation + on $\mathbb{Z}_e$ = the set of all even integers.
4. Verify (i) closure property, (ii) commutative property, (iii) associative property, (iv) existence of identity, and (v) existence of inverse for following operation on the given set.
   $m * n = m + n - mn;\ \ m, n \in \mathbb{Z}$
5. Let $A = \begin{bmatrix} 0 & 1 \\ 1 & 1 \end{bmatrix}, B = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$ be any two Boolean matrices of the same type. Find $A \vee B$ and $A \wedge B$.
6. Verify (i) closure property, (ii) commutative property, (iii) associative property, (iv) existence of identity, and (v) existence of inverse for the operation $+_5$ on $\mathbb{Z}_5$ using table corresponding to addition modulo 5.
7. Verify (i) closure property, (ii) commutative property, (iii) associative property, (iv) existence of identity, and (v) existence of inverse for the operation $\times_{11}$ on a subset A = {1,3,4,5,9} of the set of remainders {0,1,2,3,4,5,6,7,8,9,10}.
8. Using the equivalence property, show that $p \leftrightarrow q \equiv (p \wedge q) \vee (\neg p \wedge \neg q)$.
9. (i) Define an operation$*$ on $\mathbb{Q}$ as follows: $a * b = \left(\frac{a+b}{2}\right); a, b \in \mathbb{Q}$. Examine the closure, commutative, and associative properties satisfied by $*$ on $\mathbb{Q}$.

(ii) Define an operation$*$ on $\mathbb{Q}$ as follows: $a * b = \left(\frac{a+b}{2}\right); a, b \in \mathbb{Q}b$ Examine the existence of identity and the existence of inverse for the operation $*$ on $\mathbb{Q}$.

10.(i) Let $M = \left\{\begin{pmatrix} x & x \\ x & x \end{pmatrix} : x\epsilon R - \{0\}\right\}$ and let $*$ be the matrix multiplication. Determine whether M is closed under $*$. If so, examine the commutative and associative properties satisfied by $*$ on M.

(i) Let $M = \left\{\begin{pmatrix} x & x \\ x & x \end{pmatrix} : x\epsilon R - \{0\}\right\}$ and let $*$ be the matrix multiplication. Determine whether M is closed under $*$. If so, examine the existence of identity, existence of inverse properties for the operation $*$ on M.

11.(i) Let A be $\mathbb{Q}\backslash\{1\}$. Define $*$ on A by $x * y = x + y - xy$. Is $*$ binary on A? If so, examine the commutative and associative properties satisfied by$*$ on A.

(ii) Let A be $\mathbb{Q}\backslash\{1\}$. Define $*$ on A by $x * y = x + y - xy$. Is $*$ binary on A? If so, examine the existence of identity, existence of inverse properties for the operation $*$ on A.

12. Verify whether the following compound propositions are tautologies or contradictions or contingency $(p \to q) \leftrightarrow (\neg p \to q)$

13. Using truth table check whether the statements $\neg(p \vee q) \vee (\neg p \wedge q)$ and $\neg p$ are logically equivalent.

14.Prove that $p \to (\neg q \vee r) \equiv \neg p \vee (\neg q \vee r)$ using truth table.

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Example 1.1 If $A = \begin{bmatrix} 8 & -6 & 2 \\ -6 & 7 & -4 \\ 2 & -4 & 3 \end{bmatrix}$ **verify the result** $A(adj\ A) = (adj\ A)A = |A|\ I_3$

Solution:

$$|A| = \begin{vmatrix} 8 & -6 & 2 \\ -6 & 7 & -4 \\ 2 & -4 & 3 \end{vmatrix}$$

$= 8(21-16) - (-6)(-18+8) + 2(24-14)$

$= 40 - 60 + 20 = 0.$

$$adjA = \begin{bmatrix} 5 & 10 & 10 \\ 10 & 20 & 20 \\ 10 & 20 & 20 \end{bmatrix}^T = \begin{bmatrix} 5 & 10 & 10 \\ 10 & 20 & 20 \\ 10 & 20 & 20 \end{bmatrix}$$

$$A(adj\ A) = \begin{bmatrix} 8 & -6 & 2 \\ -6 & 7 & -4 \\ 2 & -4 & 3 \end{bmatrix}\begin{bmatrix} 5 & 10 & 10 \\ 10 & 20 & 20 \\ 10 & 20 & 20 \end{bmatrix}$$

$$= \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} = 0\ I_3 = |A|\ I_3$$

$$(adj\ A)A = \begin{bmatrix} 5 & 10 & 10 \\ 10 & 20 & 20 \\ 10 & 20 & 20 \end{bmatrix}\begin{bmatrix} 8 & -6 & 2 \\ -6 & 7 & -4 \\ 2 & -4 & 3 \end{bmatrix}$$

$$= \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} = 0\ I_3 = |A|\ I_3$$

Hence, $A(adj\ A) = (adj\ A)A = |A|\ I_3$.

Example 1.10 If $A = \begin{bmatrix} 4 & 3 \\ 2 & 5 \end{bmatrix}$, **find x and y such that $A^2 + xA + y\ I_2 = O_2$. Hence, find A^{-1}.**

Solution:

Since, $A^2 = A \cdot A = \begin{bmatrix} 4 & 3 \\ 2 & 5 \end{bmatrix}\begin{bmatrix} 4 & 3 \\ 2 & 5 \end{bmatrix} = \begin{bmatrix} 22 & 27 \\ 18 & 31 \end{bmatrix}$

$A^2 + xA + y\ I_2 = O_2$

$$\Rightarrow \begin{bmatrix} 22 & 27 \\ 18 & 31 \end{bmatrix} + \begin{bmatrix} 4x & 3x \\ 2x & 5x \end{bmatrix} + \begin{bmatrix} y & 0 \\ 0 & y \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$$

$$\begin{bmatrix} 22+4x+y & 27+3x+0 \\ 18+2x+0 & 31+5x+y \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$$

$27 + 3x = 0 \quad \Rightarrow \quad x = -9$

$31 + 5x + y = 0 \quad \Rightarrow \quad y = 14$

$A^2 + xA + y\ I_2 = O_2 \Rightarrow A^2 - 9A + 14\ I_2 = O_2$

Post-multiplying this equation by A^{-1}

$A - 9\ I_2 + 14A^{-1} = O_2$

$A^{-1} = \frac{1}{14}(9I_2 - A)$

$$A^{-1} = \frac{1}{14}\left(\begin{bmatrix} 9 & 0 \\ 0 & 9 \end{bmatrix} - \begin{bmatrix} 4 & 3 \\ 2 & 5 \end{bmatrix}\right) = \frac{1}{14}\begin{bmatrix} 5 & -3 \\ -2 & 4 \end{bmatrix}$$

Example 1.12 If $A = \frac{1}{7}\begin{bmatrix} 6 & -3 & a \\ b & -2 & 6 \\ 2 & c & 3 \end{bmatrix}$ is orthogonal, find a, b and c, and hence A^{-1}.

Solution:

A is **orthogonal** if and only if A is non-singular and $A^{-1} = A^T$

If A is orthogonal, then $AA^T = A^T A = I_3$

$A\,A^T = I_3$

$$\Rightarrow \frac{1}{7}\begin{bmatrix} 6 & -3 & a \\ b & -2 & 6 \\ 2 & c & 3 \end{bmatrix}\frac{1}{7}\begin{bmatrix} 6 & -3 & a \\ b & -2 & 6 \\ 2 & c & 3 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$\begin{bmatrix} 45+a^2 & 6b+6+6a & 12-3c+3a \\ 6b+6+6a & b^2+40 & 2b-2c+18 \\ 12-3c+3a & 2b-2c+18 & c^2+13 \end{bmatrix} = 49\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$\begin{bmatrix} 45+a^2 & 6b+6+6a & 12-3c+3a \\ 6b+6+6a & b^2+40 & 2b-2c+18 \\ 12-3c+3a & 2b-2c+18 & c^2+13 \end{bmatrix} = \begin{bmatrix} 49 & 0 & 0 \\ 0 & 49 & 0 \\ 0 & 0 & 49 \end{bmatrix}$$

Comparing the corresponding elements,

$45 + a^2 = 49$	$b^2 + 40 = 49$	$c^2 + 13 = 49$
$a^2 = 4$	$b^2 = 9$	$c^2 = 36$
$a = 2$	$b = -3$	$c = 6$

$$A = \frac{1}{7}\begin{bmatrix} 6 & -3 & 2 \\ -3 & -2 & 6 \\ 2 & 6 & 3 \end{bmatrix} \Rightarrow A^{-1} = A^T = \frac{1}{7}\begin{bmatrix} 6 & -3 & 2 \\ -3 & -2 & 6 \\ 2 & 6 & 3 \end{bmatrix}$$

EXERCISE 1.1

3. If $F(\alpha) = \begin{bmatrix} \cos\alpha & 0 & \sin\alpha \\ 0 & 1 & 0 \\ -\sin\alpha & 0 & \cos\alpha \end{bmatrix}$ **show that** $[F(\alpha)]^{-1} = F(-\alpha)$.

Solution:

$$F(\alpha) = \begin{bmatrix} \cos\alpha & 0 & \sin\alpha \\ 0 & 1 & 0 \\ -\sin\alpha & 0 & \cos\alpha \end{bmatrix}$$

$$|F(\alpha)| = cos^2\alpha - 0 + sin^2\alpha = 1 \neq 0$$

$$adj\ F(\alpha) = \begin{bmatrix} \cos\alpha & 0 & -\sin\alpha \\ 0 & 1 & 0 \\ \sin\alpha & 0 & \cos\alpha \end{bmatrix}$$

$$[F(\alpha)]^{-1} = \frac{1}{|F(\alpha)|} adj\ F(\alpha) = \begin{bmatrix} \cos\alpha & 0 & -\sin\alpha \\ 0 & 1 & 0 \\ \sin\alpha & 0 & \cos\alpha \end{bmatrix} \ldots(1)$$

$$F(\alpha) = \begin{bmatrix} \cos\alpha & 0 & \sin\alpha \\ 0 & 1 & 0 \\ -\sin\alpha & 0 & \cos\alpha \end{bmatrix}$$

$$\Rightarrow F(-\alpha) = \begin{bmatrix} \cos\alpha & 0 & -\sin\alpha \\ 0 & 1 & 0 \\ \sin\alpha & 0 & \cos\alpha \end{bmatrix} \ldots (2)$$

From (1) and (2) $\quad [F(\alpha)]^{-1} = F(-\alpha)$

4. If $A = \begin{bmatrix} 5 & 3 \\ -1 & -2 \end{bmatrix}$, show that $A^2 - 3A - 7\ I_2 = O_2$. Hence find A^{-1}.

Solution:

$$A^2 = A \cdot A = \begin{bmatrix} 5 & 3 \\ -1 & -2 \end{bmatrix}\begin{bmatrix} 5 & 3 \\ -1 & -2 \end{bmatrix} = \begin{bmatrix} 22 & 9 \\ -3 & 1 \end{bmatrix}$$

$$A^2 - 3A - 7I_2 = O_2$$

$$\Rightarrow \begin{bmatrix} 22 & 9 \\ -3 & 1 \end{bmatrix} + \begin{bmatrix} -15 & -9 \\ 3 & 6 \end{bmatrix} + \begin{bmatrix} -7 & 0 \\ 0 & -7 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$$

Post-multiplying this equation by A^{-1}

$$A - 3I_2 - 7A^{-1} = O_2$$

$$A^{-1} = \frac{1}{7}(A - 3\ I_2)$$

$$A^{-1} = \frac{1}{7}\left(\begin{bmatrix} 5 & 3 \\ -1 & -2 \end{bmatrix} - \begin{bmatrix} 3 & 0 \\ 0 & 3 \end{bmatrix}\right)$$

$$A^{-1} = \frac{1}{7}\begin{bmatrix} 2 & 3 \\ -1 & -5 \end{bmatrix}$$

14. If $A = \begin{bmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{bmatrix}$, show that

$A^{-1} = \frac{1}{2}(A^2 - 3I)$.

Solution:

$$|A| = 0 - 1(-1) + 1(1) = 2$$

$$adj\ A = \begin{bmatrix} -1 & 1 & 1 \\ 1 & -1 & 1 \\ 1 & 1 & -1 \end{bmatrix}$$

$$A^{-1} = \frac{1}{2}\begin{bmatrix} -1 & 1 & 1 \\ 1 & -1 & 1 \\ 1 & 1 & -1 \end{bmatrix} \text{.....................(1)}$$

$$A^2 = A\,A = \begin{bmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{bmatrix}\begin{bmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{bmatrix} = \begin{bmatrix} 2 & 1 & 1 \\ 1 & 2 & 1 \\ 1 & 1 & 2 \end{bmatrix}$$

$$\frac{1}{2}(A^2 - 3I) = \frac{1}{2}\left(\begin{bmatrix} 2 & 1 & 1 \\ 1 & 2 & 1 \\ 1 & 1 & 2 \end{bmatrix} - \begin{bmatrix} 3 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 3 \end{bmatrix}\right)$$

$$\frac{1}{2}(A^2 - 3I) = \frac{1}{2}\begin{bmatrix} -1 & 1 & 1 \\ 1 & -1 & 1 \\ 1 & 1 & -1 \end{bmatrix} \text{............(2)}$$

From (1) and (2), $A^{-1} = \frac{1}{2}(A^2 - 3I)$

Example 1.21 Find the inverse of $A = \begin{bmatrix} 2 & 1 & 1 \\ 3 & 2 & 1 \\ 2 & 1 & 2 \end{bmatrix}$ by Gauss-Jordan method.

Solution:

$$(A|I_3) = \left(\begin{array}{ccc|ccc} 2 & 1 & 1 & 1 & 0 & 0 \\ 3 & 2 & 1 & 0 & 1 & 0 \\ 2 & 1 & 2 & 0 & 0 & 1 \end{array}\right)$$

$$\sim \left(\begin{array}{ccc|ccc} 1 & 1 & 0 & -1 & 1 & 0 \\ 3 & 2 & 1 & 0 & 1 & 0 \\ 0 & 0 & 1 & -1 & 0 & 1 \end{array}\right) \begin{matrix} R_1 \to R_2 - R_1 \\ R_3 \to R_3 - R_1 \end{matrix}$$

$$\sim \left(\begin{array}{ccc|ccc} 1 & 1 & 0 & -1 & 1 & 0 \\ 0 & -1 & 1 & 3 & -2 & 0 \\ 0 & 0 & 1 & -1 & 0 & 1 \end{array}\right) R_2 \to R_2 - 3R_1$$

$$\sim \left(\begin{array}{ccc|ccc} 1 & 1 & 0 & -1 & 1 & 0 \\ 0 & -1 & 0 & 4 & -2 & -1 \\ 0 & 0 & 1 & -1 & 0 & 1 \end{array}\right) R_2 \to R_2 - R_3$$

$$(I_3|A^{-1}) = \left(\begin{array}{ccc|ccc} 1 & 0 & 0 & 3 & -1 & -1 \\ 0 & 1 & 0 & -4 & 2 & 1 \\ 0 & 0 & 1 & -1 & 0 & 1 \end{array}\right) \begin{matrix} R_1 \to R_1 + R_2 \\ R_2 \to -R_2 \end{matrix}$$

$$A^{-1} = \begin{bmatrix} 3 & -1 & -1 \\ -4 & 2 & 1 \\ -1 & 0 & 1 \end{bmatrix}$$

EXERCISE 1.2

3. Find the inverse of each of the following by Gauss-Jordan method:

(ii) $\begin{bmatrix} 1 & -1 & 0 \\ 1 & 0 & -1 \\ 6 & -2 & -3 \end{bmatrix}$ (iii) $\begin{bmatrix} 1 & 2 & 3 \\ 2 & 5 & 3 \\ 1 & 0 & 8 \end{bmatrix}$

Solution:

(ii) Let $A = \begin{bmatrix} 1 & -1 & 0 \\ 1 & 0 & -1 \\ 6 & -2 & -3 \end{bmatrix}$

$$[A|\,I_3] = \left[\begin{array}{ccc|ccc} 1 & -1 & 0 & 1 & 0 & 0 \\ 1 & 0 & -1 & 0 & 1 & 0 \\ 6 & -2 & -3 & 0 & 0 & 1 \end{array}\right]$$

$$\sim \left[\begin{array}{ccc|ccc} 1 & -1 & 0 & 1 & 0 & 0 \\ 0 & 1 & -1 & -1 & 1 & 0 \\ 0 & 4 & -3 & -6 & 0 & 1 \end{array}\right] \begin{matrix} R_2 \to R_2 - R_1 \\ R_3 \to R_3 - 6R_1 \end{matrix}$$

$$\sim \left[\begin{array}{ccc|ccc} 1 & -1 & 0 & 1 & 0 & 0 \\ 0 & 1 & -1 & -1 & 1 & 0 \\ 0 & 0 & 1 & -2 & -4 & 1 \end{array}\right] R_3 \to R_3 - 4R_2$$

$$\sim \left[\begin{array}{ccc|ccc} 1 & -1 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & -3 & -3 & 1 \\ 0 & 0 & 1 & -2 & -4 & 1 \end{array}\right] R_2 \to R_2 + R_3$$

$$(I_3|A^{-1}) = \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & -2 & -3 & 1 \\ 0 & 1 & 0 & -3 & -3 & 1 \\ 0 & 0 & 1 & -2 & -4 & 1 \end{array}\right] R_1 \to R_1 + R_2$$

$$A^{-1} = \begin{bmatrix} -2 & -3 & 1 \\ -3 & -3 & 1 \\ -2 & -4 & 1 \end{bmatrix}$$

(iii) Let $A = \begin{bmatrix} 1 & 2 & 3 \\ 2 & 5 & 3 \\ 1 & 0 & 8 \end{bmatrix}$

$$[A|\,I_3] = \left[\begin{array}{ccc|ccc} 1 & 2 & 3 & 1 & 0 & 0 \\ 2 & 5 & 3 & 0 & 1 & 0 \\ 1 & 0 & 8 & 0 & 0 & 1 \end{array}\right]$$

$$\sim \left[\begin{array}{ccc|ccc} 1 & 2 & 3 & 1 & 0 & 0 \\ 0 & 1 & -3 & -2 & 1 & 0 \\ 0 & -2 & 5 & -1 & 0 & 1 \end{array}\right] \begin{array}{l} R_2 \to R_2 - 2R_1 \\ R_3 \to R_3 - R_1 \end{array}$$

$$\sim \left[\begin{array}{ccc|ccc} 1 & 2 & 3 & 1 & 0 & 0 \\ 0 & 1 & -3 & -2 & 1 & 0 \\ 0 & 0 & -1 & -5 & 2 & 1 \end{array}\right] R_3 \to R_3 + 2R_2$$

$$\sim \left[\begin{array}{ccc|ccc} 1 & 2 & 3 & 1 & 0 & 0 \\ 0 & 1 & 0 & 13 & -5 & -3 \\ 0 & 0 & -1 & -5 & 2 & 1 \end{array}\right] R_2 \to R_2 - 3R_3$$

$(I_3|A^{-1}) =$

$$\left[\begin{array}{ccc|ccc} 1 & 0 & 0 & -40 & 16 & 9 \\ 0 & 1 & 0 & 13 & -5 & -3 \\ 0 & 0 & 1 & 5 & -2 & -1 \end{array}\right] \begin{array}{l} R_1 \to R_1 - 2R_2 + 3R_3 \\ R_3 \to (-1)R_3 \end{array}$$

$$A^{-1} = \begin{bmatrix} -40 & 16 & 9 \\ 13 & -5 & -3 \\ 5 & -2 & -1 \end{bmatrix}$$

Example 1.23 Solve the following system of equations, using matrix inversion method:
$2x_1 + 3x_2 + 3x_3 = 5, \quad x_1 - 2x_2 + x_3 = -4,$
$3x_1 - x_2 - 2x_3 = 3.$

Solution:

$$A = \begin{bmatrix} 2 & 3 & 3 \\ 1 & -2 & 1 \\ 3 & -1 & -2 \end{bmatrix}, \quad X = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}, \quad B = \begin{bmatrix} 5 \\ -4 \\ 3 \end{bmatrix}$$

$|A| = 2(4+1) - 3(-2-3) + 3(-1+6) = 40$

$$adjA = [A_{ij}]^T \quad \Rightarrow adjA = \begin{bmatrix} 5 & 3 & 9 \\ 5 & -13 & 1 \\ 5 & 11 & -7 \end{bmatrix}$$

$$A^{-1} = \frac{1}{|A|}(adj\ A) \quad \Rightarrow A^{-1} = \frac{1}{40}\begin{bmatrix} 5 & 3 & 9 \\ 5 & -13 & 1 \\ 5 & 11 & -7 \end{bmatrix}$$

$$X = A^{-1}B \quad X = \frac{1}{40}\begin{bmatrix} 5 & 3 & 9 \\ 5 & -13 & 1 \\ 5 & 11 & -7 \end{bmatrix}\begin{bmatrix} 5 \\ -4 \\ 3 \end{bmatrix}$$

$$X = \frac{1}{40}\begin{bmatrix} 25 - 12 + 27 \\ 25 + 52 + 3 \\ 25 - 44 - 21 \end{bmatrix}$$

$$X = \frac{1}{40}\begin{bmatrix} 40 \\ 80 \\ -40 \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \\ -1 \end{bmatrix}$$

Hence, $x_1 = 1, \quad x_2 = 2, \quad x_3 = -1$

Example 1.24 If $A = \begin{bmatrix} -4 & 4 & 4 \\ -7 & 1 & 3 \\ 5 & -3 & -1 \end{bmatrix}$, $B = \begin{bmatrix} 1 & -1 & 1 \\ 1 & -2 & -2 \\ 2 & 1 & 3 \end{bmatrix}$, find the products AB and BA and hence solve the system of equations

$x - y + z = 4, \quad x - 2y - 2z = 9, \quad 2x + y + 3z = 1.$

Solution:

$$A = \begin{bmatrix} -4 & 4 & 4 \\ -7 & 1 & 3 \\ 5 & -3 & -1 \end{bmatrix}, B = \begin{bmatrix} 1 & -1 & 1 \\ 1 & -2 & -2 \\ 2 & 1 & 3 \end{bmatrix}$$

$$AB = \begin{bmatrix} -4 & 4 & 4 \\ -7 & 1 & 3 \\ 5 & -3 & -1 \end{bmatrix}\begin{bmatrix} 1 & -1 & 1 \\ 1 & -2 & -2 \\ 2 & 1 & 3 \end{bmatrix}$$

$$= \begin{bmatrix} -4+4+8 & 4-8+4 & -4-8+12 \\ -7+1+6 & 7-2+3 & -7-2+9 \\ 5-3-2 & -5+6-1 & 5+6-3 \end{bmatrix}$$

$$= \begin{bmatrix} 8 & 0 & 0 \\ 0 & 8 & 0 \\ 0 & 0 & 8 \end{bmatrix} = 8\ I_3$$

$$BA = \begin{bmatrix} 1 & -1 & 1 \\ 1 & -2 & -2 \\ 2 & 1 & 3 \end{bmatrix}\begin{bmatrix} -4 & 4 & 4 \\ -7 & 1 & 3 \\ 5 & -3 & -1 \end{bmatrix}$$

$$= \begin{bmatrix} -4+7+5 & 4-1-3 & 4-3-1 \\ -4+14-10 & 4-2+6 & 4-6+2 \\ -8-7+15 & 8+1-9 & 8+3-3 \end{bmatrix}$$

$$= \begin{bmatrix} 8 & 0 & 0 \\ 0 & 8 & 0 \\ 0 & 0 & 8 \end{bmatrix} = 8\ I_3$$

$AB = BA = 8\ I_3$

$\Rightarrow \left(\frac{1}{8}A\right)B = B\left(\frac{1}{8}A\right) = I_3$

$\Rightarrow \quad B^{-1} = \frac{1}{8}A$

The given system of equations in matrix form,

$$\begin{bmatrix} 1 & -1 & 1 \\ 1 & -2 & -2 \\ 2 & 1 & 3 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 4 \\ 9 \\ 1 \end{bmatrix}$$

ie) $BX = C \quad X = B^{-1}C$

$$\Rightarrow \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \left(\frac{1}{8}A\right)\begin{bmatrix} 4 \\ 9 \\ 1 \end{bmatrix}$$

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \frac{1}{8}\begin{bmatrix} -4 & 4 & 4 \\ -7 & 1 & 3 \\ 5 & -3 & -1 \end{bmatrix}\begin{bmatrix} 4 \\ 9 \\ 1 \end{bmatrix} = \frac{1}{8}\begin{bmatrix} -16+36+4 \\ -28+9+3 \\ 20-27-1 \end{bmatrix}$$

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \frac{1}{8}\begin{bmatrix} 24 \\ -16 \\ -8 \end{bmatrix} = \begin{bmatrix} 3 \\ -2 \\ -1 \end{bmatrix}$$

Hence, $x = 3, \quad y = -2, \quad z = -1$

EXERCISE 1.3

1. (iii) $2x+3y-z=9,\ x+y+z=9,\ 3x-y-z=-1$

$$A=\begin{bmatrix}2&3&-1\\1&1&1\\3&-1&-1\end{bmatrix},\quad X=\begin{bmatrix}x\\y\\z\end{bmatrix},\quad B=\begin{bmatrix}9\\9\\-1\end{bmatrix}$$

$|A|=2(-1+1)-3(-1-3)+(-1)(-1-3)$

$|A|=16\neq 0$

$$\Rightarrow\quad adjA=\begin{bmatrix}0&4&4\\4&1&-3\\-4&11&-1\end{bmatrix}$$

$$A^{-1}=\frac{1}{|A|}(adj\ A)=\frac{1}{16}\begin{bmatrix}0&4&4\\4&1&-3\\-4&11&-1\end{bmatrix}$$

$$X=A^{-1}B\ \Rightarrow X=\frac{1}{16}\begin{bmatrix}0&4&4\\4&1&-3\\-4&11&-1\end{bmatrix}\begin{bmatrix}9\\9\\-1\end{bmatrix}$$

$$\Rightarrow X=\frac{1}{16}\begin{bmatrix}0+36-4\\36+9+3\\-36+99+1\end{bmatrix}$$

$$X=\frac{1}{16}\begin{bmatrix}32\\48\\64\end{bmatrix}=\begin{bmatrix}2\\3\\4\end{bmatrix}$$

Hence, $x=2,\ \ y=3,\ \ z=4$

1. (iv) $x+y+z-2=0,$

$6x-4y+5z-31=0,\ \ 5x+2y+2z=13$

$$A=\begin{bmatrix}1&1&1\\6&-4&5\\5&2&2\end{bmatrix},\quad X=\begin{bmatrix}x\\y\\z\end{bmatrix},\quad B=\begin{bmatrix}2\\31\\13\end{bmatrix}$$

$|A|=1(-8-10)-1(12-25)+1(12+20)=27$

$$adjA=\begin{bmatrix}-18&0&9\\13&-3&1\\32&3&-10\end{bmatrix}$$

$$A^{-1}=\frac{1}{|A|}(adj\ A)=\frac{1}{27}\begin{bmatrix}-18&0&9\\13&-3&1\\32&3&-10\end{bmatrix}$$

$X=A^{-1}B\ \Rightarrow$

$$X=\frac{1}{27}\begin{bmatrix}-18&0&9\\13&-3&1\\32&3&-10\end{bmatrix}\begin{bmatrix}2\\31\\13\end{bmatrix}$$

$$\Rightarrow X=\frac{1}{27}\begin{bmatrix}-36+0+117\\26-93+13\\64+93-130\end{bmatrix}$$

$$X=\frac{1}{27}\begin{bmatrix}81\\-54\\27\end{bmatrix}=\begin{bmatrix}3\\-2\\1\end{bmatrix}$$

Hence, $x=3,\ \ y=-2,\ \ z=1.$

2. If $A=\begin{bmatrix}-5&1&3\\7&1&-5\\1&-1&1\end{bmatrix}, B=\begin{bmatrix}1&1&2\\3&2&1\\2&1&3\end{bmatrix}$, find the products AB and BA and hence solve the system of equations $x+y+2z=1$, $3x+2y+z=7,\ \ 2x+y+3z=2.$

Solution:

$$AB=\begin{bmatrix}-5&1&3\\7&1&-5\\1&-1&1\end{bmatrix}\begin{bmatrix}1&1&2\\3&2&1\\2&1&3\end{bmatrix}$$

$$=\begin{bmatrix}-5+3+6&-5+2+3&-10+1+9\\7+3-10&7+2-5&14+1-15\\1-3+2&1-2+1&2-1+3\end{bmatrix}$$

$$=\begin{bmatrix}4&0&0\\0&4&0\\0&0&4\end{bmatrix}=4\ I_3$$

$$BA=\begin{bmatrix}1&1&2\\3&2&1\\2&1&3\end{bmatrix}\begin{bmatrix}-5&1&3\\7&1&-5\\1&-1&1\end{bmatrix}$$

$$=\begin{bmatrix}-5+7+2&1+1-2&3-5+2\\-15+14+1&3+2-1&9-10+1\\-10+7+3&2+1-3&6-5+3\end{bmatrix}$$

$$=\begin{bmatrix}4&0&0\\0&4&0\\0&0&4\end{bmatrix}=4\ I_3$$

$AB=BA=4\ I_3$

$$\Rightarrow\quad \left(\tfrac{1}{4}A\right)B=B\left(\tfrac{1}{4}A\right)=\ I_3$$

$$\Rightarrow\quad B^{-1}=\frac{1}{4}A$$

The given system of equations in matrix form, we get

$$\begin{bmatrix}1&1&2\\3&2&1\\2&1&3\end{bmatrix}\begin{bmatrix}x\\y\\z\end{bmatrix}=\begin{bmatrix}1\\7\\2\end{bmatrix}$$

ie) $BX=C\ \Rightarrow\ X=B^{-1}C\Rightarrow\begin{bmatrix}x\\y\\z\end{bmatrix}=\left(\tfrac{1}{4}A\right)\begin{bmatrix}1\\7\\2\end{bmatrix}$

$$\begin{bmatrix}x\\y\\z\end{bmatrix}=\frac{1}{4}\begin{bmatrix}-5&1&3\\7&1&-5\\1&-1&1\end{bmatrix}\begin{bmatrix}1\\7\\2\end{bmatrix}$$

$$\begin{bmatrix}x\\y\\z\end{bmatrix}=\frac{1}{4}\begin{bmatrix}-5+7+6\\7+7-10\\1-7+2\end{bmatrix}$$

$$\begin{bmatrix}x\\y\\z\end{bmatrix}=\frac{1}{4}\begin{bmatrix}8\\4\\-4\end{bmatrix}=\begin{bmatrix}2\\1\\-1\end{bmatrix}$$

Hence, $x=2,\ \ y=1,\ \ z=-1.$

5. The prices of three commodities A, B and C are $Rs.\ x$, y and z per units respectively. A person P purchases 4 units of B and sells two units of A and 5 units of C. Person Q purchases 2 units of C and sells 3 units of A and one unit of B. Person R purchases one unit of A and sells 3 unit of B and one unit of C. In the process, P, Q and R earn $Rs.$ 15,000, $Rs.$ 1,000 and $Rs.$ 4,000 respectively. Find the prices per unit of A, B and C. (Use matrix inversion method to solve the problem.)

Solution:

Let the prices of one unit of A, B and C are Rs.x, y and z respectively.

From the given data,

$2x - 4y + 5z = 15000 \rightarrow (1)$

$3x + y - 2z = 1000 \rightarrow (2)$

$-x + 3y + z = 4000 \rightarrow (3)$

$$\begin{bmatrix} 2 & -4 & 5 \\ 3 & 1 & -2 \\ -1 & 3 & 1 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 15000 \\ 1000 \\ 4000 \end{bmatrix} \quad AX = B$$

$$A = \begin{bmatrix} 2 & -4 & 5 \\ 3 & 1 & -2 \\ -1 & 3 & 1 \end{bmatrix}, \quad X = \begin{bmatrix} x \\ y \\ z \end{bmatrix}, \quad B = \begin{bmatrix} 15000 \\ 1000 \\ 4000 \end{bmatrix}$$

$|A| = 2(1+6) - 4(3+2) + 5(9+1) = 68 \neq 0$

$$adj A = [A_{ij}]^T = \begin{bmatrix} 7 & 19 & 3 \\ -1 & 7 & 19 \\ 10 & -2 & 14 \end{bmatrix}$$

$$A^{-1} = \frac{1}{|A|}(adj\ A) = \frac{1}{68}\begin{bmatrix} 7 & 19 & 3 \\ -1 & 7 & 19 \\ 10 & -2 & 14 \end{bmatrix}$$

$$X = A^{-1}B \implies X = \frac{1}{68}\begin{bmatrix} 7 & 19 & 3 \\ -1 & 7 & 19 \\ 10 & -2 & 14 \end{bmatrix}\begin{bmatrix} 15000 \\ 1000 \\ 4000 \end{bmatrix}$$

$$X = \frac{1000}{68}\begin{bmatrix} 7 & 19 & 3 \\ -1 & 7 & 19 \\ 10 & -2 & 14 \end{bmatrix}\begin{bmatrix} 15 \\ 1 \\ 4 \end{bmatrix}$$

$$= \frac{1000}{68}\begin{bmatrix} 105+19+12 \\ -15+7+76 \\ 150-2+56 \end{bmatrix}$$

$$X = \frac{1000}{68}\begin{bmatrix} 136 \\ 68 \\ 204 \end{bmatrix} = \begin{bmatrix} 2000 \\ 1000 \\ 3000 \end{bmatrix}$$

Hence, $x = 2000, \quad y = 1000, \quad z = 3000.$

The price of one unit of A, B and C are Rs.2000, 1000 and 3000 respectively.

Example 1.25 Solve, by Cramer's rule, the system of equations $x_1 - x_2 = 3$, $2x_1 + 3x_2 + 4x_3 = 17$, $x_2 + 2x_3 = 7$.

Solution:

$$\Delta = \begin{vmatrix} 1 & -1 & 0 \\ 2 & 3 & 4 \\ 0 & 1 & 2 \end{vmatrix}$$

$\Delta = 1(6-4) - (-1)(4-0) + 0 = 6$

$$\Delta_{x_1} = \begin{vmatrix} 3 & -1 & 0 \\ 17 & 3 & 4 \\ 7 & 1 & 2 \end{vmatrix}$$

$\Delta_{x_1} = 3(6-4) - (-1)(34-28) + 0 = 12$

$$\Delta_{x_2} = \begin{vmatrix} 1 & 3 & 0 \\ 2 & 17 & 4 \\ 0 & 7 & 2 \end{vmatrix}$$

$\Delta_{x_2} = 1(34-28) - 3(4-0) + 0 = -6$

$$\Delta_{x_3} = \begin{vmatrix} 1 & -1 & 3 \\ 2 & 3 & 17 \\ 0 & 1 & 7 \end{vmatrix}$$

$\Delta_{x_3} = 1(21-17) - (-1)(14-0) + 3(2-0) = 24$

B Cramer's rule, we get

$x_1 = \frac{\Delta_{x_1}}{\Delta} = \frac{12}{6} = 2$

$x_2 = \frac{\Delta_{x_2}}{\Delta} = \frac{-6}{6} = -1$

$x_3 = \frac{\Delta_{x_3}}{\Delta} = \frac{24}{6} = 4$

So, the solution is $(x_1 = 2, \quad x_2 = -1, \quad x_3 = 4)$

Example 1.26 In a T20 match, Chennai Super Kings needed just 6 runs to win with 1 ball left to go in the last over. The last ball was bowled and the batsman at the crease hit it high up. The ball traversed along a path in a vertical plane and the equation of the path is $y = ax^2 + bx + c$ with respect to an $x\,y$ - coordinate system in the vertical plane and the ball traversed through the points $(10,8), (20,16), (40,22)$, can you conclude that Chennai Super Kings won the match? Justify your answer. (All distances are measured in metres and the meeting point of the plane of the path with the farthest boundary line is $(70,0)$.)

Solution:

The path $y = ax^2 + bx + c$ passes through the points $(10,8), (20,16), (40,22)$. So, we get the system of equations

$$100a + 10b + c = 8,$$
$$400a + 20b + c = 16,$$
$$1600a + 40b + c = 22.$$

$$\Delta = \begin{vmatrix} 100 & 10 & 1 \\ 400 & 20 & 1 \\ 1600 & 40 & 1 \end{vmatrix} = 1000\begin{vmatrix} 1 & 1 & 1 \\ 4 & 2 & 1 \\ 16 & 4 & 1 \end{vmatrix}$$

$$\Delta = 1000(-2 + 12 - 16) = -6000 \neq 0$$

$$\Delta_a = \begin{vmatrix} 8 & 10 & 1 \\ 16 & 20 & 1 \\ 22 & 40 & 1 \end{vmatrix} = 20\begin{vmatrix} 4 & 1 & 1 \\ 8 & 2 & 1 \\ 11 & 4 & 1 \end{vmatrix}$$

$$\Delta_a = 20(-8 + 3 + 10) = 100$$

$$\Delta_b = \begin{vmatrix} 100 & 8 & 1 \\ 400 & 16 & 1 \\ 1600 & 22 & 1 \end{vmatrix} = 200\begin{vmatrix} 1 & 4 & 1 \\ 4 & 8 & 1 \\ 16 & 11 & 1 \end{vmatrix}$$

$$\Delta_b = 200(-3 + 48 - 84) = -7800$$

$$\Delta_c = \begin{vmatrix} 100 & 10 & 8 \\ 400 & 20 & 16 \\ 1600 & 40 & 22 \end{vmatrix} = 2000\begin{vmatrix} 1 & 1 & 4 \\ 4 & 2 & 8 \\ 16 & 4 & 11 \end{vmatrix}$$

$$\Delta_c = 2000(-10 + 84 - 64) = 20000.$$

By Cramer's rule, we get

$$a = \frac{\Delta_a}{\Delta} = \frac{100}{-6000} = \frac{-1}{60}$$

$$b = \frac{\Delta_b}{\Delta} = \frac{-7800}{-6000} = \frac{13}{10}$$

$$c = \frac{\Delta_c}{\Delta} = \frac{20000}{-6000} = \frac{-10}{3}$$

So, the equation of the path is

$$y = ax^2 + bx + c \implies y = \frac{-1}{60}x^2 + \frac{13}{10}x + \frac{-10}{3}$$

$$x = 70 \implies y = \frac{-1}{60}(70)^2 + \frac{13}{10}(70) + \frac{-10}{3}$$

$$\implies \quad y = \frac{-1}{60}(4900) + \frac{910}{10} + \frac{-10}{3} = 6$$

When $x = 70$ we get $y=6$

EXERCISE 1.4

1. Solve the systems of linear equations by Cramer's rule:

(iii) $3x + 3y - z = 11, \quad 2x - y + 2z = 9, \quad 4x + 3y + 2z = 25$

Solution:

$$\Delta = \begin{vmatrix} 3 & 3 & -1 \\ 2 & -1 & 2 \\ 4 & 3 & 2 \end{vmatrix} = -22 \neq 0$$

$$\Delta_x = \begin{vmatrix} 11 & 3 & -1 \\ 9 & -1 & 2 \\ 25 & 3 & 2 \end{vmatrix} = -44$$

$$\Delta_y = \begin{vmatrix} 3 & 11 & -1 \\ 2 & 9 & 2 \\ 4 & 25 & 2 \end{vmatrix} = -66$$

$$\Delta_z = \begin{vmatrix} 3 & 3 & 11 \\ 2 & -1 & 9 \\ 4 & 3 & 25 \end{vmatrix} = -88$$

By Cramer's rule, we get

$$x = \frac{\Delta_x}{\Delta} = \frac{-44}{-22} = 2 \qquad y = \frac{\Delta_y}{\Delta} = \frac{-66}{-22} = 3$$

$$z = \frac{\Delta_z}{\Delta} = \frac{-88}{-22} = 4$$

So, the solution is $(x = 2, \ y = 3, \ z = 4)$

(iv) $\frac{3}{x}-\frac{4}{y}-\frac{2}{z}-1=0,$

$\frac{1}{x}+\frac{2}{y}+\frac{1}{z}-2=0,$

$\frac{2}{x}-\frac{5}{y}-\frac{4}{z}+1=0$

Solution:

Let $a=\frac{1}{x}$ $b=\frac{2}{y}$ $c=\frac{1}{z}$

$3a-4b-2c=1$

$a+2b+c=2$

$2a-5b-4c=-1$

$\Delta=\begin{vmatrix}3&-4&-2\\1&2&1\\2&-5&-4\end{vmatrix}=3(-3)+4(-6)-2(-9)=-15\neq 0$

$\Delta_a=\begin{vmatrix}1&-4&-2\\2&2&1\\-1&-5&-4\end{vmatrix}=1(-3)+4(-7)-2(-8)=-15$

$\Delta_b=\begin{vmatrix}3&1&-2\\1&2&1\\2&-1&-4\end{vmatrix}=3(-7)-1(-6)-2(-5)=-5$

$\Delta_c=\begin{vmatrix}3&-4&1\\1&2&2\\2&-5&-1\end{vmatrix}=3(8)+4(-5)+1(-9)=-5$

By Cramer's rule, we get

$$a=\frac{\Delta_a}{\Delta}=\frac{-15}{-15}=1$$

$$b=\frac{\Delta_b}{\Delta}=\frac{-5}{-15}=\frac{1}{3}$$

$$c=\frac{\Delta_c}{\Delta}=\frac{-5}{-15}=\frac{1}{3}$$

$x=\frac{1}{a}=1$; $y=\frac{1}{b}=3$; $z=\frac{1}{c}=3$

So, the solution is $(x=1,\ y=3,\ z=3)$

5. A family of 3 people went out for dinner in a restaurant. The cost of two dosai, three idlies and two vadais is $Rs.$ 150. The cost of the two dosai, two idlies and four vadais is $Rs.$ 200. The cost of five dosai, four idlies and two vadais is $Rs.$ 250. The family has $Rs.$ 350 in hand and they ate 3 dosai and six idlies and six vadais. Will they be able to manage to pay the bill within the amount they had ?

Solution:

Let Rs.x, y and z be the cost of I dosai, 1 idly and I vadai respectively. From the given data,

$2x+3y+2z=150$ ……………….(1)

$2x+2y+4z=200$

$(\div 2)\Rightarrow x+y+2z=100$ ………………….(2)

$5x+4y+2z=250$ ………………(3)

$\Delta=\begin{vmatrix}2&3&2\\1&1&2\\5&4&2\end{vmatrix}=2(-6)-3(-4)+2(-1)=10\neq 0$

$\Delta_x=\begin{vmatrix}150&3&2\\100&1&2\\250&4&2\end{vmatrix}=50\begin{vmatrix}3&3&2\\2&1&2\\5&4&2\end{vmatrix}$

$=50[3(-6)-3(-6)+2(3)]=50(6)=300$

$\Delta_y=\begin{vmatrix}2&150&2\\1&100&2\\5&250&2\end{vmatrix}=50\begin{vmatrix}2&3&2\\1&2&2\\5&5&2\end{vmatrix}$

$=50[2(-6)-3(-8)+2(-5)]=50(2)=100$

$\Delta_z=\begin{vmatrix}2&3&150\\1&1&100\\5&4&250\end{vmatrix}=50\begin{vmatrix}2&3&3\\1&1&2\\5&4&5\end{vmatrix}$

$=50[2(-3)-3(-5)+3(-1)]=50(6)=300$

By Cramer's rule, we get

$x=\frac{\Delta_x}{\Delta}=\frac{300}{10}=30$

$y=\frac{\Delta_y}{\Delta}=\frac{100}{10}=10$

$z=\frac{\Delta_z}{\Delta}=\frac{300}{10}=30$

So, the solution is $(x=30,\ y=10,\ z=30)$

The cost of 3 dosai , 6 idlies and 6 vadais

$=3x+6y+6z$

$=3(30)+6(10)+6(30)$

$=90+60+180=330<350$

Hence they can manage to pay the bill.

Example 1.27 Solve the following system of linear equations, by Gaussian elimination method : $4x+3y+6z=25$, $x+5y+7z=13$, $2x+9y+z=1$.

Solution:

The matrix form of the system is AX=B, where

$$A=\begin{bmatrix}4&3&6\\1&5&7\\2&9&1\end{bmatrix},\quad X=\begin{bmatrix}x\\y\\z\end{bmatrix},\quad B=\begin{bmatrix}25\\13\\1\end{bmatrix}$$

Transforming the augmented matrix to echelon form, we get

$$[A,B]=\left[\begin{array}{ccc|c}4&3&6&25\\1&5&7&13\\2&9&1&1\end{array}\right]$$

$$\sim\left[\begin{array}{ccc|c}1&5&7&13\\4&3&6&25\\2&9&1&1\end{array}\right] R_1\leftrightarrow R_2$$

$$\sim\left[\begin{array}{ccc|c}1&5&7&13\\0&-17&-22&-27\\0&-1&-13&-25\end{array}\right] \begin{array}{l}R_2\to R_2-4R_1\\R_3\to R_3-2R_1\end{array}$$

$$\sim\left[\begin{array}{ccc|c}1&5&7&13\\0&-17&-22&-27\\0&0&199&398\end{array}\right] R_3\to -17R_3+R_2$$

The equivalent system is written by using the echelon form:

$$x+5y+7z=13\ \ldots\ldots\ldots\ldots(1)$$

$$-17y-22z=-27\ \ldots\ldots\ldots(2)$$

$$199z=398\ \ldots\ldots\ldots(3)$$

From (3) $z=2$

From (2) $-17y-44=-27$

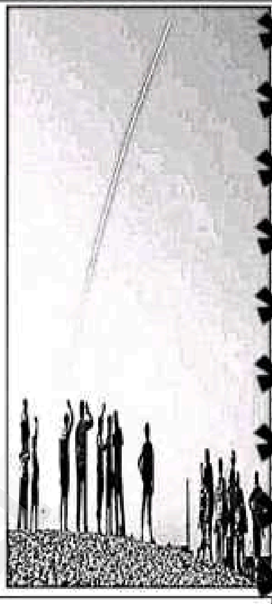
$$\Rightarrow y=\frac{-27+44}{-17}=-1$$

From (1) $x-5+14=13 \Rightarrow x=13-9=4$

So, the solution is $(x=4,\quad y=-1,\quad z=2)$

Example 1.28

The upward speed $v(t)$ of a rocket at time t is approximated by $v(t)=at^2+bt+c$, $0\le t\le 100$, where a, b and c are constants. It has been found that the speed at times $t=3$, $t=6$ and $t=9$ seconds are respectively, 64, 133 and 208 miles per second respectively. Find the speed at time $t=15$ seconds. (Use Gaussian elimination method.)



Solution:

Since $v(3)=64, v(6)=133$ and $v(9)=208$, we get the following system of linear equations

$$9a+3b+c=64$$

$$36a+6b+c=133$$

$$81a+9b+c=208$$

$$A=\begin{bmatrix}9&3&1\\36&6&1\\81&9&1\end{bmatrix},\quad X=\begin{bmatrix}a\\b\\c\end{bmatrix},\quad B=\begin{bmatrix}64\\133\\208\end{bmatrix}$$

$$[A,B]=\left[\begin{array}{ccc|c}9&3&1&64\\36&6&1&133\\81&9&1&208\end{array}\right]$$

$$\sim\left[\begin{array}{ccc|c}9&3&1&64\\0&-6&-3&-123\\0&-18&-8&-368\end{array}\right] \begin{array}{l}R_2\to R_2-4R_1\\R_3\to R_3-9R_1\end{array}$$

$$\sim\left[\begin{array}{ccc|c}9&3&1&64\\0&-6&-3&-123\\0&0&1&1\end{array}\right] R_3\to R_3-3R_2$$

The equivalent system is written by using the echelon form:

$$9a+3b+c=64\ \ldots\ldots\ldots\ldots(1)$$

$$-6b-3c=-123\ \ldots\ldots\ldots(2)$$

$$c=1\ \ldots\ldots\ldots\ldots\ldots(3)$$

From (2) $b=\frac{-123+3}{-6}=20$

From (1) $a=\frac{64-1-60}{9}=\frac{1}{3}$

So, the solution is $\left(a=\frac{1}{3},\ b=20,\ c=1\right)$

$$v(t)=at^2+bt+c=\frac{t^2}{3}+20t+1$$

$t=15$ $\qquad v(15)=75+300+1=376.$

EXERCISE 1.5

1. Solve the following systems of linear equations by Gaussian elimination method:
(i) $2x-2y+3z=2,\quad x+2y-z=3,\quad 3x-y+2z=1$

Solution:

$$A=\begin{bmatrix}2&-2&3\\1&2&-1\\3&-1&2\end{bmatrix},\quad X=\begin{bmatrix}x\\y\\z\end{bmatrix},\quad B=\begin{bmatrix}2\\3\\1\end{bmatrix}$$

Transforming the augmented matrix to echelon form, we get

$$[A,B]=\left[\begin{array}{ccc|c}2&-2&3&2\\1&2&-1&3\\3&-1&2&1\end{array}\right]$$

$$\sim\left[\begin{array}{ccc|c}1&2&-1&3\\2&-2&3&2\\3&-1&2&1\end{array}\right]\ R_1\leftrightarrow R_2$$

$$\sim\left[\begin{array}{ccc|c}1&2&-1&3\\0&-6&5&-4\\0&-7&5&-8\end{array}\right]\ \begin{array}{l}R_2\to R_2-2R_1\\R_3\to R_3-3R_1\end{array}$$

$$\sim\left[\begin{array}{ccc|c}1&2&-1&3\\0&-6&5&-4\\0&0&-5&-20\end{array}\right]\ R_3\to 6R_3-7R_2$$

$$\begin{array}{rcccc|c}6R_3&\to&0&-42&30&-40\\-7R_2&\to&0&42&-35&28\\6R_3-7R_2&\to&0&0&-5&-20\end{array}$$

$x+2y-z=3\ \ldots\ldots\ldots(1)$

$-6y+5z=-4\ \ldots\ldots\ldots(2)$

$-5z=-20\ \ldots\ldots(3)$

From (3) $\quad z=4$

From (2) $\quad y=\frac{-4-20}{-6}=4$

From (1) $\quad x=3+4-8=-1$

So, the solution is $(x=-1,\ y=4,\ z=4)$

(ii) $2x+4y+6z=22,\ 3x+8y+5z=27,\ -x+y+2z=2$

Solution:

$$A=\begin{bmatrix}2&4&6\\3&8&5\\-1&1&2\end{bmatrix},\quad X=\begin{bmatrix}x\\y\\z\end{bmatrix},\quad B=\begin{bmatrix}22\\27\\2\end{bmatrix}$$

Transforming the augmented matrix to echelon form, we get

$$[A,B]=\left[\begin{array}{ccc|c}2&4&6&22\\3&8&5&27\\-1&1&2&2\end{array}\right]$$

$$\sim\left[\begin{array}{ccc|c}1&2&3&11\\3&8&5&27\\-1&1&2&2\end{array}\right]\ R_1\to\frac{R_1}{2}$$

$$\sim\left[\begin{array}{ccc|c}1&2&3&11\\0&2&-4&-6\\0&3&5&13\end{array}\right]\ \begin{array}{l}R_2\to R_2-3R_1\\R_3\to R_3+R_1\end{array}$$

$$\sim\left[\begin{array}{ccc|c}1&2&3&11\\0&1&-2&-3\\0&3&5&13\end{array}\right]\ R_2\to\frac{R_2}{2}$$

$$\sim\left[\begin{array}{ccc|c}1&2&3&11\\0&1&-2&-3\\0&0&11&22\end{array}\right]\ R_3\to R_3-3R_2$$

$x+2y+3z=11\ \ldots\ldots\ldots(1)$

$y-2z=-3\ \ldots\ldots\ldots(2)$

$11z=22\ \ldots\ldots\ldots(3)$

From (3) $\quad z=2$

From (2) $\quad y=-3+4=1$

From (1) $\quad x=11-6-2=3$

So, the solution is $(x=3,\ y=1,\ z=2)$

2. If $ax^2 + bx + c$ is divided by $x+3$, $x-5$ and $x-1$, the remainders are 21, 61 and 9 respectively. Find *a*, *b* and c (Use Gaussian elimination method.),

Solution:

$x+3=0,$	$x-5=0,$	$x-1=0,$
$x=-3$	$x=5$	$x=1$
$p(-3)=21$	$p(5)=61$	$p(1)=9$

$\Rightarrow \quad x = -3, 5, 1$

$p(x) = ax^2 + bx + c$

$x = -3 \quad \Rightarrow \quad p(-3) = a(-3)^2 + b(-3) + c$

$9a - 3b + c = 21 \ldots\ldots\ldots(1)$

$x = 5 \quad \Rightarrow \quad p(5) = a(5)^2 + b(5) + c$

$25a + 5b + c = 61 \ldots\ldots\ldots(2)$

$x = -3 \Rightarrow p(1) = a(1)^2 + b(1) + c$

$a + b + c = 9 \ldots\ldots\ldots(3)$

The matrix form of the system is AX=B, where

$$A = \begin{bmatrix} 9 & -3 & 1 \\ 25 & 5 & 1 \\ 1 & 1 & 1 \end{bmatrix}, \quad X = \begin{bmatrix} a \\ b \\ c \end{bmatrix}, \quad B = \begin{bmatrix} 21 \\ 61 \\ 9 \end{bmatrix}$$

$$[A, B] = \left[\begin{array}{ccc|c} 9 & -3 & 1 & 21 \\ 25 & 5 & 1 & 61 \\ 1 & 1 & 1 & 9 \end{array}\right]$$

$$\sim \left[\begin{array}{ccc|c} 1 & 1 & 1 & 9 \\ 25 & 5 & 1 & 61 \\ 9 & -3 & 1 & 21 \end{array}\right] \quad R_1 \leftrightarrow R_3$$

$$\sim \left[\begin{array}{ccc|c} 1 & 1 & 1 & 9 \\ 0 & -20 & -24 & -164 \\ 0 & -12 & -8 & -60 \end{array}\right] \quad \begin{array}{l} R_2 \to R_2 - 25R_1 \\ R_3 \to R_3 - 9R_1 \end{array}$$

$$\sim \left[\begin{array}{ccc|c} 1 & 1 & 1 & 9 \\ 0 & 5 & 6 & 41 \\ 0 & 3 & 2 & 15 \end{array}\right] \quad \begin{array}{l} R_2 \to \frac{R_2}{-4} \\ R_3 \to \frac{R_3}{-4} \end{array}$$

$$\sim \left[\begin{array}{ccc|c} 1 & 1 & 1 & 9 \\ 0 & 5 & 6 & 41 \\ 0 & 0 & -8 & -48 \end{array}\right] \quad R_3 \to 5R_3 - 3R_2$$

$a + b + c = 9 \ldots\ldots\ldots(1)$

$5b + 6c = 41 \ldots\ldots\ldots(2)$

$-8c = -48 \ldots\ldots\ldots(3)$

From (3) $\quad c = 6$

From (2) $\quad b = \frac{41-36}{5} = 1$

From (1) $\quad a = 9 - 6 - 1 = 2$

So, the solution is $(a = 2, \ b = 1, \ c = 6)$.

3. An amount of $Rs.$ 65,000 is invested in three bonds at the rates of 6%, 8% and 9% per annum respectively. The total annual income is $Rs.$ 4,800. The income from the third bond is $Rs.$ 600 more than that from the second bond. Determine the price of each bond. (Use Gaussian elimination method.)

Solution:

Let the price of three bond be x, y and z respectively.

$x + y + z = 65000 \ldots\ldots\ldots(1)$

$6\% \text{ of } x + 8\% \text{ of } y + 9\% \text{ of } z = 4800$

$\Rightarrow \quad 6x + 8y + 9z = 480000 \ldots\ldots(2)$

Income from third bond = Income from second bond + Rs.600

$9\% \text{ of } z = 8\% \text{ of } y + 600$

$\Rightarrow \quad 9z = 8y + 60000$

$-8y + 9z = 60000 \ldots\ldots\ldots(3)$

The matrix form of the system is AX=B, where

$$A = \begin{bmatrix} 1 & 1 & 1 \\ 6 & 8 & 9 \\ 0 & -8 & 9 \end{bmatrix}, \quad X = \begin{bmatrix} x \\ y \\ z \end{bmatrix}, \quad B = \begin{bmatrix} 65000 \\ 480000 \\ 60000 \end{bmatrix}$$

Transforming the augmented matrix to echelon form, we get

$$[A, B] = \left[\begin{array}{ccc|c} 1 & 1 & 1 & 65000 \\ 6 & 8 & 9 & 480000 \\ 0 & -8 & 9 & 60000 \end{array}\right]$$

$$\sim \left[\begin{array}{ccc|c} 1 & 1 & 1 & 65000 \\ 0 & 2 & 3 & 90000 \\ 0 & -8 & 9 & 60000 \end{array}\right] \quad R_2 \to R_2 - 6R_1$$

$$\sim \left[\begin{array}{ccc|c} 1 & 1 & 1 & 65000 \\ 0 & 2 & 3 & 90000 \\ 0 & 0 & 21 & 420000 \end{array}\right] \quad R_3 \to R_3 + 4R_2$$

$x + y + z = 65000 \ldots\ldots\ldots(1)$

$2y + 3z = 90000 \ldots\ldots\ldots(2)$

$21z = 420000 \ldots\ldots\ldots(3)$

From (3) $\quad z = 20000$

From (2) $\quad y = \frac{90000 - 60000}{2} = 15000$

From (1) $\quad x = 65000 - 20000 - 15000 = 30000$

So, the solution is

$(x = 30000, \ y = 15000, \ z = 20000)$

The price of 6%, 8% and 9% are Rs.30000, Rs.15000 and Rs.20000 respectively.

4. A boy is walking along the path $y = ax^2 + bx + c$ through the points $(-6,8), (-2,-12)$ and $(3,8)$. He wants to meet his friend at $P(7,60)$. Will he meet his friend? (Use Gaussian elimination method.)

Solution:

$$y = ax^2 + bx + c$$

The points $(-6,8)$

$\Rightarrow 8 = a(-6)^2 + b(-6) + c$

$\Rightarrow \quad 36a - 6b + c = 8 \ldots\ldots\ldots(1)$

The points $(-2,-12)$

$\Rightarrow -12 = a(-2)^2 + b(-2) + c$

$\Rightarrow \quad 4a - 2b + c = -12 \ldots\ldots(2)$

The points $(3,8)$

$\Rightarrow 8 = a(3)^2 + b(3) + c$

$\Rightarrow \quad 9a + 3b + c = 8 \ldots\ldots\ldots(3)$

The matrix form of the system is AX=B, where

$$A = \begin{bmatrix} 36 & -6 & 1 \\ 4 & -2 & 1 \\ 9 & 3 & 1 \end{bmatrix}, \quad X = \begin{bmatrix} a \\ b \\ c \end{bmatrix}, \quad B = \begin{bmatrix} 8 \\ -12 \\ 8 \end{bmatrix}$$

Transforming the augmented matrix to echelon form, we get

$$[A,B] = \left[\begin{array}{ccc|c} 36 & -6 & 1 & 8 \\ 4 & -2 & 1 & -12 \\ 9 & 3 & 1 & 8 \end{array}\right]$$

$$\sim \left[\begin{array}{ccc|c} 36 & -6 & 1 & 8 \\ 0 & -12 & 8 & -116 \\ 0 & 18 & 3 & 24 \end{array}\right] \begin{array}{l} R_2 \to 9R_2 - R_1 \\ R_3 \to 4R_3 - R_1 \end{array}$$

$$\sim \left[\begin{array}{ccc|c} 36 & -6 & 1 & 8 \\ 0 & 3 & -2 & 29 \\ 0 & 6 & 1 & 8 \end{array}\right] \begin{array}{l} R_2 \to \frac{R_2}{-4} \\ R_3 \to \frac{R_3}{3} \end{array}$$

$$\sim \left[\begin{array}{ccc|c} 36 & -6 & 1 & 8 \\ 0 & 3 & -2 & 29 \\ 0 & 0 & 5 & -50 \end{array}\right] R_3 \to R_3 - 2R_1$$

$36a - 6b + c = 8 \ldots\ldots\ldots(1)$

$3b - 2c = 29 \quad \ldots\ldots \quad (2)$

$5c = -50 \ldots\ldots(3)$

From (3) $\qquad c = -10$

From (2) $\quad b = \frac{29-20}{3} = 3$

From (1) $\quad a = \frac{8+10+18}{36} = 1$

So, the solution is $(a = 1, \ b = 3, \ c = -10)$

$y = ax^2 + bx + c \quad \Rightarrow \quad y = x^2 + 3x - 10$

$x = 7 \quad \Rightarrow \quad y = 49 + 21 - 10 = 60$

The point (7,60) satisfies the equation

$y = x^2 + 3x - 10$, hence the boy will meet friend at (7, 60).

EXERCISE 2.4

7. Show that

(i) $(2+i\sqrt{3})^{10}-(2-i\sqrt{3})^{10}$ is purely imaginary

(ii) $\left(\frac{19-7i}{9+i}\right)^{12}-\left(\frac{20-5i}{7-6i}\right)^{12}$ is real.

Solution:

(i) z is purely imaginary if and only if $z=-\bar{z}$

Let $z=(2+i\sqrt{3})^{10}-(2-i\sqrt{3})^{10}$

$$\bar{z}=\overline{(2+i\sqrt{3})^{10}-(2-i\sqrt{3})^{10}}$$

$$=\overline{(2+i\sqrt{3})^{10}}-\overline{(2-i\sqrt{3})^{10}}$$

$$=(2-i\sqrt{3})^{10}-(2+i\sqrt{3})^{10}$$

$z=-\bar{z}$ (z is purely Imaginary)

(ii) z is real imaginary if and only if $z=\bar{z}$

Let $z=\left(\frac{19-7i}{9+i}\right)^{12}+\left(\frac{20-5i}{7-6i}\right)^{12}$

$=\left(\frac{19-7i}{9+i}\times\frac{9-i}{9-i}\right)^{12}+\left(\frac{20-5i}{7-6i}\times\frac{7+6i}{7+6i}\right)^{12}$

$=\left(\frac{(171-7)+i(-63-19)}{82}\right)^{12}+\left(\frac{(140+30)+i(120-35)}{85}\right)^{12}$

$=\left(\frac{164-82i}{82}\right)^{12}+\left(\frac{170+85i}{85}\right)^{15}$

$z=(2-i)^{12}+(2+i)^{12}$

$\bar{z}=\overline{(2-i)^{12}+(2+i)^{12}}$

$=\overline{(2-i)^{12}}+\overline{(2+i)^{12}}$

$=(2+i)^{12}+(2-i)^{12}$

$\bar{z}=z$ (z is purely real) .

Example 2.15 Let Z_1, Z_2 and Z_3 be complex numbers such that $|z_1|=|z_2|=|z_3|=r>0$ and $z_1+z_2+z_3\neq 0$ prove that $\left|\frac{z_1z_2+z_2z_3+z_1z_3}{z_1+z_2+z_3}\right|=$ r

Solution:

$$|z_1|=|z_2|=|z_3|=r$$

$$z_1\bar{z}_1=z_2\bar{z}_2=z_3\bar{z}_3=r^2$$

$\Rightarrow\quad z_1=\frac{r^2}{\bar{z}_1}$	$z_2=\frac{r^2}{\bar{z}_2}$	$z_3=\frac{r^2}{\bar{z}_3}$

$$z_1+z_2+z_3=\frac{r^2}{\bar{z}_1}+\frac{r^2}{\bar{z}_2}+\frac{r^2}{\bar{z}_3}$$

$$=r^2\left(\frac{\bar{z}_1\bar{z}_2+\bar{z}_2\bar{z}_3+\bar{z}_1\bar{z}_3}{\bar{z}_1\bar{z}_2\bar{z}_3}\right)$$

$$|z_1+z_2+z_3|=r^2\left|\frac{\overline{z_1z_2+z_2z_3+z_1z_3}}{\overline{z_1z_2z_3}}\right|$$

$$=r^2\frac{|z_1z_2+z_2z_3+z_1z_3|}{|z_1|\,|z_2|\,|z_3|}$$

$$=r^2\frac{|z_1z_2+z_2z_3+z_1z_3|}{r^3}$$

$$|z_1+z_2+z_3|=\frac{|z_1z_2+z_2z_3+z_1z_3|}{r}$$

$$\left|\frac{z_1z_2+z_2z_3+z_1z_3}{z_1+z_2+z_3}\right|=r$$

EXERCISE 2.5

7. If z_1, z_2 and z_3 be three complex numbers such that $|z_1|=1$, $|z_2|=2$, $|z_3|=3$, and $|z_1+z_2+z_3|=1$ show that $|9z_1z_2+4z_1z_3+z_2z_3|=6$.

Solution:

$|z_1|=1,\quad |z_1|^2=1,\quad z_1\bar{z}_1=1\qquad z_1=\frac{1}{\bar{z}_1}$

$|z_2|=2,\quad |z_2|^2=4\quad z_2\bar{z}_2=4\qquad z_2=\frac{4}{\bar{z}_2}$

$|z_3|=3\quad |z_3|^2=9\quad z_3\bar{z}_3=9\qquad z_3=\frac{9}{\bar{z}_3}$

$$z_1+z_2+z_3=\frac{1}{\bar{z}_1}+\frac{4}{\bar{z}_2}+\frac{9}{\bar{z}_3}$$

$$=\frac{\bar{z}_3\bar{z}_2+4\bar{z}_2\bar{z}_1+9\bar{z}_1\bar{z}_2}{\bar{z}_1\bar{z}_2\bar{z}_3}$$

$$|z_1+z_2+z_3|=\left|\frac{\overline{z_3z_2+4z_2z_1+9z_1z_2}}{\overline{z_1z_2z_3}}\right|$$

$$=\frac{|9z_1z_2+4z_1z_3+z_2z_3|}{|z_1|\,|z_2|\,|z_3|}$$

$$1=\frac{|9z_1z_2+4z_1z_3+z_2z_3|}{6}$$

$$|9z_1z_2+4z_1z_3+z_2z_3|=6$$

EXERCISE 2.6

2. If $z = x + iy$ is a complex number such that $Im\left(\frac{2z+1}{iz+1}\right) = 0$ show that the locus of z is $2x^2 + 2y^2 + x - 2y = 0$.

Solution:

Let $z = x + iy$

$$Im\left[\frac{2(x+iy)+1}{xi-y+1}\right] = 0$$

$$Im\left[\frac{(2x+1)+i2y}{(1-y)+ix} \times \frac{(1-y)-ix}{(1-y)-ix}\right] = 0$$

$$\frac{-2x^2 - x - 2y^2 + 2y}{(1-y)^2 + x^2} = 0$$

$-2x^2 - x - 2y^2 + 2y = 0.$

The locus of P is $2x^2 + 2y^2 + x - 2y = 0$

Mathematics is the Queen of Science

Example 3.6 Form the equation whose roots are the squares of the roots of the cubic equation $\boldsymbol{x^3 + ax^2 + bx + c = 0}$.

Solution:8

Since α , β , and γ are the roots of the equation $x^3 + ax^2 + bx + c = 0$

$\sum_1 = \alpha + \beta + \gamma = -a \qquad \sum_2 = \alpha\beta + \beta\gamma + \gamma\alpha = b \qquad \sum_3 = \alpha\beta\gamma = -c$

We have to form the equation whose roots are α^2, β^2 and γ^2

$\sum_1 = \alpha^2 + \beta^2 + \gamma^2 = (\alpha + \beta + \gamma)^2 - 2(\alpha\beta + \beta\gamma + \gamma\alpha)$

$= (-a)^2 - 2b = a^2 - 2b$

$\sum_2 = \alpha^2\beta^2 + \beta^2\gamma^2 + \gamma^2\alpha^2 = (\alpha\beta + \beta\gamma + \gamma\alpha)^2 - 2[(\alpha\beta)(\beta\gamma) + (\beta\gamma)(\gamma\alpha) + (\gamma\alpha)(\alpha\beta)]$

$= b^2 - 2(-c)(-a) = b^2 - 2ac$

$\sum_3 = \alpha^2\beta^2\gamma^2 = (\alpha\beta\gamma)^2 = c^2$

Hence, the required equation is

$x^3 - (\alpha^2 + \beta^2 + \gamma^2)x^2 + (\alpha^2\beta^2 + \beta^2\gamma^2 + \gamma^2\alpha^2)x - \alpha^2\beta^2\gamma^2 = 0$

$$x^3 - (a^2 - 2b)x^2 + (b^2 - 2ac)x - c^2 = 0$$

EXERCISE 3.1

4. Solve the equation $\boldsymbol{3x^3 - 16x^2 + 23x - 6 = 0}$. If the product of two roots is 1.

Solution:

Let α, β and γ are the roots of the cubic equation

$3x^3 - 16x^2 + 23x - 6 = 0 \qquad (3 \div) \Rightarrow \qquad x^3 - \frac{16}{3}x^2 + \frac{23}{3}x - 2 = 0$

The product of two roots is 1. $\quad \alpha\beta = 1 \Rightarrow \quad \beta = \frac{1}{\alpha}$

$\sum_3 = \alpha\beta\gamma$ $= -(-2) = 2$	$\sum_2 = \alpha\beta + \beta\gamma + \gamma\alpha = \frac{23}{3}$	$\sum_1 = \alpha + \beta + \gamma = -\left(-\frac{16}{3}\right) = \frac{16}{3}$
$\alpha\beta\gamma = 2$ $\Rightarrow \alpha\left(\frac{1}{\alpha}\right)\gamma = 2$ $\Rightarrow \quad \gamma = 2$	$\alpha + \beta + \gamma = \frac{16}{3}$ $\alpha + \frac{1}{\alpha} = \frac{16}{3} - 2 = \frac{10}{3}$ $\frac{\alpha^2 + 1}{\alpha} = \frac{10}{3}$ $3\alpha^2 - 10\alpha + 3 = 0$ $(\alpha - 3)(3\alpha - 1) = 0$ $\alpha = \frac{1}{3} \quad and \quad \alpha = 3$	When $\alpha = \frac{1}{3} \quad \beta = \frac{1}{\alpha} = 3$ When $\alpha = 3 \quad \beta = \frac{1}{\alpha} = \frac{1}{3}$

Thus the roots are 3, $\frac{1}{3}$ and 2 (or) $\frac{1}{3}$, 3, 2.

6. Solve the equation $x^3 - 9x^2 + 14x + 24 = 0$ if it is given that two of its roots are in the ratio $3:2$

Solution:

Let α, β and γ are the roots of the cubic equation $x^3 - 9x^2 + 14x + 24 = 0$

Two of its roots are in the ratio 3: 2 α, $\beta = 3p$ and $\gamma = 2p$

$a = 1$ $b = -9$

$c = 14$ $d = 24$

$\sum_1 = \alpha + \beta + \gamma = -(-9) = 9$	$\sum_2 = \alpha\beta + \beta\gamma + \gamma\alpha = 14$	$\sum_3 = \alpha\beta\gamma = -24$
$\alpha + 3p + 2p = 9$ $\alpha + 5p = 9$ $\alpha = 9 - 5p$ When $p = 2$, $\alpha = 9 - 10 = -1$	$3p\alpha + 6p^2 + 2p\alpha = 14$ $6p^2 + 5\,\alpha\, p = 14$ $6p^2 + 5(9 - 5p)p + 6p^2 = 14$ $6p^2 + 45p - 25p^2 = 14$ $-19p^2 + 45p - 14 = 0$ $19p^2 - 45p + 14 = 0$ $(p - 2)(19p - 7) = 0$ $p = 2$ and $p = \frac{7}{19}$ $(not\ possible)$	$\alpha\beta\gamma = -24$ $\Rightarrow \alpha(3p)(2p) = -24$ $\Rightarrow$ $6\alpha\, p^2 = -24$

Thus the roots are $-1, 6, 4$

Example 3.15 If $2 + i$ and $3 - \sqrt{2}$ are roots of the equation
$x^6 - 13x^5 + 62x^4 - 126x^3 + 65x^2 + 127x - 140 = 0$.
Find all roots.

Solution:

$2 + i$ and $3 - \sqrt{2}$ are roots of the given equation.

$x = 2 + i$

$(x-2)^2 = -1 \qquad x^2 - 4x + 5 = 0$

$x = 3 - \sqrt{2} \Rightarrow (-(x-3))^2 = 2 \Rightarrow x^2 - 6x + 7 = 0$

$x^6 - 13x^5 + 62x^4 - 126x^3 + 65x^2 + 127x - 140$

$= (x^2 - px - 4)(x^2 - 4x + 5)(x^2 - 6x + 7)$

Comparing the coefficient of x term,

$127 = -35p + (-4)(-4)(7) + (-4)(5)(-6)$

$127 = -35p + 112 + 120$

$-35p = -105 \qquad p = 3$

$x^2 - px - 4 = 0 \Rightarrow x^2 - 3x - 4 = 0$

$x = -1 \quad and\ x = 4$

The roots are

$2 + i,\ 2 - i,\ 3 + \sqrt{2},\ 3 - \sqrt{2}, -1\ and\ 4.$

Example 3.21 If the roots of $x^3 + px^2 + qx + r = 0$ are in H.P., prove that $9pqr = 27r^2 + 2q^3$.

Solution:

Let the roots be in H.P. Then, their reciprocals are in A.P. and roots of the equation

$$x^3 + px^2 + qx + r = 0$$

$$\left(\frac{1}{x}\right)^3 + p\left(\frac{1}{x}\right)^2 + q\left(\frac{1}{x}\right) + r = 0$$

$rx^3 + qx^2 + px + 1 = 0$……….(1)

Since the roots of (1) are in A.P., we can assume them as a-d, a a+d

Applying the Vieta's formula, we get

$\sum_1 = a - d + a + a + d = 3a = -\frac{q}{r}$

$a = -\frac{q}{3r}$

Put $x = a \quad rx^3 + qx^2 + px + 1 = 0$

$r\left(-\frac{q}{3r}\right)^3 + q\left(-\frac{q}{3r}\right)^2 + p\left(-\frac{q}{3r}\right) + 1 = 0$

$9pqr = 2q^3 + 27r^2$

EXERCISE 3.3

4. Determine k and solve the equation $2x^3 - 6x^2 + 3x + k = 0$.if one of its roots is twice the sum of the other two roots.

Solution:

Let α, β, γ be the roots of the given equation.

One of its roots is twice the sum of the other two roots. $\quad \alpha = 2(\beta + \gamma)$

$\sum_1 = \alpha + \beta + \gamma = -(-3) = 3$

$2\alpha + 2(\beta + \gamma) = 6 \qquad 3\alpha = 6 \qquad \alpha = 2$

2	2	-6	3	k
		4	-4	-2
	2	-2	-1	0

The value of k is

$k - 2 = 0 \quad k = 2 \qquad 2x^2 - 2x - 1 = 0$

$x = \frac{2 \pm \sqrt{4+8}}{4} = \frac{2 \pm 2\sqrt{3}}{4} = \frac{1 \pm \sqrt{3}}{2}$

Solution of the given equation $\frac{1+\sqrt{3}}{2}, \frac{1-\sqrt{3}}{2}, 2.$

5. Find all zeros of the polynomial $x^6 - 3x^5 - 5x^4 + 22x^3 - 39x^2 - 39x + 135 = 0$, if it is known that $1 + 2i$ and $\sqrt{3}$ are two of its zeros.

Solution:

$1 + 2i$ and $\sqrt{3}$ are roots of the given equation.

$x = 1 + 2i \qquad (x-1)^2 = 4(-1)$

$x^2 - 2x + 5 = 0$

$x = \sqrt{3} \qquad x^2 = 3 \qquad x^2 - 3 = 0$

$x^6 - 3x^5 - 5x^4 + 22x^3 - 39x^2 - 39x + 139$

$= (x^2 - px - 9)(x^2 - 2x + 5)(x^2 - 3)$

Comparing the coefficient of x term,

$-39 = 15p + (-9)(-2)(-3)$

$-39 = 15p - 54 \qquad 15p = 15 \qquad p = 1$

$x^2 - px - 9 = 0 \qquad x^2 - x - 9 = 0$

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a} = \frac{1 \pm \sqrt{1 + 36}}{2(1)} = \frac{1 \pm \sqrt{37}}{2}$$

The roots are

$1 + 2i,\ 1 - 2i,\ \sqrt{3}, -\sqrt{3}, \frac{1+\sqrt{37}}{2}\ and\ \frac{1-\sqrt{37}}{2}.$

Example 3.25 Solve the equation $x^3 - 5x^2 - 4x + 20 = 0$.

Solution:

-2	1	-5	-4	20
		-2	14	-20
	1	-7	10	0

$x^2 - 7x + 10 = 0 \quad (x-5)(x-2) = 0$

The roots are $x = -2, 2, 5$

Example 3.26 Find the roots of $2x^3 + 3x^2 + 2x + 3$.

Solution:

$2x^3 + 3x^2 + 2x + 3 = 0$

$2x^3 + 2x + 3x^2 + 3 = 0$

$2x(x^2+1) + 3(x^2+1) = 0$

$(2x+3)(x^2+1) = 0 \quad 2x + 3 = 0 \quad x = -\frac{3}{2}$

$x^2 + 1 = 0 \quad x = \pm i$

The roots are $x = -\frac{3}{2}, -i, i$

Example 3.28 Solve the following equation: $x^4 - 10x^3 + 26x^2 - 10x + 1 = 0$.

Solution:

$x^4 + 1 - 10x^3 - 10x + 26x^2 = 0$

$x^2\left[x^2 + \frac{1}{x^2} - 10x - \frac{10}{x} + 26\right] = 0$

$\left(x^2 + \frac{1}{x^2}\right) - 10\left(x + \frac{1}{x}\right) + 26 = 0$

Let $y = x + \frac{1}{x}$

$\Rightarrow y^2 = \left(x + \frac{1}{x}\right)^2 \Rightarrow y^2 - 2 = x^2 + \frac{1}{x^2}$

$y^2 - 2 - 10y + 26 = 0$

$y^2 - 10y + 24 = 0 \quad (y-6)(y-4) = 0$

When $y = 4 \quad 4 = x + \frac{1}{x} \quad x^2 - 4x + 1 = 0$

When $y = 6 \quad 6 = x + \frac{1}{x} \quad x^2 - 6x + 1 = 0$

$(x-2)^2 = -1 + (-2)^2 \quad (x-3)^2 = -1 + (-3)^2$

$x = \pm\sqrt{3} + 2 \quad x = \pm 2\sqrt{2} + 3$

The roots are

$\sqrt{3} + 2, -\sqrt{3} + 2, \ 2\sqrt{2} + 3, \ -2\sqrt{2} + 3.$

EXERCISE 3.5

5. Solve the equations

(i) $6x^4 - 35x^3 + 62x^2 - 35x + 6 = 0$

(ii) $x^4 + 3x^3 - 3x - 1 = 0$

Solution:

(i) $6x^4 - 35x^3 + 62x^2 - 35x + 6 = 0$

2	6	-35	62	-35	6
		12	-46	32	-6
3	6	-23	16	-3	0
		18	-15	3	
	6	-5	1	0	

$6x^2 - 5x + 1 = 0$

$(3x-1)(2x-1) = 0$

The roots are $x = 2, 3, \frac{1}{3}, \frac{1}{2}$

(ii) $x^4 + 3x^3 - 3x - 1 = 0$

1	1	3	0	-3	-1
		1	4	4	1
-1	1	4	4	1	0
		-1	-3	-1	
	1	3	1	0	

$x^2 - 3x + 1 = 0$

$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a} = \frac{-(-3) \pm \sqrt{9-4}}{2(1)} \quad x = \frac{3 \pm \sqrt{5}}{2}$

The roots are $x = \frac{3-\sqrt{5}}{2}, \frac{3+\sqrt{5}}{2}, 1, -1.$

7. Solve the equation **$6x^4 - 5x^3 - 38x^2 - 5x + 6 = 0$** if it is known that $\frac{1}{3}$ is a solution.

Solution:

One root $= \frac{1}{3}$ Other root $= 3$

$\frac{1}{3}$	6	-5	-38	-5	6
		2	-1	-13	-6
3	6	-3	-39	-18	0
		18	45	18	
	6	15	6	0	

$6x^2 + 15x + 6 = 0$

$2x^2 + 5x + 2 = 0 \quad (2x+1)(x+2) = 0$

$x = \frac{-1}{2}$ and $x = -2$

The roots are $x = -2, 3, \frac{-1}{2}, , \frac{1}{3}$

Example 4.4 Find the domain of $sin^{-1}(2-3x^2)$

Solution:

$y = \sin^{-1} x$ if and only if $x = \sin y$ for $-1 \le x \le 1$.

Let $y = sin^{-1}(2-3x^2)$

$$-1 \le 2-3x^2 \le 1$$
$$-1-2 \le -3x^2 \le 1-2$$
$$-3 \le -3x^2 \le -1$$
$$3 \ge 3x^2 \ge 1$$
$$\frac{1}{3} \le x^2 \le 1$$
$$\pm\frac{1}{\sqrt{3}} \le |x| \le \pm 1$$
$$x \in \left[-1, \frac{-1}{\sqrt{3}}\right] \cup \left[\frac{1}{\sqrt{3}}, 1\right]$$

EXERCISE 4.1

6. Find the domain of the following

(i) $f(x) = sin^{-1}\left(\frac{x^2+1}{2x}\right)$

Solution:

(i) $f(x) = sin^{-1}\left(\frac{x^2+1}{2x}\right)$

$y = \sin^{-1} x$ if and only if $x = \sin y$ for $-1 \le x \le 1$.

$$-1 \le \frac{x^2+1}{2x} \le 1$$
$$-2x \le x^2+1 \le 2x$$

$-2x \le x^2+1$	$x^2+1 \le 2x$
$0 \le x^2+2x+1$	$x^2-2x+1 \le 0$
$0 \le (x+1)^2$	$(x-1)^2 \le 0$
$-1 \le x$	$x \le 1$

The domain of $f(x) = sin^{-1}\left(\frac{x^2+1}{2x}\right)$ is $[-1, 1]$

EXERCISE 4.3

4. Find the value of

(ii) $sin\left(tan^{-1}\left(\frac{1}{2}\right) - cos^{-1}\left(\frac{4}{5}\right)\right)$

(iii) $cos\left(sin^{-1}\left(\frac{4}{5}\right) - tan^{-1}\left(\frac{3}{4}\right)\right)$

Solution:

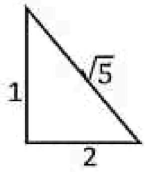
(ii) $sin\left(tan^{-1}\left(\frac{1}{2}\right) - cos^{-1}\left(\frac{4}{5}\right)\right)$

Let $x = tan^{-1}\left(\frac{1}{2}\right)$

$\tan x = \frac{1}{2}$

$\sin x = \frac{1}{\sqrt{5}}$

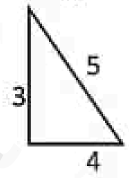
$\cos x = \frac{2}{\sqrt{5}}$



Let $y = cos^{-1}\left(\frac{4}{5}\right)$

$\cos y = \frac{4}{5}$

$\sin y = \frac{3}{5}$



$\sin(x-y) = \sin x \cos y - \cos x \sin y$

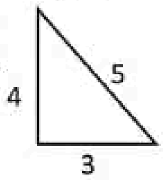
$$\sin(x-y) = \left(\frac{1}{\sqrt{5}} \times \frac{4}{5}\right) - \left(\frac{2}{\sqrt{5}} \times \frac{3}{5}\right)$$
$$= \frac{4-6}{5\sqrt{5}} = \frac{-2}{5\sqrt{5}} = \frac{-2\sqrt{5}}{25}$$

(iii) $cos\left(sin^{-1}\left(\frac{4}{5}\right) - tan^{-1}\left(\frac{3}{4}\right)\right)$

Let $x = sin^{-1}\left(\frac{4}{5}\right)$

$\sin x = \frac{4}{5}$

$\cos x =$

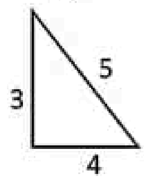


Let $y = tan^{-1}\left(\frac{3}{4}\right)$

$\tan y = \frac{3}{4}$

$\sin y = \frac{3}{5}$

$\cos y = \frac{4}{5}$



$\cos(x-y) = \cos x \cos y + \sin x \sin y$

$$\cos(x-y) = \left(\frac{3}{5} \times \frac{4}{5}\right) + \left(\frac{4}{5} \times \frac{3}{5}\right) = \frac{12+12}{25} = \frac{24}{25}$$

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Example 5.2 Find the equation of the circle described on the chord $3x + y + 5 = 0$ of the circle $x^2 + y^2 = 16$ as diameter.

Solution:

Equation of the circle passing through the points of intersection of the chord

$$x^2 + y^2 - 16 + \lambda(3x + y + 5) = 0$$
$$x^2 + y^2 + 3\lambda x + \lambda y + 5\lambda - 16 = 0$$

Centre $(-g, -f) = \left(\frac{-3\lambda}{2}, \frac{-\lambda}{2}\right)$ lies on the chord.

$$3\left(\frac{-3\lambda}{2}\right) - \frac{\lambda}{2} + 5 = 0$$
$$\frac{-9\lambda - \lambda}{2} + 5 = 0$$

$-10\lambda = -10$ $\lambda = 1$

The equation of the circle is

$x^2 + y^2 + 3x + y - 11 = 0.$

Example 5.10 Find the equation of the circle passing through the points (1,1), (2,-1), and (3,2).

Solution:

Let the general equation of the circle be

$x^2 + y^2 + 2gx + 2fy + c = 0 \dots\dots\dots (1)$

It passes through (1, 1)

$1^2 + 1^2 + 2g(1) + 2f(1) + c = 0$

$$2g + 2f + c + 2 = 0 \dots\dots (2)$$

It passes through (2, -1)

$$2^2 + (-1)^2 + 2g(2) + 2f(-1) + c = 0$$
$$4g - 2f + c + 5 = 0 \dots\dots (3)$$

It passes through (3, 2)

$3^2 + 2^2 + 2g(3) + 2f(2) + c = 0$

$$6g + 4f + c + 13 = 0 \dots\dots (4)$$

(2) – (3) gives $-2g + 4f - 3 = 0 \dots (5)$

(4) – (3) gives $2g + 6f + 8 = 0 \dots (6)$

(5) + (6) gives $f = -\frac{1}{2}$

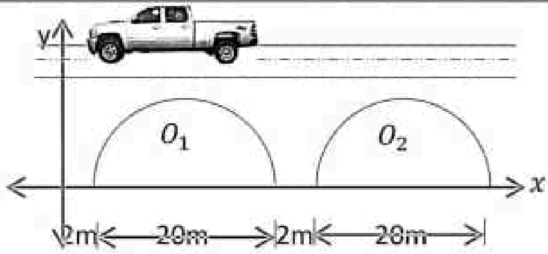
Substituting $f = -\frac{1}{2}$ in eqn(6) $g = -\frac{5}{2}$

Substituting $f = -\frac{1}{2}$ and $g = -\frac{5}{2}$ in eqn(2), $c = 4$

The required equation is

$$x^2 + y^2 + 2gx + 2fy + c = 0$$
$$x^2 + y^2 + 2\left(-\frac{5}{2}\right)x + 2\left(-\frac{1}{2}\right)y + 4 = 0$$
$$x^2 + y^2 - 5x - y + 4 = 0$$

Example 5.13 A road bridge over an irrigation canal have two semi-circular vents each with a span of 20m and the supporting pillars of width 2m. Use to write the equations that model the arches.



Solution:

Let O_1, O_2 be the centres of the two semi circular vents.

First vent with centre $O_1(12, 0)$ and r =10 yields equation to first semicircle as

$$(x - 12)^2 + (y - 0)^2 = 10^2$$
$$x^2 - 24x + 144 + y^2 = 100$$
$$x^2 + y^2 - 24x + 44 = 0$$

Second vent with centre $O_2(34, 0)$and r =10 yields equation to second vent as

$$(x - 34)^2 + (y - 0)^2 = 10^2$$
$$x^2 - 68x + 11156 + y^2 = 100$$
$$x^2 + y^2 - 68x + 1056 = 0$$

EXERCISE 5.1

4. Find the equation of the circle with centre (2, 3) and passing through the intersection of the lines $3x - 2y - 1 = 0$ and $4x + y - 27 = 0$.

Solution:

Given: Centre $(h, h) = (2, 3)$

The required equation is, $(x - h)^2 + (y - k)^2 = r^2$

$$(x - 2)^2 + (y - 3)^2 = r^2$$

Point of intersection:

$3x - 2y - 1 = 0$ $\quad$ $3x - 2y - 1 = 0$

$4x + y - 27 = 0$ $\quad$ $8x + 2y - 54 = 0$

$11x - 55 = 0$ $\quad$ $x = 5$

Substitution $x = 5$ in eqn(1) $3(5) - 2y - 1 = 0$

$-2y = -14$ $\quad$ $y = 7$

Point of intersection $(x, y) = (5, 7)$

The required equation passes through (5, 7)

$(5 - 2)^2 + (7 - 3)^2 = r^2$ $\quad$ $r = 5$

The equation of the circle is $(x - 2)^2 + (y - 3)^2 = 5^2$

$$x^2 + y^2 - 4x - 6y + 13 = 25$$
$$x^2 + y^2 - 4x - 6y - 12 = 0$$

6. Find the equation of the circle through the points (1,0), (-1,0) and (0,1).

Solution:

Given: centre (0,0) radius $r = 1$

The required equation is

$(x-0)^2 + (y-0)^2 = 1^2 \qquad x^2 + y^2 = 1$

Example 5.17 Find the vertex, focus, directrix, and length of the lactus rectum of the parabola $x^2 - 4x - 5y - 1 = 0$.

Solution:

The parabola $x^2 - 4x - 5y - 1 = 0$

$x^2 - 4x = 5y + 1 \qquad (x-2)^2 = 5y + 1 + (2)^2$

$(x-2)^2 = 5(y+1)$ is open upwards.

	X, Y	$x = X + 2$, $y = Y - 1$
Vertex (0,0)	(0, 0)	(2, -1)
Focus $(0, a)$	$\left(0, \frac{5}{4}\right)$	$\left(2, \frac{1}{4}\right)$
Directrix $y = -a$	$Y = -\frac{5}{4}$	$y = \frac{-5}{4} - 1 = \frac{-9}{4}$
Length of lactus rectum	$4a = 5$	$4a = 5$

Example 5.19 Find the equation of the ellipse whose eccentricity is $\frac{1}{2}$, one of the foci is $(2,3)$ and a directrix is $x = 7$. Also find the length of the major and minor axes of the ellipse.

Solution:

$\frac{FM}{PM} = e \qquad FM^2 = PM^2e^2$

$$(x-ae)^2 + (y-0)^2 = e^2\left[\left(x - \frac{a}{e}\right)^2 + (0-0)^2\right]$$

Foci $(ae, 0) = (2,3) \quad e = \frac{1}{2} \quad$ Directrix $x = \frac{a}{e} = 7$

$$(x-2)^2 + (y-3)^2 = \left(\frac{1}{2}\right)^2 (x-7)^2$$

$$3x^2 - 2x + 4y^2 - 24y + 3 = 0$$

$$3\left(x^2 - \frac{2}{3}x\right) + 4(y^2 - 6y) = -3$$

$$3\left(x - \frac{1}{3}\right)^2 + 4(y-3)^2 = 3\left(\frac{1}{9}\right) + 4(9) - 3$$

$$3\left(x - \frac{1}{3}\right)^2 + 4(y-3)^2 = \frac{100}{3}$$

$$\frac{\left(x - \frac{1}{3}\right)^2}{\frac{100}{9}} + \frac{(y-3)^2}{\frac{100}{12}} = 1$$

Therefore, the length of major axis = 2a

$$= 2\sqrt{\frac{100}{9}} = \frac{20}{3}$$

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You Keep moving result in Sucuess

Example 5.20 Find the foci, vertices and length of major and minor axis of the conic $4x^2 + 36y^2 + 40x - 288y + 532 = 0$

Solution:

$$4x^2 + 36y^2 + 40x - 288y + 532 = 0$$

Dividing by $(\div 4)$

$$x^2 + 9y^2 + 10x - 72y + 133 = 0$$

$$[x^2 + 10x] + [9y^2 - 72y] = -133$$

$$[x^2 + 10x] + 9[y^2 - 8y] = -133$$

$$(x+5)^2 + 9(y-4)^2 = -133 + 1(5)^2 + 9(-4)^2$$

$$(x+5)^2 + 9(y-4)^2 = -133 + 25 + 144$$

$$(x+5)^2 + 9(y-4)^2 = 36$$

$\div 36$ $\quad \frac{(x+5)^2}{36} + \frac{(y-4)^2}{4} = 1 \quad \Rightarrow \quad \frac{X^2}{36} + \frac{Y^2}{4} = 1$

Here $a^2 = 36$, $a = 6$	$b^2 = 4$, $b = 2$	$X = x + 5 \Rightarrow x = X - 5$
$(ae)^2 = c^2 = a^2 - b^2 = 36 - 4$; $(ae)^2 = c^2 = 32$; $c = ae = 4\sqrt{2}$		$Y = y - 4 \Rightarrow y = Y + 4$

Major axis : X- axis

ELLIPSE	Referred to X, Y	Referred to x, y	
		$x = X - 5$	$y = Y + 4$
Centre	$C(0,0)$	$C(-5, 4)$	
Foci $(\pm ae, 0)$	$(4\sqrt{2}, 0)$	$F_1(4\sqrt{2} - 5,\ 4)$	
	$(-4\sqrt{2}, 0)$	$F_2(-4\sqrt{2} - 5,\ 4)$	
Vertices $(\pm a,\ 0)$	$(6, 0)$	$A(1,\ 4)$	
	$(-6, 0)$	$A'(-11,\ 4)$	
Length of the major axis	$2a$	12 units	
Length of the minor axis	$2b$	4 unit	

Example 5.21For the ellipse $4x^2 + y^2 + 24x - 2y + 21 = 0$, find the centre, vertices, and the foci. Also prove that the length of latus rectum is 2.

Solution:

$$4x^2 + y^2 + 24x - 2y + 21 = 0$$

$$[4x^2 + 24x] + [y^2 - 2y] = -21$$

$$4[x^2 + 6x] + [y^2 - 2y] = -21$$

$$4(x+3)^2 + (y-1)^2 = -21 + 4(3)^2 + 1(-1)^2$$

$$4(x+3)^2 + (y-1)^2 = -21 + 36 + 1$$

$$4(x+3)^2 + (y-1)^2 = 16$$

$\div 16$ $\quad \frac{(x+3)^2}{4} + \frac{(y-1)^2}{16} = 1 \quad \Rightarrow \quad \frac{X^2}{4} + \frac{Y^2}{16} = 1$

Here $a^2 = 16$, $a = 4$	$b^2 = 4$, $b = 2$	$X = x + 3 \Rightarrow x = X - 3$
$(ae)^2 = c^2 = a^2 - b^2 = 16 - 4$; $(ae)^2 = c^2 = 12$; $c = ae = \pm 2\sqrt{3}$		$Y = y - 1 \Rightarrow y = Y + 1$

Major axis : Y- axis

ELLIPSE	Referred to X, Y	Referred to x, y	
		$x = X - 3$	$y = Y + 1$
Centre	$C(0,0)$	$C(-3, 1)$	
Foci $(\pm ae,\ 0)$	$(0, 2\sqrt{3})$	$F_1(-3, 1 + 2\sqrt{3})$	
	$(0, -2\sqrt{3})$	$F_2(-3, 1 - 2\sqrt{3})$	
Vertices $(0, \pm a)$	$(0, 4)$	$A(-3, 5)$	
	$(0, -4)$	$A'(-3, -3)$	
Length of the major axis	$2a$	8 units	
Length of the minor axis	$2b$	4 unit	
Length of Lactus rectum	$\frac{2b^2}{a}$	2 units	

Example 5.24 Find the centre, foci, and eccentricity of the hyperbola $11x^2 - 25y^2 - 44x + 50y - 256 = 0$

Solution:

$$11x^2 - 25y^2 - 44x + 50y - 256 = 0$$

$$11x^2 - 44x - 25y^2 + 50y - 256 = 0$$

$$11(x^2 - 4x) - 25(y^2 - 2y) - 256 = 0$$

$$11(x-2)^2 - 25(y-1)^2 = 256 + 11(-2)^2 - 25(-1)^2$$

$$11(x-2)^2 - 25(y-1)^2 = 256 + 44 - 25$$

$$11(x-2)^2 - 25(y-1)^2 = 275$$

$$\frac{(x-2)^2}{25} - \frac{(y-1)^2}{11} = 1$$

Centre (2,1), $c^2 = a^2 + b^2 = 25 + 11 = 36$

$ae = c = \pm 6$ and $e = \frac{c}{a} = \frac{6}{5}$

Foci $(ae, 0) = (6+2,\ 0+1) = (8,1)$

$(-ae, 0) = (-6+2,\ 0+1) = (-4,1)$

The coordinates of foci are (8,1) and (-4,1).

EXERCISE 5.2

4. Find the vertex, focus, equation of directrix and length of the latus rectum of the following:

(iv) $x^2 - 2x + 8y + 17 = 0$

(v) $y^2 - 4y - 8x + 12 = 0$

(iv) $x^2 - 2x + 8y + 17 = 0$

$$x^2 - 2x = -8y - 17$$

$$(x-1)^2 = -8y - 17 + 1$$

$$(x-1)^2 = -8(y+2)$$

$$X^2 = -8Y$$

Where $a = 2$ $X = x - 1$ $Y = y + 2$

$x = X + 1$ $y = Y - 2$

$X, \quad Y$	$x, \quad y$
Vertex: (0,0)	$(h,k) = (1,-2)$
Focus: (0, -a) = (0, -2)	Focus: (1, -4)
Directrix: Y=a= 2	Directrix: y = 0
Length of L.R. (4a)	4a = 8

(v) $y^2 - 4y - 8x + 12 = 0$

$y^2 - 4y = 8x - 12$ $(y-2)^2 = 8x - 12 + 2^2$

$(y-2)^2 = 8(x-1)$ $Y^2 = 8X$

Where $a = 2$ $X = x - 1$ $Y = y - 2$

$x = X + 1$ $y = Y + 2$

$X, \quad Y$	$x, \quad y$
Vertex: (0,0)	$(h,k) = (1,2)$
Focus: (a, 0) = (2, 0)	Focus: (3, 2)
Directrix: $X = -a = -2$	Directrix: $x = -1$
Length of L.R. (4a)	4a = 8

8. Identify the type of conic and find centre, foci, vertices, and directrices of each of the following :

(i) $\frac{(x-3)^2}{225}+\frac{(y-4)^2}{289}=1$

$$\frac{(x-3)^2}{225}+\frac{(y-4)^2}{289}=1$$

$$\Rightarrow \quad \frac{X^2}{225}+\frac{Y^2}{289}=1$$

It is an ellipse. Type II. Major axis : Y- axis

$a^2=289 \quad b^2=225$ $a=17$		$X=x-3$ $\Rightarrow x=X+3$ $Y=y-4$ $\Rightarrow y=Y+4$
$e=\sqrt{\frac{a^2-b^2}{a^2}}$ $=\sqrt{\frac{289-225}{289}}$ $=\frac{8}{17}$	ae $=17\left(\frac{8}{17}\right)$ $=8$	

ELLIPSE	Referred to X, Y	Referred to x, y	
		$x=X+3$	$y=Y+4$
Centre	$C(0,0)$	$C(3,4)$	
Foci $(0,\ \pm ae)$	$(0,8)$	$F_1(3,12)$	
	$(0,-8)$	$F_2(3,-4)$	
Vertices $(0, \pm a)$	$(0,17)$	$A(3,21)$	
	$(0,-17)$	$A'(3,-13)$	
Directrices $Y=\pm\frac{a}{e}$	$Y=\pm\frac{289}{8}$	$y=\pm\frac{289}{8}+4$	

(ii) $\frac{(x+1)^2}{100}+\frac{(y-2)^2}{64}=1$

$$\frac{(x+1)^2}{100}+\frac{(y-2)^2}{64}=1 \quad \Rightarrow \quad \frac{X^2}{100}+\frac{Y^2}{64}=1$$

It is an ellipse. Type I. Major axis : X- axis

$a^2=100 \quad a=10$ $b^2=64$		$\frac{a}{e}$ $=\frac{10}{6/10}$ $=\frac{100}{6}$ $\frac{a}{e}=\frac{50}{3}$	$X=x+1$ $\Rightarrow x=X-1$ $Y=y-2$ $\Rightarrow y=Y+2$
$e=\sqrt{\frac{a^2-b^2}{a^2}}$ $=\sqrt{\frac{100-64}{100}}$ $e=\frac{6}{10}=\frac{3}{5}$	ae $=10\left(\frac{6}{10}\right)$ $ae=6$		

ELLIPSE	Referred to X, Y	Referred to x, y	
		$x=X-1$	$y=Y+2$
Centre	$C(0,0)$	$C(-1,2)$	
Foci $(\pm ae,\ 0)$	$(6,0)$	$F_1(5,2)$	
	$(-6,0)$	$F_2(-7,2)$	
Vertices $(\pm a, 0)$	$(10,0)$	$A(9,2)$	
	$(-10,0)$	$A'(-11,2)$	
Directrices $X=\pm\frac{a}{e}$	$X=\frac{50}{3}$	$x=\frac{50}{3}-1=\frac{47}{3}$	
	$X=\frac{-50}{3}$	$x=\frac{-50}{3}-1=\frac{-53}{3}$	

$$(a+b)^2=a^2+b^2+2ab$$

(iii) $\frac{(x+3)^2}{225}-\frac{(y-4)^2}{64}=1$

$$\frac{(x+3)^2}{225}-\frac{(y-4)^2}{64}=1 \quad\Rightarrow\quad \frac{X^2}{225}-\frac{Y^2}{64}=1$$

It is an Hyperbola. Type I.

Transverses axis : X- axis

$a^2=225$ $a=15$	b^2 $=64$	$ae=15\left(\frac{17}{15}\right)$ ae $=17$	$\frac{a}{e}=\frac{15}{17/15}$ $\frac{a}{e}=\frac{225}{17}$
$e=\sqrt{\frac{a^2+b^2}{a^2}}=\sqrt{\frac{225+64}{225}}=\frac{17}{15}$			
$X=x+3 \Rightarrow x=X-3$ $Y=y-4 \Rightarrow y=Y+4$			

ELLIPSE	Referred to X , Y	Referred to x, y	
		$x=X-3$	$y=Y+4$
Centre	$C(0,0)$	$C(-3,4)$	
Foci $(\pm ae,\ 0)$	$(17,0)$	$F_1(14,4)$	
	$(-17,0)$	$F_2(-20,4)$	
Vertices $(\pm a,0)$	$(15,0)$	$A(12,4)$	
	$(-15,0)$	$A'(-18,4)$	
Directrices $X=\pm\frac{a}{e}$	$X=\frac{225}{17}$	$x=\frac{225}{17}-3=\frac{174}{17}$	
	$X=\frac{-225}{17}$	$x=-\frac{225}{17}-3=\frac{-276}{17}$	

(iv) $\frac{(y-2)^2}{25}-\frac{(x+1)^2}{16}=1$

$$\frac{(y-2)^2}{25}-\frac{(x+1)^2}{16}=1 \quad\Rightarrow\quad \frac{Y^2}{25}-\frac{X^2}{16}=1$$

It is a Hyperbola. Type II.

Transverses axis : Y- axis

$a^2=25$ $a=5$	$b^2=16$	$ae=5\left(\frac{\sqrt{41}}{5}\right)$ ae $=\sqrt{41}$	$\frac{a}{e}=\frac{5}{\sqrt{41}/5}$ $\frac{a}{e}=\frac{25}{\sqrt{41}}$
$e=\sqrt{\frac{a^2+b^2}{a^2}}=\sqrt{\frac{25+16}{25}}=\frac{\sqrt{41}}{5}$			
$X=x+1 \Rightarrow x=X-1$ $Y=y-2 \Rightarrow y=Y+2$			

ELLIPSE	Referred to X , Y	Referred to x, y	
		$x=X-1$	$y=Y+2$
Centre	$C(0,0)$	$C(-1,2)$	
Foci $(0,\ \pm ae)$	$(0,\sqrt{41})$	$F_1(-1,\sqrt{41}+2)$	
	$(0,-\sqrt{41})$	$F_2(-1,-\sqrt{41}+2)$	
Vertices $(0,\ \pm a)$	$(0,5)$	$A(-1,7)$	
	$(0,-5)$	$A'(-1,-3)$	
Directrices $Y=\pm\frac{a}{e}$	$Y=\frac{25}{\sqrt{41}}$	$y=\frac{25}{\sqrt{41}}+2$	
	$Y=-\frac{25}{\sqrt{41}}$	$y=-\frac{25}{\sqrt{41}}+2$	

$$(a-b)^2=a^2+b^2-2ab$$

(v) $18x^2 + 12y^2 - 144x + 48y + 120 = 0$

Solution:

$$18x^2 + 12y^2 - 144x + 48y + 120 = 0$$

$$18x^2 - 144x + 12y^2 + 48y = -120$$

$$18\,[x^2 - 8x] + 12[y^2 + 4y] = -120$$

$$18(x-4)^2 + 12(y+2)^2 = -120 + 18(-4)^2 + 12(2)^2$$

$$18(x-4)^2 + 12(y+2)^2 = -120 + 288 + 48$$

$$18\,(x-4)^2 + 12(y+2)^2 = 216$$

$$\div 216 \quad \frac{(x-4)^2}{12} + \frac{(y+2)^2}{18} = 1 \Rightarrow \frac{X^2}{12} + \frac{Y^2}{18} = 1$$

It is an ellipse. Type II. Major axis : Y-axis.

$a^2 = 18$ $a = 3\sqrt{2}$	$b^2 = 12$	$ae = 3\sqrt{2}\left(\frac{1}{\sqrt{3}}\right)$ $ae = \sqrt{6}$	$\frac{a}{e} = \frac{3\sqrt{2}}{1/\sqrt{3}}$ $= 3\sqrt{6}$
$e = \sqrt{\frac{a^2 - b^2}{a^2}} = \sqrt{\frac{18-12}{18}} = \frac{1}{\sqrt{3}}$			
$X = x - 4 \quad \Rightarrow x = X + 4$ $Y = y + 2 \quad \Rightarrow y = Y - 2$			

ELLIPSE	Referred to X , Y	Referred to x, y	
		$x = X + 4$	$y = Y - 2$
Centre	$C(0,0)$	$C(4, \ -2)$	
Foci $(\pm ae,\ 0)$	$(0, \sqrt{6})$	$F_1(4,\ \sqrt{6} - 2)$	
	$(0, -\sqrt{6})$	$F_2(4,\ -\sqrt{6} - 2)$	
Vertices $(0,\ \pm a)$	$(0, 3\sqrt{2})$	$A(4, 3\sqrt{2} - 2)$	
	$(0, -3\sqrt{2})$	$A'(4, -3\sqrt{2} - 2)$	
Directrices: $Y = \pm\frac{a}{e}$	$Y = 3\sqrt{6}$	$y = 3\sqrt{6} - 2$	
	$Y = -3\sqrt{6}$	$y = -3\sqrt{6} - 2$	

(vi) $9x^2 - y^2 - 36x - 6y + 18 = 0$

Solution:

$$9x^2 - 36x - y^2 - 6y + 18 = 0$$

$$9[x^2 - 4x] - [y^2 + 6y] = -18$$

$$9(x-2)^2 - (y+3)^2 = -18 + 9\,(-2)^2 - (3)^2$$

$$9(x-2)^2 - (y+3)^2 = -18 + 36 - 9$$

$$9(x-2)^2 - (y+3)^2 = 9$$

$$\div (9) \ \frac{(x-2)^2}{1} - \frac{(y+3)^2}{9} = 1 \Rightarrow \frac{X^2}{1} - \frac{Y^2}{9} = 1$$

Transverse axis : X – axis

$a^2 = 1$ $a = 1$	$b^2 = 9$	$ae = 1(\sqrt{10})$ $ae = \sqrt{10}$	$\frac{a}{e} = \frac{1}{\sqrt{10}}$
$e = \sqrt{\frac{a^2 + b^2}{a^2}} = \sqrt{\frac{1+9}{1}} = \sqrt{10}$			
$X = x - 2 \quad \Rightarrow x = X + 2$ $Y = y + 3 \quad \Rightarrow y = Y - 3$			

HYPERBOLA	Referred to X , Y	Referred to x, y	
		$x = X + 2$	$y = Y - 3$
Centre	$C(0,0)$	$C(2, -3)$	
Foci $(\pm ae,\ 0)$	$(\sqrt{10}, 0)$	$F_1(\sqrt{10} + 2, -3)$	
	$(-\sqrt{10}, 0)$	$F_2(-\sqrt{10} + 2, -3)$	
Vertices $(\pm a,\ 0)$	$(1, 0)$	$A(3, -3)$	
	$(-1, 0)$	$A'(1, -3)$	
Directrices $X = \pm\frac{a}{e}$	$X = \frac{1}{\sqrt{10}}$	$x = \frac{1}{\sqrt{10}} + 2$	
	$X = -\frac{1}{\sqrt{10}}$	$x = -\frac{1}{\sqrt{10}} + 2$	

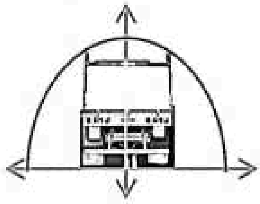
Example 5.30 A semielliptical archway over a one-way road has a height of 3m and a width of 12m. The truck has a width of 3m and a height of 27.m. Will the truck clear the opening of the archway?

Solution:

Width of the road is 12. $a = 6$

The height of the truck is 3. $b = 3$

From the given data, the ellipse is $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$



$$\frac{(1.5)^2}{36} + \frac{y^2}{9} = 1$$

$$y^2 = 9\left(1 - \frac{3^2}{144}\right)$$

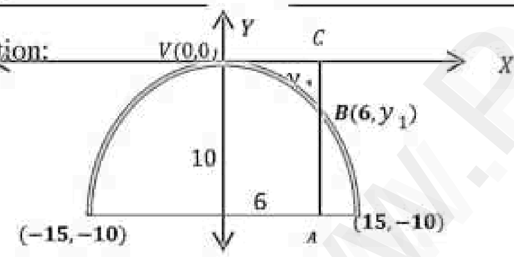
$$y^2 = \frac{9}{144}(135) = \frac{135}{16} = 2.90$$

Thus the height of arch way 15.m from the centre is approximately 2.90m . Since the truck's height is 2.7 m, the truck will clear the archway.

EXERCISE 5.5

1. A bridge has a parabolic arch that is 10m high in the centre and 30m wide at the bottom. Find the height of the arch 6m from the centre, on either sides.

Solution:



Let the equation of the parabola be $x^2 = -4ay$.

$B(15, -10)$ is a point on the parabola $x^2 = -4ay$.

$15^2 = -4a(-10)$ $\quad 4a = \frac{225}{10}$

The parabola is $x^2 = -\frac{225}{10}y$.

$(6, y_1)$ lies on the parabola $\quad 36 = -\frac{225}{10}y_1$

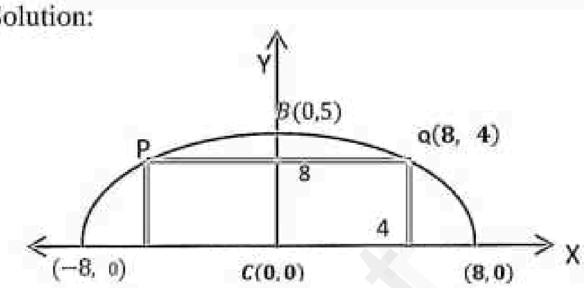
$$y_1 = \frac{-360}{225} = -1.6$$

Height of the arch 6m from the centre is

CD=CE-DE=10-1.6=8.4m

2. A tunnel through a mountain for a four lane highway is to have a elliptical opening. The total width of the highway (not the opening) is to be 16m, and the height at the edge of the road must be sufficient for a truck 4m high to clear if the highest point of the opening is to be 5m approximately . How wide must the opening be?

Solution:



Let the equation of the ellipse be $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$

$b = 5$

(8, 4) lies on the ellipse. $\frac{8^2}{a^2} + \frac{4^2}{5^2} = 1$

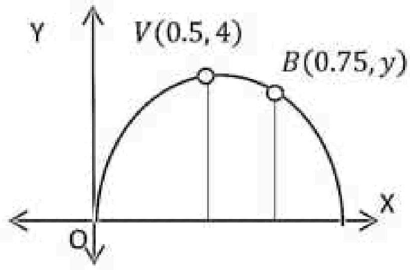
$$\frac{64}{a^2} = 1 - \frac{16}{25}$$

$a^2 = \frac{25}{9}(64) \qquad a = \frac{40}{3}$

Width $2a = 2\left(\frac{40}{3}\right) = \frac{80}{3} = 26.66m$

3. At a water fountain, water attains a maximum height of 4m at horizontal distance of 0.5m from its origin. If the path of water is a parabola, find the height of water at a horizontal distance of 0.75m from the point of origin.

Solution:



From the data the parabola is open downwards.

$$(x - h)^2 = -4a(y - k)$$

Vertex (0.5, 4) $\qquad (x - 0.5)^2 = -4a(y - 4)$

It passes through (0,0) $\quad 4a = \frac{0.25}{4}$

$$(x - 0.5)^2 = -\frac{0.25}{4}(y - 4)$$

(0.75, y) lies on the parabola

$$(0.75 - 0.5)^2 = -\frac{0.25}{4}(y - 4)$$

$$(0.25)^2 = -\frac{0.25}{4}(y - 4)$$

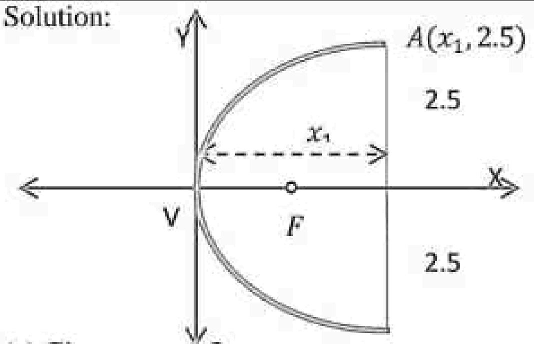
$0.25 \times (-4) = y - 4$

$-1 = y - 4 \qquad y = 3$

The required height is 3mm.

4. An engineer designs a satellite dish with a parabolic cross section. The dish is 5m wide at the opening, and the focus is placed 1.2m from the vertex
(a) Position a coordinate system with the origin at the vertex and the x -axis on the parabola's axis of symmetry and find an equation of the parabola.
(b) Find the depth of the satellite dish at the vertex.

Solution:



(a) Given: $a = 1.2m$

Equation of the parabola $y^2 = 4ax$ $y^2 = 4.8\,x$

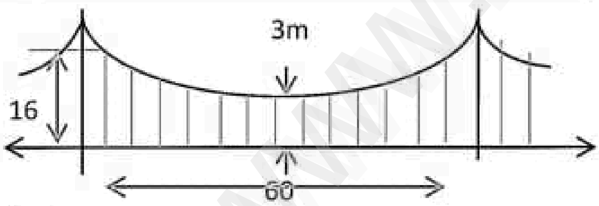
(b) let $VA = x$

(x, 2.5) which lies on the parabola $y^2 = 4.8\,x$

$(2.5)^2 = 4.8\,x$ $x = \frac{2.5\times2.5}{4\times1.2} = 1.3$

The depth of the satellite dish is 1.3m

5. Parabolic cable of a 60m portion of the roadbed of a suspension bridge are positioned as shown below. Vertical Cables are to be spaced every 6m along this portion of the roadbed. Calculate the lengths of first two of these vertical cables from the vertex.



Solution:

From the given data, the parabola is open upwards, $x^2 = 4ay$ and it passes through (30, 13).

$30^2 = 4a(13)$ $4a = \frac{30^2}{13}$

Equation of the parabola is $x^2 = \frac{30^2}{13}y$

(i) (6, y) lies on the parabola $36 = \frac{30^2}{13}y$

$y = \frac{36\times13}{30\times30} = \frac{78}{150} = 0.52$

Cable from the road = 3+0.52=3.52m.

(ii) (12, y) lies on the parabola $12^2 = \frac{30^2}{13}y$

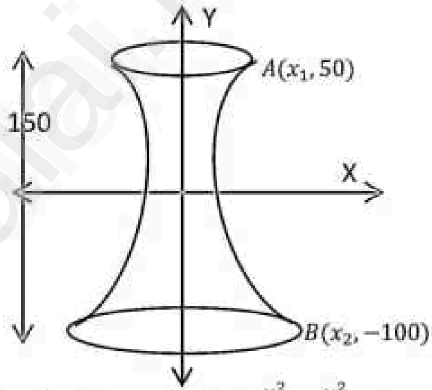
$y = \frac{12\times12\times13}{30\times30} = \frac{52}{25} = 2.08$

Cable from the road = 3+2.08=5.08m

The height of the first two vertical cables from the vertex are 3.52m and 2.08m.

6. Cross section of a Nuclear cooling tower is in the shape of a hyperbola with equation $\frac{x^2}{30^2} - \frac{y^2}{44^2} = 1$. The tower is 150m tall and the distance from the top of the tower to the centre of the hyperbola is half the distance from the base of the tower to the centre of the hyperbola. Find the diameter of the top and base of the tower.

Solution:



The equation of the hyperbola be $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$

Given: $\frac{x^2}{30^2} - \frac{y^2}{44^2} = 1$

It is given that if CP= $\frac{1}{2}y$

Then $CQ = \frac{3}{2}y = 150$ $y = 100$

Let S and R be the points on the hyperbola at the top and bottom of the tower respectively.

Hence S is $(x_1, 50)$ and R is $(x_2, -100)$.

$\frac{x_1^2}{30^2} - \frac{50^2}{44^2} = 1$ $x_1^2 = 30^2\left(1 + \frac{50^2}{44^2}\right)$

$x_1 = \frac{30}{44}(66.6) = 45.41m$

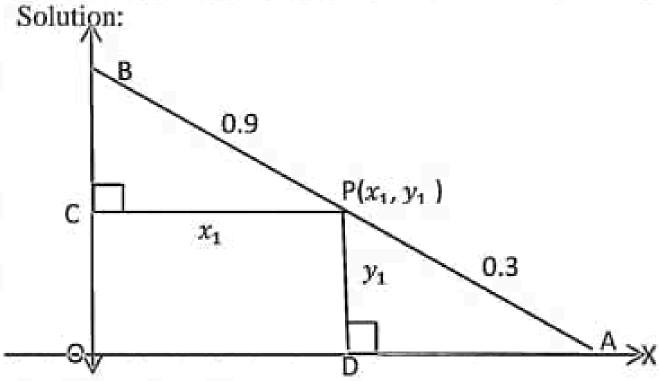
$\frac{x_2^2}{30^2} - \frac{100^2}{44^2} = 1$ $x_1^2 = 30^2\left(1 + \frac{100^2}{44^2}\right)$

$x_1 = \frac{30}{44}(109.25) = 74.45m$

Diameter of the top is 90.92m and bottom is 148.90m.

7. A rod of length 1.2 m moves with its ends always touching the coordinate axes. The locus of a point P on the rod, which is 0.3 m from the end in contact with x -axis is an ellipse. Find the eccentricity.

Solution:



Let AB be the rod.

Let P b any point on the rod which is 0.3m.

AB = 1.2m AP = 0.3m PB = 0.9m

Draw PD ⊥ X – axis PC ⊥ Y – axis

ΔAOP similarly ΔPCB

$$\sin\theta = \frac{y_1}{0.3} = \frac{y_1}{3/10}$$

$$\cos\theta = \frac{x_1}{0.9} = \frac{x_1}{9/10}$$

We know that $\cos^2\theta + \sin^2\theta = 1$

$$\frac{{x_1}^2}{81/100} + \frac{{y_1}^2}{9/100} = 1$$

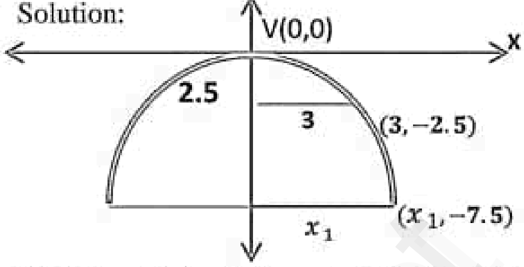
$a^2 = \frac{81}{100}$ $b^2 = \frac{9}{100}$

$$e = \sqrt{1 - \frac{b^2}{a^2}} = \sqrt{1 - \frac{\frac{9}{100}}{\frac{81}{100}}}$$

$$e = \sqrt{1 - \frac{1}{9}} = \sqrt{\frac{8}{9}} \qquad e = \frac{2\sqrt{2}}{3}$$

8. Assume that water issuing from the end of a horizontal pipe, 7.5 m above the ground, describes a parabolic path. The vertex of the parabolic path is at the end of the pipe. At a position 2.5 m below the line of the pipe, the flow of water has curved outward 3m beyond the vertical line through the end of the pipe. How far beyond this vertical line will the water strike the ground?

Solution:



From the data the parabola is open downwards.

$$x^2 = -4a\,y$$

It passes through $(3, -2.5)$

$(3)^2 = -4a\,(-2.5)$ $4a = \frac{9}{2.5}$

It required equation is $x^2 = -\frac{9}{2.5}\,y$

It passes through $(x_1, -7.5)$

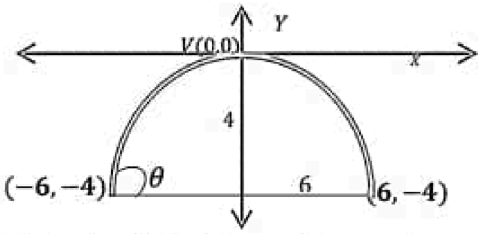
${x_1}^2 = -\frac{9}{2.5}\,(-7.5)$

$x_1 = 3\sqrt{3}\,m$

The water strikes the ground $3\sqrt{3}\,m$ beyond the vertical line.

9. On lighting a rocket cracker it gets projected in a parabolic path and reaches a maximum height of 4m when it is 6m away from the point of projection. Finally it reaches the ground 12m away from the starting point. Find the angle of projection.

Solution:



From the data the parabola is open downwards.

$x^2 = -4a\,y$

It passes through $(-6, -4)$ $(-6)^2 = -4a\,(-4)$

$4a = 9$

It required equation is $x^2 = -9\,y$

Differentiate with respect to x,

$2x = -9\frac{dy}{dx}$ $\frac{dy}{dx} = \frac{2x}{-9}$

$$m = \left.\frac{dy}{dx}\right]_{(-6,-4)} = \frac{2(-6)}{-9} = \frac{4}{3}$$

$$\tan\theta = \frac{4}{3}$$

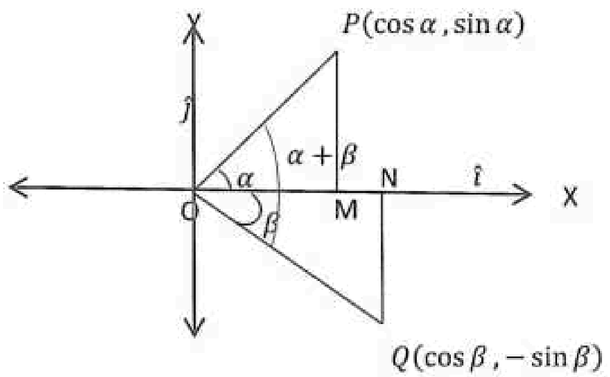
$$\theta = \tan^{-1}\left(\frac{4}{3}\right)$$

The angle of projection is $\tan^{-1}\left(\frac{4}{3}\right)$

Example 6.3 By vector method, prove that

$\cos(\alpha+\beta) = \cos\alpha\cos\beta - \sin\alpha\sin\beta$.

Proof:



Take the unit vectors $\hat{a} = \overrightarrow{OP}$ and $\hat{b} = \overrightarrow{OQ}$ which make angles α and β respectively, with positive x – axis. Draw MP and QN perpendicular to the x -axis.

$\lfloor XOP = \alpha$, $\lfloor XOQ = \beta$, $\lfloor POQ = \alpha + \beta$

Take the unit vectors $\hat{\imath}$ *and* $\hat{\jmath}$ along the X and Y axes.

$$\overrightarrow{OP} = \cos\alpha\,\hat{\imath} + \sin\alpha\,\hat{\jmath}$$

$$\overrightarrow{OQ} = \cos\beta\,\hat{\imath} - \sin\beta\,\hat{\jmath}$$

$\overrightarrow{\boldsymbol{OQ}}\cdot\overrightarrow{\boldsymbol{OP}} = \mathbf{cos}\,\alpha\,\mathbf{cos}\,\beta - \mathbf{sin}\,\alpha\,\mathbf{sin}\,\beta$……..(1)

By the definition,

$\overrightarrow{OQ}\cdot\overrightarrow{OP} = |\overrightarrow{OQ}|\;|\overrightarrow{OP}|\cos(\alpha+\beta)$

$\overrightarrow{\boldsymbol{OQ}}\cdot\overrightarrow{\boldsymbol{OP}} = \mathbf{cos}(\alpha+\beta)$ …..(2)

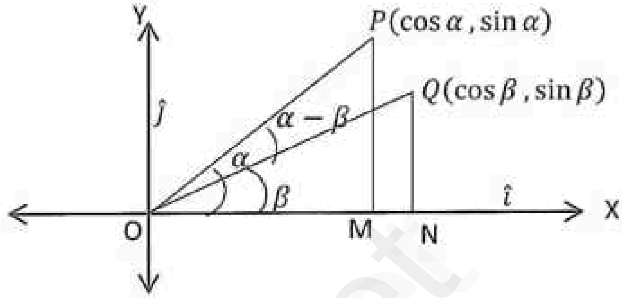
From (1) and (2),

$$\mathbf{cos}(\alpha+\beta) = \mathbf{cos}\,\alpha\,\mathbf{cos}\,\beta - \mathbf{sin}\,\alpha\,\mathbf{sin}\,\beta$$

Example 6.5 Prove by vector method that

$\boldsymbol{\sin(\alpha-\beta) = \sin\alpha\cos\beta - \cos\alpha\sin\beta}$.

Proof:



Take the unit vectors $\hat{a} = \overrightarrow{OP}$ and $\hat{b} = \overrightarrow{OQ}$ which make angles α and β respectively, with positive x – axis.

Draw MP and QN perpendicular to the x -axis.

$\lfloor XOP = \alpha$, $\lfloor XOQ = \beta$, $\lfloor POQ = \alpha - \beta$

$$\overrightarrow{OP} = \cos\alpha\,\hat{\imath} + \sin\alpha\,\hat{\jmath}$$

$$\overrightarrow{OQ} = \cos\beta\,\hat{\imath} + \sin\beta\,\hat{\jmath}$$

Take the unit vectors $\hat{\imath}$ *and* $\hat{\jmath}$ along the X and Y axes.

$$\overrightarrow{OQ}\times\overrightarrow{OP} = \begin{vmatrix} \hat{\imath} & \hat{\jmath} & \hat{k} \\ \cos\beta & \sin\beta & 0 \\ \cos\alpha & \sin\alpha & 0 \end{vmatrix}$$

$\overrightarrow{\boldsymbol{OQ}}\times\overrightarrow{\boldsymbol{OP}} = \hat{k}\,(\mathbf{sin}\,\alpha\,\mathbf{cos}\,\beta - \mathbf{cos}\,\alpha\;\mathbf{sin}\,\beta)$ ……..(1)

By the definition,

$\overrightarrow{OQ}\times\overrightarrow{OP} = |\overrightarrow{OQ}|\;|\overrightarrow{OP}|\sin(\alpha-\beta)\,\hat{k}$

$\overrightarrow{\boldsymbol{OQ}}\times\overrightarrow{\boldsymbol{OP}} = \hat{k}\,\mathbf{sin}(\alpha-\beta)$ ……. (2)

From (1) and (2),

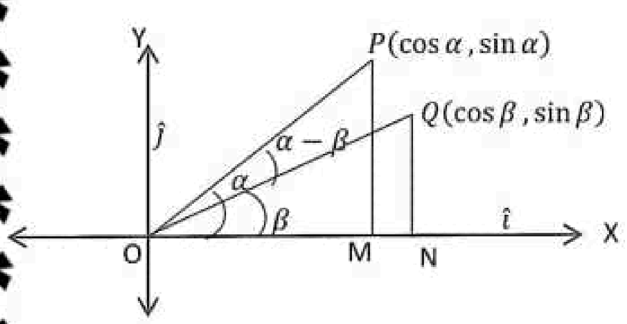
$$\sin(\alpha-\beta) = \sin\alpha\cos\beta - \cos\alpha\sin\beta$$

EXERCISE 6.1

9. Using vector method, prove that

$$\boldsymbol{\cos(\alpha-\beta)=\cos\alpha\cos\beta+\sin\alpha\sin\beta}$$

Proof:



Take the unit vectors $\hat{a}=\overrightarrow{OP}$ *and* $\hat{b}=\overrightarrow{OQ}$ which make angles α and β, respectively, with positive x – axis.

Draw MP and QN perpendicular to the x -axis.

$\lfloor XOP=\alpha,\ \lfloor XOQ=\beta,\ \lfloor POQ=\alpha-\beta$

$$\overrightarrow{OP}=\cos\alpha\,\hat{\imath}+\sin\alpha\,\hat{\jmath}$$

$$\overrightarrow{OQ}=\cos\beta\,\hat{\imath}+\sin\beta\,\hat{\jmath}$$

Take the unit vectors $\hat{\imath}$ *and* $\hat{\jmath}$ along the X and Y axes.

$\boldsymbol{\overrightarrow{OQ}\cdot\overrightarrow{OP}=\cos\alpha\cos\beta+\sin\alpha\sin\beta}$........(1)

By the definition,

$\overrightarrow{OQ}\cdot\overrightarrow{OP}=|\overrightarrow{OQ}|\ |\overrightarrow{OP}|\cos(\alpha-\beta)$

$\boldsymbol{\overrightarrow{OQ}\cdot\overrightarrow{OP}=\cos(\alpha-\beta)}$ ......(2)

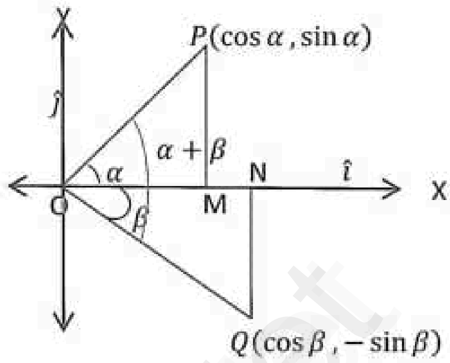
From (1) and (2),

$\boldsymbol{\cos(\alpha-\beta)=\cos\alpha\cos\beta+\sin\alpha\sin\beta}$

10. Prove by vector method that

$$\boldsymbol{\sin(\alpha+\beta)=\sin\alpha\cos\beta+\cos\alpha\sin\beta}$$

Proof:



Take the unit vectors $\hat{a}=\overrightarrow{OP}$ *and* $\hat{b}=\overrightarrow{OQ}$ which make angles α and β, respectively, with positive x – axis.

Draw MP and QN perpendicular to the x -axis.

$\lfloor XOP=\alpha,\ \lfloor XOQ=\beta,\ \lfloor POQ=\alpha+\beta$

Take the unit vectors $\hat{\imath}$ *and* $\hat{\jmath}$ along the X and Y axes.

$$\overrightarrow{OP}=\cos\alpha\,\hat{\imath}+\sin\alpha\,\hat{\jmath}$$

$$\overrightarrow{OQ}=\cos\beta\,\hat{\imath}-\sin\beta\,\hat{\jmath}$$

$$\overrightarrow{OQ}\times\overrightarrow{OP}=\begin{vmatrix}\hat{\imath} & \hat{\jmath} & \hat{k}\\ \cos\beta & -\sin\beta & 0\\ \cos\alpha & \sin\alpha & 0\end{vmatrix}$$

$\boldsymbol{\overrightarrow{OQ}\times\overrightarrow{OP}=\hat{k}\,(\sin\alpha\cos\beta+\cos\alpha\ \sin\beta)}$........(1)

By the definition,

$\overrightarrow{OQ}\times\overrightarrow{OP}=|\overrightarrow{OQ}|\ |\overrightarrow{OP}|\sin(\alpha+\beta)\,\hat{k}$

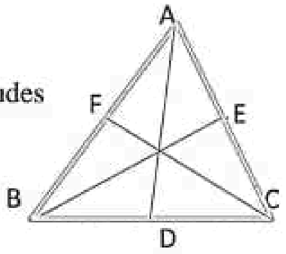
$\boldsymbol{\overrightarrow{OQ}\times\overrightarrow{OP}=\sin(\alpha+\beta)\,\hat{k}}$ ....(2)

From (1) and (2),

$$\sin(\alpha+\beta)=\sin\alpha\cos\beta+\cos\alpha\sin\beta$$

Example 6.7 Prove by vector method that the perpendiculars (attitudes) from the vertices to the opposite sides of a triangle are concurrent.

Solution:



- Let ABC be a triangle.
- Let AD, BE are its two altitudes intersecting at O.
- To prove: Third altitudes CF is also intersecting at O.
- It is enough to prove that $\overrightarrow{OC} \perp \overrightarrow{BA}$

Now, AD⊥BC $\Rightarrow \overrightarrow{OA} \perp \overrightarrow{BC} \quad \Rightarrow \overrightarrow{OA} \cdot \overrightarrow{BC} = 0$

$\Rightarrow \overrightarrow{OA} \cdot (\overrightarrow{OC} - \overrightarrow{OB}) = 0$

$\Rightarrow \overrightarrow{OA} \cdot \overrightarrow{OC} - \overrightarrow{OA} \cdot \overrightarrow{OB} = 0 \ldots\ldots\ldots (1)$

BE ⊥ CA $\Rightarrow \overrightarrow{OB} \perp \overrightarrow{CA} \quad \Rightarrow \overrightarrow{OB} \cdot \overrightarrow{CA} = 0$

$\Rightarrow \overrightarrow{OB} \cdot (\overrightarrow{OA} - \overrightarrow{OC}) = 0$

$\Rightarrow \overrightarrow{OA} \cdot \overrightarrow{OB} - \overrightarrow{OB} \cdot \overrightarrow{OC} = 0 \ldots\ldots\ldots (2)$

From (1) + (2),

$\overrightarrow{OA} \cdot \overrightarrow{OC} - \overrightarrow{OA} \cdot \overrightarrow{OB} + \overrightarrow{OA} \cdot \overrightarrow{OB} - \overrightarrow{OB} \cdot \overrightarrow{OC} = 0$

$\overrightarrow{OA} \cdot \overrightarrow{OC} - \overrightarrow{OB} \cdot \overrightarrow{OC} = 0$

$\overrightarrow{OC}(\overrightarrow{OA} - \overrightarrow{OB}) = 0$

$\overrightarrow{OC} \cdot \overrightarrow{BA} = 0 \qquad \overrightarrow{OC} \perp \overrightarrow{BA} \qquad CF \perp AB$

Hence altitudes of triangle are concurrent.

Example 6.23 If $\vec{a} = \hat{\imath} - \hat{\jmath}$, $\vec{b} = \hat{\imath} - \hat{\jmath} - 4\hat{k}$, $\vec{c} = 3\hat{\jmath} - \hat{k}$ and $\vec{d} = 2\hat{\imath} + 5\hat{\jmath} + \hat{k}$, verify that

(i) $(\vec{a} \times \vec{b}) \times (\vec{c} \times \vec{d}) = [\vec{a}, \vec{b}, \vec{d}]\vec{c} - [\vec{a}, \vec{b}, \vec{c}]\vec{d}$

(ii) $(\vec{a} \times \vec{b}) \times (\vec{c} \times \vec{d}) = [\vec{a}, \vec{c}, \vec{d}]\vec{b} - [\vec{b}, \vec{c}, \vec{d}]\vec{a}$

Solution:

(i) $(\vec{a} \times \vec{b}) \times (\vec{c} \times \vec{d}) = [\vec{a}, \vec{b}, \vec{d}]\vec{c} - [\vec{a}, \vec{b}, \vec{c}]\vec{d}$

LHS $(\vec{a} \times \vec{b}) \times (\vec{c} \times \vec{d})$

$$\vec{a} \times \vec{b} = \begin{vmatrix} \hat{\imath} & \hat{\jmath} & \hat{k} \\ 1 & -1 & 0 \\ 1 & -1 & -4 \end{vmatrix} = 4\hat{\imath} + 4\hat{\jmath}$$

$$\vec{c} \times \vec{d} = \begin{vmatrix} \hat{\imath} & \hat{\jmath} & \hat{k} \\ 0 & 3 & -1 \\ 2 & 5 & 1 \end{vmatrix} = 8\hat{\imath} - 2\hat{\jmath} - 6\hat{k}$$

$$(\vec{a} \times \vec{b}) \times (\vec{c} \times \vec{d}) = \begin{vmatrix} \hat{\imath} & \hat{\jmath} & \hat{k} \\ 4 & 4 & 0 \\ 8 & -2 & -6 \end{vmatrix}$$

$= -24\hat{\imath} + 24\hat{\jmath} - 40\hat{k}$ ……..(1)

RHS $[\vec{a}, \vec{b}, \vec{d}]\vec{c} - [\vec{a}, \vec{b}, \vec{c}]\vec{d}$

$$[\vec{a}, \vec{b}, \vec{d}] = \begin{vmatrix} 1 & -1 & 0 \\ 1 & -1 & -4 \\ 2 & 5 & 1 \end{vmatrix} = 28$$

$$[\vec{a}, \vec{b}, \vec{c}] = \begin{vmatrix} 1 & -1 & 0 \\ 1 & -1 & -4 \\ 0 & 3 & -1 \end{vmatrix} = 12$$

$[\vec{a}, \vec{b}, \vec{d}]\vec{c} - [\vec{a}, \vec{b}, \vec{c}]\vec{d}$

$= 28(3\hat{\jmath} - \hat{k}) - 12(2\hat{\imath} + 5\hat{\jmath} + \hat{k})$

$= -24\hat{\imath} + 24\hat{\jmath} - 40\hat{k}$ ……..(2)

From (1) and (2) $(\vec{a} \times \vec{b}) \times (\vec{c} \times \vec{d}) = [\vec{a}, \vec{b}, \vec{d}]\vec{c} - [\vec{a}, \vec{b}, \vec{c}]\vec{d}$

(ii) $(\vec{a} \times \vec{b}) \times (\vec{c} \times \vec{d}) = [\vec{a}, \vec{c}, \vec{d}]\vec{b} - [\vec{b}, \vec{c}, \vec{d}]\vec{a}$

LHS $(\vec{a} \times \vec{b}) \times (\vec{c} \times \vec{d})$

$$\vec{a} \times \vec{b} = \begin{vmatrix} \hat{\imath} & \hat{\jmath} & \hat{k} \\ 1 & -1 & 0 \\ 1 & -1 & -4 \end{vmatrix} = 4\hat{\imath} + 4\hat{\jmath}$$

$$\vec{c} \times \vec{d} = \begin{vmatrix} \hat{\imath} & \hat{\jmath} & \hat{k} \\ 0 & 3 & -1 \\ 2 & 5 & 1 \end{vmatrix} = 8\hat{\imath} - 2\hat{\jmath} - 6\hat{k}$$

$$(\vec{a} \times \vec{b}) \times (\vec{c} \times \vec{d}) = \begin{vmatrix} \hat{\imath} & \hat{\jmath} & \hat{k} \\ 4 & 4 & 0 \\ 8 & -2 & -6 \end{vmatrix}$$

$= -24\hat{\imath} + 24\hat{\jmath} - 40\hat{k}$ ……..(3)

RHS $[\vec{a}, \vec{c}, \vec{d}]\vec{b} - [\vec{b}, \vec{c}, \vec{d}]\vec{a}$

$$[\vec{a}, \vec{c}, \vec{d}] = \begin{vmatrix} 1 & -1 & 0 \\ 0 & 3 & -1 \\ 2 & 5 & 1 \end{vmatrix} = 10$$

$$[\vec{b}, \vec{c}, \vec{d}] = \begin{vmatrix} 1 & -1 & -4 \\ 0 & 3 & -1 \\ 2 & 5 & 1 \end{vmatrix} = 34$$

$[\vec{a}, \vec{c}, \vec{d}]\vec{b} - [\vec{b}, \vec{c}, \vec{d}]\vec{a} = 10(\hat{\imath} - \hat{\jmath} - 4\hat{k}) - 34(\hat{\imath} - \hat{\jmath})$

$= -24\hat{\imath} + 24\hat{\jmath} - 40\hat{k}$ ……..(4)

From (3) and (4) $(\vec{a} \times \vec{b}) \times (\vec{c} \times \vec{d}) = [\vec{a}, \vec{c}, \vec{d}]\vec{b} - [\vec{b}, \vec{c}, \vec{d}]\vec{a}$

Life is a Good Circle,
You Choose the Best Radius

Example 6.27 Find the vector equation in parametric form and Cartesian equations of a straight passing through the points (-5, 7, -4) and (13, -5, 2). Find the point where the straight line crosses the xy -plane.

Solution:

Let $\vec{a} = -5\hat{\imath} + 7\hat{\jmath} - 4\hat{k}$ and $\vec{b} = 13\hat{\imath} - 5\hat{\jmath} + 2\hat{k}$

$\vec{b} - \vec{a} = 18\hat{\imath} - 12\hat{\jmath} + 6\hat{k}$

$(x_1, y_1, z_1) = (-5, 7, -4), (x_2, y_2, z_2) = (13, -5, 2)$

Parametric form of vector equation:

$\vec{r} = (-5\hat{\imath} + 7\hat{\jmath} - 4\hat{k}) + t(18\hat{\imath} - 12\hat{\jmath} + 6\hat{k})\, t \in R$

Cartesian form of equation:

$\frac{x+5}{18} = \frac{y-7}{-12} = \frac{z+4}{6} \dots\dots (1)$

From eqn (1) crosses the xy –plane, ie) $z = 0$

$$\frac{x+5}{18} = \frac{y-7}{-12} = \frac{4}{6}$$

$\frac{x+5}{18} = \frac{4}{6} \Rightarrow x + 5 = 12 \Rightarrow\Rightarrow x = 7$

$\frac{y-7}{-12} = \frac{4}{6} \Rightarrow y - 7 = -8 \Rightarrow\Rightarrow y = -1$

Point of intersection: $(x, y, z) = (7, -1, 0)$

Example 6.33 Find the point of intersection of the lines $\frac{x-1}{2} = \frac{y-2}{3} = \frac{z-3}{4}$ and $\frac{x-4}{5} = \frac{y-1}{2} = z$.

Solution:

$\frac{x-1}{2} = \frac{y-2}{3} = \frac{z-3}{4} = s$ (x, y, z) $= (2s+1, 3s+2, 4s+3)$ $\dots\dots (1)$	$\frac{x-4}{5} = \frac{y-1}{2} = z = t$ (x, y, z) $= (5t+4, 2t+1, t)$ $\dots\dots (2)$

From (1) and (2)

$(2s+1, 3s+2, 4s+3) = (5t+4, 2t+1, t)$

Comparing the corresponding terms

$2s + 1 = 5t + 4 \dots\dots (3)$

$4s + 3 = t \dots\dots\dots\dots (4)$

$2s - 5t = 3 \qquad 4s - 10t = 6$

$4s - t = -3 \qquad 4s - t = -3$

We get $t = -1$ and $s = -1$.

The point of intersection $(-1, -1, -1)$.

Example 6.34 Find the equation of a straight line passing through the point of intersection of the straight lines
$\vec{r} = (\hat{\imath} + 3\hat{\jmath} - \hat{k}) + t(2\hat{\imath} + 3\hat{\jmath} + 2\hat{k})$
and $\frac{x-2}{1} = \frac{y-4}{2} = \frac{z+3}{4}$ and perpendicular to both straight lines.

Solution:

$\vec{r} = (\hat{\imath} + 3\hat{\jmath} - \hat{k})$ $+t(2\hat{\imath} + 3\hat{\jmath} + 2\hat{k})$ $\frac{x-1}{2} = \frac{y-3}{3} = \frac{z+1}{2} = s$ (x, y, z) $= (2s+1, 3s+3, 2s-1)$ $\dots\dots (1)$	$\frac{x-2}{1} = \frac{y-4}{2} = \frac{z+3}{4} = t$ (x, y, z) $= (t+2, 2t+4, 4t-3)$ $\dots\dots (2)$

From (1) and (2)

$(2s+1, 3s+3, 2s-1) = (t+2, 2t+4, 4t-3)$

Comparing the corresponding terms,

$2s + 1 = t + 2 \quad 3s + 3 = 2t + 4 \quad 2s - 1 = 4t - 3$

$2s - t = 1 \dots\dots (3)$

$3s - 2t = 1 \dots\dots (4)$

$2s - 4t = -2 \dots\dots (5)$

We get $t = 1$ and $s = 1$

Point of intersection $(3, 6, 1)$.

Let $\vec{b} = 2\hat{\imath} + 3\hat{\jmath} + 2\hat{k}$ and $\vec{d} = \hat{\imath} + 2\hat{\jmath} + 4\hat{k}$

$$\vec{b} \times \vec{d} = \begin{vmatrix} \hat{\imath} & \hat{\jmath} & \hat{k} \\ 2 & 3 & 2 \\ 1 & 2 & 4 \end{vmatrix} = 8\hat{\imath} - 6\hat{\jmath} + \hat{k}$$

The required straight line passing through (3,6,1).

Equation of the required straight line is

$\vec{r} = (3\hat{\imath} + 6\hat{\jmath} + \hat{k}) + m(8\hat{\imath} - 6\hat{\jmath} + \hat{k}) \quad m \in \mathbb{R}$

Example 6.36 Find the shortest distance between the two given straight lines
$\vec{r} = (2\hat{\imath} + 3\hat{\jmath} + 4\hat{k}) + t(-2\hat{\imath} + \hat{\jmath} - 2\hat{k})$ and
$\frac{x-3}{2} = \frac{y}{-1} = \frac{z+2}{2}$.

Solution:

The parametric form of vector equations of the given straight lines are

$\vec{r} = (2\hat{\imath} + 3\hat{\jmath} + 4\hat{k}) + t(-2\hat{\imath} + \hat{\jmath} - 2\hat{k}) \quad \vec{r} = \vec{a} + t\vec{b}$

$\vec{r} = (3\hat{\imath} + 0\hat{\jmath} - 2\hat{k}) + t(2\hat{\imath} - \hat{\jmath} + 2\hat{k}) \qquad \vec{r} = \vec{c} + s\,\vec{d}$

Let $\vec{a} = 2\hat{\imath} + 3\hat{\jmath} + 4\hat{k}$, $\vec{b} = -2\hat{\imath} + \hat{\jmath} - 2\hat{k}$,

$\vec{c} = 3\hat{\imath} + 0\hat{\jmath} - 2\hat{k}$, $\vec{d} = 2\hat{\imath} - \hat{\jmath} + 2\hat{k}$

$\vec{b}$ **and** $\vec{d}$ two vectors are parallel.

$\vec{c} - \vec{a} = \hat{\imath} - 3\hat{\jmath} - 6\hat{k}$

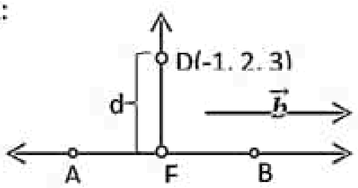
$$(\vec{c} - \vec{a}) \times \vec{b} = \begin{vmatrix} \hat{\imath} & \hat{\jmath} & \hat{k} \\ 1 & -3 & -6 \\ -2 & 1 & -2 \end{vmatrix} = 12\hat{\imath} + 14\hat{\jmath} - 5\hat{k}$$

The shortest distance between the two parallel lines

$$d = \frac{|(\vec{c}-\vec{a})\times\vec{b}|}{|\vec{b}|} = \frac{\sqrt{12^2+14^2+(-5)^2}}{\sqrt{(-2)^2+1^2+(-2)^2}} = \frac{\sqrt{365}}{3} \text{ units.}$$

Example 6.37 Find the coordinates of the foot of the perpendicular drawn from the point (-1, 2, 3) to the straight line $\vec{r} = (\hat{\imath} - 4\hat{\jmath} + 3\hat{k}) + t(2\hat{\imath} + 3\hat{\jmath} + \hat{k})$. Also, find the shortest distance from the point to the straight line.

Solution:



Let $\overrightarrow{OD} = -\hat{\imath} + 2\hat{\jmath} + 3\hat{k}$,

Let F be the foot of the perpendicular from D to the straight line.

$\vec{r} = (\hat{\imath} - 4\hat{\jmath} + 3\hat{k}) + t(2\hat{\imath} + 3\hat{\jmath} + \hat{k}) \qquad \vec{r} = \vec{a} + t\vec{b}$

Cartesian form: $\frac{x-1}{2} = \frac{y+4}{3} = \frac{z-3}{1} = \lambda$

$$\overrightarrow{OF} = (2\lambda+1)\hat{\imath} + (3\lambda-4)\hat{\jmath} + (\lambda+3)\hat{k}$$

F is of the form $((2\lambda+1), (3\lambda-4), (\lambda+3))$

$$\overrightarrow{DF} = \overrightarrow{OF} - \overrightarrow{OD} = (2+2\lambda)\hat{\imath} + (-6+3\lambda)\hat{\jmath} + \lambda\hat{k}$$

$\vec{b} \perp \overrightarrow{DF} \qquad \vec{b} \cdot \overrightarrow{DF} = 0$

$$(2\hat{\imath} + 3\hat{\jmath} + \hat{k}) \cdot [(2+2\lambda)\hat{\imath} + (-6+3\lambda)\hat{\jmath} + \lambda\hat{k}] = 0$$

$4 + 4\lambda - 18 + 9\lambda + \lambda = 0 \qquad 14\lambda = 14 \qquad \lambda = 1$

Therefore, the coordinate of F is $(3, -1, 4)$

$\overrightarrow{DF} = 4\hat{\imath} - 3\hat{\jmath} + \hat{k}$

Now, $|\overrightarrow{DF}| = \sqrt{4^2 + (-3)^2 + 1^2} = \sqrt{26}$ units

EXERCISE 6.5

1. Find the parametric form of vector equation and Cartesian equations of a straight line passing through (5, 2, 8) and is perpendicular to the straight lines $\vec{r} = (\hat{\imath} + \hat{\jmath} - \hat{k}) + s(2\hat{\imath} - 2\hat{\jmath} + \hat{k})$ and $\vec{r} = (2\hat{\imath} - \hat{\jmath} - 3\hat{k}) + t(\hat{\imath} + 2\hat{\jmath} + 2\hat{k})$.

Solution:

Given: $\vec{r} = (\hat{\imath} + \hat{\jmath} - \hat{k}) + s(2\hat{\imath} - 2\hat{\jmath} + \hat{k})$ and

$\vec{r} = (2\hat{\imath} - \hat{\jmath} - 3\hat{k}) + t(\hat{\imath} + 2\hat{\jmath} + 2\hat{k})$

Let $\vec{b} = 2\hat{\imath} - 2\hat{\jmath} + \hat{k}$ and $\vec{d} = \hat{\imath} + 2\hat{\jmath} + 2\hat{k}$

$\vec{b}$ and $\vec{d}$ are perpendicular.

$$\vec{b} \times \vec{d} = \begin{vmatrix} \hat{\imath} & \hat{\jmath} & \hat{k} \\ 2 & -2 & 1 \\ 1 & 2 & 2 \end{vmatrix} = -6\hat{\imath} - 3\hat{\jmath} + 6\hat{k}$$

The equation of the straight line passing through (5, 2, 8) and parallel to $-6\hat{\imath} - 3\hat{\jmath} + 6\hat{k}$.

Parametric form of vector equation:

$$\vec{r} = (5\hat{\imath} + 2\hat{\jmath} + 8\hat{k}) + m(-6\hat{\imath} - 3\hat{\jmath} + 6\hat{k}), m \in R$$

Cartesian equations: $\frac{x-5}{-6} = \frac{y-2}{-3} = \frac{z-8}{6}$

$$(a+b)(a-b) = a^2 - b^2$$

2. Show that the lines

$\vec{r} = (6\hat{i} + \hat{j} + 2\hat{k}) + s(\hat{i} + 2\hat{j} - 3\hat{k})$ and

$\vec{r} = (3\hat{i} + 2\hat{j} - 2\hat{k}) + t(2\hat{i} + 4\hat{j} - 5\hat{k})$ are skew lines and hence find the shortest distance between them.

Solution:

Comparing the given two equations with

$\vec{r} = \vec{a} + s\vec{b}$ and $\vec{r} = \vec{c} + t\vec{d}$

Let $\vec{a} = 6\hat{i} + \hat{j} + 2\hat{k}$, $\vec{b} = \hat{i} + 2\hat{j} - 3\hat{k}$,

$\vec{c} = 3\hat{i} + 2\hat{j} - 2\hat{k}$, $\vec{d} = 2\hat{i} + 4\hat{j} - 5\hat{k}$

$\vec{b}$ and $\vec{d}$ two vectors are not parallel.

$$\vec{b} \times \vec{d} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 1 & 2 & -3 \\ 2 & 4 & -5 \end{vmatrix} = 2\hat{i} - \hat{j} + 0\hat{k}$$

$$(\vec{c} - \vec{a}) \cdot (\vec{b} \times \vec{d}) = (-3\hat{i} + \hat{j} - 4\hat{k}) \cdot (2\hat{i} - \hat{j} + 0\hat{k})$$

$$= -6 - 1 + 0 = -7$$

The shortest distance between the two skew lines

$$\delta = \frac{|(\vec{c}-\vec{a})\cdot(\vec{b}\times\vec{d})|}{|\vec{b}\times\vec{d}|} = \frac{|-7|}{\sqrt{2^2+(-1)^2}} = \frac{7}{\sqrt{5}} \text{ units.}$$

4. Show that the lines $\frac{x-3}{3} = \frac{y-3}{-1}$, $z - 1 = 0$ and $\frac{x-6}{2} = \frac{z-1}{3}$, $y - 2 = 0$ intersect. Also find the point of intersection.

Solution:

Let $\frac{x-3}{3} = \frac{y-3}{-1} = s$, $z - 1 = 0$	Let $\frac{x-6}{2} = \frac{z-1}{3} = t$, $y - 2 = 0$
(x, y, z)	(x, y, z)
$= (3s + 3, \; -s + 3, \; 1)$ ……(1)	$= (2t + 6, \; 2, \; 3t + 1)$ ……(2)

From (1) and (2)

$(3s + 3, \; -s + 3, \; 1) = (2t + 6, \; 2, \; 3t + 1)$

Comparing the corresponding terms,

$3s + 3 = 2t + 6 \qquad -s + 3 = 2 \qquad 1 = 3t + 1$

We get $t = 0$ and $s = 1$

Point of intersection is $(6, 2, 1)$.

5. Show that the straight lines $x + 1 = 2y = -12z$ and $x = y + 2 = 6z - 6$ are skew and hence find the shortest distance between them.

Solution:

$x + 1 = 2y = -12z$	$x = y + 2 = 6z - 6$
$\frac{x+1}{-12} = \frac{y}{-6} = \frac{z}{1}$	$\frac{x}{6} = \frac{y+2}{6} = \frac{z-1}{1}$
$\vec{a} = -\hat{i} + 0\hat{j} + 0\hat{k}$,	$\vec{c} = 0\hat{i} - 2\hat{j} + \hat{k}$,
$\vec{b} = -12\hat{i} - 6\hat{j} + \hat{k}$	$\vec{d} = 6\hat{i} + 6\hat{j} + \hat{k}$

$\vec{b}$ and $\vec{d}$ two vectors are not parallel.

$\vec{c} - \vec{a} = \hat{i} - 2\hat{j} + \hat{k}$

$$\vec{b} \times \vec{d} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ -12 & -6 & 1 \\ 6 & 6 & 1 \end{vmatrix}$$

$$= -12\hat{i} + 18\hat{j} - 36\hat{k} = 6(-2\hat{i} + 3\hat{j} - 6\hat{k})$$

$|\vec{b} \times \vec{d}| = 6\sqrt{(-2)^2 + (3)^2 + (-6)^2} = 6(7) = 42$

$(\vec{c} - \vec{a}) \cdot (\vec{b} \times \vec{d})$

$= (\hat{i} - 2\hat{j} + \hat{k}) \cdot (-12\hat{i} + 18\hat{j} - 36\hat{k})$

$= -12 - 36 - 36 = -84$

The shortest distance between the two skew lines

$$\delta = \frac{|(\vec{c}-\vec{a})\cdot(\vec{b}\times\vec{d})|}{|\vec{b}\times\vec{d}|} = \frac{|-84|}{42} = \frac{84}{42} = 2 \text{ units.}$$

6. Find the parametric form of vector equation of the straight line passing through (-1, 2, 1) and parallel to the straight line $\vec{r} = (2\hat{i} + 3\hat{j} - \hat{k}) + t(\hat{i} - 2\hat{j} + \hat{k})$ and hence find the shortest distance between the lines.

Solution:

The straight line passing through (-1, 2, 1).

Let $\vec{c} = -\hat{i} + 2\hat{j} + \hat{k}$.

Parallel to the straight line

$\vec{r} = (2\hat{i} + 3\hat{j} - \hat{k}) + t(\hat{i} - 2\hat{j} + \hat{k})$.

$\vec{a} = 2\hat{i} + 3\hat{j} - \hat{k}$, $\vec{b} = \hat{i} - 2\hat{j} + \hat{k}$

$\vec{c} - \vec{a} = -3\hat{i} - \hat{j} + 2\hat{k}$

$$(\vec{c} - \vec{a}) \times \vec{b} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ -3 & -1 & 2 \\ 1 & -2 & 1 \end{vmatrix} = 3\hat{i} + 5\hat{j} + 7\hat{k}$$

The shortest distance between the two parallel lines

$$d = \frac{|(\vec{c}-\vec{a})\times\vec{b}|}{|\vec{b}|} = \frac{\sqrt{3^2+5^2+7^2}}{\sqrt{1^2+(-2)^2+1^2}} = \frac{\sqrt{83}}{\sqrt{6}} \text{ units.}$$

7. Find the foot of the perpendicular drawn from the point (5, 4, 2) to the line $\frac{x+1}{2}=\frac{y-3}{3}=\frac{z-1}{-1}$. Also, find the equation of the perpendicular.

Solution:

Let $\vec{a}=-\hat{\imath}+3\hat{\jmath}+\hat{k}$, $\vec{b}=2\hat{\imath}+3\hat{\jmath}-\hat{k}$ and

$\overrightarrow{OD}=5\hat{\imath}+4\hat{\jmath}+2\hat{k}$

Let F be the foot of the perpendicular from D to the straight line.

$$\frac{x+1}{2}=\frac{y-3}{3}=\frac{z-1}{-1}=\lambda$$

F is of the form $((2\lambda-1),(3\lambda+3),(-\lambda+1))$

$\overrightarrow{DF}=\overrightarrow{OF}-\overrightarrow{OD}=(2\lambda-6)\hat{\imath}+(3\lambda-1)\hat{\jmath}+(-\lambda-1)\hat{k}$

$\vec{b}$ is perpendicular to $\overrightarrow{DF}$ $\quad \vec{b}\cdot\overrightarrow{DF}=0$

$(2\hat{\imath}+3\hat{\jmath}-\hat{k})\cdot((2\lambda-6)\hat{\imath}+(3\lambda-1)\hat{\jmath}+(-\lambda-1)\hat{k})=0$

$2(2\lambda-6)+3(3\lambda-1)-(-\lambda-1)=0$

$-12+4\lambda-3+9\lambda+1+\lambda=0$

$14\lambda=14 \qquad \lambda=1$

Therefore, the coordinate of F is

$((2\lambda-1),(3\lambda+3),(-\lambda+1))=(1,6,0)$

Equation of $\overrightarrow{DF}=-4\hat{\imath}+2\hat{\jmath}-2\hat{k}$

Parametric form of vector equation:

$\vec{r}=(5\hat{\imath}+4\hat{\jmath}+2\hat{k})+m(-4\hat{\imath}+2\hat{\jmath}-2\hat{k}), m\in R$

Cartesian equations: $\frac{x-5}{-4}=\frac{y-4}{2}=\frac{z-2}{-2}$

EXERCISE 6.6

4. A plane passes through the point (-1, 1, 2) and the normal to the plane of magnitude $3\sqrt{3}$ makes equal acute angles with the coordinate axes. Find the equation of the plane.

Solution:

Given $\vec{a}=-\hat{\imath}+\hat{\jmath}+2\hat{k}$ $\quad |\vec{n}|=3\sqrt{3}$

Let α,β,γ be the acute angles made by $\vec{n}$ with coordinate axes and let l,m,n be its directional cosines.

$\cos\alpha=\cos\beta=\cos\gamma \implies l=m=n$

$\implies l^2+m^2+n^2=1 \implies l^2=\frac{1}{3} \implies l=\pm\frac{1}{\sqrt{3}}$

$\hat{n}=l\hat{\imath}+m\hat{\jmath}+n\hat{k} \implies \hat{n}=\frac{1}{\sqrt{3}}\hat{\imath}+\frac{1}{\sqrt{3}}\hat{\jmath}+\frac{1}{\sqrt{3}}\hat{k}$

$$=\pm\frac{1}{\sqrt{3}}(\hat{\imath}+\hat{\jmath}+\hat{k})$$

$\hat{n}=\frac{\vec{n}}{|\vec{n}|} \implies \vec{n}=|\vec{n}|\ \hat{n} \implies \vec{n}=3\sqrt{3}\ \times\pm\frac{1}{\sqrt{3}}(\hat{\imath}+\hat{\jmath}+\hat{k})$

$\vec{n}=3\hat{\imath}+3\hat{\jmath}+3\hat{k}$

$\vec{r}\cdot\vec{n}=\vec{a}\cdot\vec{n}$

$\vec{r}\cdot(3\hat{\imath}+3\hat{\jmath}+3\hat{k})=(-\hat{\imath}+\hat{\jmath}+2\hat{k})\cdot(3\hat{\imath}+3\hat{\jmath}+3\hat{k})$

Vector equation of the plane is

$\vec{r}\cdot(3\hat{\imath}+3\hat{\jmath}+3\hat{k})=-3+3+6=6$

$\vec{r}\cdot(\hat{\imath}+\hat{\jmath}+\hat{k})=2$

Cartesian equation of the plane $\quad x+y+z=2$

Example 6.43 Find the non-parametric form of vector equation, and Cartesian equation of the plane passing through the point (0, 1, -5) and parallel to the straight lines $\vec{r}=(\hat{\imath}+2\hat{\jmath}-4\hat{k})+s(2\hat{\imath}+3\hat{\jmath}+6\hat{k})$ and $\vec{r}=(\hat{\imath}-3\hat{\jmath}+5\hat{k})+t(\hat{\imath}+\hat{\jmath}-\hat{k})$.

Solution:

Given $\vec{a}=0\hat{\imath}+\hat{\jmath}-5\hat{k}$ $\quad \vec{b}=2\hat{\imath}+3\hat{\jmath}+6\hat{k}$ and

$\vec{c}=\hat{\imath}+\hat{\jmath}-\hat{k}$

One point and two parallel lines:

Cartesian form of equation:

$$\begin{vmatrix} x-x_1 & y-y_1 & z-z_1 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix}=0 \quad \begin{vmatrix} x-0 & y-1 & z+5 \\ 2 & 3 & 6 \\ 1 & 1 & -1 \end{vmatrix}=0$$

$x(-3-6)-(y-1)(-2-6)+(z+5)(2-3)=0$

$-9x+8y-8-z-5=0$

$-9x+8y-z=13$ or $9x-8y+z+13=0$

Non-parametric form of vector equation:

$[\vec{r}-\vec{a} \quad \vec{b} \quad \vec{c}]=0 \quad \vec{r}\cdot(-9\hat{\imath}+8\hat{\jmath}-\hat{k})=13$

Example 6.44 Find the vector parametric, vector non-parametric and Cartesian form of the equation of the plane passing through the points (-1, 2, 0), (2, 2, -1) and parallel to the straight line $\frac{x-1}{1}=\frac{2y+1}{2}=\frac{z+1}{-1}$.

Solution:

Given $\vec{a}=-1\hat{\imath}+2\hat{\jmath}+0\hat{k}$ $\quad \vec{b}=2\hat{\imath}+2\hat{\jmath}-\hat{k}$

and $\vec{c}=\hat{\imath}+\hat{\jmath}-\hat{k}$

Two points and one parallel line:

Parametric form of vector equation:

$\vec{r}=(1-s)\vec{a}+s\vec{b}+t\vec{c}$

$\vec{r}=(1-s)(-1\hat{\imath}+2\hat{\jmath}+0\hat{k})+s(2\hat{\imath}+2\hat{\jmath}-\hat{k})$
$+t(\hat{\imath}+\hat{\jmath}-\hat{k})$

Cartesian form of equation:

$$\begin{vmatrix} x-x_1 & y-y_1 & z-z_1 \\ x_2-x_1 & y_2-y_1 & z_2-z_1 \\ c_1 & c_2 & c_3 \end{vmatrix}=0$$

$$\begin{vmatrix} x+1 & y-2 & z-0 \\ 2-(-1) & 2-2 & -1-0 \\ 1 & 1 & -1 \end{vmatrix}=0$$

$$\begin{vmatrix} x+1 & y-2 & z-0 \\ 3 & 0 & -1 \\ 1 & 1 & -1 \end{vmatrix}=0$$

$(x+1)(1)-(y-2)(-2)+3z=0$

$x+1+2y-4+3z=0$

$x+2y+3z=3$

Non-parametric form of vector equation:

$[\vec{r}-\vec{a} \quad \vec{b}-\vec{a} \quad \vec{c}]=0$

$\vec{r}\cdot(\hat{\imath}+2\hat{\jmath}+3\hat{k})=3$

EXERCISE 6.7

1. Find the non-parametric form of vector equation, and Cartesian equation of the plane passing through the point (2,3,6) and parallel to the straight lines $\frac{x-1}{2}=\frac{y+1}{3}=\frac{z-3}{1}$ and $\frac{x+3}{2}=\frac{y-3}{-5}=\frac{z+1}{-3}$.

Solution:

Given $\vec{a}=2\hat{\imath}+3\hat{\jmath}+6\hat{k}$ $\vec{b}=2\hat{\imath}+3\hat{\jmath}+\hat{k}$ and $\vec{c}=2\hat{\imath}-5\hat{\jmath}-3\hat{k}$

One point and two parallel lines:

Cartesian form of equation:

$$\begin{vmatrix} x-x_1 & y-y_1 & z-z_1 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix}=0$$

$$\begin{vmatrix} x-2 & y-3 & z-6 \\ 2 & 3 & 1 \\ 2 & -5 & -3 \end{vmatrix}=0$$

$(x-2)(-4)-(y-3)(-8)+(z-6)(-16)=0$

$-4x+8+8y-24-16z+96=0$

$-4x+8y-16z+80=0$ or

$x-2y+4z-20=0$

Non-parametric form of vector equation:

$[\vec{r}-\vec{a}\quad \vec{b}\quad \vec{c}]=0$

$\vec{r}\cdot(\hat{\imath}-2\hat{\jmath}+4\hat{k})=20$

2. Find the parametric form of vector equation, and Cartesian equations of the plane passing through the points (2, 2, 1), (9, 3, 6) and perpendicular to the plane $2x+6y+6z=9$.

Solution:

Given $\vec{a}=2\hat{\imath}+2\hat{\jmath}+\hat{k}$ $\vec{b}=9\hat{\imath}+3\hat{\jmath}+6\hat{k}$ and $\vec{c}=2\hat{\imath}+6\hat{\jmath}+6\hat{k}$

Two points and one parallel line:

Parametric form of vector equation:

$\vec{r}=(1-s)\vec{a}+s\vec{b}+t\vec{c}\qquad s,t\in R$

$\vec{r}=(1-s)(2\hat{\imath}+2\hat{\jmath}+\hat{k})+s(9\hat{\imath}+3\hat{\jmath}+6\hat{k})$
$+t(2\hat{\imath}+6\hat{\jmath}+6\hat{k})$

Cartesian form of equation:

$$\begin{vmatrix} x-x_1 & y-y_1 & z-z_1 \\ x_2-x_1 & y_2-y_1 & z_2-z_1 \\ c_1 & c_2 & c_3 \end{vmatrix}=0$$

$$\begin{vmatrix} x-2 & y-2 & z-1 \\ 7 & 1 & 5 \\ 2 & 6 & 6 \end{vmatrix}=0$$

$(x-2)(-24)-(y-2)(32)+(z-1)(40)=0$

$-24x+48-32y+64+40z-40=0$

$-24x-32y+40z+72=0$

$3x+4y-5z-9=0$

Non-parametric form of vector equation:

$[\vec{r}-\vec{a}\quad \vec{b}-\vec{a}\quad \vec{c}]=0$

$\vec{r}\cdot(3\hat{\imath}+4\hat{\jmath}-5\hat{k})-9=0$

3. Find parametric form of vector equation and Cartesian equations of the plane passing through the points (2, 2, 1), (1, -2, 3) and parallel to the straight line passing through the points (2, 1, -3) and (-1, 5, -8).

Solution:

Let $(x_1,y_1,z_1)=(2,1,-3)$ and

$(x_2,y_2,z_2)=(-1,5,-8)$

$$\frac{x-x_1}{x_2-x_1}=\frac{y-y_1}{y_2-y_1}=\frac{z-z_1}{z_2-z_1}$$

The required line is $\frac{x-2}{-3}=\frac{y-1}{4}=\frac{z+3}{-5}$

$\vec{c}=-3\hat{\imath}+4\hat{\jmath}-5\hat{k}$

$\vec{a}=2\hat{\imath}+2\hat{\jmath}+\hat{k}$ and $\vec{b}=\hat{\imath}-2\hat{\jmath}+3\hat{k}$

Two points and one parallel line:

Parametric form of vector equation:

$\vec{r}=(1-s)\vec{a}+s\vec{b}+t\vec{c}\qquad s,t\in R$ (OR)

$\vec{r}=\vec{a}+s(\vec{b}-\vec{a})+t\vec{c}$

$\vec{r}=(2\hat{\imath}+2\hat{\jmath}+\hat{k})+s(-\hat{\imath}-4\hat{\jmath}+2\hat{k})+t(-3\hat{\imath}+4\hat{\jmath}-5\hat{k})$

Cartesian form of equation:

$$\begin{vmatrix} x-x_1 & y-y_1 & z-z_1 \\ x_2-x_1 & y_2-y_1 & z_2-z_1 \\ c_1 & c_2 & c_3 \end{vmatrix}=0$$

$$\begin{vmatrix} x-2 & y-2 & z-1 \\ -1 & -4 & 2 \\ -3 & 4 & -5 \end{vmatrix}=0$$

$(x-2)(12)-(y-2)(11)+(z-1)(-16)=0$

$12x-24-11y+22-16z+16=0$

$12x-11y-16z+14=0$

Non-parametric form of vector equation:

$[\vec{r}-\vec{a}\quad \vec{b}-\vec{a}\quad \vec{c}]=0$

$\vec{r}\cdot(12\hat{\imath}-11\hat{\jmath}-16\hat{k})+14=0$

4. Find the non-parametric form of vector equation of the plane passing through the point (1, -2, 4) and perpendicular to the plane $x+2y-3z=11$ and parallel to the line $\frac{x+7}{3}=\frac{y+3}{-1}=\frac{z}{1}$.

Solution:

Given $\vec{a}=\hat{\imath}-2\hat{\jmath}+4\hat{k}$ $\vec{b}=\hat{\imath}+2\hat{\jmath}-3\hat{k}$ and $\vec{c}=3\hat{\imath}-\hat{\jmath}+\hat{k}$

One point and two parallel lines:

Cartesian form of equation:

$$\begin{vmatrix} x-x_1 & y-y_1 & z-z_1 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix}=0$$

$$\begin{vmatrix} x-1 & y+2 & z-4 \\ 1 & 2 & -3 \\ 3 & -1 & 1 \end{vmatrix}=0$$

$(x-1)(-1)-(y+2)(10)+(z-4)(-7)=0$

$-x+1-10y-20-7z+28=0$

$-x-10y-7z+9=0$ (OR) $x+10y+7z-9=0$

Non-parametric form of vector equation:

$[\vec{r}-\vec{a}\quad \vec{b}\quad \vec{c}]=0$

$\vec{r}\cdot(-\hat{\imath}-10\hat{\jmath}-7\hat{k})+9=0$ (OR)

$\vec{r}\cdot(\hat{\imath}+10\hat{\jmath}+7\hat{k})-9=0$

5. Find the parametric form of vector equation, and Cartesian equations of the plane containing the line $\vec{r} = (\hat{\imath} - \hat{\jmath} + 3\hat{k}) + t(2\hat{\imath} - \hat{\jmath} + 4\hat{k})$ and perpendicular to plane $\vec{r} \cdot (\hat{\imath} + 2\hat{\jmath} + \hat{k}) = 8$.

Solution:

Given $\vec{a} = \hat{\imath} - \hat{\jmath} + 3\hat{k}$ $\quad \vec{b} = 2\hat{\imath} - \hat{\jmath} + 4\hat{k}$ and $\vec{c} = \hat{\imath} + 2\hat{\jmath} + \hat{k}$

Parametric form of vector equation:

$\vec{r} = \vec{a} + s\vec{b} + t\vec{c}$

$\vec{r} = (\hat{\imath} - \hat{\jmath} + 3\hat{k}) + s(2\hat{\imath} - \hat{\jmath} + 4\hat{k}) + t(\hat{\imath} + 2\hat{\jmath} + \hat{k})$

One point and two parallel lines:

Cartesian form of equation:

$$\begin{vmatrix} x - x_1 & y - y_1 & z - z_1 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix} = 0$$

$$\begin{vmatrix} x - 1 & y + 1 & z - 3 \\ 2 & -1 & 4 \\ 1 & 2 & 1 \end{vmatrix} = 0$$

$(x-1)(-9) - (y+1)(-2) + (z-3)(5) = 0$

$-9x + 9 + 2y + 2 + 5z - 15 = 0$

$-9x + 2y + 5z - 4 = 0$ or $9x - 2y - 5z + 4 = 0$

6. Find the parametric vector, non-parametric vector and Cartesian form of the equations of the plane passing through the points (3, 6, -2), (-1, -2, 6), and (6, 4, -2).

Solution:

Given $\vec{a} = 3\hat{\imath} + 6\hat{\jmath} - 2\hat{k}$ $\quad \vec{b} = -\hat{\imath} - 2\hat{\jmath} + 6\hat{k}$ and

$\vec{c} = 6\hat{\imath} + 4\hat{\jmath} - 2\hat{k}$

Parametric form of vector equation:

$$\vec{r} = (1 - s - t)\vec{a} + s\vec{b} + t\vec{c}$$

$\vec{r} = (1-s-t)(3\hat{\imath} + 6\hat{\jmath} - 2\hat{k}) + s(-\hat{\imath} - 2\hat{\jmath} + 6\hat{k}) + t(6\hat{\imath} + 4\hat{\jmath} - 2\hat{k})$

Cartesian form of equation:

$$\begin{vmatrix} x - x_1 & y - y_1 & z - z_1 \\ x_2 - x_1 & y_2 - y_1 & z_2 - z_1 \\ x_3 - x_1 & y_3 - y_1 & z_3 - z_1 \end{vmatrix} = 0$$

$$\begin{vmatrix} x - 3 & y - 6 & z + 2 \\ -4 & -8 & 8 \\ 3 & -2 & 0 \end{vmatrix} = 0$$

$(x-3)(16) - (y-6)(-24) + (z+2)(32) = 0$

$16x - 48 + 24y - 144 + 64z + 64 = 0$

$16x + 24y + 32z - 128 = 0$

$2x + 3y + 4z - 16 = 0$

Non-parametric form of vector equation:

$[\vec{r} - \vec{a} \quad \vec{b} - \vec{a} \quad \vec{c} - \vec{a}] = 0$

$\vec{r} \cdot (2\hat{\imath} + 3\hat{\jmath} + 4\hat{k}) - 16 = 0$

7. Find the non-parametric form of vector equation, and Cartesian equations of the plane $\vec{r} = (6\hat{\imath} - \hat{\jmath} + \hat{k}) + s(-\hat{\imath} + 2\hat{\jmath} + \hat{k}) + t(-5\hat{\imath} - 4\hat{\jmath} - 5\hat{k})$.

Solution:

Parametric form of vector equation:

$\vec{r} = \vec{a} + s\vec{b} + t\vec{c}$

$\vec{r} = (6\hat{\imath} - \hat{\jmath} + \hat{k}) + s(-\hat{\imath} + 2\hat{\jmath} + \hat{k}) + t(-5\hat{\imath} - 4\hat{\jmath} - 5\hat{k})$

Given $\vec{a} = 6\hat{\imath} - \hat{\jmath} + \hat{k}$ $\quad \vec{b} = -\hat{\imath} + 2\hat{\jmath} + \hat{k}$ and $\vec{c} = -5\hat{\imath} - 4\hat{\jmath} - 5\hat{k}$

Cartesian form of equation:

$$\begin{vmatrix} x - x_1 & y - y_1 & z - z_1 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix} = 0$$

$$\begin{vmatrix} x - 6 & y + 1 & z - 1 \\ -1 & 2 & 1 \\ -5 & -4 & -5 \end{vmatrix} = 0$$

$(x-6)(-6) - (y+1)(10) + (z-1)(14) = 0$

$-6x + 36 - 10y - 10 + 14z - 14 = 0$

$-6x - 10y + 14z + 12 = 0$

$3x + 5y - 7z - 6 = 0$

Non-parametric form of vector equation:

$[\vec{r} - \vec{a} \quad \vec{b} \quad \vec{c}] = 0$

$\vec{r} \cdot (3\hat{\imath} + 5\hat{\jmath} - 7\hat{k}) - 6 = 0$

EXERCISE 6.8

3. If the straight lines $\frac{x-1}{1} = \frac{y-2}{2} = \frac{z-3}{m^2}$ and $\frac{x-3}{1} = \frac{y-2}{m^2} = \frac{z-1}{2}$ are coplanar, find the distinct real values of m.

Solution:

$\frac{x-1}{1} = \frac{y-2}{2} = \frac{z-3}{m^2}$ and $\frac{x-3}{1} = \frac{y-2}{m^2} = \frac{z-1}{2}$

Let $\vec{a} = \hat{\imath} + 2\hat{\jmath} + 3\hat{k}$ $\quad \vec{b} = \hat{\imath} + 2\hat{\jmath} + m^2\hat{k}$,

$\vec{c} = 3\hat{\imath} + 2\hat{\jmath} + \hat{k}$ and $\vec{d} = \hat{\imath} + m^2\hat{\jmath} + 2\hat{k}$

$\vec{c} - \vec{a} = 2\hat{\imath} + 0\hat{\jmath} - 2\hat{k}$

$$(\vec{c} - \vec{a}) \cdot (\vec{b} \times \vec{d}) = \begin{vmatrix} 2 & 0 & -2 \\ 1 & 2 & m^2 \\ 1 & m^2 & 2 \end{vmatrix}$$

$= (4 - m^4)(2) - (2 - m^2)(0) + (m^2 - 2)(-2)$

$(\vec{c} - \vec{a}) \cdot (\vec{b} \times \vec{d}) = 0$

$(4 - m^4)2 - (2 - m^2)0 + (m^2 - 2)(-2) = 0$

$(4 - m^4)2 = (m^2 - 2)2$

$4 - m^4 = m^2 - 2$

$m^4 + m^2 - 6 = 0 \qquad (m^2 + 3)(m^2 - 2) = 0$

$m = \pm\sqrt{2}$ and $m = -3$ (not possible)

4. If the straight lines $\frac{x-1}{2}=\frac{y+1}{\lambda}=\frac{z}{2}$ and $\frac{x+1}{5}=\frac{y+1}{2}=\frac{z}{\lambda}$ are coplanar, find λ and equations of the planes containing these two lines.

Solution:

$\frac{x-1}{2}=\frac{y+1}{\lambda}=\frac{z}{2}$ and $\frac{x+1}{5}=\frac{y+1}{2}=\frac{z}{\lambda}$

Let $\vec{a}=\hat{\imath}-\hat{\jmath}+0\,\hat{k}$ $\quad \vec{b}=2\,\hat{\imath}+\lambda\,\hat{\jmath}+2\,\hat{k}$,

$\vec{c}=-\hat{\imath}-\hat{\jmath}+0\,\hat{k}$ and $\vec{d}=5\,\hat{\imath}+2\,\hat{\jmath}+\lambda\,\hat{k}$

$\vec{c}-\vec{a}=-2\,\hat{\imath}+0\,\hat{\jmath}+0\,\hat{k}$

$$(\vec{c}-\vec{a})\cdot(\vec{b}\times\vec{d})=\begin{vmatrix}-2 & 0 & 0\\ 2 & \lambda & 2\\ 5 & 2 & \lambda\end{vmatrix}$$

$=(\lambda^2-4)(-2)-(2\lambda-10)(0)+(4-5\lambda)(0)$

$(\vec{c}-\vec{a})\cdot(\vec{b}\times\vec{d})=0$

$(-2)(\lambda^2-4)=0 \qquad \lambda^2=4 \qquad \lambda=\pm 2$

Cartesian equation:

$$\begin{vmatrix}x-1 & y+1 & z-0\\ 2 & \lambda & 2\\ 5 & 2 & \lambda\end{vmatrix}=0$$

When $\lambda=-2$

$$\begin{vmatrix}x-1 & y+1 & z-0\\ 2 & -2 & 2\\ 5 & 2 & -2\end{vmatrix}=0$$

$(x-1)(4-4)-(y+1)(-4-10)+z(4+10)=0$

$14(y+1)+14z=0$

$\div(14) \qquad y+1+z=0$

When $\lambda=2$

$$\begin{vmatrix}x-1 & y+1 & z-0\\ 2 & 2 & 2\\ 5 & 2 & 2\end{vmatrix}=0$$

$(x-1)(4-4)-(y+1)(4-10)+z(4-10)=0$

$6(y+1)-6z=0$

$\div(6) \qquad y+1-z=0$

Cartesian equation is

$y+z+1=0$ and $y-z+1=0$

Example 6.50 Find the distance of the point (5, -5, -10) from the point of intersection of a straight line passing through the points A(4, 1, 2) and B(7, 5, 4) with the plane $x-y+z=5$.

Solution:

The Cartesian equation of the straight line joining A and B is

$$\frac{x-x_1}{x_2-x_1}=\frac{y-y_1}{y_2-y_1}=\frac{z-z_1}{z_2-z_1}$$

The required line is $\frac{x-4}{3}=\frac{y-1}{4}=\frac{z-2}{2}$

$$\frac{x-4}{3}=\frac{y-1}{4}=\frac{z-2}{2}=t$$

$(x,y,z)=(3t+4,4t+1,2t+2)$ ……….(1)

$x-y+z=5 \Rightarrow 3t+4-4t-1+2t+2=5$

$\Rightarrow t=0$

Point of intersection of the straight line is $(4,1,2)$.

Now, the distance between the two points $(4,1,2)$and$(5,-5,-10)$is

$$d=\sqrt{(5-4)^2+(-5-1)^2+(-10-2)^2}$$

$=\sqrt{1+36+144}=\sqrt{181}$ units.

Example 7.1 For the function $f(x) = x^2$, $x \in [0, 2]$ compute the average rate of changes in the subintervals $[0,\ 0.5], [0.5,\ 1], [1,\ 1.5], [1.5,\ 2]$ and the instantaneous rate of changes at the points $x = 0.5,\ 1,\ 1.5,\ 2$.

Solution:

The average rate of change in an interval $[a, b]$ is $\frac{f(b)-f(a)}{b-a}$ whereas, the instantaneous rate of change at a point x is $f'(x)$ for the given function. They are respectively, $b + a$ and 2x.

Rate of changes

a	b	x	Average rate is $\frac{f(b)-f(a)}{b-a} = b+a$	Instantaneous rate is $f'(x) = 2x$
0	0.5	0.5	0.5	1
0.5	1	1	1.5	2
1	1.5	1.5	2.5	3
1.5	2	2	3.5	4

Example 7.5 A particle is fired straight up from the ground to reach a height of s feet in t seconds, where $s(t) = 128t - 16t^2$.
(1) Compute the maximum height of the particle reached.
(2) What is the velocity when the particle hits the ground?

Solution:

(1) At the maximum height, the velocity $v(t)$ of the particle is zero.

$V(t) = \frac{ds}{dt} = 128 - 32t$

$v(t) = 0$

$128 - 32t = 0 \qquad t = 4.$

At $t = 4$ is

$s(4) = 128(4) - 16(4)^2 = 512 - 256 = 256$ft.

(2) When the particle hits the ground then $s = 0$.

$s(t) = 128t - 16t^2 = 0$

$t = 0, 8$ seconds.

The particle hits the ground at t =8 seconds.

The velocity when it hits the ground is

$v(8) = 128 - 32(8) = 128 - 256 = -128\ ft/s.$

Example 7.6 A particle moves along a horizontal line such that its position at any time $t \geq 0$ is given by $s(t) = t^3 - 6t^2 + 9t + 1$, where s is measured in metres and t in seconds?
(1) At what time the particle is at rest?
(2) At what time the particle changes direction?
(3) Find the total distance travelled by the particle in the first 2 seconds.

Solution:

$s(t) = t^3 - 6t^2 + 9t + 1$	S(0) = 0+1=1
$v(t) = 3t^2 - 12t^2 + 9$	S(1) = 1-6+9+1=5
$a(t) = 6t - 24$	S(2) = 8-24+18+1=3

(1) when $v = 0$, $\quad 3t^2 - 12t^2 + 9 = 0$

$t = 1, \qquad t = 3$

(2) The particle changes direction when $v(t)$ changes its sign. Now.

If $0 \leq t < 1$ then both $(t-3) < 0$ *and* $(t-1) < 0$ and hence $v(t) > 0$.

If $1 \leq t < 3$ then $(t-3) < 0$ *and* $(t-1) > 0$. and hence $v(t) < 0$.

If $t > 3$ then both $(t-3) > 0$ *and* $(t-1) > 0$ and hence $v(t) > 0$.

Therefore, the particle changes direction when t =1 and $t = 3$.

(3) The total distance travelled by the particle from time $t = 0$ to $t = 2$ is given by,

$|s(0) - s(1)| + |s(1) - s(2)|$

$= |1-5| + |5-3| = 6$ metres.

Example 7.7 If we blow air into a balloon of spherical shape at a rate of $1000\ cm^3$ per second. At what rate the radius of the baloon changes when the radius is 7cm? Also compute the rate at which the surface area changes.

Solution:

Given: $\frac{dV}{dt} = 1000, \quad r = 7$

The volume of the baloon of radius r is

$V = \frac{4}{3}\pi r^3.$

$\frac{dV}{dt} = 4\pi r^2 \times \frac{dr}{dt}$

$\frac{dr}{dt} = \frac{1000}{4\pi r^2} = \frac{1000}{4\pi \times 49} = \frac{250}{49\pi}$

The surface area S of the baloon is $S = 4\pi r^2$

$\frac{ds}{dt} = 8\pi \times r \times \frac{dr}{dt} = 8\pi \times 7 \times \frac{250}{49\pi} = \frac{2000}{7}$

$\frac{dr}{dt} = \frac{250}{49\pi}$ cm/sec and $\frac{ds}{dt} = \frac{2000}{7}\ cm^2/sec.$

Example 7.8 The price of a product is related to the number of units available (supply) by the equation $Px + 3P - 16x = 234$, where P is the price of the product per unit in Rupees and x is the number of units. Find the rate at which the price is changing with respect to time when 90 units are available and the supply is increasing at a rate of 15 units/week.

Solution:

$$Px + 3P - 16x = 234$$
$$Px + 3P = 234 + 16x$$
$$P(x+3) = 234 + 16x$$
$$P = \frac{234+16x}{x+3} = \frac{16x+48+186}{x+3}$$
$$P = \frac{16(x+3)+186}{x+3} = 16 + \frac{186}{x+3}$$

$$\frac{dP}{dt} = 0 + \frac{(-186)}{(x+3)^2} \times \frac{dx}{dt}$$

When $x = 90$, $\frac{dx}{dt} = 15$ we get

$$\frac{dP}{dt} = \frac{-186}{(93)^2} \times 15 = -\frac{10}{31} \approx -0.32$$

The price is changing, in fact decreasing at the rate of Rs. 0.32 per unit.

Example 7.9 Salt is poured from a conveyer belt at a rate of 30 cubic metre per minute forming a conical pile with a circular base whose height and diameter of base are always equal. How fast is the height of the pile increasing when the pile is 10 metre high?

Solution:

Given: $\frac{dV}{dt} = 30\ mtr^3/min$ and

$h^2 = 100\ metre$

Let h and r be the height and the base radius.

Therefore $h = 2r$, $r = \frac{h}{2}$, $r^2 = \frac{h^2}{4}$

Let V be the volume of the salt cone.

$$V = \frac{1}{3}\pi r^2 h = \frac{1}{3}\pi\left(\frac{h^2}{4}\right)h$$
$$V = \frac{1}{12}\pi h^3$$
$$\frac{dV}{dt} = \frac{1}{12}\left(3\pi h^2\right)\frac{dh}{dt}$$
$$30 = \frac{1}{4}\left(\pi h^2\right)\frac{dh}{dt}$$
$$\frac{dh}{dt} = \frac{120}{100\pi} = \frac{6}{5\pi} \text{ mtr/ min}$$

Example 7.10 (Two variable related rate problem) A road running north to south crosses a road going east to west at the point P. Car A is driving north along the first road, and car B is driving east along the second road. At a particular time car A is 10 kilometres to the north of P and traveling at 80 km/hr, while car B is 15 kilometres to the east of P and traveling at 100 km/hr. How fast is the distance between the two cars changing?

Solution:

Let $a(t)$ be the distance of car A north of P at time t, and $b(t)$ the distance of car B east of P at time t, and $c(t)$ be the distance from car A to car B at time t.

By the Pythagorean Theorem,

$$c^2 = a^2 + b^2$$
$$c = \sqrt{a^2+b^2}$$
$$2cc' = 2aa' + 2bb'$$
$$c' = \frac{aa'+bb'}{c} = \frac{aa'+bb'}{\sqrt{a^2+b^2}}$$
$$c' = \frac{10(80)+15(100)}{\sqrt{10^2+15^2}} = \frac{460}{\sqrt{13}}$$
$$c' \approx 127.6 \text{ km/hr}$$

EXERCISE 7.1

1. A point moves along a straight line in such a way that after t seconds its distance from the origin is $s = 2t^2 + 3t$ metres.

(i) Find the average velocity of the points between $t = 3$ and $t = 6$ seconds.

(ii) Find the instantaneous velocities at $t = 3$ and $t = 6$ seconds.

Solution:

(i) Average velocity $= \frac{s(6)-s(3)}{6-3} = \frac{90-27}{3}$

$$= \frac{63}{3} = 21 \text{ m/s}$$

(ii) $v = 4t + 3$

$v(3) = 15$ m/s $\qquad v(6) = 27$ m/s

2. A camera is accidentally knocked off an edge of a cliff 400 ft high. The camera falls a distance of $s = 16t^2$ in t seconds.
(i) How long does the camera fall before it hits the ground?
(ii) What is the average velocity with which the camera falls during the last 2 seconds?
(iii) What is the instantaneous velocity of the camera when it hits the ground?

Solution:

(i) when $s = 400$

$16t^2 = 400 \quad t^2 = 25 \qquad t = 5$ sec

(ii) Average velocity $= \frac{s(5)-s(3)}{5-3} = \frac{400-144}{2}$

$= \frac{256}{2} = 128$ ft/sec.

(iii) velocity $v = 32t$

When $t = 5$, we get $v = 160$ ft/sec

3. A particle moves along a line according to the law $s(t) = 2t^3 - 9t^2 + 12t - 4$, where $t \geq 0$.
(i) At what times the particle changes direction?
(ii) Find the total distance travelled by the particle in the first 4 seconds.
(iii) Find the particle's acceleration each time the velocity is zero.

Solution:

$s(t) = 2t^3 - 9t^2 + 12t - 4$ $v(t) = 6t^2 - 18t + 12$ $a(t) = 12t - 18$	S(0) = 0 – 4 = - 4 S(1) = 2 - 9+12 - 4= 1 S(2) = 16 – 36 + 24 - 4=0 S(4) = 128-144+48-4=28

(i) When $v(t) = 0$

$6(t-1)(t-2) = 0 \qquad t = 1, 2$

(ii) The total distance travelled by the particle in the first 4 seconds is

$|s(0) - s(1)| + |s(1) - s(2)| + |s(2) - s(4)|$

$= 5 + 1 + 28 = 34$ metres.

(iii) The particle's acceleration each time the velocity is zero.

$a(t) = 12t - 18$

$a(1) = -6\ m/s^2$ and

$a(2) = 6\ m/s^2$

7. A beacon makes one revolution every 10 seconds. It is located on a ship which is anchored 5 km from a straight shore line. How fast is the beam moving along the shore line when it makes an angle of 45° with the shore?

Solution:

Given: $\frac{d\theta}{dt} = \frac{2\pi}{10} = \frac{\pi}{5}$ rad/sec

$x = 5\tan\theta \qquad \frac{dx}{dt} = 5\sec^2\theta\frac{d\theta}{dt}$

$\left.\frac{dx}{dt}\right]_{\theta=\frac{\pi}{4}} = 5\left(\sec^2\frac{\pi}{4}\right)\frac{\pi}{5} = \left(\sqrt{2}\right)^2\pi = 2\pi$ km/sec

8. A conical water tank with vertex down of 12 metres height has a radius of 5 metres at the top. If water flows into the tank at a rate 10 cubic m/min, how fast is the depth of the water increases when the water is 8 metres deep?

Solution:

Given: $\frac{dV}{dt} = 10$ and $h = 8$

$12r = 5h \qquad r = \frac{5h}{12}$

Volume of the cone $V = \frac{1}{3}\pi r^2 h$

$V = \frac{1}{3}\pi\left(\frac{5h}{12}\right)^2 h = \frac{25}{3\times144}\pi h^3$

$\frac{dV}{dt} = \frac{25}{3\times144}\pi(3h^2)\frac{dh}{dt}$

$\frac{dh}{dt} = \frac{144\times10}{25\times\pi\times64} = \frac{9}{10\pi}$ m/min

9. A ladder 17 metre long is leaning against the wall. The base of the ladder is pulled away from the wall at a rate of 5 m/s. When the base of the ladder is 8 metres from the wall. (i) How fast is the top of the ladder moving down the wall? (ii) At what rate, the area of the triangle formed by the ladder, wall, and the floor, is changing?

Solution:

Let x be the distance between the wall and the base, y be the distance between top of the ladder and bottom of the wall.

Given : $\frac{dx}{dt} = 5$

$x^2 + y^2 = 17^2 \qquad 2x\frac{dx}{dt} + 2y\frac{dy}{dt} = 0$

$\frac{dy}{dt} = -\frac{x}{y}\frac{dx}{dt} \qquad \frac{dy}{dt} = -\frac{5x}{y}$

When $x = 8$,

$y^2 = 17^2 - 8^2 = 225 \qquad y = 15$

When $x = 8$, $\frac{dy}{dt} = -\frac{5(8)}{15} = -\frac{8}{3}$ m/s.

(ii) $A = \frac{1}{2}xy$

$\frac{dA}{dt} = \frac{1}{2}\left(x\frac{dy}{dt} + y\frac{dx}{dt}\right) = \frac{1}{2}\left(8 \times \frac{-8}{3} + 15 \times 5\right)$

$= \frac{1}{6}(-64 + 225) = \frac{161}{6} = 26.83\ m^2/s$

10, A police jeep, approaching an orthogonal intersection from the northern direction, is chasing a speeding car that has turned and moving straight east. When the jeep is 0.6 km north of the intersection and the car is 0.8 km to the east. The police determine with a radar that the distance between them and the car is increasing at 20 km/hr. If the jeep is moving at 60 km/hr at the instant of measurement, what is the speed of the car?

Solution:

Let x and y be the positions of the car ad jeep, z be the distance between them at any time t.

Given: $\frac{dy}{dt} = -60, \quad \frac{dz}{dt} = 20$ and

$x = 0.8\ and\ y = 0.6, z = 1$

$x^2 + y^2 = z^2 \qquad 2x\frac{dx}{dt} + 2y\frac{dy}{dt} = 2z\frac{dz}{dt}$

$x\frac{dx}{dt} + y\frac{dy}{dt} = z\frac{dz}{dt} \qquad x\frac{dx}{dt} = z\frac{dz}{dt} - y\frac{dy}{dt}$

$(0.8)\frac{dx}{dt} = (1)(20) - (0.6)(-60)$

$\frac{dx}{dt} = \frac{20+36}{0.8} = 70$ km/hr.

Example 7.13 Find the equation of the tangent and normal to the Lissajous curve given by $x = 2cos3t$ and $y = 3sin\,2t, t \in \mathbb{R}$.

Solution:

$y = 3sin\,2t \qquad x = 2cos3t$

$\frac{dy}{dt} = 6\,cos2t \qquad \frac{dx}{dt} = -6sin\,3t$

$m = \frac{dy}{dx} = \frac{dy/dt}{dx/dt} = \frac{6\cos 2t}{-6\sin 3t} = \frac{-\cos 2t}{\sin 3t}$

The tangent at any point is $y - y_1 = m(x - x_1)$

$y - 3\sin 2t = \frac{-\cos 2t}{\sin 3t}(x - 2\cos 3t)$

$y\sin 3t - 3\sin 3t\sin 2t = -x\cos 2t + 2\cos 2t\cos 3t$

$x\cos 2t + y\sin 3t = 3\sin 3t\sin 2t + 2\cos 2t\cos 3t$

Hence, the equation of the normal is

$y - y_1 = \frac{-1}{m}(x - x_1)$

$y - 3\sin 2t = \frac{\sin 3t}{\cos 2t}(x - 2\cos 3t)$

$y\cos 2t - 3\cos 2t\sin 2t = x\sin 3t - 2\sin 3t\cos 3t$

$x\sin 3t - y\cos 2t = 2\sin 3t\cos 3t - 3\cos 2t\sin 2t$

$x\sin 3t - y\cos 2t = \sin 6t - \frac{3}{2}\sin 4t$

Example 7.14 Find the acute angle between $y = x^2$ and $y = (x-3)^2$.

Solution:

Given : $y = x^2$ .... (1) and $y = (x-3)^2$ ... (2)

From (1) and (2) $\quad x^2 = (x-3)^2$

$x^2 = x^2 - 6x + 9$

$-6x + 9 = 0 \qquad x = \frac{3}{2}$.

Substitute in eqn (1) $\quad y = \left(\frac{3}{2}\right)^2 = \frac{9}{4}$.

The point of intersection is $(x, y) = \left(\frac{3}{2}, \frac{9}{4}\right)$

$y = x^2$	$y = (x-3)^2$
$\frac{dy}{dx} = 2x$	$\frac{dy}{dx} = 2(x-3)$
$\left.\frac{dy}{dx}\right]_{\left(\frac{3}{2},\frac{9}{4}\right)} = 2\left(\frac{3}{2}\right) = 3$	$\left.\frac{dy}{dx}\right]_{\left(\frac{3}{2},\frac{9}{4}\right)} = 2\left(\frac{3}{2} - 3\right) = -3$

$\tan\theta = \left|\frac{3-(-3)}{1+3(-3)}\right| = \frac{3}{4} \qquad \theta = \tan^{-1}\left(\frac{3}{4}\right)$

Example 7.15 Find the acute angle between the curves $y = x^2$ and $x = y^2$ at their points of intersection $(0,0), (1,1)$.

Solution:

Let θ_1 and θ_2 be the acute angles at (0,0) and (1,1) respectively.

$y = x^2$	$x = y^2$
$m_1 = \frac{dy}{dx} = 2x$	$m_2 = \frac{dy}{dx} = \frac{1}{2y}$
At (0,0) $m_1 = 0$	At (0,0) $m_1 = \infty$
At (1, 1) $m_1 = 2$	At (1, 1) $m_1 = \frac{1}{2}$
$\tan\theta_1 = \lim_{(x,y)\to(0,0)} \left\|\frac{m_1 - m_2}{1 + m_1 m_2}\right\|$	$\tan\theta_2 = \lim_{(x,y)\to(1,1)} \left\|\frac{m_1 - m_2}{1 + m_1 m_2}\right\|$
$\tan\theta_1 = \left\|\frac{0-\infty}{1+0(\infty)}\right\|$	$\tan\theta_2 = \left\|\frac{2-\frac{1}{2}}{1+2\left(\frac{1}{2}\right)}\right\| = \frac{3/2}{2}$
$\theta_1 = \tan^{-1}\infty = \frac{\pi}{2}$	$\theta_2 = \tan^{-1}\left(\frac{3}{4}\right)$

Example 7.16 Find the angle of intersection of the curve $y = \sin x$ with the positive x -axis.

Solution:

The curve $y = sinx$ intersects the positive x -axis. When $y = 0$ which gives

$x = n\pi$, n =1,2,3,…… .

$y = sinx$ $\quad \frac{dy}{dx} = \cos x \quad \cos(n\pi) = (-1)^n$

The required angle of intersection is

$$\tan^{-1}(-1)^n = \begin{cases} \frac{\pi}{4}, & when\ n\ is\ even \\ \frac{3\pi}{4}, & when\ n\ is\ odd \end{cases}$$

Example 7.17 If the curves $ax^2 + by^2 = 1$ and $c\,x^2 + d\,y^2 = 1$ intersect each other orthogonally then, $\frac{1}{a} - \frac{1}{b} = \frac{1}{c} - \frac{1}{d}$.

Solution:

The two curves intersect at a point (x_0, y_0) if

$$(a-c){x_0}^2 + (b-d){y_0}^2 = 0 \ldots\ldots (1)$$

$ax^2 + by^2 = 1$	$c\,x^2 + d\,y^2 = 1$
$2ax + 2by\frac{dy}{dx} = 0$	$2cx + 2dy\frac{dy}{dx} = 0$
$\frac{dy}{dx} = \frac{-ax}{by}$	$\frac{dy}{dx} = \frac{-cx}{dy}$

Now, two curves cut orthogonally,

$$\left(\frac{-ax_0}{by_0}\right)\left(\frac{-cx_0}{dy_0}\right) = -1$$

$$ac\,{x_0}^2 + bd\,{y_0}^2 = 0 \ldots\ldots (2)$$

From (1) and (2), $\frac{a-c}{ac} = \frac{b-d}{bd}$

$$\Rightarrow \frac{1}{c} - \frac{1}{a} = \frac{1}{d} - \frac{1}{b} \quad \Rightarrow \frac{1}{a} - \frac{1}{b} = \frac{1}{c} - \frac{1}{d}.$$

Example 7.18 Prove that the ellipse $x^2 + 4y^2 = 8$ and the hyperbola $x^2 - 2y^2 = 4$ intersect orthogonally.

Solution:

The two curves intersect at a point (a, b),

$x^2 + 4y^2 = 8$	$x^2 - 2y^2 = 4$
$a^2 + 4b^2 = 8 \ldots (1)$	$a^2 - 2b^2 = 4 \ldots (2)$
$2x + 8y\frac{dy}{dx} = 0$	$2x - 4y\frac{dy}{dx} = 0$
$\frac{dy}{dx} = \frac{-x}{4y}$	$\frac{dy}{dx} = \frac{x}{2y}$
At (a, b) $\left.\frac{dy}{dx}\right]_{(a,b)} = \frac{-a}{4b}$	At (a, b) $\left.\frac{dy}{dx}\right]_{(a,b)} = \frac{a}{2b}$

From(1)-(2) $\quad 6b^2 = 4 \quad b^2 = \frac{2}{3}$

Substitute in eqn (1) $\quad a^2 = 8 - 4\left(\frac{2}{3}\right) = 8 - \frac{8}{3} = \frac{16}{3}$

$$m_1 m_2 = \left(\frac{-a}{4b}\right)\left(\frac{a}{2b}\right) = \frac{-a^2}{8b^2} = \frac{-16/3}{16/3} = -1$$

Hence, the curves cut orthogonally.

EXERCISE 7.2

6. Find the equations of the tangents to the curve $y = 1 + x^3$ for which the tangent is orthogonal with the line $x + 12y = 12$.

Solution:

$y = 1 + x^3$	$x + 12y = 12$
$m_1 = \frac{dy}{dx} = 3x^2$	$1 + 12\frac{dy}{dx} = 0$
	$m_2 = \frac{dy}{dx} = \frac{-1}{12}$

The tangent is orthogonal $m_1 m_2 = -1$

$$3x^2\left(\frac{-1}{12}\right) = -1$$

$3x^2 = 12 \quad x^2 = 4 \quad x = \pm 2$

When $x = 2$ $\quad y = 1 + 8 = 9$

When $x = -2$ $\quad y = 1 - 8 = -7$

Equation of tangent at $(2, 9)$ is

$y - 9 = 12(x - 2) \Rightarrow \quad 12x - y - 15 = 0$

Equation of normal at $(-2, -7)$ is

$y + 7 = 12(x + 2) \Rightarrow \quad 12x - y + 17 = 0$

7. Find the equations of the tangents to the curve $y = \frac{x+1}{x-1}$ which are parallel to the line $x+2y = 6$.

Solution:

$y = \frac{x+1}{x-1}$ $\frac{dy}{dx} = \frac{(x-1)-(x+1)}{(x-1)^2}$ $= \frac{-2}{(x-1)^2}$	$x+2y = 6$ $1 + 2\frac{dy}{dx} = 0$ $\frac{dy}{dx} = \frac{-1}{2}$

The tangent is parallel $\quad m_1 = m_2$

$\frac{-2}{(x-1)^2} = \frac{-1}{2} \qquad (x-1)^2 = 4$

$x - 1 = \pm 2 \qquad x = 3 \text{ and } -1$

When $x = 3 \qquad y = \frac{3+1}{3-1} = 2$

When $x = -1 \qquad y = \frac{-1+1}{-1-1} = 0$

The points are $(-1, 0)$ *and* $(3,2)$

The equation of tangent at $(-1, 0)$ is

$y - 0 = \frac{-1}{2}(x+1) \implies x + 2y + 1 = 0$

The equation of normal at $(3, 2)$ is

$y - 2 = \frac{-1}{2}(x-3) \implies x + 2y - 7 = 0$

8. Find the equation of tangent and normal to the curve given by $x = 7\cos t$ and $y = 2\sin t, t \in \mathbb{R}$ at any point on the curve.

Solution:

$x = 7\cos t$ $\frac{dx}{dt} = -7\sin t$	$y = 2\sin t$ $\frac{dy}{dt} = 2\cos t$

$\frac{dy}{dx} = \frac{dy/dt}{dx/dt} = \frac{2\cos t}{-7\sin t} = \frac{-2}{7}\cot t$

The point is $(7\cos t, 2\sin t)$.

The equation of tangent at $(7\cos t, 2\sin t)$ is

$y - 2\sin t = \frac{2\cos t}{-7\sin t}(x - 7\cos t)$

$-7y\sin t + 14\sin^2 t = 2x\cos t - 14\cos^2 t$

$2x\cos t + 7y\sin t = 14$

The equation of normal at $(7\cos t, 2\sin t)$ is

$y - 2\sin t = \frac{7\sin t}{2\cos t}(x - 7\cos t)$

$2y\cos t - 4\cos t\sin t = 7x\sin t - 49\sin t\cos t$

$7x\sin t - 2y\cos t = 45\sin t\cos t$

9. Find the angle between the rectangular hyperbola $xy = 2$ and the parabola $x^2 + 4y = 0$

Solution:

$x^2 + 4y = 0 \ldots (1) \qquad xy = 2 \ldots\ldots (2)$

From (1) $\qquad y = \frac{-x^2}{4}$

Substitute in eqn (2) $\quad x^3 = -8 \qquad x = -2$

When $x = -2 \qquad y = -1$

The point of intersection

$xy = 2$ $x\frac{dy}{dx} + y(1) = 0$ $\frac{dy}{dx} = \frac{-y}{x}$ $\left.\frac{dy}{dx}\right]_{(-2,-1)} = \frac{-1}{2}$	$x^2 + 4y = 0$ $\frac{dy}{dx} = \frac{-2x}{4} = \frac{-x}{2}$ $\left.\frac{dy}{dx}\right]_{(-2,-1)} = 1$

$\tan\theta = \left|\frac{1-\left(\frac{-1}{2}\right)}{1+1\left(\frac{-1}{2}\right)}\right| = \left|\frac{3/2}{1/2}\right| = 3$

$\theta = \tan^{-1}(3)$

Example 7.44 Evaluate : $\lim_{x\to\infty}(1+2x)^{\frac{1}{2\log x}}$

Solution:

$\lim_{x\to\infty}(1+2x)^{\frac{1}{2\log x}} \quad \infty^0$ **form**

Taking the logarithm, we get

$\log\left(\lim_{x\to\infty}(1+2x)^{\frac{1}{2\log x}}\right)$

$= \lim_{x\to\infty}\frac{\log(1+2x)}{2\log x} \quad \left(\frac{\infty}{\infty}\right)$ *form*

By Using *l'Hôpital Rule*

$\log\left(\lim_{x\to\infty}(1+2x)^{\frac{1}{2\log x}}\right)$

$= \lim_{x\to\infty}\frac{x}{(1+2x)} \quad \left(\frac{\infty}{\infty}\right)$ *form*

By Using *l'Hôpital Rule*

$\log\left(\lim_{x\to\infty}(1+2x)^{\frac{1}{2\log x}}\right) = \lim_{x\to\infty}\frac{1}{2} = \frac{1}{2}$

Hence by exponentiating,

we get $\lim_{x\to\infty}(1+2x)^{\frac{1}{2\log x}} = e^{1/2} = \sqrt{e}.$

EXERCISE 7.5

12. If an initial amount A_0 of money is invested at an interest rate r compounded n times a year, the value of the investment after t years is $A = A_0\left(1+\frac{r}{n}\right)^{nt}$. If the interest is compounded continuously, (that is as $n \to \infty$), show that the amount after t years is $A = A_0\, e^{rt}$

Solution:

$$\text{Let } y = \left(1+\frac{r}{n}\right)^{nt}$$

$$\log y = nt\log\left(1+\frac{r}{n}\right)$$

$$\log y = \frac{\log\left(1+\frac{r}{n}\right)}{1/nt}$$

$$\lim_{n\to\infty}\log y = \lim_{n\to\infty}\frac{\log\left(1+\frac{r}{n}\right)}{1/nt} \quad \left(\frac{0}{0}\right) form$$

$$\log\left(\lim_{n\to\infty} y\right) = \lim_{n\to\infty}\frac{\frac{1}{\left(1+\frac{r}{n}\right)}\left(\frac{-r}{n^2}\right)}{-1/n^2t} = rt$$

Hence by exponentiating, we get $\lim_{n\to\infty} y = e^{rt}$.

$$\lim_{n\to\infty} A = \lim_{n\to\infty} A_0\left(1+\frac{r}{n}\right)^{nt} = A_0\, e^{rt}$$

$$A = A_0\, e^{rt}$$

Example 7.48 Find the absolute maximum and absolute minimum values of the function $f(x) = 2x^3 + 3x^2 - 12x$ on $[-3, 2]$.

Solution:

$$f(x) = 2x^3 + 3x^2 - 12x$$
$$f'(x) = 6x^2 + 6x - 12$$
$$f'(x) = 6(x+2)(x-1)$$

$f'(x) = 0$ $\quad x = -2$ *and* 1.

When x = -3

$$f(-3) = 2(-3)^3 + 3(-3)^2 - 12(-3) = 9$$

When x = 2

$$f(2) = 2(2)^3 + 3(2)^2 - 12(2) = 4$$

When x = -2

$$f(-2) = 2(-2)^3 + 3(-2)^2 - 12(-2) = 20$$

When x = 1

$$f(1) = 2(1)^3 + 3(1)^2 - 12(1) = -7$$

Absolute maximum is 20 which occurs at $x = -2$,

Absolute minimum is -7 which occurs at $x = 1$.

Example 7.53 Discuss the monotonicity and local extrema of the function $f(x) = \log(1+x) - \frac{x}{1+x}$, $x > -1$, and hence find the domain where, $\log(1+x) > \frac{x}{1+x}$.

Solution:

$$f(x) = \log(1+x) - \frac{x}{1+x}$$
$$= \log(1+x) - \frac{(1+x)-1}{1+x}$$
$$f(x) = \log(1+x) - 1 + \frac{1}{1+x}$$
$$f'(x) = \frac{1}{1+x} - 0 - \frac{1}{(1+x)^2}$$
$$f'(x) = \frac{x}{(1+x)^2}$$

$f'(x) = 0 \quad \frac{x}{(1+x)^2} = 0 \qquad$ x=0 and -1

Intervals are $(-1, 0)$ *and* $(0, \infty)$.

Intervals	Sign of $f'(x)$	Monotonicity
$(-1, 0)$	$-$	Strictly decreasing
$(0, \infty)$	$+$	Strictly increasing

The local minimum at $x = 0$ which is $f(0)=0$.

$x > 0 \quad f(x) > f(0) \quad \log(1+x) - \frac{x}{1+x} > 0$

$$\log(1+x) > \frac{x}{1+x} \quad on\ (0, \infty)$$

The domain is $(0, \infty)$.

Example 7.54 Find the intervals of monotonicity and local extrema of the function $f(x) = x\log x + 3x$.

Solution:

The given function is defined and is differentiable at all $x \in (0, \infty)$.

$f(x) = x\log x + 3x \qquad f'(x) = \log x + 1 + 3$

$f'(x) = 0 \qquad \log x + 4 = 0 \qquad x = e^{-4}$

Intervals are $(0, e^{-4})$ *and* (e^{-4}, ∞).

Intervals	Sign of $f'(x)$	Monotonicity
$(0, e^{-4})$	$-$	Strictly decreasing
(e^{-4}, ∞)	$+$	Strictly increasing

The local minimum at $x = e^{-4}$ which is

$$f(e^{-4}) = e^{-4}\log(e^{-4}) + 3e^{-4} = -4e^{-4} + 3e^{-4}.$$
$$f(e^{-4}) = -e^{-4}$$

Example 7.55 Find the intervals of monotonicity and local extrema of the function $f(x) = \frac{1}{1+x^2}$.

Solution:

$f(x) = \frac{1}{1+x^2}$ $\quad f'(x) = \frac{2x}{(1+x^2)^2}$

$f'(x) = 0$ $\quad \frac{2x}{(1+x^2)^2} = 0$ $\quad x = 0$

Intervals are $(-\infty, 0)$ and $(0, \infty)$.

Intervals	Sign of $f'(x)$	Monotonicity
$(-\infty, 0)$	+	Strictly increasing
$(0, \infty)$	−	Strictly decreasing

The local minimum at $x = 0$ which is

$f(0) = \frac{1}{1+0} = 1.$

Example 7.56 Find the intervals of monotonicity and local extrema of the function $f(x) = \frac{x}{1+x^2}$.

Solution:

$f(x) = \frac{x}{1+x^2}$

$$f'(x) = \frac{(1+x^2)(1) - x(2x)}{(1+x^2)^2} = \frac{1-x^2}{(1+x^2)^2}$$

$f'(x) = 0 \Rightarrow 1 - x^2 = 0 \Rightarrow x^2 = 1 \Rightarrow x = \pm 1$

Intervals are $(-\infty, -1), (-1, 1)$ and $(1, \infty)$.

Intervals	Sign of $f'(x)$	Monotonicity
$(-\infty, -1)$	−	Strictly decreasing
$(-1, 1)$	+	Strictly increasing
$(1, \infty)$	−	Strictly decreasing

The local maximum at $x = 1$ which is $f(1) = \frac{1}{2}$.

The local minimum at $x = -1$ which is

$f(-1) = \frac{-1}{2}$.

EXERCISE 7.6

1. (iv) $f(x) = 2\cos x + \sin 2x$; $\left[0, \frac{\pi}{2}\right]$

Solution:

$$f(x) = 2\cos x + \sin 2x$$
$$f'(x) = -2\sin x + 2\cos 2x$$
$$f'(x) = 2(\cos 2x - \sin x)$$

$f'(x) = 0$ $\quad 2(\cos 2x - \sin x) = 0$

$\cos 2x - \sin x = 0$ $\quad \cos 2x = \sin x$

$$1 - 2\sin^2 x = \sin x$$
$$2\sin^2 x + \sin x - 1 = 0$$
$$(2\sin x - 1)(\sin x + 1) = 0$$

$\sin x = \frac{1}{2}$ $\quad x = \frac{\pi}{6} \in \left[0, \frac{\pi}{2}\right]$

$\sin x = -1$ $\quad x = \frac{3\pi}{2} \notin \left[0, \frac{\pi}{2}\right]$

When x = 0 $\quad f(0) = 2\cos(0) + \sin 2(0) = 2$

When x = $\frac{\pi}{2}$ $\quad f\left(\frac{\pi}{2}\right) = 2\cos\left(\frac{\pi}{2}\right) + \sin 2\left(\frac{\pi}{2}\right) = 0$

When x = $\frac{\pi}{6}$ $\quad f\left(\frac{\pi}{6}\right) = 2\cos\left(\frac{\pi}{6}\right) + \sin 2\left(\frac{\pi}{6}\right) = \frac{3\sqrt{3}}{2}$

Absolute maximum is $\frac{3\sqrt{3}}{2}$.

Absolute minimum is 0 .

2. Find the intervals of monotonicities and hence find the local extremum for the following functions:

(i) $f(x) = 2x^3 + 3x^2 - 12x$ (ii) $f(x) = \frac{x}{x-5}$

(iii) $f(x) = \frac{e^x}{1-e^x}$ (iv) $f(x) = \frac{x^3}{3} - \log x$

(v) $f(x) = \sin x \cos x + 5$, $x \in [0, 2\pi]$

Solution:

(i) $f(x) = 2x^3 + 3x^2 - 12x$

$$f'(x) = 6x^2 + 6x - 12$$

$f'(x) = 0$ $\quad 6(x^2 + x - 2) = 0$

$$(x+2)(x-1) = 0$$
$$x = -2 \text{ and } 1.$$

Intervals are $(-\infty, -2), (-2, 1)$ and $(1, \infty)$

Intervals	Sign of $f'(x)$	Monotonicity
$(-\infty, -2)$	+	Strictly increasing
$(-2, 1)$	−	Strictly decreasing
$(1, \infty)$	+	Strictly increasing

The local maximum at $x = -2$ which is

f $(-2) = -16 + 12 + 24 = 20$.

The local minimum at $x = 1$ which is

$f(1) = 2 + 3 - 12 = -7$

(iv) $f(x) = \frac{x^3}{3} - \log x$

$f'(x) = x^2 - \frac{1}{x}$

$f'(x) = 0$ $\quad x^3 = 1$ $\quad x = 1.$

Intervals are $(0, 1)$ and $(1, \infty)$

Intervals	Sign of $f'(x)$	Monotonicity
$(0, 1)$	−	Strictly decreasing
$(1, \infty)$	+	Strictly increasing

The local minimum at $x = 1$ which is $f(1) = \frac{1}{3}$

(v) $f(x) = \sin x \cos x + 5,\ x \in [0, 2\pi]$

$f'(x) = \cos x(\cos x) + \sin x(-\sin x)$

$= \cos^2 x - \sin^2 x = \cos 2x$

$f'(x) = 0 \qquad \cos 2x = 0$

$2x = \frac{\pi}{2}, \frac{3\pi}{2}, \frac{5\pi}{2}, \ldots\ldots\ldots$

$x = \frac{\pi}{4}, \frac{3\pi}{4}, \frac{5\pi}{4}, \frac{7\pi}{4} \ldots\ldots\ldots$

$x = \frac{\pi}{4}, \frac{3\pi}{4}, \frac{5\pi}{4}, \frac{7\pi}{4} \in [0, 2\pi]$

Intervals are

$\left(0, \frac{\pi}{4}\right), \left(\frac{\pi}{4}, \frac{3\pi}{4}\right), \left(\frac{3\pi}{4}, \frac{5\pi}{4}\right), \left(\frac{5\pi}{4}, \frac{7\pi}{4}\right)$ and $\left(\frac{7\pi}{4}, 2\pi\right)$

Intervals	Sign of $f'(x)$	Monotonicity
$\left(0, \frac{\pi}{4}\right)$	+	Strictly increasing
$\left(\frac{\pi}{4}, \frac{3\pi}{4}\right)$	−	Strictly decreasing
$\left(\frac{3\pi}{4}, \frac{5\pi}{4}\right)$	+	Strictly increasing
$\left(\frac{5\pi}{4}, \frac{7\pi}{4}\right)$	−	Strictly decreasing
$\left(\frac{7\pi}{4}, 2\pi\right)$	+	Strictly increasing

The local maximum at $x = \frac{\pi}{4}$ and $\frac{5\pi}{4}$ which is $f\left(\frac{\pi}{4}\right) = f\left(\frac{5\pi}{4}\right) = \frac{11}{2}$.

The local minimum at $x = \frac{3\pi}{4}$ and $\frac{7\pi}{4}$ which is $f\left(\frac{3\pi}{4}\right) = f\left(\frac{7\pi}{4}\right) = \frac{9}{2}$

Example 7.57 Determine the intervals of concavity of the curve $f(x) = (x-1)^3 \cdot (x-5), x \in \mathbb{R}$ and, points of inflection if any.

Solution:

$f(x) = (x-1)^3 \cdot (x-5)$

$= (x-1)^3 \cdot [(x-1) - 4]$

$= (x-1)^4 - 4(x-1)^3$

$f'(x) = 4(x-1)^3 - 12(x-1)^2$

$f''(x) = 12(x-1)^2 - 24(x-1)$

$f''(x) = 12(x-1)(x-1-2)$

$f''(x) = 12(x-1)(x-3)$

$f''(x) = 0 \qquad x = 1$ and 3.

Intervals are $(-\infty, 1), (1, 3)$ and $(3, \infty)$.

Intervals	Sign of $f''(x)$	Concavity
$(-\infty, 1)$	+	Up ward
$(1, 3)$	−	Down ward
$(3, \infty)$	+	Up ward

Points of inflection:

When $x = 1 \qquad f(1) = 0$

When $x = 3 \qquad f(3) = 8 \times -2 = -16$

Inflection point is $(1, 0)$ and $(3, -16)$.

Example 7.58 Determine the intervals of concavity of the curve $y = 3 + \sin x$.

Solution:

$y = f(x) = 3 + \sin x$

$y' = f'(x) = \cos x$

$y'' = f''(x) = -\sin x$

$f''(x) = 0 \qquad -\sin x = 0 \qquad x = 0, \pi, 2\pi, \ldots\ldots\ldots n\pi, n \in Z$

$x = 0 \in (-\pi, \pi)$

Intervals are $(-\pi, 0)$ and $(0, \pi)$

Intervals	Sign of $f''(x)$	Concavity
$(-\pi, 0)$	+	Up ward
$(0, \pi)$	−	Down ward

Points of inflection:

When $x = 0 \qquad f(0) = 3 + \sin 0 = 3$

Inflection point is $(0, 3)$.

In general intervals $[n\pi, (n + 1\pi)],\ n \in Z$.

The point of inflection is $(n\pi, 3)$.

EXTREMA USING SECOND DERIVATIVE TEST:

Theorem 7.13 (The Second Derivative Test)

Suppose that c is a critical point at which $f'(c) = 0$, that $f'(x)$ exists in a neighborhood of c, and that $f'(c)$ exists. Then f has a relative maximum value at c if $f''(c) < 0$ and a relative minimum value at c if $f''(c) > 0$.If $f''(c) = 0$, the test is not informative.

Example 7.59 Find the local extremum of the function $f(x) = x^4 + 32x$.

Solution:

$f(x) = x^4 + 32x$

$f'(x) = 4x^3 + 32 \qquad f'(x) = 0$

$4(x^3 - 8) = 0 \qquad x = -2$

$f''(x) = 12x^2$

$f''(-2) = 12(-2)^2 > 0.$

Local minimum at $x = -2$.

$f(-2) = (-2)^4 + 32(-2) = -48$

The local minimum value is -48.

The extreme point (- 2, - 48) .

Example 7.60 Find the local extrema of the function $f(x) = 4x^6 - 6x^4$.

Solution:

$f(x) = 4x^6 - 6x^4$

$f'(x) = 24x^5 - 24x^3 \qquad f'(x) = 0$

$24x^3(x^2 - 1) = 0 \qquad x = 0 \;\; and\; x = \pm 1$

$f''(x) = 120x^4 - 72x^2$

When $x = 0 \qquad f''(x) = 120(0)^4 - 72(0)^2 = 0$

Does not give any information about local extrema at $x = 0$.

When $x = 1 \qquad f''(x) = 120(1)^4 - 72(1)^2 > 0$

When $x = -1 \quad f''(x) = 120(-1)^4 - 72(-1)^2 > 0$

The intervals are

$(-\infty, -1), (-1, 0), (0, 1), (1, \infty)$.

Intervals	Sign of $f'(x)$	Monotonicity
$(-\infty, -1)$	$-$	Strictly decreasing
$(-1,0)$	$+$	Strictly increasing
$(0,1)$	$-$	Strictly decreasing
$(1, \infty)$	$+$	Strictly increasing

When x = 0 $\qquad f(0) = 4x^6 - 6x^4 = 0$

When x = –1 $\qquad f(-1) = 4x^6 - 6x^4 = -2$

When x = 1 $\qquad f(1) = 4x^6 - 6x^4 = -2$

Local minimum value is -2.

Local maximum value is 0.

Example 7.61 Find the local maximum and minimum of the function x^2y^2 on the line $x + y = 10$.

Solution:

Let $f(x) = x^2y^2$	$x + y = 10$ $y = 10 - x$

$f(x) = x^2(10 - x)^2 = x^2(100 - 20x + x^2)$

$f(x) = 100x^2 - 20x^3 + x^4$

$f'(x) = 200x - 60x^2 + 4x^3$

$f'(x) = 4x(x^2 - 15x + 50)$

$f'(x) = 4x(x - 10)(x - 5)$

$f'(x) = 0 \qquad x = 0, 10 \; and\; 5$

$f''(x) = 200 - 120x + 12x^2$

When $x = 0 \qquad f''(x) = 200 > 0$

When $x = 5 \quad f''(x) = 12(5)^2 - 120(5) + 200 < 0$

When $x = 10 \quad f''(x) = 12(10)^2 - 120(10) + 200 > 0$

The intervals are $(-\infty, 0), (0, 5), (5, 10), (10, \infty)$.

Intervals	Sign of $f'(x)$	Monotonicity
$(-\infty, 0)$	$-$	Strictly decreasing
$(0,5)$	$+$	Strictly increasing
$(5,10)$	$-$	Strictly decreasing
$(10, \infty)$	$+$	Strictly increasing

When x = 0 $\quad f(0) = 0^2(10 - 0)^2 = 100$

When x = 5 $\qquad f(5) = 5^2(10 - 5)^2 = 625$

When x = 10 $\qquad f(10) = 10^2(10 - 10)^2 = 0$

Local minimum value is 0.

Local maximum value is 625.

EXERCISE 7.7

1. Find intervals of concavity and points of inflexion for the following functions:

(i) $f(x) = x\,(x - 4)^3$

(ii) $f(x) = \sin x + \cos x,\;\; 0 < x < 2\pi$

(iii) $f(x) = \frac{1}{2}(e^x - e^{-x})$

Solution:

(i) $f(x) = x\,(x - 4)^3 = (x - 4 + 4)\,(x - 4)^3$

$f(x) = (x - 4)^4 + 4(x - 4)^3$

$f'(x) = 4(x - 4)^3 + 12(x - 4)^2$

$f''(x) = 12(x - 4)^2 + 24(x - 4)$

$f''(x) = 12(x - 4)(x - 4 + 2)$

$f''(x) = 12(x - 4)(x - 2)$

$f''(x) = 0 \qquad x = 4 \;\; and\; 2.$

Intervals are $(-\infty, 2), (2, 4)$ *and* $(4, \infty)$.

Intervals	Sign of $f''(x)$	Concavity
$(-\infty, 2)$	$+$	Up ward
$(2, 4)$	$-$	Down ward
$(4, \infty)$	$+$	Up ward

When x = 2 $\qquad f(2) = 2\,(2 - 4)^3 = -16$

When x = 4 $\qquad f(5) = 4\,(4 - 4)^3 = 0$

Inflection points $(2, -16)$ *and* $(4, 0)$.

(ii) $f(x) = \sin x + \cos x,\;\; 0 < x < 2\pi$

$f'(x) = \cos x - \sin x$

$f''(x) = -\sin x - \cos x$

$f''(x) = 0 \qquad \sin x = -\cos x \qquad \tan x = -1$

$x = n\pi - \frac{\pi}{4} \qquad x = \frac{3\pi}{4}, \frac{7\pi}{4} \in (0, 2\pi).$

Intervals are $\left(0, \frac{3\pi}{4}\right), \left(\frac{3\pi}{4}, \frac{7\pi}{4}\right)$ *and* $\left(\frac{7\pi}{4}, 2\pi\right)$.

Intervals	Sign of $f''(x)$	Concavity
$\left(0, \frac{3\pi}{4}\right)$	$-$	Down ward
$\left(\frac{3\pi}{4}, \frac{7\pi}{4}\right)$	$+$	Up ward
$\left(\frac{7\pi}{4}, 2\pi\right)$	$-$	Down ward

When x = $\frac{3\pi}{4}$ $\qquad f\left(\frac{3\pi}{4}\right) = \sin\frac{3\pi}{4} + \cos\frac{3\pi}{4} = 0$

When x = $\frac{7\pi}{4}$ $\qquad f\left(\frac{7\pi}{4}\right) = \sin\frac{7\pi}{4} + \cos\frac{7\pi}{4} = 0$

Inflection points $\left(\frac{3\pi}{4}, 0\right)$ *and* $\left(\frac{7\pi}{4}, 0\right)$.

(iii) $f(x)=\frac{1}{2}(e^x-e^{-x})$

$f(x)=\frac{1}{2}(e^x-e^{-x})$ $\quad f'(x)=\frac{1}{2}(e^x+e^{-x})$

$$f''(x)=\frac{1}{2}(e^x-e^{-x})$$

$f''(x)=0$ $\quad \frac{1}{2}(e^x-e^{-x})=0$ $\quad e^x=e^{-x}$

$x=-x$ $\quad 2x=0$ $\quad x=0$

Intervals are $(-\infty,0)$ *and* $(0,\infty)$.

Intervals	Sign of $f''(x)$	Concavity
$(-\infty,0)$	$-$	Down ward
$(0,\infty)$	$+$	Up ward

When x = 0 $\quad f(0)=\frac{1}{2}(e^0-e^{-0})=0$

Inflection points (0,0) .

2. Find the local extrema for the following functions using second derivative test :
(i) $f(x)=-3x^5+5x^3$
(iii) $f(x)=x^2e^{-2x}$

Solution:

(i) $f(x)=-3x^5+5x^3$

$f'(x)=-15x^4+15x^2$

$f'(x)=0$ $\quad -15x^2(x^2-1)=0$

$x=0$ *and* ± 1.

$f''(x)=-60x^3+30x$

When $x=0$, $\quad f''(0)=-60(0)^3+30(0)=0$

When $x=1$, $\quad f''(1)=-60(1)^3+30(1)=-30<0$

When $x=-1$, $\quad f''(-1)=-60(-1)^3+30(-1)=30>0$

The local minimum at $x=-1$

which is $f(-1)=-2$

The local maximum at $x=1$ which is $f(1)=2$.

$f''(x)$ does not give any information about local extrema at $x=0$.

(iii) $f(x)=x^2e^{-2x}$

$f'(x)=-2x^2e^{-2x}+2xe^{-2x}=2xe^{-2x}(1-x)$

$f'(x)=0$ $\quad 2xe^{-2x}(1-x)=0$ $\quad x=0$ *and* 1.

$f''(x)=2xe^{-2x}(-1)+(1-x)(2e^{-2x}-4xe^{-2x})$

$f''(x)=2e^{-2x}(-x+(1-x)(1-2x))$

$f''(x)=2e^{-2x}(-x+1-2x-x+2x^2)$

$f''(x)=2e^{-2x}(2x^2-4x+1)$

When $x=0$, $\quad f''(0)=2(1)(0-0+1)=2>0$

When $x=1$, $\quad f''(1)=2e^{-2}(2-4+1)=-2e^{-2}<0$

The local maximum at $x=1$ which is

$f(1)=(1)^2e^{-2}=e^{-2}(or)\frac{1}{e^2}$.

The local minimum at $x=0$ which is

$f(0)=(0)^2e^{-0}=0$.

3. For the function $f(x)=4x^3+3x^2-6x+1$ find the intervals of monotonicity, local extrema, intervals of concavity and points of inflection.

Solution:

$f(x)=4x^3+3x^2-6x+1$

$f'(x)=12x^2+6x-6$

$f'(x)=0$ $\quad 6(x^2+x-1)=0$

$(x+1)(2x-1)=0$ $\quad x=-1$ *and* $\frac{1}{2}$

Intervals are $(\infty,-1),\left(-1,\frac{1}{2}\right)$ *and* $\left(\frac{1}{2},\infty\right)$.

Intervals	Sign of $f'(x)$	Monotonicity
$(\infty,-1)$	$+$	Strictly increasing
$\left(-1,\frac{1}{2}\right)$	$-$	Strictly decreasing
$\left(\frac{1}{2},\infty\right)$	$+$	Strictly increasing

The local maximum at $x=-1$ which is

$f(-1)=4(-1)^3+3(-1)^2-6(-1)+1$

$=-4+3+6+1=6$.

The local minimum at $x=\frac{1}{2}$ which is

$f\left(\frac{1}{2}\right)=4\left(\frac{1}{2}\right)^3+3\left(\frac{1}{2}\right)^2-6\left(\frac{1}{2}\right)+1=\frac{-6}{8}=\frac{-3}{4}$.

Example 7.62 We have a 12 square unit piece of thin material and want to make an open box by cutting small squares from the corners of our material and folding the sides up. The question is, which cut roduces the box of maximum volume?

Solution:

Let x = length of the cut on each side of the little squares. V = the volume of the folded box.

Length of the base $=12-2x$

Depth of the box $=x$

Volume of box $=l\times b\times h=(12-2x)(12-2x)(x)$

$V(x)=x(12-2x)^2$

$V'(x)=-4x(12-2x)+(12-2x)^2(1)$

$V'(x)=(12-2x)(-4x+12-2x)$

$V'(x)=(12-2x)(12-6x)$

$V'(x)=0$ $\quad x=2\in(0,6)$ *and* $x=6\notin(0,6)$.

$V''(x)=(12-2x)(-6)+(12-6x)(-2)$

$=-72+12x-24+12x$

$=24x-96$

When $x=2$ $\quad V''(2)=24(2)-96<0$

Maximum volume at $x=2$.

Volume of box $=x(12-2x)^2=2(12-4)^2$

$=2(64)=128$ *cu.* Units

Example 7.63 Find the points on the unit circle $x^2 + y^2 = 1$ nearest and farthest from $(1,1)$.

Solution:

Let $P(x, y)$ be any point on the circle.

Distance from the two points $d = \sqrt{(x_2-x_1)^2 + (y_2-y_1)^2}$

$$D = d^2 = (x-1)^2 + (y-1)^2$$

$$D = d^2 = x^2 + y^2 - 2x - 2y + 2$$

$$D = d^2 = 3 - 2x - 2y$$

$$\frac{dD}{dx} = -2 - 2\frac{dy}{dx}$$

$$\frac{dD}{dx} = -2 - 2\left(\frac{-x}{y}\right) = -2 + 2\left(\frac{x}{y}\right)$$

$$\frac{dD}{dx} = 0 \quad -2 + 2\left(\frac{x}{y}\right) = 0$$

$$\left(\frac{x}{y}\right) = 1 \quad x = y$$

$$x^2 + y^2 = 1$$

$$2x + 2y\frac{dy}{dx} = 0$$

$$\frac{dy}{dx} = \frac{-x}{y}$$

$$x^2 + x^2 = 1$$

$$2x^2 = 1$$

$$x = \pm\frac{1}{\sqrt{2}}$$

$$\frac{d^2D}{dx^2} = 2\left(\frac{y(1) - x\frac{dy}{dx}}{y^2}\right)$$

$$\frac{d^2D}{dx^2} = 2\left(\frac{y - x\left(\frac{-x}{y}\right)}{y^2}\right)$$

$$\frac{d^2D}{dx^2} = 2\left(\frac{y - x\left(\frac{-x}{y}\right)}{y^2}\right)$$

$$\frac{d^2D}{dx^2} = 2\left(\frac{x^2+y^2}{y^3}\right) = 2\left(\frac{1}{y^3}\right)$$

The points are $\left(\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}\right)$ and $\left(\frac{-1}{\sqrt{2}}, \frac{-1}{\sqrt{2}}\right)$

$\left.\frac{d^2D}{dx^2}\right]_{\left(\frac{1}{\sqrt{2}},\frac{1}{\sqrt{2}}\right)} > 0$ and $\left.\frac{d^2D}{dx^2}\right]_{\left(\frac{-1}{\sqrt{2}},\frac{-1}{\sqrt{2}}\right)} < 0$

When $\left(\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}\right)$

$D = d^2 = \left(\frac{1}{\sqrt{2}} - 1\right)^2 + \left(\frac{1}{\sqrt{2}} - 1\right)^2 = \left(\sqrt{2} - 1\right)^2 \quad d = \sqrt{2} - 1$

When $\left(\frac{-1}{\sqrt{2}}, \frac{-1}{\sqrt{2}}\right)$

$D = d^2 = \left(\frac{-1}{\sqrt{2}} - 1\right)^2 + \left(\frac{-1}{\sqrt{2}} - 1\right)^2 = \left(\sqrt{2} + 1\right)^2 \quad d = \sqrt{2} + 1$

The nearest and farthest distance are $\sqrt{2} - 1$ and $\sqrt{2} + 1$

Example 7.64 A steel plant is capable of producing x tonnes per day of a low-grade steel and y tonnes per day of a high-grade steel, where $y = \frac{40-5x}{10-x}$. If the fixed market price of low-grade steel is half that of high-grade steel, then what should be optimal productions in low-grade steel and high-grade steel in order to have maximum receipts.

Solution:

Let the price of low-grade steel be Rs.p per tonne. Then the price of high-grade steel is Rs. $2p$ per tonne.

The total receipt per day $R = px + 2py$

$$y = \frac{40 - 5x}{10 - x}$$

$$R = px + 2p\left(\frac{40 - 5x}{10 - x}\right) = p\left(x + \frac{80 - 10x}{10 - x}\right)$$

$$= p\left(x + \frac{80 - 10x + 20 - 20}{10 - x}\right)$$

$$R(x) = p\left(x + \frac{10(10 - x) - 20}{10 - x}\right)$$

$$= p\left(x + 10 - \frac{20}{10 - x}\right)$$

$$R'(x) = p\left(1 + \frac{20}{(10-x)^2}\right)$$

$$R'(x) = 0 \qquad p\left(1 + \frac{20}{(10-x)^2}\right) = 0$$

$$(10 - x)^2 + 20 = 0 \qquad (x - 10)^2 = 20$$

$$x - 10 = \pm 2\sqrt{5} \qquad x = 10 \pm 2\sqrt{5}$$

$$R''(x) = p\left(\frac{-2 \times 20}{(10 - x)^3}\right) = p\left(\frac{-40}{(10 - x)^3}\right)$$

At $x = 10 - 2\sqrt{5}$ $\quad R'(10 - 2\sqrt{5}) < 0$

Maximum at $x = 10 - 2\sqrt{5}$

When $x = 10 - 2\sqrt{5}$

$$y = \frac{40 - 5(10 - 2\sqrt{5})}{10 - (10 - 2\sqrt{5})} = 5 - \sqrt{5}$$

The steel plant must produce low-grade

$x = 10 - 2\sqrt{5}$.

High grade $\quad y = 5 - \sqrt{5}$

Example 7.65 Prove that among all the rectangles of the given area square has the least perimeter.

Solution:

Let x, y be the sides of the rectangle.

Area of the rectangle $A = l \times b$

$xy = k \qquad y = \frac{k}{x}$

Perimeter of the rectangle

$$P = 2(l + b) = 2(x + y) = 2\left(x + \frac{k}{x}\right)$$

$$P'(x) = 2\left(1 + \frac{-k}{x^2}\right)$$

$$P''(x) = 2\left(\frac{2k}{x^3}\right) = \frac{4k}{x^3}$$

$P'(x) = 0 \qquad 2\left(1 + \frac{-k}{x^2}\right) = 0 \qquad \frac{-k}{x^2} = -1$

$x^2 = k \qquad x = \sqrt{k} > 0$

At $x = \sqrt{k} \qquad P''(x) = \frac{4k}{(\sqrt{k})^3} > 0$

Perimeter is minimum at $x = \sqrt{k}$

When $x = \sqrt{k} \qquad y = \frac{k}{\sqrt{k}} = \sqrt{k}$

$x = y = \sqrt{k}$ It is square.

EXERCISE 7.8

1. Find two positive numbers whose sum is 12 and their product is maximum.

Solution:

Let x and y be the two positive integers.

Given: $x + y = 12 \qquad y = 12 - x$

Let A be the product of the numbers.

$A = xy \qquad A = x(12 - x) = 12x - x^2$

$A' = 12 - 2x$

$A'(x) = 0 \qquad 12 - 2x = 0 \qquad x = 6$

$A''(x) = -2$

At $x = 6 \qquad A''(6) = -2 < 0$

Maximum value at x = 6 $\qquad y = 12 - 6 = 6$

Product of the maximum = $6 \times 6 = 36$.

2. Find two positive numbers whose product is 20 and their sum is minimum.

Solution:

Let x and y be the two positive integers.

Given: $xy = 20 \qquad y = \frac{20}{x}$

Let A be the sum of the numbers.

$A = x + y \qquad A = x + \frac{20}{x} \qquad A' = 1 - \frac{20}{x^2}$

$A'(x) = 0 \qquad x^2 = 20 \qquad x = \pm 2\sqrt{5}$

$A''(x) = \frac{40}{x^3}$

At $x = 2\sqrt{5} \qquad A''(2\sqrt{5}) = \frac{40}{(2\sqrt{5})^3} > 0$

Minimum value at x = $2\sqrt{5} \qquad y = \frac{20}{2\sqrt{5}} = 2\sqrt{5}$

Sum of the maximum = $2\sqrt{5} + 2\sqrt{5} = 4\sqrt{5}$.

3. Find the smallest possible value of $x^2 + y^2$ given that $x + y = 10$.

Solution:

Given: $x + y = 10 \qquad y = 10 - x$

Let $f(x) = x^2 + y^2 \qquad f(x) = x^2 + (10 - x)^2$

$f'(x) = 2x + 2(10 - x)(-1)$

$= 2(x - 10 + x)$

$f'(x) = 2(2x - 10) = 4(x - 5)$

$f'(x) = 0 \qquad x - 5 = 0 \qquad x = 5$

$f''(x) = 4(1) = 4$

At $x = 5 \qquad f''(5) = 4 > 0$

Smallest values at $x = 5 \qquad y = 10 - 5 = 5$

Value of $x^2 + y^2 = 25 + 25 = 50$.

The smallest possible value is 50.

4. A garden is to be laid out in a rectangular area and protected by wire fence. What is the largest possible area of the fenced garden with 40 metres of wire.

Solution:

Let x and y be the length and breadth of the rectangle.

Total length of the fencing is 40m.

Perimeter of the rectangle = $40m$

$2(x + y) = 40 \qquad x + y = 20 \qquad y = 20 - x$

Area of the rectangle

$A(x) = x(20 - x) = 20x - x^2$

$A'(x) = 20 - 2x \qquad A'(x) = 0 \qquad x = 10$

$A''(x) = -2$

At $x = 10 \qquad A''(10) = -2 < 0$.

Area of maximum at $x = 10 \qquad y = 20 - 10 = 10$

The largest possible area = $xy = 100m^2$.

5. A rectangular page is to contain 24 cm^2 of print. The margins at the top and bottom of the page are 1.5 cm and the margins at other sides of the page is 1 cm. What should be the dimensions of the page so that the area of the paper used is minimum.

Solution:

Let x and y be the length and breadth of the rectangle paper in printed area.

Area $A = xy = 24$ $\qquad y = \frac{24}{x}$

Dimensions of the paper length and breadth is $x + 2$ and $y + 3$.

Area of the paper

$A(x) = (x + 2)(y + 3) = xy + 3x + 2y + 6$

$A(x) = 24 + 6 + 3x + 2\left(\frac{24}{x}\right) = 30 + 3x + \frac{48}{x}$

$A'(x) = 3 - \frac{48}{x^2}$ $\qquad A''(x) = \frac{96}{x^3}$

$A'(x) = 0$ $\qquad 3x^2 = 48$ $\qquad x = \pm 4$

At $x = 4$ $\qquad A''(4) = \frac{96}{4^3} > 0.$

Area minimum at $x = 4$.

When $x = 4$ $\qquad y = \frac{24}{4} = 6$

Dimensions of the paper length and breadth are 4+2=6 and 6+3=9.

6. A farmer plans to fence a rectangular pasture adjacent to a river. The pasture must contain 1,80,000 sq.mtrs in order to provide enough grass for herds. No fencing is needed along the river. What is the length of the minimum needed fencing material?

Solution:

Let x and y be thee length and breadth of the rectangular pasture.

Given: Total area = 180000 sq.m

$Area = xy = 180000$ $\qquad y = \frac{180000}{x}$

Length of the perimeter (or) fencing

$L(x) = x + 2y = x + 2\left(\frac{180000}{x}\right) = x + \frac{360000}{x}$

$L'(x) = 1 - \frac{360000}{x^2}$

$L'(x) = 0$ $\qquad x^2 = 360000$ $\qquad x = \pm 600$

$L''(x) = \frac{720000}{x^3}$

At $x = 600$ $\qquad L''(600) = \frac{720000}{(600)^3} > 0$

Minimum length at $x = 600$.

When $x = 600$ $\qquad y = \frac{180000}{600} = 300$

Minimum length of the fencing

$x + 2y = 600 + 600 = 1200m$

7. Find the dimensions of the rectangle with maximum area that can be inscribed in a circle of radius 10 cm.

Solution:

Let $2x$ and $2y$ be the length and breadth of the rectangle.

Let θ be made by OP in x-axis.

$\cos\theta = \frac{x}{r}$	$\sin\theta = \frac{y}{r}$
$x = r\cos\theta$	$y = r\sin\theta$
When $r = 10$	When $r = 10$
$x = 10\cos\theta$	$y = 10\sin\theta$

Area of the rectangle

$A = 2x \times 2y = 2(10\cos\theta) \times 2(10\sin\theta)$

$A(\theta) = 400\cos\theta\sin\theta = 200\sin 2\theta$

$A'(\theta) = 400\cos 2\theta$

$A'(\theta) = 0$ $\qquad \cos 2\theta = 0$ $\qquad \theta = \frac{\pi}{4} \in \left(0, \frac{\pi}{2}\right)$

$A''(\theta) = -800\sin 2\theta$

At $\theta = \frac{\pi}{4}$ $\qquad A''\left(\frac{\pi}{4}\right) = -800\sin 2\left(\frac{\pi}{4}\right) < 0$

Area is maximum at $\theta = \frac{\pi}{4}$

Length $2x = 2\left(10\cos\frac{\pi}{4}\right) = 20 \times \frac{1}{\sqrt{2}} = 10\sqrt{2}$ cm

Breadth $2y = 2\left(10\sin\frac{\pi}{4}\right) = 20 \times \frac{1}{\sqrt{2}} = 10\sqrt{2}$ cm

8. Prove that among all the rectangles of the given perimeter, the square has the maximum area.

Solution:

Let x and y be the length and breadth of the rectangle.

Let L be the perimeter of the rectangle.

Perimeter $L(x) = 2x + 2y$ $\qquad y = \frac{L-2x}{2}$

Let A be the area of the rectangle. $\qquad A = xy$

$A(x) = x\left(\frac{L-2x}{2}\right) = \frac{xL - 2x^2}{2} = \frac{xL}{2} - x^2$

$A'(x) = \frac{L}{2} - 2x$

$A'(x) = 0$ $\qquad \frac{L}{2} - 2x = 0$ $\qquad x = \frac{L}{4}$

$A''(x) = -2$ $\qquad$ At $x = \frac{L}{4}$ $\qquad A''\left(\frac{L}{4}\right) = -2 < 0$

Area is maximum at $x = \frac{L}{4}$ $\qquad y = \frac{4x-2x}{2} = x$

It is square.

The rectangle is square has the maximum area.

9. Find the dimensions of the largest rectangle that can be inscribed in a semi-circle of radius r cm.

Solution:
Let 2x and y be the length and breadth of the rectangle.
Let θ be the angle made up OP with x-axis.
Semicircle radius is r.

$\cos\theta = \frac{x}{r}$	$\sin\theta = \frac{y}{r}$
$x = r\cos\theta$	$y = r\sin\theta$

Area of the rectangle
$A = 2xy = 2r^2\cos\theta\sin\theta = r^2\sin 2\theta$
$A(\theta) = r^2\sin 2\theta \qquad A'(\theta) = 2r^2\cos 2\theta$
$A'(\theta) = 0 \qquad 2r^2\cos 2\theta = 0 \qquad \theta = \frac{\pi}{4} \in \left(0, \frac{\pi}{2}\right)$
$A''(\theta) = -4r^2\sin 2\theta$
At $\theta = \frac{\pi}{4} \qquad A''\left(\frac{\pi}{4}\right) = -4r^2\sin 2\left(\frac{\pi}{4}\right) < 0$
Area maximum at $\theta = \frac{\pi}{4}$.
When $\theta = \frac{\pi}{4} \qquad x = r\cos\left(\frac{\pi}{4}\right) = \frac{r}{\sqrt{2}}$
$y = r\sin\frac{\pi}{4} = \frac{r}{\sqrt{2}}$
Length $2x = \sqrt{2}r$ and Breadth $y = \frac{r}{\sqrt{2}}$
Area of the rectangle $= l \times b = 2xy = r^2$

10. A manufacturer wants to design an open box having a square base and a surface area of 108 sq.cm. Determine the dimensions of the box for the maximum volume.

Solution:
Let x, x and y be the length, breadth and height of the box.
Area of the base, one side and 4 side are x^2, $x \times y$ and $4xy$ respectively.
Given: surface area = 108
$x^2 + 4xy = 108$
$4xy = 108 - x^2 \qquad y = \frac{108 - x^2}{4x}$
Volume of the box $= V(x) = l \times b \times h$
$= x^2y = x^2\left(\frac{108 - x^2}{4x}\right) = 27x - \frac{x^3}{4}$
$V'(x) = 27 - \frac{3x^2}{4} \qquad V'(x) = 0$
$27 - \frac{3x^2}{4} = 0 \qquad -3x^2 = -108$
$x^2 = 36 \qquad x = \pm 6$
$V''(x) = -\frac{6x}{4}$ At $x = 6 \qquad V''(6) = -\frac{36}{4} < 0$
Maximum volume at $x = 6 \qquad y = \frac{108 - 36}{24} = 3$
Dimension of the box are length = 6cm and breadth =3 cm.

11. The volume of a cylinder is given by the formula $V = \pi r^2 h$. Find the greatest and least values of V if $r + h = 6$.

Solution:
Given: $r + h = 6 \qquad h = 6 - r$
Volume of the cylinder
$V = \pi r^2 h = \pi r^2(6 - r) = \pi(6r^2 - r^3)$
$V(r) = \pi(6r^2 - r^3)$
$V'(r) = \pi(12r - 3r^2)$
$V'(r) = 0 \qquad 12r - 3r^2 = 0 \qquad 3r(4 - r) = 0$
$r = 0$ and 4.
$V''(r) = \pi(12 - 6r)$
At $r = 0 \qquad V''(0) = 12\pi > 0$
volume is minimum at $r = 0$
At $r = 4 \qquad V''(4) = \pi(12 - 24) = -12\pi < 0$
Volume is maximum at $r = 4$.
When $r = 4 \qquad h = 6 - 4 = 2$
When $r = 0 \qquad h = 6 - 0 = 6$
When $r = 4$ and $h = 2$
Maximum Volume $V = \pi r^2 h = \pi(4)^2(2) = 32\pi$
When $r = 0$ and $h = 6$
Minimum Volume $V = \pi r^2 h = \pi(0)^2(2) = 0$

12. A hollow cone with base radius a cm and height b cm is placed on a table. Show that the volume of the largest cylinder that can be hidden underneath is $\frac{4}{9}$ times volume of the cone.

Solution:
Let a and b be the radius and height of the cone.
Let x and y be the radius and distance from the top of the cone to the inscribed the cylinder.

Cone	Cylinder
Radius = a	Radius = x
Height = b	Height h= b-y

Volume of the cylinder $V = \pi r^2 h = \pi x^2(b - y)$
$\frac{y}{x} = \frac{b}{a} \qquad y = \frac{xb}{a}$
$V(x) = \pi x^2\left(b - \frac{xb}{a}\right) = \pi b\left(x^2 - \frac{x^3}{a}\right)$
$V'(x) = \pi b\left(2x - \frac{3x^2}{a}\right) \qquad V'(x) = 0$
$2x - \frac{3x^2}{a} = 0 \qquad 3x^2 = 2ax \qquad x = \frac{2a}{3}$
$V''(x) = \pi b\left(2 - \frac{6x}{a}\right)$
At $x = \frac{2a}{3} \qquad V''\left(\frac{2a}{3}\right) = \pi b\left(2 - \frac{12}{3}\right) < 0$
Maximum volume at $x = \frac{2a}{3}$.
Volume of cylinder $V(x) = \pi b\left(\left(\frac{2a}{3}\right)^2 - \frac{\left(\frac{2a}{3}\right)^3}{a}\right)$
$= \pi b\left(\frac{4a^2}{9} - \frac{8a^2}{27}\right) = \pi b\left(\frac{12a^2 - 8a^2}{27}\right) = \pi b\left(\frac{4a^2}{27}\right)$
$= \frac{4}{9}\left(\frac{1}{3}\pi a^2 b\right) = \frac{4}{9}$[volume of cone].

EXERCISE 8.2

5. The trunk of a tree has diameter 30 cm. During the following year, the circumference grew 6 cm.
(i) Approximately, how much did the tree's diameter grow?
(ii) What is the percentage increase in area of the tree's cross-section?

Solution:

Given: $r = 15$ cm and $D = 30$ cm

Circumference of the circle $S(r) = 2\pi r$

$= 2\pi(15) = 30\pi$ cm

Differentiating = 6 cm

$2\pi r_2 - 2\pi r_1 = 6$ $\quad dr = r_2 - r_1 = \frac{3}{\pi}$

(i)

Approximate error

$= A(r + \Delta r) + A(r)$

$\approx A'(r)dr$

$= 90\ cm^2 (increasing)$

Area of the circle

$A(r) = \pi r^2$

$dA = 2\pi r\ dr$

$dA = 30\pi\left(\frac{3}{\pi}\right) = 90$

(ii) Percentage increasing:

Actual value (Area) = $\pi r^2 = 225\pi$

Actual error = $\frac{Approximate\ value}{Actual\ value} \times 100 = \frac{90}{225\pi} \times 100 = \frac{40}{\pi}$ %

Excercise	Example
Ex : 8.3 -3 Ex : 8.4 -2(i), (ii), (iii), 6,7,8 Ex 8.6 : -4,6,7,8,9 Ex : 8.7 - 3,6	8.8,8.9, 8.10 8.13,8.14 8.22

Example 9.10 Evaluate : $\int_1^2 \frac{x}{(x+1)(x+2)} dx$

Solution:

$$\frac{x}{(x+1)(x+2)} = \frac{A}{x+1} + \frac{B}{x+2}$$

$x = A(x+2) + B(x+1)$

When $x = -1 \quad A = -1$

When $x = -2 \quad B = 2$

$$\frac{x}{(x+1)(x+2)} = \frac{-1}{x+1} + \frac{2}{x+2}$$

$$\int_1^2 \frac{x}{(x+1)(x+2)} dx = \int_1^2 \left[\frac{-1}{x+1} + \frac{2}{x+2}\right] dx$$

$$= [-\log(x+1) + 2\log(x+2)]_1^2$$

$$= \left[\log\left|\frac{(x+2)^2}{x+1}\right|\right]_1^2 = \log\left(\frac{16}{3}\right) - \log\left(\frac{9}{2}\right)$$

$$= \log\left(\frac{16}{3} \times \frac{2}{9}\right) = \log\frac{32}{27}$$

Example 9.11 Evaluate : $\int_0^{\frac{\pi}{2}} \frac{\cos\theta}{(1+\sin\theta)(2+\sin\theta)} d\theta.$

Solution:

$$\int_0^{\frac{\pi}{2}} \frac{\cos\theta}{(1+\sin\theta)(2+\sin\theta)} d\theta = \int_1^2 \frac{1}{u(u+1)} du$$

$u = 1 + \sin\theta$ $u + 1 = 2 + \sin\theta$ $du = \cos\theta \ d\theta$	When $x = 0 \quad u = 1$ $x = \frac{\pi}{2} \quad u = 2$

$$= \int_1^2 \frac{u+1-u}{u(u+1)} du$$

$$= \int_1^2 \left[\frac{1}{u} - \frac{1}{(u+1)}\right] du$$

$$= [\log|u| - \log|u+1|]_1^2$$

$$= \left[\log\left|\frac{u}{u+1}\right|\right]_1^2$$

$$= \log\left(\frac{2}{3}\right) - \log\left(\frac{1}{2}\right) = \log\left(\frac{2}{3} \times 2\right)$$

$$= \log\frac{4}{3}$$

Example 9.12 Evaluate : $\int_0^{\frac{1}{\sqrt{2}}} \frac{\sin^{-1}x}{(1-x^2)^{3/2}} dx$

Solution:

$u = \sin^{-1}x$ $x = \sin u$ $1 - x^2$ $= 1 - \sin^2 u$ $= \cos^2 u$	$du = \frac{1}{\sqrt{1-x^2}} dx$ $x = 0 \quad u = 0$ $x = \frac{1}{\sqrt{2}} \quad u = \frac{\pi}{4}$

$$= \int_0^{\frac{\pi}{4}} \frac{u}{\cos^2 u} du = \int_0^{\frac{\pi}{4}} u \sec^2 u \, du$$

$$= [u \tan u]_0^{\pi/4} - \int_0^{\frac{\pi}{4}} \tan u \, du$$

$$= [u \tan u]_0^{\pi/4} - [\log|\cos u|]_0^{\pi/4}$$

$$= \frac{\pi}{4} + \log\frac{1}{\sqrt{2}} = \frac{\pi}{4} - \frac{1}{2}\log 2.$$

Example 9.13

Evaluate : $\int_0^{\frac{\pi}{2}} (\sqrt{\tan x} + \sqrt{\cot x}) dx.$

Solution:

$$I = \int_0^{\frac{\pi}{2}} (\sqrt{\tan x} + \sqrt{\cot x}) dx$$

$$= \int_0^{\frac{\pi}{2}} \left(\sqrt{\frac{\sin x}{\cos x}} + \sqrt{\frac{\cos x}{\sin x}}\right) dx$$

$$= \int_0^{\frac{\pi}{2}} \left(\frac{\sqrt{\sin x}}{\sqrt{\cos x}} + \frac{\sqrt{\cos x}}{\sqrt{\sin x}}\right) dx$$

$$I = \int_0^{\frac{\pi}{2}} \left(\frac{\sin x + \cos x}{\sqrt{\sin x \cos x}}\right) dx$$

$$= \sqrt{2} \int_0^{\frac{\pi}{2}} \left(\frac{\sin x + \cos x}{\sqrt{2\sin x \cos x}}\right) dx$$

$$= \sqrt{2} \int_0^{\frac{\pi}{2}} \left(\frac{\sin x + \cos x}{\sqrt{1 - (\sin x - \cos x)^2}}\right) dx$$

$$= \sqrt{2} \int_{-1}^{1} \frac{du}{\sqrt{1-u^2}} = \sqrt{2}[\sin^{-1} u]_{-1}^{1}$$

$$= \sqrt{2}\left[\frac{\pi}{2} - \left(\pi - \frac{\pi}{2}\right)\right] = \pi\sqrt{2}$$

Example 9.16 show that $\int_0^{\frac{\pi}{2}} \frac{dx}{4+5\sin x} = \frac{1}{3}\log_e 2$

Solution:

$\sin x = \frac{2\tan\frac{x}{2}}{1+tan^2\frac{x}{2}}$ $u = \tan\frac{x}{2}$ $\sin x = \frac{2u}{1+u^2}$ $x = 0 \quad u = 0$ $x = \frac{\pi}{2} \quad u = 1$	$du = \frac{1}{2}sec^2\frac{x}{2}dx$ $dx = \frac{2\,du}{sec^2\frac{x}{2}}$ $dx = \frac{2\,du}{1+tan^2\frac{x}{2}}$ $dx = \frac{2\,du}{1+u^2}$

$$\int_0^{\frac{\pi}{2}} \frac{dx}{4+5\sin x} = \int_0^1 \frac{dx}{4+5\left(\frac{2u}{1+u^2}\right)}$$

$$= \int_0^1 \frac{\left(\frac{2}{1+u^2}\right)du}{\left(\frac{4u^2+10u+4}{1+1+u^2}\right)}$$

$$= \int_0^1 \frac{2\,du}{2(2u^2+5u+2)}$$

$$= \int_0^1 \frac{du}{2\left(u^2+\frac{5}{2}u+1\right)}$$

$$= \int_0^1 \frac{du}{2\left(\left(u+\frac{5}{4}\right)^2-\frac{25}{16}+1\right)} = \frac{1}{2}\int_0^1 \frac{du}{\left(u+\frac{5}{4}\right)^2-\left(\frac{3}{4}\right)^2}$$

$$= \frac{1}{2}\times\frac{4}{2\times 3}\left[log\left(\frac{4u+2}{4u+8}\right)\right]_0^1$$

$$= \frac{1}{3}\left[log\left(\frac{6}{12}\right) - log\left(\frac{2}{8}\right)\right]$$

$$= \frac{1}{3}\left[log\left(\frac{6}{12}\times\frac{8}{2}\right)\right]$$

$$= \frac{1}{3}\log_e 2$$

Example 9.17 Prove that $\int_0^{\frac{\pi}{4}} \frac{\sin 2x\ dx}{\sin^4 x+\cos^4 x} = \frac{\pi}{4}$

Solution:

$$a^2+b^2 = (a+b)^2 - 2ab$$

$$\sin^4 x + \cos^4 x = (sin^2 x + cos^2 x)^2 - 2\sin^2 x\cos^2 x$$

$$\sin^4 x + \cos^4 x = 1 - \frac{1}{2}(2\sin x\cos x)^2$$

$$\sin^4 x + \cos^4 x = \frac{2-(\sin 2x)^2}{2}$$

$u = \cos 2x \qquad du = -2\sin 2x\ dx$

$-du = 2\sin 2x\ dx$

When $x = 0 \quad u = 1 \quad and\ x = \frac{\pi}{4} \quad u = 0$

$$\int_0^{\frac{\pi}{4}} \frac{\sin 2x\ dx}{\sin^4 x+\cos^4 x} = \int_0^{\frac{\pi}{4}} \frac{2\sin 2x\,dx}{2-(\sin 2x)^2}$$

$$= \int_0^{\frac{\pi}{4}} \frac{2\sin 2x\,dx}{2-sin^2 2x} = \int_0^{\frac{\pi}{4}} \frac{2\sin 2x\,dx}{2-(1-cos^2 2x)}$$

$$= \int_0^{\frac{\pi}{4}} \frac{2\sin 2x\,dx}{1+cos^2 2x}$$

$$= \int_1^0 \frac{-du}{1+u^2} = \int_0^1 \frac{du}{1+u^2} = [\tan^{-1} u]_0^1 = \frac{\pi}{4}$$

Example 9.18 Prove that

$\int_0^{\frac{\pi}{4}} \frac{dx}{a^2\sin^2 x+b^2\cos^2 x} = \frac{1}{ab}\tan^{-1}\left(\frac{a}{b}\right)$,

where a, b > 0.

Solution:

$$\int_0^{\frac{\pi}{4}} \frac{dx}{a^2\sin^2 x+b^2\cos^2 x} = \int_0^{\frac{\pi}{4}} \left(\frac{1}{\cos^2 x}\right)\frac{dx}{a^2\tan^2 x+b^2}$$

$u = \tan x$ $x = 0 \quad u = 0$ $x = \frac{\pi}{4} \quad u = 1$	$du = \sec^2 x\ dx$

$$= \int_0^{\frac{\pi}{4}} \frac{\sec^2 x\ dx}{a^2\tan^2 x+b^2} = \int_0^1 \frac{du}{a^2u^2+b^2}$$

$$= \frac{1}{a^2}\int_0^1 \frac{du}{u^2+\left(\frac{b}{a}\right)^2} = \frac{1}{a^2}\left[\frac{a}{b}\tan^{-1}\left(\frac{au}{b}\right)\right]_0^1$$

$$= \frac{1}{ab}\tan^{-1}\left(\frac{a}{b}\right).$$

Example 9.19 Evaluate : $\int_0^{\frac{\pi}{4}} \frac{1}{\sin x+\cos x} dx$

Solution:

$$\int_0^{\frac{\pi}{4}} \frac{1}{\sin x + \cos x} dx = \int_0^{\frac{\pi}{4}} \frac{1}{\sqrt{2}\left(\frac{1}{\sqrt{2}}\sin x + \frac{1}{\sqrt{2}}\cos x\right)} dx$$

$$= \frac{1}{\sqrt{2}} \int_0^{\frac{\pi}{4}} \frac{1}{\cos\left(\frac{\pi}{4} - x\right)} dx = \frac{1}{\sqrt{2}} \int_0^{\frac{\pi}{4}} \frac{1}{\cos x} dx$$

$$= \frac{1}{\sqrt{2}} \int_0^{\frac{\pi}{4}} \sec x \, dx$$

$$= \frac{1}{\sqrt{2}} [\log [\sec x + \tan x]]_0^{\pi/4}$$

$$= \frac{1}{\sqrt{2}} [\log(\sqrt{2}+1) - \log(1+0)] = \frac{1}{\sqrt{2}} \log(\sqrt{2}+1)$$

Example 9.21 Evaluate : $\int_0^{\pi} \frac{x}{1+\sin x} dx$

Solution:

$$\int_0^a x f(x)\, dx = \frac{a}{2}\int_0^a f(x)\, dx \text{ if } f(a-x) = f(x)$$

$$f(x) = \frac{1}{1+\sin x},$$

$$f(\pi - x) = \frac{1}{1+\sin(\pi - x)} = \frac{1}{1+\sin x} = f(x)$$

$$\int_0^{\pi} x \frac{1}{1+\sin x} dx = \frac{\pi}{2}\int_0^{\pi} \frac{1}{1+\sin x} dx$$

$$= 2\int_0^{\frac{\pi}{2}} \frac{1}{1+\sin\left(\frac{\pi}{2} - x\right)} dx$$

$$= 2\int_0^{\frac{\pi}{2}} \frac{1}{1+\cos x} dx$$

$$= 2\int_0^{\frac{\pi}{2}} \frac{1}{2\cos^2 \frac{x}{2}} dx$$

$$= \int_0^{\frac{\pi}{2}} \sec^2 \frac{x}{2} dx$$

$$= \left[2\tan\frac{x}{2}\right]_0^{\frac{\pi}{2}} = 2[1-0] = 2$$

Example 9.23 If $f(x) = f(a+x)$, **then** $\int_0^{2a} f(x)\, dx = 2\int_0^a f(x)\, dx$

Solution:

$$\int_0^{2a} f(x)\, dx = \int_0^a f(x)\, dx + \int_a^{2a} f(x)\, dx$$

Take $\int_a^{2a} f(x)\, dx$

Let $x = a + u \quad dx = du$

when $x = a,\ u = 0$ and $x = 2a,\ u = a$

$$\int_0^{2a} f(x)\, dx = \int_0^a f(x)\, dx + \int_0^a f(a+u)\, du$$

$$\int_0^{2a} f(x)\, dx = \int_0^a f(x)\, dx + \int_0^a f(x)\, dx$$

$$= 2\int_0^a f(x)\, dx \quad (\because \text{given } f(x) = f(a+x))$$

Hence proved.

Example 9.25 Evaluate : $\int_{-\log 2}^{\log 2} e^{-|x|} dx$

Solution:

Let $f(x) = e^{-|x|}$

$$f(-x) = e^{-|(-x)|} = e^{-|x|} = f(x)$$

$f(x) = e^{-|x|}$ **is an even function.**

$$\int_{-\log 2}^{\log 2} e^{-|x|} dx = 2\int_0^{\log 2} e^{-|x|} dx$$

$$= 2\left[\frac{e^{-x}}{-1}\right]_0^{\log 2}$$

$$= -2[e^{-\log 2} - e^0] = -2\left[\frac{1}{2} - 1\right] = -2\left(\frac{-1}{2}\right) = 1$$

Example 9.27

Prove that $\int_0^{\frac{\pi}{4}} \log(1+\tan x)\, dx = \frac{\pi}{8}\log 2.$

Solution:

$$I = \int_0^{\frac{\pi}{4}} \log(1+\tan x)\, dx \ldots\ldots\ldots\ldots (1)$$

$$I = \int_0^{\frac{\pi}{4}} \log\left(1+\tan\left(\frac{\pi}{4} - x\right)\right) dx$$

$$= \int_0^{\frac{\pi}{4}} \log\left(1 + \frac{1 - \tan x}{1 + \tan x}\right) dx$$

$$I = \int_0^{\frac{\pi}{4}} \log\left(\frac{2}{1 + \tan x}\right) dx$$

$$= \int_0^{\frac{\pi}{4}} [\log 2 - \log(1 + \tan x)]\, dx \ldots\ldots (2)$$

From (1) + (2)

$$2I = \int_0^{\frac{\pi}{4}} \log(1 + \tan x) + \log 2 - \log(1 + \tan x)\, dx$$

$$= \int_0^{\frac{\pi}{4}} \log 2\, dx = \frac{\pi}{4} \log 2$$

$$I = \frac{\pi}{8} \log 2$$

Example 9.28 Show that

$\int_0^1 (\tan^{-1} x + \tan^{-1}(1 - x))\, dx = \frac{\pi}{2} - \log_e 2.$

Solution:

$$I = \int_0^1 (\tan^{-1} x + \tan^{-1}(1 - x))\, dx$$

$$= \int_0^1 \tan^{-1} x\, dx + \int_0^1 \tan^{-1}(1 - x)\, dx$$

$$= \int_0^1 \tan^{-1} x\, dx + \int_0^1 \tan^{-1}(1 - (1 - x))\, dx$$

$$I = \int_0^1 (\tan^{-1} x + \tan^{-1} x)\, dx$$

$$= 2\int_0^1 \tan^{-1} x\, dx$$

$u = \tan^{-1} x$ $du = \frac{1}{1 + x^2} dx$	$dv = dx$ $v = x$

$$I = 2\left[x \tan^{-1} x - \int_0^1 \frac{x}{1 + x^2} dx\right]_0^1$$

$$= 2\left[x \tan^{-1} x - \frac{1}{2}\int_0^1 \frac{2x}{1 + x^2} dx\right]_0^1$$

$$= 2\left[x \tan^{-1} x - \frac{1}{2} \log(1 + x^2)\right]_0^1$$

$$= 2\left[\frac{\pi}{4} - \frac{1}{2}\log 2 - \left(0 - \frac{1}{2}\log 1\right)\right]$$

$$= 2\left[\frac{\pi}{4} - \frac{1}{2}\log 2\right] = \frac{\pi}{2} - \log 2$$

Example 9.29 Evaluate : $\int_2^3 \frac{\sqrt{x}}{\sqrt{5-x}+\sqrt{x}} dx$

Solution:

$$I = \int_2^3 \frac{\sqrt{x}}{\sqrt{5 - x} + \sqrt{x}} dx \ldots\ldots\ldots (1)$$

$$I = \int_2^3 \frac{\sqrt{(3 + 2) - x}}{\sqrt{5 - (3 + 2 - x)} + \sqrt{(3 + 2 - x)}} dx$$

$$= \int_2^3 \frac{\sqrt{5 - x}}{\sqrt{x} + \sqrt{5 - x}} dx \ldots\ldots (2)$$

From (1) + (2),

$$2I = \int_2^3 \frac{\sqrt{x} + \sqrt{5 - x}}{\sqrt{x} + \sqrt{5 - x}} dx = [x]_2^3 = [3 - 2] = 1$$

$$I = \frac{1}{2}$$

Example 9.30 Evaluate : $\int_{-\pi}^{\pi} \frac{\cos^2 x}{1+a^x} dx$

Solution:

$$\int_{-\pi}^{\pi} \frac{\cos^2 x}{1 + a^x} dx \ldots\ldots\ldots (1)$$

$$= \int_{-\pi}^{\pi} \frac{\cos^2(\pi - \pi - x)}{1 + a^{\pi - \pi - x}} dx$$

$$= \int_{-\pi}^{\pi} \frac{\cos^2(-x)}{1 + a^{-x}} dx = \int_{-\pi}^{\pi} a^x \left(\frac{\cos^2(x)}{1 + a^x}\right) dx \ldots\ldots (2)$$

From (1) + (2),

$$2I = \int_{-\pi}^{\pi} (a^x + 1)\left(\frac{\cos^2(x)}{1+a^x}\right) dx$$

$$2\,\mathrm{I} = 2\int_0^{\pi} \cos^2 x\, dx$$

$$I = \int_0^{\pi} \frac{1 + \cos 2x}{2} dx = \frac{1}{2}\left[x - \frac{\sin 2x}{2}\right]_0^{\pi}$$

$$I = \frac{1}{2}\left[\pi - \frac{\sin 2\pi}{2}\right]$$

$$I = \frac{1}{2}[\pi - 0] = \frac{\pi}{2}$$

EXERCISE 9.3

1. (iii) $\int_0^1 \sqrt{\frac{1-x}{1+x}}\,dx$

Solution:

$$\int_0^1 \sqrt{\frac{1-x}{1+x}}\,dx = \int_0^1 \sqrt{\frac{1-x}{1+x}\times\frac{1-x}{1-x}}\,dx = \int_0^1 \frac{1-x}{\sqrt{1-x^2}}\,dx$$

$$= \int_0^1 \frac{1}{\sqrt{1-x^2}}\,dx + \frac{1}{2}\int_0^1 \frac{-2x}{\sqrt{1-x^2}}\,dx$$

$$= [\sin^{-1} x]_0^1 + \frac{1}{2}\times 2\left[\sqrt{1-x^2}\right]_0^1$$

$$\int \frac{f'(x)}{\sqrt{f(x)}}\,dx = 2\sqrt{f(x)} + c$$

$$= [\sin^{-1}(1) - \sin^{-1}(0)] + [0-1]$$

$$= \frac{\pi}{2} - 1$$

1. (iv) $\int_0^{\frac{\pi}{2}} e^x\left(\frac{1+\sin x}{1+\cos x}\right)dx$

Solution:

$$\int_0^{\frac{\pi}{2}} e^x\left(\frac{1+\sin x}{1+\cos x}\right)dx = \int_0^{\frac{\pi}{2}} e^x\left(\frac{1}{1+\cos x} + \frac{\sin x}{1+\cos x}\right)dx$$

$$= \int_0^{\frac{\pi}{2}} e^x\left(\frac{1}{2\cos^2\frac{x}{2}} + \frac{2\sin\frac{x}{2}\cos\frac{x}{2}}{2\cos^2\frac{x}{2}}\right)dx$$

$$= \int_0^{\frac{\pi}{2}} e^x\left(\frac{1}{2}\sec^2\frac{x}{2} + \tan\frac{x}{2}\right)dx$$

$$\int e^x(f'(x) + f(x))dx = e^x f(x) + c$$

$$= \left[e^x \tan\frac{x}{2}\right]_0^{\frac{\pi}{2}} = e^{\frac{\pi}{2}}$$

(vi) $\int_0^1 \frac{1-x^2}{(1+x^2)^2}\,dx$

Solution:

$$\int_0^1 \frac{1-x^2}{(1+x^2)^2}\,dx$$

$$= \int_0^{\frac{\pi}{4}} \frac{1-\tan^2 u}{(1+\tan^2 u)^2}(\sec^2 u\,du)$$

$$= \int_0^{\frac{\pi}{4}} \frac{1-\tan^2 u}{(1+\tan^2 u)^2}(1+\tan^2 u)\,du$$

$$= \int_0^{\frac{\pi}{4}} \frac{1-\tan^2 u}{(1+\tan^2 u)}\,du$$

$$= \int_0^{\frac{\pi}{4}} \cos 2u\,du = \left[\frac{\sin 2u}{2}\right]_0^{\frac{\pi}{4}} = \frac{1}{2}[1-0] = \frac{1}{2}$$

2. (iv) $\int_0^{2\pi} x\log\left(\frac{3+\cos x}{3-\cos x}\right)dx$

Solution:

$$f(x) = \log\left(\frac{3+\cos x}{3-\cos x}\right)$$

$$f(2\pi - x) = \log\left(\frac{3+\cos(2\pi-x)}{3-\cos(2\pi-x)}\right) = \log\left(\frac{3+\cos x}{3-\cos x}\right) = f(x)$$

$$\int_0^{2\pi} x\log\left(\frac{3+\cos x}{3-\cos x}\right)dx$$

$$= \frac{2\pi}{2}\int_0^{2\pi} \log\left(\frac{3+\cos x}{3-\cos x}\right)dx$$

$$= \pi\int_0^{2\pi} \log\left(\frac{3+\cos x}{3-\cos x}\right)dx$$

$$\int_0^{2a} f(x)\,dx = 0 \quad if \quad f(2a-x) = -f(x)$$

$$f(\pi - x) = \log\left(\frac{3+\cos(\pi-x)}{3-\cos(\pi-x)}\right) = \log\left(\frac{3-\cos x}{3+\cos x}\right) = -f(x)$$

$$= \pi\times 2\int_0^{\pi} \log\left(\frac{3+\cos x}{3-\cos x}\right)dx = 0$$

(v) $\int_0^{2\pi} \sin^4 x\cos^3 x\;dx$

Solution:

Let $f(x) = \sin^4 x\cos^3 x$

$f(2\pi - x) = \sin^4 x\cos^3 x = f(x)$

Let $I = \int_0^{2\pi} \sin^4 x\cos^3 x\;dx =$
$2\int_0^{\pi} \sin^4 x\cos^3 x\;dx$

Let $f(x) = 2\sin^4 x\cos^3 x$

$f(\pi - x) = 2\sin^4 x\cos^3 x = -f(x)$

$$I = 2\int_0^{\pi} \sin^4 x\cos^3 x\;dx = 0$$

(vii) $\int_0^{\sin^2 x} \sin^{-1}\sqrt{t}\,dt + \int_0^{\cos^2 x} \cos^{-1}\sqrt{t}\,dt$

Solution:

Let $t = \sin^2\theta$

$dt = 2\sin\theta\cos\theta\,d\theta \quad = \sin 2\theta\,d\theta$

When $t = \sin^2 x$ $\sin^2 x = \sin^2\theta$ $\theta = x$	When $t = \cos^2 x$ $\cos^2 x = \sin^2\theta$ $\theta = \frac{\pi}{2} - x$	When $t = 0$ $0 = \sin^2\theta$ $\theta = 0$

$$\int_0^{\sin^2 x} \sin^{-1}\sqrt{t}\,dt + \int_0^{\cos^2 x} \cos^{-1}\sqrt{t}\,dt$$

$$= \int_0^{x} \sin^{-1}\sqrt{\sin^2\theta}\,(\sin 2\theta d\theta) + \int_0^{\frac{\pi}{2}-x} \cos^{-1}\sqrt{\sin^2\theta}\,(\sin 2\theta d\theta)$$

$$= \int_0^{x} \theta\,(\sin 2\theta d\theta) + \int_0^{\frac{\pi}{2}-x} \cos^{-1}(\sin\theta)\,(\sin 2\theta d\theta)$$

$$= \int_0^{x} \theta\,(\sin 2\theta d\theta) + \int_0^{\frac{\pi}{2}-x} \cos^{-1}\left(\cos\left(\frac{\pi}{2}-\theta\right)\right)(\sin 2\theta d\theta)$$

$$= \int_0^x \theta \sin 2\theta \, d\theta + \int_0^{\frac{\pi}{2}-x} \left(\frac{\pi}{2}-\theta\right) \sin 2\theta \, d\theta$$

$$= \left[-\frac{\theta \cos 2\theta}{2} + \frac{\sin 2\theta}{4}\right]_0^x + \left[-\left(\frac{\pi}{2}-\theta\right)\frac{\cos 2\theta}{2} + \frac{(-1)\sin 2\theta}{4}\right]_0^{\frac{\pi}{2}-x}$$

$$= -\frac{x\cos 2x}{2} + \frac{\sin 2x}{4} + \left[-\left(\frac{\pi}{2}-\frac{\pi}{2}+x\right)\frac{\cos 2\left(\frac{\pi}{2}-x\right)}{2} + \frac{(-1)\sin 2\left(\frac{\pi}{2}-x\right)}{4}\right] - \left[-\left(\frac{\pi}{2}-0\right)\frac{\cos 0}{2} - 0\right]$$

$$= -\frac{x\cos 2x}{2} + \frac{\sin 2x}{4} + \left[\frac{x\cos 2x}{2} - \frac{\sin 2x}{4}\right] + \frac{\pi}{4}$$

$$= \frac{\pi}{4}$$

(viii) $\int_0^1 \frac{\log(1+x)}{1+x^2} dx$

Solution:

$x = tan\theta$	$dx = sec^2\theta d\theta$
When $x = 0;\ \theta = 0$	$x = 1;\ \theta = \frac{\pi}{4}$

$$\int_0^1 \frac{\log(1+x)}{1+x^2} dx = \int_0^{\frac{\pi}{4}} \frac{\log(1+\tan\theta)}{1+tan^2\theta} sec^2\theta d\theta$$

$$I = \int_0^{\frac{\pi}{4}} \log(1+\tan\theta)\, d\theta \ldots\ldots\ldots (1)$$

$$I = \int_0^{\frac{\pi}{4}} \log\left(1+\tan\left(\frac{\pi}{4}-\theta\right)\right) d\theta$$

$$= \int_0^{\frac{\pi}{4}} \log\left(1+\frac{1-tan\,\theta}{1+tan\,\theta}\right) dx$$

$$I = \int_0^{\frac{\pi}{4}} \log\left(\frac{2}{1+tan\,\theta}\right) d\theta$$

$$= \int_0^{\frac{\pi}{4}} [\log 2 - log\,(1+tan\,\theta)]\, d\theta \ldots\ldots (2)$$

From (1) + (2)

$$2I = \int_0^{\frac{\pi}{4}} \log(1+\tan x) + log\,2 - log\,(1+tan\,x)\, dx$$

$$= \int_0^{\frac{\pi}{4}} \log 2\, dx = \frac{\pi}{4} log\,2$$

$$I = \frac{\pi}{8} log\,2$$

(ix) $\int_0^{\pi} \frac{x\sin x}{1+\sin x} dx$

Solution:

Let $I = \int_0^{\pi} \frac{x\sin x}{1+\sin x} dx$

$$= \int_0^{\pi} \frac{(\pi-x)\sin(\pi-x)}{1+\sin(\pi-x)} dx$$

$$I = \pi\int_0^{\pi} \frac{\sin x}{1+\sin x} dx - \int_0^{\pi} \frac{x\sin x}{1+\sin x} dx$$

$$I = \pi\int_0^{\pi} \frac{\sin x}{1+\sin x} dx - I$$

$$2I = \pi\int_0^{\pi} \frac{\sin x}{1+\sin x} dx$$

$$I = \frac{\pi}{2}\int_0^{\pi} \frac{\sin x\,(1+\sin x)}{1-\sin^2 x} dx$$

$$= \frac{\pi}{2}\int_0^{\pi} \frac{\sin x - \sin^2 x}{\cos^2 x} dx$$

$$= \frac{\pi}{2}\int_0^{\pi} \left[\frac{\sin x}{\cos^2 x} - \frac{\sin^2 x}{\cos^2 x}\right] dx$$

$$= \frac{\pi}{2}\int_0^{\pi} \left[sec\,x\ tan\,x - \frac{1-\cos^2 x}{\cos^2 x}\right] dx$$

$$= \frac{\pi}{2}\int_0^{\pi} [sec\,x\ tan\,x - \sec^2 x + 1]\ dx$$

$$= \frac{\pi}{2}[sec\,x - tanx + x]_0^{\pi}$$

$$= \frac{\pi}{2}[(0-1)-(0+1)+\pi] = \frac{\pi}{2}(\pi-2)$$

$$= \frac{\pi^2 - 2\pi}{2}$$

(x) $\int_{\frac{\pi}{8}}^{\frac{3\pi}{8}} \frac{1}{1+\sqrt{\tan x}}\, dx$

Solution:

$$I = \int_{\frac{\pi}{8}}^{\frac{3\pi}{8}} \frac{1}{1+\sqrt{\tan x}}\, dx \;\dots\dots (1)$$

$$I = \int_{\frac{\pi}{8}}^{\frac{3\pi}{8}} \frac{1}{1+\sqrt{\tan\left(\frac{3\pi}{8}+\frac{\pi}{8}-x\right)}}\, dx$$

$$= \int_{\frac{\pi}{8}}^{\frac{3\pi}{8}} \frac{1}{1+\sqrt{\cot x}}\, dx$$

$$= \int_{\frac{\pi}{8}}^{\frac{3\pi}{8}} \frac{\sqrt{\tan x}}{1+\sqrt{\tan x}}\, dx \;\dots\dots (2)$$

From (1) + (2),

$$2I = \int_{\frac{\pi}{8}}^{\frac{3\pi}{8}} \frac{1+\sqrt{\tan x}}{1+\sqrt{\tan x}}\, dx = \int_{\frac{\pi}{8}}^{\frac{3\pi}{8}} dx = \frac{3\pi}{8} - \frac{\pi}{8} = \frac{2\pi}{8}$$

$$I = \frac{2\pi}{8\times 2} = \frac{\pi}{8}$$

(xi) $\int_0^{\pi} x[\sin^2(\sin x) + \cos^2(\cos x)]\, dx$

Solution:

$$I = \int_0^{\pi} x[\sin^2(\sin x) + \cos^2(\cos x)]\, dx$$

$$f(x) = \sin^2(\sin x) + \cos^2(\cos x)$$

$$f(\pi - x) = \sin^2(\sin(\pi - x)) + \cos^2(\cos(\pi - x)) = f(x)$$

$$I = \int_0^{\pi} x[\sin^2(\sin x) + \cos^2(\cos x)]\, dx$$

$$= \frac{\pi}{2}\int_0^{\pi} [\sin^2(\sin x) + \cos^2(\cos x)]\, dx$$

$$= \frac{\pi}{2} \times 2\int_0^{\frac{\pi}{2}} [\sin^2(\sin x) + \cos^2(\cos x)]\, dx$$

$$I = \pi\int_0^{\frac{\pi}{2}} [\sin^2(\sin x) + \cos^2(\cos x)]\, dx \;\dots\dots\dots (1)$$

$$= \pi\int_0^{\frac{\pi}{2}} \left[\sin^2\left(\sin\left(\frac{\pi}{2}-x\right)\right) + \cos^2\left(\cos\left(\frac{\pi}{2}-x\right)\right)\right] dx$$

$$I = \pi\int_0^{\frac{\pi}{2}} [\sin^2(\cos x) + \cos^2(\sin x)]\, dx \;\dots\dots\dots (2)$$

From (1) + (2),

$$2I = \pi\int_0^{\frac{\pi}{2}} [\sin^2(\cos x) + \cos^2(\sin x) + \sin^2(\sin x) + \cos^2(\cos x)]\, dx$$

$$= \pi\int_0^{\frac{\pi}{2}} 2\, dx$$

$$2I = 2\pi \times \frac{\pi}{2}$$

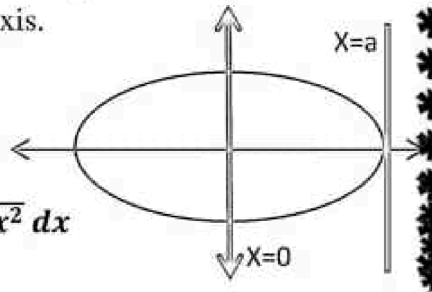
$$I = \frac{\pi^2}{2}$$

Example 9.49 Find the area of the region bounded by the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$.

Solution:

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1 \qquad y = \frac{b}{a}\sqrt{a^2 - x^2}$$

$x = 0, x = a$ and x-axis.



Area $= 4\int_c^d y\, dx$

$$A = 4\int_0^a \frac{b}{a}\sqrt{a^2 - x^2}\, dx$$

$$= \frac{4\,b}{a}\left[\frac{x}{2}\sqrt{a^2 - x^2} + \frac{a^2}{2}\sin^{-1}\left(\frac{x}{a}\right)\right]_0^a$$

$$= \frac{4\,b}{a}\left(\frac{a^2}{2}\sin^{-1}(1) - 0\right)$$

$$= \frac{4\,b}{a} \times \frac{a^2}{2} \times \frac{\pi}{2} = \pi ab$$

$Area = \pi ab$ sq.units

Example 9.50 Find the area of the region bounded between the parabola $y^2 = 4ax$ and its latus rectum.

Solution:

$y^2 = 4ax$

$y = 2\sqrt{a}\sqrt{x}$ x-axis, $x = 0$ and $x = a$.

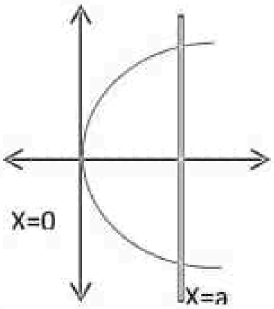
Area $= 2\int_c^d y\,dx$

$= 2\int_0^a 2\sqrt{a}\sqrt{x}\,dx$

$= 4\sqrt{a}\int_0^a \sqrt{x}\,dx$

$= 4\sqrt{a}\left[\frac{2}{3}x\sqrt{x}\right]_0^a = \frac{8\sqrt{a}}{3}(a\sqrt{a})$

$Area = \frac{8a^2}{3}$ sq.units



Example 9.51 Find the area of the region bounded by the y -axis and the parabola $x = 5 - 4y - y^2$.

Solution:

$-x = -5 + 4y + y^2$

$5 - x = 4y + y^2$

$5 - x = 4 + 4y + y^2 - 4$

$5 - x = (y+2)^2 - 4$

$9 - x = (y+2)^2 \qquad x = 9 - (y+2)^2$

$x = 5 - 4y - y^2$

y-axis, $y = -5$ *and* $y = 1$

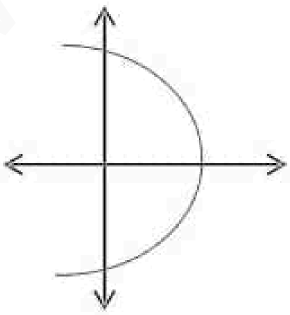
Area $= \int_a^b x\,dy$

$= \int_{-5}^{1} \left(9 - (y+2)^2\right)\,dy$

$= \left[9y - \frac{(y+2)^3}{3}\right]_{-5}^{1}$

$= \left(9 - \frac{27}{3}\right) - \left(-45 + \frac{27}{3}\right)$

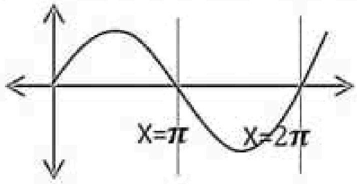
$= (9 - 9) + (45 - 9) = 36$



Example 9.52 Find the area of the region bounded by x-axis, the sine curve $y = \sin x$, the lines $x = 0$ and $x = 2\pi$.

Solution:

$y = \sin x$, x-axis $x = 0$ and $x = 2\pi$.



$$Area = \int_0^{\pi} y\,dx + \int_{\pi}^{2\pi} (-y)\,dx$$

$$= \int_0^{\pi} \sin x\,dx - \int_{\pi}^{2\pi} \sin x\,dx$$

X	0	0	5
Y	-5	1	0

$= [-\cos x]_0^{\pi} - [-\cos x]_{\pi}^{2\pi}$

$= -[(-1) - (1)] + (1 - (-1))$

$= 2 + 2 = 4$

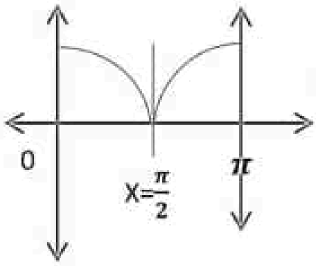
Example 9.53 Find the area of the region bounded by x-axis, the curve $y = |\cos x|$, the lines $x = 0$ and $x = \pi$.

Solution:

$$y = |\cos x| = \begin{cases} \cos x\ ; & if\ 0 \le x \le \frac{\pi}{2} \\ -\cos x\ ; & if\ \frac{\pi}{2} \le x \le \pi \end{cases}$$

$Area = \int_0^{\frac{\pi}{2}} y\,dx + \int_{\frac{\pi}{2}}^{\pi} (-y)\,dx$

$= \int_0^{\frac{\pi}{2}} \cos x\,dx - \int_{\frac{\pi}{2}}^{\pi} \cos x\,dx$



$= [\sin x]_0^{\frac{\pi}{2}} - [\sin x]_{\frac{\pi}{2}}^{\pi}$

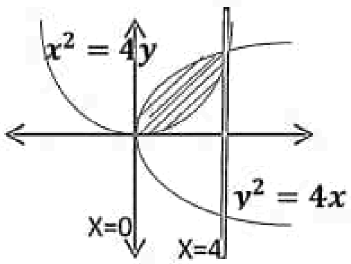
$= [1 - 0] - (0 - 1) = 2$

Example 9.54 Find the area of the region bounded between the parabolas $y^2 = 4x$ and $x^2 = 4y$.

Solution:

$y^2 = 4x$ $y = 2\sqrt{x}$	$x^2 = 4y$ $y = \frac{x^2}{4}$	$\left(\frac{x^2}{4}\right)^2 = 4x$ $x^4 = 64x$ $x(x^3 - 16) = 0$ $x = 0$ *and* $x = 4$

x	0	4
$y^2 = 4x$	0	4



$$Area \;\; A = \int_a^b [Y_U - Y_L]\; dx = \int_0^4 \left[2\sqrt{x} - \frac{x^2}{4}\right] dx$$

$$= \left[2\left(\frac{2x\sqrt{x}}{3}\right) - \frac{x^3}{12}\right]_0^4$$

$$= \left[\frac{4x\sqrt{x}}{3} - \frac{x^3}{12}\right]_0^4$$

$$= \left[\frac{32}{3} - \frac{64}{12}\right] - [0 - 0]$$

$$= \frac{128 - 64}{12}$$

$$= \frac{64}{12}$$

$$= \frac{16}{3}.$$

Example 9.55 Find the area of the region bounded between the parabola $x^2 = y$ and the curve $y = |x|$.

Solution:

$x^2 = y$	$y = \lvert x\rvert = \begin{cases} x & if\ x \geq 0 \\ -x & if\ x < 0 \end{cases}$	X	1	-1
		$x^2 = y$	1	1

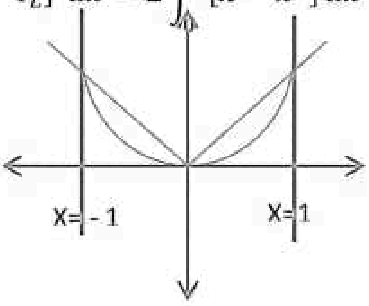
$$Area \quad A = \int_a^b [Y_U - Y_L]\ dx = 2\int_0^1 [x - x^2]\,dx$$

$$= 2\left[\frac{x^2}{2} - \frac{x^3}{3}\right]_0^1$$

$$= 2\left[\frac{1}{2} - \frac{1}{3}\right]$$

$$= 2\left(\frac{1}{6}\right)$$

$$= \frac{1}{3}.$$



Example 9.56 Find the area of the region bounded by $y = \cos x$, $y = \sin x$, the lines $x = \frac{\pi}{4}$ and $x = \frac{5\pi}{4}$

Solution:

$y = \cos x$	$y = \sin x$	$\cos x = \sin x$ $\frac{\pi}{4} \leq x \leq \frac{5\pi}{4}$

$$Area \quad A = \int_a^b [Y_U - Y_L]\ dx$$

$$= \int_{\frac{\pi}{4}}^{\frac{5\pi}{4}} [\sin x - \cos x]\,dx$$

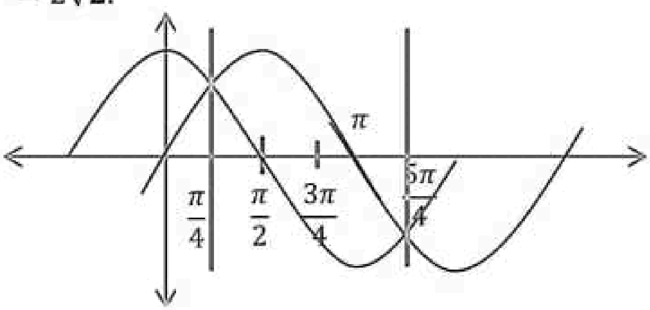
$$= \int_{\frac{5\pi}{4}}^{\frac{\pi}{4}} [\cos x - \sin x]\,dx$$

$$= [\sin x + \cos x]_{\frac{5\pi}{4}}^{\frac{\pi}{4}}$$

$$= \left[\frac{1}{\sqrt{2}} + \frac{1}{\sqrt{2}}\right] - \left[\frac{-1}{\sqrt{2}} + \frac{-1}{\sqrt{2}}\right]$$

$$= [\sqrt{2} + \sqrt{2}]$$

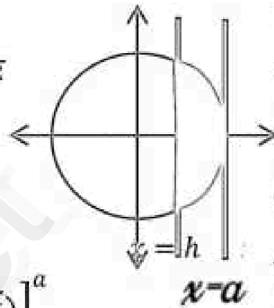
$$= 2\sqrt{2}.$$



Example 9.57 The region enclosed by the circle $x^2 + y^2 = a^2$ is divided into two segments by the line $x = h$. Find the area of the smaller segment.

Solution:

$$x^2 + y^2 = a^2 \qquad y = \sqrt{a^2 - x^2}$$



$$Area \quad A = 2\int_h^a \sqrt{a^2 - x^2}\ dx$$

$$= 2\left[\frac{x}{2}\sqrt{a^2 - x^2} + \frac{a^2}{2}\sin^{-1}\left(\frac{x}{a}\right)\right]_h^a$$

$$= 2\left(\left[0 + \frac{a^2}{2}\sin^{-1}(1)\right] - \left[\frac{h}{2}\sqrt{a^2 - h^2} + \frac{a^2}{2}\sin^{-1}\left(\frac{h}{a}\right)\right]\right)$$

$$= 2\left(\left[\frac{a^2}{2}\left(\frac{\pi}{2}\right)\right] - \frac{1}{2}\left[h\sqrt{a^2 - h^2} + a^2\sin^{-1}\left(\frac{h}{a}\right)\right]\right)$$

$$= \frac{\pi a^2}{2} - h\sqrt{a^2 - h^2} - a^2\sin^{-1}\left(\frac{h}{a}\right)$$

$$= a^2\left[\frac{\pi}{2} - \sin^{-1}\left(\frac{h}{a}\right)\right] - h\sqrt{a^2 - h^2}$$

$$= a^2\cos^{-1}\left(\frac{h}{a}\right) - h\sqrt{a^2 - h^2}$$

Example 9.58 Find the area of the region in the first quadrant bounded by the parabola $y^2 = 4x$, the line $x + y = 3$ and y -axis.

Solution:

The line $x + y = 3$ meets the y -axis at (0,3) .

$y^2 = 4x$ $(3-x)^2 = 4x$ $9 - 6x + x^2 - 4x = 0$ $x^2 - 10x + 9 = 0$ $x = 9$ *and* 1	$x + y = 3$ $y = 3 - x$	X: 1, 9; $y = 3 - x$: 2, -6 $x = \begin{cases} \frac{y^2}{4} & if\ 0 \le y \le 2 \\ 3 - y & if\ 2 \le y \le 3 \end{cases}$

$$Area\quad A = \int_0^2 x\,dy + \int_2^3 x\,dy$$

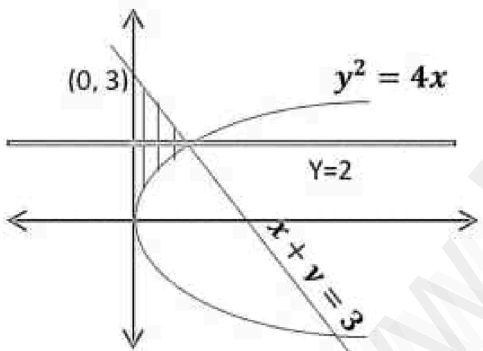
$$= \int_0^2 \frac{y^2}{4}dy + \int_2^3 (3-y)\,dy$$

$$= \left[\frac{y^3}{12}\right]_0^2 + \left[\frac{(3-y)^2}{2\times(-1)}\right]_2^3$$

$$= \frac{8}{12} + 0 - \frac{(1)^2}{(-2)}$$

$$= \frac{2}{3} + \frac{1}{2}$$

$$= \frac{4+3}{6} = \frac{7}{6}.$$



Example 9.59 Find, by integration, the area of the region bounded by the lines $5x - 2y = 15$, $x + y + 4 = 0$ and the x-axis.

Solution:

$5x - 2y - 15 = 0$ ....(1) $x + y + 4 = 0$ ....(2) From (1)+2(2) $7x - 7 = 0 \quad x = 1$ When $x = 1$ $y = -5$	$5x - 2y = 15$ meets the x-axis at (3,0). $x + y + 4 = 0$ meets the x-axis at $(-4,0)$.

$$Area\ \ A = \int_{-4}^{1} (-y)\ dx + \int_{1}^{3} (-y)\ dx$$

$$= \int_{-4}^{1} (x+4)\ dx + \int_{1}^{3} \left(\frac{15-5x}{2}\right) dx$$

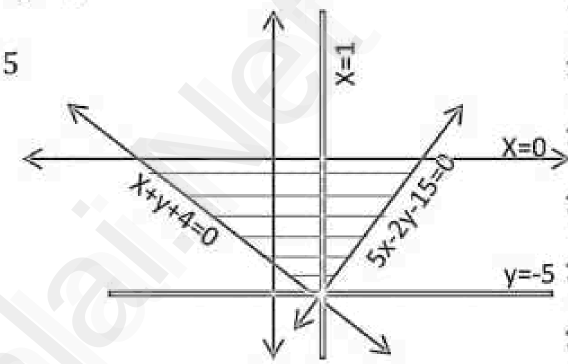
$$= \int_{-4}^{1} (x+4)\ dx + \frac{5}{2}\int_{1}^{3} (3-x)\ dx$$

$$= \left[\frac{(x+4)^2}{2}\right]_{-4}^{1} + \frac{5}{2}\left[\frac{(3-x)^2}{2\times -1}\right]_1^3$$

$$= \left[\frac{25}{2} - 0\right] + \frac{5}{2}(2)$$

$$= \frac{25}{2} + 5$$

$$= \frac{35}{2}$$



Example 9.60 Using integration find the area of the region bounded by triangle ABC, whose vertices A, B, and C are $(-1,1)$,(3,2), and (0,5)respectively.

Solution:

Equation of AB is $\frac{y-1}{2-1} = \frac{x+1}{3+1}$ or $y = \frac{1}{4}(x+5)$

Equation of BC is $\frac{y-5}{2-5} = \frac{x-0}{3-0}$ or $y = -x + 5$

Equation of CA is $\frac{y-1}{5-1} = \frac{x+1}{0+1}$ or $y = 4x + 5$

Area of ΔABC =Area of $DACO$ + Area of $OCBE$ − Area of $DABE$

$$= \int_{-1}^{0} (4x+5)\,dx + \int_{0}^{3} (5-x)\,dx - \frac{1}{4}\int_{-1}^{3} (x+5)\,dx$$

$$= \left[\frac{(4x+5)^2}{2\times 4}\right]_{-1}^{0} + \left[\frac{(5-x)^2}{2\times -1}\right]_0^3 - \frac{1}{4}\left[\frac{(x+5)^2}{2}\right]_{-1}^{3}$$

$$= \frac{25}{8} - \frac{1}{8} + \left[\frac{-4}{2} + \frac{25}{2}\right] - \frac{1}{4}\left[\frac{64}{2} - \frac{16}{2}\right]$$

$$= \frac{25-1}{8} + \frac{-4+25}{2} - \left[\frac{64-16}{8}\right]$$

$$= \frac{24}{8} + \frac{21}{2} - \frac{48}{8} = \frac{24+84-48}{8} = \frac{60}{8} = \frac{15}{2}$$

Example 9.61 Using integration, find the area of the region which is bounded by x-axis, the tangent and normal to the circle $x^2+y^2=4$ drawn at $(1,\sqrt{3})$.

Solution:

The equation of the tangent to the circle $x^2+y^2=4$ at (x_1,y_1) is $xx_1+yy_1=a^2$.

So, the equation of the tangent to the circle $x^2+y^2=4$ at $(1,\sqrt{3})$ is $x+\sqrt{3}y=4$ or $y=-\frac{1}{\sqrt{3}}(x-4)$. The tangent meets the x-axis at the point (4,0).

The equation of the normal is $y-\sqrt{3}=\sqrt{3}(x-1)$ or $y=\sqrt{3}x$ and it passes through the origin.

Area $=\int_0^1 y\,dx+\int_1^4 y\,dx$

$=\int_0^1 \sqrt{3}x\,dx+\int_1^4 -\frac{1}{\sqrt{3}}(x-4)\,dx$

$=\int_0^1 \sqrt{3}x\,dx-\frac{1}{\sqrt{3}}\int_1^4 (x-4)\,dx$

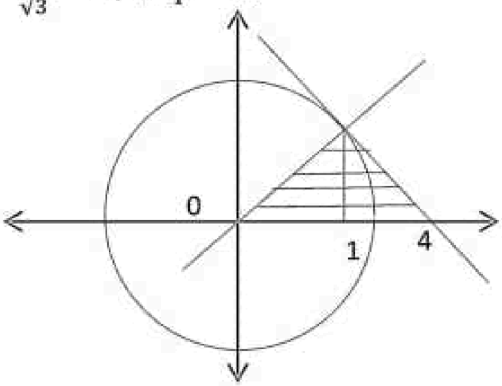
$=\sqrt{3}\left[\frac{x^2}{2}\right]_0^1-\frac{1}{\sqrt{3}}\left[\frac{(x-4)^2}{2}\right]_1^4$

$=\frac{\sqrt{3}}{2}-\frac{1}{\sqrt{3}}\left[0-\frac{9}{2}\right]$

$=\frac{\sqrt{3}}{2}+\frac{9}{2\sqrt{3}}$

$=\frac{3+9}{2\sqrt{3}}$

$=\frac{6}{\sqrt{3}}=2\sqrt{3}$ sq.units.



EXERCISE 9.8

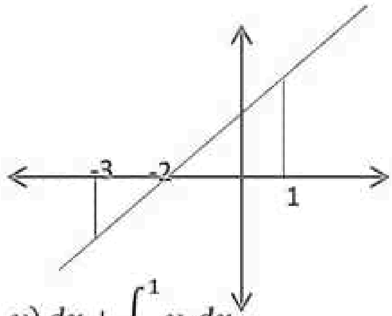
1. Find the area of the region bounded by $3x-2y+6=0, x=-3, x=1$ and x-axis.

Solution:

$3x-2y+6=0$

$2y=3x+6$

$y=\frac{3}{2}(x+2)$



$Area\quad A=\int_{-3}^{-2}(-y)\,dx+\int_{-2}^{1}y\,dx$

$=\int_{-2}^{-3}y\,dx+\int_{-2}^{1}y\,dx$

$=\frac{3}{2}\int_{-2}^{-3}(x+2)\,dx+\frac{3}{2}\int_{-2}^{1}(x+2)\,dx$

$=\frac{3}{2}\left[\frac{(x+2)^2}{2}\right]_{-2}^{-3}+\frac{3}{2}\left[\frac{(x+2)^2}{2}\right]_{-2}^{1}$

$=\frac{3}{2}\left[\frac{1}{2}-0+\frac{9}{2}-0\right]$

$=\frac{15}{2}$ sq. units

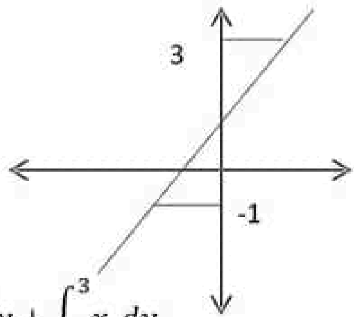
2. Find the area of the region bounded by $2x-y+1=0, y=-1, y=3$ and y-axis.

Solution:

$2x-y+1=0$

$2x=y-1$

$x=\frac{1}{2}(y-1)$



$Area\quad A=\int_{-1}^{1}(-x)\,dy+\int_{1}^{3}x\,dy$

$=\int_{1}^{-1}x\,dy+\int_{1}^{3}x\,dy$

$=\frac{1}{2}\int_{1}^{-1}(y-1)\,dy+\frac{1}{2}\int_{1}^{3}(y-1)\,dy$

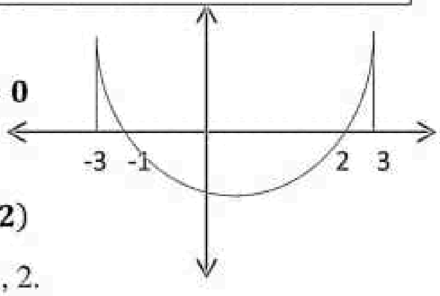
$=\frac{1}{2}\left[\frac{(y-1)^2}{2}\right]_{1}^{-1}+\frac{1}{2}\left[\frac{(y-1)^2}{2}\right]_{1}^{3}$

$=\frac{1}{2}\left[\frac{4}{2}-0+\frac{4}{2}-0\right]$

$=\frac{1}{2}(4)=2$ sq. units.

3. Find the area of the region bounded by the curve $2 + x - x^2 + y = 0$, x-axis, $x = -3$ and $x = 3$.

Solution:



$2 + x - x^2 + y = 0$

$y = x^2 - x - 2$

$y = (x+1)(x-2)$

X intercepts are -1, 2.

$$Area\ \ A = \int_{-3}^{-1} y\ dx + \int_{-1}^{2} (-y)\,dx + \int_{2}^{3} y\ dx$$

$$= \int_{-3}^{-1} y\ dx + \int_{2}^{-1} y\,dx + \int_{2}^{3} y\ dx$$

$$= \int_{-3}^{-1} (x^2 - x - 2)\ dx + \int_{2}^{-1} (x^2 - x - 2)\,dx$$

$$+ \int_{2}^{3} (x^2 - x - 2)\ dx$$

$$= \left[\frac{x^3}{3} - \frac{x^2}{2} - 2x\right]_{-3}^{-1} + \left[\frac{x^3}{3} - \frac{x^2}{2} - 2x\right]_{2}^{-1}$$

$$+ \left[\frac{x^3}{3} - \frac{x^2}{2} - 2x\right]_{2}^{3}$$

$$= \left[\frac{-1}{3} - \frac{1}{2} + 2\right] - \left[\frac{-27}{3} - \frac{9}{2} + 6\right] + \left[\frac{-1}{3} - \frac{1}{2} + 2\right]$$

$$- \left[\frac{8}{3} - \frac{4}{2} - 4\right] + \left[\frac{27}{3} - \frac{9}{2} - 6\right]$$

$$- \left[\frac{8}{3} - \frac{4}{2} - 4\right]$$

$$= \frac{-1 + 27 - 1 - 8 + 27 - 8}{3}$$

$$+ \frac{-1 + 9 - 1 + 4 - 9 + 4}{2}$$

$$+ (2 - 6 + 2 + 4 - 6 + 4)$$

$$= \frac{54 - 18}{3} + \frac{8 - 2}{2} + 0$$

$A\ = \frac{36}{3} + \frac{6}{2} = 12 + 3 = 15$ sq. units

4. Find the area of the region bounded by the line $y = 2x + 5$ and the parabola $y = x^2 - 2x$.

Solution:

$y = 2x + 5 \qquad y = x^2 - 2x$

$x^2 - 2x = 2x + 5$

$x^2 - 4x - 5 = 0$

$(x+1)(x-5) = 0$

$x = -1, 5$

X	-1	5
Y	3	15

$$Area \quad A = \int_a^b [Y_U - Y_L]\,dx$$

$$= \int_{-1}^{5} [(2x + 5) - (x^2 - 2x)]\,dx$$

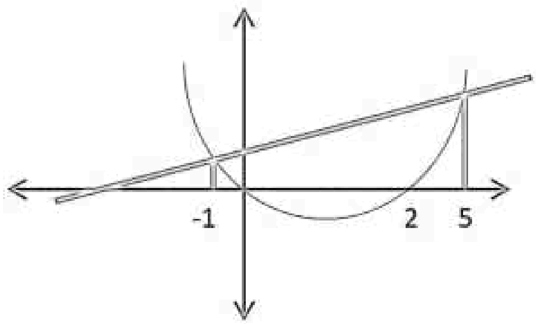
$$= \int_{-1}^{5} (5 + 4x - x^2)\,dx$$

$$= \left[5x + 4\left(\frac{x^2}{2}\right) - \frac{x^3}{3}\right]_{-1}^{5}$$

$$= \left[5x + 2x^2 - \frac{x^3}{3}\right]_{-1}^{5}$$

$$= (25 + 5) + 2(25 - 1) - \left(\frac{125 + 1}{3}\right)$$

$= 30 + 48 - 42 = 36$ sq.units.



5. Find the area of the region bounded between the curves $y = \sin x$ and $y = \cos x$ and the lines $x = 0$ and $x = \pi$.

Solution:

$y = \cos x$	$y = \sin x$	$\cos x = \sin x$ $\frac{\pi}{4} \le x \le \frac{5\pi}{4}$

$$Area\ \ A = \int_a^b [Y_U - Y_L]\ dx$$

$$= \int_0^{\frac{\pi}{4}} [\cos x - \sin x]\, dx + \int_{\frac{\pi}{4}}^{\pi} [\sin x - \cos x]\, dx$$

$$= [\sin x + \cos x]_0^{\frac{\pi}{4}} + [-\cos x - \sin x]_{\frac{\pi}{4}}^{\pi}$$

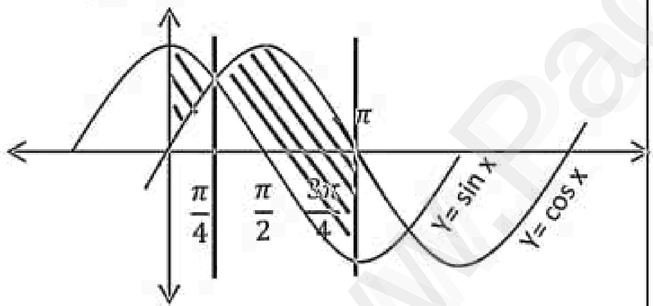
$$= [\sin x + \cos x]_0^{\frac{\pi}{4}} - [\cos x + \sin x]_{\frac{\pi}{4}}^{\pi}$$

$$= \left[\frac{1}{\sqrt{2}} + \frac{1}{\sqrt{2}} - 0 - 1\right] - \left[-1 - 0 - \frac{1}{\sqrt{2}} - \frac{1}{\sqrt{2}}\right]$$

$$= \frac{2}{\sqrt{2}} + \frac{2}{\sqrt{2}}$$

$$= [\sqrt{2} + \sqrt{2}]$$

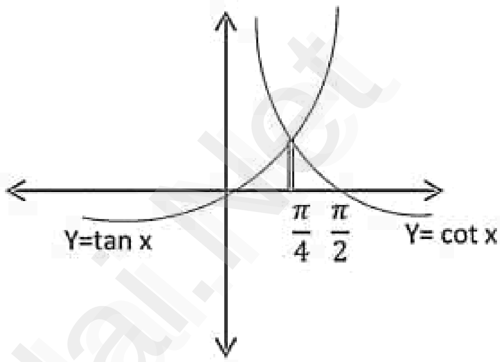
$$= 2\sqrt{2} \text{ sq. units.}$$



6. Find the area of the region bounded by $y = \tan x$, $y = \cot x$ and the lines $x = 0, x = \frac{\pi}{2}$.

Solution:

$y = \tan x$ $y = \cot x$	$\tan x = \cot x$ $x = \frac{\pi}{4} \in \left[0, \frac{\pi}{2}\right]$



$$Area = \int_0^{\frac{\pi}{4}} y\ dx + \int_{\frac{\pi}{4}}^{\frac{\pi}{2}} y\, dx$$

$$= \int_0^{\frac{\pi}{4}} \tan x\ dx + \int_{\frac{\pi}{4}}^{\frac{\pi}{2}} \cot x\, dx$$

$$= [\log \sec x]_0^{\pi/4} + [\log \sin x]_{\pi/4}^{\pi/2}$$

$$= \log\sqrt{2} - \log 1 + \log 1 - \log\frac{1}{\sqrt{2}}$$

$$= \log\sqrt{2} - \log\frac{1}{\sqrt{2}}$$

$$= \log\sqrt{2} - [\log 1 - \log\sqrt{2}]$$

$$= \log\sqrt{2} + \log\sqrt{2}$$

$$= 2\log\sqrt{2}$$

$$= \log 2 \text{ sq.units.}$$

$$(a+b+c)^2 = a^2 + b^2 + c^2 + 2(ab+bc+ca)$$

7. Find the area of the region bounded by the parabola $y^2 = x$ and the line $y = x - 2$.

Solution:

The line $y = x-2$ meets the y-axis at (0,-2).

$y^2 = x$

$y = x - 2$

$(x-2)^2 = x$

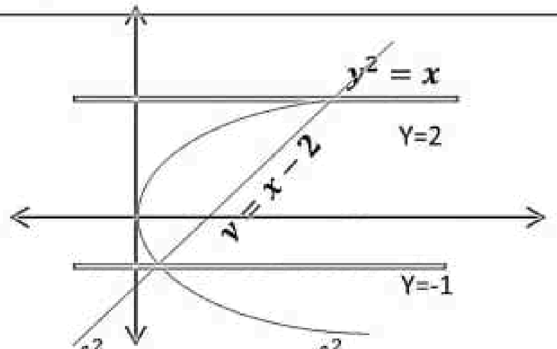
$x^2 - 4x + 4 - x = 0$

$x^2 - 5x + 4 = 0$

$x = 4 \ \ and \ \ 1$

X	1	4
$y = x - 2$	-1	2

$$x = \begin{cases} y^2 & if\ 0 \le y \le 2 \\ y+2 & if\ 2 \le y \le 3 \end{cases}$$



$$Area = \int_{-1}^{2} [X_U - X_L]\, dy = \int_{-1}^{2} [(y+2) - y^2]\, dy$$

$$= \left[\frac{(y+2)^2}{2} - \frac{y^3}{3}\right]_{-1}^{2}$$

$$= \frac{16}{2} - \frac{8}{3} - \frac{1}{2} - \frac{1}{3}$$

$$= \frac{15}{2} - \frac{9}{3} = \frac{45-18}{6}$$

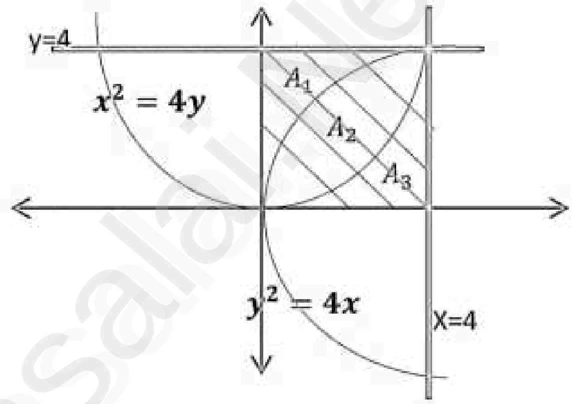
$$= \frac{27}{6}$$

$= \frac{9}{2}$ sq.units.

8. Father of a family wishes to divide his square field bounded by $x = 0$, $x = 4$, $y = 4$ and $y = 0$ along the curve $y^2 = 4x$ and $x^2 = 4y$ into three equal parts for his wife, daughter and son. Is it possible to divide? If so, find the area to be divided among them.

Solution:

The shade part is to be divided into three equal parts.



$Area \quad A = A_1 = A_2 = A_3$

$$A = \int_0^4 y\, dx = \int_0^4 (Y_U - Y_L)\, dx = \int_0^4 x\, dy$$

$$A = \int_0^4 \left(\frac{x^2}{4}\right) dx = \int_0^4 \left(\sqrt{4x} - \frac{x^2}{4}\right) dx = \int_0^4 \left(\frac{y^2}{4}\right) dy$$

$$A = \left[\frac{x^3}{12}\right]_0^4 = \int_0^4 \left(2\sqrt{x} - \frac{x^2}{4}\right) dx = \left[\frac{y^3}{12}\right]_0^4$$

$$A = \frac{16}{3} = \left[\frac{4x\sqrt{x}}{3} - \frac{x^3}{12}\right]_0^4 = \frac{16}{3}$$

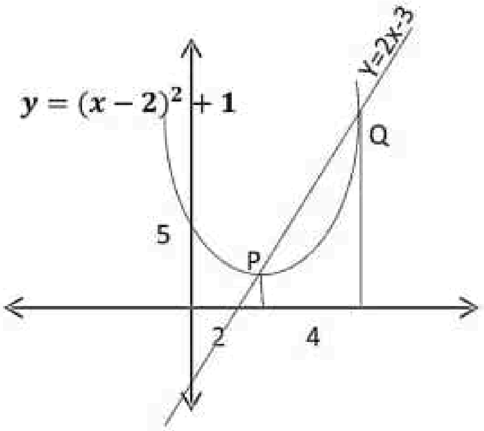
$$A = \frac{16}{3} = \frac{16}{3} = \frac{16}{3}$$

Yes, it is possible to divide among them and the area is $\frac{16}{3}$ sq.units.

9. The curve $y = (x-2)^2 + 1$ has a minimum point at P. A point Q on the curve is such that the slope of PQ is 2. Find the area bounded by the curve and the chord PQ.

Solution:

$y = (x-2)^2 + 1$ $(x-2)^2 = y - 1$ **It is open upwards.** **Vertex P is (2, 1).**	The straight line form: Slope 2 and point (2,1). $y - y_1 = m(x - x_1)$ $y - 1 = 2(x-2)$ $y = 2x - 3$	$(x-2)^2 = 2x - 4$ $x^2 - 4x + 4 = 2x - 4$ $x^2 - 6x + 8 = 0$ $(x-2)(x-4) = 0$ Point Q is 4.



$$Area \quad A = \int_a^b (Y_U - Y_L)\, dx$$

$$= \int_2^4 (2x - 3 - (x-2)^2 - 1)\, dx$$

$$= \int_2^4 (2x - (x-2)^2 - 4)\, dx$$

$$= \int_2^4 \left((2x-4) - (x-2)^2\right) dx$$

$$= \left[\frac{(2x-4)^2}{2 \times 2} - \frac{(x-2)^3}{3}\right]_2^4$$

$$= \frac{(4)^2}{4} - \frac{(2)^3}{3}$$

$$= 4 - \tfrac{8}{3} = \frac{12-8}{3} = \frac{4}{3} \text{ sq.units.}$$

10. Find the area of the region common to the circle $x^2 + y^2 = 16$ and the parabola $y^2 = 6x$.

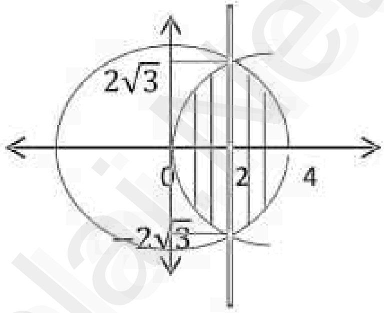
Solution:

$x^2 + y^2 = 16$

$x^2 + 6x = 16$ $\qquad$ $x^2 + 6x - 16 = 0$

$(x+8)(x-2) = 0$ $\qquad$ $x = -8 \text{ and } 2$

When x= 2 $\quad$ Y= $2\sqrt{3}$



$$Area \quad A = \int_a^b [X_R - X_L]\, dy$$

$$= \int_{-2\sqrt{3}}^{2\sqrt{3}} \left[\sqrt{16 - y^2} - \frac{y^2}{6}\right] dy$$

$$= 2\left[\frac{y}{2}\sqrt{16 - y^2} + \frac{16}{2}\sin^{-1}\left(\frac{y}{4}\right) - \frac{y^3}{18}\right]_0^{2\sqrt{3}}$$

$$= 2\left(\frac{2\sqrt{3}}{2}\sqrt{16 - 12} + 8\sin^{-1}\left(\frac{2\sqrt{3}}{4}\right) - \frac{24\sqrt{3}}{18}\right)$$

$$= 2\left(\sqrt{3}(2) + 8\sin^{-1}\left(\frac{\sqrt{3}}{2}\right) - \frac{4\sqrt{3}}{3}\right)$$

$$= 2\left(2\sqrt{3} + \frac{8\pi}{3} - \frac{4\sqrt{3}}{3}\right)$$

$$= \frac{4}{3}(3\sqrt{3} + 4\pi - 2\sqrt{3})$$

$$= \tfrac{4}{3}(4\pi + \sqrt{3}) \text{ sq.units.}$$

EXERCISE 10.5

1. If F is the constant force generated by the motor of an automobile of mass M , its velocity V is given by $M\frac{dV}{dt} = F - kV$ where k is a constant. Express V in terms of t given that $V = 0$ when $t = 0$.

Solution:

$$M\frac{dV}{dt} = F - kV$$

$$-\frac{M}{k}\frac{-k\,dV}{F-kV} = dt$$

Integrating on both sides,

$$-\frac{M}{k}\log(F-kV) = t + c \dots\dots(1)$$

$t = 0$ and $V = 0$ $\qquad -\frac{M}{k}\log F = c$

From (1) $\quad -\frac{M}{k}\log(F-kV) = t - \frac{M}{k}\log F$

$$-\frac{M}{k}\log(F-kV) + \frac{M}{k}\log F = t$$

$$\frac{M}{k}\log\left(\frac{F}{F-kV}\right) = t$$

$$\log\left(\frac{F}{F-kV}\right) = \frac{k\,t}{M}$$

$$\frac{F}{F-kV} = e^{\frac{k\,t}{M}}$$

$$F = (F-kV)e^{\frac{k\,t}{M}}$$

$$F - kV = Fe^{-\frac{k\,t}{M}}$$

$$kV = F\left(1 - e^{-\frac{k\,t}{M}}\right)$$

$$V = \frac{F}{k}\left(1 - e^{-\frac{k\,t}{M}}\right)$$

2. The velocity v , of a parachute falling vertically satisfies the equation $v\frac{dv}{dx} = g\left(1 - \frac{v^2}{k^2}\right)$ where g and k are constants. If v and x are both initially zero, find v in terms of x .

Solution:

$v\frac{dv}{dx} = g\left(1 - \frac{v^2}{k^2}\right) \qquad v\frac{dv}{dx} = \frac{g}{k^2}(k^2 - v^2)$

$\frac{v\,dv}{k^2 - v^2} = \frac{g}{k^2}dx \qquad \frac{-1}{2}\left(\frac{-2v}{k^2 - v^2}\right)dv = \frac{g}{k^2}dx$

Integrating on both sides,

$$\frac{-1}{2}\log\left(k^2 - v^2\right) = \frac{g}{k^2}x + c \dots\dots(1)$$

$v = 0$ and $x = 0 \qquad \frac{-1}{2}\log(k^2) = c$

From (1), $\quad \frac{-1}{2}\log\left(k^2 - v^2\right) = \frac{g}{k^2}x \frac{-1}{2}\log k^2$

$$\frac{1}{2}\log k^2 + \frac{-1}{2}\log\left(k^2 - v^2\right) = \frac{g}{k^2}x$$

$$\frac{1}{2}\log\left(\frac{k^2}{k^2 - v^2}\right) = \frac{g}{k^2}x$$

$$\log\left(\frac{k^2}{k^2 - v^2}\right) = \frac{2g}{k^2}x$$

$\frac{k^2}{k^2 - v^2} = e^{\frac{2g}{k^2}x} \qquad k^2 e^{-\frac{2g}{k^2}x} = k^2 - v^2$

$$v^2 = k^2 - k^2 e^{\frac{-2g}{k^2}x}$$

$$v^2 = k^2\left(1 - e^{\frac{-2g}{k^2}x}\right)$$

3. Find the equation of the curve whose slope is $\frac{y-1}{x^2+x}$ and which passes through the point (1,0).

Solution:

$$\frac{dy}{dx} = \frac{y-1}{x^2+x}$$

$$\frac{dy}{y-1} = \frac{dx}{x^2+x}$$

$$\frac{dy}{y-1} = \frac{dx}{x(1+x)}$$

$$\frac{dy}{y-1} = \frac{1+x-x}{x(1+x)}dx$$

$$\frac{dy}{y-1} = \left[\frac{1}{x} - \frac{1}{(1+x)}\right]dx$$

$$\log|y-1| = \log|x| - \log|1+x| + \log c$$

$$\log|y-1| = \log\left|\frac{cx}{1+x}\right|$$

$y - 1 = \frac{cx}{1+x} \qquad y = 1 + \frac{cx}{1+x} \dots\dots(1)$

Since this curve passes through (1, 0), we have $c = -2$

From (1), $\quad y = 1 - \frac{2x}{1+x} \qquad y = \frac{1-x}{1+x}$

The required equation is $y = \frac{1-x}{1+x}$

Example 10.18 Solve
$\left(y+\sqrt{x^2+y^2}\right)dx - xdy = 0, \quad y(1)=0$

Solution:

$$\left(y+\sqrt{x^2+y^2}\right)dx - xdy = 0$$

$$\left(y+\sqrt{x^2+y^2}\right)dx = xdy$$

$$\frac{dy}{dx}=\frac{y+\sqrt{x^2+y^2}}{x}$$

Let $y=vx$ and $v=\frac{y}{x}$ $\quad \frac{dy}{dx}=v+x\frac{dv}{dx}$

$$v+x\frac{dv}{dx}=\frac{vx+\sqrt{x^2+v^2x^2}}{x}$$

$$x\frac{dv}{dx}=\frac{vx+x\sqrt{1+v^2}}{x}-v$$

$$x\frac{dv}{dx}=v+\sqrt{1+v^2}-v$$

$$\frac{dv}{\sqrt{1+v^2}}=\frac{dx}{x}$$

Integrating on both sides,

$$\log\left|v+\sqrt{1+v^2}\right| = \log|x|+\log|c|$$

$$v+\sqrt{1+v^2}=\pm xc$$

$$\frac{y}{x}+\sqrt{1+\left(\frac{y}{x}\right)^2}=\pm xc$$

$y+\sqrt{x^2+y^2}=\pm cx^2$ (OR)

$$y+\sqrt{x^2+y^2}=Cx^2$$

When $x=1$ and $y=0$ then $C=1$

The required equation is $y+\sqrt{x^2+y^2}=x^2$

Example 10.19 Solve
$(2x+3y)dx+(y-x)dy=0$

Solution:

$$(2x+3y)dx+(y-x)dy=0$$

$$(y-x)dy=-(2x+3y)dx$$

$$\frac{dy}{dx}=-\frac{2x+3y}{y-x}$$

Put $y=vx$ $\quad v=\frac{y}{x}$ $\quad \frac{dy}{dx}=v+x\frac{dv}{dx}$

$$v+x\frac{dv}{dx}=-\frac{2x+3vx}{vx-x}=\frac{2+3v}{1-v}$$

$$x\frac{dv}{dx}=\frac{2+3v}{1-v}-v$$

$$x\frac{dv}{dx}=\frac{2+3v+v^2-v}{1-v}=\frac{(v+1)^2+1}{1-v}$$

$$\frac{(1-v)dv}{(v+1)^2+1}=\frac{dx}{x}$$

$$-\frac{1}{2}\left[\frac{2v+2}{(v+1)^2+1}-\frac{4}{(v+1)^2+1}\right]dv=\frac{dx}{x}$$

Integrating on both sides,

$$-\tfrac{1}{2}\left[\log\left|(v+1)^2+1\right|-4\tan^{-1}(v+1)\right]$$

$$=\log|x|+\log|c|$$

$$\left[\log\left|(v+1)^2+1\right|-4\tan^{-1}(v+1)\right]$$

$$=-2\log|x|-2\log|c|$$

$$\left[\log\left|(v+1)^2+1\right|+\log\left|x^2\right|-4\tan^{-1}(v+1)\right]$$

$$=-2\log|c|$$

$$\log\left[\left((v+1)^2+1\right)x^2\right]-4\tan^{-1}(v+1)=k$$

$$\log\left[y^2+2xy+2x^2\right]-4\tan^{-1}\left(\frac{x+y}{x}\right)=k$$

Example 10.21 Solve
$\left(1+2e^{x/y}\right)dx+2e^{x/y}\left(1-\frac{x}{y}\right)dy=0$

Solution:

$$\left(1+2e^{x/y}\right)dx+2e^{x/y}\left(1-\frac{x}{y}\right)dy=0$$

$$\frac{dx}{dy}=-\frac{2e^{x/y}\left(1-\frac{x}{y}\right)}{\left(1+2e^{x/y}\right)} \quad \ldots\ldots\ldots(1)$$

Put $x=vy$ $\quad v=\frac{x}{y}$ $\quad \frac{dx}{dy}=v+y\frac{dv}{dy}$

Sub (2) in (1) $\quad v+y\frac{dv}{dy}=-\frac{2e^v(1-v)}{(1+2e^v)}$

$$y\frac{dv}{dy}=-\frac{2e^v(1-v)}{(1+2e^v)}-v$$

$$y\frac{dv}{dy}=\frac{-2e^v+2ve^v-v-2ve^v}{(1+2e^v)}=-\frac{(v+2e^v)}{(1+2e^v)}$$

$$\int\frac{(1+2e^v)}{(v+2e^v)}dv=-\int\frac{dy}{y}$$

$$\log(v+2e^v)=-\log y+\log c$$

$$\log\left(\frac{x}{y}+2e^{x/y}\right)=\log\left(\frac{c}{y}\right)$$

$$\frac{x}{y}+2e^{x/y}=\pm\frac{c}{y}\ x+2ye^{x/y}=k$$

EXERCISE 10.6

2. $(x^3+y^3)dy - x^2y\,dx = 0$

Solution:

$$(x^3+y^3)dy - x^2y\,dx = 0$$
$$(x^3+y^3)dy = x^2y\,dx$$
$$\frac{dx}{dy} = \frac{x^3+y^3}{x^2y} = \frac{x}{y} + \frac{y^2}{x^2}$$

Put $x = vy \quad v = \frac{x}{y} \quad \frac{dx}{dy} = v + y\frac{dv}{dy}$

$$v + y\frac{dv}{dy} = v + \frac{1}{v^2} \qquad y\frac{dv}{dy} = \frac{1}{v^2}$$
$$v^2\,dv = \frac{dy}{y}$$

Integrating on both sides,

$$\frac{v^3}{3} = \log|y| + \log|c|$$
$$\frac{v^3}{3} = \log|yc| \qquad \frac{x^3}{3y^3} = \log|yc|$$
$$yc = e^{\frac{x^3}{3y^3}} \qquad y = k\,e^{\frac{x^3}{3y^3}}$$

4. $2xy\,dx + (x^2+2\,y^2)dy = 0$

Solution:

$$(x^2+2\,y^2)dy = -2xy\,dx$$
$$2xy\,dx = -(x^2+2\,y^2)dy$$
$$\frac{dx}{dy} = -\frac{(x^2+2\,y^2)}{2xy}$$

Put $x = vy \quad v = \frac{x}{y} \quad \frac{dx}{dy} = v + y\frac{dv}{dy}$

$$v + y\frac{dv}{dy} = -\frac{(v^2y^2+2\,y^2)}{2(vy)y} = \frac{(v^2+2\,)}{2v}$$
$$y\frac{dv}{dy} = -\frac{(v^2+2\,)}{2v} - v$$
$$y\frac{dv}{dy} = \frac{-v^2-2-2v^2}{2v} = \frac{-3v^2-2}{2v}$$
$$\frac{2v\,dv}{3v^2+2} = -\frac{dy}{y}$$
$$\frac{1}{3}\frac{6v\,dv}{3v^2+2} = \frac{-dy}{y}$$

Integrating on both sides,

$$\frac{1}{3}\log|3v^2+2| = -\log|y| + \log|c|$$
$$3v^2+2 = \left(\frac{c}{y}\right)^3$$
$$3\left(\frac{x}{y}\right)^2 + 2 = \left(\frac{c}{y}\right)^3$$
$$3x^2y + 2y^3 = c$$

5. $(y^2-2xy)dx = (x^2-2xy)dy$

Solution:

$$(y^2-2xy)dx = (x^2-2xy)dy$$
$$\frac{dx}{dy} = \frac{y^2-2xy}{x^2-2xy}$$

Put $y = vx \quad v = \frac{y}{x} \quad \frac{dy}{dx} = v + x\frac{dv}{dx}$

$$v + x\frac{dv}{dx} = \frac{v^2x^2-2x(vx)}{x^2-2x(vx)} = \frac{v^2-2v}{1-2v}$$
$$x\frac{dv}{dx} = \frac{v^2-2v}{1-2v} - v$$
$$x\frac{dv}{dx} = \frac{v^2-2v-v+2v^2}{1-2v} = \frac{3(v^2-v)}{1-2v}$$
$$\frac{(2v-1)dv}{(v^2-v)} = -3\frac{dx}{x}$$
$$\log|v^2-v| = -3\log|x| + \log|c|$$
$$\log|v^2-v| = \log\left|\frac{c}{x^3}\right|$$
$$v^2 - v = \frac{c}{x^3}$$
$$\left(\frac{y}{x}\right)^2 - \frac{y}{x} = \frac{c}{x^3} \quad \text{or } xy^2 - x^2y = c$$

7. $\left(1+3\,e^{\frac{y}{x}}\right)dy + 3\,e^{\frac{y}{x}}\left(1-\frac{y}{x}\right)dx = 0$, given that $y = 0$ when $x = 1$.

Solution:

$$\left(1+3\,e^{\frac{y}{x}}\right)dy + 3\,e^{\frac{y}{x}}\left(1-\frac{y}{x}\right)dx = 0$$
$$\frac{dy}{dx} = -\frac{3\,e^{\frac{y}{x}}\left(1-\frac{x}{y}\right)}{\left(1+3e^{\frac{y}{x}}\right)} \quad \ldots\ldots\ldots(1)$$

Put $y = vx \quad v = \frac{y}{x} \qquad \frac{dy}{dx} = v + x\frac{dv}{dx} \quad \ldots\ldots(2)$

Sub (2) in (1) $\qquad v + x\frac{dv}{dx} = -\frac{3\,e^v(1-v)}{(1+3\,e^v)}$

$$x\frac{dv}{dx} = -\frac{3\,e^v(1-v)}{(1+3\,e^v)} - v$$
$$x\frac{dv}{dx} = \frac{-3e^v + 3ve^v - v - 3ve^v}{(1+\,3\,e^v)}$$
$$\int\frac{(1+3\,e^v)}{(v+3e^v)}dv = -\int\frac{dx}{x}$$
$$\log(v+3e^v) = -\log x + \log c$$

$$\log\left(\frac{y}{x}+3e^{\frac{y}{x}}\right)=\log\left(\frac{c}{x}\right)$$

$$\frac{y}{x}+3e^{\frac{y}{x}}=\pm\frac{c}{x}$$

$$y+3\,x\,e^{\frac{y}{x}}=k\;\ldots\ldots(3)$$

When $y=0$ and $x=1$ we get $k=3$

From (3), $y+3\,x\,e^{\frac{y}{x}}=3$

8. $(x^2+y^2)dy=xy\,dx$. It is given that $y(1)=1$ and $y(x_0)=e$. Find the value of x_0.

Solution:

$$(x^2+y^2)dy=xy\,dx$$

$$\frac{dx}{dy}=\frac{x^2+y^2}{xy}=\frac{x}{y}+\frac{y}{x}$$

Put $x=vy$ & $v=\frac{x}{y}$ $\quad \frac{dx}{dy}=v+y\frac{dv}{dy}$

$$v+y\frac{dv}{dy}=v+\frac{1}{v}$$

$$v+y\frac{dv}{dy}=v+\frac{1}{v}$$

$$y\frac{dv}{dy}=\frac{1}{v}$$

$$v\,dv=\frac{dy}{y}$$

Integrating on both sides,

$$\frac{v^2}{2}=\log|y|+\log|c|$$

$$\frac{x^2}{2y^2}=\log|yc| \qquad yc=e^{\frac{x^2}{2y^2}}\ldots\ldots(1)$$

When $x=1,\ y=1$ we get $c=\sqrt{e}$

From (1), $\quad y\sqrt{e}=e^{\frac{x^2}{2y^2}}$

When $x=x_0,\ y=e$

From (2), $\quad e\sqrt{e}=e^{\frac{x_0^2}{2e^2}}$

$$e^{\frac{3}{2}}=e^{\frac{x_0^2}{2e^2}}$$

$$\frac{3}{2}=\frac{x_0^2}{2e^2} \qquad x_0^2=3e^2 \qquad x_0=\pm\sqrt{3}e$$

Example 10.23 Solve

$[y(1-x\tan x)+x^2\cos x]dx-x\,dy=0$

Solution:

$$[y(1-x\tan x)+x^2\cos x]dx-x\,dy=0$$

$$[y(1-x\tan x)+x^2\cos x]dx=x\,dy$$

$$\frac{dy}{dx}+\frac{-1+x\tan x}{x}y=x\cos x$$

Linear in y. $\quad P=\frac{-1+x\tan x}{x}$ and $Q=x\cos x$

$$y(I.F)=\int Q(I.F)dx+c$$

$$I.F=e^{\int Pdx}=e^{\int\frac{-1+x\tan x}{x}dx}$$

$$=e^{\int\left(\frac{-1}{x}+\tan x\right)dx}$$

$$=e^{-\log|x|-\log|\cos x|}$$

$$=e^{-\log(x\cos x)}$$

$$=\frac{1}{x\cos x}$$

$$y\left(\frac{1}{x\cos x}\right)=\int x\cos x\left(\frac{1}{x\cos x}\right)dx+c$$

$$y\left(\frac{1}{x\cos x}\right)=\int dx+c$$

$$y\left(\frac{1}{x\cos x}\right)=x+c$$

$$y=x^2\cos x+c\,x\cos x$$

Example 10.25 Solve:

$(1+x^3)\frac{dy}{dx}+6x^2y=1+x^2$.

Solution:

$$(1+x^3)\frac{dy}{dx}+6x^2y=1+x^2$$

$$\frac{dy}{dx}+\frac{6x^2}{1+x^3}y=\frac{1+x^2}{1+x^3}$$

Linear in y. $\quad P=\frac{6x^2}{1+x^3}$ and $Q=\frac{1+x^2}{1+x^3}$

$$\text{I.F}=e^{\int Pdx}=e^{\int 2\frac{3x^2}{1+x^3}dx}=e^{2\log|1+x^3|}$$

$$=(1+x^3)^2$$

Differential equation is

$$y(I.F)=\int Q(I.F)dx+c$$

$$y(1+x^3)^2=\int\frac{1+x^2}{1+x^3}(1+x^3)^2\,dx+c$$

$$y(1+x^3)^2=\int(1+x^2)(1+x^3)\,dx+c$$

$$y(1+x^3)^2=\int(1+x^3+x^2+x^5)\,dx+c$$

$$y(1+x^3)^2=x+\frac{x^3}{3}+\frac{x^4}{4}+\frac{x^6}{6}+c \text{ or}$$

$$y=\frac{1}{(1+x^3)^2}\left(x+\frac{x^3}{3}+\frac{x^4}{4}+\frac{x^6}{6}+C\right)$$

EXERCISE 10.7

3. $\frac{dy}{dx}+\frac{y}{x}=\sin x$

Solution:

$$\frac{dy}{dx}+\frac{y}{x}=\sin x$$

Linear in y. $P=\frac{1}{x}$ *and* $Q=\sin x$

I.F $= e^{\int Pdx} = e^{\int\frac{1}{x}dx} = e^{\log|x|} = x$

Differential equation is

$$y(I.F)=\int Q(I.F)dx+c$$

$$y\,x=\int(\sin x)\,x\,dx+c$$

$$y\,x=\int\frac{1}{\sqrt{1-x^2}}\,dx+c$$

$$xy=-x\cos x+\sin x+c$$

7. $\left(y-e^{\sin^{-1}x}\right)\frac{dx}{dy}+\sqrt{1-x^2}=0$

Solution:

$$\left(y-e^{\sin^{-1}x}\right)\frac{dx}{dy}+\sqrt{1-x^2}=0$$

$$\left(y-e^{\sin^{-1}x}\right)\frac{dx}{dy}=-\sqrt{1-x^2}$$

$$\frac{dx}{dy}=\frac{-\sqrt{1-x^2}}{y-e^{\sin^{-1}x}}$$

$$\frac{dy}{dx}=\frac{y-e^{\sin^{-1}x}}{-\sqrt{1-x^2}}=\frac{-y}{\sqrt{1-x^2}}+\frac{e^{\sin^{-1}x}}{\sqrt{1-x^2}}$$

$$\frac{dy}{dx}+\frac{1}{\sqrt{1-x^2}}y=\frac{e^{\sin^{-1}x}}{\sqrt{1-x^2}}$$

Linear in y. $P=\frac{1}{\sqrt{1-x^2}}$ *and* $Q=\frac{e^{\sin^{-1}x}}{\sqrt{1-x^2}}$

I.F $= e^{\int Pdx} = e^{\int\frac{1}{\sqrt{1-x^2}}dx} = e^{\sin^{-1}x}$

Differential equation is $y(I.F)=\int Q(I.F)dx+c$

$$y\left(e^{\sin^{-1}x}\right)=\int\left(\frac{e^{\sin^{-1}x}}{\sqrt{1-x^2}}\right)\left(e^{\sin^{-1}x}\right)dx+c$$

$$y\left(e^{\sin^{-1}x}\right)=\int\frac{\left(e^{2\sin^{-1}x}\right)}{\sqrt{1-x^2}}dx+c$$

$$y\left(e^{\sin^{-1}x}\right)=\int e^{2\sin^{-1}x}\,d\left(\sin^{-1}x\right)+c$$

$$y\left(e^{\sin^{-1}x}\right)=\frac{e^{2\sin^{-1}x}}{2}+c$$

8. $\frac{dy}{dx}+\frac{y}{(1-x)\sqrt{x}}=1-\sqrt{x}$

Solution:

$$\frac{dy}{dx}+\frac{y}{(1-x)\sqrt{x}}=1-\sqrt{x}$$

Linear in y. $P=\frac{1}{(1-x)\sqrt{x}}$ *and* $Q=1-\sqrt{x}$

I.F $= e^{\int Pdx} = e^{\int\frac{1}{(1-x)\sqrt{x}}dx}$

$$=e^{\int\frac{1}{(\sqrt{x}-x)}dx}=\frac{1+\sqrt{x}}{1-\sqrt{x}}$$

Differential equation is $y(I.F)=\int Q(I.F)dx+c$

$$y\left(\frac{1+\sqrt{x}}{1-\sqrt{x}}\right)=\int(1-\sqrt{x})\left(\frac{1+\sqrt{x}}{1-\sqrt{x}}\right)dx+c$$

$$y\left(\frac{1+\sqrt{x}}{1-\sqrt{x}}\right)=\int(1+\sqrt{x})\,dx+c$$

$$y\left(\frac{1+\sqrt{x}}{1-\sqrt{x}}\right)=x+\frac{2}{3}x\sqrt{x}+c$$

9. $(1+x+xy^2)\frac{dy}{dx}+(y+y^3)=0$

Solution:

$$(1+x+xy^2)\frac{dy}{dx}+(y+y^3)=0$$

$$(1+x+xy^2)\frac{dy}{dx}=-(y+y^3)$$

$$\frac{dy}{dx}=-\frac{(y+y^3)}{(1+x+xy^2)}$$

$$\frac{dx}{dy}=-\frac{1+x(1+y^2)}{(y+y^3)}=-\frac{1}{y+y^3}-\frac{x(1+y^2)}{y(1+y^2)}$$

$$\frac{dx}{dy}=-\frac{1}{y+y^3}-\frac{x}{y}$$

$$\frac{dx}{dy}+\frac{x}{y}=-\frac{1}{y+y^3}$$

Linear in x. $P=\frac{1}{y}$ *and* $Q=\frac{-1}{y+y^3}=\frac{-1}{y(1+y^2)}$

I.F $= e^{\int\frac{1}{y}dy} = e^{\log y} = y$

Differential equation is $x(I.F)=\int Q(I.F)\,dy+c$

$$x(y)=\int\frac{-1}{y(1+y^2)}(y)\,dy+c$$

$$xy=-\int\frac{1}{1+y^2}dy+c$$

$$xy=-\tan^{-1}y+c$$

14. $x\frac{dy}{dx}+2y-x^2\log x=0$

Solution:

$$x\frac{dy}{dx}+2y-x^2\log x=0$$

$$x\frac{dy}{dx}+2y=x^2\log x$$

$(x\div)\Rightarrow$ $$\frac{dy}{dx}+\frac{2}{x}y=\frac{x^2\log x}{x}$$

$$\frac{dy}{dx}+\frac{2}{x}y=x\log x$$

Linear in y. $P=\frac{2}{x}$ and $Q=x\log x$

I.F $= e^{\int P dx} = e^{\int\frac{2}{x}dx} = e^{2\log|x|} = x^2$

Differential equation is $y(I.F)=\int Q(I.F)dx+c$

$$x^2y=\int x\log x\,(x^2)\;dx+c$$

$$x^2y=\int x^3\log x\;dx+c$$

$$x^2y=(\log x)\frac{x^4}{4}-\int\frac{x^4}{4}\left(\frac{1}{x}\right)dx+c$$

$$x^2y=(\log x)\frac{x^4}{4}-\int\frac{x^3}{4}\,dx+c$$

$$x^2y=\frac{x^4}{4}(\log x)-\frac{x^4}{16}+c$$

Example 10.27 The growth of a population is proportional to the number present. If the population of a colony doubles in 50 years, in how many years will the population become triple?

Solution:

Let A be the population at any time t.

$$\frac{dA}{dt}\propto A$$

$$\frac{dA}{dt}=kA$$

$$A=Ce^{kt}$$

Let A_0 *be the initial population.*

When $t=0$, $A=A_0 \Rightarrow c=A_0$

When $t=50$, $A=2A_0 \Rightarrow e^{50k}=2$

$$k=\frac{1}{50}\log 2$$

Assume that the population is tripled in t_1 years.

When $t=t_1$, $A=3A_0$ $3A_0=A_0e^{t_1k}$

$e^{t_1k}=3$ $e^{\frac{1}{50}\log 2\;t_1}=3$

Taking log on both sides,

$$t_1\frac{1}{50}\log 2=\log 3$$

$$50\;t_1=\frac{\log 3}{\log 2}$$

$$t_1=50\left(\frac{\log 3}{\log 2}\right)\text{ years}$$

Example 10.28 A radioactive isotope has an initial mass 200mg, which two years later is 50mg. Find the expression for the amount of the isotope remaining at any time. What is its half-life? (half-life means the time taken for the radioactivity of a specified isotope to fall to half its original value).

Solution:

Let A be the mass of the isotope remaining after t years, and let $-k$ be the constant of proportionality, where $k > 0$.

$$\frac{dA}{dt} \propto A$$

$$\frac{dA}{dt} = -kA$$

$$A = Ce^{-kt}$$

When $t = 0$, $A = 200 \rightrightarrows c = 200$

$$A = 200\, e^{-kt}$$

When $t = 2$, $A = 150 \rightrightarrows$

$$e^{2k} = \frac{200}{150} \qquad k = \frac{1}{2}log\left(\frac{4}{3}\right)$$

Let t_1 be the half-year.

When $t = t_1$, $A = 100 \rightrightarrows 100 = 200\, e^{-t_1 k}$

$$e^{-t_1 k} = \frac{1}{2}$$

$$e^{-\frac{1}{2}log\left(\frac{4}{3}\right)\ t_1} = \frac{1}{2}$$

Taking log on both sides,

$$-\frac{1}{2}log\left(\frac{4}{3}\right)\ t_1 = log\left(\frac{1}{2}\right)$$

$$t_1 = \frac{2\,log\left(\frac{1}{2}\right)}{-log\left(\frac{4}{3}\right)} = \frac{2\,log\left(\frac{1}{2}\right)}{log\left(\frac{3}{4}\right)} \text{ years.}$$

Example 10.29

In a murder investigation, a corpse was found by a detective at exactly 8 p.m. Being alert, the detective also measured the body temperature and found it to be 70^0F. Two hours later, the detective measured the body temperature again and found it to be 60^0F. If the room temperature is 50^0F, and assuming that the body temperature of the person before death was 98.6^0F, at what time did the murder occur?

$\left[\begin{matrix}\log(2.43) = 0.88789, \\ \log(0.5) = -0.69315\end{matrix}\right]$

Solution:

$$\frac{dT}{dt} \propto (T - S)$$

$$\frac{dT}{dt} = k(T - S)$$

$$T - S = Ce^{kt}$$

When $t = 0$, $T = 70$, $s = 50 \rightrightarrows c = 20$

When $t = 2$, $T = 60$, $s = 50$

$$e^{2k} = \frac{10}{20} = 0.5 \qquad 2\,k = log(0.5)$$

$$k = \frac{1}{2}(-0.69315)$$

Let t_1 be the elapsed time after the death,

When $t = t_1$, $T = 98.6$, $\quad 20\, e^{k\,t_1} = 48.6$

$$k\ t_1 = log\left(\frac{48.6}{20}\right) = \log(2.43)$$

$$t_1 = \frac{-2 \times 0.88789}{0.69315} \approx -2.561 \text{ (4 hour 30 mins)}$$

The time of death = 10 – 4. 30 = 5.30

It appears that the person was murdered at about 5.30 p.m.

Example 10.30 A tank contains 1000 litres of water in which 100 grams of salt is dissolved. Brine (Brine is a high-concentration solution of salt (usually sodium chloride) in water) runs in a rate of 10 litres per minute, and each litre contains 5grams of dissolved salt. The mixture of the tank is kept uniform by stirring. Brine runs out at 10 litres per minute. Find the amount of salt at any time t .

Solution:

Let $x(t)$ denote the amount of salt in the tank at time t . Its rate of change is

$$\frac{dx}{dt} = \text{in flow rate} - \text{out flow rate}$$

In flow rate = $5 \times 10 = 50 grams$

Out flow rate = $(10/1000) \times x = 0.01x\ grams$

$$\frac{dx}{dt} = 50 - 0.01x = -0.01(x - 5000)$$

$$\frac{dx}{x-5000} = -0.01\,dt$$

Integrating on both sides,

$$\log|x - 5000| = -0.01\,t + \log c$$

$$x - 5000 = c\,e^{-0.01\,t}$$

$$x = 5000 + c\,e^{-0.01\,t}$$

Initially, when $t = 0$, $x = 100$,

$$100 = 5000 + c \qquad c = -4900$$

Hence, the amount of the salt in the tank at time t is

$$x = 5000 - 4900\,e^{-0.01\,t}$$

EXERCISE 10.8

1. The rate of increase in the number of bacteria in a certain bacteria culture is proportional to the number present. Given that the number triples in 5 hours, find how many bacteria will be present after 10 hours?

Solution:

Let A be the number of bacteria at any time t.

$$\frac{dA}{dt} \propto A \qquad \frac{dA}{dt} = kA \qquad A = Ce^{kt}$$

Let A_0 be the initial amount of bacteria.

When $t = 0$, $A = A_0 \rightrightarrows c = A_0$

When $t = 5$, $A = 3A_0 \rightrightarrows e^{5k} = 3$

When $t = 10$, $A = ?$ $A = A_0\,e^{10k}$

$$= A_0\left(e^{5k}\right)^2$$

$$= A_0(3)^2$$

$$A = 9A_0$$

Hence after 10 hours the number of bacteria is 9 times the original number of bacteria.

2. Find the population of a city at any time t, given that the rate of increase of population is proportional to the population at that instant and that in a period of 40 years the population increased from 3,00,000 to 4,00,000.

Solution:

Let A be the population of a city at any time t.

$$\frac{dA}{dt} \propto A \qquad \frac{dA}{dt} = kA \quad A = Ce^{kt} \ \ldots\ldots(1)$$

When $t = 0$, $A = 300000 \rightrightarrows c = 300000$

When $t = 40$, $A = 400000 \rightrightarrows$

$$e^{40\,k} = \frac{400000}{300000} = \frac{4}{3} \quad e^{k} = \left(\frac{4}{3}\right)^{1/40}$$

From (1) $$A = 300000\left(\frac{4}{3}\right)^{t/40}$$

3. The equation of electromotive force for an electric circuit containing resistance and self-inductance is $E = Ri + L\frac{di}{dt}$ where E is the electromotive force is given to the circuit, R the resistance and L, the coefficient of induction. Find the current i at time t when E = 0.

Solution:

$$E = Ri + L\frac{di}{dt}$$

$$\frac{di}{dt} + \frac{R}{L}i = \frac{E}{L}$$

Linear in *i*. $P = \frac{R}{L}$ and $Q = \frac{E}{L}$

I.F $= e^{\int P dt} = e^{\int \frac{R}{L} dt} = e^{\frac{R}{L}t}$

The solution is $i(I.F) = \int Q\,(I.F)dt + c$

$$i\left(e^{\frac{R}{L}t}\right) = \int \frac{E}{L}\left(e^{\frac{R}{L}t}\right)dt + c$$

$$i\left(e^{\frac{R}{L}t}\right) = \frac{E}{L} \times \frac{L}{R} e^{\frac{R}{L}t} + c$$

$$i\left(e^{\frac{R}{L}t}\right) = \frac{E}{R} e^{\frac{R}{L}t} + c$$

$$i = \frac{E}{R} + ce^{-\frac{R}{L}t}$$

When $E = 0$, we get $i = ce^{-\frac{R}{L}t}$

Sun

Tuition

Center

4. The engine of a motor boat moving at 10 m / s is shut off. Given that the retardation at any subsequent time (after shutting off the engine) equal to the velocity at that time. Find the velocity after 2 seconds of switching off the engine.

Solution:

Let *v* be the velocity of the motor boat at any time t.

$$-\frac{dv}{dt} = v$$

$$\frac{dv}{v} = -dt$$

Integrating on both sides, $\log|v| = -t + c$

When $t = 0$, $v = 10\ m/s \Rightarrow \log|10| = c$

$c = \log 10$

$$\log|v| = -t + \log 10$$

$$t = \log 10 - \log v = \log\left(\frac{10}{v}\right)$$

$$\frac{10}{v} = e^t$$

$$v = 10\,e^{-t}$$

When t = 2 sec, we get $v = 10\,e^{-2}\ m/s$

5. Suppose a person deposits 10,000 Indian rupees in a bank account at the rate of 5% per annum compounded continuously. How much money will be in his bank account 18 months later?

Solution:

Let A be the principal at any time t and r be the rate of interest compounded continuously.

$$\frac{dA}{dt} \propto A$$

$$\frac{dA}{dt} = kA$$

Here k = 0.05 $A = Ce^{0.05\,t}$

When $t = 0$, $A = 10000 \Rightarrow c = 10000$

When $t = 18 months\ (or)\ t = \frac{3}{2}$,

$$\Rightarrow A = 10000\,e^{0.05\left(\frac{3}{2}\right)}$$

$$A = 10000\,e^{0.075}$$

6. Assume that the rate at which radioactive nuclei decay is proportional to the number of such nuclei that are present in a given sample. In a certain sample 10% of the original number of radioactive nuclei have undergone disintegration in a period of 100 years. What percentage of the original radioactive nuclei will remain after 1000 years?

Solution:

Let A be the amount of radioactive nuclei present at any time t.

$$\frac{dA}{dt} \propto A$$

$$\frac{dA}{dt} = -kA \qquad k > 0$$

$$A = Ce^{-kt}$$

When $t = 0, \quad A = A_0 \rightrightarrows \; c = A_0$

When $t = 100, \; A = 90\% \, 0f \, A_0$

$$\frac{90}{100} A_0 = A_0 e^{-100k}$$

$$e^{-100k} = \frac{9}{10}$$

When $t = 1000, \quad A = ?$

$$A = A_0 e^{-1000k}$$

$$A = A_0 \left(e^{-10\,k}\right)^{10}$$

$$A = A_0 \left(\frac{9}{10}\right)^{10} \%$$

Hence the radioactive nuclei after 1000 years is

$$A = A_0 \left(\frac{9}{10}\right)^{10} (100)\%$$

$$A = A_0 \left(\frac{9^{10}}{10^{8}}\right) \%$$

7. Water at temperature 100^0C cools in 10 minutes to 80^0C in a room temperature of 25^0C. Find

(i) The temperature of water after 20 minutes

(ii) The time when the temperature is 40^0C

$$\left[\log_e \frac{11}{15} = -0.3101; \;\; \log_e 5 = 1.6094\right]$$

Solution:

Let T be the temperature of water at any time t.

$$\frac{dT}{dt} \propto (T - S)$$

$$\frac{dT}{dt} = k(T - S)$$

$$T - S = Ce^{kt}$$

When $t = 0, \; T = 100, \quad s = 25 \;\; \rightrightarrows \; c = 75$

When $t = 10, \; T = 80, \quad s = 25$

$e^{10\,k} = \frac{55}{75} = \frac{11}{15} \qquad k = \frac{1}{10} log \left(\frac{11}{15}\right)$

(i) When $t = 20, \;\; T = ?, \quad s = 25$

$$T - 25 = 75\,(e^{10\,t})^2$$

$$T = 25 + 75\left(\frac{11}{15}\right)^2$$

$$T = 25 + 75\frac{11 \times 11}{15 \times 15} = 25 + \frac{121}{3}$$

$$= 25 + 40.33 = 65.33^0C$$

The temperature of water after 20 minutes is 65.33^0C (approximately)

(ii) when $T = 40 \qquad 40 - 25 = 75e^{kt}$

$$75e^{kt} = 15$$

$$e^{kt} = \frac{1}{5}$$

$$kt = \log\left(\frac{1}{5}\right)$$

$$t = \frac{\log\left(\frac{1}{5}\right)}{k} = 10\frac{\log\left(\frac{1}{5}\right)}{log\left(\frac{11}{15}\right)}$$

$$t = -10\frac{1.6094}{-0.3101} \approx 51.899$$

The time when the temperature 40^0C is 51.9 minutes (approximately).

8. At 10.00 A.M. a woman took a cup of hot instant coffee from her microwave oven and placed it on a nearby Kitchen counter to cool. At this instant the temperature of the coffee was 180^0F, and 10 minutes later it was 160^0F. Assume that constant temperature of the kitchen was 70^0F.

(i) What was the temperature of the coffee at 10.15A.M.?

(ii) The woman likes to drink coffee when its temperature is between 130^0F and 140^0F. between what times should she have drunk the coffee $\left[log\frac{9}{11} = -0.2006,\ log\ \frac{6}{11} = -0.606135\right]$

Solution:

Let T be the temperature of coffee at any time t.

$\frac{dT}{dt} \propto (T-S) \qquad \frac{dT}{dt} = k(T-S) \qquad T-S = Ce^{kt}$

When 10 $A.M$ as

$t = 0,\ T = 180,\ \ s = 70\ \rightrightarrows\ c = 110$

When $t = 10,\ T = 160,\ \ s = 70$

$e^{10\,k} = \frac{90}{110} = \frac{9}{11}$

$$k = \frac{1}{10} log\left(\frac{9}{11}\right) = -0.02006$$

When $t = 15,\ T = ?,\ \ s = 70$

$$T - 70 = 110\,(e^{10\,t})^{\frac{3}{2}}$$

$$T = 70 + 110\left(\frac{9}{11}\right)^{\frac{3}{2}}$$

$$T = 70 + 110\frac{9\times 3}{11\times\sqrt{11}}$$

$$= 70 + \frac{270}{\sqrt{11}} \approx 70 + 81.4 \approx 151.4\,F$$

The temperature of the coffee at 10.15A.M is $151.4\ F$ (approximately)

(ii) When $T = 130,\ t_1 = ?$

$$130 - 70 = 110\,e^{-0.02006\,t_1}$$

$$e^{-0.02006\,t_1} = \frac{60}{110}$$

$t_1 = \frac{0.606135}{0.02006} = 30.216$ mins

When $T = 140,\ t_2 = ?$

$$140 - 70 = 110\,e^{-0.02006\,t_2}$$

$$e^{-0.02006\,t_2} = \frac{70}{110}$$

$t_2 = \frac{0.45198}{0.02006} = 22.53$ mins

Temperature 130^0F to 140^0F. comes between 10.22 AM to 10.30 AM.

9. A pot of boiling water at 100^0C is removed from a stove at time t = 0 and left to cool in the kitchen. After 5 minutes, the water temperature has decreased to 80^0C, and another 5 minutes later it has dropped to 65^0C. Determine the temperature of the kitchen.

Solution:

Let T be the temperature of water at any time t.

$\frac{dT}{dt} \propto (T-S) \rightrightarrows \frac{dT}{dt} = k(T-S)$

$\rightrightarrows T - S = Ce^{kt}$

When $t = 0,\ T = 100,\ \ \rightrightarrows\ c = 100 - S$

When $t = 5,\ T = 80,\ \ 80 - S = (100 - S)e^{5k}$

$$e^{5k} = \frac{80-S}{100-S}$$

When $t = 10,\ T = 65,$

$$65 - S = (100-S)\left(e^{5\,k}\right)^2$$

$$65 = S + \frac{(80-S)^2}{100-S}$$

$$65(100-S) = S(100-S) + (80-S)^2$$

$$S = 20^0C$$

The temperature of the kitchen is 20^0C.

10. A tank initially contains 50 litres of pure water. Starting at time $t = 0$ a brine containing with 2 grams of dissolved salt per litre flows into the tank at the rate of 3 litres per minute. The mixture is kept uniform by stirring and the well-stirred mixture simultaneously flows out of the tank at the same rate. Find the amount of salt present in the tank at any time $t > 0$.

Solution:

Let x(t) denotes the amount of salt in the tank at time t.

$\frac{dx}{dt}$ = in flow rate – out flow rate

In flow rate = $2 \times 3 = 6\ grams$

Out flow rate = $(3/50) \times x\ \ grams$

$$\frac{dx}{dt} = 6 - \frac{3}{50}x = -\frac{3}{50}\left(x - 6\times\frac{50}{3}\right)$$

$$\frac{dx}{dt} = -\frac{3}{50}(x-100)$$

$$\frac{dx}{x-100} = -\frac{3}{50}\,dt$$

Integrating on both sides,

$\log|x-100| = -\frac{3}{50}\,t + \log\ c$

Take exponential on both sides

$$x - 100 = c\,e^{-\frac{3}{50}t}$$

$$x = 100 + c\,e^{-\frac{3}{50}t}$$

Initially, when $t = 0,\ x = 0,\ \ c = -100$

Hence, the amount of the salt in the tank at time t is

$$x = 100 - 100\,e^{-\frac{3}{50}t}$$

$$x = 100\left(1 - e^{-\frac{3}{50}t}\right)$$

Example 11.8 A six sided die is marked '1' on one face, '2' on two of its faces, and '3' on remaining three faces. The die is rolled twice. If X denotes the total score in two throws.

(i) Find the probability mass function. **(ii) Find the cumulative distribution function.**

(iii) Find $P(3 \le X < 6)$ **(iv) Find $(X \ge 4)$.**

Solution:

Number of Sample space n(S) = **36**

$$S = \begin{Bmatrix} (1,1), (1,2), (1,2), (1,3), (1,3), (1,3) \\ (2,1), (2,2), (2,2), (2,3), (2,3), (2,3) \\ (2,1), (2,2), (2,2), (2,3), (2,3), (2,3) \\ (3,1), (3,2), (3,2), (3,3), (3,3), (3,3) \\ (3,1), (3,2), (3,2), (3,3), (3,3), (3,3) \\ (3,1), (3,2), (3,2), (3,3), (3,3), (3,3) \end{Bmatrix}$$

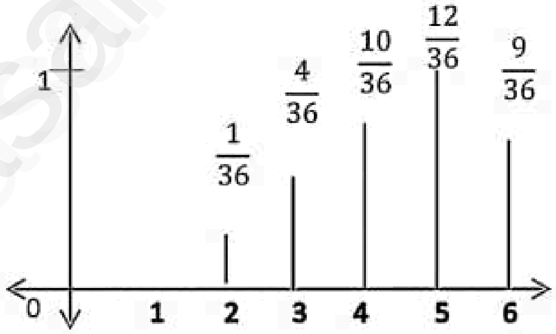
Let X is a random variables takes the values 2, 3, 4, 5 and 6.

Let X (ω) denotes the total score in two throws, this gives

Values of Random Variables	2	3	4	5	6	Total
Number of points in inverse image	1	4	10	12	9	36

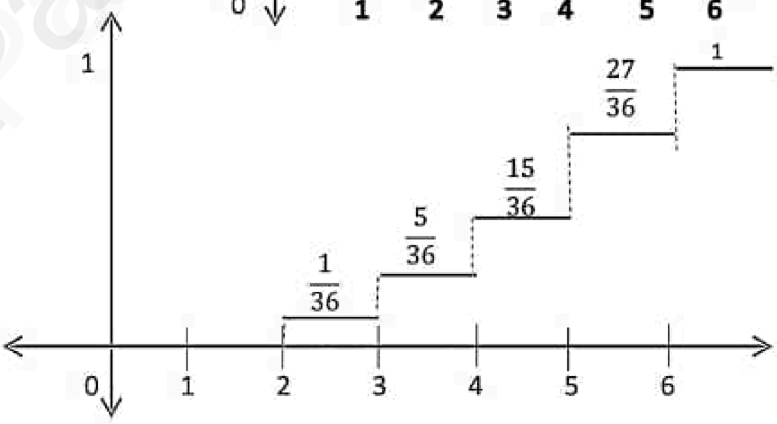
(i) The probability mass function is given by

x	2	3	4	5	6
$f(x)$	$\frac{1}{36}$	$\frac{4}{36}$	$\frac{10}{36}$	$\frac{12}{36}$	$\frac{9}{36}$



(ii) Cumulative distribution function is

$$F(x) = \begin{cases} 0; & -\infty < x < 2 \\ \frac{1}{36}; & 2 \le x < 3 \\ \frac{5}{36}; & 3 \le x < 4 \\ \frac{15}{36}; & 4 \le x < 5 \\ \frac{27}{36}; & 5 \le x < 6 \\ 1; & 6 < x < \infty \end{cases}$$



(iii) $P(3 \le X < 6) = P(X = 3) + P(X = 4) + P(X = 5) = \frac{26}{36}$

(iv) $P(X \ge 4) = P(X = 4) + P(X = 5) + P(X = 6) = \frac{31}{36}$

Example 11.10

A random variable X has the following probability mass function.

x	1	2	3	4	5	6
$f(x)$	k	2k	6k	5k	6k	10k

Find (i) $P(2 < X < 6)$ (ii) $P(2 \le X < 5)$ (iii) $P(X \le 4)$ (iv) $P(3 < X)$

Solution:

Since the given function is a probability mass function, the total probability is one. That is $\sum_x f(x) = 1$

$$k + 2k + 6k + 5k + 6k + 10k = 1 \quad \Rightarrow \quad 30k = 1 \quad \Rightarrow \quad k = \frac{1}{30}.$$

Probability mass function is

x	1	2	3	4	5	6
$f(x)$	$\frac{1}{30}$	$\frac{2}{30}$	$\frac{6}{30}$	$\frac{5}{30}$	$\frac{6}{30}$	$\frac{10}{30}$

(i) $P(2 < X < 6) = f(3) + f(4) + f(5) = \frac{6}{30} + \frac{5}{30} + \frac{6}{30} = \frac{17}{30}$

(ii) $P(2 \le X < 5) = f(2) + f(3) + f(4) = \frac{2}{30} + \frac{6}{30} + \frac{5}{30} = \frac{13}{30}$

(iii) $P(X4) = f(1) + f(2) + f(3) + f(4) = \frac{1}{30} + \frac{2}{30} + \frac{6}{30} + \frac{5}{30} = \frac{14}{30} = \frac{7}{15}$

(iv) $P(3 < X) = f(4) + f(5) + f(6) = \frac{5}{30} + \frac{6}{30} + \frac{10}{30} = \frac{21}{30} = \frac{7}{10}$

EXERCISE 11.2

2. A six sided die is marked '1' on one face, '3' on two of its faces, and '5' on remaining three faces. The die is thrown twice. If X denotes the total score in two throws, find

(i) the probability mass function **(ii) the cumulative distribution function**

(iii) $P(4 \le X < 10)$ **(iv) $P(X \ge 6)$**

Solution:

Number of Sample space n(S) = **36**

$$S = \begin{Bmatrix} (1,1),(1,3),(1,3),(1,5),(1,5),(1,5) \\ (3,1),(3,3),(3,3),(3,5),(3,5),(3,5) \\ (3,1),(3,3),(3,3),(3,5),(3,5),(3,5) \\ (5,1),(5,3),(5,3),(5,5),(5,5),(5,5) \\ (5,1),(5,3),(5,3),(5,5),(5,5),(5,5) \\ (5,1),(5,3),(5,3),(5,5),(5,5),(5,5) \end{Bmatrix}$$

Let X is a random variables takes the values 2, 3, 4, 5 and 6.

Let X (ω) denotes the total score in two throws, this gives

Values of Random Variables	2	4	6	8	10
Number of points in inverse image	1	4	10	12	9
(i) Probability mass function $f(x) = P(X = x)$	$\frac{1}{36}$	$\frac{4}{36}$	$\frac{10}{36}$	$\frac{12}{36}$	$\frac{9}{36}$
(ii) Cumulative distribution function $F(x) = P(X \le x)$	$\frac{1}{36}$	$\frac{5}{36}$	$\frac{15}{36}$	$\frac{27}{36}$	$\frac{36}{36}$

(iii) $P(4 \le X < 10) = P(X = 4) + P(X = 6) + P(X = 8) = \frac{4}{36} + \frac{10}{36} + \frac{12}{36} = \frac{26}{36} = \frac{13}{18}$

(iv) $P(X \ge 6) = P(X = 6) + P(X = 8) + P(X = 10) = \frac{10}{36} + \frac{12}{36} + \frac{9}{36} = \frac{31}{36}$

4. Suppose a discrete random variable can only take the values 0, 1, and 2. The probability mass function is defined by $f(x) = \begin{cases} \frac{x^2+1}{k} & ; \; for\ x = 0, 1, 2 \\ 0 & ; \; otherwise \end{cases}$

Find (i) the value of k (ii) cumulative distribution function (iii) $P(X \geq 1)$.

Solution:

Since the given function is a probability mass function, the total probability is one. That is $\sum_x f(x) = 1$

$\frac{1}{k} + \frac{2}{k} + \frac{5}{k} = 1 \quad \Rightarrow \quad \frac{8}{k} = 1 \quad \Rightarrow \quad k = 8.$

Values of Random Variables	0	1	2
(i) Probability mass function $f(x) = P(X = x)$	$\frac{1}{8}$	$\frac{2}{8}$	$\frac{5}{8}$
(ii) Cumulative distribution function $F(x) = P(X \leq x)$	$\frac{1}{8}$	$\frac{3}{8}$	$\frac{8}{8}$

(iii) $P(X \geq 1) = 1 - P(X < 1) = 1 - \frac{1}{8} = \frac{7}{8}$

6. A random variable X has the following probability mass function.
Find (i) the value of k (ii) $P(2 \leq X < 5)$ (iii) $P(3 < X)$

Solution:

(i) Since the given function is a probability mass function, the total probability is one. That is $\sum_x f(x) = 1$

$k^2 + 2k^2 + 3k^2 + 2k + 3k = 1 \quad \Rightarrow \quad 6k^2 + 5k = 1 \quad \Rightarrow \quad 6k^2 + 5k - 1 = 0.$

$(6k - 1)(k + 1) = 0 \qquad k = -1\ and\ k = \frac{1}{6}$ since $0 \leq f(x) \leq 1, \quad k = \frac{1}{6}$

(ii) $P(2 \leq X < 5) = P(X = 2) + P(X = 3) + P(X = 4) = 2k^2 + 3k^2 + 2k = \frac{2}{36} + \frac{3}{36} + \frac{2}{6} = \frac{17}{36}$

(iii) $P(3 < X) = P(X > 3) = P(X = 4) + P(X = 5) = 2k + 3k = 5k = \frac{5}{6}$

7. The cumulative distribution function of a discrete random variable is given by

$$F(x) = \begin{cases} 0 & ; \; -\infty < x < 0 \\ \frac{1}{2} & ; \; 0 \leq x < 1 \\ \frac{3}{5} & ; \; 1 \leq x < 2 \\ \frac{4}{5} & ; \; 2 \leq x < 3 \\ \frac{9}{10} & ; \; 3 \leq x < 4 \\ 1 & ; \; 4 \leq x < \infty \end{cases}$$

Find (i) the probability mass function (ii) $P(X < 3)$ and (iii) $P(X \geq 2)$.

Solution:

(i) The probability mass function

X	0	1	2	3	4
$F(x) = P(X \leq x)$	$\frac{1}{2} = \frac{5}{10}$	$\frac{3}{5} = \frac{6}{10}$	$\frac{4}{5} = \frac{8}{10}$	$\frac{9}{10}$	1
$f(x) = P(X = x) = F(x) - F(\bar{x})$	$\frac{5}{10}$	$\frac{1}{10}$	$\frac{2}{10}$	$\frac{1}{10}$	$\frac{1}{10}$

(ii) $P(X < 3) = P(X = 0) + P(X = 1) + P(x = 2) = \frac{5}{10} + \frac{1}{10} + \frac{2}{10} = \frac{8}{10} = \frac{4}{5}$

(iii) $(X \geq 2) = P(X = 2) + P(X = 3) + P(X = 4) = \frac{2}{10} + \frac{1}{10} + \frac{1}{10} = \frac{4}{10} = \frac{2}{5}$

Example 11.11 **Find the constant C such that the function** $f(x) = \begin{cases} Cx^2 ; & 1 < x < 4 \\ 0 ; & Othrewise \end{cases}$ **is a density function, and compute (i)** $P(1.5 < X < 3.5)$ **(ii)** $P(X \le 2)$ **(iii)** $P(3 < X)$.

Solution:

Since the given function is a probability density function, $\int_{-\infty}^{\infty} f(x)\ dx = 1.$

$\int_1^4 Cx^2\ dx = 1 \qquad C\left[\frac{x^3}{3}\right]_1^4 = 1 \qquad C\left[\frac{64}{3} - \frac{1}{3}\right] = 1 \qquad C = \frac{1}{21}$

(i) $P(1.5 < X < 3.5) \qquad \int_{1.5}^{3.5} Cx^2\,dx = C\left[\frac{x^3}{3}\right]_{1.5}^{3.5} = \frac{1}{21}\left[\frac{(3.5)^3}{3} - \frac{(1.5)^3}{3}\right] = \frac{79}{126}$

(ii) $P(X \le 2) \qquad \int_{-\infty}^{2} Cx^2\,dx = \int_1^2 Cx^2\,dx = C\left[\frac{x^3}{3}\right]_1^2 = \frac{1}{21}\left[\frac{(2)^3}{3} - \frac{(1)^3}{3}\right] = \frac{7}{63}$

(iii) $P(3 < X) = P(X > 3) \qquad \int_3^{\infty} Cx^2\,dx = \int_3^4 Cx^2\,dx = C\left[\frac{x^3}{3}\right]_3^4 = \frac{1}{21}\left[\frac{(4)^3}{3} - \frac{(3)^3}{3}\right] = \frac{37}{63}$

Example 11.12 **If X is the random variable with probability density function** $f(x)$ **given by,**

$f(x) = \begin{cases} x-1 ; & 1 \le x < 2 \\ -x+3 ; & 2 \le x < 3 \\ 0 ; & Otherwise \end{cases}$ **find (i) the distribution function** $F(x)$ **(ii)** $P(1.5 \le X \le 2.5)$.

Solution:

By the definition, $F(x) = P(X \le x) = \int_{-\infty}^{x} f(u)\ du, \quad -\infty < u < \infty.$

When $x < 1 \qquad F(x) = P(X \le x) = \int_{-\infty}^{x} 0\ du = 0$

When $1 \le x < 2 \qquad F(x) = P(X \le x) = \int_{-\infty}^{1} 0\ du + \int_1^x (u-1)\ du = \left[\frac{(u-1)^2}{2}\right]_1^x = \frac{(x-1)^2}{2}$

When $2 \le x < 3 \qquad F(x) = P(X \le x) = \int_{-\infty}^{1} 0\ du + \int_1^2 (u-1)\ du + \int_2^x (-u+3)\ du$

$$F(x) = P(X \le x) = \int_{-\infty}^{1} 0\ du + \int_1^2 (u-1)\ du - \int_2^x (u-3)\ du$$

$$= \left[\frac{(u-1)^2}{2}\right]_1^2 - \left[\frac{(u-3)^2}{2}\right]_2^x = \frac{1}{2} + 0 - \frac{(x-3)^2}{2} + \frac{1}{2} = 1 - \frac{(3-x)^2}{2}$$

When $x \ge 3 \qquad F(x) = P(X \le x) = \int_{-\infty}^{1} 0\ du + \int_1^2 (u-1)\ du + \int_2^3 (3-u)\ du + \int_3^x 0\ du$

$$= \left[\frac{(u-1)^2}{2}\right]_1^2 - \left[\frac{(u-3)^2}{2}\right]_2^3 = \frac{1}{2} - \left(-\frac{1}{2}\right) = 1$$

$$F(x) = \begin{cases} 0, & -\infty \le x < 1 \\ \frac{(x-1)^2}{2}, & 1 \le x < 2 \\ 1 - \frac{(3-x)^2}{2}, & 2 \le x < 3 \\ 1, & 3 \le x < \infty \end{cases}$$

(ii) $P(1.5 \le X \le 2.5) = F(2.5) - F(1.5) = \left(1 - \frac{(3-2.5)^2}{2}\right) - \left(\frac{(1.5-1)^2}{2}\right) = 0.75$

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Example 11.14 The probability density function of random variable X is given by

$$f(x) = \begin{cases} k & ; \ 0 \le x < 5 \\ 0 & ; \ Otherwise \end{cases}$$

Find (i) Distribution function (ii) $P(X < 3)$ (iii) $P(2 < X < 4)$ (iv) $P(3 \le X)$

Solution:

(i) Since the given function is a probability density function, $\int_{-\infty}^{\infty} f(x)\ dx = 1.$

$\int_{-\infty}^{\infty} k\ dx = 1$ $\quad k[x]_0^5 = 1$ $\quad k = \frac{1}{4}.$ $\quad f(x) = \begin{cases} \frac{1}{4} & ; \ 0 \le x < 5 \\ 0 & ; \ Otherwise \end{cases}$

By the definition, $F(x) = P(X \le x) = \int_{-\infty}^{x} f(u)\ du, \quad -\infty < u < \infty.$

When $x < 1$ $\quad F(x) = P(X \le x) = \int_{-\infty}^{x} 0\ du = 0$

When $1 \le x < 5$ $\quad F(x) = P(X \le x) = \int_{-\infty}^{1} 0\ du + \int_{1}^{x} \frac{1}{4}\ du = \frac{1}{4}[u]_1^x = \frac{1}{4}(x-1)$

When $x \ge 3$ $\quad F(x) = P(X \le x) = \int_{-\infty}^{1} 0\ du + \int_{1}^{5} \frac{1}{4}\ du + \int_{5}^{x} 0\ du = \frac{1}{4}[u]_1^5 = \frac{1}{4}[4] = 1$

$$F(x) = \begin{cases} 0, & -\infty \le x < 1 \\ \frac{1}{4}(x-1), & 1 \le x \le 5 \\ 1, & 5 < x \end{cases}$$

(ii) $P(X < 3) = P(x \le 3) = F(3) = \frac{1}{4}(3-1) = \frac{1}{2}$

(iii) $P(2 < X < 4) = F(4) - F(2) = \frac{1}{4}(3-1) = \frac{1}{2}$

(iv) $P(3 \le X) = 1 - P(X<3) = 1 - \frac{1}{2} = \frac{1}{2}$

Example 11.15 Let X be a random variable denoting the life time of an electrical equipment having probability density function $f(x) = \begin{cases} k\, e^{-2x} & for\ x > 0 \\ 0 & for\ x \le 0 \end{cases}$ Find (i) the value of k

(ii) Distribution function

(iii) $P(X < 2)$ (iv) calculate the probability that X is at least for four unit of time

(v) $P(X = 3)$.

Solution:

(i) Since the given function is a probability density function, $\int_{-\infty}^{\infty} f(x)\ dx = 1.$

$\int_{0}^{\infty} k\, e^{-2x}\ dx = 1$ $\quad k\left[\frac{e^{-2x}}{-2}\right]_0^{\infty} = 1$ $\quad k\left[\frac{e^{-\infty} - e^{0}}{-2}\right] = 1$ $\quad k = 2$

$$f(x) = \begin{cases} 2\, e^{-2x} & for\ x > 0 \\ 0 & for\ x \le 0 \end{cases}$$

(ii) By the definition, $F(x) = P(X \le x) = \int_{-\infty}^{x} f(u)\ du, \quad -\infty < u < \infty.$

When $x \le 0$ $\quad F(x) = P(X \le x) = \int_{-\infty}^{x} 0\ du = 0$

When $x < 0$ $\quad F(x) = P(X \le x) = \int_{-\infty}^{0} 0\ du + \int_{0}^{x} 2\, e^{-2x}\ du = 2\left[\frac{e^{-2x}}{-2}\right]_0^x = \frac{e^{-2x} - e^{0}}{-1} = 1 - e^{-2x}$

$$F(x) = \begin{cases} 0, & for\ x \le 0 \\ 1 - e^{-2x} & for\ x > 0 \end{cases}$$

(iii) $P(X < 2) = F(2) = 1 - e^{-4}$

(iv) Calculate the probability that X is at least for four unit of time is

$$P(X \ge 4) = 1 - F(4) = 1 - (1 - e^{-8}) = e^{-8}$$

(v) In the continuous case, $f(x)$ at $x = a$ is not the probability that X takes the value a, that is $f(x)$ at $x = a$ is not equal to $P(X = a)$. If X is continuous type, $P(X = a) = 0$ for $a \in \mathbb{R}$. ie) $P(X = 3) = 0$.

EXERCISE 11.3

2. The probability density function of X is $f(x) = \begin{cases} x & ; \ 0 < x < 1 \\ 2-x & ; \ 1 \le x < 2 \\ 0 & ; \ otherwise \end{cases}$

Find (i) $P(0.2 \le X < 0.6)$ **(ii)** $P(1.2 \le X < 1.8)$ **(iii)** $P(0.5 \le X < 1.5)$

Solution:

Given: $f(x) = \begin{cases} x & ; \ 0 < x < 1 \\ 2-x & ; \ 1 \le x < 2 \\ 0 & ; \ otherwise \end{cases}$

By the definition, $F(x) = P(X \le x) = \int_{-\infty}^{x} f(u)\ du, \quad -\infty < u < \infty.$

(i) $P(0.2 \le X < 0.6) = \int_{0.2}^{0.6} x\ dx = \left[\frac{x^2}{2}\right]_{0.2}^{0.6} = \frac{1}{2}[(0.6)^2 - (0.2)^2] = \frac{1}{2}(0.32) = 0.16$

(ii) $P(1.2 \le X < 1.8) = \int_{1.2}^{1.8} (2-x)\ dx = \left[\frac{(2-x)^2}{-2}\right]_{1.2}^{1.8} = \frac{-1}{2}[(0.2)^2 - (0.8)^2] = \frac{1}{2}(0.6) = 0.3$

(iii) $P(0.5 \le X < 1.5) = \int_{0.5}^{1.0} x\ dx + \int_{1.0}^{1.5} (2-x)\ dx = \left[\frac{x^2}{2}\right]_{0.5}^{1.0} + \left[\frac{(2-x)^2}{-2}\right]_{1.0}^{1.5}$

$= \frac{1}{2}\left((1.0)^2 - (0.5)^2 - (0.5)^2 + (1.0)^2\right) = \frac{1}{2}(2 - 0.5) = 0.75$

3. Suppose the amount of milk sold daily at a milk booth is distributed with a minimum of 200 litres and a maximum of 600 litres with probability density function

$f(x) = \begin{cases} k & ; \ for\ 200 \le x \le 600 \\ 0 & ; \ otherwise \end{cases}$ **. Find (i) the value of k (ii) the distribution function**

(iii) the probability that daily sales will fall between 300 litres and 500 litres?

Solution:

(i) Since $f(x)$ is a probability density function, $f(x) \ge 0$ and $\int_{-\infty}^{\infty} f(x)\,dx = 1$.

$\int_{200}^{600} k\,dx = 1 \qquad k\,[600 - 200] = 1 \qquad k = \frac{1}{400}$

(ii) By the definition, $F(x) = P(X \le x) = \int_{-\infty}^{x} f(u)\ du, \quad -\infty < u < \infty.$

When $x < 200$ $\qquad F(x) = P(X \le x) = \int_{-\infty}^{x} 0\ du = 0$

When $x \le 600$ $\qquad F(x) = P(X \le x) = \int_{-\infty}^{200} 0\ du + \int_{200}^{x} \frac{1}{400}\ du = \frac{1}{400}[u]_{200}^{x} = \frac{1}{400}(x - 200)$

When $x > 600$ $\qquad F(x) = P(X \le x) = \int_{-\infty}^{200} 0\ du + \int_{200}^{600} \frac{1}{400}\ du + \int_{600}^{x} 0\ du$

$$= \frac{1}{400}[u]_{200}^{600} = \frac{1}{400}[600 - 200] = 1$$

$$F(x) = \begin{cases} 0, & -\infty \le x < 200 \\ \frac{1}{400}(x - 200), & 200 \le x \le 600 \\ 1, & 600 < x \end{cases}$$

(iii) $P(300 \le X \le 500) = F(500) - F(300) = \frac{1}{400}[(500 - 200) - (300 - 200)] = \frac{200}{400} = \frac{1}{2}$

4. The probability density function of X is given by $f(x)=\begin{cases} k\,e^{-\frac{x}{3}}; & for\ x>0 \\ 0\quad ; & for\ x\le 0\end{cases}$. Find (i) the value of k (ii) the distribution function (iii) $P(X<3)$ (iv) $P(5\le X)$ (v) $P(X\le 4)$.

Solution:

(i) Since the given function is a probability density function, $\int_{-\infty}^{\infty} f(x)\ dx = 1$.

$$\int_0^{\infty} k\,e^{-\frac{x}{3}}\ dx = 1 \quad k\left[\frac{e^{-\frac{x}{3}}}{-1/3}\right]_0^{\infty} = 1 \qquad -3k[e^{-\infty}-e^0]=1 \qquad k=\frac{1}{3}$$

$$f(x)=\begin{cases}\frac{1}{3}e^{-\frac{x}{3}} & for\ x>0\\ 0 & for\ x\le 0\end{cases}$$

(ii) By the definition, $F(x)=P(X\le x)=\int_{-\infty}^{x} f(u)\ du,\quad -\infty<u<\infty.$

When $x\le 0$ $\qquad F(x)=P(X\le x)=\int_{-\infty}^{x} 0\ du = 0$

When $x<0$ $\quad F(x)=P(X\le x)=\int_{-\infty}^{0} 0\ du+\int_0^x \frac{1}{3}e^{-\frac{x}{3}}\ du=\frac{1}{3}\left[\frac{e^{-\frac{x}{3}}}{-1/3}\right]_0^x=\frac{e^{-\frac{x}{3}}-e^0}{-1}=1-e^{-\frac{x}{3}}$

$$F(x)=\begin{cases}0, & \text{for } x\le 0\\ 1-e^{-\frac{x}{3}} & \text{for } x>0\end{cases}$$

(iii) $P(X<3)=\mathrm{F}(3)=1-e^{-1}$

(iv) $P(X\ge 5)=1-F(5)=1-\left(1-e^{-\frac{5}{3}}\right)=e^{-\frac{5}{3}}$

(v) $P(X\le 4)=F(4)=1-e^{-\frac{4}{3}}$

5. If X is the random variable with probability density function $f(x)$ given by, $f(x)=\begin{cases}x+1 & ;\ -1\le x<0\\ -x+1 & ;\ 0\le x<1\\ 0 & ;\ otherwise\end{cases}$ then find (i)the distribution function F(x) (ii) $P(-0.5\le X\le 0.5)$

Solution:

(i) By the definition, $F(x)=P(X\le x)=\int_{-\infty}^{x} f(u)\ du,\quad -\infty<u<\infty.$

When $x<-1$ $\qquad F(x)=P(X\le x)=\int_{-\infty}^{x} 0\ du=0$

When $-1\le x<0$ $\qquad F(x)=P(X\le x)=\int_{-\infty}^{-1} 0\ du+\int_{-1}^{x}(x+1)\ dx=\left[\frac{(x+1)^2}{2}\right]_{-1}^{x}=\frac{(x+1)^2}{2}$

When $0\le x<1$ $\qquad F(x)=P(X\le x)=\int_{-\infty}^{-1} 0\ du+\int_{-1}^{0}(x+1)\ dx+\int_0^x(-x+1)\ dx$

$$F(x)=P(X\le x)=\int_{-\infty}^{-1} 0\ du+\int_{-1}^{0}(x+1)\ dx-\int_0^x(x-1)\ dx$$

$$=\left[\frac{(x+1)^2}{2}\right]_{-1}^{0}-\left[\frac{(x-1)^2}{2}\right]_0^x=\frac{1}{2}-\frac{(x-1)^2}{2}+\frac{1}{2}=1-\frac{(x-1)^2}{2}=\frac{1}{2}+x-\frac{x^2}{2}$$

When $x\ge 1$

$$F(x)=P(X\le x)=\int_{-\infty}^{-1} 0\ du+\int_{-1}^{0}(x+1)\ dx+\int_0^1(1-x)\ dx+\int_1^x 0\ dx$$

$$=\left[\frac{(x+1)^2}{2}\right]_{-1}^{0}-\left[\frac{(x-1)^2}{2}\right]_0^1=\frac{1}{2}+\frac{1}{2}=1$$

$$F(x)=\begin{cases}0, & -\infty\le x<-1\\ \frac{(x+1)^2}{2}, & -1\le x<0\\ 1-\frac{(x-1)^2}{2}, & 0\le x<1\\ 1, & 1\le x<\infty\end{cases}$$

(ii) $P(-0.5\le X\le 0.5)=F(0.5)-F(-0.5)=\left(\frac{(0.5+1)^2}{2}\right)-\left(\frac{(-0.5+1)^2}{2}\right)=0.75$

Example 12.2 Verify the (i) closure property, (ii) commutative property, (iii) associative property (iv) existence of identity and (v) existence of inverse for the arithmetic operation + on $\mathbb{Z}$.

Solution:

Let $G = Z = set\ of\ all\ integers$

Binary operator = +

❖ Closure axiom: $\forall\, m,n \in G \Rightarrow m+n \in G$

closure axiom is ture.

❖ ***Commutative axiom***:

$\forall\, m,n \in G$ $m+n = n+m$

commutative axiom is true.

❖ Associative axiom: $\forall\, m,n \in G$

$$m + (n+p) = (m+n) + p$$

Associative axiom is ture.

❖ Existence of Identity axiom:

Let $e = 0 \in G$ be the identity element.

$m + 0 = 0 + m = m$

$\boldsymbol{m} \in G$ Identity axiom is true.

❖ Existence of Inverse axiom:

$-m \in G$ *be the inverse element.*

$$m + (-m) = (-m) + m = 0$$

Inverse axiom is true.

Example 12.3 Verify the (i) closure property, (ii) commutative property, (iii) associative property (iv) existence of identity and (v) existence of inverse for the arithmetic operation − on $\mathbb{Z}$.

Solution:

Let $G = Z = set\ of\ all\ integers$

Binary operator '−'

❖ Closure axiom: $\forall\, m,n \in G \Rightarrow m-n \in G$

closure axiom is ture.

❖ ***Commutative axiom***: $\forall\, m,n \in G$

$m - n \neq n - m$ commutative axiom is not true.

❖ Associative axiom:

$\forall\, m,n \in G$ $m - (n-p) \neq (m-n) - p$

Associative axiom is not ture.

❖ Existence of Identity axiom:

Let $e = \mathbf{0} \in G$ be the identity element.

$m - 0 = m,\ \ 0 - m = -m$ $\boldsymbol{m} \in G$

$\boldsymbol{m - 0 \neq 0 - m}$ Identity axiom is not true.

❖ Existence of Inverse axiom:

$-m \in G$ *be the inverse element.*

$m - (-m) = (-m) - m \neq 0$ Inverse axiom is not true.

Hence proved.

Example 12.4 Verify the (i) closure property, (ii) commutative property, (iii) associative property (iv) existence of identity and (v) existence of inverse for the arithmetic operation + on $\mathbb{Z}_e$ = the set of all even integers.

Solution:

Let $G = \mathbb{Z}_e$ = the set of all even integers.

Binary operator = +

❖ Closure axiom: $\forall\, m,n \in G$

let $m = 2a\ and\ n = 2b$

$\Rightarrow m + n = 2(a+b) \in G$

closure axiom is ture.

❖ ***Commutative axiom***:

$\forall\, m,n \in G$ let $m = 2a\ and\ n = 2b$

$m + n = 2(a+b) = 2(b+a) = n + m$

commutative axiom is true.

❖ Associative axiom: $\forall\, m,n \in G$.

Let $m = 2a\ and\ n = 2b$ and $p = 2c$

$$m + (n+p) = 2(a + (b+c))$$

$$(m+n) + p = 2((a+b) + c)$$

Associative axiom is ture.

❖ Existence of Identity axiom: let $\boldsymbol{m = 2a}$

Let $e \in G$ be the identity element.

$m + e = m$ $2a + e = 2a$ $e = 0$ $\mathbf{0} \in G$

Identity axiom is true.

❖ Existence of Inverse axiom:

$m' \in G$ *be the inverse element.*

Let $m = 2a$ $2a + m' = 0$

$m' = -2a$ $-2a \in G$

Inverse axiom is ture.

Example 12.7 Verify (i) closure property, (ii) commutative property, (iii) associative property, (iv) existence of identity, and (v) existence of inverse for following operation on the given set.

$\boldsymbol{m * n = m + n - mn;}$ $\boldsymbol{m,n} \in \mathbb{Z}$

Solution:

❖ Closure axiom: $\forall\, \boldsymbol{m,n} \in \mathbb{Z}$

$\boldsymbol{m * n = m + n - mn;}$ $\boldsymbol{m * n} \in \mathbb{Z}$

closure axiom is ture.

❖ ***Commutative axiom***: $\forall\, \boldsymbol{m,n} \in \mathbb{Z}$

$\boldsymbol{m * n = m + n - mn = n + m - nm}$

$\boldsymbol{m * n = n * m}$ Commutative axiom is true.

❖ Associative axiom: $\forall\, \boldsymbol{m,n,p} \in \mathbb{Z}$.

$\boldsymbol{m * (n * p) = m * (n + p - np)}$

$\boldsymbol{= m + (n + p - np) - m(n + p - np)}$

$= m + n + p - np - mn - mp + mnp$ ... (1)

$\boldsymbol{(m * n) * p = (m + n - mn) * p}$

$\boldsymbol{= (m + n - mn) + p - (m + n - mn)p}$

$= m + n + p - mn - mp - np + mnp$ ... (2)

From (1) and (2)

Associative axiom is ture.

- Existence of Identity axiom: $\forall \boldsymbol{m}, \boldsymbol{e} \in \mathbb{Z}$

Let e be the identity element.

$m * e = mm + e - me = m$

$e = \frac{0}{(1-m)} = 0 \in \mathbb{Z}$

Identity axiom is true.

- Existence of Inverse axiom: $\forall \boldsymbol{m}, \boldsymbol{m'} \in \mathbb{Z}$

m' be the inverses elements.

$m * m' = e \qquad m + m' - mm' = 0$

$m' = \frac{m}{m-1} \notin \mathbb{Z}$

Inverse axiom is not true.

Example 12.9

Verify (i) closure property, (ii) commutative property, (iii) associative property, (iv) existence of identity, and (v) existence of inverse for the operation $+5$ on $\mathbb{Z}_5$ using table corresponding to addition modulo 5.

Solution:

Let $G = Z_5 = \{[0], [1], [2], [3], [4]\}$

We take reminders {0,1,2,3,4} to represent the classes {[0],[1],[2],[3],[4]}.

+5	0	1	2	3	4
0	0	1	2	3	4
1	1	2	3	4	0
2	2	3	4	0	1
3	3	4	0	1	2
4	4	0	1	2	3

From the table,

- CLOSURE AXIOM is true. Since each box has a unique element of G.
- COMMUTATIVE AXIOM is true. Since the entries are symmetrical about the main diagonal.
- ASSOCIATIVE AXIOM is true.
- Identity element is $0 \in Z_5$. The entries of both the row and column headed by the element 1 are identical. IDENTITY AXIOM is true.

Inverse of 0 is 0

Inverse of 1 is 4

Inverse of 2 is 3

Inverse of 3 is 2

Inverse of 4 is 1

- INVERSE AXIOM is true.

Example 12.10 Verify (i) closure property, (ii) commutative property, (iii) associative property, (iv) existence of identity, and (v) existence of inverse for the operation $\times \mathbf{11}$ on a subset A = {1,3,4,5,9} of the set of remainders {0,1,2,3,4,5,6,7,8,9,10}.

Solution:

Let $G = A = \{1,3,4,5,9\}$.

·11	1	3	4	5	9
1	1	3	4	5	9
3	3	9	1	4	5
4	4	1	5	9	3
5	5	4	9	3	1
9	9	5	3	1	4

From the table,

- CLOSURE AXIOM is true. (Since each box has an unique element of G)
- COMMUTATIVE AXIOM is true. (Since The entries are symmetrical about the main diagonal.)
- ASSOCIATIVE AXIOM is true.
- Identity element is $1 \epsilon\ G$. (The entries of both the row and column headed by the element 1 are identical.) IDENTITY AXIOM is true.

Inverse of 1 is 1

Inverse of 3 is 4

Inverse of 4 is 3

Inverse of 5 is 9

Inverse of 9 is 5

- INVERSE AXIOM is true.

EXERCISE 12.1

5. Define an operation $*$ on $\mathbb{Q}$ as follows:

$a * b = \left(\frac{a+b}{2}\right)$; $a, b \in \mathbb{Q}$. Examine the closure, commutative, and associative, the existence of identity and the existence of inverse for the operation $*$ on $\mathbb{Q}$.

Solution:

Let $G = \mathbb{Q} = \{All\ rational\ numbers\}$

Binary operater: $* \rightarrow a * b = \left(\frac{a+b}{2}\right) \in G$

- Closure axiom: $\forall\, a, b \in G$

$$\Rightarrow a * b = \left(\frac{a+b}{2}\right) \in G$$

closure axiom is ture.

- Commutative axiom: $\forall\, a, b \in G$

$$\Rightarrow a * b = \left(\frac{a+b}{2}\right) = \left(\frac{b+a}{2}\right) = b * a$$

commutative axiom is ture.

- Associative axiom: $\forall\, a, b, c \in G$

$a * (b * c)$	$(a * b) * c$
$= a * \left(\frac{b+c}{2}\right)$	$= \left(\frac{a+b}{2}\right) * c$
$= \frac{a + \left(\frac{b+c}{2}\right)}{2}$	$= \frac{\left(\frac{a+b}{2}\right) + c}{2}$
$a * (b * c) = \frac{2a+b+c}{4}$ --------(1)	$(a * b) * c = \frac{a+b+2c}{4}$ --------(2)

From (1) and (2) $\quad a * (b * c) \neq (a * b) * c$

Associative axiom is ture.

- Identity axiom: $\quad a * e = e * a = a$

$$\Rightarrow \quad a * e = \frac{a+e}{2} = a \quad \Rightarrow e = a \in G$$

The identity element is unique. Therefore Identity axiom is not true.

- Inverse axiom:

If identity axiom is not satisfied we can't discuss inverse axiom. Inverse axiom is not true.

9. (i) Let $M = \left\{\begin{pmatrix} x & x \\ x & x \end{pmatrix} : x \in \mathbb{R} - \{0\}\right\}$ and let $*$ be the matrix multiplication. Determine whether M is closed under $*$. If so, examine the commutative and associative , existence of identity, existence of inverse properties for the operation $*$ on M .

Solution:

Let $G = M = \left\{\begin{pmatrix} x & x \\ x & x \end{pmatrix} / x \in R - \{0\}\right\}$

- Closure axiom:

$\forall\, A = \begin{pmatrix} x & x \\ x & x \end{pmatrix}, B = \begin{pmatrix} y & y \\ y & y \end{pmatrix} \in G$

$$AB = \begin{pmatrix} x & x \\ x & x \end{pmatrix}\begin{pmatrix} y & y \\ y & y \end{pmatrix} = \begin{pmatrix} 2xy & 2xy \\ 2xy & 2xy \end{pmatrix} \in G$$

Closure axiom is true.

- Commutative axiom:

$$B * A = \begin{pmatrix} y & y \\ y & y \end{pmatrix}\begin{pmatrix} x & x \\ x & x \end{pmatrix} = A * B$$

Commutative axiom is true.

- Associative axiom: $\quad A(BC) = (AB)C$

Matrix multiplication is always associative.

Associative axiom is true.

- Identity axiom:

Let E= $\begin{pmatrix} e & e \\ e & e \end{pmatrix} \in G$ *be the identity elements.*

$$AE = A \Rightarrow \begin{pmatrix} x & x \\ x & x \end{pmatrix}\begin{pmatrix} e & e \\ e & e \end{pmatrix} = \begin{pmatrix} x & x \\ x & x \end{pmatrix}$$

$$\begin{pmatrix} 2xe & 2xe \\ 2xe & 2xe \end{pmatrix} = \begin{pmatrix} x & x \\ x & x \end{pmatrix} \qquad 2xe = x$$

$$e = \frac{1}{2} \in R - \{0\} \qquad E = \begin{pmatrix} 1/2 & 1/2 \\ 1/2 & 1/2 \end{pmatrix} \in G$$

Identity axiom is true.

- Inverse axiom:

Let $A^{-1} = \begin{pmatrix} z & z \\ z & z \end{pmatrix} \in G$ be the inverrse elements.

$$AA^{-1} = E \Rightarrow \begin{pmatrix} x & x \\ x & x \end{pmatrix}\begin{pmatrix} z & z \\ z & z \end{pmatrix} = \begin{pmatrix} 1/2 & 1/2 \\ 1/2 & 1/2 \end{pmatrix}$$

$$\begin{pmatrix} 2xz & 2xz \\ 2xz & 2xz \end{pmatrix} = \begin{pmatrix} 1/2 & 1/2 \\ 1/2 & 1/2 \end{pmatrix} \; 2xz = \frac{1}{2}$$

$$z = \frac{1}{4x} \in R - \{0\} \qquad A^{-1} = \begin{pmatrix} 1/4x & 1/4x \\ 1/4x & 1/4x \end{pmatrix} \in G$$

Inverse axiom is true.

10. (i) Let A be $\mathbb{Q}\backslash\{1\}$. Define $*$ on A by $x * y = x + y - xy$. Is $*$ binary on A ? If so, examine the commutative and associative , the existence of identity, existence of inverse properties for the operation $*$ on A .

Solution:

Let $G = A = \mathbb{Q}\backslash\{1\} = \{All\ rational\ numbers expect\ \ 1\}$

Binary operater: $* \ \rightarrow a * b = a + b - ab$

❖ Closure axiom: $\forall\, a, b\ \in G$

$a \neq 1,\ \ b \neq 1 \Longrightarrow\ \ a - 1 \neq 0,\ \ b - 1 \neq 0$

$(a-1)(b-1) \neq 0\ \ a + b - ab \neq 1$

$\Longrightarrow a + b - ab \in G \quad \Rightarrow a * b \in G$

closure axiom is ture.

❖ ***Commutative axiom***: $\forall\, a, b\ \in G$

$a * b = a + b - ab = b + a - ba$

$a * b = b * a$ commutative axiom is true.

❖ Associative axiom: $\forall\, a, b, c\ \in G$

$a * (b * c) = a + b + c - ab - bc - ac + abc$ -------(1)

$(a * b) * c = a + b + c - ab - bc - ac + abc$ ----(2)

From (1) and (2) $\quad a * (b * c) = (a * b) * c$

Associative axiom is ture.

❖ Identity axiom:

Let $e \in G$ be the identity element.

$a * e = e * a = a \Longrightarrow a * e = a + e - ae = a$

$\Longrightarrow \quad \boldsymbol{e(1-a) = 0 \Longrightarrow} \quad \boldsymbol{e = 0} \in G$

Identity axiom is true.

❖ Inverse axiom: *Let* $a^{-1} \in$ *G be the inverse element.*

$a * a^{-1} = a^{-1} * a = e = 0$

$a * a^{-1} = 0 \Longrightarrow \quad a + a^{-1} - a\,a^{-1} = 0$

$\Rightarrow \boldsymbol{a^{-1}(1-a) = -a} \qquad \Rightarrow \boldsymbol{a^{-1}} = \frac{-a}{1-a} \in G$

❖ Inverse axiom is true.

Example 12.19 Using the equivalence property, show that $\boldsymbol{p \leftrightarrow q \equiv (p \wedge q) \vee (\neg p \wedge \neg q)}$.

Solution:

Method 1:

p	q	$\neg p$	$\neg q$	$p \wedge q$	$\neg p \wedge \neg q$	$p \leftrightarrow q$	$(p \wedge q) \vee (\neg p \wedge \neg q)$
T	T	F	F	T	F	T	T
T	F	F	T	F	F	F	F
F	T	T	F	F	F	F	F
F	F	T	T	F	T	T	T
						.....(1)	........(2)

Method 2:

$p \leftrightarrow q \equiv (\neg p \vee q) \wedge\ (\neg q \vee p)$ ... (1)

$\equiv (\neg p \vee q) \wedge (p \vee \neg q)$ (by Commutative Law) ... (2)

$\equiv (\neg p \wedge (p \vee \neg q)) \vee (q \wedge (p \vee \neg q))$ (by Distributive Law)

$\equiv (\neg p \wedge p) \vee (\neg p \wedge \neg q) \vee (q \wedge p) \vee (q \wedge \neg q)$

(by Distributive Law)

$\equiv \mathbb{F} \vee (\neg p \wedge \neg q) \vee (q \wedge p) \vee \mathbb{F}$;

(by Complement Law)

$\equiv (\neg p \wedge \neg q) \vee (q \wedge p)$; (by Identity Law)

$\equiv (p \wedge q) \vee (\neg p \wedge \neg q)$; (by Commutative Law)

Finally (1) becomes, $p \leftrightarrow q \equiv (p \wedge q) \vee (\neg p \wedge \neg q)$

EXERCISE 12.2

6. (iv) $\boldsymbol{(\neg p \rightarrow r) \wedge (p \leftrightarrow q)}$

(iv) $(\neg p \rightarrow r) \wedge (p \leftrightarrow q)$

p	q	r	$\neg p$	$\neg p \rightarrow r$	$p \leftrightarrow q$	$(\neg p \rightarrow r) \wedge (p \leftrightarrow q)$
T	T	T	F	T	T	T
T	T	F	F	T	T	T
T	F	T	F	T	F	F
T	F	F	F	T	F	F
F	T	T	T	T	F	F
F	T	F	T	F	F	F
F	F	T	T	T	T	T
F	F	F	T	F	T	F

13. Using truth table check whether the statements $\neg(p \vee q) \vee (\neg p \wedge q)$ and $\neg p$ are logically equivalent.

Solution:

p	q	$p \vee q$	$\neg(p \vee q)$	$\neg p$	$\neg p \wedge q$	$\neg(p \vee q) \vee (\neg p \wedge q)$
T	T	T	F	F	F	F
T	F	T	F	F	F	F
F	T	T	F	T	T	T
F	F	F	T	T	F	T
				.(1)		..(2)

From (1) and (2), $\neg(p \vee q) \vee (\neg p \wedge q) \equiv \neg p$

15. Prove that $p \to (\neg q \vee r) \equiv \neg p \vee (\neg q \vee r)$ using truth table.

Solution:

p	q	r	$\neg p$	$\neg q$	$\neg q \vee r$	$\neg p \vee (\neg q \vee r)$	$p \to (\neg q \vee r)$
T	T	T	F	F	T	T	T
T	T	F	F	F	F	F	F
T	F	T	F	T	T	T	T
T	F	F	F	T	T	T	T
F	T	T	T	F	T	T	T
F	T	F	T	F	F	T	T
F	F	T	T	T	T	T	T
F	F	F	T	T	T	T	T
						.....(1)	.....(2)

From (1) and (2), $p \to (\neg q \vee r) \equiv \neg p \vee (\neg q \vee r)$