MOUNT CARMEL MAT. H.S. SCHOOL - KALLAKURICHI.

Class : 12

Register Number

COMMON HALF YEARLY EXAMINATION - 2023 - 24

MATHEMATICS

Time Allowed : 3.00 Hours] [Max. Marks : 90

PART – A (Answer All the questions) 20 X 1=20

1. If $A = \begin{bmatrix} 2 & 0 \\ 1 & 5 \end{bmatrix}$ and $B = \begin{bmatrix} 1 & 4 \\ 2 & 0 \end{bmatrix}$ then $|adj(AB)| =$
   [1]-40 [2]-80 [3]-60 [4]-20
2. If f and g are polynomials of degrees m and n respectively, and if $h(x) = (f \circ g)(x)$, then the degree of h is
   [1]mn [2]$m+n$ [3]m^n [4]n^m
3. A balloon rises straight up at 10 m/s. An observer is 40 m away from the spot where the balloon left the ground. Find the rate of change of the balloon's angle of elevation in radian per second when the balloon is 30 metres above the ground.
   [1]$\frac{3}{25}$ radians/sec [2]$\frac{4}{25}$ radians/sec [3]$\frac{1}{5}$ radians/sec [4]$\frac{1}{3}$ radians/sec
4. The value of $\sum_{i=1}^{13}(i^n + i^{n-1})$ is
   [1]$1+i$ [2]i [3]1 [4]0
5. $\sin^{-1}(2\cos^2 x - 1) + \cos^{-1}(1 - 2\sin^2 x) =$
   [1]$\frac{\pi}{2}$ [2]$\frac{\pi}{3}$ [3]$\frac{\pi}{4}$ [4]$\frac{\pi}{6}$
6. Tangents are drawn to the hyperbola $\frac{x^2}{9} - \frac{y^2}{4} = 1$ parallel to the straight line $2x - y = 1$. One of the points of contact of tangents on the hyperbola is
   [1]$\left(\frac{9}{2\sqrt{2}}, \frac{-1}{\sqrt{2}}\right)$ [2]$\left(\frac{-9}{2\sqrt{2}}, \frac{1}{\sqrt{2}}\right)$ [3]$\left(\frac{9}{2\sqrt{2}}, \frac{1}{\sqrt{2}}\right)$ [4]$(3\sqrt{3}, -2\sqrt{2})$
7. Linear approximation for $g(x) = \cos x$ at $x = \frac{\pi}{2}$ is
   [1]$x + \frac{\pi}{2}$ [2]$-x + \frac{\pi}{2}$ [3]$x - \frac{\pi}{2}$ [4]$-x - \frac{\pi}{2}$
8. If $\vec{a}, \vec{b}, \vec{c}$ are three non-coplanar vectors such that $\vec{a} \times (\vec{b} \times \vec{c}) = \frac{\vec{b} + \vec{c}}{\sqrt{2}}$, then the angle between $\vec{a}$ and $\vec{b}$ is
   [1]$\frac{\pi}{2}$ [2]$\frac{3\pi}{4}$ [3]$\frac{\pi}{4}$ [4]π
9. If $\left|z - \frac{3}{z}\right| = 2$, then the least value of $|z|$ is
   [1]1 [2]2 [3]3 [4]5
10. The tangent to the curve $y^2 - xy + 9 = 0$ is vertical when
    [1]$y = 0$ [2]$y = \pm\sqrt{3}$ [3]$y = \frac{1}{2}$ [4]$y = \pm 3$
11. If $f(x) = \int_1^x \frac{e^{\sin u}}{u} du, x > 1$ and $\int_1^3 \frac{e^{\sin x^2}}{x} dx = \frac{1}{2}[f(a) - f(1)]$, then one of the possible value of a is
    [1]3 [2]6 [3]9 [4]5
12. The distance between two foci of the ellipse equation $9x^2+5y^2=180$ is
    [1]4 [2]6 [3]8 [4]2

13. If $(\lambda, 0, 3), (1, 3, -1)$ $(-5, -3, 7)$ are coplanar points then the value of λ is
[1] -2 [2] 3 [3] -3 [4] 2

14. The differential equation of $y=ke^{\lambda x}$ is
[1] $\frac{dy}{dx} = \lambda y$ [2] $\frac{dy}{dx} = ky$ [3] $\frac{dy}{dx} + ky = 0$ [4] $\frac{dy}{dx} = e^{\lambda x}$

15. The solution of the differential equation $\frac{dy}{dx} = 2xy$ is
[1]$y = Ce^{x^2}$ [2]$y = 2x^2 + C$ [3]$y^2 + 2\sin^{-1}x = C$[4]$y = x^2 + C$

16. The domain of the function $f(x) = \sin^{-1}(3x+5)$ is
[1]$\left[2, \frac{4}{3}\right]$ [2] $\left[-2, -\frac{4}{3}\right]$ [3]$[0, 2]$ [4] $\left[-2, \frac{4}{3}\right]$

17. Consider a game where the player tosses a six-sided fair die. If the face that comes up is 6 the player wins Rs.36, otherwise he loses Rs.k^2, where is the face that comes up $k = \{1, 2, 3, 4, 5\}$. The expected amount to win at this game in Rs. is
[1]$\frac{19}{6}$ [2]$-\frac{19}{6}$ [3]$\frac{3}{2}$ [4]$-\frac{3}{2}$

18. In the set $\mathbb{R}$ of real numbers '$*$' is defined as follows. Which one of the following is not a binary operation on $\mathbb{R}$?
[1] $a * b = \min(a.b)$ [2] $a * b = \max(a, b)$[3] $a * b = a$ [4] $a * b = a^b$

19. The value of $\int_{-1}^{2}|x|dx$ is
[1]$\frac{1}{2}$ [2]$\frac{3}{2}$ [3]$\frac{5}{2}$ [4]$\frac{7}{2}$

20. If A=$\begin{bmatrix} 3 & 1 \\ 5 & 2 \end{bmatrix}$ then the inverse matrix of A is
[1] $\begin{bmatrix} 2 & -1 \\ -5 & 3 \end{bmatrix}$ [2] $\begin{bmatrix} -2 & 5 \\ 1 & -3 \end{bmatrix}$ [3 $\begin{bmatrix} 3 & -1 \\ -5 & 2 \end{bmatrix}$ [4] $\begin{bmatrix} -3 & 5 \\ 1 & -2 \end{bmatrix}$

PART –B Answer Any Seven Question. Question No. 30 is compulsory **7 X 2 = 14**

21. Prove that $\begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}$ is orthogonal.

22. Find the Value of $\left|(1+i)\frac{(2+i)}{(3+2i)}\right|$

23. Find the domain of $\tan^{-1}(\sqrt{9-x^2})$

24. Obtain the equation of the circle for which $(-4, -2)$and $(1, 1)$ are the ends of a diameter.

25. If the vectors $\vec{a}, \vec{b}, \vec{c}$ are coplanar, then prove that the vectors $\vec{a}+\vec{b}, \vec{b}+\vec{c}, \vec{c}+\vec{a}$ are also coplanar.

26. Evaluate: $\lim_{x \to 0^+} x \log x$

27. Evaluate: $\int_0^1 [2x]\, dx$ Where $[\cdot]$ is the greatest integer function.

28. From the differential equation by eliminating the arbitrary constants A and B from $y = A\cos x + B\sin x$

29. Let $A = \begin{bmatrix} 0 & 1 \\ 1 & 1 \end{bmatrix}, B = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$ be any two Boolean matrices of the same type. Find $A \vee B$ and $A \wedge B$.

30. A sphere is made of ice having radius 5 cm. Its radius decreases from 5 cm to 4.7 cm. Find the change in the volume approximation.?

PART – C Answer Any Seven Question. Question No. 40 is compulsory 7 X 3 = 21

31. Solve the linear equation by Cramer's rule. $\frac{3}{x}+2y=12, \frac{2}{x}+3y=13$

32. If $z=x+iy$ is a complex number such that $\left|\frac{z-4i}{z+4i}\right|=1$. Show that the locus of z is real axis.

33. Show that, if p,q,r are rational, the roots of the equation
$x^2-2px+p^2-q^2+2qr-r^2=0$ are rational.

34. Find the vertices and Foci of the equation $49y^2-16x^2=784$

35. The volume of the parallelepiped whose coterminus edges are $7\hat{\imath}+\lambda\hat{\jmath}-3\hat{k}$, $\hat{\imath}+2\hat{\jmath}-\hat{k}, -3\hat{\imath}+7\hat{\jmath}+5\hat{k}$ is 90 cubic units. Find the value of λ.

36. If $v(x,y)=x^2-xy+\frac{1}{4}y^2+7$, $x,y\in\mathbb{R}$, find the differential dv.

37. Evaluate: $\int_0^{\frac{\pi}{2}}\begin{vmatrix}\cos^4 x & 7\\ \sin^5 x & 3\end{vmatrix}dx$

38. A random variable X has the following probability mass function.

x	1	2	3	4	5
$f(x)$	k^2	$2k^2$	$3k^2$	$2k$	$3k$

Find (i) the value of k (ii) $P(2\le X<5)$

39. Show that $y=ae^{-3x}+b$, where a and b are arbitary constants, is a solution of the differential equation $\frac{d^2y}{dx^2}+3\frac{dy}{dx}=0$.

40. Prove, using mean value theorem, that $|\sin\alpha-\sin\beta|\le|\alpha-\beta|, \alpha,\beta\in\mathbb{R}$

PART – D (Answer All the Questions) 7 X5=35

41. a) Find the parametric form of vector equation, and Cartesian equations of the plane containing the line) $\frac{x-1}{2}=\frac{-y}{3}=\frac{z+3}{1}$ and perpendicular to plane $\vec{r}.(\hat{\imath}-2\hat{\jmath}+3\hat{k})=2$

(OR)

b) Salt is poured from a conveyer belt at a rate of 30 cubic metre per minute forming a conical pile with a circular base whose height and diameter of base are always equal. How fast is the height of the pile increasing when the pile is 10 metre high?

42. a) By vector method, prove that $\cos(\alpha+\beta)=\cos\alpha\cos\beta-\sin\alpha\sin\beta$.

(OR)

b) Show that $p\leftrightarrow q\equiv(p\wedge q)\vee(\neg p\wedge\neg q)$

43. a) Find the constant C such that the function $f(x)=\begin{cases}Cx^2 & 1<x<4\\ 0 & \text{Otherwise}\end{cases}$ is a density function, and compute (i)$P(1.5<X<3.5)$ (ii)$(X\le 2)$ (iii)$P(3<X)$

(OR)

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b) On lighting a rocket cracker it gets projected in a parabolic path and reaches a maximum height of $4m$ when it is $6m$ away from the point of projection. Finally it reaches the ground $12m$ away from the starting point. Find the angle of projection.

44. a) Prove that among all the rectangles of the given area square has the least perimeter.

(OR)

b) Find the population of a city at time $t = 60$, given that the rate of increase of population is proportional to the population at that instant and that in a period of 30 years the population increased from 1,30,000 to 1,60,000.

($\log_e\left(\frac{16}{13}\right) = 0.2070$, $e^{0.42} = 1.52$)

45. a) Find the area of the region bounded between the curves $y = \sin x$ and $y = \cos x$ and the lines $x = 0$ and $x = \pi$.

(OR)

b) Find the Value of $\cot^{-1}[1] + \sin^{-1}\left(-\frac{\sqrt{3}}{2}\right) - \sec^{-1}(-\sqrt{2})$

46. a) If $3 + i$ is a root of a polynomial equation $x^4 - 8x^3 + 24x^2 - 32x + 20$ the find the remaining roots of the equation.

(OR)

b) The upward speed $v(t)$ of a rocket at time t is approximated by $v(t) = at^2 + bt + c$, $0 \le t \le 100$ where a, b and c are constant. It has been found that the speed at times $t = 3, t = 6$ and $t = 9$ seconds are respectively, 64, 133 and 208 miles per second respectively. Find the speed at time $t = 15$ seconds. (Use Gaussian elimination method)

47. a) Solve the Equation $z^3 + 8i = 0$ where $z \in \mathbb{C}$.

(OR)

b) Solve the Equation : $(x^2 - 3y^2)dx + 2xydy = 0$

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KEY ANSWERS

PART: A

ONE MARKS

1. (2) -80
2. (1) mn
3. (2) $\frac{4}{25}$ radians/sec
4. (1) $1+i$
5. (1) $\frac{\pi}{2}$
6. (3) $\left(\frac{9}{2\sqrt{2}}, \frac{1}{\sqrt{2}}\right)$
7. (2) $-x+\frac{\pi}{2}$
8. (2) $\frac{3\pi}{4}$
9. (1) 1
10. (4) $y=\pm 3$
11. (3) 9
12. (3) 8

Answer:-

$$9x^2+5y^2=180$$

$$\frac{9x^2}{180}+\frac{5y^2}{180}=\frac{180}{180}$$

$$\frac{x^2}{20}+\frac{y^2}{36}=1$$

$a^2=20$, $b^2=36$

$b^2>a^2$

$$e=\sqrt{1-a^2/b^2}$$

$$=\sqrt{1-\frac{20}{36}}$$

$$e=\frac{2}{3}$$

Distance b/w two foci is $2be$. $\Rightarrow 2be=2\times\cancel{6}^{2}\times\frac{2}{\cancel{3}}$

$\therefore 2be=8$.

15. (1) $y=ce^{x^2}$
16. (2) $\left[-2,-\frac{4}{3}\right]$

Answer:

$$-1\le 3x+5\le 1$$

$$-6\le 3x\le 1-5$$

$$-6\le 3x\le -4$$

$$-\frac{\cancel{6}^{2}}{\cancel{3}}\le x\le -\frac{4}{3}$$

$$-2\le x\le -\frac{4}{3}$$

$\therefore$ The domain is $\left[-2,-\frac{4}{3}\right]$

17. (2) $-\frac{19}{6}$
18. (4) $a*b=a^b$
19. (3) $\frac{5}{2}$
20. (1) $\begin{bmatrix}2 & -1\\ -5 & 3\end{bmatrix}$

Answer:

$$A^{-1}=\frac{1}{|A|}\operatorname{adj}A$$

$$=\frac{1}{6-5}\begin{bmatrix}2 & -1\\ -5 & 3\end{bmatrix}$$

$$= \frac{1}{1}\begin{bmatrix} 2 & -1 \\ -5 & 3 \end{bmatrix}$$

$$\therefore A^{-1} = \begin{bmatrix} 2 & -1 \\ -5 & 3 \end{bmatrix}$$

(13) (1) -2

Answer:

Since three vertices are coplanar, we have

$$\begin{vmatrix} \lambda & 0 & 3 \\ 1 & 3 & -1 \\ -5 & -3 & 7 \end{vmatrix} = 0$$

$$\lambda(21-3) - 0(7-5) + 3(-3+15) = 0$$

$$\lambda(18) + 3(12) = 0$$

$$18\lambda + 36 = 0$$

$$18\lambda = -36$$

$$\boxed{\lambda = -2}$$

(14) (1) $\frac{dy}{dx} = \lambda y$

Answer:

$y = ke^{\lambda x}$; k is arbitrary constant

Diff. w.r.to x

$$\frac{dy}{dx} = k\,e^{\lambda x} \cdot \lambda$$

$$dy = k\lambda e^{\lambda x} \cdot dx$$

$$\frac{dy}{dx} = \lambda(ke^{\lambda x})$$

$$\therefore \frac{dy}{dx} = \lambda y \quad (\because y = ke^{\lambda x})$$

PART - B 2 MARKS

(21)

Let $A = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}$

$$A^T = \begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix}$$

$$AA^T = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix} \begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix}$$

$$AA^T = \begin{bmatrix} \cos^2\theta + \sin^2\theta & \cos\theta\sin\theta - \sin\theta\cos\theta \\ \sin\theta\cos\theta - \cos\theta\sin\theta & \sin^2\theta + \cos^2\theta \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = I_2$$

Similarly, we get $A^TA = I_2$, Hence $AA^T = A^TA = I$ is orthogonal.

(23)

Let $f(x) = \tan^{-1}\sqrt{9-x^2}$

$\sqrt{9-x^2} \in R$ but $\sqrt{9-x^2} \geq 0$

$$\therefore 9 - x^2 \geq 0$$

$$x^2 - 9 \leq 0$$

$(x+3)(x-3)=0$

$+$ $\quad$ $-$ $\quad$ $+$

$-\infty$ $\quad$ -3 $\quad$ 3 $\quad$ ∞

Domain is $[-3, 3]$

(25)

Since the vectors $\vec{a},\vec{b},\vec{c}$ are coplanar, we have $[\vec{a},\vec{b},\vec{c}]=0$.

Using the properties of the scalar triple product we get

$[\vec{a}+\vec{b},\ \vec{b}+\vec{c},\ \vec{c}+\vec{a}]$

$=[\vec{a},\vec{b}+\vec{c},\vec{c}+\vec{a}]+[\vec{b},\vec{b}+\vec{c},\vec{c}+\vec{a}]$

$=[\vec{a},\vec{b},\vec{c}+\vec{a}]+[\vec{a},\vec{c},\vec{c}+\vec{a}]+[\vec{b},\vec{b},\vec{c}+\vec{a}]+[\vec{b},\vec{c},\vec{c}+\vec{a}]$

$=[\vec{a},\vec{b},\vec{c}]+[\vec{a},\vec{b},\vec{a}]+[\vec{a},\vec{c},\vec{c}]+[\vec{a},\vec{c},\vec{a}]+[\vec{b},\vec{b},\vec{c}]+[\vec{b},\vec{b},\vec{a}]+[\vec{b},\vec{c},\vec{c}]+[\vec{b},\vec{c},\vec{a}]$

$=[\vec{a},\vec{b},\vec{c}]+[\vec{a},\vec{b},\vec{c}]$

$=2[\vec{a},\vec{b},\vec{c}]=0$

$\therefore$ Hence vectors $\vec{a}+\vec{b}$, $\vec{b}+\vec{c}$, $\vec{c}+\vec{a}$ are coplanar.

(26)

$\lim_{x\to 0^+} x\log x$

This is an indeterminate form $(0\times\infty)$

Apply L' Hôpital Rule

$\lim_{x\to 0^+} x\log x=\lim_{x\to 0^+}\left(\dfrac{\log x}{\frac{1}{x}}\right)$ $\left(\dfrac{\infty}{\infty}\text{ form}\right)$

$=\lim_{x\to 0^+}\left(\dfrac{\frac{1}{x}}{-\frac{1}{x^2}}\right)$

$=\lim_{x\to 0^+}(-x)=0$.

(27)

$\int_0^1 [2x]\,dx$

$=\int_0^{1/2}[2x]\,dx+\int_{1/2}^1[2x]\,dx$

$=\int_0^{1/2}0\,dx+\int_{1/2}^1 1\,dx$

$=0+[x]_{1/2}^{1}$

$=[1-\frac{1}{2}]$

$=\frac{1}{2}$.

$\therefore\int_0^1[2x]\,dx=\frac{1}{2}$.

(28)

Given that:

$y=A\cos x+B\sin x$

Diff. w.r.to 'x'.

$\dfrac{dy}{dx}=-A\sin x+B\cos x$

Diff. again w.r.to x

$$\frac{d^2y}{dx^2} = -A\cos x - B\sin x$$

$$= -[A\cos x + B\sin x]$$

$$\frac{d^2y}{dx^2} = -y$$

$\frac{d^2y}{dx^2} + y = 0$ is the required differential equation.

(29) $A = \begin{bmatrix} 0 & 1 \\ 1 & 1 \end{bmatrix}$, $B = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$

$$A \vee B = \begin{bmatrix} 0 & 1 \\ 1 & 1 \end{bmatrix} \vee \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 0\vee 1 & 1\vee 1 \\ 1\vee 0 & 1\vee 1 \end{bmatrix}$$

$$A \vee B = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}$$

$(\because a\vee b = \max\{a,b\})$

$$A \wedge B = \begin{bmatrix} 0 & 1 \\ 1 & 1 \end{bmatrix} \wedge \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 0\wedge 1 & 1\wedge 1 \\ 1\wedge 0 & 1\wedge 1 \end{bmatrix}$$

$$A \wedge B = \begin{bmatrix} 0 & 1 \\ 0 & 1 \end{bmatrix}$$

$(\because a\wedge b = \min\{a,b\})$

(22) $\left| (1+i) \frac{(2+i)}{(3+2i)} \right|$

Sol

$$|1+i| \frac{|2+i|}{|3+2i|} = \sqrt{1^2+1^2} \times \frac{\sqrt{2^2+1^2}}{\sqrt{3^2+2^2}}$$

$$= \sqrt{2} \times \frac{\sqrt{5}}{\sqrt{13}}$$

$$= \frac{\sqrt{10}}{\sqrt{13}} \times \frac{\sqrt{13}}{\sqrt{13}}$$

$$= \frac{\sqrt{130}}{13}$$

$$\approx 0.877058.$$

(24) Given end points of diameter are $(-4,-2)$ $(1,1)$
$\quad x_1, y_1 \quad x_2, y_2$

Equation of the circle is

$$(x-x_1)(x-x_2) + (y-y_1)(y-y_2) = 0$$

$$(x+4)(x-1) + (y+2)(y-1) = 0$$

$$x^2 - x + 4x - 4 + y^2 - y + 2y - 2 = 0$$

$$x^2 + y^2 + 3x + y - 6 = 0.$$

PART - C 3 MARKS

(31) Let $\frac{1}{x} = z$.

$3z + 2y = 12,$

$2z + 3y = 13$

$\Delta = \begin{vmatrix} 3 & 2 \\ 2 & 3 \end{vmatrix} = 9 - 4 = 5$

$\Delta_1 = \begin{vmatrix} 12 & 2 \\ 13 & 3 \end{vmatrix} = 36 - 26$
$= 10$

$\Delta_2 = \begin{vmatrix} 3 & 12 \\ 2 & 13 \end{vmatrix} = 39 - 24 = 15.$

$z = \frac{\Delta_1}{\Delta} = \frac{10}{5} = 2$

$\therefore \frac{1}{x} = 2 \Rightarrow x = \frac{1}{2}$

$y = \frac{\Delta_2}{\Delta} = \frac{15}{5} = 3$

$\therefore x = \frac{1}{2}, \ y = 3.$

(32) Given: $z = x + iy$

consider:

$\left| \frac{z - 4i}{z + 4i} \right| = 1$

$\left| \frac{x + iy - 4i}{x + iy + 4i} \right| = 1$

$\left| \frac{x + i(y-4)}{x + i(y+4)} \right| = 1$

$\Rightarrow \frac{\sqrt{x^2 + (y-4)^2}}{\sqrt{x^2 + (y+4)^2}} = 1$

$\sqrt{x^2 + (y-4)^2} = \sqrt{x^2 + (y+4)^2}$

Squaring both sides,

$x^2 + (y-4)^2 = x^2 + (y+4)^2$

$\cancel{x^2} + \cancel{y^2} + \cancel{16} - 8y$
$= \cancel{x^2} + \cancel{y^2} + \cancel{16} + 8y$

$-8y - 8y = 0$

$-16y = 0.$

$y = 0.$

$\therefore y = 0$ is the equation of real axis

$\therefore$ Locus of z is real axis.

(33)

The roots are rational if $\Delta = b^2 - 4ac$

$= (2p)^2 - 4(p^2 - q^2 + 2qr - r^2)$

$= \cancel{4p^2} - \cancel{4p^2} + 4q^2 - 8qr + 4r^2$

$= 4(q^2 - 2qr + r^2)$

(or)

$4(q-r)^2$ which is perfect square.

Hence the roots are rational.

35

Let $\vec{a} = 7\hat{i} + \lambda\hat{j} - 3\hat{k}$, $\vec{b} = \hat{i} + 2\hat{j} - \hat{k}$ and $\vec{c} = -3\hat{i} + 7\hat{j} + 5\hat{k}$

$\therefore$ volume of the parallelepiped $= \vec{a}\cdot(\vec{b}\times\vec{c})$

Given: $\vec{a}\cdot(\vec{b}\times\vec{c}) = 90$

$$\begin{vmatrix} 7 & \lambda & -3 \\ 1 & 2 & -1 \\ -3 & 7 & 5 \end{vmatrix} = 90$$

$7(10+7) - \lambda(5-3) - 3(7+6) = 90$

$7(17) - \lambda(2) - 3(13) = 90$

$119 - 2\lambda - 39 = 90$

$119 - 39 - 90 = 2\lambda$

$-10 = 2\lambda$

$\boxed{\lambda = -5}$

36

Given:

$v(x,y) = x^2 - xy + \frac{1}{4}y^2 + 7$, $x, y \in R$

$dv = 2x\,dx - (x\,dy + y\,dx) + \frac{1}{4}(2y)\,dy + 0$

$dv = (2x - y)\,dx + (-x + \frac{1}{2}y)\,dy$

37

$$\int_0^{\pi/2} \begin{vmatrix} \cos^4 x & 7 \\ \sin^5 x & 3 \end{vmatrix} dx$$

$$I = \int_0^{\pi/2} (3\cos^4 x - 7\sin^5 x)\,dx$$

$$= 3\int_0^{\pi/2} \cos^4 x\,dx - 7\int_0^{\pi/2} \sin^5 x\,dx$$

$$= 3\left[\frac{4-1}{4}\cdot\frac{4-3}{4-2}\cdot\frac{\pi}{2}\right] - 7\left[\frac{5-1}{5}\cdot\frac{5-3}{5-2}\times 1\right]$$

$\because$ If n is even: $\int_0^{\pi/2} \cos^n x\,dx$ $\frac{n-1}{n}\cdot\frac{n-3}{n-2}\cdots\frac{1}{2}\times\frac{\pi}{2}$

If n is odd $\frac{n-1}{n}\cdot\frac{n-3}{n-2}\cdots\frac{2}{3}\times 1$

$$= 3\times\frac{3}{4}\times\frac{1}{2}\times\frac{\pi}{2} - 7\,\frac{4}{5}\times\frac{2}{3}$$

$$= \frac{9\pi}{16} - \frac{56}{15}$$

$$\therefore \int_0^{\pi/2} \begin{vmatrix} \cos^4 x & 7 \\ \sin^5 x & 3 \end{vmatrix} dx = \frac{9\pi}{16} - \frac{56}{15}$$

(38)

Given Probability Mass function is

x	1	2	3	4	5
$f(x)$	K^2	$2K^2$	$3K^2$	$2K$	$3K$

(i) Since $f(x)$ is p.m.f is $\sum_{i=1}^{5} f(x_i) = 1$

$\Rightarrow K^2 + 2K^2 + 3K^2 + 2K + 3K = 1$

$\Rightarrow 6K^2 + 5K = 1$

$\Rightarrow 6K^2 + 5K - 1 = 0$

$(K+1)(6K-1) = 0$

$K = -1, \; K = \frac{1}{6}$

$K = -1$ is not possible.

$\therefore \boxed{K = \frac{1}{6}}$

(ii) $P(2 \leq x < 5)$

$= P(x=2) + P(x=3) + P(x=4)$

$= 2K^2 + 3K^2 + 2K$

$= 5K^2 + 2K$

$= 5\left(\frac{1}{6}\right)^2 + 2\left(\frac{1}{6}\right)$

$= \frac{5}{36} + \frac{2}{6}$

$= \frac{5+12}{36} = \frac{17}{36}$

$\therefore P(2 \leq x < 5) = \frac{17}{36}$.

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(39) Given:

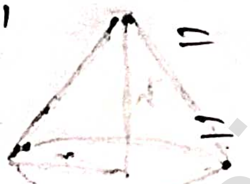
$y = ae^{-3x} + b$

where, a and b are arbitrary constants.

Diff. w.r.to 'x'

$\frac{dy}{dx} = -3ae^{-3x} + 0$

$\frac{d^2y}{dx^2} = (-3)(-3)a\,e^{-3x}$



$= 9ae^{-3x}$

$= -3\left(\frac{dy}{dx}\right)$

$\frac{d^2y}{dx^2} + 3\frac{dy}{dx} = 0$

Hence, $y = ae^{-3x} + b$ is a solution of the differential equation $\frac{d^2y}{dx^2} + 3\frac{dy}{dx} = 0$.

(40) Let $f(x) = \sin x$

Consider an interval $[\alpha, \beta]$.

Applying the mean value theorem there exists $c \in (\alpha, \beta)$ such that

$\frac{\sin\beta - \sin\alpha}{\beta - \alpha} = f'(c) = \cos c$

$\therefore \left|\frac{\sin\alpha - \sin\beta}{\alpha - \beta}\right| = |\cos c| \leq 1$

Hence $|\sin\alpha - \sin\beta| \leq |\alpha - \beta|$

PART-D

5 MARKS

(41)

(b) Let h and r be the height and base radius.

Therefore $h = 2r$.

Let 'v' be the volume of the salt cone.

$V = \frac{1}{3}\pi r^2 h$

$V = \frac{1}{12}\pi h^3$; $(\because h = 2r)$

given:

$\frac{dv}{dt} = 30\ m^3/min$

$V = \frac{1}{12}\pi r h^3$

Diff. w.r.to 't'

$\frac{dv}{dt} = \frac{1}{12}\pi\, 3h^2 \frac{dh}{dt}$

$= \frac{1}{4}\pi (10)^2 \frac{dh}{dt}$

$30 = \frac{1}{4}\pi(100)\frac{dh}{dt}$

$\frac{30\times 4}{100\pi} = \frac{dh}{dt}$

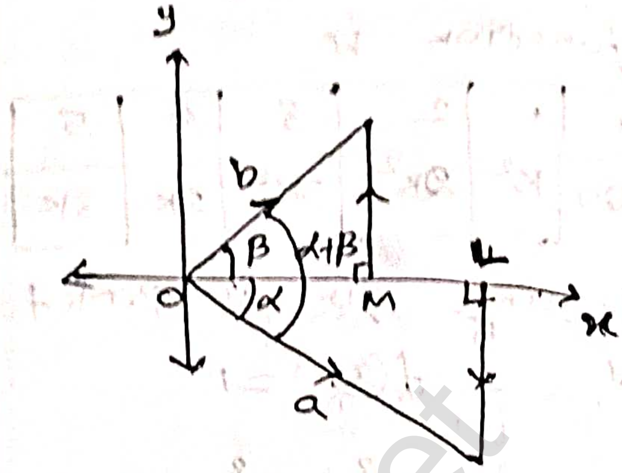
$\frac{12}{10\pi} = \frac{dh}{dt}$

$\frac{6}{5\pi} = \frac{dh}{dt}$

$\therefore \frac{dh}{dt} = \frac{6}{5\pi}\ m/min.$

(42)

(a)



Let $\vec{a} = \vec{OA}$; $\vec{b} = \vec{OB}$ be the unit vectors and which make angles α and β respectively

$|\vec{OL}|\hat{i} = \cos\alpha\,\hat{i}$

$|\vec{LA}| = \sin\alpha(-\hat{j})$

$\vec{a} = \vec{OA} = \vec{OL} + \vec{LA}$

$= \cos\alpha\,\hat{i} - \sin\alpha\,\hat{j}$ — (1)

Similarly,

$\vec{b} = \cos\beta\,\hat{i} + \sin\beta\,\hat{j}$ — (2)

The angle b/w $\vec{a}$ and $\vec{b}$ is $\alpha+\beta$ and so,

$\hat{a}\cdot\hat{b} = |\hat{a}||\hat{b}|\cos(\alpha+\beta)$

$= \cos(\alpha+\beta)$ — (3)

On the other hand from (1) & (2)

$\hat{a}\cdot\hat{b} = \cos\alpha\cos\beta - \sin\alpha\sin\beta$ — (4)

from (3) & (4) $\Rightarrow$

$\cos(\alpha+\beta) = \cos\alpha\cos\beta - \sin\alpha\sin\beta$.

42 (b)

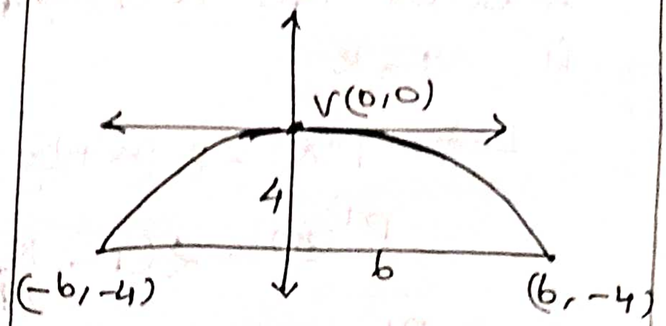
p	q	$p\leftrightarrow q$	$p\wedge q$	$\neg p$	$\neg q$	$\neg p\wedge\neg q$	$(p\wedge q)\vee(\neg p\wedge\neg q)$
T	T	T	T	F	F	F	T
T	F	F	F	F	T	F	F
F	T	F	F	T	F	F	F
F	F	T	F	T	T	T	T

(1) ≡ (2)

$\therefore p\leftrightarrow q \equiv (p\wedge q)\vee(\neg p\wedge\neg q)$

43 (b)

By taking the vertex at the origin, the parabola is open downward.



The eqn. is $x^2 = -4ay$ —(1)

It passes through $(6,-4)$

$36 = -4a(-4)$

$4a = \frac{36}{4} = 9$

(1) $\Rightarrow x^2 = -9y$ —(2)

To find the slope at $(-6,-4)$

Diff. (2) w.r.to 'x'

$2x = -9\frac{dy}{dx}$

$\frac{dy}{dx} = -\frac{2x}{9}$

At $(-6,-4) \Rightarrow \frac{-2(-6)}{9}$

$= \frac{12}{9} = \frac{4}{3}$

$\tan\theta = \frac{4}{3}$

$\theta = \tan^{-1}\left(\frac{4}{3}\right)$.

$\therefore$ The angle of projection is $\tan^{-1}\left(\frac{4}{3}\right)$.

44 (a)

Let x, y be the sides of the rectangle.

Area of the rectangle is $xy = k$

Let $P(x) = 2\left(x + \frac{k}{x}\right)$

$P'(x) = 2\left(1 - \frac{k}{x^2}\right)$

$P'(x) = 0 \Rightarrow$

$x = \pm\sqrt{k}$

As x, y are sides of the rectangle,

$x = \sqrt{k}$ is a critical numbers.

$P''(x) = \frac{4k}{x^3}$

$P''(\sqrt{k}) > 0.$

Sub. $x = \sqrt{k}$ in $xy = k$ we get

$y = \sqrt{k}.$

$\therefore$ The minimum perimeter rectangle of given area is square.

(45) (a)

Equation of the given curves are $y = \sin x \rightarrow (1)$

$y = \cos x \rightarrow (2)$

from (1) and (2)

$\sin x = \cos x$

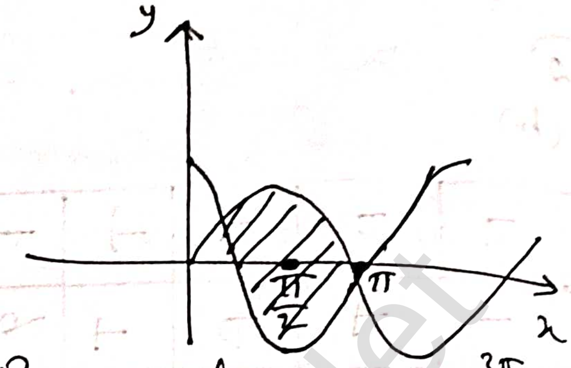
$x = \frac{\pi}{4}$

$y = \sin x$

x	0	$\pi/2$
y	0	1

$y = \cos x$

x	0	$\pi/2$
y	1	0



Required area $= 2\int_{\pi/4}^{3\pi/4} (\sin x - \cos x)\,dx$

($\because$ The area is symmetrical about x-axis)

$= 2\left[-\cos x - \sin x\right]_{\pi/4}^{3\pi/4}$

$= -2\left[\left(\cos\frac{3\pi}{4} - \sin\frac{3\pi}{4}\right) - \left(\cos\frac{\pi}{4} + \sin\frac{\pi}{4}\right)\right]$

$= -2\left[\left(\frac{-1}{\sqrt{2}} + \frac{1}{\sqrt{2}}\right) - \left(\frac{1}{\sqrt{2}} + \frac{1}{\sqrt{2}}\right)\right]$

[$\cos\frac{3\pi}{4} = \cos 135^\circ$

$\cos 135^\circ = \cos(180^\circ - 45)$

$= -\cos 45^\circ$

$\sin 135^\circ = \sin(180^\circ - 45)$

$= -\sin 45^\circ$]

$= -2\left[\frac{-2}{\sqrt{2}}\right] = \frac{4}{\sqrt{2}} \times \frac{\sqrt{2}}{\sqrt{2}}$

$= \frac{4\sqrt{2}}{2} = 2\sqrt{2}$ sq. units

45 (b) let $\cot^{-1}(1) = x$

$\cot x = 1$

$\tan x = 1$

$\tan x = \tan\frac{\pi}{4}$

$x = \frac{\pi}{4}$ $\left[\because \frac{\pi}{4} \in -\frac{\pi}{2}, \frac{\pi}{2}\right]$

Let $\sin^{-1}\left(-\frac{\sqrt{3}}{2}\right) = y$

$\sin y = \sin\left(-\frac{\pi}{3}\right)$

$(\sin(-\theta) = -\sin\theta)$

$y = -\frac{\pi}{3}$

Let $\sec^{-1}(-\sqrt{2}) = z$

$\cos z = -\frac{1}{\sqrt{2}}$

$\cos z = -\cos\frac{\pi}{4}$

$\cos z = \cos\left(\pi - \frac{\pi}{4}\right)$

$\cos z = \cos\left(\frac{3\pi}{4}\right)$

$z = \frac{3\pi}{4}$

$\cot^{-1}(1) + \sin^{-1}\left(-\frac{\sqrt{3}}{2}\right) - \sec^{-1}(-\sqrt{2})$

$= \frac{\pi}{4} - \frac{\pi}{3} - \frac{3\pi}{4}$

$= \frac{3\pi - 4\pi - 9\pi}{12}$

$= \frac{-10\pi}{12}$

$\cot^{-1}(1) + \sin^{-1}\left(-\frac{\sqrt{3}}{2}\right) - \sec^{-1}(-\sqrt{2}) = \frac{-5\pi}{6}$ //

46 (b) Since, $V(3) = 64$

$V(6) = 133$, and $V(9) = 208$

we get,

$9a + 3b + c = 64$

$36a + 6b + c = 133$

$81a + 9b + c = 208$

$$[A|B] = \left[\begin{array}{ccc|c} 9 & 3 & 1 & 64 \\ 36 & 6 & 1 & 133 \\ 81 & 9 & 1 & 208 \end{array}\right]$$

$$\xrightarrow{R_2 \to R_2 - 4R_1,\ R_3 \to R_3 - 9R_1}$$

$$\left[\begin{array}{ccc|c} 9 & 3 & 1 & 64 \\ 0 & -6 & -3 & -123 \\ 0 & -18 & -8 & -368 \end{array}\right]$$

$$\xrightarrow{R_2 \to R_2 \div (-3),\ R_3 \to R_3 \div (-2)}$$

$$\left[\begin{array}{ccc|c} 9 & 3 & 1 & 64 \\ 0 & 2 & 1 & 41 \\ 0 & 9 & 4 & 184 \end{array}\right] \xrightarrow{R_3 \to 2R_3}$$

$$\left[\begin{array}{ccc|c} 9 & 3 & 1 & 64 \\ 0 & 2 & 1 & 41 \\ 0 & 18 & 8 & 368 \end{array}\right] \xrightarrow{R_3 \to R_3 - 9R_2}$$

$$\left[\begin{array}{ccc|c} 9 & 3 & 1 & 64 \\ 0 & 2 & 1 & 41 \\ 0 & 0 & -1 & -1 \end{array}\right] \xrightarrow{R_3 \to (-1)R_3}$$

$$\left[\begin{array}{ccc|c} 9 & 3 & 1 & 64 \\ 0 & 2 & 1 & 41 \\ 0 & 0 & 1 & 1 \end{array}\right]$$

$9a + 3b + c = 64$

$2b + c = 41$

$c = 1$

$\boxed{c=1}$, $\boxed{b = \frac{41-c}{2} = 20}$

$a = \frac{64-3b-c}{9}$

$\boxed{a = \frac{1}{3}}$

So, we get,

$v(t) = \frac{1}{3}t^2 + 20t + 1.$

$v(15) = \frac{1}{3}(225) + 20(15) + 1$

$= 75 + 300 + 1$

$v(15) = 376.$

47 (a)

Let, $z^3 + 8i = 0.$

$z^3 = -8i$

$= 8\left[\cos\left(\frac{-\pi}{2} + 2k\pi\right) + i\sin\left(\frac{-\pi}{2} + 2k\pi\right)\right]$, $k \in z.$

$\therefore z = \sqrt[3]{8}\left[\cos\left(\frac{-\pi + 4k\pi}{6}\right) + i\sin\left(\frac{-\pi + 4k\pi}{6}\right)\right]$, $k = 0, 1, 2.$

Taking, $k = 0, 1, 2$ we get

$k = 0 \Rightarrow$

$z = 2\left(\cos\left(\frac{-\pi}{6}\right) + i\sin\left(\frac{-\pi}{6}\right)\right)$

$= 2\left(\frac{\sqrt{3}}{2} - i\frac{1}{2}\right)$

$= \sqrt{3} - i$

$k = 1 \Rightarrow$

$z = 2\left(\cos\left(\frac{\pi}{2}\right) + i\sin\left(\frac{\pi}{2}\right)\right)$

$= 2(0 + i) = 0 + 2i$

$= 2i$

$k = 2 \Rightarrow$

$z = 2\left(\cos\frac{7\pi}{6} + i\sin\frac{7\pi}{6}\right)$

$= 2\left(\cos\left(\pi + \frac{\pi}{6}\right) + i\sin\left(\pi + \frac{\pi}{6}\right)\right)$

$= 2\left(-\cos\left(\frac{\pi}{6}\right) - i\sin\left(\frac{\pi}{6}\right)\right)$

$= 2\left(-\frac{\sqrt{3}}{2} - i\frac{1}{2}\right)$

$= -\sqrt{3} - i.$

The values of z are $\sqrt{3} - i$, $2i$ and $-\sqrt{3} - i$.

47 (b)

We know that the given equation is homogeneous.

$(x^2 - 3y^2)\,dx + 2xy\,dy = 0$

$x^2 - 3y^2\,dx = -2xy\,dy$

$\frac{dy}{dx} = \frac{3y}{2x} - \frac{x}{2y}$

Taking, $y = vx$

we have

$v + x\frac{dv}{dx} = \frac{3v}{2} - \frac{1}{2v}$

(or) $x\frac{dv}{dx} = \frac{v^2-1}{2v}$.

Separating the variables

$$\frac{2v\,dv}{v^2-1} = \frac{dx}{x}$$

On integration, we get

$\log|v^2-1| = \log|x| + \log|c|$

Hence $|v^2-1| = |cx|$

where c is an arbitrary constant.

Replace v by $\frac{y}{x}$

to get $\left|\frac{y^2}{x^2}-1\right| = |cx|$

$|y^2-x^2| = |cx^3|$

Hence, $y^2-x^2 = \pm cx^3$

or

$y^2-x^2 = kx^3$

gives the general solution.

(30)

volume of sphere is

$\Rightarrow v = \frac{4}{3}\pi r^3$

$r = 5$ cm ; $\frac{dr}{dt} = -0.3$ cm

change in the volume

$\Rightarrow v = \frac{4}{3}\pi r^3$

Diff. w.r.to 't'

$\frac{dv}{dt} = \frac{4}{\cancel{3}}\pi \cancel{3}r^2 \frac{dr}{dt}$

$= 4\pi r^2 \frac{dr}{dt}$

$= 4\pi (5)^2 (-0.3)$

$= 100\pi (-0.3)$

$\frac{dv}{dt} = -30\pi \text{ cm}^3$

$\therefore$ volume decreases by 30π cm^3.

(34) $49y^2 - 16x^2 = 784$

$\div 784 \Rightarrow$

$\frac{\cancel{49}y^2}{\cancel{784}_{16}} - \frac{\cancel{16}x^2}{\cancel{784}_{49}} = \frac{\cancel{784}}{\cancel{784}}$

$\frac{y^2}{16} - \frac{x^2}{49} = 1$

$a^2 = 16, \quad b^2 = 49$

with the transverse axis is along y-axis

vertices are; $A(h, k+a)$

$A'(h, k-a)$

$\therefore A(0, 4)\ (0, -4)$

Foci $S(h, (k+c))$,

$S'(h, (k-c))$

$c^2 = a^2 + b^2$

$= 16 + 49 = 65$

$c^2 = 65 \Rightarrow c = \pm\sqrt{65}$

Foci are $S(0, \sqrt{65})$, $S'(0, -\sqrt{65})$

41 (a)

The plane containing the line,

$\frac{x-1}{2} = \frac{-y}{3} = \frac{z+3}{1}$ is of the form

$\frac{x-x_1}{b_1} = \frac{y-y_1}{b_2} = \frac{z-z_1}{b_3}$

$(b_1, b_2, b_3) = (2, -3, 1)$

$(x_1, y_1, z_1) = (1, 0, -3)$

$\vec{a} = \hat{i} + 0\hat{j} - 3\hat{k}$

& $\vec{b} = 2\hat{i} - 3\hat{j} + \hat{k}$

Also the plane is $\perp^r$ to the plane

$\vec{r}\cdot(\hat{i} - 2\hat{j} + 3\hat{k}) = 2$

$\therefore$ The required plane is parallel to the vector

$\vec{c} = \hat{i} - 2\hat{j} + 3\hat{k}$

(i) Parametric form of vector equation

$\vec{r} = \vec{a} + s\vec{b} + t\vec{c}$, where $s, t \in R$

$\Rightarrow \vec{r} = \hat{i} - 3\hat{k} + s(2\hat{i} - 3\hat{j} + \hat{k}) + t(\hat{i} - 2\hat{j} + 3\hat{k})$

(ii) Cartesian equation is

$$\begin{vmatrix} x-x_1 & y-y_1 & z-z_1 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix} = 0$$

$$\begin{vmatrix} x-1 & y-0 & z+3 \\ 2 & -3 & 1 \\ 1 & -2 & 3 \end{vmatrix} = 0$$

$\Rightarrow (x-1)(-9+2) - y(6-1) + (z+3)(-4+3) = 0$

$\Rightarrow (x-1)(-7) - y(5) + (z+3)(-1) = 0$

$-7x + 7 - 5y - z - 3 = 0$

$-7x - 5y - z + 4 = 0$

$7x + 5y + z - 4 = 0$.

The required equation of cartesian.

(44)

(b) Let P be denote the population of a city

Given:

$$\frac{dp}{dt} \propto P$$

$$\frac{dp}{dt} = kP$$

(where k is constant)

$$\frac{dp}{P} = k\,dt$$

On integrating,

$$\int \frac{dp}{P} = k \int dt$$

$$\log P = kt + c$$

$$P = e^{kt+c}.$$

$$P = e^{kt} \cdot e^{c}$$

$$(\because e^{c} = c)$$

$$P = ce^{kt} \longrightarrow (1)$$

Given:

when $t=0$, $P = 1,30,000$

$(1) \Rightarrow 1,30,000 = ce^{k(0)}$

$$\boxed{c = 1,30,000}$$

$$(\because e^{0} = 1)$$

$(1) \Rightarrow$

$$P = 1,30,000\, e^{kt} \longrightarrow (2)$$

when $t = 30$, $P = 1,60,000$

$(2) \Rightarrow 1,60,000 = 1,30,000\, e^{k(30)}$

$$\frac{16}{13} = e^{30k}$$

$$e^{k} = \left(\frac{16}{13}\right)^{1/30}$$

$$k = \log\left(\frac{16}{13}\right)^{\frac{1}{30}} \longrightarrow (3)$$

$(2) \Rightarrow P = 1,30,000\, e^{\log\left(\frac{16}{13}\right)^{\frac{1}{30}t}}$

$$P = 1,30,000\, e^{\log\left(\frac{16}{13}\right)^{\frac{1}{30}t}}$$

$$\therefore P = 1,30,000 \left(\frac{16}{13}\right)^{\frac{t}{30}}$$

$\therefore$ Hence, the population of a city after 60 years is

$$1,30,000 \left(\frac{16}{13}\right)^{t/30}.$$

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