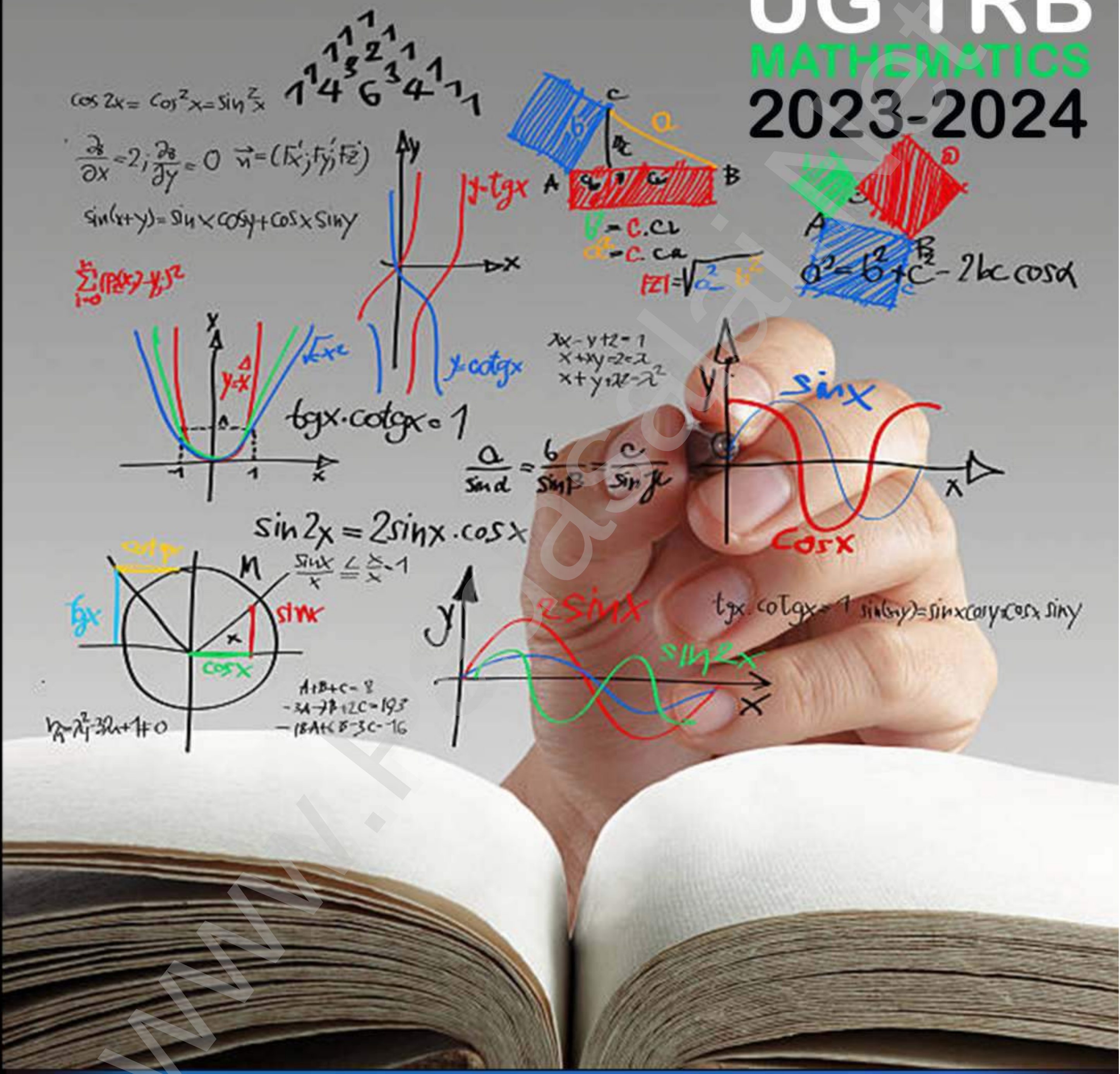
TEACHER'S CARE ACADEMY

38/23, Vaigundaperumal Koil Street, Kanchipuram-1

Mobile : 95665 35080, 97862 69980

Help Line Numbers : 9360268118,7639967359

UG TRB
MATHEMATICS
2023-2024



UNIT I
Algebra & Trigonometry

Your Success is Our Goal....

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UNIT – I – ALGEBRA AND TRIGONOMETRY

TEACHER'S CARE ACADEMY, KANCHIPURAM

TNPSC-TRB- COMPUTER SCIENCE -TET COACHING CENTER

HEAD OFFICE: NO. 38/23, VAIGUNDA PERUMAL KOIL, SANNATHI STREET, KANCHIPURAM – 1. CELL: 9566535080

B.Off 2: 65C, Thillai Ngr(West), 4th Cross St, Trichy – 620018

B.Off 3: Vijiyaraghavachariar Memorial Hall(Opp to Sundar Lodge), Salem

Trichy: 76399 67359 Salem: 93602 68118

UG TRB – MATHEMATICS – 2023-24

UNIT – I

ALGEBRA & TRIGONOMETRY

1.1. Polynomial Equations:

$$f(x)=a_0x^n+a_1x^{n-1}+a_2x^{n-2}+...+a_n$$

- Where n is a positive integers and $a_0, a_1, ..., a_n$ constant is called polynomial in x of nth degree, if $a_0 \neq 0$.

Fundamental Theorem of Algebra

1. Every polynomial equation of the nth degree has n and only n roots.
2. If $f(x)=0$ is an equation of odd degree then it has atleast one real roots.

   Whose sign opposite to that of the last term.
3. If $f(x)=0$ is an even degree another constant terms is negative. The equation has atleast one positive root and atleast one negative root.
4. If $f(x)=0$ has no real root between a and b (a < b), then f(a) and f (b) are same sign.

Exercises

1. Find the coefficient of x^n in the expansion of e^{a+bx}.

   (A) $\frac{e^a.e^b}{n!}$ (B) $\frac{e^a.e^n}{n!}$ (C) $\frac{e^a.e^n}{a!}$ (D) $\frac{e^a.e^n}{b!}$

2. The expansion of log (1+x) is

   (A) $\log(1+x)=x-\frac{x^2}{2!}+\cdots$ (B) $\log(1+x)=1-\frac{x^2}{2!}+\cdots$

(C) $\log(1+x) = x - \frac{x^3}{3!} + \cdots$ (D) None of these

3. The number of primes is

(A) finite (B) prime (C) infinite (D) None of these

4. Every polynomial equation f(x) = 0 has atleast one root real or ____

(A) imaginary (B) real (C) algebraic (D) complex

1.2. Imaginary and Irrational Roots

Solve $x^4 + 4x^3 + 5x^2 + 2x - 2 = 0$, solve $-1+i$ is a root.

Solution:

Given $-1+i$ is a root,

$-1-i$ is also a root.

$$\left[x-(-1+i)\right]\left[x-(-1-i)\right] = \left((x+1)-i\right)\left((x+1)+i\right)$$

$$=(x+1)^2 - i^2$$

$$=(x+1)^2 + 1$$

$$=x^2 + 2x + 1 + 1$$

$$=x^2 + 2x + 2$$

- When the polynomial is divided by x^2+2x+2 . The remainder is zero.
- Equating the co-efficient of x^3 - term of both side

$$\therefore x^4 + 4x^3 + 5x^2 + 2x - 2 = \left(x^2+2x+2\right)\left(x^2+ax-1\right)$$

$$2 + a = 4$$

$$a = 4 - 2$$

$$a = 2$$

$$\therefore f(x) = \left(x^2+2x+2\right)\left(x^2+2x-1\right)$$

$$\therefore x^2 + 2x - 1 = 0$$

$$= a\frac{-b \pm \sqrt{b^2-4ac}}{2a}$$

$$a\frac{-2\pm\sqrt{4-4(-1)}}{2(1)}$$

$$=\frac{-2\pm\sqrt{8}}{2}$$

$$=\frac{-2\pm2\sqrt{2}}{2}$$

$$=\frac{2\left(-1\pm\sqrt{2}\right)}{2}$$

$$=-1\pm\sqrt{2}$$

- The two roots are $\left(-1-\sqrt{2}\right),\left(-1+\sqrt{2}\right)$

Solve: $x^4-10x^3+26x^2-10x+1=0$ given that $2+\sqrt{3}$ in a root of the equations.

Solu:

- Since, $2+\sqrt{3}$ is a roots, $2-\sqrt{3}$ is also a root.

$$\left(x-\left(2+\sqrt{3}\right)\right)\left[x-\left(2-\sqrt{3}\right)\right]=\left[x\left(-2-\sqrt{3}\right)\right]\left[x\left(-2+\sqrt{3}\right)\right]$$

$$=\left[(x-2)-\sqrt{3}\right]\left[(x-2)+\sqrt{3}\right]$$

$$=(x-2)^2-\left(\sqrt{3}\right)^2$$

$$=x^2-4x+4-3$$

$$=x^2-4x+1$$

- When the polynomial is divided by x^2-4x+1 the remainder is zero.
- Equality the co-efficient of x^3 term on both side.

$$\therefore x^4-10x^3+26x^2-10x+1=\left(x^2-4x+1\right)\left(x^2-ax+1\right)$$

$$-a-4=-10$$

$$-a=-10+4$$

$$-a=-6$$

$$a=6$$

$\therefore f(x)=(x^2-4x+1)(x^2-6x+1)$

$$x=\frac{-(-6)\pm\sqrt{6^2-4\times1\times1}}{2\times1}$$

$$=\frac{6\pm\sqrt{36-4}}{2}$$

$$=\frac{6\pm\sqrt{32}}{2}$$

$$=\frac{6\pm4\sqrt{2}}{2}$$

$$=\frac{2(3\pm2\sqrt{2})}{2}$$

$$=3\pm2\sqrt{2}$$

$\therefore$ The two roots are $(3+2\sqrt{2})$ and $(3-2\sqrt{2})$

Solve: $x^3-15x^2+71x-105=0$ given that the roots of equation are in A.P.

Soln:

- Let the roots be $\alpha-d, \alpha, \alpha+d$
- Sum of roots $=\alpha-d+d+\alpha+d$

  $=3\alpha$
- General formula

  $\alpha-d+\alpha+\alpha+d=p$

  $3\alpha=+15$

  $\alpha=\frac{15}{3}$

  $\alpha=5$
- Since $x=5$ is a root, $x-5$ is a factor of $f(x)$

  $x^3-15x+71x-105=(x-5)(x^2+ax+21)$
- Equating the coefficient of x^2 term in both side

$ax-5=-15 \Rightarrow a-5=15$

$a=-15+5$

$a=10$

$x^2-10x+21=(x-7)(x-3)$

$x-7=0$ $\quad\quad$ $x-3=0$

$x=7$ $\quad\quad$ $x=3$

$\therefore$ The roots are (3, 5, 7)

Alter

Product of root = - (- 105)

$(5-d)(5)(5+d)=-(-105)$

$5(5^2-d^2)=105$

$25-d^2=\frac{105}{5}$

$25-d^2=21$

$d^2=25-21$

$d^2=4$

$d=\pm 2$

$\therefore$ The roots are $\alpha-d, \alpha, \alpha+d$ is

$\Rightarrow (5-2, 5, 5+2)$

$\Rightarrow (3,5,7)$

Solve: $x^3-19x^2+114x-216=0$ given that the roots are in G.P.

Soln.

- Let the roots be $\left(\frac{\alpha}{r}, \alpha, \alpha r\right)$
- Product of the roots = $\frac{\alpha}{r}, \alpha, \alpha r=-r$

$\alpha^3=216$

$\alpha^3 = 6^3$

$\alpha = 6$

- Since $x = 6$ is a root, $(x-6)$ is a factor $x^3 - 19x^2 + 114x - 216 = (x-6)(x^2 + ax + 36)$

$a - 6 = -19$

$a = -19 + 6$

$a = -13$

$(x^2 - 13x + 36) = (x-9)(x-4)$

$(x-9)(x-4) = 0$

$x = 9, 4$

$\therefore$ The roots are (6, 4, 9)

Solve: $6x^3 - 11x^2 + 6x - 1 = 0$ given the roots are in H.P.

Soln:

- Put $x = \frac{1}{y}$

$6x^3 - 11x^2 + 6x - 1 = 0$

$6\left(\frac{1}{y}\right)^3 - 11\left(\frac{1}{y}\right)^2 + 6\left(\frac{1}{y}\right) - 1 = 0$

$\frac{6}{y^3} - \frac{11}{y^2} + \frac{6}{y} - 1 = 0$

$6 - 11y + 6y^2 - y^3 = 0$

$y^3 - 6y^2 + 11y - 6 = 0$

- Sum of the roots

$\alpha - d + \alpha + \alpha + d = 6$

$3\alpha = 6$

$\alpha = \frac{6}{3}$

$\alpha = 2$

- Since y = 2 is a root, $(y-2)$ is a factor

  $y^3-6y^2+11y-6=(y-2)(y^2+ay+3)$

  - Equating the co-efficient of y^2 terms in both side

  $a-2=-6$

  $a=-6+2$

  $a=-4$

  $\therefore y^2-4y+3=0$

  $(y-1)(y-3)=0$

  $y=1,3$

  $\therefore$ The roots are (1, 2, 3)

- The roots of a given equation are

  $\left(1,\frac{1}{2},\frac{1}{3}\right)$

Solve: $x^4-2x^3+4x^2+6x-21=0$ given that two of its roots are equal in magnitude but opposite in sign.

Soln:

- Let the roots be α,β,γ and δ

  $\alpha+\beta=0$

  $\alpha=-\beta$

$\therefore x^4-2x^3+4x^2=6x-21=(x-\alpha)(x-\beta)(x-\gamma)(x-\delta)$

$=\left[x^2-(\alpha+\beta)x+\alpha\beta\right]\left[x^2-(\gamma-\delta)x+\gamma\delta\right]$

$=(x^2+\alpha\beta)(x^2-(\gamma+\delta)x+\gamma\delta)$

$=(x^2-a)(x^2-6x+c)$

$=x^4-6x+cx^2-ax^2+abx+ac$

- Equating the co-efficient of x^3,x^2 and x terms are both side

$-b=-2$ $\quad c-a=4$ $\quad ab=6$

$b=2$ $\quad c-3=4$ $\quad a\times 2=6$

$c=4+3$ $\quad a=3$

$c=7$

$$\left(x^2-3\right)\left(x^2-2x+7\right)$$

$$x^2-3=0$$

$$x^2=3\Rightarrow x=\pm\sqrt{3}$$

$$x^2-2x+7=0$$

$$x=\frac{-b\pm\sqrt{b^2-4ac}}{2a}$$

$$x=\frac{2\pm\sqrt{4-4\times 7}}{2}$$

$$x=\frac{2\pm\sqrt{4-28}}{2}$$

$$=\frac{2\pm\sqrt{-24}}{2}$$

$$=\frac{2\pm i2\sqrt{6}}{2}$$

$$=\frac{2\left(1\pm i\sqrt{6}\right)}{2}$$

$$x=1\pm i\sqrt{6}$$

Exercises

1. The series $x-\frac{x^2}{2!}+\frac{x^3}{3!}-\cdots\infty$ is

   (A) Convergent (B) Divergent

   (C) Infinite (D) Finite

2. If A and B are symmetric, then AB is symmetric iff A and B are

   (A) Symmetric (B) skew symmetric

   (C) commutative (D) Associative

3. If A and B are Hermitian then AB+BA is Hermitian and AB-BA is

(A) Hermitian (B) skew symmetric

(C) skew Hermitian (D) Non Hermitian

4. If A^*A= I, then a square matrix A is said to be

(A) unitary (B) orthogonal (C) diagonal (D) None of these

5. The roots of the equation $x + \frac{1}{x} = 1$ are

(A) 1, -1 (B) 1+i and $\frac{1}{2} + \frac{i\sqrt{3}}{2}$

(C) 1+i and 1-i (D) $\frac{1+i\sqrt{3}}{2}$ and $\frac{1-i\sqrt{3}}{2}$

6. One real root of the equation $x^3 - 7x^2 + 14x - 8 = 0$ is

(A) -2 (B) $\frac{1}{2}$ (C) - $\frac{1}{2}$ (D) 2

1.3. Relation Between Roots and Coefficients Symmetric Function of Roots In Terms Of Coefficient

If α,β,γ are the roots are the equation $x^3 + px^2 + qx = r = 0$ Find the value of

(i) $\sum\alpha^2\beta$ (ii) $\sum\alpha^2$ (iii) $\sum\alpha^3$

Soln

[i] $\sum\alpha = -p$

[ii] $\sum\alpha\beta = q$

[iii] $\sum\alpha\beta\gamma = -r$

Therefore

[i] $\sum\alpha^2\beta = \left(\sum\alpha\beta\right)\left(\sum\alpha\right) - 3\alpha\beta\gamma$

$= q(-p) - 3 - r$

$= -pq + 3r$

$= 3r - pq$

[ii] $\sum\alpha^2 = \left(\sum\alpha\right)^2 - 2\sum\alpha\beta$

$= (-p)^2 - 2(q)$

$= p^2 - 2q$

[iii] $\sum\alpha^3 = \left(\sum\alpha\right)^3 - 3\left(\sum\alpha\right)\left(\sum\alpha\beta\right) + 3\alpha\beta\gamma$

$= -p^3 + (3p)(q) - r$

$= -p^3 + 3pq - r$

Prove that the sum of cubes of the roots $x^3 - 6x^2 + 11x - 6 = 0$ is 36

Soln:

$\sum\alpha^3 = \left(\sum\alpha\right)^3 - 3\left(\sum\alpha\right)\left(\sum\alpha\beta\right) + 3\alpha\beta\gamma$

$= (-p)^3 + 3pq + 3r$

$= (6)^3 - 3(6)(11) + 3(6)$

$= 216 - 198 + 18$

Here P = 6, q = 11 , r = 6

$= 216 - 198 - 18$

$= 234 - 198$

$\sum\alpha^3 = 36$

- Hence, they proved $x^3 - 6x^2 + 11x - 6 = 0$ for cube roots is 36.
- If α, β, γ are the roots $x^3 - x - 1 = 0$ for equation where roots are $\frac{1}{\alpha^3}, \frac{1}{\beta^3}, \frac{1}{\gamma^3}$

Soln:

- Let $y = \frac{1}{\alpha^3} = \frac{1}{x^3}, y = \frac{1}{x^3}$

$\therefore x^3 = \frac{1}{y}$

Hence, $x^3 - x - 1 = 0$ _________ (1)

$$\frac{1}{y}-x-1=0$$

$$\frac{1}{y}=x+1$$

$$x=\frac{1-y}{y}$$

- Put $x=\frac{1-y}{y}$ in (1)

$$\left(\frac{1-y}{y}\right)^3-\left(\frac{1-y}{y}\right)-1=0$$

$$\frac{(1-y)^3}{y^3}-\frac{(1-y)}{y}-1=0$$

$$(1-y)^3-(1-y)y^2-y^3=0$$

$$1-3y+3y^2-y^3-y^2+y^3-y^3=0$$

$$13y+3y^2-y^3-y^2=0$$

$$1-3y+3y^2-y^3-y^2=0$$

$$-y^3+2y^2-3y+1=0$$

$$y^3-2y^2+3y-1=0$$

$\therefore$ This are corresponding equation.

If α,β,γ the roots of $x^3-3ax+6=0$ show that $\sum(\alpha-\beta)(\alpha-\gamma)=9a$

Soln:

- We have $\sum\alpha=0, \sum\alpha\beta=-3a, \alpha\beta\gamma=-6$

$$\sum\alpha^2=\left(\sum\alpha\right)^2-2\sum\alpha\beta$$

$$=0-2(-3a)$$

$$=6a$$

$$\sum(\alpha-\beta)(\alpha-\gamma)=\sum\left[\alpha^2-\alpha\gamma-\alpha\beta+\beta\gamma\right]$$

$$=\sum\alpha^2-\sum\alpha\gamma\sum\alpha\beta+\sum\beta\gamma$$

$$=6a-(-3a)-(-3a)+(-3a)$$

$$=6a+3a+3a-3a$$

$$=9a$$

$$\therefore\sum(\alpha-\beta)(\alpha-\gamma)=9a$$

Exercises

1. Choose the wrong answer from the following choices

   Every nth degree equation f(x) = 0 has _________.

   (A) atleast n roots (B) atmost n roots

   (C) exactly n roots (D) atleast one real root

2. If the equation $x^3-4x^2+4x-16=0$ has two roots 2i and -2i then the other root is

   (A) 1+i (B) 1-i (C) 2-i (D) 4

3. If α,β,γ are the roots of $x^3+2x-6=0$ then the value of $\alpha\beta\gamma$ is

   (A) 0 (B) 2 (C) 6 (D) - 6

4. If the product of the roots of $3x^4-4x^3+2x^2+x+a=0$ is 21 then the value of a is

   (A) 7 (B) -7 (C) - 63 (D) 63

1.4. Transformation of Equation:

- Transform the equation $x^4-8x^3-x^2+68x-60=0$ into 1 which does not contain the terms in x^3 hence the solve the equation.

Soln:

Given: $x^4-8x^3-x^2+68x-60=0$ __________ (1)

Take $h=\dfrac{-a_1}{na_0}=\dfrac{8}{4}=2$

$h=2$

Diminish the root by 2

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Mobile : 95665 35080, 97862 69980
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UNIT II
Differential Calculus, Integral Calculus & Analytical Geometry

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UNIT – II – DIFFERENTIATION

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TNPSC-TRB- COMPUTER SCIENCE -TET COACHING CENTER
HEAD OFFICE: NO. 38/23, VAIGUNDA PERUMAL KOIL,
SANNATHI STREET, KANCHIPURAM – 1. CELL: 9566535080
B.Off 2: 65C, Thillai Ngr(West), 4th Cross St, Trichy – 620018
B.Off 3:266-C, Advaitha Ashram Road,(Opp to New Bus Stand),Salem-636 004.
Trichy: 76399 67359 Salem: 93602 68118

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UNIT II : DIFFERENTIATION

2.1. SUCCESSIVE DIFFERENTIATION nth DERIVATIVES

- If y is a function of x, its derivative $\frac{dy}{dx}$ will be some other function of x and the differentiation of this function with respect to x is called second derivative and is denoted by $\frac{d^2y}{dx^2}$.

  i.e., $\frac{d}{dx}\left(\frac{dy}{dx}\right)=\frac{d^2y}{dx^2}$

- Similarly, the third derivative is denoted by $\frac{d^3y}{dx^3}$

  i.e., $\frac{d}{dx}\left(\frac{d^2y}{dx^2}\right)=\frac{d^3y}{dx^3}$

- Thus, if we differentiate y twice with respect to x, we get the second derivative. If y is differentiated thrice with respect to x we get the third derivative.

Problem:

1. If $y=\frac{ax+b}{cx+d}$ **Find** $\frac{d^2y}{dx^2}$.

Solution:

$$y=\frac{ax+b}{cx+d}$$

$$\frac{dy}{dx}=\frac{(cx+d)a-(ax+b)c}{(cx+d)^2}$$

$$=\frac{acx+ad-acx-bc}{(cx+d)^2}$$

$$=\frac{ad-bc}{(cx+d)^2}$$

$$\frac{d^2y}{dx^2}=\frac{0-(ad-bc)(2)(cx+d)c}{(cx+d)^4}$$

$$=\frac{-2c(ad-bc)}{(cx+d)^3}$$

2. If $x=a(\cos t+t\sin t)$ $y=a(\sin t-t\cos t)$ **Find** $\frac{d^2y}{dx^2}$.

Solution:

$$y=a(\sin t-t\cos t)$$

$$\frac{dy}{dx}=a(\cos t+t\sin t-\cos t)$$

$$=at\sin t$$

$$x=a(\cos t-t\sin t)$$

$$\frac{dx}{dt}=a(-\sin t+t\cos t+\sin t)$$

$$=at\cos t$$

$$\frac{dy}{dx}=\frac{dy}{dt}\cdot\frac{dt}{dx}=\frac{at\sin t}{at\cos t}=\tan t$$

$$\frac{d^2y}{dx^2}=\frac{d}{dx}\left(\frac{dy}{dx}\right)=\frac{d}{dt}\left(\frac{dy}{dx}\right)\frac{dt}{dx}$$

$$=\frac{d}{dt}(\tan t)\cdot\frac{dt}{dx}$$

$$= \sec^2 t \cdot \frac{1}{at\cos t}$$

$$= \frac{\sec^3 t}{at}$$

3. If $y = a\cos 5x + b\sin 5x$ **show that** $\frac{d^2y}{dx^2} + 25y = 0$

Solution:

$$y = a\cos 5x + b\sin 5x$$

Differentiating with respect to x,

$$\frac{dy}{dx} = -5a\sin 5x + 5b\cos 5x$$

$$\frac{d^2y}{dx^2} = -25a\cos 5x - 25b\sin 5x$$

$$= -25(a\cos 5x + b\sin 5x)$$

$$= -25y$$

$$\frac{d^2y}{dx^2} + 25y = 0$$

4. If $y = a\cos(\log x) + b\sin(\log x)$ **show that** $x^2\frac{d^2y}{dx^2} + x\frac{dy}{dx} + y = 0$

Solution:

$$y = a\cos(\log x) + b\sin(\log x)$$

Differentiating with respect to x,

$$\frac{dy}{dx} = \frac{-a\sin(\log x)}{x} + \frac{b\cos(\log x)}{x}$$

$$x\frac{dy}{dx} = -a\sin(\log x) + b\cos(\log x)$$

Again, differentiating with respect to x,

$$x\frac{d^2y}{dx^2}+\frac{dy}{dx}\cdot 1=\frac{-a\cos(\log x)}{x}-\frac{b\sin(\log x)}{x}$$

$$x^2\frac{d^2y}{dx^2}+x\frac{dy}{dx}+y=0$$

5. If $\left(y=x+\sqrt{1+x^2}\right)^m$ **show that** $(1+x^2)y_2+xy_1-m^2y=0$

Solution:

$$y=\left(x+\sqrt{1+x^2}\right)^m$$

Differentiating with respect to x,

$$\frac{dy}{dx}=m\left(x+\sqrt{1+x^2}\right)^{m-1}\left[1+\frac{2x}{2\sqrt{1+x^2}}\right]$$

$$=\frac{m\left(x+\sqrt{1+x^2}\right)^{m-1}\left[\sqrt{1+x^2}+x\right]}{\sqrt{1+x^2}}$$

$$=\frac{m\left(x+\sqrt{1+x^2}\right)^{m}}{\sqrt{1+x^2}}$$

$$=\frac{my}{\sqrt{1+x^2}}$$

Cross multiplying and squaring we get,

$$(1+x^2)\left(\frac{dy}{dx}\right)^2=m^2y^2$$

$$(1+x^2)2\frac{dy}{dx}\cdot\frac{d^2y}{dx^2}+\left(\frac{dy}{dx}\right)^2\cdot 2x=m^2\cdot 2y\cdot\frac{dy}{dx}$$

Cancelling, $2\frac{dy}{dx}$ we get,

$$(1+x^2)\frac{d^2y}{dx^2}+\frac{dy}{dx}-m^2y=0$$

$$(1+x^2)y_2+xy_1-m^2y=0$$

6. If $y = e^{a\sin^{-1}x}$ show that $\left(1+x^2\right)y_2 + xy_1 - a^2y = 0$.

Solution:

$$y = e^{a\sin^{-1}x}$$

Differentiating with respect to x,

$$\frac{dy}{dx} = e^{a\sin^{-1}x} \cdot \frac{a}{\sqrt{1-x^2}} = \frac{ay}{\sqrt{1-x^2}}$$

$$\sqrt{1-x^2}\frac{dy}{dx} = ay$$

$$\left(1-x^2\right)\left(\frac{dy}{dx}\right)^2 = a^2y^2$$

Differentiating with respect to x,

$$\left(1-x^2\right)\cdot 2\frac{dy}{dx},\frac{d^2y}{dx^2} + \frac{dy}{dx}(-2x) = a^2 \cdot 2y\frac{dy}{dx}$$

Cancelling $2\frac{dy}{dx}$ throughout,

$$\left(1-x^2\right)y_2 - xy_1 - a^2y = 0$$

7. If $y = \sin\left(m\sin^{-1}x\right)$ show that $\left(1-x^2\right)y_2 - xy_1 + m^2y = 0$

Solution:

$$y = \sin\left(m\sin^{-1}x\right)$$

$$\sin^{-1}y = m\sin^{-1}x$$

Differentiating with respect to x,

$$\frac{1}{\sqrt{1-y^2}}\frac{dy}{dx} = \frac{m}{\sqrt{1-x^2}}$$

Squaring and cross multiplying,

$$\left(1-x^2\right)\left(\frac{dy}{dx}\right)^2 = m^2\left(1-y^2\right)$$

Differentiating with respect to x we get

$$\left(1-x^2\right)2\frac{dy}{dx}\frac{d^2y}{dx^2}+\left(\frac{dy}{dx}\right)^2(-2x)=m^2\left(-2y\frac{dy}{dx}\right)$$

Cancelling $2\frac{dy}{dx}$,

$$\left(1-x^2\right)y_2-xy_1+m^2y=0$$

8. If $2x=y^{\frac{1}{m}}+y^{\frac{-1}{m}}$ **prove that** $\left(x^2-1\right)y_2+xy_1-m^2y=0$

Solution:

$$2x=y^{\frac{1}{m}}+y^{\frac{-1}{m}}$$

Differentiating with respect to x, we get,

$$2=\frac{1}{m}\cdot y^{\frac{1}{m}-1},y_1-\frac{1}{m}y^{\frac{-1}{m}-1},y_1$$

$$=\frac{y_1}{my}\left(y^{\frac{1}{m}}-y^{\frac{-1}{m}}\right)$$

$$2my=y_1\left(y^{\frac{1}{m}}-y^{\frac{-1}{m}}\right)$$

Squaring,

$$4m^2y^2=y_1^2\left(y^{\frac{1}{m}}-y^{\frac{-1}{m}}\right)^2$$

$$4m^2y^2=y_1^2\left[\left(y^{\frac{1}{m}}-y^{-\frac{1}{m}}\right)-4\right]$$

$$4m^2y^2=y_1^2\left(4x^2-4\right)$$

$$m^2y^2=y_1^2\left(x^2-1\right)$$

Differentiating with respect to x,

$$m^2\cdot 2y\frac{dy}{dx}=y_1^2 2x+\left(x^2-1\right)2y_1\cdot y_2$$

Cancelling $2y_1$, we get, $\left(x^2-1\right)y_2+xy_1-m^2y=0$

9. If $y=\frac{1}{2}\left(\sin^{-1}x\right)^2$ show that $\left(1-x^2\right)y_2-xy_1=1$

Solution:

$$y=\frac{1}{2}\left(\sin^{-1}x\right)^2$$

Differentiating with respect to x,

$$y_1=\frac{1}{2}2\left(\sin^{-1}x\right)\frac{1}{\sqrt{1-x^2}}$$

Squaring and cross multiplying we get,

$$\left(1-x^2\right)y_1^2=\left(\sin^{-1}x\right)^2$$

i.e., $\left(1-x^2\right)y_1^2=2y$

Differentiating again with respect to x,

$$\left(1-x^2\right)=2y_1y_2+y_1^2\left(-2x\right)=2y_1$$

Cancelling $2y_1$ throughout

$$\left(1-x^2\right)y_2-xy_1=1$$

10. If $x=\sin t, y=\sin pt$ obtain $\cos t\frac{dy}{dx}=p\cot p$. Now differentiating both side with Respect to x deduce $\left(1-x^2\right)\frac{d^2y}{dx^2}-x\frac{dx}{dy}+p^2y=0$.

Solution:

$$x=\sin t, y=\sin pt$$

$$\frac{dx}{dt}=\cos t, \frac{dy}{dt}=p\cos pt$$

$$\frac{dy}{dx}=\frac{dy}{dt}\cdot\frac{dt}{dx}=\frac{p\cos pt}{\cos t}$$

$$\cos t\frac{dy}{dx}=p\cos pt$$

Differentiating both sides with respect to x,

$$\cos t\frac{d^2y}{dx^2}+\frac{dy}{dx}(-\sin t)\frac{dt}{dx}=p^2(-\sin pt)\frac{dt}{dx}$$

$$\therefore \cos t\frac{dx}{dt}\frac{d^2y}{dx^2}-\sin t\frac{dy}{dx}+p^2\sin pt=0$$

$$\left(1-\sin^2 t\right)\frac{d^2y}{dx^2}-\sin t\frac{dy}{dx}+p^2\sin pt=0 \qquad \left(\text{since } \frac{dx}{dt}=\cos t\right)$$

$$\left(1-x^2\right)\frac{d^2y}{dx^2}-x\frac{dy}{dx}+p^2y=0$$

11. If $y=\left(\tan^{-1}x\right)^2$ **show that** $\left(1+x^2\right)^2 y_2+2x\left(1+x^2\right)y_1=2$

Solution:

$$y=\left(\tan^{-1}x\right)^2$$

Differentiating with respect to x,

$$y_1=\frac{2\tan^{-1}x}{1+x^2}$$

$$\left(1+x^2\right)y_1=2\tan^{-1}x$$

Again differentiating $\left(1+x^2\right)y_2+y_1 2x=\dfrac{2}{1+x^2}$

$$=\left(1+x^2\right)^2 y_2+2x\left(1+x^2\right)y_1=2$$

12. If $y=a\sin^m x$ **prove that** $\sin^2 x\cdot\dfrac{d^2y}{dx^2}=\left(m^2\cos^2 x-m\right)y$

Solution:

$$y=\sin^m x$$

Differentiating with respect to x,

$$\frac{dy}{dx}=m\sin^{m-1}x\cos x$$

$$\frac{d^2y}{dx} = m(m-1)\sin^{m-2}x \cdot \cos^2 x - m\sin^m x$$

Multiplying both sides by $\sin^2 x$,

$$\therefore \sin^2 x \frac{d^2y}{dx^2}$$

$$= m(m-1)\sin^m x\cos^2 x - m\sin^m x \cdot \sin^2 x$$

$$= m(m-1)y\cos^2 x - my\sin^2 x$$

$$= m^2 y\cos^2 x - my\cos^2 x - my\sin^2 x$$

$$= m^2 y\cos^2 x - my\left(\cos^2 x + \sin^2 x\right)$$

$$\therefore \sin^2 x \frac{d^2y}{dx^2} = m^2 y\cos^2 x - my$$

$$= \left(m\cos^2 x - m\right)y$$

13. If $y = -x^3, \log x$ prove that $x\frac{d^2y}{dx^2} - 2\frac{dy}{dx} + 3x^2 = 0$

Solution:

$$y = -x^3 \log x$$

$$\frac{dy}{dx} = -\frac{x^3}{3} - 3x^2 \log x$$

$$= -x^2 - 3x^2 \log x$$

$$\frac{d^2y}{dx^2} = -2x - \frac{3x^2}{x} - 6x\log x$$

$$x\frac{d^2y}{dx^2} = -2x^2 - 3x^2 - 6x^2 \log x$$

$$= -3x^2 - 2\left(x^2 + 3x^2 \log x\right)$$

$$\therefore x\frac{d^2y}{dx^2} - 2\frac{dy}{dx} + 3x^2 = 0$$

EXERCISES

1. If $y = a\cos 5x + b\sin 5x$ then $\frac{d^2y}{dx^2}$ is ________.

(A) 25 (B) – 25 (C) – 25 y (D) y

2. If $x = \sin t, y = \sin pt$ then $\frac{dy}{dt} =$ ________.

(A) $\sin pt$ (B) $p\sin pt$ (C) $p\cos pt$ (D) $\cos pt$

3. If $y = \left(\tan^{-1} x\right)^2$ then $y_1 =$ ________.

(A) $\frac{2\tan^{-1} x}{1+x^2}$ (B) $\tan^{-1} x$ (C) $\frac{\tan^{-1} x}{1+x^2}$ (D) $\frac{2}{1+x^2}$

4. If $x = at^2, y = 2at$ then $y_2 =$ ________.

(A) $\frac{1}{2at^3}$ (B) $\frac{-1}{2at^3}$ (C) $\frac{1}{at^3}$ (D) $\frac{-1}{t^3}$

5. If $y = \frac{1}{2}\left(\sin^{-1} x\right)^2$ then ________.

(A) $\left(1-x^2\right)y_1^2 = 2y$ (B) $y_1^2 = 2y$ (C) $\left(1-x^2\right)y_1^2 = y$ (D) none

6. The n[th] derivative of e^{ax} is

(A) $y_n = a^n$ (B) $y_n = a^n e^{ax}$ (C) $y_n = e^{ax}$ (D) $y_n = ne^{ax}$

7. The n[th] derivative of $\sin(ax+b)$ is

(A) $a^n \sin(ax+b)$ (B) $a^n \sin\left(ax+b+\frac{\pi}{2}\right)$

(C) $\sin\left(ax+b+\frac{n\pi}{2}\right)$ (D) $n\sin\left(ax+b+\frac{n\pi}{2}\right)$

8. If $y = \tan^{-1}\left(\frac{x}{a}\right)$ then $y_1 =$ ________.

(A) $\frac{a}{x^2+a^2}$ (B) $\frac{1}{x^2+a^2}$ (C) $\frac{1}{x^2-a^2}$ (D) $\frac{a}{x^2-a^2}$

9. If nth derivative of $\frac{1}{(2x+3)^2}$ is __________.

(A) $\frac{(-1)^n (n+1)!}{(2x+3)^{n+2}}$ (B) $\frac{(-1)^n 2^n (n+1)!}{(2x+3)^{n+2}}$ (C) $\frac{(n+1)!}{(2x+3)^{n+2}}$ (D) $\frac{(-1)2^n}{(2x+3)^{n+2}}$

10. The nth derivative of $\sin 2x$ is __________.

(A) $2^n \sin\left(2x+\frac{n\pi}{2}\right)$ (B) $2^n \sin 2x$ (C) $\sin\left(2x+\frac{n\pi}{2}\right)$ (D) none

2.2. STANDARD nth DERIVATIVE

1. nth derivative of e^{ax}.

Solution:

$$y = e^{ax}$$

$$y_1 = e^{ax} \cdot a$$

$$y_2 = e^{ax} \cdot a^2$$

$$y_3 = e^{ax} \cdot a^3$$

$$\therefore y_n = a^n e^{ax}$$

2. nth derivative of $\frac{1}{ax+b}$

Solution:

$$y = \frac{1}{ax+b} = (ax+b)^{-1}$$

$$y_1 = -1(ax+b)^{-2} a$$

$$y_2 = (-1)(-2)(ax+b)^{-3} \cdot a^2$$

$$y_3 = (-1)(-2)(-3)(ax+b)^{-4} \cdot a^3$$

$$\therefore \quad y_n = (-1)(-2)(-3)\ldots(-n)(ax+b)^{-(n+1)} \cdot a^n$$

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$$= \frac{(-1)^n n! a^n}{(ax+b)^{n+1}}$$

3. n^th derivative of $\frac{1}{(ax+b)^2}$

$$y = (ax+b)^{-2}$$

$$y_1 = (-2)(ax+b)^{-3} \cdot a$$

$$y_2 = (-2)(-3)(ax+b)^{-4} \cdot a^2$$

$$y_3 = (-2)(-3)(-4)(ax+b)^{-5} \cdot a^3$$

.

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$$y_n = (-2)(-3)(-4)\ldots(\overline{-n+1})(ax+n)^{-(n+2)} \cdot a^n$$

$$= \frac{(-1)^n (n+1)! a^n}{(ax+b)^{n+2}}$$

4. n^th derivative of $\log(ax+b)$

$$y = \log(ax+b)$$

$$y_1 = \frac{1}{ax+b} \cdot a$$

$$\therefore y_n = \frac{(-1)^{n-1} (n-1)! a^n}{(ax+b)^n}$$

5. n^th derivative of $\sin(ax+b)$

$$y = \sin(ax+b)$$

TEACHER'S CARE ACADEMY

38/23, Vaigundaperumal Koil Street, Kanchipuram-1
Mobile : 95665 35080, 97862 69980
Help Line Numbers : 9360268118,7639967359

UG TRB
MATHEMATICS
2023-2024

UNIT III
Differential Equations & Laplace Transformations

Your Success is Our Goal….

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UG TRB - MATHEMATICS – 2023-24

UNIT – III

DIFFERENTIAL EQUATIONS AND LAPLACE TRANSFORMATION

3.1. Ordinary Differential Equations:

- An ordinary differential equation is an equation which is defined for one or more functions of one independent variable and its derivations. It is abbreviated as ODE. Example $\frac{dy}{dx} = x+3$
- When the function involved in the equation depends on only a single variable, its derivatives are ordinary derivatives and the differential equation is classed as an ordinary differential equation.
- On the other hand, if the function depends on several independent variables the differential equation is classed as a partial differential equation.

Order and Degree of Ordinary Differential Equations:

- The order of differential equation is the highest derivative in the equation is the **highest derivative** in the equation. The degree of the diffi. equation of the highest power to which the derivative is raised.

Problems:

1. **Solve:** $9yy' + 4x = 0$

Solution:

$$9y\frac{dy}{dx} = -4x \Rightarrow 9ydy = -4xdx$$

Integrating we get,

$$\frac{9y^2}{2} = \frac{-4x^2}{2} + c$$

$$\Rightarrow \frac{y^2}{4} = \frac{-x^2}{9} + c \Rightarrow \frac{x^2}{9} + \frac{y^2}{4} = c$$

2. Solve: $\frac{dy}{dx} = \frac{2x+y-1}{4x+2y-4}$

Solution:

Let $V = \frac{2x+y}{2}$,

The D.E. becomes

$$\therefore \frac{dy}{dx} = 1 + \frac{1}{2}\left(\frac{2v-1}{4v-4}\right)$$

$$\Rightarrow \frac{8v-8}{10v-9} \quad v = dx \Rightarrow \left[\frac{8v - \frac{36}{5} - \frac{4}{5}}{10v-9}\right] dv = dx$$

$$\Rightarrow \left[\frac{4}{5} - \frac{4}{5}\left[\frac{1}{10v-9}\right]\right] dv = dx$$

Int. we get

$$\frac{4v}{5} - \frac{2}{25}\log(10v-9) + c = x$$

$$\Rightarrow \frac{2}{5}(2x+y) - \frac{2}{25}\log(10x+5y-9) + c = x$$

$$\Rightarrow \frac{x}{5} + \frac{2y}{5} - \frac{2}{25}\log(10x+5y-9) + c = 0$$

Exercises

1. An ordinary differential equation is an equation which is defined for one or more functions of _______independent variables.

(A) several (B) one (C) two (D) more than one

2. The order of this equation is ______ $\left[\frac{d^2y}{dx^2}+2y\right]^3+\frac{d^3y}{dx^3}+y=0$

(A) 2 (B) 1 (C) 3 (D) none

3. A general solution of the equation $y'=\cos x$ is ______.

(A) $y=\sin x+c$ (B) $y=\operatorname{cosec} x+c$ (C) $y=\cos x+c$ (D) $y=\sec x+c$

3.2. Homogeneous Differential Equations:

Homogeneous Function:

- A function $f(x,y)$ in x and y is said to be a homogenous function if the degree of each term in the function is constant. In general, a homo function $f(x,y)$ of degree n is expressible as

$$f(x,y)=\lambda^n f\left(\frac{y}{x}\right)$$

Homogeneous Differential Equation

- A differential Equation in which all the functions are of the same degree is called a homogenous differential equation

Example:

$\frac{dy}{dx}=\frac{x^2-y^2}{xy}$ is a homogeneous differential equation.

- Homogenous differential equations are differential equations with homogeneous functions. They are equations containing a differentiation operator, a function and a set of variable. The general form of the homogenous differential equation is $f(x,y)dy+g(x,y)dx=0$, where $f(x,y)$ and $g(x,y)$ is a homo. function.
- Homo. functions are defined as functions in which the total power of all the terms of the function is constant.
- Homo. function and homogenous differential equation are represented in the below form.

Homo. function: $f(x,y)=\lambda^n f\left(\frac{y}{x}\right)$

Homo. Differential equation: $\frac{dy}{dx}=f(x,y)$

Exercises

4. The solution of the differential equation $xy^2dy-\left(x^3+y^3\right)dx=0$ is _____.

(A) $y^3=3x^3+c$ (B) $y^3=3x^3\log(cx)$

(C) $y^3=3x^3+\log(cx)$ (D) none

5. The solution of differential equation $\cos(x+y)dy=dx$ is _____.

(A) $y=x\sec\left(\frac{y}{x}\right)+c$ (B) $y+\cos^{-1}\left(\frac{y}{x}\right)=c$

(C) $y=\tan\left(\frac{x+y}{2}\right)+c$ (D) $y=\cot\left(\frac{x+y}{2}\right)+c$

3.3. Exact Differential Equation:

- A differential equation is said to be exact if it can be derived directly from its primitive without any further operation of elimination or reduction. Thus the differential equation

$$M(x,y)dx+N(x,y)dy=0 \qquad \text{\_\_\_\_\_\_\_ (1)}$$

it exact if it can be derived by equating the differential of some function $V(x,y)$ to zero.

Let $v(x,y)=c$ be the solution

Differentiating this we get

$$\frac{\partial u}{\partial x}dx+\frac{\partial v}{\partial y}dy=0 \qquad \text{\_\_\_\_\_\_\_ (2)}$$

(1) and (2) are identical

$$M=\frac{\partial u}{\partial x}, N=\frac{\partial u}{\partial y}$$

If we eliminate v between there by means of the equivalence of the relation

$$\frac{\partial}{\partial x}\left(\frac{\partial u}{\partial y}\right)=\frac{\partial^2 u}{\partial x\partial y}=\frac{\partial}{\partial y}\left(\frac{\partial u}{\partial x}\right) \text{ we get}$$

Thus, the condition for $Mdx+Ndy=0$ to be an exact equation is

$$\frac{\partial M}{\partial y}=\frac{\partial N}{\partial x}$$

Rule for solving $Mdx + N\,dy = 0$ when it is exact

(i) First integrate M w.r.to x regarding y as a constant.

(ii) Then integrate w.r.to y those terms in N which do not contain x.

(iii) The sum of the expressions obtained in (i) and (ii), when equated to an arbitrary constant, will be the solution.

Problems:

1. Solve $\left(\sin x\cos y + e^{2x}\right)dx + \left(\cos x\sin y + \tan y\right)dy = 0$

Solution:

Here $M = \sin x\cos y + e^{2x}$

$N = \cos x\sin y + \tan y$

$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$, the equation is exact, integrating M w.r.to x regarding y as a constant we get

$$\left[-\cos x\,\cos y + \frac{1}{2}e^{2x}\right]$$

In N, the term not involving x namely *tan y* is integrated w.r.to y giving $\log\sec y$

$\therefore$ the solution is

$$-\cos x\cos y + \frac{e^{2x}}{2} + \log\sec y = c$$

2. Solve $\left(ye^{xy} - 2y^3\right)dx + \left(xe^{xy} - 6xy^2 - 2y\right)dy = 0$

Solution:

$$M = ye^{xy} - 2y^3, \frac{\partial M}{\partial y} = e^{xy} + xye^{xy} - 6y^2$$

$$N = xe^{xy} - 6xy^2 - 2y, \frac{\partial N}{\partial x} = e^{xy} + xye^{xy} - 6y^2$$

Since $\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$, the equation is exact

$$\int Mdx = \int\left(ye^{xy} - 2y^3\right)dx$$

$$= y\frac{e^{xy}}{y} - 2xy^3 = e^{xy} - 2xy^3$$

Integrating those terms in N which do not contain *x*, with respect to *y*, we get $\int N\,dy = \int -2y\,dy = -y^2$, omitting terms involving x in N.

$\therefore$ The solution is $e^{xy} - 2xy^3 = y^2 = c$

3. Solve $y\left(2x^2y + e^x\right)dx - \left(e^x + y^3\right)dy = 0$

Solution:

$$M = 2x^2y^2 + ye^x : \frac{\partial M}{\partial y} = 4x^2y + e^x$$

$$N = -\left(e^x + y^3\right) \quad : \frac{\partial N}{\partial x} = -e^x$$

As $\frac{\partial M}{\partial y} \neq \frac{\partial N}{\partial x}$, the equation is not exact, However, we can rearrange the equation as

$$ye^x dx - e^x dy + \left(2x^2 dx - ydy\right)y^2 = 0$$

Now dividing by y^2, we have

$$\frac{ye^x dx - e^x dy}{y^2} + 2x^2 dx - ydy = 0$$

$$d\left(\frac{e^x}{y}\right) + 2x^2 dx - ydy = 0$$

Integrating, we find the solution as $\frac{e^x}{y} + \frac{2x^3}{3} - \frac{y^2}{2} = c$

4. Solve $\left(\log x + y\right)dx - xdy = 0$

Solution:

Observing that the equation is not proper we arrange it as

$$\log xdx + ydx - xdy = 0$$

Dividing by x^2 (an integrating factor) we get

$$\frac{1}{x^2}\log x dx + \left(\frac{ydx - xdy}{x^2}\right) = 0$$

$$\int \frac{1}{x^2}\log x dx + \int d\left(\frac{-y}{x}\right) = 0$$

$$\int \log x d\left(\frac{-1}{x}\right) - \frac{y}{x} = c$$

$$\frac{-\log x}{x} + \int x^{-2} dx - \frac{y}{x} = c$$

$$-\frac{\log x}{x} - \frac{1}{x} - \frac{y}{x} = c$$

or $cx + y + \log x + 1 = 0$ is the solution.

Exercises

6. A differential equation of the form $M(x, y)dx + N(x, y)dy = 0$ is said to be ______ if it can be directly obtained from its primitive by differentiation.

   (A) Linear equation (B) Separable equation

   (C) Exact equation (D) Lagrange's equation

7. The solution of $\left[\sec x \tan x \tan y - e^x\right]dx + \left[\sec x \sec^2 y\right]dy = 0$ is ________.

   (A) $\tan y - e^x = c$ (B) $\sec x \tan y - e^x = c$

   (C) $\tan x \sec y - e^x = c$ (D) $\sec x \tan y = c$

8. The exact condition value of $\left(x^3 + 3xy^2\right)dx + \left(3x^2 y + y^3\right)dy = 0$ is _______.

   (A) $6xy$ (B) $3xy$ (C) $2xy$ (D) $12xy$

9. The diff. equation $2ydx - (3y - 2x)dy = 0$ is

   (A) exact and homogenous but not linear

   (B) exact, homogenous and linear

   (C) exact and linear but not homogeneous

   (D) homogenous and linear but not exact

3.4. Integrating Factors:

Rule 1:

- When $Mx+Ny\neq 0$, and the equation is a homogenous one, $\frac{1}{Mx+Ny}$ is an integrating factor.

Problems:

1. Solve $x^2ydx-\left(x^3+y^3\right)dy=0$

Solution:

The equation is not exact and $Mx+Ny=y^4\neq 0$. Hence $-\frac{1}{y^4}$ can be used as an I.F. then

$$-\frac{x^2}{y^3}dx+\left(\frac{x^3+y^3}{y^4}\right)dy=0$$

$$\frac{\partial M}{\partial y}=\frac{3x^2}{y^4};\ \frac{\partial N}{\partial x}=\frac{3x^2}{y^4}$$

Hence the equation has become exact

$$\int Mdx=\int\frac{x^2}{y^3}dx=-\frac{x^3}{3y^3}$$

In N, integrating the term not containing x, namely $\frac{1}{y}$ w.r.to y we get $\log y$

$\therefore$ the solution is $-\frac{x^3}{3y^3}+\log y=c$

Rule 2:

If the equation is of the form $f_1(xy)dx+xf_2(xy)dy=0$ and $Mx-Ny\neq 0$, then $\frac{1}{Mx-Ny}$ is an I.F.

2. Solve $y\left(x^2y^2+xy+1\right)dx+x\left(x^2y^2-xy+1\right)dy=0$

Solution:

The equation is not exact since $\frac{\partial M}{\partial y}\neq\frac{\partial N}{\partial x}$

$$Mx - Ny = x^3y^3 + x^2y^2 + xy - x^3y^3 + x^2y^2 - xy = 2x^2y^2 \neq 0$$

Using $\frac{1}{Mx - Ny} = \frac{1}{2x^2y^2}$ as an I.F. we get

$$\left(\frac{x^2y^2 + xy + 1}{2x^2y}\right)dx + \left(\frac{x^2y^2 - xy + 1}{2xy^2}\right)dy = 0$$

$$\left(y + \frac{1}{x} + \frac{1}{x^2y}\right)dx + \left(x - \frac{1}{y} + \frac{1}{xy^2}\right)dy = 0$$

Now $\frac{\partial M}{\partial y} = 1 - \frac{1}{x^2y^2}$ and $\frac{\partial N}{\partial x} = 1 - \frac{1}{x^2y^2}$

$\therefore$ The equation is exact and the solution is

$$\int\left(y + \frac{1}{x} + \frac{1}{x^2y}\right)dx + \int -\frac{1}{y}dy = c$$

$$xy + \log x - \frac{1}{xy} - \log y = c$$

Rule 3:

(i) If $\frac{1}{N}\left(\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x}\right)$ is a function of x alone, say $f(x)$, then $e^{\int f(x)dx}$ is an integration factor.

(ii) If $\frac{1}{M}\left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right)$ is a function of y alone, say g(y), then $e^{\int g(y)dy}$ is an integrating factor.

3. Solve $\left(xy^3 + y\right)dx + 2\left(x^2y^2 + x + y + y^4\right)dy = 0$

Solution:

The equation is not exact and $\frac{\partial M}{\partial y} = 3xy^2 + 1$, $\frac{\partial N}{\partial x} = 4xy^2 + 2$

$$\frac{1}{M}\left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right) = \frac{1}{y} = g(y)$$

$e^{\int g(y)dy} = e^{\log y} = y$ is an integrating factor multiplying by y we get the equation

$$\left(xy^4 + y^2\right)dx + 2\left(x^2y^3 + xy + y^5\right)dy = 0$$

Now the equation is exact and the solution is

$$\int\left(xy^4+y^2\right)dx+2\int y^5dy=c$$

$$3y^4x^2+6xy^2+2y^6=c$$

Rule 4:

If the equation $Mdx+Ndy=0$ is of the form

$x^a y^6\left(mydx+nxdy\right)+x^r y^s\left(pydx+qxdy\right)=0$ where a, b, m, n, r, s, p, q are constants, then $x^h y^k$, is an integrating factor, where h and k are determined using the condition that after multiplication by $x^h y^k$, the equation becomes exact.

4. Solve $\left(y^3-2yx^3\right)dx+\left(2xy^2-x^3\right)dy=0$

Solution:

The equation is not an exact one and it can be rewritten as

$$y\left(y^2-2x^2\right)dx+x\left(2y^2-x^2\right)dy=0$$

$$y^2\left(ydx+2xdy\right)+x^2\left(-2ydx-xdy\right)=0$$

So that is of the form mentioned in rule IV above Multiplying the equation by $x^h y^k$ we get

$$\left(x^h y^{3+k}-2x^{h+2}y^{k+1}\right)dx+\left(2x^{h+1}x^{h+3}y^k\right)dy=0$$

Now $\frac{\partial M}{\partial y}=\left(3+k\right)x^h y^{k+2}-2\left(k+1\right)x^{h+2}y^k$ and $\frac{\partial N}{\partial k}=2\left(h+1\right)x^h y^{k+2}-\left(h+3\right)x^{h+2}y^k$

Using the condition $\frac{\partial M}{\partial y}=\frac{\partial N}{\partial x}$ and equating the coefficients of like lowered terms on both sides, we get

$$3+k=2\left(h+1\right)$$

$$2k+2=h+3$$

Solving them we get k = 1, h = 1so that the integrating factor is *xy*

The equation (1) for these values of h and k becomes

$$\left(xy^4-2x^3y^2\right)+\left(2x^2y^3-x^4y\right)dy=0$$

At this equation is exact, the solution is

$$\int \left(xy^4 - 2x^3y^2\right)dx = c, \frac{x^2y^4}{2} - \frac{2x^4y^2}{4} = c$$

$$x^2y^4 - x^4y^2 = k$$

Exercises

10. If the equation is of the form $f_1(x,y)dx + f_2(x,y)dy = 0$, when $Mx + Ny \neq 0$ then the integrating factor is _______.

(A) $\frac{1}{Mx+Ny}$ (B) $Mx+Ny$ (C) $\frac{1}{Mx-Ny}$ (D) $Mx-Ny$

11. For the equation $(xy^3 + y)dx + 2(x^2y^2 + x + y + y^4)dy = 0$ the integrating factor is _____.

(A) x (B) $2x$ (C) y (D) $-y$

12. If $\frac{1}{M}\left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right)$ is a function of y alone, say g (y) then ______ is an integrating factor.

(A) $e^{\int f(x)dx}$ (B) $e^{\int g(y)dy}$ (C) $e^{\int g(y)dx}$ (D) none

3.5. Linear Equations:

- A differential equation of the form $\frac{dy}{dx} + Py = Q$ where P and Q are function of x, is said to be a linear equation in y.

Multiplying both sides by $Qe^{\int pdx}$ we get

$$e^{\int pdx}\left(\frac{dy}{dx} + Py\right) = Qe^{\int Pdx}$$

$$\frac{d}{dx}\left(ye^{\int pdx}\right) = Qe^{\int pdx}$$

Integrating we get the solution as $ye^{\int Pdx} = \int Qe^{\int Pdx}dx + c$

Problems

1. Solve $\frac{dy}{dx} + y\cot x = 4x \quad \text{cosec } x$ **given that y = 0 when** $x = \frac{\pi}{2}$**.**

Solution:

Comparing with $\frac{dy}{dx} + Py = Q$ we find that

$$P = \cot x, Q = 4x\,\text{cosec}\,x$$

$$\int Pdx = \int \cot x\,dx = \log\sin x$$

$$e^{\int pdx} = e^{\log}\sin x$$

Solution is $y\sin = \int 4x\,\text{cosec}\,x\sin x\,dx$

$$= \int 4xdx = 2x^2 + c$$

$y = 0$ when $x = \frac{\pi}{2}$ gives $c = \frac{\pi^2}{2}$

$\therefore$ The solution is

$$y\sin x = 2x^2 - \frac{\pi^2}{2}$$

2. Solve $\left(1+y^2\right)dx = \left(\tan^{-1}y - x\right)dy$

Solution:

$$\frac{dx}{dy} + \frac{1}{1+y^2} = \frac{\tan^{-1}y}{1+y^2}$$

This is an equation of the type

$\frac{dx}{dy} + Px = Q$, which is linear in x,

$$P = \frac{1}{1+y^2};\; Q = \frac{\tan^{-1}y}{1+y^2}$$

$$\int Pdy = \int \frac{dy}{1+y^2} = \tan^{-1}y$$

$$e^{\int Pdy} = e^{\tan^{-1} y}$$

$\therefore$ the solution is $xe^{\int Pdy} = \int Qe^{\int Pdy} dy + c$

$$xe^{\tan^{-1} y} = \int e^{\tan^{-1} y} \frac{\tan^{-1} y}{1+y^2} dy + c$$

Putting $t = \tan^{-1} y$ on the R.H.S, we get

$$xe^{\tan^{-1} y} = \int te^t dt + c$$

$$= te^t - e^t + c$$

$\therefore$ Solution is $xe^{\tan^{-1} y} = e^{\tan^{-1} y}\left(\tan^{-1} y - 1\right) + c$ or $x = \tan^{-1} y - 1 + ce^{-\tan^{-1} y}$

Exercises

13. Solving the differential equation $\frac{dy}{dx} + \frac{y}{x} = 4x^2$ we get the solution ________.

(A) $x^2 + c$ (B) $x^3 + \frac{c}{x}$ (C) $x^2 + \frac{c}{x}$ (D) $x^3 + c$

14. The solution of the differential equation $x\frac{dy}{dx} - y = 3$ represents a family of ________.

(A) straight line (B) circle (C) ellipse (D) parabola

15. A differential equation of the form $\frac{dy}{dx} + Py = Q$ has the solution as ______.

(A) $ye^{\int Pdx} = \int Qdx + c$ (B) $ye^{\int Pdx} = \int Qe^{\int Pdx} dx + c$

(C) $y = \int Qe^{\int Pdx} dx + c$ (D) $ye^{\int Pdx} = \int e^{\int Pdx} dx + c$

3.6. Reduction of Order:

Equations Reducible to Linear Form

Consider the equation $\frac{dy}{dx} + Py = Qy^n$

Where P and Q are functions of x

Dividing by y^n we get

$$y^{-n}\frac{dy}{dx}+y^{1-n}P=Q$$

Putting $V=y^{1-n},\frac{dv}{dx}=(1-n)y^{-n}\frac{dy}{dx}$

Using the above equation

$$\frac{dv}{dx}+(1-n)vP=(1-n)Q$$

which is a linear equation in v and hence can be solved by the previous method.

Problems

1. Solve $\frac{dy}{dx}+x\sin 2y=x^3\cos^2 y$

Solution:

Dividing by $\cos^2 y$ we get

$$\sec^2 y\frac{dy}{dx}+2x\tan y=x^3$$

Let $v=\tan y$ then $\frac{dv}{dx}=\sec^2 y\frac{dy}{dx}$ ______ (1)

$\therefore$ (1) becomes $\frac{dv}{dx}+2vx=x^3$

$$P=2x;\ Q=x^3$$

$$\int Pdx=\int 2xdx=x^2 \text{ and } e^{\int Pdx}=e^{x^2}$$

Solution is $ve^{\int Pdx}=\int Qe^{\int Pdx}dx+c$

$$\therefore ve^{x^2}=\int x^3e^{x^2}dx+c=\int xx^2e^{x^2}dx+c$$

Put $t=x^2;\ dt=2x\,dx$

$$\therefore ve^{x^2}=\frac{1}{2}\int te^t dt$$

$$ve^{x^2} = \frac{1}{2}\left(te^t - e^t\right) + c = \frac{1}{2}\left(x^2e^{x^2} - e^{x^2}\right) + c$$

$$v = \frac{1}{2}\left(x^2 - 1\right) + ce^{-x^2}$$

The solution is

$$\tan y = \frac{1}{2}\left(x^2 - 1\right) + ce^{-x^2}$$

2. Solve $\cos x \frac{dy}{dx} - y\sin x = y^3 \cos^2 x$

Solution:

Dividing by y^3, we get $\frac{1}{y^3}\frac{dy}{dx}\cos x - \frac{1}{y^2}\sin x = \cos^2 x$

Dividing by $\cos x$ we get $\frac{dy}{dx}\frac{1}{y^3} - \frac{1}{y^2}\tan x = \cos x$

Substituting $v = \frac{1}{y^2}$ gives $\frac{dv}{dx} = \frac{-2}{y^3}\frac{dy}{dx}$

Now the above equation becomes $-\frac{1}{2}\frac{dv}{dx} - v\tan x = \cos x$

or $\frac{dv}{dx} + 2v\tan x = -2\cos x$

$P = 2\tan x,\ Q = -2\cos x$

$\int P dx = 2\int \tan x\, dx = 2\log(\sec x)$

$e^{\int P dx} = e^{2(\log \sec x)} = \sec^2 x$

$v\sec^2 x = -\int 2\cos x \sec^2 x\, dx$

$= -2\int \sec x\, dx$

$= -2\log(\sec x + \tan x) + c$

$\frac{\sec^2 x}{y^2} = c - 2\log(\sec x + \tan x)$ is the solution

TEACHER'S CARE ACADEMY
38/23, Vaigundaperumal Koil Street, Kanchipuram-1
Mobile : 95665 35080, 97862 69980
Help Line Numbers : 9360268118,7639967359

UG TRB
MATHEMATICS
2023-2024

UNIT IV
Vector Calculus
& Fourier Series, Fourier Transforms

Your Success is Our Goal....

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UNIT – II – DIFFERENTIATION

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TNPSC-TRB- COMPUTER SCIENCE -TET COACHING CENTER

HEAD OFFICE: NO. 38/23, VAIGUNDA PERUMAL KOIL, SANNATHI STREET, KANCHIPURAM – 1. CELL: 9566535080

B.Off 2: 65C, Thillai Ngr(West), 4th Cross St, Trichy – 620018

B.Off 3:266-C, Advaitha Ashram Road,(Opp to New Bus Stand),Salem-636 004.

Trichy: 76399 67359 Salem: 93602 68118

UG TRB – MATHEMATICS – 2023-24

UNIT - IV - VECTOR

4.1. VECTOR DIFFERENTIATION

- Vector function: If for each value of scalar variable u there corresponds a vector $\vec{f}$, then $\vec{f}$ is said to be a vector function of the scalar variable n. It written as $\vec{f}(u)$

Constant Function:

- A vector whose magnitude is constant and whose direction is in a fixed direction is a constant vector.

  Note:

  - A scalar function has only a magnitude while a vector function has both magnitude and direction.

Derivative of a Vector Function

- It is denoted by $\Delta \vec{f}$, then

$$\frac{d\bar{f}}{du} = \lim_{\Delta u \to 0} \frac{\Delta \bar{f}}{\Delta u}$$

4.2. VELOCITY OF A PARTICLE

- The displacement of the particle in time interval is Δt. So the rate of displacement of the particle at P is

$$\lim_{\Delta t \to 0} \frac{\Delta \vec{r}}{\Delta t} \text{ (or) } \frac{d\vec{r}}{dt}$$

- But the rate of displacement is the velocity of the particle. It is denoted by v.

(i.e.,) $\bar{v} = \frac{d\bar{r}}{dt}$

4.3. VECTOR VALUED FUNCTION AND SCALAR POTENTIAL

- Vector point Function: Suppose, in a physical situation for every point (x, y, z), there corresponds a vector.
- $f(x,y,z)i + g(x,y,z)j + h(x,y,z)k$, then this vector function is said to be a vector point function.

Scalar Point Function:

- In a physical situation, for every point (x, y, z), there corresponds a scalar $\phi(x, y, z)$. Then $\phi(x, y, z)$ is said to be a scalar point function.

Level Surfaces:

- If $\phi(x, y, z)$ is a scalar, then the equation $\phi(x, y, z) = c$, where c is a varying constant, represents surface called level surfaces. Thus, the value ϕ is a constant.

4.4.GRADIENT OF A SCALAR POINT FUNCTION

- If ϕ is a scalar point function, then $\bar{i}\frac{\partial \phi}{\partial x} + \bar{j}\frac{\partial \phi}{\partial y} + \bar{k}\frac{\partial \phi}{\partial z}$ is called the gradient of ϕ at (x, y, z)

Notation:

- Gradient of ϕ is denoted by grad ϕ or $\nabla\phi$ where ∇ is the operator which stands for $\bar{i}\frac{\partial}{\partial x} + \bar{j}\frac{\partial}{\partial y} + \bar{k}\frac{\partial}{\partial z}$
- Thus ϕ is a scalar but $\nabla\phi$ is a vector

4.5.DIVERGENCE AND CURL OF A VECTOR POINT FUNCTION

Divergence:

- The scalar point functions

$$\frac{\partial v_1}{\partial x}+\frac{\partial v_2}{\partial y}+\frac{\partial v_3}{\partial z}$$

- is called the divergence of the vector point function $V_1 i + V_2 j + V_3 k$

Notation:

- Divergence of V or $div\ V$ or $\nabla \cdot \dot{V}$

$$\nabla \cdot V = \left(i\frac{\partial}{\partial x} + j\frac{\partial}{\partial y} + k\frac{\partial}{\partial z} \right) \cdot \left(V_1 i + V_2 j + V_3 k \right)$$

$$= \frac{\partial V_1}{\partial x}+\frac{\partial V_2}{\partial y}+\frac{\partial V_3}{\partial z}$$

Curl:

- The vector point function

$$i\left(\frac{\partial V_3}{\partial y}-\frac{\partial V_2}{\partial z}\right)+j\left(\frac{\partial V_1}{\partial z}-\frac{\partial V_3}{\partial x}\right)+k\left(\frac{\partial V_2}{\partial x}-\frac{\partial V_1}{\partial y}\right)$$

- is called the curl of the vector point function $v_1\bar{i}+v_2\bar{j}+v_3\bar{k}$

Notation:

- Curl of V is curl V (or) $\nabla \times \dot{V}$

$$\nabla \times \bar{V} = \begin{vmatrix} \bar{i} & \bar{j} & \bar{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ V_1 & V_2 & V_3 \end{vmatrix}$$

Particular cases of $\nabla \times \bar{V}$ and $\nabla \times \bar{V}$

Solenoidal Vector:

- If $\nabla \cdot \bar{V} = 0$, then V said to be solenoidal.

Irrotational Vector:

- If $\nabla \cdot \bar{V} = 0$, then V is said to be irrotational.

4.6. DIRECTIONAL DERIVATIVE OF A SCALAR POINT FUNCTION

- Suppose $\phi(x,y,z)$ is a scalar point function and $\phi(p)$ is the value of ϕ at P. If P' is any point, then $\lim_{p'\to p}\frac{\phi(p')-\phi(p)}{pp'}$.
- is called the directional derivative of ϕ. The directional derivative is a scalar. Actually it is the rate of change of ϕ in the given direction.

4.7. UNIT NORMAL

- This directional derivative of ϕ in the direction specified by the unit vector $\hat{e}$ having direction cosines l, m, n is $(\nabla\phi)\cdot\hat{e}$.
- The unit vector normal to the surface $\phi(x,y,z)=c$ at any point (x,y,z) is

$$\hat{n}=\frac{\nabla\phi}{|\nabla\phi|}$$

4.8. LAPLIACIAN OPERATOR

- The operator ∇^2 defined by

$$\nabla^2=\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}+\frac{\partial^2}{\partial z^2}$$

- is called Laplacian differential operator, when it operator on a scalar pint function, it results in a scalar. When it operates on a vector point function, it results in a vector.

4.9. HARMONIC FUNCTION:

- For every scalar point function, having continuous second partials, $\nabla\times(\nabla\phi)=0$.
- In words curl of a gradient vanishes.
- For every vector point function $\bar{A}$, having continuous second partials,

  $\nabla\cdot(\nabla\times\bar{A})=0$. In words

- Divergence of a curl vanishes.

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4.1 to 4.9 – EXAMPLES

PROBLEMS

1. A particle moves along the curve $x = e^{-t}, y = 2\cos 3t, z = 2\sin 3t$. Determine the velocity and acceleration at any time t and their magnitudes at t = 0

Soln:

$$r = xi + yj + zk$$

$$= e^{-t} i + 2\cos 3t\, j + 2\sin 3t\, k$$

$$\frac{d\vec{r}}{dt} = -e^{-t}\vec{i} - 6\sin 3t j + 6\cos 3t\vec{k}$$

$$\frac{d\vec{r}}{dt}_{(t=0)} = -\vec{i} + 6k \quad \text{(velocity at time)}$$

Magnitude of the velocity $= \sqrt{1+36} = \sqrt{37}$

$$\bar{a} = \frac{d^2\bar{r}}{dt^2} = e^{-t}\bar{i} - 18\cos 3t\bar{j} - 18\sin 3t\bar{k}$$

$$\frac{d^2\vec{r}}{dt^2} = \vec{i} - 18\vec{j} \quad = \text{acceleration at time t} = 0$$

$$|\vec{a}| = \sqrt{1+324} = \sqrt{325} = 5\sqrt{13}$$

2. A particle moves along the curve $x = t^3 + 1, y = t^2, z = 2t + 5$ where t is the time. Find the components of its velocity and acceleration at t = 1 in the direction $i + \vec{j} + 3k$

Soln:

$$r = xi + yj + zk$$

$$= (t^3 + 1)\vec{i} + t^2\vec{j} + (2t + 5)\vec{k}$$

$$\vec{v} = \frac{d\vec{r}}{dt} = 3t^2\vec{i} + 2t\vec{j} + 2\vec{k}$$

Velocity at t = 1 is $V = 3i + 2\vec{j} + 2k$

$$a = \frac{dr^2}{dt^2} = 6ti + 2j$$

Acceleration at t = 1 is $\bar{a} = 6\bar{i} + 2\bar{j}$

$$b = i + j + 3k$$

$$= \frac{\bar{v} \cdot \bar{b}}{|\bar{b}|}$$

$$= (3i + 2j + 2k)\frac{i + j + 3k}{\sqrt{11}}$$

$$= \frac{3+2+6}{\sqrt{11}} = \frac{11}{\sqrt{11}} = \sqrt{11}$$

Acceleration component in the direction of $b \cdot at \ t = 1$

$$= \frac{\bar{a} \cdot \bar{b}}{|\bar{b}|}$$

$$= (6\bar{i} + 2\bar{j})\frac{(\bar{i} + \bar{j} + 3\bar{k})}{\sqrt{11}}$$

$$= \frac{6+2}{\sqrt{11}}$$

$$= \frac{8}{\sqrt{11}}$$

3. If $\phi(x,y,z) = x^2y + y^2x + z^2$ find $\nabla\phi$ at the point (1, 1, 1)

Soln:

$$\phi(x,y,z) = x^2y + y^2x + z^2$$

$$\frac{\partial\phi}{\partial x} = 2xy + y^2$$

$$\frac{\partial\phi}{\partial y} = x^2 + 2xy$$

$$\frac{\partial\phi}{\partial z} = 2z$$

$$\nabla\phi = \bar{i}\frac{\partial\phi}{\partial x} + \bar{j}\frac{\partial\phi}{\partial y} + \bar{k}\frac{\partial\phi}{\partial z}$$

$$= \left(2xy + y^2\right)i + \left(x^2 + 2xy\right)\dot{j} + 2z\dot{k}$$

$$\nabla\phi_{(1,1,1)} = 3\dot{i} + 3\dot{j} + 2\dot{k}$$

4. If $r = xi + yj + zk$ and $r = |\dot{r}|$ prove that

(i) $\nabla_r = \frac{1}{r}\dot{r}$ **(ii)** $\nabla\left(\frac{1}{r}\right) = \frac{-\dot{r}}{r^3}$

(iii) $\nabla r^n = nr^{n-2}\dot{r}$ **(iv)** $\nabla f(r) = f'(r)\frac{\dot{r}}{r} = f'(r)\nabla_r$

(v) $\nabla(\log r) = \frac{\dot{r}}{r^2}$ **(vi)** $\nabla f(r) \times \dot{r} = 0$

Soln:

(i) $r = x\hat{i} + y\hat{j} + z\hat{k}$ $\therefore |\dot{r}| = \sqrt{x^2 + y^2 + z^2} = r$

$$r^2 = x^2 + y^2 + z^2$$

Differentiating partially with respect to x, we get

$$2r = \frac{\partial r}{\partial x} = 2x \quad \therefore \frac{\partial r}{\partial x} = \frac{x}{r}$$

$$\frac{\partial r}{\partial y} = \frac{y}{r} \text{ and } \frac{\partial r}{\partial z} = \frac{z}{r}$$

$$\nabla_r = \bar{i}\frac{\partial r}{\partial x} + \bar{j}\frac{\partial r}{\partial y} + \bar{k}\frac{\partial r}{\partial z}$$

$$= \frac{x\bar{i} + y\bar{j} + z\bar{k}}{r}$$

$$= \frac{\dot{r}}{r}$$

(ii) $\nabla\left(\frac{1}{r}\right) = \dot{i}\frac{\partial}{\partial x}\left(\frac{1}{r}\right) + j\frac{\partial}{\partial y}\left(\frac{1}{r}\right) + \dot{k}\frac{\partial}{\partial z}\left(\frac{1}{r}\right)$

$$= \frac{-1}{r^2}\left[\dot{i}\frac{\partial r}{\partial x} + j\frac{\partial r}{\partial y} + \dot{k}\frac{\partial r}{\partial z}\right]$$

$$= \frac{-1}{r^2}\left(i\frac{x}{r}+j\frac{y}{r}+k\frac{z}{r}\right)$$

$$= \frac{-1}{r^2}\left(\frac{\dot{r}}{r}\right) = \frac{-r}{r^3}$$

(iii) $\nabla r^n = \bar{i}\frac{\partial}{\partial x}(r^n)+\bar{j}\frac{\partial}{\partial y}(r^n)+\bar{k}\frac{\partial}{\partial z}(r^n)$

$$= nr^{n-1}\left[\dot{i}\frac{\partial r}{\partial x}+j\frac{\partial r}{\partial y}+\dot{k}\frac{\partial r}{\partial z}\right]$$

$$= nr^{n-1}\left[\frac{x\bar{i}+y\bar{j}+z\bar{k}}{r}\right]$$

$$= nr^{n-2}\dot{r}$$

(iv) $\nabla f(r) = \bar{i}\frac{\partial}{\partial x}f(r)+\bar{j}\frac{\partial}{\partial y}f(r)+\bar{k}\frac{\partial}{\partial z}f(r)$

$$= f'(r)\left[\dot{i}\frac{\partial r}{\partial x}+j\frac{\partial r}{\partial y}+\dot{k}\frac{\partial r}{\partial z}\right]$$

$$= f'(r)\frac{(x\dot{i}+yj+z\dot{k})}{r}$$

$$= f'(r)\frac{\dot{r}}{r}$$

$$= f'(r)\nabla(r)$$

(v) $\nabla(\log r) = \bar{i}\frac{\partial}{\partial x}(\log r)+\bar{j}\frac{\partial}{\partial y}(\log r)+\bar{k}\frac{\partial r}{\partial z}(\log r)$

$$= \frac{1}{r}\left[\dot{i}\frac{\partial r}{\partial x}+j\frac{\partial r}{\partial y}+\dot{k}\frac{\partial r}{\partial z}\right]$$

$$= \frac{1}{r}\frac{x\hat{i}+y\hat{j}+z\hat{k}}{r}$$

$$= \frac{\dot{r}}{r^2}$$

(vi) $\nabla f(r)\times\dot{r}$

$$\nabla f(r) = \frac{f'(r)}{r}\bar{r}$$

$$\nabla f(r) \times \bar{r} = \frac{f'(r)}{r}\bar{r} \times \bar{r} = 0 \text{ since } r \times r = 0$$

5. If $u = x + y + z$

$v = x^2 + y^2 + z^2$

$w = yz + zx + xy$ prove that grad u × grad v × grad w = 0

Soln:

$$\text{grad u} = \nabla u = \bar{i}\frac{\partial u}{\partial x} + \bar{j}\frac{\partial u}{\partial y} + \bar{k}\frac{\partial u}{\partial z}$$

$$\text{grad v} = \nabla u = \bar{i}\frac{\partial v}{\partial x} + \bar{j}\frac{\partial v}{\partial y} + \bar{k}\frac{\partial v}{\partial z}$$

$$= 2\left(x\dot{i} + y\dot{j} + z\dot{k}\right)$$

$$\text{grad w} = \bar{i}\frac{\partial w}{\partial x} + \bar{j}\frac{\partial w}{\partial y} + \bar{k}\frac{\partial w}{\partial z}$$

$$= (y+z)\dot{i} + (z+x)j + (x+y)\dot{k}$$

$$(grad\ u)(grad\ v \times grad\ w) = \begin{vmatrix} 1 & 1 & 1 \\ 2x & 2y & 2z \\ y+z & z+x & x+y \end{vmatrix}$$

$$= 2\begin{vmatrix} 1 & 1 & 1 \\ x & y & z \\ y+z & z+x & x+y \end{vmatrix}$$

$$= 2\begin{vmatrix} 1 & 1 & 1 \\ x & y & z \\ x+y+z & x+y+z & x+y+z \end{vmatrix}$$

$$= 2(x+y+z)\begin{vmatrix} 1 & 1 & 1 \\ x & y & z \\ 1 & 1 & 1 \end{vmatrix}$$

= 0 since two rows are identical

6. Find the directional derivative of $xyz - xy^2z^3$ at the point (1, 2, - 1) in the direction of the vector $i - j - 3k$.

Soln:

$$\phi = xyz - xy^2z^3$$

$$\nabla\phi = \bar{i}\frac{\partial\phi}{\partial x} + \bar{j}\frac{\partial\phi}{\partial y} + \bar{k}\frac{\partial\phi}{\partial z}$$

$$= \left(yz - y^2z^3\right)\dot{i} + \left(xz - 2xyz^3\right)j + \left(xy - 3xy^2z^2\right)\dot{k}$$

$$\dot{n} = \frac{\dot{i} - \dot{j} - 3\dot{k}}{\sqrt{11}}$$

$\frac{d\phi}{dn} = \nabla\phi\cdot\dot{n} =$ directional derivative of ϕ in the direction of the vector $i - \dot{j} - 3k$

$$= \frac{\left[\left(yz - y^2z^3\right) - \left(xz - 2xyz^3\right) - 3\left(xy - 3xy^2z^2\right)\right]}{\sqrt{11}}$$

$$\nabla\phi\dot{n}_{(1,2,-1)} = \frac{(-2+4)-(-1+4)-3(2-12)}{\sqrt{11}} = \frac{29}{\sqrt{11}}$$

7. Show that (i) grad $(\dot{r}\cdot\dot{a}) = a$ (ii) grad $\left[\dot{r},\dot{a},\dot{b}\right] = \dot{a}\times b$ where a and $\dot{b}$ are constant vectors and $r = xi + yj + zk$

Soln:

Let $\dot{a} = a_1\dot{i} + a_2\dot{j} + a_3\dot{k}$

$\dot{b} = b_1\dot{i} + b_2 j + b_3 k$

(i) $\bar{a}\cdot\bar{r} = a_1x + a_2y + a_3z$

grad $(\dot{a}\cdot r) = \left[\dot{i}\frac{\partial}{\partial x} + j\frac{\partial}{\partial y} + \dot{k}\frac{\partial}{\partial z}\right](a_1x + a_2y + a_3z)$

$= a_1\dot{i} + a_2j + a_3\dot{k} = \dot{a}$ ___ (1)

(ii) grad $\left[\dot{r},\dot{a},\dot{b}\right] = grad\left(\dot{r}\cdot\dot{a}\times\dot{b}\right)$

$= \dot{a}\times\dot{b}$ using (1)

8. Find the unit vector normal to the surface $x^2+3y^2+2z^2=6$ **at the point (2, 0, 1)**

Soln:

$$\phi = x^2+3y^2+2z^2$$

$$\nabla\phi = 2xi + 6yj + 4zk$$

$$\nabla\phi_{(2,0,1)} = 4\hat{i}+4\hat{k}$$

$$\hat{n} = \frac{\nabla\phi}{|\nabla\phi|} = \frac{4\dot{i}+4\dot{k}}{4\sqrt{2}} = \frac{i+\dot{k}}{\sqrt{2}}$$

the unit normal vector at the point (2, 0, 1) to the given surface

$$= \frac{1}{\sqrt{2}}\left(\bar{i}+\bar{k}\right)$$

9. Find the maximum value of the directional derivative of the function $\phi = 2x^2+3y^2+5z^2$ **at the point (1, 1, - 4)**

Soln:

$$\phi = 2x^2+3y^2+5z^2$$

$$\nabla\phi = \bar{i}\frac{\partial\phi}{\partial x} + \bar{j}\frac{\partial\phi}{\partial y} + \bar{k}\frac{\partial\phi}{\partial z}$$

$$= 4xi + 6yj + 10zk$$

$$\nabla\phi_{(1,1,-4)} = 4\dot{i} + 6j - 40\dot{k}$$

Maximum value of the directional derivative at the point (1, 1, - 4)

$$= \sqrt{16+36+1600} = \sqrt{1652}$$

10. Find the angle between the normal to the surface $xy-z^2=0$ **at the point (1, 4, - 2) and** $(-3,-3,3)$

Soln:

$$\phi = xy - z^2$$

$$\nabla\phi = \bar{i}\frac{\partial\phi}{\partial x} + \bar{j}\frac{\partial\phi}{\partial y} + \bar{k}\frac{\partial\phi}{\partial z}$$

$$= yi + xj - 2zk$$

$$\nabla\phi_{(1,4,-2)} = 4i + j + 4k$$

$$\nabla x_{(-3,-3,3)} = -3i - 3j - 6k$$

Unit normal vector to the surface at the point (1, 4, - 2) is

$$\dot{n}_1 = \frac{\nabla\phi}{|\nabla\phi|} = \frac{4i + j + 4k}{\sqrt{33}}$$

Unit normal vector at the point (-3, -3, 3) is

$$\dot{n}_2 = \frac{-3i - 3j - 3k}{\sqrt{9+9+36}} = \frac{-3i - 3j - 3k}{\sqrt{54}}$$

If θ is the angle between the normal then

$$\cos\theta = \bar{n}_1\bar{n}_2 = \frac{-12-3-24}{\sqrt{33}\sqrt{54}} = \frac{-39}{9\sqrt{22}} = \frac{-3}{3\sqrt{22}}$$

$$\therefore \quad \theta = \cos^{-1}\left(\frac{-13}{3\sqrt{22}}\right)$$

11. Show that the surface $5x^2 - 2yz - 9x = 0$ **and** $4x^2y + z^3 - 4 = 0$ **are orthogonal at** $(1,-1,-2)$

Soln:

Let $\phi_1 = 5x^2 - 2yz - 9x$

$$\phi_2 = 4x^2y + z^3 - 4$$

$$\nabla\phi_1(10x-9) = i - 2zj - 2yk$$

$$\nabla\phi_1(1,-1,2) = \hat{i} - 4\hat{j} + 2\hat{k}$$

$$\nabla\phi_2 = 8xyi + 4x^2j + 3z^2k$$

$$\nabla\phi_2(1,-1,2) = -8\hat{i} + 4\hat{j} + 12\hat{k}$$

$$\nabla\phi_1 \cdot \nabla\phi_2 = -8 - 16 + 24 = 0$$

$\therefore$ The surface are orthogonal at the point $(1,-1,2)$

12. Find ϕ if $\nabla\phi = (6xy+z^3)\vec{i} + (3x^2 - z)j + (3xz^2 - y)\vec{k}$

Soln:

$$\nabla\phi = \bar{i}\frac{\partial\phi}{\partial x} + \bar{j}\frac{\partial\phi}{\partial y} + \bar{k}\frac{\partial\phi}{\partial z} \qquad \text{\_\_ (1)}$$

Also $\nabla\phi = (6xy+z^3)\vec{i} + (3x^2 - z)j + (3xz^2 - y)\vec{k}$ __ (2)

Comparing (1) and (2), we get

$$\frac{\partial\phi}{\partial x} = 6xy + z^3 \qquad \text{\_\_ (1)}$$

$$\frac{\partial\phi}{\partial y} = 3x^2 - z \qquad \text{\_\_ (2)}$$

$$\frac{\partial\phi}{\partial z} = 3xz^2 - y \qquad \text{\_\_ (3)}$$

Integrating (1), (2) and (3), w.r.t. x,y, z respectively we get,

$$\phi = 3x^2y + xz^3 + f_1(y,z) \qquad \text{\_\_ (4)}$$

$$\phi = 3x^2y - yz + f_2(x,z) \qquad \text{\_\_ (5)}$$

$$\phi = xz^3 - yz + f_3(x,y) \qquad \text{\_\_ (6)}$$

From (4), (5) and (6) $\phi = 3x^2 + xz^3 - yz + c$ where c is an arbitrary constant.

13. Find ϕ if $\nabla\phi = (y+\sin z)\vec{i} + xj + x\cos z\vec{k}$

Soln:

$$\nabla\phi = \bar{i}\frac{\partial\phi}{\partial x} + \bar{j}\frac{\partial\phi}{\partial y} + \bar{k}\frac{\partial\phi}{\partial z} \qquad \text{\_\_ (1)}$$

$$= (y+\sin z)\vec{i} + xj + x\cos z\vec{k} \qquad \text{\_\_ (2)}$$

Comparing (1) and (2) we get,

TEACHER'S CARE ACADEMY
38/23, Vaigundaperumal Koil Street, Kanchipuram-1
Mobile : 95665 35080, 97862 69980
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UG TRB
MATHEMATICS
2023-2024

UNIT V
Algebraic Structures

Your Success is Our Goal....

UG TRB 2022-23 MATHEMATICS

UNIT - V – ALGEBRAIC STRUCTURE

INDEX

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HEAD OFFICE: NO. 38/23, VAIGUNDA PERUMAL KOIL, SANNATHI STREET, KANCHIPURAM – 1. CELL: 9566535080

B.Off 2: 65C, Thillai Ngr(West), 4th Cross St, Trichy – 620018

B.Off 3: Vijiyaraghavachariar Memorial Hall(Opp to Sundar Lodge), Salem

Trichy: 76399 67359 Salem: 93602 68118

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UNIT - V

ALGEBRAIC STRUCTURE

5.1 GROUPS

5.1.1. BINARY OPERATIONS:

Binary operation means "way of putting two things together.

Eg. The set of all natural number under addition

Closure Property Under".":

Let A be a set with binary operation ".". Thus operation is said to be closure if $a,b \in A \Rightarrow a \cdot b \in A$

Associative Property Under ".":

Let A be a set with binary operation ".". Thus operation is said to be associative

if $a,b,c \in A \Rightarrow (a \cdot b) \cdot c = a \cdot (b \cdot c)$

Identity Element Under".":

Let a be a set with binary operation ".". An element e is said to be identify element if

$a \cdot e = e \cdot a = a \quad \forall a \in A$

Inverse Element Under ".":

Let A be a set with binary operation '.'. Suppose that A contains an identity element e.

If $a \in A$ and if $a^{-1} \in A \ni a \cdot a^{-1} = a^{-1} \cdot a = e$

Where a^{-1} is called inverse element of A.

5.1.2. GROUP UNDER ".":

A non-empty set G with binary operation "." is called a group if it satisfies the following conditions.

(i)Closure:

If a, b, $\in G \Rightarrow a \cdot b \in G \; \forall \; a, b \in G$

(ii) Associative:

$a \cdot (b \cdot c) = (a \cdot b) \cdot c \;\; \forall a, b, c \in G$ -

(iii) Identify:

If an element $e \; e \; G \ni a \cdot e = e \cdot a = a \;\; \forall a \; e \; G$

(iv) Inverse:

$\forall \; a \in G$ if an element $a^{-1} \in G$

$\Rightarrow a \cdot \leq a^{-1} = e = a^{-1} \cdot a$

Where a^{-1} is the inverse element of G.

5.1.3. GROUP UNDER "+":

A non- empty set G with binary operations '+' is called a group if it satisfies the following conditions.

(i) closure:

If $a, b \in G \Rightarrow a + b, \in G \quad \forall a, b \in G$

(ii) Associative:

$a + (b + c) = (a + b) + c \quad \forall a, b, c \in G$

(iii) Identify:

If an element e $(a + e) = e + a = a \quad \forall a \in G$

(iv) Inverse

$\forall a \in G$, if an element $a^{-1} \in G$

$a + a^{-1} = e = a^{-1} + a$

Where a^{-1} is the inverse element of G of G

Commutative Property:

Let A be a set with binary operation "." If a.b = b.a $\forall a,b \in A$, then A satisfies commutative property.

5.1.4. ABELIAN GROUP:

If (G, .) is a group than (G, .) is abelian, if the group of the operation "." is commutative.

(i.e.,) a . b = b. a $\forall\ a,b \in G$

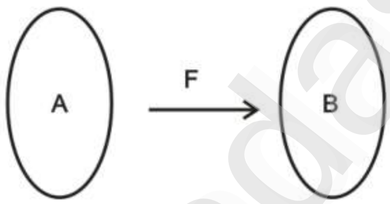
5.1.5. NON-ABELIAN GROUP:

A group which is not abelian is called non-abelian group.

5.1.6. TYPES OF FUNCTIONS

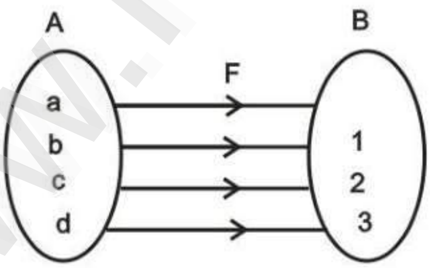
One – To – One Function:

A function $f : A \to B$ is said to be a one-to-one function if distinct element of A have distinct image of B.



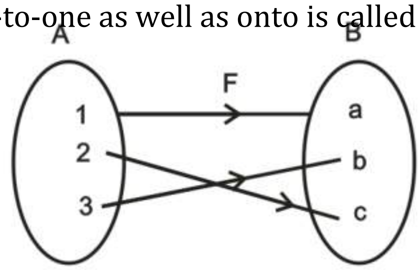
Onto function:

A function $f : A \to B$ is said to be onto if every element of B has atleast one – preimage in A.



Bijective Function:

A function which is one-to-one as well as onto is called bijective function.



5.1.7. ORDER OF A GROUP:

The number of elements in a group G is called order of a group, it is denoted by O(G).

Eg. $G = \{1, -1, i, -i\}$

$O(G) = 4$

5.1.8. FINITE GROUP:

A group G is called finite if it consists of only finite number of elements and we say that the group is of finite order.

PROPLEMS:

1. Prove that (S, .) is a group where S is the set of all 4th roots of unity.

Solution:

Let $S = \{1, -1, i, -i\}$

.	1	- 1	i	- i
1	1	- 1	i	- i
- 1	- 1	1	- i	i
i	i	- i	- 1	1
- i	- i	i	1	- 1

Closure:

Let $1, i \in S$

$\Rightarrow 1 \cdot i \in S$

$\therefore (S, \cdot)$ satisfies closure property.

Associative:

Let $1, -1, i \in S$

$1 \cdot (-1, i) = (1 \cdot (-1)) \cdot i$

$1 \cdot (-i) = (-1) \cdot i$

$-i = -i$

$\therefore (S, \cdot)$ satisfies associative property.

Identity:

$a \cdot e = e \cdot a = a$

$1 \cdot i = i \cdot 1 = i$

$1.(-i) = (-i) \cdot 1 = -i$

$1 \cdot 1 = 1 \cdot 1 = 1$

$1 \cdot (-1) = (-1) \cdot 1 = -1$

$\therefore$ 1 is the identity element of S.

Inverse Law:

$a \cdot a^{-1} = a^{-1} \cdot a = e$

Inverse of 1 = 1 Inverse of i = - i

Inverse of – 1 = 1 Inverse of – i = i

$\therefore$ Inverse exists

$\therefore (S, .)$ is as group.

2. Find the residue group of integers under addition modulo 5.

Solution:

Let $z_5 = \{[0],[1],[2],[3],[4]\}$

$\oplus_5$	[0]	[1]	[2]	[3]	[4]
[0]	[0]	[1]	[2]	[3]	[4]
[1]	[1]	[2]	[3]	[4]	[0]
[2]	[2]	[3]	[4]	[0]	[1]
[3]	[3]	[4]	[1]	[1]	[2]
[4]	[4]	[0]	[1]	[2]	[3]

(i) Closure

Let $[0],[1] \in z_5$

$\Rightarrow [0] \oplus_5 [1] = [1] \in z_5$

$\therefore (z_5, \oplus_5)$ satisfies closure property.

(ii) Associative

Let $[2]\ [3]\ [4] \in z_5$

$$[2]\oplus_5([3]+[4]) = ([2]\oplus_5[3])\oplus_5[4]$$

$$[2]\oplus_5[2] = [0]\oplus_5[4]$$

$$[4] = [4]$$

$\therefore (z_5, \oplus_5)$ satisfies property.

Identity Law:

$$a\oplus e = e\oplus a = a$$

$$[0]\oplus_5[0] = [0]$$

$$[1]\oplus_5[0] = [1]$$

$$[2]\oplus_5[0] = [2]$$

$$[3]\oplus_5[0] = [3]$$

$$[4]\oplus_5[0] = [4]$$

$\therefore [0]$ is the identity element.

Inverse Law:

$$a\oplus a^{-1} = a^{-1}\oplus a = e$$

$$[0]\oplus_5[0] = [0]$$

$$[1]\oplus_5[4] = [0]$$

$$[2]\oplus_5[3] = [0]$$

$$[3]\oplus_5[2] = [0]$$

$$[4]\oplus_5[1] = [0]$$

$\therefore$ Inverse exists

$\therefore (z_5, \oplus_5)$ is a group.

Associate property:

For any $a,b\in \Rightarrow a*(b*c)*c$

Here, for any $a,b\in \Rightarrow Z_5 \Rightarrow a*(b*c)=(a*c)=(a*b)*c$

Let us take [1], [3], [4] $\in Z_5$

Consider

$$[1]\oplus_5([3]\oplus_5[4])=[1]\oplus_5[2]=[3]$$

Consider

$$([1]\oplus_5[3])\oplus_5[4]=[4]\oplus_5[4]=[3]$$

$\therefore (Z,\oplus_5)$ satisfies associative property

Identify Property:

In the table, [0] is an identity element in Z_5

$$[1]\oplus_5[0]=[0]$$

$$[1]\oplus_5[0]=[1]$$

$$[2]\oplus_5[0]=[2]$$

$$[3]\oplus_5[0]=[3]$$

$$[4]\oplus_5[0]=[4]$$

Inverse Property:

$$[1]\oplus_5[0]=[0]$$

$$[1]\oplus_5[4]=[1]$$

$$[2]\oplus_5[3]=[0]$$

$$[3]\oplus_5[2]=[0]$$

$$[4]\oplus_5[1]=[0]$$

Inverse element is exist.

Hence, $(Z,\oplus_5)$ is a group.

Problem – 3:

Find the residue class of integers under addition modulo 7 and prove that it is a group.

Solution:

Let $Z_7 = \{0,1,2,3,4,5,6\}$ be the set of all residue class of integer for Z_7 under addition.

To prove: $= (Z, \oplus_7)$ is a group.

Closure property:

$\oplus_7$	[0]	[1]	[2]	[3]	[4]	[5]	[6]
[0]	[0]	[1]	[2]	[3]	[4]	[5]	[6]
[1]	[1]	[2]	[3]	[4]	[5]	[6]	[0]
[2]	[2]	[3]	[4]	[5]	[6]	[0]	[1]
[3]	[3]	[4]	[5]	[6]	[0]	[1]	[2]
[4]	[4]	[5]	[6]	[0]	[1]	[2]	[3]
[5]	[5]	[6]	[0]	[1]	[2]	[3]	[4]
[6]	[6]	[0]	[1]	[2]	[3]	[4]	[5]

for any $a, b \in G \Rightarrow a * b \in G$

In the above table, any two elements in Z_7 their addition is in Z_7 .

Associative Property:

for any $a, b \in G \Rightarrow a * b \in G$

Here, for any $a, b \in Z_7 \Rightarrow a*(b*c) = (a*c)*c$

Let us take [1], [3], [4] $\in Z_7$

Consider

$$[1] \oplus_7 ([3] \oplus_7 [4]) = [1] \oplus_7 [0] = [1]$$

Consider

$$([1] \oplus_7 [3]) \oplus_7 [4] = [4] \oplus_7 [4] = [1]$$

Identity Property:

In the table, [0] is an identity element in Z_7

$[0] \oplus_7 [0] = [0]$ $\quad$ $[1] \oplus_7 [0] = [1]$

$[2] \oplus_7 [0] = [2]$ $\quad$ $[3] \oplus_7 [0] = [3]$

$[4] \oplus_7 [0] = [4]$ $\quad$ $[5] \oplus_7 [0] = [5]$

$[6] \oplus_7 [0] = [6]$

Inverse Property:

$[0] \oplus_7 [0] = [0]$ $\quad$ $[1] \oplus_7 [6] = [0]$

$[2] \oplus_7 [5] = [0]$ $\quad$ $[3] \oplus_7 [4] = [0]$

$[4] \oplus_7 [3] = [0]$ $\quad$ $[5] \oplus_7 [2] = [0]$

$[6] \oplus_7 [1] = [0]$

Inverse element is exist for each element of Z_7 and in Z_7

Hence $(Z, \oplus_7)$ is a group.

Problem – 4:

Find the residue class of integers under multiplication modulo 7 and prove that it is a group.

Solution:

Let $Z_7 = \{1,2,3,4,5,6\}$ be the set of all residue class of integer for Z_7 under addition

To prove: $Z_7 = (Z, \oplus_7)$ is a group.

$\oplus_7$	[1]	[2]	[3]	[4]	[5]	[6]
[1]	[1]	[2]	[3]	[4]	[5]	[6]
[2]	[2]	[4]	[6]	[1]	[3]	[5]
[3]	[3]	[6]	[2]	[5]	[1]	[4]
[4]	[4]	[1]	[5]	[2]	[6]	[3]
[5]	[5]	[3]	[1]	[6]	[4]	[2]
[6]	[6]	[5]	[4]	[3]	[2]	[1]

Closure Property:

For any $a,b \in G \Rightarrow a*b \in G$

In the above table, any two elements in Z_7 their addition is in Z_7.

Associative Property:

For any $a,b \in G \Rightarrow a*(b*c)=(a*c)=(a*c)*c$

Here, for any $a,b \in Z_7 \Rightarrow a*(b*c)*c$

Let us take, $[1],[3],[4] \in Z_7$

Consider

$$[1]\oplus_7([3]\oplus_7[4])=[1]\times_7[5]=[5]$$

Consider $([1]\oplus_7[3])\oplus_7[4]=[3]\oplus_7[4]=[5]$

$\therefore$ $(Z,\oplus_7)$ satisfies associative property

Identity property:

In the table, [1] is an identity element in Z_7

$[1]\oplus_7[1]=[1]$ $[2]\oplus_7[1]=[2]$

$[3]\oplus_7[3]=[3]$ $[4]\oplus_7[1]=[4]$

$[5]\oplus_7[5]=[5]$ $[6]\oplus_7[1]=[6]$

Inverse property:

$[1]\oplus_7[1]=[1]$ $[2]\oplus_7[4]=[1]$

$[3]\oplus_7[5]=[1]$ $[4]\oplus_7[2]=[1]$

$[5]\oplus_7[3]=[1]$ $[6]\oplus_7[6]=[1]$

Inverse element is exist for each element of Z_7 and in Z_7

Hence $(Z,\oplus_7)$ is a group.

Problem – 5:

Prove that $(S,.)$ where S is the set of all fourth roots of unity is a group

Solutions:

Let S be set of al fourth root of unity

(i.e.,) $S=\{1,-1,i,-1\}$

	1	– 1	i	$-i$
1	1	– 1	i	$-i$
– 1	– 1	1	$-i$	i
i	i	$-i$	-1	1
$-i$	$-i$	i	1	– 1

Closure property:

for any $a,b\in G \Rightarrow a*b\in G$

In the above table, any two elements in S their addition is in S.

Associative property:

for any $a,b\in G \Rightarrow a*b\in G$

Let us take, $1\cdot -1\cdot i\in S$

Consider,

$1\cdot(-1\cdot i)=1\cdot(-i)=-i$

Consider,

$\therefore(S_1.)$ satisfies associative property.

Identity Property:

Here, 1 is in identity element

1	.	1	=	1
– 1	.	1	=	– 1
i	.	1	=	i
$-i$	.	1	=	$-i$

Inverse Property:

1	.	1	=	1
– 1	.	– 1	=	1
i	.	$-i$	=	1
i	.	$-i$	=	1
$-i$	.	i	=	1

Inverse element is exist for each element of S and in S

Hence $(S,.)$ is a group.

Problem – 6:

Show that the set of all rational numbers except 1 is a group under the binary operation *defined as $a*b=a+b-ab$ is group.

Solution:

Let $Q-\{1\}=\left\{\frac{p}{q}\middle| p,q\in N \;\&\; p,q\neq 0.1\right\}$

Closure property:

For any $a,b\in Q-\{1\}$

$\Rightarrow a*b=a+b-ab\in Q-\{1\}$

$\therefore a*b\in Q-\{1\}$

Associative Property:

For any $a,b,c\in Q-\{1\}$

$\Rightarrow a*(b*c)=(a*b)*b$ consider,

$\Rightarrow a*(b*c)=a*(b+c-bc)$

$=a+b+c-bc-a(b+c-bc)$

$=a+b+c-bc-ab-ac+abc$

Consider

$$(a*b)*c=(a+b-ab)*c$$

$$=a+b+c-ab-(a+b-ab)c$$

$$=a+b+c-ab-ac-bc+abc$$

$\therefore Q-\{1\}$ satisfies associative property.

Identity Property:

For any $a\in G,\exists e\in G$ such that $a*e=e*a=a$

$a*e=a$

$a+e-ae=a$

$e-ae=0$

$e(1-a)=0$

$e=0$

$\therefore 0\in Q-\{1\}$

Inverse Property:

for each $a\in G,\exists a^{-1}\in G$ such that $a*a^{-1}=a^{-1}*a=e$

Consider,

$$a*a^{-1}=0$$

$$a+a^{-1}-aa^{-1}=0$$

$$a^{-1}(1-a)=-a$$

$$a^{-1}=\frac{-a}{1-a}$$

$$a^{-1}=\frac{a}{a-1}$$

Inverse element exists in Q – {1} for each a

Hence set of all rational numbers except 1 is a group under the binary operations * defined as $a*b=a+b-ab$ is group.

Problem – 7:

Prove that $(Q,*)$ is group with respect to $*$ as defined as $a*b=\frac{ab}{2}\ \forall\ a,b\in Q$

Closure property:

For any $a,b\in Q$

$a*b=\frac{ab}{2}\in Q$

$\therefore a*b\in Q$

Associative property:

For any $a,b,c\in Q$

$a*(b*c)=(a*b)*b$

Consider,

$$a*(b*c)=a*\left(\frac{bc}{2}\right)$$

$$=\frac{abc}{4}$$

Consider,

$$(a*b)*c=\left(\frac{ab}{2}\right)*c$$

$$=\frac{abc}{4}$$

Hence associative property is satisfied

Identity Property:

For any $a\in G,\exists\, e\in G$ such that $a*e=e*a=a$

Consider,

$a*e=a$

$\frac{ae}{2}=a$

$ae = 2a$

$e = 2$

$\therefore 2 \in Q$

Hence identity elements in Q

Inverse Property:

for each $a \in G, \exists\, a^{-1} \in G$ such that $a * a^{-1} = a^{-1} * a = e$

Consider,

$a * a^{-1} = 2$

$\frac{aa^{-1}}{2} = 2$

$\frac{aa^{-1}}{2} = 4$

$a^{-1} = \frac{4}{a}$

Inverse element exist in Q for each a.

Hence $(Q,*)$ is group with respect to $*$

Problem – 8:

Prove that $(Z,*)$ is group with respect to $*$ as defined as $a*b = a+b+1\ \forall\ a,b \in Q$

Solution

Closure property:

For any $a,b \in Z$

$a*b = a+b+1 \in Z$

$\therefore a*b \in Z$

Associative Property:

for any $a,b,c \in Z \Rightarrow a*(b*c) = (a*b)*b$

5.32. ALGEBRA STRUCTURE - MCQ

1. A group G is said to be _______ if for every $a,b \in G$, $a \cdot b = b \cdot a$

A) semigroup B) abelian

C) monoid D) quasi group

2. Let $G = \{a^i\}, i = 0,1,2,\ldots,n-1$ where $a^0 = a^n = e$, $a^{i+j} = \begin{cases} a^{i+j^2} & if \quad i+j < n \\ a^{i+j-n} & if \quad i+j \geq n \end{cases}$, the G is a

A) cyclic group of order n – 1 B) cyclic group of order 2n

C) cyclic group of order n D) cyclic group of order n + 1

3. Every subgroup of _______is normal.

A) cyclic group B) Abelian group

C) Cyclic or abelian D) cyclic and abelian

4. Let G be the set of all 2×2 matrices $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$, where a, b, c, d $\in R$ such that ad – bc = 1, then G is _________.

A) finite abelian group B) finite non-abelian group

C) infinite abelian group D) infinite non-abelian group

5. Which of the following is incorrect?

A) The identity G is unique B) Every $a \in G$ has a unique inverse in G

C) For every $a \in G$, $\left(a^{-1}\right)^{-1} = a$ D) for all $a,b \in G$ $\left(a \cdot b\right)^{-1} = a^{-1}b^{-1}$

6. G is a finite group of order 4 and $a \in G$, then $a^4 =$

A) 4 B) 2 C) e D) 1

7. If G has a element $a \neq e$ such that $a^2 = e$, then G is a group of

A) odd order B) even order

C) finite order D) infinite order

8. For any _______ construct a non-abelian group of order 2n

A) n > 1 B) $n \geq 2$ C) $n \geq 1$ D) $n > 2$

9. Let G be the set of all 2×2 matrices $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$ where a, b, c, d are integers modulo 2, such that ad – bc = 1 is a group under multiplication, then a (G) =

A) 6 B) 48 C) 4 D) 3

10. Let G be the set of all 2×2 matrices $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$ where ad – bc = 1, a, b, c, d are integers mod 3, forms group under multiplication then O(G) =

A) 48 B) 6 C) 4 D) 9

11. A non-empty subset H of a group G is a subgroup of G of

A) $a,b\in H \Rightarrow ab\in H$ B) $a\in H \Rightarrow a^{-1}\in H$

C) $a,b\in H \Rightarrow ab^{-1}\in H$ D) all A, B, C

12. If H is a non-empty ________ of a group G and H is closed under multiplication, there H is a subgroup of G

A) infinite subset B) finite subset

C) proper subset D) improper subset

13. Let $G=(z,+)$ Let H be a subset consisting of all multiples of m (Hn) then H is ________ of G.

A) subgroup B) not subgroup

C) may be subgroup D) none of these

14. If H is a subgroup of G, then index of H if no. of ________ of H in G.

A) all right cosets of G B) distinct right cosets

C) distinct left cosets D) both c and b

15. If G is a finite group and $a\in G$, then $a^{0(G)} =$

A) 0A) B) 0 (G)

C) e D) 0

16. If n is a +ve integer and a is relatively prime onto n, then $a^{\phi(n)} \equiv 1$ mod n is

A) Euler theorem B) Fermat theorem

C) Sylow's theorem D) Cayley's theorem

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UNIT VI
Real Analysis

Your Success is Our Goal....

UG TRB 2022-23 MATHEMATICS

UNIT - VI – REAL ANALYSIS

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UNIT - VI

REAL ANALYSIS

6.1. SETS

DEFINITION OF SETS:

- A set is a collection of objects chosen from some universe
  - Example: $\{1,2,3,4\}$ is a set of numbers

6.1.1. Order of Sets:

- The order of a set defines the number of elements a set is having. It describes the size of a set. The order of a sets is also known as the cardinality.

6.1.2. Types of Sets:

(i) Empty set - A set which doesn't contain any element. It is denoted by $\{\ \}$ or ϕ

(ii) Singleton set - A set which contains a single element.

(iii) Finite set - A set which consists of a definite number of elements.

(iv) Infinite set - A set which is not finite.

(v) Equivalent set - If the number of elements is the same for two different sets, then they are called equivalent sets.

(vi) Equal sets - The two sets A and B are said to be equal if they have exactly the same elements, the order of elements do not matter.

(vii) Disjoint sets - Two sets are said to be disjoint if the sets does not contain any common element.

(viii) Subsets - A sets 'A' is said to be a sub sets of B if every element of A is also an element of B, denoted as $A \subseteq B$.

(ix) proper subset - If $A \subseteq B$ and $A \neq B$, then A is called the proper subset of B and it can be written as $A \subset B$.

(x) superset - Sets A is said to be the suspect of B if all the elements of sets B are the elements of set A. it is represented as $A \supset B$

(xi) universal set - A set which contains all the sets relevant to a certain condition is called the universal set. It is the set of all possible values.

6.1.3. Operations of Set:

(i) Union Sets:

If set A and set B are two sets, then A union B is the set that contains all the elements of a set A and set B. It is denoted as $A \cup B$.

- **Example:**

  $A = \{1,2,3\}$ and $B = \{4,5,6\}$

  $A \cup B = \{1,2,3,4,5,6\}$

(ii) Intersection of Sets:

If sets A and set B are two sets, then A intersection B is the set that contains only the common elements between set A and set B . If denoted as $A \cap B$

- **Example:**

  $A = \{1,2,3\}$ and $B = \{4,5,6\}$

  $A \cap B = \{\ \}$ or ϕ

(iii) Complement of Sets:

The complement of sets of any set, say p is the set of all elements in the universal set that are not in set P. If is denoted by 'p'

- Properties of complements sets

  a) $P \cup P' = \cup$

  b) $P \cap P' = \phi$

  c) $(P')' = P$

  d) $\phi' = \cup$ and $\cup' = \phi$

(iv) Cartesian product of sets:

If set A and set B are two sets then the Cartesian product of set A and set B is a set of all ordered pairs (a,b) such that a is an element of A and b is an element of B. It is denoted by $A \times B$

$$A \times B = \{(a,b); a \in A \ \ and \ \ b \in B\}$$

(v) Difference of sets:

If set A and set B are two, then set A different set B is a set which has element of A but no elements of B. It denoted as $A - B$

- **Example:**

  $A = \{1,2,3\}$ and $B = \{3,2,4\}$

  $A - B = \{1\}$

6.1.4. Properties of Sets:

(i) commutative property : $A \cup B = B \cup A$ and $A \cap B = B \cap A$

(ii) Associative property : $A \cup (B \cup C) = (A \cup B) \cup C$

$A \cap (B \cap C) = (A \cap B) \cap C$

(iii) Distributive property : $A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$

$A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$

(iv) De Morgan's law: Law of union : $(A \cup B)' = A' \cap B'$

Law of intersection : $(A \cap B)' = A' \cup B'$

(v) complement law : $A \cup A' = A' \cup A = \cup$ and $A \cap A' = \phi$

(vi) Idempotent law and law of null and universal set for any finite set A,

(a) $A \cup A = A$

(b) $A \cap A = A$

(c) $\phi' = \cup$

(d) $\phi = \cup'$

Ex:

- The set $f = \{< x, x^2 > -\infty < x < \infty\}$ is the function defined by
  - $f(x) = x^2 \ (-\infty < x < \infty)$
  - $f(1) = 1 \quad f(-1) = 1$
  - $f(2) = 4 \quad f(-2) = 4$

Define: Image and Range:

- Let 'f' be a function from X to Y for any $x \in X, f(x) = y \in Y$ here $f(x) = y$ is called an image of 'x' under f. Let 'f' be a function from X to Y define, $f(x) = \{y / y = f(x);\ f \ or \ some \ x \in X\}$ is called a range of 'f'.

Define: Inverse Image:

- Let 'f' is a function $f : X \to Y$ such that $f(x) = y \Rightarrow x = f^{-1}(y)$, here $f(x)$ is called an image of y under 'f'. $f^{-1}(y)$ is called an inverse image of x under 'f'.

  Let B be a subset of Y. i.e., $B \subset Y$

  $f^{-1}(B) = \{x / f(x) = y;\ for \ \ y \in B\}$

Define: One-One function (or) Injective:

- A function $f : X \to Y$ is said to be a one-one function if for any $x_1, x_2 \in X$. Such that $x_1 \neq x_2 \Rightarrow f(x_1) \neq f(x_2)$ (or) $x_1 = x_2 \Rightarrow f(x_1) = f(x_2)$
  - i.e., The distinct elements in X has distinct image in Y.

Define: Onto function (or) Surjective:

- A function $f : X \to Y$ is said to be a onto function, if the range of 'f' is equal to Y. i.e.,

  $f(x) = y$

  $f : R \to R$

Let $f_1 : R \to (0, \infty)$

$f_1(x) = x^2$

$f_1(-2) = 4$

$f_1(-1) = 1$

$f_1(0) = 0$

$f_1(1) = 1$

$f_1(2) = 4$

Range of $f_1(0, \infty) \subset R$. It is a onto function but not into

Let $f_2 : R \to R$ $\qquad f_2(x) = x$

Range of $f_2(-\infty, \infty) = R$

Define 1 – 1 Correspondence (or) Bijective:

- If the function f is both one-one and onto then we say that the function f is 1 – 1 Correspodance (or) Bijective.

Define: Constant function:

- The function f is said to be constant function, if all the images are same. i.e., $f(x) = k$ for all x in domain

Define: Inverse function:

- Let 'f' be a function from X to Y, such that f is one-one and onto function.

  $\therefore$ The function $f^{-1} : Y \to X$ is called a inverse function of 'f'.

Define Characteristic function:

- If $A \subset S$ then the characteristic function ψ_A is defined as,

$$0.\psi_A(x) = \begin{cases} 1 & if \quad x \in A \\ 0 & if \quad x \in A' \end{cases}$$

Theorem – 1:

- If $f : A \to B$ and $X \subset B, Y \subset B$ Then $f^{-1}(X \cup Y) = f^{-1}(X) \cup f^{-1}(Y)$ (or) The inverse image of the union of two sets is the union of the inverse images.

Proof:

- Given that $f : A \to B$ and $X \subset B, Y \subset B$

  To prove: $f^{-1}(X \cup Y) = f^{-1}(X) \cup f^{-1}(Y)$

  Let $b \in X \cup Y$

  Since $f : A \to B$

  $\therefore f(a) = b$ such that $a \in A, b \in B$ and hence $X \subset B,\ Y \subset B$

For some $a \in A$,

$$f(a) \in X \cup Y \to (1)$$

$\therefore f(a)(or)\ f(a) \in Y$

$a \in f^{-1}(X)\ (or)\ a \in f^{-1}(Y)$

$$\Rightarrow a \in f^{-1}(X) \cup f^{-1}(Y)$$

From (1), $f(a) \in X \cup Y$

$$a \in f^{-1}(X \cup Y)$$

$\Rightarrow f^{-1}(X \cup Y) \subseteq f^{-1}(X) \cup f^{-1}(Y) \to (*)$

Now, let $a \in f^{-1}(X) \cup f^{-1}(Y)$

$a \in f^{-1}(X)$ (or) $a \in f^{-1}(Y)$

$f(a) \in X$ (or) $f(a) \in Y$

$f(a) \in X \cup Y$

$a \in f^{-1}(X \cup Y)$

$\therefore f^{-1}(X) \cup f^{-1}(Y) \subseteq f^{-1}(X \cup Y) \rightarrow (**)$

From (*) and (**)

$f^{-1}(X \cup Y) = f^{-1}(X) \cup f^{-1}(Y)$

Hence proved

Theorem – 2:

If $f : A \rightarrow B, X \in A, Y \in A$ then $f(X \cup Y) = f(X) \cup f(Y)$

Proof:

Given that $f : A \rightarrow B, X \in A, Y \in A$

To prove:

$f(X \cup Y) = f(X) \cup f(Y)$

Suppose $b \in f(X \cup Y)$

Since f is a function from A to B

$\therefore b = f(a)$, for some $a \in X \cup Y$

$\Rightarrow a \in X \ (or) \ a \in Y$

$\Rightarrow f(a) \in f(X)$ (or) $\Rightarrow f(a) \in f(Y)$

$\Rightarrow f(a) \in f(X) \cup f(Y)$

$\Rightarrow b \in f(X) \cup f(Y)$

$\therefore f(X \cup Y) \subseteq f(X) \cup f(Y)$ ___ (*)

Since f is a function from A to B

$\therefore v = f(a)$; for some $a \in X \cup Y$

Suppose, $b \in f(X) \cup f(Y)$

$b \in f(X)(or) f(Y)$

From (*) and (**)

$f^{-1}(X \cap Y) = f^{-1}(X) \cap f^{-1}(Y)$

Hence proved.

Define: Real Valued Function

- If $f : X \to R$ then f is called a Real valued function. If $x \in X$ then $f(x)$ is also called the value of f at x.

Ex.

1. $f(x) = x^2$ or $(-\infty < x < \infty)$ it is a real valued function.

2. $f : Z \to C$

$f(x) = ix$

It is not a real valued function but it is a complex valued function.

Note:

1. If $A \subset B$ then every element of A is an element of B.

2. If A is a proper subset of B then $A \subset B$ and $A \neq B$.

3. If A is an improper subset of B then $A \subset B$ and $A = B$.

4. If $A \subseteq B$ and $B \subseteq A \Rightarrow A = B$

5. If $a \in A$ and $a \in B$ here a is an arbitrary then $A \subseteq B$

Operations on real valued function:

Let $f : A \to T$, $g : B \to R$

We define, $f + g$ as the function whose value at $x \in A$ is equal to $f(x) + g(x)$

i.e., $(f+g)(x) = f(x) + g(x), (x \in A)$

Similarly, $(f-g)(x) = f(x) - g(x), (x \in A)$

$(fg)(x) = f(x)g(x), (x \in A)$

$(cf)(x) = cf(x), (x \in A)$ and c – constant

$$\left(\frac{f}{g}\right)(x) = \frac{f(x)}{g(x)}, (x \in A)$$

$$|f|(x) = |f(x)|, (x \in A)$$

$$Max(f, g)(x) = Max((f(x), g(x))), (x \in A)$$

$$Min(f, g)(x) = Max((f(x), g(x))), (x \in A)$$

Define: Composition of function:

- Let X, Y, Z are three non-empty sets. Let us define function, $f: X \to Y$ and $g: Y \to Z$. The function f composition of g is denoted by $g\,o\,f: X \to Y \to Z$

  $$\Rightarrow g\,o\,f: X \to Z$$

- It is defined by, for any $x \in X$ such that $(g\,o\,f)(x) = g[f(x)]$. The composition function if possible only if, the co-domain of f is equal to the domain of g.

  - **Ex.**

    Let $f(x) = 1 + \sin x$ on $(-\infty < x < \infty)$

    $g(x) = x^2$ on $(-\infty < x < \infty)$

    The find $(g\,o\,f)(x)$

Solution:

By the definition of composition function

$$g\,o\,f(x) = g[f(x)]$$

$$= g[1 + \sin x]$$

$$= (1 + \sin x)^2$$

$$= 1 + \sin^2 x + 2\sin x \quad \text{on} \ (-\infty < x < \infty)$$

Define: Equivalent set

- If there exist a 1 – 1 corresponds between the sets A and B then we say that A and B are equivalence sets of equivalent sets.

Note:

1. Any two sets containing exactly same number of elements are equivalent
2. Every set A is equivalent to itself.
3. If A and B are equivalent. Then B and A are equivalent
4. If A and B are equivalent and B and C are equivalent then A and C also equivalent

Define: Equivalent function:

- Two sets A and B are said to be equivalent sets is there exist a one-one and onto functions from A to B.
  - **Ex.**

    $f : Z \to 2Z \cup \{0\}$

    $f(z) = 2x$

  Here f is one-one on to function therefore Z and 2Z $\cup \{0\}$ are equivalent set.

Exceise Questions:

1. How many elements are there in the complement of set A?

   A) 0 B) 1

   C) All the elements of A D) None of tehse

2. Empty set is a __________.

   A) Infinite set B) Finite set

   C)unknown set D) universal set

3. Order of the power set P(A) of a set A of order n is equal to

   A) n B) 2n C) 2^n D) n^2

4. The cardinality of the power set of $\{x : x \in N,\ x \le 10\}$ is ________.

   A) 1024 B) 1023 C) 2048 D) 2043

5. The range of the function $f(x) = 3x - 2$, is:

   A) $(-\infty, \infty)$ B) $R - \{3\}$ C) $(-\infty, 0)$ D) $(0, -\infty)$

6.37. REAL ANALYSIS – IMPORTANT MCQ

Choose the Correct Answer:

1. The cardinal number of empty is

   (A) $n(\phi)=\infty$ (B) $n(\phi)=1$ (C) $n(\phi)=0$ (D) $n(\phi)=-\infty$

2. which one is countable set

   (A) Algebraic number (B) Transcendental number

   (C) Cantor set (D) irrational number

3. The element of a_{41} is

   (A) 4 (B)5 (C)3 (D) 2

4. Every bounded and infinite set has a

   (A) Interior point (B) limit point

   (C) Derived set (D) Neighborhoods points

5. Which one is an closed set

   (A) ϕ (B) ϕ' (C)N (D) (a,b)

6. The set of all real number is

   (A) uncountable (B) countable (C)finite (D) none of these

7. The interval $[0,1]$ is

   (A) uncountable (B) countable

   (C)finite (D) at most countable

8. The cardinality of the set $x=\{a,e,i,o,u\}$ if______________

   (A) $n(x)=5$ (B) $n(x)=\infty$

   (C) $n(x)=2$ (D) $n(x)=4$

9. The extended real line $\overline{R}$ =__________________

(A) R (B) $\overline{R}$ (C) $R \cup \{-\infty, \infty\}$ (D) $R \cap \{-\infty, \infty\}$

10. If $S = [0,1)$ then exterior of s=__________________

(A) $(0,1)$ (B) $(-\infty,0) \cup (1,\infty)$

(C) $(-\infty,0)$ (D) $(1,\infty)$

11. If S is such that $S \cap S^1 = \phi$, then

(A) S is uncountable (B) S is countable

(C) S is compact (D) S is not closed

12. The Lévesque measure of cantor set C is

(A) 1 (B)0 (C)4 (D) prime no

13. The continuity on a set A implies uniform continuity if A is

(A) complete (B) compact (C) open (D) closed

14. Compact implies

(A) bounded only (B) closed only

(C) closed and bounded (D) none of these

15. If $\lim_{n} x_n = l$, then $\lim_{n} \frac{x_1 + x_2 + . \quad . \quad . + x_n}{n} =$ _________

(A) l (B) $l+n$ (C) $\frac{l}{n}$ (D) $l-n$

16. The series $\sum_{n=1}^{\infty} ar^{n-1}$ _________

(A) converges if $|r| < 1$ (B) diverges to if $r \geq 1$

(C)oscillates if $r < -1$ (D) all are true

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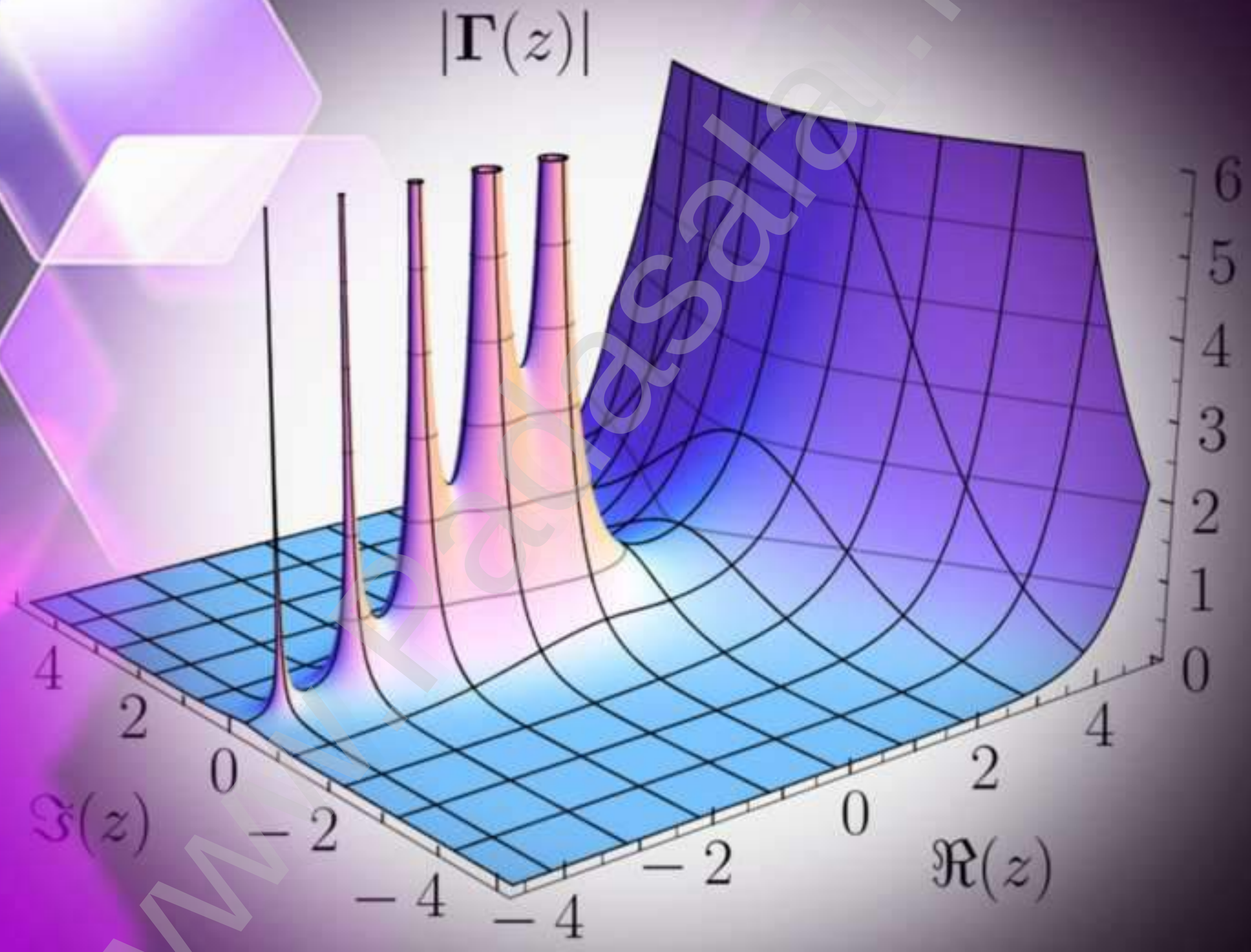

UNIT VII
Complex Analysis

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UG TRB – MATHEMATICS – 2022-23

UNIT - VII

COMPLEX ANALYSIS

ALGEBRA OF COMPLEX NUMBERS

7.1. FUNCTION OF A COMPLEX VARIABLE:

- We use the letters z and w to denote complex variables. Thus, to denote a complex valued function of a complex variable we use the notation $w = f(z)$.Throughout this chapter we shall consider functions whose domain of definition is a region of the complex plane.
- The function $w = iz + 3$ is defined in the entire complex plane.
- The function $w = \frac{1}{z^2+1}$ is defined at all points of complex plane except at $z = \pm i$
- The function $w = |z|$ is defined in the entire complex plane and this is a real values function of the complex variable z.
- If $a_0, a_1, ...a_n$ are complex constants the function $p(z) = a_0 + a_1 z + ... + a_n z^n$ is defined in the entire complex plane and is called a polynomial in z.
- If P(Z) and Q(Z) are polynomials the quotient $\frac{P(Z)}{Q(Z)}$ is called a rational function and it is defined for all z with $Q(Z) \neq 0$
- The function $f(z) = x^4 + y^4 + i(x^2 + y^2)$ is defined over the entire complex plane.

- In general if $u(x,y)$ and $v(x,y)$ are real valued functions of two variables both defined on region S of the complex plane then $f(z)=u(x,y)+i\,v(x,y)$ is a complex values function defined on S.
- Conversely each complex function $w=f(z)$ can be put in the form

  $$w=f(z)=u(x,y)+iv(x,y)$$

- When u and v are real valued functions of the real variables x and y $u(x,y)$ is called the real part and $v(x,y)$ is called the imaginary part of the function $f(z)$

For Example:

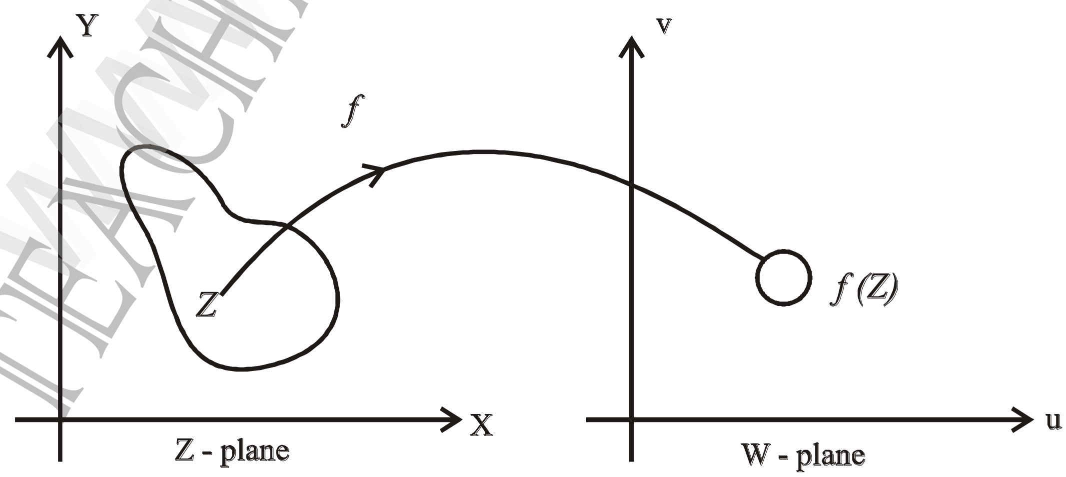
$$f(x)=z^2=(x+iy)^2$$

$$=x^2+2ixy+y^2(i^2)$$

$$=(x^2-y^2)+i(2xy)$$

So that $u(x,y)=x^2-y^2$ and $u(x,y)=2xy$

- Thus, a complex function $w=f(z)$ can be viewed as a function of the complex variable z or as a function of two real variables x and y.
- To have a geometric representation of the function $w=f(z)$ it is convenient to draw separate complex planes for the variables z and w so that corresponding to each point z = x + iy of the z-plane there is a point w = u + iv in the w-plane.



Exercise Questions:

1. The value of (iota) is _______.

A) – 1 B) 1 C) $(-1)^{\frac{1}{2}}$ D) $(-1)^{\frac{1}{4}}$

2. Is i(iota) a root of $1+x^2=0$?

A) True B) False

3. In $z=4+i$, what is the real part?

A) 4 B) I C) 1 D) $4+i$

4. In $z=4+i$, what is the imaginary part?

A) 4 B) I C) 1 D) $4+i$

5. $(x+3)+i(y-2)=5+i2$, find the values of x and y.

A) $x=8$ and $y=4$ B) $x=2$ and $y=4$

C) $x=2$ and $y=0$ D) $x=8$ and $y=0\backslash$

6. Find the domain of the function defined by $f(z)=\frac{z}{(z+\bar{z})}$

A) $\text{Im}(z)\neq 0$ B) $\text{Re}(z)\neq 0$ C) $\text{Im}(z)=0$ D) $\text{Re}(z)=0$

7. Let $f(z)=z+\frac{1}{z}$ what will be the definition of this function in polar form.

A) $\left(r+\frac{1}{r}\right)\cos\theta+i\left(r-\frac{1}{r}\right)\sin\theta$ B) $\left(r-\frac{1}{r}\right)\cos\theta+i\left(r+\frac{1}{r}\right)\sin\theta$

C) $\left(r+\frac{1}{r}\right)\sin\theta+i\left(r-\frac{1}{r}\right)\cos\theta$ D) $\left(r+\frac{1}{r}\right)\sin\theta-i\left(r-\frac{1}{r}\right)\cos\theta$

8. For the function $f(z)=z^i$, what is the value of $|f(w)|+Arg\, f(\omega),\omega$ being the cube root of unity with $\text{Im}(\omega)>0$?

A) $e^{-2\pi/3}$ B) $e^{2\pi/3}$ C) $e^{-2\pi/3}+2\pi/3$ D) $e^{-2\pi/3}-2\pi/3$

9. Let $f(z)=\left(z^2-z-1\right)^7$. If $a^2+a+1=0$ and $\operatorname{Im}(\alpha)>0$, then find $f(\alpha)$

A) $128\ \alpha$ B) -128α C) $128\alpha^2$ D) $-128\alpha^2$

10. For all complex numbers z satisfying $\operatorname{Im}(z)\neq 0$, if $f(z)=z^2+z+1$ is a real value function the find its range

A) $(-\infty,-1]$ B) $\left(-\infty,\frac{1}{3}\right)$ C) $\left(-\infty,\frac{1}{2}\right)$ D) $\left(-\infty,\frac{3}{4}\right)$

7.2. LIMITS

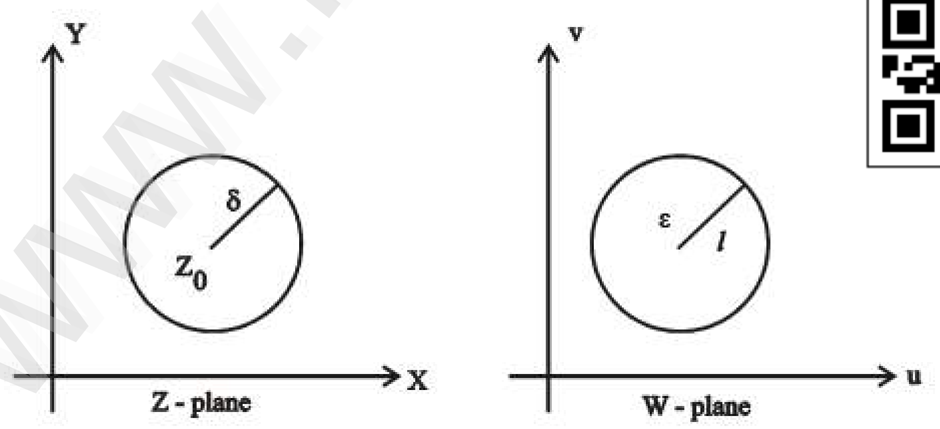
Definition:

- A function w = f(z) is said to have the limit l as z tends to z_0 if given $\varepsilon>0$ there exists $\delta>0$ such that $0<|z-z_0|<\delta$

  $$\Rightarrow |f(z)-l|<\varepsilon$$

  In this case we write $\lim_{z\to z_0} f(z)=l$

- Geometrically the definition states that given any open disc with centre l and radius ε, there exists an open disc with centre z_0 and radius δ such that for every point $z(\neq z_0)$ in the disc $|z-z_0|<\delta$ the image $w=f(z)$ lies in the disc $|w-l|<\varepsilon$



Lemma:

- When the limit of a function $f(z)$ exists as z tends to z_0 then the limit has a unique value.

Proof:

Suppose that $\lim_{z \to z_0} f(z)$ has two values l_1 and l_2

Then given $\varepsilon > 0$ there exists δ_1 and $\delta_2 > 0$ such that

$$0 < |z - z_0| < \delta_1 \Rightarrow |f(z) - l_1| < \frac{\varepsilon}{2} \text{ and}$$

$$0 < |z - z_0| < \delta_2 \Rightarrow |f(z) - l_2| < \frac{\varepsilon}{2}$$

Now let $\delta = \min\{\delta_1, \delta_2\}$

Then if $0 < |z - z_0| < \delta$ we have

$$|l_1 - l_2| = |l_1 - f(z) + f(z) - l_2|$$

$$\leq |f(z) - l_1| + |f(z) - l_2|$$

$$< \frac{\varepsilon}{2} + \frac{\varepsilon}{2}$$

$= \varepsilon$ (Using triangle inequalities)

Since $\varepsilon < 0$ is arbitrary $|l_1 - l_2| = 0$

So that $l_1 = l_2$

Example – 1:

Let $f(z) = \begin{cases} z^2 & if \quad z \neq i \\ 0 & if \quad z = i \end{cases}$

As z approaches i, $f(z)$ approaches $i^2 = -1$

Hence, we expect that $\lim_{z \to i} f(z) = -1$

To prove that the given $\varepsilon > 0$ there exists $\delta > 0$ such that $0 < |z - i| < \delta$

$\Rightarrow |z^2 + 1| < \varepsilon$

Now, $|z^2 + 1| = |(z+i)(z-i)| \Rightarrow |z+i||z-i|$ _____ (1)

Note that if we can find a $\delta > 0$ satisfying the requirements of the definition then we can choose another $\delta \le 1$ satisfying the requirements of the definition.

Now $0 < |z-i| < 1 \Rightarrow |z+i| = |z-i+2i|$

$$\le |z-i| + |2i|$$

$$< 1 + 2 = 3$$

$$\therefore |z+i| < 3$$

Using this in (1) we obtain $0 < |z-i| < 1$

$$\Rightarrow |z^2+1| < 3|z-i|$$

Hence if we choose $\delta = \min\left\{1, \frac{\varepsilon}{3}\right\}$ we get

$$0 < |z-i| < \delta$$

$$\Rightarrow |z^2+1| < \varepsilon$$

$$\therefore \lim_{z \to i} f(z) = -1$$

Example – 2:

$$\lim_{z \to 2} \frac{z^2-4}{z-2} = 4$$

Let $f(z) = \frac{z^2-4}{z-2}$

Hence $f(z)$ is not defined at z = 2 and when $z \ne 2$ we have

$$f(z) = \frac{(z+2)(z-2)}{z-2}$$

$$= z+2$$

$$\therefore |f(z)-4| = |z+2-4|$$

$$= |z-2| \text{ when } \quad z \ne 2$$

Now given $\varepsilon > 0$, we choose $\delta = \varepsilon$

Then $0 < |z-2| < \delta \Rightarrow |f(z)-4| < \varepsilon$

$$\therefore \lim_{z \to 2} f(z) = 4$$

Example – 3:

The function $f(z) = \frac{\bar{z}}{z}$ does not have a limit as $z \to 0$.

$$f(z) = \frac{\bar{z}}{z} = \frac{x-iy}{x+iy}$$

Suppose $z \to 0$ along the path $y = mx$

Along this path $f(z) = \frac{x-imx}{y+imx}$

$$= \frac{1-im}{1+im} \text{ as } x \neq 0$$

Hence if $z \to 0$ along the path $y = mx$, $f(z)$ tends to $\frac{1-im}{1+im}$ which is different for values of m.

Hence $f(z)$ does not have a limit as $z \to 0$

7.3. MAPPINGS

The mapping $w = z^2$

The transformation $w = z^2$ is conformed at all points except z = 0

Put $w = u + iv$ and $z = x + iy$

$$u + iv = (x+iy)^2$$

$$u + iv = x^2 - y^2 + i2xy$$

Equating real and imaginary parts, we get

$$u = x^2 - y^2 \qquad v = 2xy$$

Now we discuss the following cases,

Case (i):

The equation of real axis y = 0 in the z – plane

When y = 0, we have $u = x^2$ $\qquad v = 0$

The real axis y = 0 in the z-plane is mapped to positive u-axis in the w-plane

Case (ii):

The equation of imaginary axis x = 0 in the z-plane

When x = 0, we have $u = -y^2$ $\qquad v = 0$

∴ The imaginary axis x = 0 in the z-plane is mapped to negative u-axis in the w-plane

Case (iii):

The equation of the line parallel to x-axis in the z-plane is y = 0

Then, we have $u = x^2 - c^2;\ v = 2xc$

$$\Rightarrow x = \frac{v}{2c}$$

$$\therefore u = \frac{v^2}{4c^2} - c^2$$

$$u = \frac{v^2 - 4c^4}{4c^2}$$

$$4uc^2 \neq 4c^4 = v^2$$

$$4c^2\left(u + c^2\right) = v^2$$

This is a parabola with focus at the origin in the w-plane and u-axis as its axis.

For different values of c, we obtain a family of confocal parabola with u-axis as the axes.

Case (iv):

The equation of the line parallel to y-axis (i.e.,) $x = d$ we have

$$u = d^2 - y^2 \qquad v = 2dy$$

$$\Rightarrow y = \frac{v}{2d}$$

$$u = d^2 - \frac{v^2}{4d^2}$$

$$4d^2 u = 4d^4 - v^2$$

$$v^2 = -4d^2 u + 4d^4$$

$$v^2 = -4d^2\left[u - d^2\right]$$

- This is also a parabola with focus at the origin and u-axis as its axes in the w-plane.
- For different values of d, we get a family of focal parabola with u-axis as the axes and the common focus at the origin.

The mapping $w = \sin z$

Put $w = u + iv$ and $z = x + iy$

$$u + iv = \sin(x + iy)$$

$$= \sin x \cos iy + \cos x \sin iy$$

$$= \sin x \cosh y + \cos x(i \sinh y)$$

$$u + iv = \sin x \cosh y + i \cos x \sinh y$$

Equating real and imaginary parts, we get

$$u = \sin x \cosh y \qquad v = \cos x \sinh y$$

Case (i):

The equation of real axis y = 0 in the z – plane

When y = 0, we have u = sin x, v = 0

Since, $\sin x$ takes values between – 1 and 1, the image of the real axis y = 0 is the line segment $-1 \le u \le 1$ of the u – axis.

Case (ii):

The equation of imaginary axis x = 0 in the z-plane

When x = 0, we have u = 0, v = sin hy

If y = 0, sin hy is positive and if y < 0, sin hy is negative

- Hence the upper – half of the imaginary axis in the z-plane maps into the upper half of the imaginary axis of the w-plane, while the lower halves of both corresponds with one another.

Case (iii):

The equation of any line parallel to x-axis in the z-plane is y = c

From $u = \sin x \cosh y$ $\qquad v = \cos x \sinh y$

$$\Rightarrow \sin x = \frac{u}{\cosh y}, \cos x = \frac{v}{\sinh y}$$

W.K.T $\sin^2 x + \cos^2 x = 1$

$$\frac{u^2}{\cosh^2 y} + \frac{v^2}{\sinh^2 y} = 1$$

Put y = c in above equation

$$\frac{u^2}{\cosh^2 c} + \frac{v^2}{\sinh^2 c} = 1$$

When $c \neq 0$ the above equation represent ellipse with semi-axes $\cosh c$ and $\sinh c$

Case (iv):

The equation of any line parallel to y-axis in the z-plane is $x = d$

From $u = \sin x \cosh y,\ v = \cos x \sinh y$

$$\cosh y = \frac{u}{\sin x}, \sinh y = \frac{v}{\cos x}$$

W.K.T $\cosh^2 y - \sinh^2 y = 1$

$$\frac{u^2}{\sin^2 x} - \frac{v^2}{\cos^2 x} = 1$$

Put $x = d$ in above equation

$$\frac{u^2}{\sin^2 d} - \frac{v^2}{\cos^2 d} = 1$$

- The above equation represents a system of hyperbola. Hence, the lines parallel to the imaginary axis of the z-plane map into confocal hyperbola.

The mapping $w = e^z$

The given transformation, $w = e^z$

Since $\frac{dw}{dx} = e^z \neq 0$

For any values of z, the mapping $w = e^z$ is conformal at all the points in z-plane.

Replace $z = x + iy$ and $w = u + iv$ in the mapping, we get

$$u + iv = e^{x+iy}$$

$$= e^x \cdot e^{iy}$$

$$u + iv = e^x (\cos y + i \sin y)$$

$$u + iv = e^x \cos y + ie^x \sin y$$

Equating real and imaginary parts we have

$$u = e^x \cos y \qquad v = e^x \sin y$$

Eliminating y from the above equation, we get

$$u^2 + v^2 = e^{2x} \cos^2 y + e^{2x} \sin^2 y$$

$$= e^{2x} \text{c}\left(\text{os}^2 y + \sin^2 y\right)$$

$$u^2 + v^2 = e^{2x} \qquad \_\_\_\_\_\_ (1)$$

Eliminating x from the above equation, we have

$$\frac{v}{u} = \frac{e^x \sin y}{e^x \cos y}$$

$$\frac{v}{u} = \tan y$$

$$u \tan y = v \qquad \_\_\_\_\_\_ (2)$$

- Which represent a system of concentric circles with the origin.
- In particular, $x = 0$ transforms into a circle of unit radius with centre at the origin in the w-plane.

- Hence the lines parallel to y-axis transform into concentric circles with the centre and w = 0

When y = constant

- The equation (2) represent a line through the origin in the w-plane
- Hence the line parallel to x-axis Transforms into radial line

1. When y = 0 from the equation $u = e^x \cos y$ and $v = e^x \sin y$, we have $u = e^x, v = 0$

   Since e^x is always positive for u > 0, v = 0. Hence x-axis transforms into positive u-axis in the w plane.

2. When $y = \frac{\pi}{2}$, we have u = 0 and $v = e^x$ Hence the line $y = \frac{\pi}{2}$, transforms into the v-axis in the w-plane.

3. When $y = \pi, v = 0$ and $u = -e^x < 0$

   Hence the lines $y = \pi$ transforms into negative u-axis.

4. When $y = \frac{3\pi}{2}$, $u = 0$ and $v = -e^x < 0$

   Hence the lines $y = \frac{3\pi}{2}$ transforms into the negative v-axis, in the w-plane.

5. When $y = 2\pi, v = 0$ and $u = e^x > 0$

   Hence the lines $y = 2\pi$ transforms into the positive side of the u-axis in the w-plane.
   Hence a ny horizontal strip of the z-plane of height 2π will cover the entire w-plane.

The mapping $w = z + d$

The transformation $w = z + d$, where d is complex constant, represent a translation,

Let $z = x + iy$ and $u + iv = w,\ d = a + ib$, then transformation becomes,

$$u + iv = x + iy + a + ib$$

$$u + iv = (x + a) + i(y + b)$$

Equating real and imaginary part

We get

$$u = x + a \qquad v = y + b$$

- The point (x, y) in the z-plane is mapped onto the point $(x+a, y+b)$ in the w-plane.
- If we impose the w-plane on the z-plane, the figure of the w-plane is shifted to constant vector.
- Also, the region in the z and w planes will have the same shape, size and orientation.
- In particular, this transformations maps circles into circles.

Exercise Questions:

1. The function $f : N^+ \to N^+$, define on the set of (+ve) integers N^+, satisfies the following properties

   $f(n) = f(n/2)$, if n is even

   $f(n) = f(n/5)$ if n is odd

   Let $R = \{i / \exists j; f(j) = i\}$ be the set of distinct values that f takes. The maximum possible size of R is

   A) 5 B) 2 C) 0 D) – 1

2. The value of the limit $\lim_{x\to 0}(\cos x)^{\cot 2x}$ is

   A) 1 B) e C) $e^{\frac{1}{2}}$ D) $e^{-\frac{1}{2}}$

3. The value of the limit $\lim_{x\to 0}\{\sin(a+x)-\sin(a-x)\}/x$ is

   A) 0 B) 1 C) $2\cos a$ D) $2\sin a$

4. $\lim_{x\to -1}\left[1+x+x^2+\dots+x^{10}\right]$ is

   A) 0 B) 1 C) – 1 D) 2

5. The principal argument of $\frac{1}{2+3i}$ is ______.

   A) $\tan^{-1}(1.5)$ B) $\tan^{-1}(0.5)$ C) $\tan^{-1}(2.5)$ D) $\tan^{-1}(3.5)$

7.32. MULTIPLE CHOICE QUESTIONS

1. If $Z_1 = x_1 + iy_1$ and $Z_2 = x_2 + iy_2 \neq 0$ then $\frac{Z_1}{Z_2} = ?$

A) $\frac{x_1x_2 - y_1y_2}{x_2^2 - y_2^2} + i\frac{y_1x_2 + x_1y_2}{x_2^2 + y_2^2}$ B) $\frac{x_1x_2 + y_1y_2}{x_2^2 + y_2^2} + i\frac{y_1x_2 - x_1y_2}{x_2^2 + y_2^2}$

C) $\frac{x_1x_2 + y_1y_2}{x_2^2 + y_2^2} - i\frac{y_1x_2 - x_1y_2}{x_2^2 + y_2^2}$ D) $\frac{x_1x_2 + y_1y_2}{x_2^2 - y_2^2} + i\frac{y_1x_2 - x_1y_2}{x_2^2 - y_2^2}$

2. $\left[\frac{1+i}{1-i}\right]^5 - \left[\frac{1-i}{1+i}\right] = ?$

A) i B) – i C) 2i D) – 2i

3. The absolute value of $\frac{2+i}{4i(1+i)^2}$

A) $\sqrt{2}$ B) $\sqrt{5}$ C) $\frac{\sqrt{5}}{b}$ D) $\frac{b}{\sqrt{5}}$

4. One value of arg Z when $Z = \frac{-2}{1+i\sqrt{3}}$

A) $\frac{2\pi}{3}$ B) $\frac{\pi}{2}$ C) $-\frac{\pi}{2}$ D) $-\frac{2\pi}{3}$

5. The values of $(-i)^{\frac{1}{3}}$

A) $\pm(1+i)$ B) $i, \frac{\sqrt{3}-i}{2}$ C) $i, \pm\frac{\sqrt{3}-i}{2}$ D) $i, \pm\frac{\sqrt{3}+i}{2}$

6. Find the complex numbers represented by the points $\left(\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}, 0\right)$

A) i B) – i C) 1 D) $\frac{1+i}{\sqrt{2}}$

7. Find the value of $\lim_{z \to -i} \frac{\bar{z} + z^2}{1 - \bar{z}}$

A) 1 B) i C) – 1 D) – i

8. $f(z)=\cos x(\cosh y+a\sinh y)+i\sin x(\cosh y+b\sinh y)$

A) $a=1, b=1$ B) $a=-1, b=-1$

C) $a=1, b=-1$ D) $a=-1, b=1$

9. Which one is incorrect?

A) If f is analytic at ever point of a region D then f is said to be analytic in D

B) A function which is analytic at every point of the complex plant is called an entire function or integral function

C) Any polynomial is an entire function

D) $f(z)=|z|^2$ f is differentiable at z = 0 but not analytic at $z \neq 0$

10. Which one is not an analytic function?

A) z^3+z B) $e^x(\cos y+i\sin y)$

C) $e^x(\cos y-i\sin y)$ D) $e^{-x}(\cos y-i\sin y)$

11. The power series $\sum_{n=0}^{\infty} z^n = 1+z+z^2+\ldots+z^{n-1}+\ldots$

A) diverges if $|z|<1$ and converges if $|z|\geq 1$

B) diverges if $|z|\geq 1$ and converges if $|z|<1$

C) diverges if $|z|>1$ and converges if $|z|\leq 1$

D) None of these

12. Consider the power series is convergence if

A) $z=\pm 1$ B) $z=1$ C) $z=-1$ D) $z=a$

13. The radius of convergence of the series

$$\frac{1}{2}z+\frac{1}{2}\cdot\frac{3}{5}z^2+\frac{1}{2}\cdot\frac{3}{5}\cdot\frac{5}{8}z^3+\ldots$$

A) 3 B) 2 C) 2/3 D) 3/2

14. Which one is wrong?

A) $e^{iz} = 1 + \frac{iz}{1!} - \frac{z^2}{2!} - \frac{iz^3}{3!} + \ldots$

B) $e^{iz} = 1 - \frac{iz}{1!} + \frac{z^2}{2!} - \frac{iz^3}{3!} + \ldots$

C) $\cos z = 1 - \frac{z^2}{2!} + \frac{z^4}{4!} \ldots$

D) $\sin z = z - \frac{z^3}{3!} + \frac{z^5}{5!} \ldots$

15. Which one is wrong?

A) $\cos z = \frac{e^{iz} + e^{-iz}}{2}$

B) $\sin z = \frac{e^{iz} + e^{-iz}}{2!}$

C) $\cosh z = \frac{e^{z} + e^{-z}}{2}$

D) $\sinh z = \frac{e^{z} - e^{-z}}{2}$

16. The function $f(x)$ is said to be continuous at a iff

A) $\lim_{x \to a} f(x) = f(a)$

B) $\lim_{x \to a^+} f(x) = f(a)$

C) $\lim_{x \to a} f(x)^{-1} = f(a)$

D) $\lim_{x \to a} f(x) = 0$

17. The function u with satisfies Laplace equation $\Delta u = 0$ is said to be

A) Homorphic B) Analytic C) Harmonic D) Conjugate

18. If $u = x^2 - y^2$ then the analytic function $f(z) =$

A) $2xy + c$ B) $z^3 + ic$ C) $z^2 + ic$ D) $z^3 - ic$

19. If g(w) and f(z) are analytic function then

A) g(z) is analytic

B) g(f(z)) is analytic

C) f(g(z)) is analytic

D) g(f(w)) is analytic

20. The function f(z) and $f(\overline{z})$ are

A) harmonic B) conjugate C) analytic D) constant

21. The Bilinear Transformation which map $\operatorname{Im} Z \geq 0$ onto $|w| \leq 1$ are of the form

A) $w = e^{i\lambda} \frac{z - z_1}{z - \overline{z}_1}$ B) $w = \frac{z - \overline{z}_1}{z - \overline{z}_1}$ C) $w = e^{-i\lambda} \frac{z - z_1}{z - z_1}$ D) $w = \frac{z + z_1}{z + \overline{z}_1}$

TEACHER'S CARE ACADEMY
38/23, Vaigundaperumal Koil Street, Kanchipuram-1
Mobile : 95665 35080, 97862 69980
Help Line Numbers : 9360268118,7639967359

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MATHEMATICS
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UNIT VIII
Mechanics

Your Success is Our Goal….

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UNIT –VIII – MECHANICS: STATICS & DYNAMICS

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SANNATHI STREET, KANCHIPURAM – 1. CELL: 9566535080
B.Off 2: 65C, Thillai Ngr(West), 4th Cross St, Trichy – 620018
B.Off 3:266-C, Advaitha Ashram Road,(Opp to New Bus Stand),Salem-636 004.
Trichy: 76399 67359 Salem: 93602 68118

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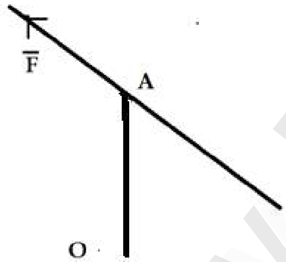
UNIT VIII : MECHANICS – STATICS & DYNAMICS

STATICS– PART - I

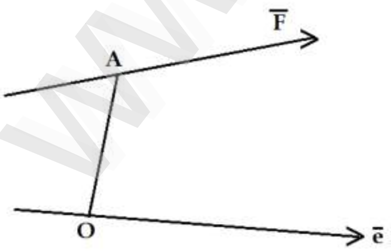
UNIT I - FORCES ON A RIGID BODY

8.1. MOMENT OF A FORCE

- Let $\overline{F}$ be a force and A, a point on its on line of action. Let O be a point in space, then the vector
- $\overline{OA} \times \overline{F}$ is called the moment of $\overline{F}$ about O.



Moment of a Force About a Line:

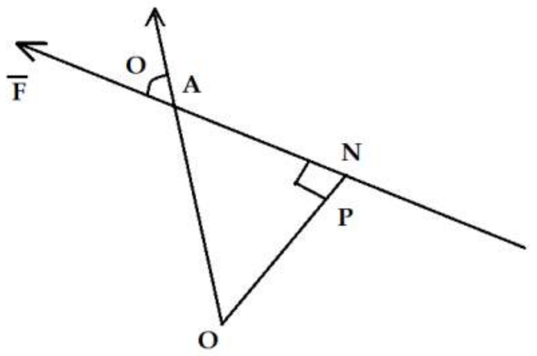


- Let $\overline{F}$ be a force and A, a point on its line of a action. Let l be a directed line through a point O, the direction of the line being specified by $\dot{e}$, then the scalar triple product

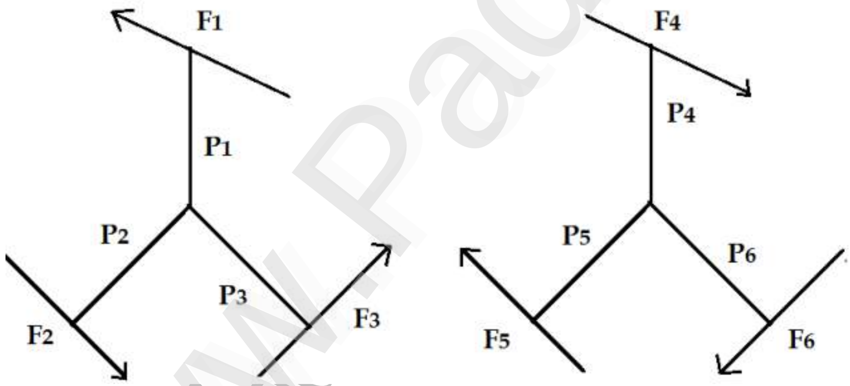
- $(\overline{OA}\times\overline{F}).\dot{e}$ is called the moment of the force $\overline{F}$ about ⸪.

Scalar Moment:

- Let $\overline{F}$ be a force in a plane. Let A be a point on its line of action and O, any point in the plane. Let ON be the perpendicular from O to the line and ON = P then the moment $\overline{F}$ about O is
- $\overline{OA}\times\overline{F} = OA.F \sin\theta\hat{n} = PF\dot{n}$



- Where θ is the angle between $\overline{OA}$ and $\overline{F}$, and $\dot{n}$ is the unit vector perpendicular to $\overline{OA}$, $\overline{F}$ such that $\overline{OA}, \overline{F}, \hat{n}$ from a right handed biad. Now we call p^F of the scalar moment of $\overline{F}$ about O.



- the scalar moments of F_1, F_2, F_3 in the first figure are

  P_1F_1, P_2P_2, P_3F_3

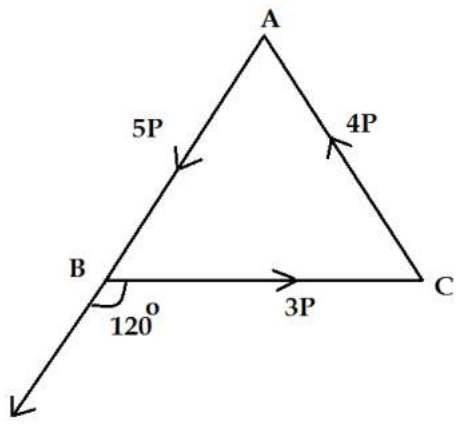
- which are positive and the moments of F_4, F_5, F_6 in the second figure are

  $-P_4F_4, -P_5F_5, -P_6F_6$

- Which are negative, the first three forces are such as to cause on a rigiid body a rotational motion in the anticlockwise sense and the other three to cause a rotational motion in the clockwise sense.

Example

- Forces of a magnitudes 3P, 4P, 5P, act along the sides BC, CA, AB of an equilateral triangle of side a. Find the moment of the resultant about A,



- the moment of the resultant about A equals the sum of the moments of the individual forces about A. But the forces 4P, 5P pass through A. So their moments about A are zero, the moment of 3P which passes through B is

$$\overline{AB}\times\left(3P\hat{BC}\right)=AB.2P\sin120^{\circ}\hat{n}$$

$$=a.3P.\frac{\sqrt{3}}{2}\hat{n}$$

- So, this is the moment of the resultant about A.

Exercise - 1

1. If three parallel forces are in equilibrium then each is proportional to the

   (A) Angle between the other two

   (B) n Distance between the other two

   (C) Cosine of the angle between the other two

   (D) None of these

2. S is the circumcentre of a triangle ABC.Forces of magnitudes P, Q, R acting along SA, SB, SC respectively are in equilibrium. Then P, Q, R are in the ratio

   (A) $\cos\frac{A}{2}:\cos\frac{B}{2}:\cos\frac{C}{2}$ (B) $a:b:c$

   (C) $\sin2A:\sin2B:\sin2C$ (D) SA:SB:SC

3. Maximum range on an inclined plane of inclination β is

(A) $\dfrac{u^2}{g(1+\cos\alpha)}$ (B) $\dfrac{u^2}{g(1+\sin\beta)}$

(C) $\dfrac{u^2}{g(1-\cos\alpha)}$ (D) $\dfrac{u^2}{g(1-\sin\beta)}$

8.2. GENERAL MOTION OF A RIGID BODY

- In this section we extend the Newton's laws of motion, N.1, N.2, N.3 to the motion of a rigid body.

Rigid Body:

- A system of particles such that the distance between any two of them is always constant, is called a rigid body.

Applied Forces:

- Forces applied on a body by external agencies are called applied forces on the body

Effective Forces:

- If a particle of mass m has an acceleration $\ddot{r}$, then the quantity $m\ddot{r}$ is called the effective force of the particle. With the nomenclature are have that the equation of motion of the particle, $m\ddot{r} = \bar{F}$, is that the effective force on a particle = the applied force on a particle

Exercise - 2

1. If a particle is projected with a velocity of 490 meters/sec at an elevation of $30°$ then the time of flight.

   (A) 5 seconds (B) 25 seconds (C) 50 seconds (D) 100 seconds

2. A particle is thrown vertically upwards with a velocity u. The time taken by it toreach the maximum height is ______

   (A) $\frac{u^2}{g}$ (B) $\frac{2u}{g}$ (C) $\frac{u^2}{2g}$ (D) $\frac{u}{g}$

3. Two forces of magnitude 7 and 8 act a point. If the magnitude of the resultant force is 13. Then angle between the two forces is ______

   (A) 30° (B) 45° (C) 60° (D) 90°

8.3. EQUIVALENT (OR EQUI POLENT)

Systems of Forces

- Two systems of forces, which produce the same motion on a given rigid body are equivalent or equipotent so, from the equations of the motion of the mass centre and motion of the body about the mass centre , we get that two systems of forces are equivalent or equipolent.

  (i) If the vector sum of the forces of one system equals the Vector sum of forces of the other system and

  (ii) If the Vector sum of the moments of the forces of one system, about any fixed point or other mass centre, equals the Vector sum of the moments of the forces of the other system about the same point on the mass centre

- In symbols, the system of forced $\bar{F}_i$ acting at $\bar{r}_i$ on a rigid body is equivalent to the system of forces $\bar{F}_j$ acting at $\bar{r}'_j$ on the rigid body if

$$\sum_i \bar{F}_i = \sum_j \bar{F}'_j,$$

$$\sum_i \bar{r}_i \times \bar{F}_i = \sum_j \bar{r}'_j \times \bar{F}'_j$$

Exercise - 3

1. The centre of parallel forces is ______

   (A) Not a unique paint (B) Not a multi point

   (C) a multi point (D) a unique point

2. The ratio of the limiting friction to the normal reaction is called the ____

   (A) coefficient of friction (B) angle of friction

   (C) cone of friction (D) None of these

3. Two couples in the same plane whose moments are equal and of the same sign are _____

   (A) not equivalent to one another

   (B) equivalent to one another

   (C) equivalent to a force

   (D) None of these

8.4. PARALLEL FORCES

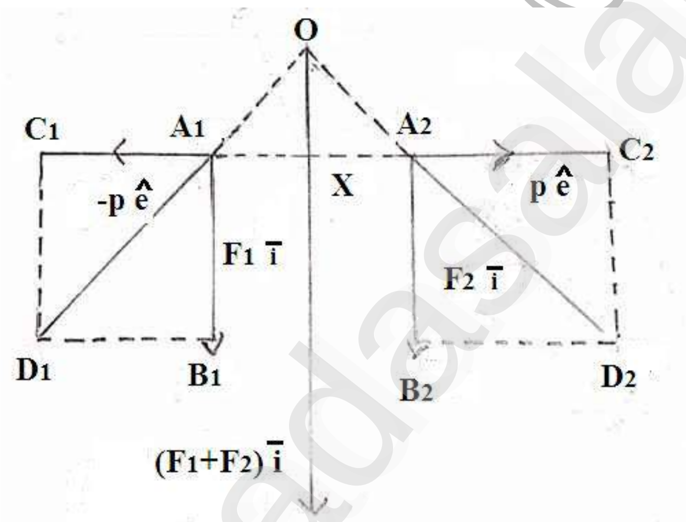
- Forces whose lines of action are parallel are called parallel forces. If their directions are in the same sense, then they are called like parallel forces otherwise they are called unlike parallel forces.

Book Work

- To find the resultant of two parallel forces acting on a rigid body

Case (i)

- Let the forces be like parallel forces, namely $F_1\bar{i}$ and $F_2\bar{i}$ acting at A_1 and A_2 respectively, where $\bar{i}$ is the unit vector in the direction of the forces,



- Let $\hat{e}$ be the unit Vector in the direction of $\overline{A_1A_2}$. Introduce a force $-p\hat{e}$ at
- A_1 and a force $p\hat{e}$ at A_2 .Since these two forces are equal in magnitude and opposite in direction and act along the same line , their introduction will not affect the effects of the given two forces,

Let $\overline{A_1B_1} = F_1\bar{i}, \overline{A_2B_2} = F_2\bar{i}, \overline{A_1C_1}$

$= -p\hat{e}, \overline{A_2C_2}$

$= p\hat{e}$

- Complete the parallelogram A,B,C,D and $A_2B_2D_2C_2$ then the resultantof two forces $F_1\bar{i}$ and $-p\hat{e}$ acting at A_1 is

$$\overline{A_1D_1} = F_1\hat{i} - p\hat{e}$$

and the resultant of the forces $F_2\hat{i}$ and $p\hat{e}$ acting at A_2 is

$$\overline{A_2D_2} = F_2\hat{i} + p\hat{e}$$

If the lines A, D, and A_2D_2 intersect at O, then the resultant of these two resultants is

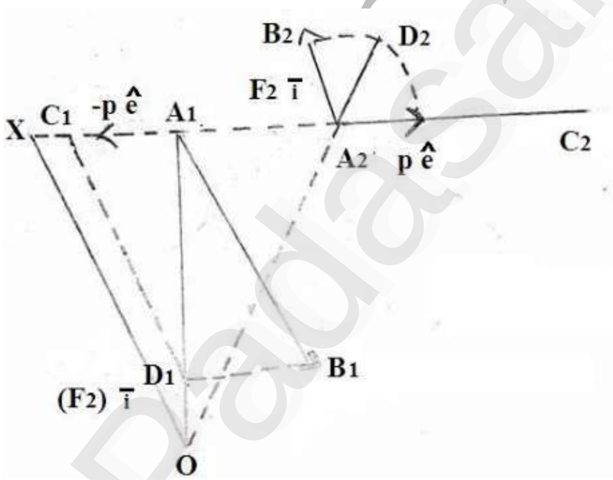
$$\overline{A_1D_1} + \overline{A_2D_2} = \left(F_1\bar{i} - p\dot{e}\right) + \left(F_2\bar{i} + p\bar{e}\right)$$

$$= \left(F_1 + F_2\right)\bar{i}$$

acting at C. Note that their resultant is parallel to the original forces.

Cases (ii)

- Let the given forces be unlike parallel forces $F_1\bar{i}$ and $F_2\left(\bar{i}\right), \left(F_1 > F_2\right)$, acing at A_1 and A_2 respectively.



- If we adopt the procedure followed in case (i), we see that the steps of case (i) repeat with the only difference that instead of F_2 they have $-F_2$ their we get that the resultant of the forces $F_1\bar{i}$ and $-F_2\bar{i}$ acting at A_1 and A_2 is $\left\{F_1 + \left(-F_2\right)\right\}\bar{i}$ acting at the point which divides A_1A_2 in the ratio $\left(-F_2\right):F_1$, that is,at the point which divides A_1A_2 externally in the ratio $F_2 : F_1$.

Example

- Two like parallel forces of magnitudes P, Q act on a rigid body. If Q is changed to $\frac{P^2}{Q}$, with the line of action being the same, show that line of the action of the resultant will be the same as it would be,if the forces were simply interchanged.

Solution

- If the forces, P and $\frac{P^2}{Q}$, act at A, B, then their resultant divides AB.
- Internally in the ratio

$$\frac{P^2}{Q} : P \ (\text{or}) \ \frac{P}{Q} = 1 \ (\text{or}) \ P : Q$$

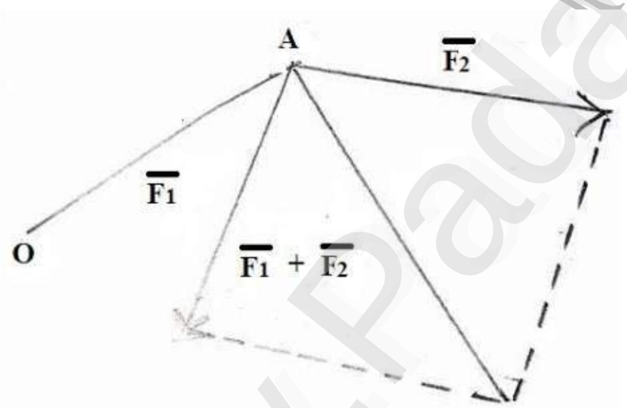
- For the second case also, the ratio is the same P: Q. Further all the involved forces and the resultants are parallel to one another.

Varignon's Theorem

- The sum of the moments of two intersecting or parallel force about any point in equal to the moment of the resultant of the forces about the same point

Intersecting Forces

Case (i)



- Let the lines of action of the forces $\overline{F}_1$ and $\overline{F}_2$ intersect at A, then the moment of $\overline{F}_1$ and $\overline{F}_2$ about any point O are

$$\overline{OA} \times \overline{F}_1 , \overline{OA} \times \overline{F}_2$$

- and their sum is

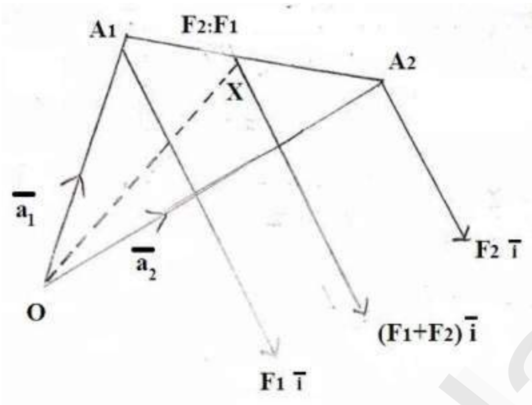
$$\overline{OA} \times \overline{F}_1 + \overline{OA} \times \overline{F}_2$$

- But the resultant of $\overline{F}_1$ and $\overline{F}_2$ acting at A, so it moment about O is $\overline{OA} \times \left(\overline{F}_1 + \overline{F}_2\right)$
- Since $\overline{OA} \times \overline{F}_1 + \overline{OA} \times \overline{F}_2 = \overline{CA} \times \left(\overline{F}_1 + \overline{F}_2\right)$ the theorem follows for the intersecting forces

Case (ii)

Parallel Forces

- Let the parallel forces be $\bar{F}_1 = F_1\bar{i}$ and $\bar{F}_2 = F_2\bar{i}$ acting at A_1 and A_2. Let $\bar{a}_1, \bar{a}_2$ be the P.V's of A_1, A_2 with respect to O, then the moment of $\bar{F}_1, \bar{F}_2$ about O are



$$\bar{a}_1 \times F_1\bar{i} . \bar{a}_2 \times F_2\bar{i}$$

their sum

$$\bar{a}_1 \times F_1\bar{i} + \bar{a}_2 \times F_2\bar{i} = \left(F_1\bar{a}_1 + F_2\bar{a}_2\right) \times \bar{i}$$

- But the resultant of $F_1\bar{i}$ and $F_2\bar{i}$ is $(F_1 + F_2)\bar{i}$ acting at x, where x divides A_1A_2 internally in the rate $F_2 : F_1$ to the P.V of x is

$$\frac{F_1\bar{a}_1 + F_2\bar{a}_2}{F_1 + F_2} ------(1)$$

- So, the moment of the resultant about O is

$$\overline{OX} \times (F_1 + F_2)\bar{i} = \frac{F_1\bar{a}_1 + F_2\bar{a}_2}{F_1 + F_2} \times (F_1 + F_2)\bar{i}$$

$$\left(F_1\bar{a}_1 + F_2\bar{a}_2\right) \times \bar{i} ------(2)$$

- From (1) and (2) we get the theorem for parallel forces.

Example

- Three like parallel forces P, Q, R act at the vertices of a triangle ABC, show that their resultant passes through

(i) The centroid if P = Q = R, (ii) the in centre if $\frac{P}{a} = \frac{Q}{b} = \frac{R}{c}$

Let $\bar{a}, \bar{b}, \bar{c}$ be the P. V's of A, B, C, then the resultant passes through the pointwhose P.V is

$$\frac{P\bar{a}+Q\bar{b}+R\bar{c}}{P+Q+R}$$

(i) If P=Q=R, then

$$\frac{P\bar{a}+Q\bar{b}+R\bar{c}}{P+Q+R}=\frac{\bar{a}+\bar{b}+\bar{c}}{3}$$

Which is the P.V of the centroid

(ii) If $\frac{P}{a}=\frac{Q}{b}=\frac{R}{c}=k$, then

$$\frac{P\bar{a}+Q\bar{b}+R\bar{c}}{P+Q+R}=\frac{R\left(a\bar{a}+b\bar{b}+c\bar{c}\right)}{k\left(a+b+c\right)}$$

$$\frac{a\bar{a}+b\bar{b}+c\bar{c}}{a+b+c}$$

Which is the P.V of the incentre

Exercise - 4

1. The centre of gravity of a triangle is _____

   (A) orthocentre (B) incentre

   (C) centroid (D) circumcentre

2. On lighting a rocket cracker it gets projected in a parabolic path and reaches a maximum height of 4m. When its is 6m away from the point of projection. Finally it reaches the ground 12m away from the starting point. The angle of projection at the origin is _____

   (A) $\tan^{-1}\left(\frac{3}{4}\right)$ (B) $\tan^{-1}\left(\frac{1}{3}\right)$

   (C) $\tan^{-1}\left(\frac{1}{4}\right)$ (D) $\tan^{-1}\left(\frac{4}{3}\right)$

3. A particle is tossed up vertically with velocity of 19.6 m/sec. The time taken to reach the maximum height is_____

   (A) 4 secs (B) 1 sec

   (C) 2 secs (D) 2/3 sec

8.4.1. FORCES ALONG THE SIDES OF A TRIANGLE

Example

- Three forces P, Q, R act along the sides BC, CA, AB of a triangle ABC. If their resultant passes through the incentre and cenbioid, the show that

$$\frac{P}{a(b-c)}=\frac{Q}{b(c-a)}=\frac{R}{c(a-b)}$$

Since the resultant passes through the incentre and cenbioid. We have respectively

$$P+Q+R=0-----(1)$$

$$\frac{P}{a}+\frac{Q}{b}+\frac{R}{c}=0-----(2)$$

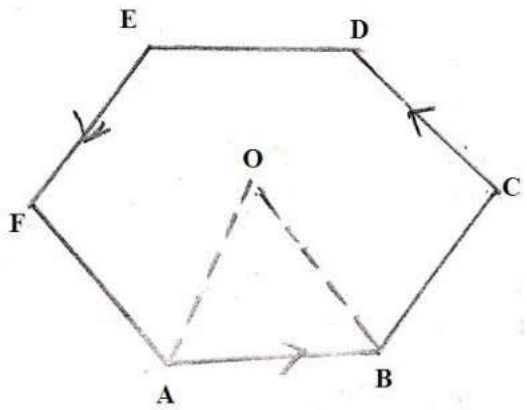
Solving (1) and (2)

$$\frac{P}{\begin{vmatrix}1 & 1\\ \frac{1}{b} & \frac{1}{c}\end{vmatrix}}=\frac{Q}{\begin{vmatrix}1 & 1\\ \frac{1}{c} & \frac{1}{a}\end{vmatrix}}=\frac{R}{\begin{vmatrix}1 & 1\\ \frac{1}{a} & \frac{1}{b}\end{vmatrix}}$$

$$\frac{P}{\frac{1}{c}-\frac{1}{b}}=\frac{Q}{\frac{1}{a}-\frac{1}{c}}=\frac{R}{\frac{1}{b}-\frac{1}{a}}$$

8.5. RESULTANT OF SEVERAL COPLANAR FORCES

- Show that the forces $\overline{AB},\overline{CD},\overline{EF}$ acting respectively at A, C, E of a regular hexagon ABCDEF, are equivalent to a couple of moment equal to the area of the hexagon.



- Let O be the centre of the hexagon
- Now the sum of the forces is

  $\overline{AB}+\overline{CD}+\overline{EF}$

  $\overline{AB}+\overline{BD}+\overline{OA}$

- It is evident that is zero, so either the system is in equilibrium or it reduces to a couple.
- Multiplying the denominators by abc, we get the result.
- But the moment of $\overline{AB}$ about O is

  $\overline{OA}\times\overline{AB}=OA.AB\sin OAB\,\overline{k}$

  $=2\Delta\overline{k}$

- Where Δ is the area of ΔAOB. By symmetry the sum of the moments of all the forces is $3(2\Delta)\overline{k}\,(or)\,6\Delta\overline{k}$, so the system reduces to a couple of moment 6Δ. But the area of the hexagon also is 6Δ.

Exercise - 5

1. The resolved part of a force in its own direction is the force itself ___

   (A) when $\theta=\pi$ (B) when $\theta=0$

   (C) when $\theta=\frac{\pi}{2}$ (D) when $\theta=\frac{3\pi}{2}$

2. O is the orthocentre and S is the circumcentre of a triangle AB. The resultant of forces OA, OB, OC is

   (A) AB (B) BC

   (C) OS (D) 2OS

3. Three like parallel forces P, Q, R act at the corners of a triangle ABC. Then their centre is the orthocentre of the triangle if

   (A) $\frac{P}{OA}=\frac{Q}{OB}=\frac{R}{OC}$

   (B) $\frac{P}{\tan A}=\frac{Q}{\tan B}=\frac{R}{\tan C}$

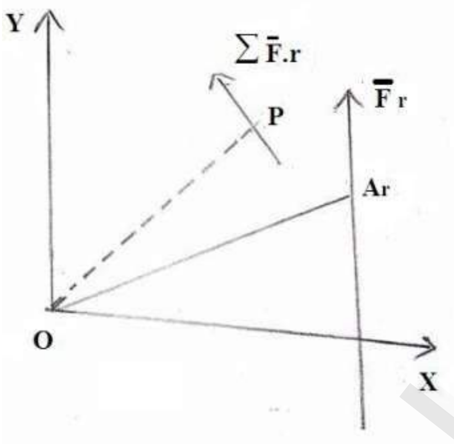
   (C) $P\tan A=Q\tan B=R\tan C$

   (D) $\frac{P}{\sin(Q,R)}=\frac{Q}{\sin(P,R)}=\frac{R}{\sin(P,Q)}$

8.6. EQUATION OF THE LINE OF ACTION OF THE RESULTANT

Book Work

- When a system of Coplanar forces $\bar{F}_1, \bar{F}_2, ...\bar{F}_n$, acting at $A_1, A_2, ...\Delta_n$, reduce to a single force, to find the equation of line of action



- Choose any two perpendicular linesOx, Oy in the plane of the forces as the x, y axes and let $\bar{i}, \bar{j}$
- Let the unit Vectors in their direction. Let p(x, y) be any point on the line of action of the resultant force $\sum \bar{F}_r$ of the system. Then any relation is x,y is the equation of the line . Now

$$\overline{OP} = x\bar{i} + y\bar{j}$$

- Let P_r, Q_r be the components of $\bar{F}_r$ in the $\bar{i}, \bar{j}$ directions, then

$$\bar{F}_r = P_r\bar{i} + Q_r\bar{j}$$

- since the sum of the moments of the forces about any point, say O, equals the moment of their resultant about O,

$$\sum\left(\overline{OA}_r, \bar{F}_r\right) = \overline{OP} \times \left(\sum \bar{F}_r\right)$$

(or)

$$\overline{OP} \times \left(\sum \bar{F}_r\right) - \sum\left(\overline{OA}_r \times \bar{F}_r\right) = 0$$

i.e.,

$$\left(x\bar{i} + y\bar{j}\right) \times \sum\left(P_r\bar{i} + Q_r\bar{j}\right) - \sum\left(\overline{OA}_r \times \bar{F}_r\right) = \bar{O}$$

i.e.,

$$(x\bar{i}+y\bar{j})\times\{(\sum P_r)\bar{i}+(\sum Q_r)\bar{j}\}-\sum(\overline{OA}_r\times\bar{F}_r)=\bar{O}$$

i.e.,

$$x(\sum Q_r)\bar{k}-y(\sum P_r)\bar{k}-(\sum P_rF_r)\bar{k}=\bar{O}$$

- The sum of the moment about with the usual meaning for $\bar{k}$ and P_r being the perpendicular distance of O from E_r such that its value is positive or negative according as the sense of rotation of $\bar{F}_r$ about O is anticlockwise or not thus the equation of the line of action of the resultant is

$$(\sum Q_r)x-(\sum P_r)y-\sum P_rF_r=0----(1)$$

(or)

$$(\sum Q_r)x-(\sum P_r)y-\sum G_r=0$$

Where $G_r=P_rF_r$

this equation can be put in the elegant form

$$Y_x-X_y-G=0-----(2)$$

Where

$X=\sum P_r$ = sum of the components of the forces in the x direction

$Y=\sum Q_r$ = sum of the component of the forces in the y direction

$G=\sum G_r=\sum P_rF_r$ = sum of the scalar moments of the forces about the origin.

Now we have that the resultant force is

$$\bar{F}_1+\bar{F}_2+....+\bar{F}_n$$

(or)

$$X\bar{i}+Y\bar{j}$$

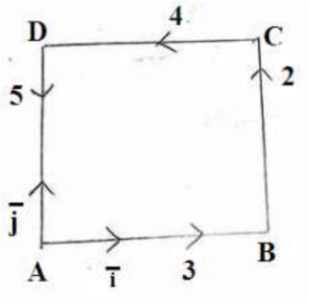
Whose magnitude is $\sqrt{X^2+Y^2}$ and the line of action is

$$y_x-X_y=G$$

the slope of the line is $\frac{Y}{X}$.

Examples

- Forces 3,2,4,5 Kg. wt. act along the sides AB, BC, CD, CA of a square. Find their resultant and its line of action.



Let $\bar{i},\bar{j}$ be the unit Vectors parallel to $\overline{AB},\overline{AD}$ and $AB=a\bar{j}$.

Let AB, AD be the x, y axes, the vector sum of the forces is

$$(3\bar{i})+(2\bar{j})+(-4\bar{i})+(-5\bar{j})=-\bar{i}-3\bar{j}$$

Let X, Y, be the sum of $\bar{i},\bar{j}$

Components of the forces and G, the sum of the moments about the origin A, then

$$X=-1, Y=-3$$

the magnitude of the resultant force is

$$\sqrt{X^2+Y^2}=\sqrt{(-1)^2+(-3)^2}=\sqrt{10}$$

$$G=0\times3+a(2)+a(4)+0\times5=6a$$

the equation of line of action of the resultant forces is

$$\begin{vmatrix} X & Y \\ x & y \end{vmatrix}+G=0$$

(or)

$$\begin{vmatrix} -1 & -3 \\ x & y \end{vmatrix}+6a=0$$

i.e.,

$$-y+3x+6a=0$$

Exercise – 6

1. A solid sphere of mass m rolls down a plane inclined to the horizon at an angle α . The acceleration is

(A) $\frac{g\sin\alpha}{7}$ (B) $\frac{3g\sin\alpha}{7}$

(C) $\frac{4g\sin\alpha}{7}$ (D) $\frac{5g\sin\alpha}{7}$

2. A 100 gm cricket ball moving horizontally at 24 m/s was hit straight back with a speed of 15 m/s. If the contact lasted $\frac{1}{20}$ second. The average force exerted by the bat is ____

(A) 78000 Dynes (B) 8000 Dynes

(c) 90000 Dynes (D) 1500 Dynes

3. Let u and v be two velocities at the point A then their resultant direction is _____

(A) $\tan\theta = \frac{v\cos\alpha}{u+v\sin\alpha}$ (B) $\tan\theta = \frac{u\cos\alpha}{v+u\sin\alpha}$

(C) $\tan\theta = \frac{v\sin\alpha}{u+v\cos\alpha}$ (D) $\tan\theta = \frac{v\sin\alpha}{v+u\sin\alpha}$

8.7. EQUILIBRIUM OF A RIGID BODY UNDER THREE COPLANAR FORCES

Book Work

- If three coplanar forces keep a rigid body in equilibrium, then either they all are parallel to one another or they are concurrent.
- Let the forces be $\bar{F}_1, \bar{F}_2, \bar{F}_3$ considering only $\bar{F}_1$ and $\bar{F}_2$, we get the following two cases

  (i) $\bar{F}_1$ and $\bar{F}_2$ are parallel

  (ii) $\bar{F}_1$ and $\bar{F}_2$ are not parallel

Case (i)

- Suppose $\bar{F}_1 = F_1\bar{i}$ and $\bar{F}_2 = F_2\bar{i}$ act at A_1 and A_2,...then their resultant is $(F_1+F_2)\bar{i}$. Consequently their resultant $(F_1+F_2)\bar{i}$ and $\bar{F}_3$ keep the body in equilibrium, this implies not only that these two forces act along the same line but also that $\bar{F}_3 = -(F_1+F_2)\bar{i}$ so $\bar{F}_3$ is parallel to $\bar{F}_1$ and $\bar{F}_2$ that is the given there forces are parallel to one another.

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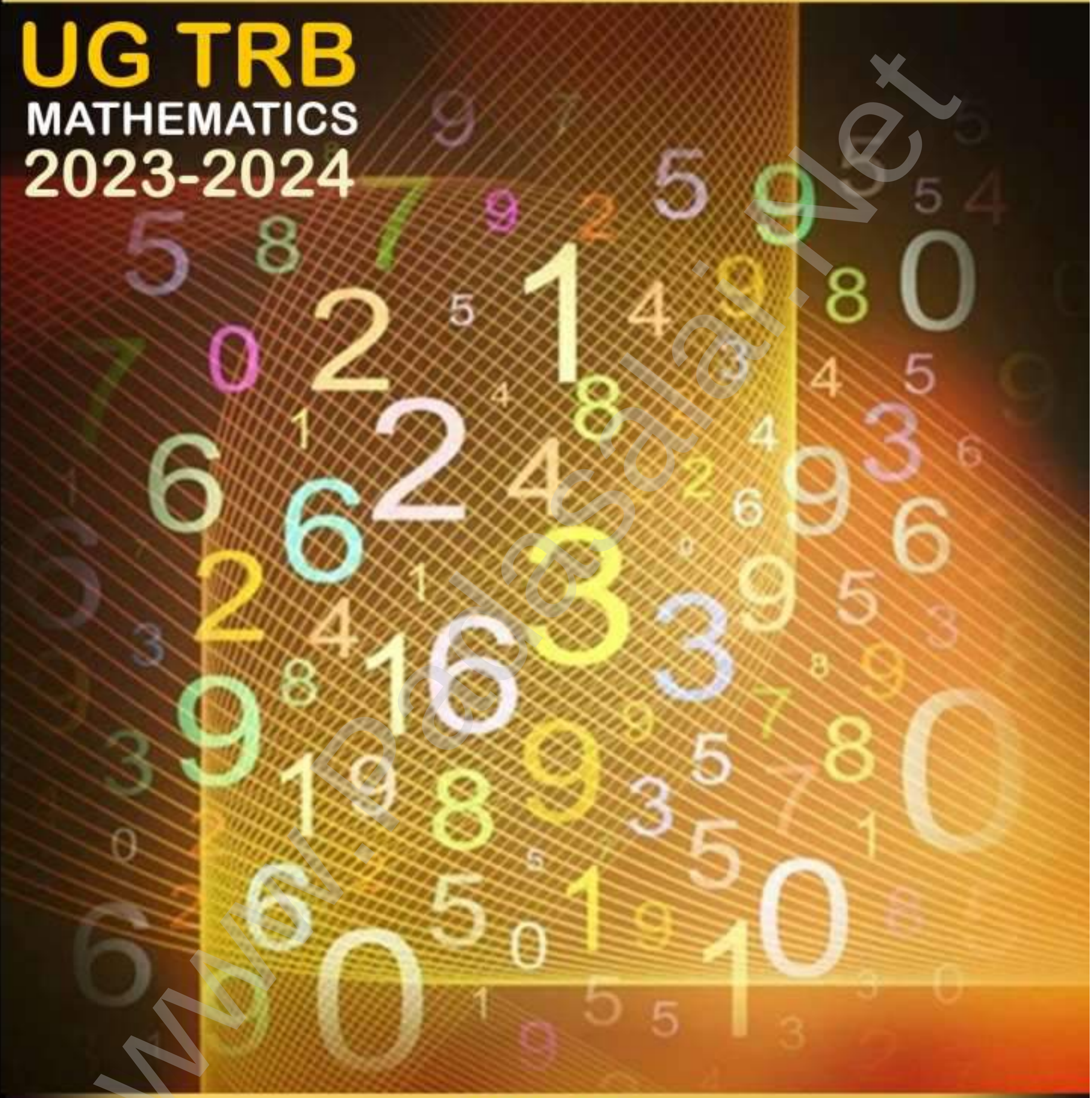

UNIT IX
Operations Research

Your Success is Our Goal....

UNIT – IX - OPERATIONS RESEARCH

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UNIT - IX

OPERATIONS RESEARCH

1.1. Introduction:

- ❖ Operations Research is the study of optimisation techniques. It is applied decision theory. The existence of optimisation techniques can be traced at least to the days of Newton and Lagrange. Rapid development and invention of new techniques occurred since the World War II essentially, because of the necessary to win the war with the limited resources available.
- ❖ Different teams had to do research on military operations in order to invent techniques to manage with available resources so as to obtain the desired objective. Hence the nomenclature Operations Research or Resource Management Techniques.

1.2. Scope or Uses or Applications of O.R.:

O.R. is useful for solving.

- Resource allocation problems.
- Inventory control problems.
- Maintenance and Replacement problems.
- Sequencing and scheduling problems.
- Assignment of jobs to applicants to maximise total profit or minimize total cost.
- Transportation problems.
- Shortest route problems like travelling sales person problems.
- Marketing Management problems.

- Finance Management problems.
- Production, planning and control problems.
- Design problems
- Queuing problems, etc. to mention a few.

1.3. Role of Operations Research In Business And Management:

1. **Marketing management Operations research techniques have definitely a role to play in**

   (a) Product selection

   (b) Competitive strategies

   (c) Advertising strategy etc

2. **Production Management:**

   (a) Production scheduling

   (b) Project scheduling

   (c) Allocation of resources

   (d) Location of factories and their sizes

   (e) Equipment replacement and maintenance

   (f) Inventory policy etc.

3. **Finance Management**

   (a) Cash flow analysis

   (b) Capital requirement

   (c) Credit policies

   (d) Credit risks etc.

4. **Personal Management**

   (a) Recruitment policies and

   (b) Assignment of jobs are some of the areas of personnel management where O.R. techniques are useful.

5. **Purchasing and procurement:**

   (a) Rules for purchasing

   (b) Determining the quality

   (c) Determining the time of purchaser are some of the areas where O.R. techniques can be applied.

6. Distribution

(a) Location of warehouses

(b) Size of the ware houses

(c) Rental outlets

(d) Transportation strategies

1.4. Classification of Models:

❖ The first thing one has to do to use O.R. techniques after formulating a practical problem is to construct a suitable model to represent practical problem. A model is a reasonably simplified representation of a real-world situation. It is an abstraction of reality. The models can broadly be classified as.

Iconic Model

❖ This is physical, or pictorial representation of various aspects of a system.

Example:

❖ Toy, Miniature model of a building, scaled up model of a cell in biology etc.

Analogue or schematic model:

❖ This uses one set of properties to represent another set of properties which a system under study has

Example:

❖ A network of water pipes to represent the flow of current in an electrical network or graphs organisational charts etc.

Mathematical model symbolic Model:

❖ This uses a set of mathematical symbols (letters, numbers, etc) to represent the decision variables of a system under consideration. These variables related by mathematical equations or inequalities which describes the properties of the system.

Example:

❖ A linear programming model, A system of equations representing an electrical network or differential equations representing dynamic systems etc.

Static model:

❖ This is a model which does not take time into account. It assumes that the values of the variables do not change with time during a certain period of time horizon.

Example:

- A linear programming problem, an assignment problem, transportation problem etc

Dynamic Model:

- This is a model which considers time as one of the important variables.

Example:

- A dynamic programming problem, A replacement problem.

Deterministic Model:

- This is a model which does not take uncertainty into account.

Example:

- A linear programming problem, an assignment problem etc.

Stochastic Model:

- This is a model which considers uncertainty as an important aspect of the problem.

Example:

- Any stochastic programming problem, stochastic inventory models etc.

Descriptive model:

- This is one which just describes a situation or system.

Example

- An opinion poll, any survey

Predictive Model:

- This is one which predicts something based on some data. Predicting election results before actually the counting is completed.

Prescriptive model:

- This is one which prescribes or suggests a course of action for a problem.

Example:

- Any programming (linear, nonlinear, dynamic, geometric etc.) problem.

Analytic model:

- This is a model in which exact solution is obtained by mathematical methods in closed form.

Simulation model:

- This is a representation of reality through the use of a model or device which will react in the same manner as reality under a given set of conditions.
- Once a simulation model is designed, it takes only a little time, in general, to run a simulation on a computer.
- It is usually less mathematical and less time consuming and generally least expensive as well, in many situations.

Example:

- Queuing problems, Inventory problems

1.5. Some Characteristics of A Good Model:

- It should be simple
- Assumptions should be as small as possible
- Number of variables should be minimum
- The models should be open to parametric treatment
- It is easy and economical to construct.

1.6. General methods for Solving O.R. Models:

(1) Analytic Procedure:

Solving models by classical mathematical techniques like differential calculus, finite differences etc. to obtain analytic solutions.

(2) Iterative Procedure:

Starts with a trial solution and a set of rules for improving it by repeating the procedure until further improvement is not possible.

(3) Monte-Carlo Technique:

Taking sample observations, computing probability distributions for the variable using random numbers and constructing some functions to determine values of the decision variables.

1.7. Main Phases of O.R.:

(i) Formulation of the Problems:

- Identifying the objective, the decision variables involved and the constraints that arise involving the decision variables.

(ii) Construction of a Mathematical Model:

- Expressing the measure of effectiveness which may be total profit, total cost, utility etc. to be optimised by a mathematical function called objective function
- Representing the constraints like budget constraints, raw materials, constraints, resource constraints, quality constraints etc, by means of mathematical equations or inequalities.

(iii) Solving the Model Constructed:

- Determining the solution by analytic or iterative or Monte-Carlo method depending upon the structure of the mathematical model.

(iv) Controlling and Updating:

- A solution which is optimum today may not be so tomorrow. The values of the variables may change, new variables may emerge. The structural relationship between the variables may also undergo a change. All these are determined in updating.
- Controls must be established to indicate the limits within which the model and its solution can be considered as reliable. This is called controlling.

(v) Testing the Model and its Solution (i.e.,) Validating the Model

- Checking as far as possible either from the past available data or by expertise and experience whether the model gives a solution which can be used in practice.

(vi) Implementation

- Implement using the solution to achieve the desired goal.

1.8. Limitation:

- Mathematical models which are the essence of OR do not take into account qualitative or emotional or some human factors which are quite real and influence the decision making.
- All such influencing factors find no place in O.R. This is the m ain limitation of O.R.
- Hence O.R is only an aid in decision making.

EXERCISES:

1. Operation research is the ____________ of providing executive with analytical and objective basic for decision

   (A) scientific method (B) economic method

   (C) both a and b (D) none of these

2. The objective of __________ is to identifies the significant factors and interrelationships.
   (A) OR (B) models (C) both a and b (D) none of these
3. ______________ model is to describe and predict the facts and relationships among the various activities of the problem.
   (A) descriptive (B) predictive (C) optimization (D) Iconic
4. ________________ models are used in predictive analysis is involving a variety of statistical techniques used to analyze the current and historical facts to make predictions about future events.
   (A) optimization (B) descriptive (C) Analogue (D) predictive
5. __________ are prescriptive in nature and develop objective decisions rules for optimum solution.
   (A) descriptive (B) predictive (C) optimization (D) Analogue
6. One set of properties to represent another set of properties which a system under study, then the model is_______________
   (A) Iconic model (B) Analogue model (C) static model (D) dynamic model
7. ___________________ is a model which does not take time into account.
   (A) Iconic model (B)symbolic model (C) dynamic model (D) static model
8. __________________ is a model which considers time as one of the important variables.
   (A) Iconic model (B) mathematical model
   (C) dynamic model (D) static model
9. __________ technique is to taking samples observations, computing probability distributions for the variable using random numbers and constructing some functions to determine values of the variables.
   (A) Monte- carlo (B) analytic (C) Iterative (D) none of these
10. If solving models by classical mathematical techniques like differential calculus, finite difference etc., to obtain analytic solution is known as _____________.
   (A) Monte- carlo technique (B) analytic procedure
   (C) Iterative procedure (D) none of these
11. If starts with a trial solution and a set of rules for improving it by repeating the procedure until further improvement is not possible is________________
   (A) Monte- carlo technique (B) analytic procedure
   (C) Iterative procedure (D) none of these

2. LINEAR PROGRAMMING FORMULATION

2.1. Introduction:

- Linear Programming problems deal with determining optimal allocations of limited resources to meet given objectives.
- The objective is usually maximizing profit. Minimizing total cost, maximizing utility etc.
- Linear programming problem deals with the optimization (maximization or minimization) of a function of decision variables known as objective function.
- Subject to a set of simultaneous linear equations (or inequalities) known as constraints.
- The term linear means that all the variables occurring in the objective function and the constraints are of the first degree in the problems under consideration and the term programming means the process of determining a particular course of action.
- Linear programming techniques are used in many industrial and economic problems.

2.2. Mathematical Formulation of L.P.P:

If $x_j\,(j=1,2,\ldots,n)$ are the n decision variables of the problem and if the system is subject to m constraints, the general mathematical model can be written in the form:

Optimize $Z = f(x_1, x_2, \ldots x_n)$

Subject to $g_i(x_1, x_2, \ldots, x_n) \leq, =, \geq b_i \; (i=1,2,\ldots,m)$ and $x_1, x_2, \ldots x_n \geq 0$

2.3. Procedure for Forming a LPP Model:

Step 1: Identify the unknown decision variables to be determined and assign symbols to them.

Step 2: Identify all the restrictions or constraints in the problem and express them as linear or inequalities of decision variables.

Step 3: Identify the objective or aim and represent it also as a linear function of decision variables.

Step 4: Express the complete formulation of LPP as a general mathematical model.

Problem 1:

- A firm manufactures two types of products A and B and sells them at a profit or Rs. 2 on type A and Rs. 3 on type B. Each product is processed on two machines M_1 and M_2. Type A requires 1 minute to processing time on M_1 and two minutes on M_2. Type B requires 1 minute on M_1 and 1 minute on M_2. Machine M_1 is available for not more than 6 hours 40 minutes while machine M_2 is available for 10 hours during any working day. Formulate the problem as a LPP so as to maximize the profit.

Solution:

Formulation of LPP is

Maximize $Z = 2x_1 + 3x_2$

Subject to the constraints

$x_1 + x_2 \le 400$

$2x_1 + x_2 \le 600$

and $x_1, x_2 \ge 0$

Problem 2:

- A company makes two types of leather products A and B. Product A is of high quality and product B is of lower quality. The respective profits are Rs. 4 and Rs. 3 per product. Each product A requires twice as much time as product B and if all products were of type B, the company could make 1000 per day. The supply of leather is sufficient for only 800 products per day (Both A and B combined), Product A requires a special spare part and only 400 per day are available. There are only 700 special spare parts a day available for product B. Formulate this as a LPP.

Solution:

Maximize $Z = 4x_1 + 3x_2$

Subject to,

$2x_1 + x_2 \le 1000$

$x_1 + x_2 \le 800$

$x_1 \le 400$

$x_2 \le 700$

and $x_1, x_2 \ge 0$

Problem 3:

❖ A firm engaged in producing two models A and B performs three operations – painting, Assembly and testing. The relevant data are as follows:

Model	Units Sale Price	Hours required for each unit		
		Assembly	Painting	Testing
A	Rs. 50	1.0	0.2	0.0
B	Rs. 80	1.5	0.2	0.1

❖ Total number of hours available are: Assembly 600, painting 100, testing 30. Determine weekly production schedule to maximize the profit.

Solution:

Maximize $Z = 50x_1 + 80x_2$

Subject to,

$x_1 + 1.5x_2 \le 600$

$0.2x_1 + 0.2x_2 \le 100$

$0.1x_2 \le 30$

and $x_1, x_2 \ge 0$

Problem 4:

❖ A person wants to decide the constituents of a diet which will fulfil his daily requirements of proteins, fats and carbohydrates at the minimum cost. The choice is to be made from four different types of foods. The yields per unit of these foods are given in the following table.

Food type	Yield/unit			Cost/unit (Rs.)
	Proteins	Fats	Carbohydrates	
1	3	2	6	45
2	4	2	4	40
3	8	7	7	85

4	6	5	4	65
Maximum Requirement	800	200	700	

❖ Formulate the L.P model for the problem

Solution:

Minimize $Z = 45x_1 + 40x_2 + 85x_3 + 65x_4$

Subject to,

$3x_1 + 4x_2 + 8x_3 + 6x_4 \geq 800$

$2x_1 + 2x_2 + 7x_3 + 5x_4 \geq 200$

$6x_1 + 4x_2 + 7x_3 + 4x_4 \geq 700$

and $x_1, x_2, x_3, x_4 \geq 0$

Problem 5:

❖ A television company operates two assembly sections, section A and section B. Each section is used to assemble the components of three types of televisions: colour, standard and Economy. The expected daily production on each section is as follows:

T.V. Model	Section A	Section B
Colour	3	1
Standard	1	1
Economy	2	6

❖ The daily running costs for two sections average Rs. 6000 for section A and Rs. 4000 for section B. It is given that the company must produce atleast 24 colours, 16 standard and 40 Economy TV sets for which an order is pending. Formulate this as a L.P.P so as to minimize the total cost.

Solution:

Maximize $Z = 6000x_1 + 4000x_2$

Subject to

$3x_1 + x_2 \geq 24$

$x_1 + x_2 \geq 16$

$2x_1 + 6x_2 \geq 40$

and $x_1, x_2 \geq 0$

Problem 6:

❖ A company produces refrigerators in Unit I and heaters in Unit II. The two products are produced and sold on a weekly basics. The weekly production cannot exceed 25 in Unit I and 36 in Unit II, due to constraints 60 workers are employed. A refrigerator requires 2 man-week of labour, while a heater requires 1 man-week of labour. The profit available is Rs. 600 per refrigerator and Rs. 400 per heater. Formulate the LPP problem.

Solution:

Maximize $Z = 600x_1 + 400x_2$

Subject to,

$2x_1 + x_2 \leq 60$

$x_1 \leq 25$

$x_2 \leq 36$

and $x_1, x_2 \geq 0$

2.4. Basic Assumptions:

The linear programming problems are formulated on the basic on the following assumptions:

1. **Proportionality:** The contribution of each variable in the objective function or its usage of the resources is directly proportional to the value of the variable.
2. **Additivity:** Sum of the resources used by different activities must be equal to the total quantity of resources used by each activity for all the resources individually or collectively.
3. **Divisibility:** The variables are not restricted to integer values.
4. **Certainty or Deterministic:** Co-efficients in the objective function and constraints are completely known and do not change during the period understudy in all the problems considered.
5. **Finiteness:** Variables and constraints are finite in number.
6. **Optimality:** In a linear programming problem we determine the decision variables so as to extremise (optimize) the objective function of the LPP.
7. The problem involves only one objective namely profit maximization or cost minimization.

2.5. Graphical Method of the Solution of a L.P.P:

- Linear programming problems involving only two variables can be effectively solved by a graphical method which provides a pictorial representation of the problems and its solutions and which gives the basic concepts used in solving general L.P.P. which may involve any finite number of variables. This method is simple to understand and easy to use.
- Graphical method is not a powerful tool of linear programming as most of the practical situations do involve more than two variables. But the method is really useful to explain the basic concepts of L.P.P to the persons who are not familiar with this. Though graphical method can deal with any number of constraints but since each constraint is shown as a line on a graph a large constraint is shown as a line on a graph, a large number of lines makes the graph difficult to read.

Problem 1:

- Solve the following L.P.P by the graphical method.

Maximize $Z = 3x_1 + 2x_2$

Subject to,

$-2x_1 + x_2 \le 1$

$x_1 \le 2$

$x_1 + x_2 \le 3$

and $x_1, x_2 \ge 0$

Solution:

- First consider the inequality constraints as equalities.

$-2x_1 + x_2 = 1$ ______ (1)

$x_1 = 2$ ______ (2)

$x_1 + x_2 = 3$ ______ (3)

and $x_1 = 0$ ______ (4)

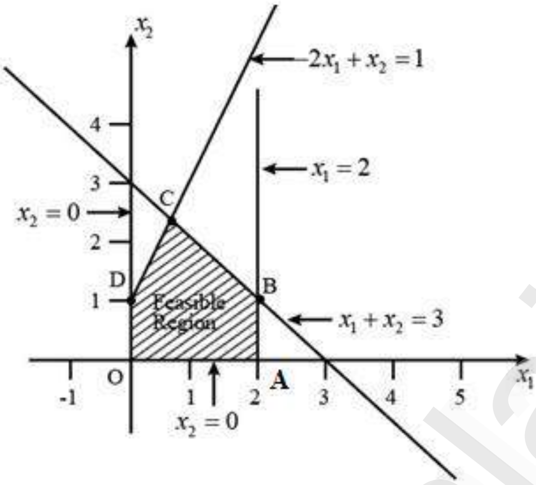
$x_2 = 0$ ______ (5)

For the line $-2x_1 + x_2 = 1$

Put $x_1 = 0 \Rightarrow x_1 = 1 \Rightarrow (0,1)$

Put $x_2 = 0 \Rightarrow -2x_1 = 1 \Rightarrow x_1 = -0.5 \Rightarrow (-0.5,0)$

❖ The vertices of the solution space are O (0, 0), A (2, 0), B (2, 1), C $\left(\frac{2}{3},\frac{7}{3}\right)$ and D $(0,1)$



❖ The value of Z at these vertices are given by $\because (z = 3x_1 + 2x_2)$

Vertex	Value of Z
O(0, 0)	0
A (2, 0)	6
B (2, 1)	8
$C\left(\frac{2}{3},\frac{7}{3}\right)$	$\frac{20}{3}$
D$(0,1)$	2

❖ Since the problem is of maximization type, the optimum solution to the L.P.P is maximum Z = 8, $x_1 = 2,\ x_2 = 1$

Problem 2:

Solve the following L.P.P by the graphical method.

Maximize $Z = 3x_1 + 5x_2$

Subject to,

$-3x_1 + 4x_2 \le 12$

$x_1 \le 4$

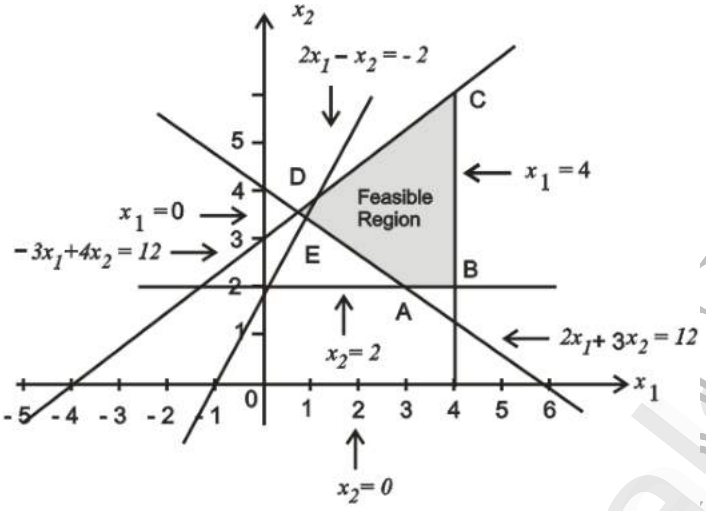
$2x_1 - x_2 \ge -2$

$x_2 \geq 2$

$2x_1 + 3x_2 \geq 12$ and $x_1, x_2 \geq 0$

Solution:

The vertices of the solution space are A (3, 2), B (4, 2), C (4, 6), D $\left(\frac{4}{5}, \frac{18}{5}\right)$ and E $\left(\frac{3}{4}, \frac{7}{2}\right)$



The value of Z at these vertices are given by $\because (z = 3x_1 + 5x_2)$

Vertex	Value of Z
A (3, 2)	19
C (4, 2)	22
C (4, 6)	42
D $\left(\frac{4}{5}, \frac{18}{5}\right)$	$\frac{102}{5}$
E $\left(\frac{3}{4}, \frac{7}{2}\right)$	$\frac{79}{4}$

Since the problem is of minimization type, the optimum solution is,

Minimum Z = 19, $x_1 = 3,\ x_2 = 2$

Problem 3:

Apply graphical method to solve the L.P.P

Maximize $Z = x_1 - 2x_2$

Subject to,

$-x_1 + x_2 \leq 1$

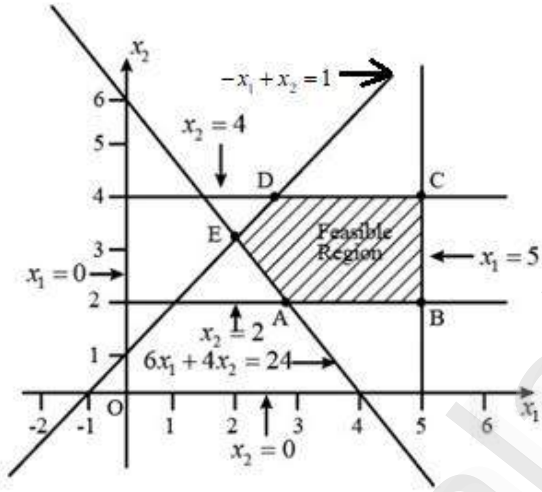
$6x_1 + 4x_2 \geq 24$

$0 \leq x_1 \leq 5$

$2 \le x_2 \le 4$

Solution:

❖ By using graphical method, the solution space is given below with shaded area ABCDE with vertices $A\left(\frac{8}{3},2\right), B(5,2), C(5,4), D(3,4)$ and $E(2,3)$



❖ The value of Z at these vertices are given by $\because (z = x_1 - 2x_2)$

Vertex	Value of Z
$A\left(\frac{8}{3},2\right)$	$-\frac{4}{3}$
B (5, 2)	1
C (5, 4)	–3
D $(3,4)$	–5
E $(2,3)$	–4

Since the problem is of maximization type, the optimum solution is,

Maximum Z = 1, $x_1 = 5$, $x_2 = 2$

2.6. Some More Cases:

The constraints generally, give region of feasible solution which may be bounded or unbounded. However, it may not be true for every problem. In general, a linear programming problem may have:

(i) A unique optimal solution (ii) an infinite number of optimal solutions (iii) an unbounded solution (iv) no solution.

EXERCISES

1. Branch and Bound method is applicable to ______________________ IPP.

   A) pure B) mixed C) both a& b D) None of these

2. If sometimes a few or all the variables of an IPP are constrained by their upper or lower bounds, then the most general method for the solution of optimization problem is called_________

   A) Branch and Bound method B) Gomary's cutting plane –method

   C) simplex method D) Big – M method

12. SET – I - ONE MARKS

1. Operations research is the application of ________ methods to arrive at the optimal solutions to the problems.

   A) economical B) scientific

   C) both (a) and (b) D) none of the above

2. In operations research the ________ are prepared for situations.

   A) mathematical models B) iconic model

   C) static model D) dynamic model

3. ________ is a physical or pictorial representation of various aspects of a system.

   A) mathematical models B) iconic model

   C) static model D) dynamic model

4. Analytic model is a model in which exact solution is obtained by ________ in closed form.

   A) static B) iconic

   C) simulation D) mathematical

5. Operations research started just before World War II in Britain with the establishment of teams of scientists to study the strategic and tactical problems involved in military operations.

   A) True B) False

6. OR can be applied only to those aspects of libraries where mathematical models can be prepared.

   A) True B) False

7. OR has a characteristic that it is done by a team of

A) Scientists B) mathematicians

C) Academics D) All the above

8. OR uses models to help the management to determine is ________.

A) Policies B) Actions

C) Both (A) and (B) D) None of the above

9. Linear programming problem deals with the _______ of a function of decision variables.

A) maximization B) minimization

C) optimization D) None of the above

10. The variables whose values determine the solution of a problem are called _______ of the problem.

A) decision variables B) objective function

C) constraints D) non-negativity restrictions

11. In LPP optimization of a function of decision variables is known as

A) decision variables B) objective function

C) constraints D) non-negativity restrictions

12. Linear programming techniques are used in many problems.

A) industrial B) economic

C) both (A) and (b) D) none of the above

13. LPP Technique requires

A) objective function B) constraints

C) non-negativity restrictions D) all the above

14. LPP involving only two variables can be effectively solved by a _____ which provides a pictorial representation of the problems.

A) formulation method B) graphical method

C) simplex method D) Big – M – method

15. In graphical method, if there exists an optimal solution of an L.P.P, it will be at one of the vertices of the ________.

A) feasible region B) unique optimal solution

C) an unbounded solution D) no solution

16. In graphical method, the problem is of maximization type and the maximum value of Z is attained at a single vertex, then the solution is _________.

A) unique optimal solution
B) an unbounded solution
C) infinite number of optimal solution
D) no solution

17. An LPP having more than one optimal solution is said to have _______ solution.

A) feasible
B) unique
C) multiple optimal
D) no solution

18. An L.P.P, the maximum value of Z occurs at infinity, then the solution is _______ solution.

A) feasible
B) unique
C) multiple optimal
D) unbounded

19. In graphical method, the given LPP cannot be solved, then the solution is ________ solution.

A) unique
B) unbounded
C) infinite
D) no feasible

20. A set of values $x_1, x_2, \ldots x_n$ which satisfies the constraints of the LPP is called its

A) feasible solution
B) solution
C) optimal solution
D) no solution

21. Any solution to a LPP which satisfies the non-negativity restrictions of the LPP is called its

A) feasible solution
B) solution
C) optimal solution
D) unique solution

22. Any feasible solution which optimizes the objective function of the LPP is called its

A) feasible solution
B) solution
C) optimal solution
D) unbounded solution

23. In simplex method, to convert the inequalities into equalities for $\leq$type constraints to introduce _____ variables.

A) optimum
B) slack
C) surplus
D) none of the above

24. In simplex method, to convert the inequalities into equalities for $\geq$type constraints to introduce variables.

A) optimum
B) slack
C) surplus
D) none of the above

TEACHER'S CARE ACADEMY
38/23, Vaigundaperumal Koil Street, Kanchipuram-1
Mobile : 95665 35080, 97862 69980
Help Line Numbers : 9360268118,7639967359

UG TRB
MATHEMATICS
2023-2024

UNIT X
Statistics / Probability

Your Success is Our Goal....

UG TRB 2022-23 MATHEMATICS

UNIT - X - STATISTICS PROBABILITY

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UG TRB – MATHEMATICS – 2022-23

UNIT - X

STATISTICS / PROBABILITY

1. MEASURES OF CENTRAL TENDENCY

- An average is a value which is typical or representative of a set of data. The measures of central tendency are also known as "measures of location".
- Various measures of central tendency are the following

  1. Arithmetic mean, 2. Median, 3. mode, 4. Geometric mean and, 5. Harmonic mean

1.1 Characteristics of An Average:

1. It should be rigidly defined
2. It should be based on all the items
3. It should not be unduly affected by extreme items.
4. It should lend itself for algebraic manipulation.
5. It should be simple to understand and easy to calculate.
6. It should have sampling stability.

1.2 Arithmetic Mean:

- Arithmetic mean is the total of the value of the items divided by their number.
- It is denoted by $\overline{x}$

Type - I: Individual observations or Raw data)

Formula: $A.M = \dfrac{Total\ of\ the\ observations}{No.\ of\ the\ observations}$

$$\text{(i.e)}\ A.M = \frac{x_1 + x_2 + . \quad . \quad . \quad . + x_n}{n} = \frac{\sum X}{n}$$

Problem: the expenditure of 10 families in rupees are given below:

Family	**A**	**B**	**C**	**D**	**E**	**F**	**G**	**H**	**I**	**J**
Expenditure	**30**	**70**	**10**	**75**	**500**	**8**	**42**	**250**	**40**	**36**

Calculate the arithmetic mean:

Solution: x- Expenditure: N=10

Family	**Expenditure (Rs)** X
A	30
B	70
C	10
D	75
E	500
F	8
G	42
H	250
I	40
J	36
TOTAL	$\sum x = 1061$

$$\overline{X} = \frac{\sum X}{n}$$

$$= \frac{1061}{10}$$

$$\overline{X} = 106.1$$

Type – II: (Discrete series)

$$\overline{X} = \frac{\sum fX}{\sum f}$$

Problem:

Calculate the mean number of persons per house

Given

No. of persons per house	2	3	4	5	6	Total
No. of houses	10	25	30	25	10	100

Solution:

x- **No. of persons per house**

f - No. of houses

No. of persons per house X	**No. of houses** f	fx
2	10	20
3	25	75
4	30	120
5	25	125
6	10	60
	$\sum f = 100$	$\sum fx = 400$

$$\overline{X} = \frac{\sum fX}{\sum f}$$

$$= \frac{400}{100}$$

$$\overline{X} = 4$$

Type – III: (Continuous Series): Exclusive class Intervals

$\overline{X} = \frac{\sum fm}{\sum f}$; m = mid point of the class interval

Problem: calculate A.M for the following

Marks	20-30	30-40	40-50	50-60	60-70	70-80
No. of students	5	8	12	15	6	4

Marks	No. of students	m	fm
20-30	5	25	125
30-40	8	35	280
40-50	12	45	540
50-60	15	55	825
60-70	6	65	390
70-80	4	75	300
	$\sum f = 50$		$\sum fm = 2460$

$$\overline{X} = \frac{\sum fm}{\sum f}$$

$$= \frac{2460}{50}$$

$$\overline{X} = 49.20$$

Continuous series: Inclusive class Intervals

Problem: The annual profits of 90 companies are given below. Find the arithmetic mean.

Annual profit (Rs. lakhs)	0-19	20-39	40-59	60-79	80-99
No. of companies	5	17	32	24	12

Solution:

Annual profit (Rs. lakhs)	No. of companies f	Mid value m	fm
0-19	5	19.5	47.5
20-39	17	29.5	501.5
40-59	32	49.5	1584.0
60-79	24	69.5	1668.0
80-99	12	89.5	1074.0
	$\sum f = 90$		$\sum fm = 4875.0$

$$\bar{X} = \frac{\sum fm}{\sum f}$$

$$= \frac{4875.0}{90}$$

$$\bar{X} = Rs.\ 54.17 \text{ lakhs}$$

Problem:

- Average rainfall of a city from Monday to Saturday was 1.2 cms. Due to heavy rainfall on Sunday, the average rainfall ons Sunday, the average rainfall increased to 2cms. What was the rain fall on Sunday?

Solution:

Total rain fall on 6 days = Number $\times$ Average

$= 6 \times 1.2$

$= 7.2$ cms

Total rain fall on 7 days= $7 \times 2 = 14$cms

Total rain fall on 7th days, Sunday = $14 - 7.2 = 6.8$ cms

Formula for combined means:

If two means are given,

$$\bar{X}_{12} = \frac{N_1\bar{X}_1 + N_2\bar{X}_2}{N_1 + N_2}$$

If three means are given, $\bar{X}_{123} = \dfrac{N_1\bar{X}_1 + N_2\bar{X}_2 + N_3X_3}{N_1 + N_2 + N_3}$

Problem:

- There are two branches of an establishment employing 100 and 80 persons respectively. If the arithmetic means of the monthly salaries paid by the two branches are Rs.275 and Rs.225 respectively. Find the arithmetic mean of the salaries of the empolyes of the establishment as a whole.

Solution:

Given $N_1 = 100, N_2 = 80, \overline{X_1} = 275, \overline{X_2} = 225$

$$\overline{X_{12}} = \frac{N_1\overline{X_1} + N_2\overline{X_2}}{N_1 + N_2}$$

$$= \frac{(100\times 275)+(80\times 225)}{100+80}$$

$$\overline{X_{12}} = Rs.252.78$$

Problem:

- The average mark in mathematics of foundation course students of three centers, Kolkata, Mumbai and Delhi is 50. The number candidates in Kolkata, Mumbai and Delhi are respectively 100,120 and 150. The average marks of Kolkata and Mumbai are 70 and 40 respectively. Find the average mark of Delhi.

Solution:

Given $\overline{X_{123}} = 50, N_1 = 100, N_2 = 120; N_3 = 150;\ \bar{X}_1 = 70,\ \bar{X}_2 = 40$

$$\overline{X_{123}} = \frac{N_1\overline{X_1} + N_2\overline{X_2} + N_3\overline{X_3}}{N_1 + N_2 + N_3}$$

$$50 = \frac{(100\times 70)+(120\times 40)+\left(150\times \overline{X_3}\right)}{100+120+150}$$

$$\overline{X_3} = \frac{6700}{150} = 44.67$$

Corrected Arithmetic Mean:

Problem:

- The mean of 20 marks is found to be 40. Later on it was discovered that a mark 53 was misread as 83, Find the correct mean.

Solution:

Given $N=20, \overline{X}_W=40, X_c=53, X_w=83$

$$\overline{X}_W=\frac{(\Sigma X)_w}{N}$$

$\therefore$ Wrong total $(\Sigma X)_W = N\overline{X}_W$

$=20\times 40=800$

$\therefore$ Correct total $(\Sigma X)_C=(\Sigma X)_W - X_W + X_C$

$=800-83+53$

$=770$

$\therefore$ Correct mean $\overline{X}_C=\dfrac{(\Sigma X)_C}{N}$

$=\dfrac{770}{20}=38.5$

Problem:

- A student found the mean of 50 items as 38.6. when checking the work he found that he had taken one item as 50 while it should correctly read as 40. Also the number of items turned out to be only 49. In the circumstances, what should be the correct mean?

Solution:

Given $N_W=50; \overline{X}_W=38.6, X_W=50, X_C=40; N_C=49$

$\therefore$ Wrong total $(\Sigma X)_W = N_w\overline{X}_W$

$=50\times 38.6=1930$

$\therefore$ Correct total $(\Sigma X)_C=(\Sigma X)_W - X_W + X_C$

$=1930-50+40=1920$

$\therefore$ Correct mean $\overline{X}_C = \frac{(\Sigma X)_c}{N}$

$= \frac{1920}{49} = 39.18$

Missing frequencies:

Problem:

Find the missing frequency from the following frequency distribution if mean is 38.

Marks	10	20	30	40	50	60	70
No. of students	8	11	20	25	-	10	3

Solution: Let the missing frequency be f

$\sum f = 8+11+20+25+f+10+3 = 77+f$

$\sum f_x = 2710 + 50f$

Consider, $\overline{X} = \frac{\sum fx}{\sum f}$

$38 = \frac{2710+50f}{77+f} \Rightarrow 38f + 2926 = 2710 + 50f$

$50f - 38f = 2926 - 2710$

$f = \frac{216}{12} = 18$

1.3 Mathematical Characteristics:

1. The algebraic sum of the deviations, of all the items from their arithmetic mean is zero.

   (ie) $\Sigma\left(X - \overline{X}\right) = 0$

2. The sum of the standard deviations of the items from mean is a minimum.

3. If all the items of a series are increased (or) decreased by any constant number, the arithmetic mean will also increase (or) decrease by the same constant.

Discrete series: (Direct method)

$$\overline{X} = \frac{\sum fX}{N}$$

$\overline{X} =$ Arithmetic mean; $\sum fX$ = the sum of product;

N= total number of items

Problem: Calculate mean from the following data

Value	1	2	3	4	5	6	7	8	9	10
Frequency	21	30	28	40	26	34	40	9	15	57

Solution:

X	**f**	f_x
1	21	21
2	30	60
3	28	84
4	40	160
5	26	130
6	34	204
7	40	280
8	9	72
9	15	135
10	57	570
	N=300	$\sum fX = 1716$

$$\overline{X} = \frac{\sum fX}{N}$$

$$= \frac{1716}{300}$$

$$\overline{X} = 5.72$$

Short Cut Method:

$$\overline{X} = A \pm \frac{\sum fd}{N}$$

$\overline{X}$ = Mean, A= Assumed mean, $\sum fd$ = sum of total deviations, N = total frequency

Problem: (solving the previous problem)

Solution:

X	f	$d=(X-A)$	fd
1	21	-4	-84
2	30	-3	-90
3	28	-2	-56
4	40	-1	-40
5	26	0	0
6	34	1	34
7	40	2	80
8	9	3	27
9	15	4	60
10	57	5	285
	$\sum f = 300$		$\sum fd = +216$

Continuous Series:

1. Direct method

$$\overline{X} = \frac{\sum fm}{\sum f};$$

$\overline{X}$ = mean, m – mid value,

Problem: From the following find out the mean profits:

Profits per shop Rs	100-200	200-300	300-400	400-500	500-600	600-700	700-800
Number of shops	10	18	20	26	30	28	18

Solution:

X	**f**	**M**	**fm**
100-200	10	150	1500
200-300	18	250	4500
300-400	20	350	7000
400-500	26	450	11700
500-600	30	550	16500
600-700	28	650	18200
700-800	18	750	13500
	$\sum f = 150$		$\sum fm = 72900$

$$\overline{X} = \frac{\sum fm}{\sum f}$$

$$= \frac{72900}{150}$$

$$\overline{X} = 486$$

2. Short cut method

$$\overline{X} = A \pm \frac{\sum fd}{N}$$

A= Assumed mean, $\sum fd$ = sum of total deviations, N =Number of items

3) step deviation method

$$\overline{X} = A \pm \frac{\sum fd'}{N}$$

$\overline{X}$ =Mean, A= Assumed mean, $\sum fd'$ = sum of total deviations, N =Number of items , C= common factor.

Note:

- If we use any method to find the arithmetic mean for continues series , we can get the same answer for same problem.

Problem:

Find mean of the following data:

Class – Interval	**0-9**	**10-19**	**20-29**	**30-39**	**40-49**	**50-59**
Frequency	2	15	10	8	3	1

Solution:

- The given problem is to be convert into exclusive class interval series (ie. Left side C.I subtract 0.5 and right side C.I add 0.5 to given data)

C.I	**True C.I**	**f**	**m**	$d' = \frac{m-34.5}{10}$	fd'
0-9	0.5-9.5	1	4.5	2	2
10-19	9.5-19.5	3	14.5	1	3
20-29	19.5-29.5	8	24.5	0	0
30-39	29.5-39.5	10	34.5	-1	-10
40-49	39.5-49.5	15	44.5	-2	-30
50-59	49.5-59.5	2	54.5	-3	-6
	$\sum f = 40$				$\sum fd' = -41$

$$\overline{X} = A \pm \frac{\sum fd'}{N} \times c$$

A=34.5, $\sum fd' = -41$,N=40,C=10

$$\overline{X} = 34.5 - \frac{(-41)}{40} \times 10$$

$$\overline{X} = 24.25$$

1.4. Merits of Arithmetic Mean:

1. It is easy to understand
2. It is easy to calculate
3. It is rigidly defined
4. It is based on the value of every item in the series
5. It provides a good basis for comparison.
6. It can be used for further analysis and algebraic treatment.
7. The mean is a more stable measure of central tendency.

1.5. Demerits (Limitations)

1. The mean is unduly affected by the extreme items.
2. It is unrealistic.
3. It may lead to a false conclusion.
4. It cannot be accurately determined even if one of the values is not known.
5. It cannot be located by observations or the graphic method.
6. It gives greater importance to bigger items of a series and lesser importance to smaller items.

1.6. Uses of Arithmetic Mean:

It is used in social economic and business problem.

1.7. Median:

- Median is the value of item that goes to divided the series into equal parts. Median may be defined as the value of that item which divides the series into two equal parts, one half containing values greater than it and the other half containing values less that it. Therefore, the series has to be arranged in ascending or descending order, before finding the median. It is also called positional average.

Individual Series:

Problem (odd number problem)

Find the median of the following series.

X : 10 15 9 25 19

Solution:

Size of the item ascending order(x)	Size of the item descending order(x)
9	25
10	19
15	15
19	10
25	9

Median = size of $\left(\frac{N+1}{2}\right)^{th}$ item

= size of $\left(\frac{5+1}{2}\right)^{th}$ item

= Size of 3^{rd} item

median = 15

Problem (even number problem)

Find the value of median from the following series.

X: 8 10 5 9 12 11

Solution:

X
5
8
9
10
11
12

Median = size of $\left(\frac{N+1}{2}\right)^{th}$ items

= size of $\left(\frac{6+1}{2}\right)^{th}$ items

= Size of 3.5th item

= Size of $\left(\frac{3^{rd}\ item + 4^{th}\ item}{2}\right)$

$= \frac{9+10}{2}$

median = 9.5

Discrete Series:

Problem: Find out the median from the following:

Size of shoes	5	5.5	6	6.5	7	7.5	8
Frequency	10	16	28	15	30	40	34

Solution:

Size of shoes	**f**	**Cf**
5	10	10
5"'[.5	16	26
6	28	54
6.5	15	69
7	30	99
7.5	40	139
8	34	173

Median = size of $\left(\frac{N+1}{2}\right)^{th}$ item

13. MULTIPLE CHOICE QUESTIONS

1. _______ is a typical value of the entire group or data.

   A) Mean B) Median

   C) Mode D) Measure of central tendency

2. Arithmetic average is also called as ______.

   A) Mean B) Median C) Mode D) G.M.

3. In continuous series, the formula for A.M. is

   A) $\bar{X} = \frac{\sum fx}{N}$ B) $\bar{X} = \frac{\sum fm}{N}$

   C) $\bar{X} = \frac{\sum X}{N}$ D) None of these

4. The sum of the deviations taken from A.M is

   A) Minimum B) Maximum C) zero D) None of these

5. The sum of squares of deviations from A.M is

   A) zero B) Maximum C) Minimum D) one

6. The best measure of central tendency is

   A) A.M B) Median C) G.M D) H.M

7. For dealing with qualitative data the best average is

   A) Mean B) Median C) Mode D) H.M

8. Median is a _____ average.

   A) Positional B) Locational C) both (a) and (b) `D) None of these

9. _____ is the most unstable average.

   A) Mean B) Median C) Mode D) G.M.

10. _____ average is affected by extreme observations.

    A) H.M B) A.M C) G.M. D) Median

11. Harmonic mean is the _______ of the arithmetic average of the reciprocal of values.

    A) reciprocal B) non-reciprocal

    C) neither a nor b D) equal

12. If the items in a distribution have the same value then,

A) $\bar{X} \neq G.M \neq H.M$ B) $\bar{X} > G.M > H.M$

C) $\bar{X} < G.M < H.M$ D) $\bar{X} = G.M = H.M$

13. ____ is the measure of the variation of the items.

A) dispersion B) range C) Q.D D) S.D

14. Range is the best measure of dispersion.

A) True B) False

15. Quartile deviation is more suitable in case of open – end distribution.

A) True B) False

16. Mean deviation can never be negative

A) True B) False

17. Formula for standard deviation in discrete series is,

A) $\sigma = \sqrt{\frac{\sum X^2}{N} - \left(\frac{\sum X}{N}\right)^2}$ B) $\sigma = \sqrt{\frac{\sum fX^2}{N} - \left(\frac{\sum fX}{N}\right)^2}$

C) $\sigma = \sqrt{\frac{\sum fm^2}{N} - \left(\frac{\sum fm}{N}\right)^2}$ D) None of thes

18. Standard deviation is always ______ than range.

A) Maximum B) Minimum C) less D) more

19. Variance is ______ of S.D.

A) equal B) square

C) both a and b D) None of these

20. Formula for combined mean is,

A) $\bar{X}_{12} = \frac{N_1\bar{X}_1 + N_2\bar{X}_2}{N_1 + N_2}$ B) $\bar{X}_{12} = \frac{N_2\bar{X}_1 + N_1\bar{X}_2}{N_1 + N_2}$

C) $\bar{X}_{12} = \frac{\bar{X}_1 + \bar{X}_2}{N_1 + N_2}$ D) $\bar{X}_{12} = \frac{N_1 + N_2}{\bar{X}_1 + \bar{X}_2}$

21. The coefficient of skewness is zero, then distribution is,

A) J-shaped B) U-shaped C) Z-shaped D) symmetrical

22. A negative coefficient of skewness implies that

A) Mean > Mode B) Mean < Mode

C) Mean = Mode D) Mean ≠ Mode

23. For a symmetrical distribution the coefficient of skewness is

A) + 1 B) – 1 C) + 3 D) – 3

24. The first central moment is always zero

A) True B) False

25. The second central moment does not indicate the variance.

A) True B) False

26. β_2 must always be positive

A) True B) False

27. If β_2 is greater than 3, then curve is called,

A) mesokurtic B) Leptokurtic C) Platykurtic D) None of these

28. If β_2 is less than 3, the curve is called

A) mesokurtic B) Leptokurtic C) Platykurtic D) None of these

29. The coefficient of correlation.

A) cannot be positive B) cannot be negative

C) can be either positive or negative D) none of these

30. The coefficient of correlation is independent of

A) change of scale only B) change of origin only

C) both change of scale and origin D) none of these

31. The study of two variables excluding some other variables is called _____ correlation.

A) positive B) negative C) multiple D) partial