12 - MATHEMATICS

SPECIAL GUIDE

KRISHNAGIRI DISTRICT

2023-24

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12-ஆம் வகுப்பு

ஒரு மதிப்பெண் வினாக்கள்

12-**ஆம் வகுப்பு பாடப்புத்தகத்தில் உள்ள ஒரு மதிப்பெண் வினாக்கள்,** *GeoGebra* **மென்பொருளின் உதவியோடு,ஒரு வினாவிற்கு சரியான விடையை தேர்வு செய்ய ,அதிகபட்சம் மூன்று வாய்ப்புகள் வழங்கி, மாணவர்களின் கற்றல் ,கற்பித்தல் திறன் அதிகரிக்கும் வகையில் வடிவமைக்கப்பட்டுள்ளது என்பதை தெரிவித்துக்கொள்கிறோம்.**

குறிப்பு : *Hi-Tech Lab*-**ல்** *QR Code* **-ஐ** *Scan* **செய்து அல்லது** *Link* **-ஐ** *click* **செய்து மாணவர்கள் பயிற்சி செய்யும் விதமாக மென்பொருள் உருவாக்கப்பட்டுள்ளது** .

தமிழ் வழி

https://www.geogebra.org/m/svp4anun

ஆங்கில வழி

https://www.geogebra.org/m/zzajah2u

உருவாக்கம் :

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அரசு மே.நி.பள்ளி, மோரன அள்ளி.

CONTENTS

VECTOR ALGEBRA

Important hints:

• $\vec{a}\cdot\vec{b} = \|\vec{a}\|\|\vec{b}\|\cos\theta$ • $\vec{a}\times\vec{b} = \|\vec{a}\|\|\vec{b}\|\sin\theta\,\hat{n}$ • Work done $W = \vec{F}\cdot\vec{d}$ • Torque $\vec{\tau} = \vec{r}\times\vec{F}$	• $\vec{a},\vec{b}$ are perpendicular vectors $\Leftrightarrow \vec{a}\cdot\vec{b} = 0$ • $\vec{a},\vec{b}$ are parallel vectors $\Leftrightarrow \vec{a}\times\vec{b} = \vec{0}$ • Volume of parallelepiped with coterminous vectors $V = \|[\vec{a},\vec{b},\vec{c}]\|$

- If $\vec{a} = a_1\hat{\imath} + a_2\hat{\jmath} + a_3\hat{k}$ and $\vec{b} = b_1\hat{\imath} + b_2\hat{\jmath} + b_3\hat{k}$
- Scalar product (or) dot product $\vec{a}\cdot\vec{b} = a_1b_1 + a_2b_2 + a_3b_3$
- The vector product (or) cross product $\vec{a}\times\vec{b} = \begin{vmatrix}\hat{\imath} & \hat{\jmath} & \hat{k}\\ a_1 & a_2 & a_3\\ b_1 & b_2 & b_3\end{vmatrix}$
- If θ is the acute angle between two straight lines $\vec{r} = \vec{a} + s\vec{b}$ and $\vec{r} = \vec{c} + t\vec{d}$, then $\theta = \cos^{-1}\left(\frac{|\vec{b}\cdot\vec{d}|}{|\vec{b}||\vec{d}|}\right)$
- The acute angle θ between the two planes $\vec{r}\cdot\vec{n}_1 = p_1$ and $\vec{r}\cdot\vec{n}_2 = p_2$ is $\theta = \cos^{-1}\left(\frac{|\vec{n}_1\cdot\vec{n}_2|}{|\vec{n}_1||\vec{n}_2|}\right)$
- If θ is the acute angle between the line $\vec{r} = \vec{a} + t\vec{b}$ and the plane $\vec{r}\cdot\vec{n} = p$, then $\theta = \sin^{-1}\left(\frac{|\vec{b}\cdot\vec{n}|}{|\vec{b}||\vec{n}|}\right)$
- Let $\vec{a} = a_1\hat{\imath} + a_2\hat{\jmath} + a_3\hat{k}$, $\vec{b} = b_1\hat{\imath} + b_2\hat{\jmath} + b_3\hat{k}$, $\vec{c} = c_1\hat{\imath} + c_2\hat{\jmath} + c_3\hat{k}$. Then,
- $\vec{a},\vec{b},\vec{c}$ are coplanar $\Leftrightarrow [\vec{a},\vec{b},\vec{c}] = 0 \Leftrightarrow \begin{vmatrix}a_1 & a_2 & a_3\\ b_1 & b_2 & b_3\\ c_1 & c_2 & c_3\end{vmatrix} = 0$
- If two lines $\frac{x-x_1}{b_1} = \frac{y-y_1}{b_2} = \frac{z-z_1}{b_3}$ and $\frac{x-x_2}{d_1} = \frac{y-y_2}{d_2} = \frac{z-z_2}{d_3}$ intersect, then $\begin{vmatrix}x_2-x_1 & y_2-y_1 & z_2-z_1\\ b_1 & b_2 & b_3\\ d_1 & d_2 & d_3\end{vmatrix} = 0$

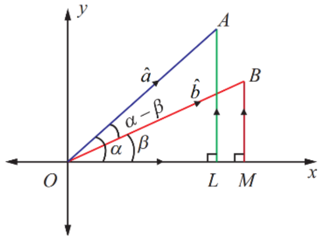
Parametric Vector Equation	Non Parametric Vector Equation	Cartiesan Equation
$\vec{r} = \vec{a} + s\vec{b} + t\vec{c}$	$(\vec{r}-\vec{a})\cdot(\vec{b}\times\vec{c}) = 0$	$\begin{vmatrix}x-x_1 & y-y_1 & z-z_1\\ b_1 & b_2 & b_3\\ c_1 & c_2 & c_3\end{vmatrix} = 0$
$\vec{r} = (1-s)\vec{a} + s\vec{b} + t\vec{c}$	$(\vec{r}-\vec{a})\cdot[(\vec{b}-\vec{a})\times\vec{c}] = 0$	$\begin{vmatrix}x-x_1 & y-y_1 & z-z_1\\ x_2-x_1 & y_2-y_1 & z_2-z_1\\ c_1 & c_2 & c_3\end{vmatrix} = 0$
$\vec{r} = (1-s-t)\vec{a} + s\vec{b} + t\vec{c}$	$(\vec{r}-\vec{a})\cdot[(\vec{b}-\vec{a})\times(\vec{c}-\vec{a})] = 0$	$\begin{vmatrix}x-x_1 & y-y_1 & z-z_1\\ x_2-x_1 & y_2-y_1 & z_2-z_1\\ x_3-x_1 & y_3-y_1 & z_3-z_1\end{vmatrix} = 0$

5 MARKS

1. By vector method, prove that $\cos(\alpha-\beta)=\cos\alpha\cos\beta+\sin\alpha\sin\beta$

Soln:

Let $\hat{a}$ and $\hat{b}$ are two unit vectors



$\hat{a}=\cos\alpha\hat{\imath}+\sin\alpha\hat{\jmath}$

$\hat{b}=\cos\beta\hat{\imath}+\sin\beta\hat{\jmath}$

$\hat{b}\cdot\hat{a}=\cos(\alpha-\beta)----\rightarrow(1)$

$\hat{b}\cdot\hat{a}=(\cos\beta\hat{\imath}+\sin\beta\hat{\jmath})\cdot(\cos\alpha\hat{\imath}+\sin\alpha\hat{\jmath})$

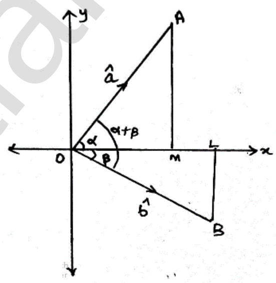
$=\cos\alpha\cos\beta+\sin\alpha\sin\beta----\rightarrow(2)$

From (1)&(2) $\cos(\alpha-\beta)=\cos\alpha\cos\beta+\sin\alpha\sin\beta$

2. By vector method, prove that $\cos(\alpha+\beta)=\cos\alpha\cos\beta-\sin\alpha\sin\beta$

Soln:

Let $\hat{a}$ and $\hat{b}$ are two unit vectors



$\hat{a}=\cos\alpha\hat{\imath}+\sin\alpha\hat{\jmath}$

$\hat{b}=\cos\beta\hat{\imath}-\sin\beta\hat{\jmath}$

$\hat{b}\cdot\hat{a}=\cos(\alpha+\beta)----\rightarrow(1)$

$\hat{b}\cdot\hat{a}=(\cos\beta\hat{\imath}-\sin\beta\hat{\jmath})\cdot(\cos\alpha\hat{\imath}+\sin\alpha\hat{\jmath})$

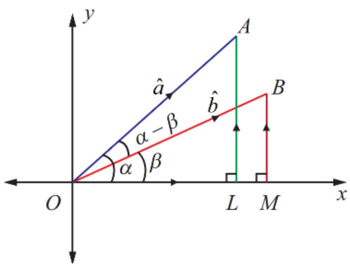
$=\cos\alpha\cos\beta-\sin\alpha\sin\beta----\rightarrow(2)$

From (1)&(2) $\cos(\alpha+\beta)=\cos\alpha\cos\beta-\sin\alpha\sin\beta$

3. By vector method, prove that $\sin(\alpha-\beta)=\sin\alpha\cos\beta-\cos\alpha\sin\beta$

Soln:

Let $\hat{a}$ and $\hat{b}$ are two unit vectors



$\hat{a}=\cos\alpha\hat{\imath}+\sin\alpha\hat{\jmath}$

$\hat{b}=\cos\beta\hat{\imath}+\sin\beta\hat{\jmath}$

$\hat{b}\times\hat{a}=\sin(\alpha-\beta)(\hat{k})--\rightarrow(1)$

$$\hat{b}\times\hat{a}=\begin{vmatrix}\hat{\imath} & \hat{\jmath} & \hat{k}\\ \cos\beta & \sin\beta & 0\\ \cos\alpha & \sin\alpha & 0\end{vmatrix}$$

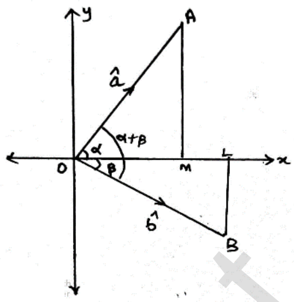
$=(\sin\alpha\cos\beta-\cos\alpha\sin\beta)(\hat{k})----\rightarrow(2)$

From (1) & (2) $\sin(\alpha-\beta)=\sin\alpha\cos\beta-\cos\alpha\sin\beta$

4. By vector method, prove that $sin(\alpha+\beta) = sin\alpha cos\beta + cos\alpha sin\beta$

<u>Soln</u>:

Let $\hat{a}$ and $\hat{b}$ are two unit vectors



$$\hat{a} = cos\alpha\hat{\imath} + sin\alpha\hat{\jmath}$$

$$\hat{b} = cos\beta\hat{\imath} - sin\beta\hat{\jmath}$$

$$\hat{b}\times\hat{a} = sin(\alpha+\beta)\hat{k} \;----\rightarrow (1)$$

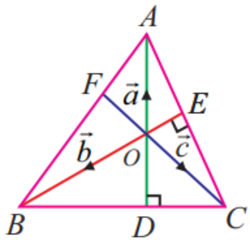
$$\hat{b}\times\hat{a} = \begin{vmatrix} \hat{\imath} & \hat{\jmath} & \hat{k} \\ cos\beta & -sin\beta & 0 \\ cos\alpha & sin\alpha & 0 \end{vmatrix}$$

$$= (sin\alpha cos\beta + cos\alpha sin\beta)\hat{k} \;----\rightarrow (2)$$

From (1)&(2) $sin(\alpha+\beta) = sin\alpha cos\beta + cos\alpha sin\beta$

5. **Prove by vector method that the perpendiculars (attitudes) from the vertices to the opposite sides of a triangle are concurrent**.

<u>Soln</u>: $\overrightarrow{OA} = \vec{a}$, $\overrightarrow{OB} = \vec{b}$, $\overrightarrow{OC} = \vec{c}$



AD⊥BC ; BE ⊥CA To prove CF⊥BA

Case:1 AD⊥BC	Case:2 BE ⊥ CA
$\overrightarrow{OA}\cdot\overrightarrow{BC} = 0$	$\overrightarrow{OB}\cdot\overrightarrow{CA} = 0$
$\overrightarrow{OA}\cdot(\overrightarrow{OC}-\overrightarrow{OB}) = 0$	$\overrightarrow{OB}\cdot(\overrightarrow{OA}-\overrightarrow{OC}) = 0$
$\vec{a}\cdot\vec{c} - \vec{a}\cdot\vec{b} = 0 \longrightarrow (1)$	$\vec{b}\cdot\vec{a} - \vec{b}\cdot\vec{c} = 0 --\longrightarrow (2)$

From (1) + (2)$\Longrightarrow$ $\vec{a}\cdot\vec{c} - \vec{b}\cdot\vec{c} = 0$

$$(\vec{a}-\vec{b})\cdot\vec{c} = 0$$

$$(\overrightarrow{OA}-\overrightarrow{OB})\cdot\overrightarrow{OC} = 0$$

$$\overrightarrow{BA}\cdot\overrightarrow{OC} = 0 \Longrightarrow \overrightarrow{BA}\cdot\overrightarrow{CF} = 0 \Longrightarrow \text{CF}\perp \text{BA}$$

Hence, the perpendiculars (attitudes) from the vertices to the opposite sides of a triangle are concurrent.

6. **If $\vec{a} = \hat{\imath}-\hat{\jmath}$, $\vec{b} = \hat{\imath}-\hat{\jmath}-4\hat{k}$, $\vec{c} = 3\hat{\jmath}-\hat{k}$, and $\vec{d} = 2\hat{\imath}+5\hat{\jmath}+\hat{k}$ verify that $(\vec{a}\times\vec{b})\times(\vec{c}\times\vec{d}) = [\vec{a},\vec{b},\vec{d}]\vec{c} - [\vec{a},\vec{b},\vec{c}]\vec{d}$**

Soln:

$$\vec{a}\times\vec{b} = \begin{vmatrix} \hat{\imath} & \hat{\jmath} & \hat{k} \\ 1 & -1 & 0 \\ 1 & -1 & -4 \end{vmatrix} = 4\hat{\imath} + 4\hat{\jmath}$$

$$\vec{c}\times\vec{d} = \begin{vmatrix} \hat{\imath} & \hat{\jmath} & \hat{k} \\ 0 & 3 & -1 \\ 2 & 5 & 1 \end{vmatrix} = 8\hat{\imath} - 2\hat{\jmath} - 6\hat{k}$$

$$(\vec{a}\times\vec{b})\times(\vec{c}\times\vec{d}) = \begin{vmatrix} \hat{\imath} & \hat{\jmath} & \hat{k} \\ 4 & 4 & 0 \\ 8 & -2 & -6 \end{vmatrix} = -24\hat{\imath}+24\hat{\jmath}-40\hat{k} \longrightarrow (1)$$

$$[\vec{a},\vec{b},\vec{d}] = \begin{vmatrix} 1 & -1 & 0 \\ 1 & -1 & -4 \\ 2 & 5 & 1 \end{vmatrix} = 28,\ [\vec{a},\vec{b},\vec{c}] = \begin{vmatrix} 1 & -1 & 0 \\ 1 & -1 & -4 \\ 0 & 3 & -1 \end{vmatrix} = 12$$

$$[\vec{a},\vec{b},\vec{d}]\vec{c} - [\vec{a},\vec{b},\vec{c}]\vec{d} = 28(3\hat{\jmath}-\hat{k}) - 12(2\hat{\imath}+5\hat{\jmath}+\hat{k}) = -24\hat{\imath}+24\hat{\jmath}-40\hat{k} \longrightarrow (2)$$

From (1), (2)

$$(\vec{\boldsymbol{a}}\times\vec{\boldsymbol{b}})\times(\vec{\boldsymbol{c}}\times\vec{\boldsymbol{d}}) = [\vec{\boldsymbol{a}},\vec{\boldsymbol{b}},\vec{\boldsymbol{d}}]\vec{\boldsymbol{c}} - [\vec{\boldsymbol{a}},\vec{\boldsymbol{b}},\vec{\boldsymbol{c}}]\vec{\boldsymbol{d}}$$

Try Yourself: $(\vec{a}\times\vec{b})\times(\vec{c}\times\vec{d}) = [\vec{a},\vec{c},\vec{d}]\vec{b} - [\vec{b},\vec{c},\vec{d}]\vec{a}$

7. If $\vec{a} = 2\hat{\imath}+3\hat{\jmath}-\hat{k}$, $\vec{b} = 3\hat{\imath}+5\hat{\jmath}+2\hat{k}$, and $\vec{c} = -\hat{\imath}-2\hat{\jmath}+3\hat{k}$ verify that $\vec{a}\times(\vec{b}\times\vec{c}) = (\vec{a}\cdot\vec{c})\vec{b} - (\vec{a}\cdot\vec{b})\vec{c}$

Soln:

$$\vec{b}\times\vec{c} = \begin{vmatrix} \hat{\imath} & \hat{\jmath} & \hat{k} \\ 3 & 5 & 2 \\ -1 & -2 & 3 \end{vmatrix} = 19\hat{\imath}-11\hat{\jmath}-\hat{k}$$

$$\vec{a}\times(\vec{b}\times\vec{c}) = \begin{vmatrix} \hat{\imath} & \hat{\jmath} & \hat{k} \\ 2 & 3 & -1 \\ 19 & -11 & -1 \end{vmatrix}$$

$$= -14\hat{\imath}-17\hat{\jmath}-79\hat{k} \text{ ----------> } (1)$$

$$\vec{a}\cdot\vec{c} = (2\hat{\imath}+3\hat{\jmath}-\hat{k})\cdot(-\hat{\imath}-2\hat{\jmath}+3\hat{k}) = -11$$

$$\vec{a}\cdot\vec{b} = (2\hat{\imath}+3\hat{\jmath}-\hat{k})\cdot(3\hat{\imath}+5\hat{\jmath}+2\hat{k}) = 19$$

$$(\vec{a}\cdot\vec{c})\vec{b} - (\vec{a}\cdot\vec{b})\vec{c} = -11(3\hat{\imath}+5\hat{\jmath}+2\hat{k}) - 19(-\hat{\imath}-2\hat{\jmath}+3\hat{k})$$

$$(\vec{a}\cdot\vec{c})\vec{b} - (\vec{a}\cdot\vec{b})\vec{c} = -14\hat{\imath}-17\hat{\jmath}-79\hat{k} \text{ ----------> } (2)$$

From (1),(2)

$$\vec{\boldsymbol{a}}\times(\vec{\boldsymbol{b}}\times\vec{\boldsymbol{c}}) = (\vec{\boldsymbol{a}}\cdot\vec{\boldsymbol{c}})\vec{\boldsymbol{b}} - (\vec{\boldsymbol{a}}\cdot\vec{\boldsymbol{b}})\vec{\boldsymbol{c}}$$

Try yourself : $(\vec{a}\times\vec{b})\times\vec{c} = (\vec{a}\cdot\vec{c})\vec{b} - (\vec{b}\cdot\vec{c})\vec{a}$

TYPE- I

8. Find the non-parametric form of Vector Equation, and Cartesian equation of the plane passing through the point (0,1,-5) and parallel to the straight lines $\vec{r} = (\hat{\imath}+2\hat{\jmath}-4\hat{k}) + s(2\hat{\imath}+3\hat{\jmath}+6\hat{k})$ and $\vec{r} = (\hat{\imath}-3\hat{\jmath}+5\hat{k}) + t(\hat{\imath}+\hat{\jmath}-\hat{k})$.

Soln: $\vec{a} = 0\hat{\imath}+\hat{\jmath}-5\hat{k}$ $\quad \vec{b} = 2\hat{\imath}+3\hat{\jmath}+6\hat{k}$ $\quad \vec{c} = \hat{\imath}+\hat{\jmath}-\hat{k}$

Vector Equation: $\vec{r} = \vec{a}+s\vec{b}+t\vec{c}$

$$\vec{r} = (0\hat{\imath}+\hat{\jmath}-5\hat{k}) + s(2\hat{\imath}+3\hat{\jmath}+6\hat{k}) + t(\hat{\imath}+\hat{\jmath}-\hat{k})$$

Cartiesian Equation: $\begin{vmatrix} x-x_1 & y-y_1 & z-z_1 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix} = 0$

$$\Rightarrow \begin{vmatrix} x-0 & y-1 & z+5 \\ 2 & 3 & 6 \\ 1 & 1 & -1 \end{vmatrix} = 0$$

$(x-0)(-3-6)-(y-1)(-2-6)+(z+5)(2-3)=0$

$-9x+8y-z-13=0$ or $9x-8y+z+13=0$

Non Parametirc Vector Equation: $(\vec{r}-\vec{a})\cdot(\vec{b}\times\vec{c})=0$

$$\boldsymbol{\vec{r}\cdot(9\hat{\imath}-8\hat{\jmath}+\hat{k})+13=0}$$

9. Find the non-parametric form of Vector Equation, and Cartesian equation of the plane passing through the point (2,3,6) and parallel to the straight lines $\frac{x-1}{2}=\frac{y+1}{3}=\frac{z-3}{1}$ and $\frac{x+3}{2}=\frac{y-3}{-5}=\frac{z+1}{-3}$

Soln: $\vec{a}=2\hat{\imath}+3\hat{\jmath}+6\hat{k}$ $\quad\vec{b}=2\hat{\imath}+3\hat{\jmath}+\hat{k}$ $\quad\vec{c}=2\hat{\imath}-5\hat{\jmath}-3\hat{k}$

Vector Equation: $\boldsymbol{\vec{r}=\vec{a}+s\vec{b}+t\vec{c}}$

$\vec{r}=(2\hat{\imath}+3\hat{\jmath}+6\hat{k})+s(2\hat{\imath}+3\hat{\jmath}+\hat{k})+t(2\hat{\imath}-5\hat{\jmath}-3\hat{k})$

Cartiesian Equation: $\begin{vmatrix} x-x_1 & y-y_1 & z-z_1 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix} = 0$

$$\Rightarrow \begin{vmatrix} x-2 & y-3 & z-6 \\ 2 & 3 & 1 \\ 2 & -5 & -3 \end{vmatrix} = 0$$

$(x-2)(-9+5)-(y-3)(-6-2)+(z-6)(-10-6)=0$

$-4x+8y-16z+80=0$ (or) $x-2y+4z-20=0$

Non Parametirc.Vector Equation: $(\vec{r}-\vec{a})\cdot(\vec{b}\times\vec{c})=0$

$$\boldsymbol{\Rightarrow \vec{r}\cdot(\hat{\imath}-2\hat{\jmath}+4\hat{k})-20=0}$$

10. Find the non-parametric form of Vector Equation and Cartesian equation of the plane passing through the point (1,-2,4) and perpendicular to the plane $x+2y-3z=11$ and parallel to the line $\frac{x+7}{3}=\frac{y+3}{-1}=\frac{z}{1}$.

Soln: $\vec{a}=\hat{\imath}-2\hat{\jmath}+4\hat{k}$ $\quad\vec{b}=\hat{\imath}+2\hat{\jmath}-3\hat{k}$ $\quad\vec{c}=3\hat{\imath}-\hat{\jmath}+\hat{k}$

Vector Equation : $\boldsymbol{\vec{r}=\vec{a}+s\vec{b}+t\vec{c}}$

$\vec{r}=(\hat{\imath}-2\hat{\jmath}+4\hat{k})+s(\hat{\imath}+2\hat{\jmath}-3\hat{k})+t(3\hat{\imath}-\hat{\jmath}+\hat{k})$

Cartiesian Equation: $\begin{vmatrix} x-x_1 & y-y_1 & z-z_1 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix} = 0$

$$\Rightarrow \begin{vmatrix} x-1 & y+2 & z-4 \\ 1 & 2 & -3 \\ 3 & -1 & 1 \end{vmatrix} = 0$$

$(x-1)(2-3)-(y+2)(1+9)+(z-4)(-1-6)=0$

$-x-10y-7z+9=0$ (or) $x+10y+7z-9=0$

Non Parametirc.Vector Equation: $\boldsymbol{(\vec{r}-\vec{a})\cdot(\vec{b}\times\vec{c})=0}$

$$\Rightarrow\ \vec{r}\cdot(\hat{\imath}+10\hat{\jmath}+7\hat{k})-9=0$$

11. Find the parametric form of Vector Equation, & Cartesian equations of the plane containing the line $\vec{r}=(\hat{\imath}-\hat{\jmath}+3\hat{k})+t(2\hat{\imath}-\hat{\jmath}+4\hat{k})$ and perpendicular to plane $\vec{r}.(\hat{\imath}+2\hat{\jmath}+\hat{k})=8$.

Soln: $\vec{a}=\hat{\imath}-\hat{\jmath}+3\hat{k}$ $\quad\vec{b}=2\hat{\imath}-\hat{\jmath}+4\hat{k}$ $\quad\vec{c}=\hat{\imath}+2\hat{\jmath}+\hat{k}$

Vector Equation : $\vec{r}=\vec{a}+s\vec{b}+t\vec{c}$

$$\vec{r}=(\hat{\imath}-\hat{\jmath}+3\hat{k})+s(2\hat{\imath}-\hat{\jmath}+3\hat{k})+t(\hat{\imath}+2\hat{\jmath}+\hat{k})$$

Cartiesian Equation: $\begin{vmatrix} x-x_1 & y-y_1 & z-z_1 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix}=0$

$$\begin{vmatrix} x-1 & y+1 & z-3 \\ 2 & -1 & 4 \\ 1 & 2 & 1 \end{vmatrix}=0$$

$$(x-1)(-1-8)-(y+1)(2-4)+(z-3)(4+1)=0$$

$$-9x+2y+5z-4=0\quad (or)\quad 9x-2y-5z+4=0$$

Non Parametirc.Vector Equation: $(\vec{r}-\vec{a})\cdot(\vec{b}\times\vec{c})=0$

$$\Rightarrow \vec{r}\cdot(9\hat{\imath}-2\hat{\jmath}-5\hat{k})+4=0$$

12. Find the non-parametric form of vector eqn, and Cartesian eqns of the plane $\vec{r}=(6\hat{\imath}-\hat{\jmath}+\hat{k})+s(-\hat{\imath}+2\hat{\jmath}+\hat{k})+t(-5\hat{\imath}-4\hat{\jmath}-5\hat{k})$.

Soln: $\vec{a}=6\hat{\imath}-\hat{\jmath}+\hat{k}$ $\quad\vec{b}=-\hat{\imath}+2\hat{\jmath}+\hat{k}$ $\quad\vec{c}=-5\hat{\imath}-4\hat{\jmath}-5\hat{k}$

Vector Equation : $\vec{r}=\vec{a}+s\vec{b}+t\vec{c}$

$$\vec{r}=(6\hat{\imath}-\hat{\jmath}+\hat{k})+s(-\hat{\imath}+2\hat{\jmath}+\hat{k})+t(-5\hat{\imath}-4\hat{\jmath}-5\hat{k})$$

Cartiesian Equation: $\begin{vmatrix} x-x_1 & y-y_1 & z-z_1 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix}=0$

$$\begin{vmatrix} x-6 & y+1 & z-1 \\ -1 & 2 & 1 \\ -5 & -4 & -5 \end{vmatrix}=0$$

$$(x-6)(-10+4)-(y+1)(5+5)+(z-1)(4+10)=0$$

$$-6x-10y+14z+12=0\quad (or)\quad 3x+5y-7z-6=0$$

Non Parametirc.Vector Equation: $(\vec{r}-\vec{a})\cdot(\vec{b}\times\vec{c})=0$

$$\Rightarrow\ \vec{r}\cdot(3\hat{\imath}+5\hat{\jmath}-7\hat{k})-6=0$$

MODEL-II

13. Find the non-parametric and Cartesian form of the eqn of the plane passing through the points (-1,2,0), (2,2,-1) and parallel to the straight line $\frac{x-1}{1}=\frac{2y+1}{2}=\frac{z+1}{-1}$

Soln: $\vec{a}=-\hat{\imath}+2\hat{\jmath}+0\hat{k}$ $\quad\vec{b}=2\hat{\imath}+2\hat{\jmath}-\hat{k}$ $\quad\vec{c}=\hat{\imath}+\hat{\jmath}-\hat{k}$

Vector Equation: $\vec{r} = (1-s)\vec{a} + s\vec{b} + t\vec{c}$

$$\vec{r} = (1-s)(-\hat{\imath} + 2\hat{\jmath}) + s(2\hat{\imath} + 2\hat{\jmath} - \hat{k}) + t(\hat{\imath} + \hat{\jmath} - \hat{k})$$

Cartiesian Equation: $\begin{vmatrix} x-x_1 & y-y_1 & z-z_1 \\ x_2-x_1 & y_2-y_1 & z_2-z_1 \\ c_1 & c_2 & c_3 \end{vmatrix} = 0$

$$\Rightarrow \begin{vmatrix} x+1 & y-2 & z-0 \\ 3 & 0 & -1 \\ 1 & 1 & -1 \end{vmatrix} = 0$$

$$(x+1)(0+1) - (y-2)(-3+1) + (z-0)(3-0) = 0$$

$$x + 2y + 3z - 3 = 0$$

Non Parametirc Vector Equation: $(\vec{r} - \vec{a}) \cdot [(\vec{b} - \vec{a}) \times \vec{c}] = 0$

$$\Rightarrow \vec{r} \cdot (\hat{\imath} + 2\hat{\jmath} + 3\hat{k}) = 3$$

14. Find the non-parametric form of vector eqn, Cartesian eqns of the plane passing through the points (2,2,1), (9,3,6) and perpendicular to the plane $2x + 6y + 6z = 9$.

<u>**Soln**</u>: $\vec{a} = 2\hat{\imath} + 2\hat{\jmath} + \hat{k}$ $\quad \vec{b} = 9\hat{\imath} + 3\hat{\jmath} + 6\hat{k}$ $\quad \vec{c} = 2\hat{\imath} + 6\hat{\jmath} + 6\hat{k}$

Vector Equation: $\vec{r} = (1-s)\vec{a} + s\vec{b} + t\vec{c}$

$$\vec{r} = (1-s)(2\hat{\imath} + 2\hat{\jmath} + \hat{k}) + s(9\hat{\imath} + 3\hat{\jmath} + 6\hat{k}) + t(2\hat{\imath} + 6\hat{\jmath} + 6\hat{k})$$

Cartiesian Equation: $\begin{vmatrix} x-x_1 & y-y_1 & z-z_1 \\ x_2-x_1 & y_2-y_1 & z_2-z_1 \\ c_1 & c_2 & c_3 \end{vmatrix} = 0$

$$\Rightarrow \begin{vmatrix} x-2 & y-2 & z-1 \\ 7 & 1 & 5 \\ 2 & 6 & 6 \end{vmatrix} = 0$$

$$(x-2)(6-30) - (y-2)(42-10) + (z-1)(42-2) = 0$$

$$-24x - 32y + 40z + 72 = 0 \text{ (or) } 3x + 4y - 5z - 9 = 0$$

Non Parametirc Vector Equation: $(\vec{r} - \vec{a}) \cdot [(\vec{b} - \vec{a}) \times \vec{c}] = 0$

$$\Rightarrow \quad \vec{r} \cdot (3\hat{\imath} + 4\hat{\jmath} - 5\hat{k}) - 9 = 0$$

15. Find parametric form of Vector Equation and Cartesian equations of the plane passing through the points (2, 2,1), (1,-2,3) and parallel to the straight line passing through the points (2, 1, −3) and (−1,5, −8).

<u>**Soln**</u>: $\vec{a} = 2\hat{\imath} + 2\hat{\jmath} + \hat{k}$ $\quad \vec{b} = \hat{\imath} - 2\hat{\jmath} + 3\hat{k}$ $\quad \vec{c} = -3\hat{\imath} + 4\hat{\jmath} - 5\hat{k}$

Vector Equation: $\vec{r} = (1-s)\vec{a} + s\vec{b} + t\vec{c}$

$$\vec{r} = (1-s)(2\hat{\imath} + 2\hat{\jmath} + \hat{k}) + s(\hat{\imath} - 2\hat{\jmath} + 3\hat{k}) + t(-3\hat{\imath} + 4\hat{\jmath} - 5\hat{k})$$

Cartiesian Equation: $\begin{vmatrix} x-x_1 & y-y_1 & z-z_1 \\ x_2-x_1 & y_2-y_1 & z_2-z_1 \\ c_1 & c_2 & c_3 \end{vmatrix} = 0$

$$\begin{vmatrix} x-2 & y-2 & z-1 \\ -1 & -4 & 2 \\ -3 & 4 & -5 \end{vmatrix} = 0$$

$$(x-2)(20-8)-(y-2)(5+6)+(z-1)(-4-12)=0$$

$$12x-11y-16z+14=0$$

Non Parametirc Vector Equation: $(\vec{\boldsymbol{r}}-\vec{\boldsymbol{a}})\cdot\left[\left(\vec{\boldsymbol{b}}-\vec{\boldsymbol{a}}\right)\times\vec{\boldsymbol{c}}\right]=\mathbf{0}$

$$\Rightarrow\vec{\boldsymbol{r}}\cdot\left(\mathbf{12}\hat{\boldsymbol{\imath}}-\mathbf{11}\hat{\boldsymbol{\jmath}}-\mathbf{16}\hat{\boldsymbol{k}}\right)+\mathbf{14}=\mathbf{0}$$

MODEL-III

16. Find the parametric vector, non-parametric vector and Cartesian form of the equations of the plane passing through the three non-Collinear points (3,6, −2), (−1, −2,6) ,and (6, 4, –2).

<u>Soln</u>: $\vec{a}=3\hat{\imath}+6\hat{\jmath}-2\hat{k}$ $\quad\vec{b}=-\hat{\imath}-2\hat{\jmath}+6\hat{k}$ $\quad\vec{c}=6\hat{\imath}+4\hat{\jmath}-2\hat{k}$

Vector Equation: $\vec{\boldsymbol{r}}=(\mathbf{1}-\boldsymbol{s}-\boldsymbol{t})\vec{\boldsymbol{a}}+\boldsymbol{s}\vec{\boldsymbol{b}}+\boldsymbol{t}\vec{\boldsymbol{c}}$

$\vec{r}=(1-s-t)(3\hat{\imath}+6\hat{\jmath}-2\hat{k})+s(-\hat{\imath}-2\hat{\jmath}+6\hat{k})+t(6\hat{\imath}-4\hat{\jmath}-2\hat{k})$

Cartiesian Equation: $\begin{vmatrix} x-x_1 & y-y_1 & z-z_1 \\ x_2-x_1 & y_2-y_1 & z_2-z_1 \\ x_3-x_1 & y_3-y_1 & z_3-z_1 \end{vmatrix}=0$

$$\Rightarrow\begin{vmatrix} x-3 & y-6 & z+2 \\ -4 & -8 & 8 \\ 3 & -2 & 0 \end{vmatrix}=0$$

$$(x-3)(0+16)-(y-6)(0-24)+(z+2)(8+24)=0$$

$$16x-48+24y-144+32z+64=0\ \ (or)\quad 16x+24y+32z-128=0$$

$$2x+3y+4z-16=0$$

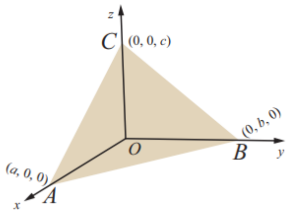
Non Parametirc Vector Equation: $(\vec{\boldsymbol{r}}-\vec{\boldsymbol{a}})\cdot\left[\left(\vec{\boldsymbol{b}}-\vec{\boldsymbol{a}}\right)\times(\vec{\boldsymbol{c}}-\vec{\boldsymbol{a}})\right]=\mathbf{0}$

$$\Rightarrow\vec{\boldsymbol{r}}\cdot\left(\mathbf{2}\hat{\boldsymbol{\imath}}+\mathbf{3}\hat{\boldsymbol{\jmath}}+\mathbf{8}\hat{\boldsymbol{k}}\right)-\mathbf{16}=\mathbf{0}$$

17. Derive the equation of the plane in the intercept form.

Soln: $A(a,0,0), B(0,b,0), C(0,0,c)$

$\vec{a}=a\hat{\imath}+0\hat{\jmath}+0\hat{k}\,,\vec{b}=0\hat{\imath}+b\hat{\jmath}+0\hat{k},\vec{c}=0\hat{\imath}+0\hat{\jmath}+c\hat{k},$



Vector Equation: $\vec{\boldsymbol{r}}=(\mathbf{1}-\boldsymbol{s}-\boldsymbol{t})\vec{\boldsymbol{a}}+\boldsymbol{s}\vec{\boldsymbol{b}}+\boldsymbol{t}\vec{\boldsymbol{c}}$

$\vec{r}=(1-s-t)a\hat{\imath}+sb\hat{\jmath}+tc\hat{k}$

Cartiesian Equation: $\begin{vmatrix} x-x_1 & y-y_1 & z-z_1 \\ x_2-x_1 & y_2-y_1 & z_2-z_1 \\ x_3-x_1 & y_3-y_1 & z_3-z_1 \end{vmatrix}=0$

$$\begin{vmatrix} x-a & y & z \\ -a & b & 0 \\ -a & 0 & c \end{vmatrix}=0$$

$$\frac{\boldsymbol{x}}{\boldsymbol{a}}+\frac{\boldsymbol{y}}{\boldsymbol{b}}+\frac{\boldsymbol{z}}{\boldsymbol{c}}=\mathbf{1}$$

18. **Show that the lines $\frac{x-1}{2}=\frac{y-2}{3}=\frac{z-3}{4}$ and $\frac{x-4}{5}=\frac{y-1}{2}=z$ intersect and hence find the point of intersection.**

Soln: Condition for intersecting lines $\begin{vmatrix} x_2-x_1 & y_2-y_1 & z_2-z_1 \\ l_1 & m_1 & n_1 \\ l_2 & m_2 & n_2 \end{vmatrix}=0\Rightarrow\begin{vmatrix} 3 & -1 & -3 \\ 2 & 3 & 4 \\ 5 & 2 & 1 \end{vmatrix}=0$

Let $\frac{x-1}{2}=\frac{y-2}{3}=\frac{z-3}{4}=s \implies (x,y,z)=(2s+1,3s+2,4s+3)$

$\frac{x-4}{5}=\frac{y-1}{2}=z=t \implies (x,y,z)=(5t+4,2t+1,t)$

At the point of intersection $(2s+1,3s+2,4s+3)=(5t+4,2t+1,t)$

$\therefore$ we get $s=-1,\ t=-1$

The point of intersection $(\boldsymbol{x},\boldsymbol{y},\boldsymbol{z})=(-\mathbf{1},-\mathbf{1},-\mathbf{1})$

Try yourself.

- ✓ Show that the lines $\frac{x-3}{3}=\frac{y-3}{-1},z-1=0$ and $\frac{x-6}{2}=\frac{z-1}{3},y-2=0$ intersect and hence find the point of intersection. **Hint:** $\frac{x-3}{3}=\frac{y-3}{-1}=\frac{z-1}{0}$ & $\frac{x-6}{2}=\frac{z-1}{3}=\frac{y-2}{0}$
- ✓ Find the parametric form of a vector equation of a straight line passing through the point of intersection of the straight lines $\vec{r}=\hat{\imath}+3\hat{\jmath}-\hat{k}+t(2\hat{\imath}+3\hat{\jmath}+2\hat{k})$ and $\frac{x-2}{1}=\frac{y-4}{2}=\frac{z+3}{4}$, and perpendicular to both straight lines. **Hint:** $\frac{x-1}{2}=\frac{y-3}{3}=\frac{z+1}{2}$ & $\frac{x-2}{1}=\frac{y-4}{2}=\frac{z+3}{4}$
- ✓ **If the two lines $\frac{x-1}{2}=\frac{y+1}{3}=\frac{z-1}{4}$ and $\frac{x-3}{1}=\frac{y-m}{2}=z$ intersect at a point, find the value of m.**

ANALYTICAL GEOMETRY

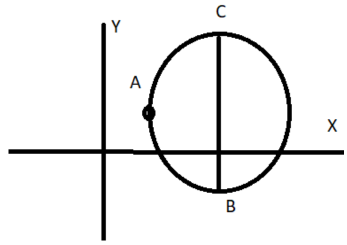
5 Marks

Hints:

- ➢ ***Ellipse*** $c^2=a^2m^2+b^2$, **point of contact** $\left(-\frac{a^2m}{c},\frac{b^2}{c}\right)$
- ➢ **Hyperbola** $c^2=a^2m^2-b^2$, **point of contact** $\left(-\frac{a^2m}{c},-\frac{b^2}{c}\right)$

1. Find the equation of the circle passing through the points $(1,1)$, $(2,-1)$, and $(3,2)$

Soln: $A(1,1), B(2,-1), C(3,2)$



$M_1=$ Slope of $AB=\frac{y_2-y_1}{x_2-x_1}=\frac{-1-1}{2-1}=-2$

$M_2=$ Slope of $AC=\frac{2-1}{3-1}=\frac{1}{2}$

$m_1\times m_2=-1\ \therefore \angle A=90^\circ$

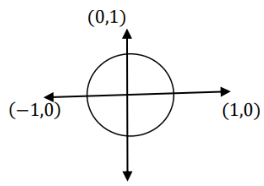
End points of diameter B, C

$(x-x_1)(x-x_2)+(y-y_1)(y-y_2)=0$

$(x-2)(x-3)+(y+1)(y-2)=0$

$\boldsymbol{x^2+y^2-5x-y+4=0}$

2. Find the equation of the circle through the points $(1,0)$, $(-1,0)$, and $(0,1)$.



Soln : End point of diameter of $(1,0),(-1,0)$

Centre$(0,0)$, radius=1

Equation of circle $\boldsymbol{x^2+y^2=1}$

3. Show that the line $x - y + 4 = 0$ is a tangent to the ellipse $x^2 + 3y^2 = 12$.Also find the coordinates of the point of contact.

Soln : $x - y + 4 = 0$ $\quad x^2 + 3y^2 = 12$ $\quad$ **hint:** $y = mx + c, \frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$

$y = x + 4$ $\quad \frac{x^2}{12} + \frac{y^2}{4} = 1$

$\boldsymbol{m = 1, c = 4}$ $\quad \boldsymbol{a^2 = 12, b^2 = 4}$

Condition:$\boldsymbol{c^2 = a^2m^2 + b^2}$

$16 = 16$

$x - y + 4 = 0$ is a tangent to $x^2 + 3y^2 = 12$

Point of contact: $\left(-\frac{a^2m}{c}, \frac{b^2}{c}\right) = (\boldsymbol{-3, 1})$

4. Show that the line $5x + 12y = 9$ is a tangent to the hyperbola $x^2 - 9y^2 = 9$,also find point of contact?

Soln : $5x + 12y = 9 \Rightarrow y = -\frac{5}{12}x + \frac{3}{4}$ $\quad , m = -\frac{5}{12}, c = \frac{3}{4}$

$x^2 - 9y^2 = 9 \Rightarrow \frac{x^2}{9} - \frac{y^2}{1} = 9$ $\quad a^2 = 9, b^2 = 1$

Condition $\boldsymbol{c^2 = a^2m^2 - b^2,}$

$\Rightarrow \boldsymbol{\frac{9}{16} = \frac{9}{16}}$

$5x + 12y = 9$ is a tangent to the $x^2 - 9y^2 = 9$

Point of contact is $\left(-\frac{a^2m}{c}, -\frac{b^2}{c}\right) = (\boldsymbol{5, -\frac{4}{3}})$

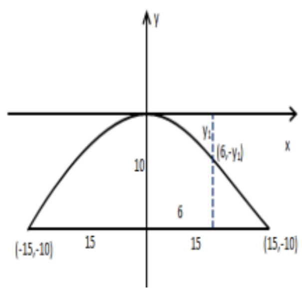
5. A bridge has a parabolic arch that is 10*m* high in the centre and 30*m* wide at the bottom. Find the height of the arch 6*m* from the centre, on either sides.

Soln: $x^2 = -4ay - - - \longrightarrow (1)$

At $(15, -10)$

$(1) \Longrightarrow (15)^2 = -4a(-10) \Longrightarrow a = \frac{225}{40}$

$(1) \Longrightarrow x^2 = -4\left(\frac{225}{40}\right)y - - - \longrightarrow (2)$



At $(6, -y_1)$

$(2) \Longrightarrow (6)^2 = -4 \times \frac{225}{40}(-y_1)$

$\frac{36 \times 40}{4 \times 225} = y_1 \Longrightarrow y_1 = 1.6$

Required height is $\boldsymbol{10 - y_1 = 10 - 1.6 = 8.4}$ **m**

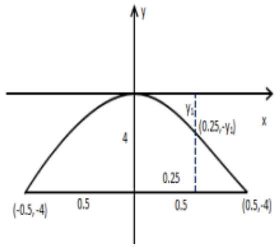
6. At a water fountain, water attains a maximum height of 4*m* at horizontal distance of 0.5 *m* from its origin. If the path of water is a parabola, find the height of water at a horizontal distance of 0.75 *m* from the point of origin.

Soln: $x^2 = -4ay - - - \longrightarrow (1)$

At $(-0.5,-4)$

$(1) \Longrightarrow (-\frac{1}{2})^2 = -4a(-4) \Longrightarrow a = \frac{1}{64}$

$(1) \Rightarrow x^2 = -4\left(\frac{1}{64}\right)y = -\left(\frac{1}{16}\right)y \longrightarrow (2)$



At $(0.25,-y_1)$

$(2) \Longrightarrow \left(\frac{1}{4}\right)^2 = -4 \times \frac{1}{64}(-y_1) \Longrightarrow \frac{64}{4\times16} = y_1 \Longrightarrow y_1 = 1$

Required distance is $4 - y_1 = 4 - 1 = 3$ m

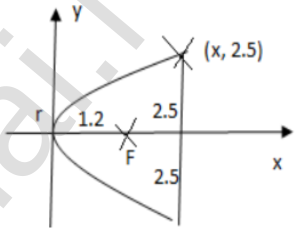
7. **An engineer designs a satellite dish with a parabolic cross section. The dish is 5*m* wide at the opening, and the focus is placed 1.2 *m* from the vertex**
   (a) Position a coordinate system with the origin at the vertex and the *x* -axis on the parabola's axis of symmetry and find an equation of the parabola.
   (b) Find the depth of the satellite dish at the vertex.

Soln: $y^2 = 4ax \longrightarrow (1)$

Given $a = 1.2$

(i)equation of parabola $y^2 = 4 \times 1.2 \times x = 4.8x$

$y^2 = 4.8x - - - \longrightarrow (2)$



(ii)At $(x_1, 2.5)$

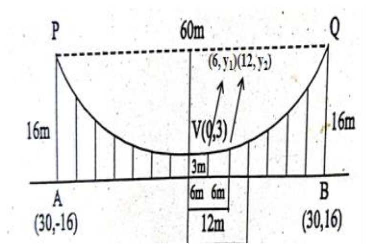
$(2) \Longrightarrow (2.5)^2 = 4.8x_1 \Longrightarrow \frac{6.25}{4.8} = x_1 \quad \therefore x_1 = 1.3$ m

8. **Parabolic cable of a 60*m* portion of the roadbed of a suspension bridge are positioned as shown below. Vertical Cables are to be spaced every 6*m* along this portion of the roadbed. Calculate the lengths of first two of these vertical cables from the vertex.**

Soln: $x^2 = 4ay - - - \longrightarrow (1)$

At $(30,13) \Longrightarrow 30^2 = 4a(13) \Longrightarrow a = \frac{900}{52}$

Equation of parabola $x^2 = 4 \times \frac{900}{52} y \Longrightarrow x^2 = \frac{900}{13} y - - - \longrightarrow (2)$



(i)At $(6, y_1)$

$(2) \Longrightarrow 6^2 = \frac{900}{13} y_1 \Longrightarrow \frac{36\times13}{900} = y_1 \Longrightarrow y_1 = 0.52$

Height of the first cable is $3 + y_1 = 3 + 0.52 = 3.52$

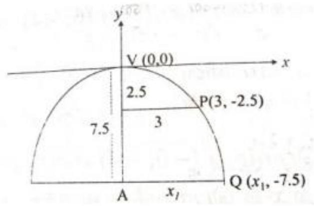
(i)At $(12, y_2)$

$(2) \Longrightarrow 12^2 = \frac{900}{13} y_2 \Longrightarrow \frac{144\times13}{900} = y_2 \Longrightarrow y_2 = 2.08$

Height of the second cable is $3 + y_2 = 3 + 2.08 = 5.08$ m

9. **Assume that water issuing from the end of a horizontal pipe, 7.5 *m* above the ground, describes a parabolic path. The vertex of the parabolic path is at the end of the pipe. At a position 2.5 *m* below the line of the pipe, the flow of water has curved outward 3*m* beyond the vertical line through the end of the pipe. How far beyond this vertical line will the water strike the ground?**

Soln: $x^2 = -4ay - - - \longrightarrow (1)$



At $(3,-2.5)$, $(1) \Longrightarrow (3)^2 = -4a(-2.5) \Rightarrow a = \frac{9}{10}$

$$(1) \Longrightarrow \quad x^2 = -4\left(\frac{9}{10}\right)y ----\longrightarrow (2)$$

At $(x_1,-7.5)$ $(2) \Longrightarrow (x_1)^2 = -4 \times \frac{9}{10}(-7.5)$

$$\Rightarrow \boldsymbol{(x_1)^2 = 9 \times 3 \Rightarrow x_1 = 3\sqrt{3}\ \mathbf{m}}$$

10. On lighting a rocket cracker it gets projected in a parabolic path and reaches a maximum height of 4*m* when it is 6*m* away from the point of projection. Finally it reaches the ground 12*m* away from the starting point. Find the angle of projection

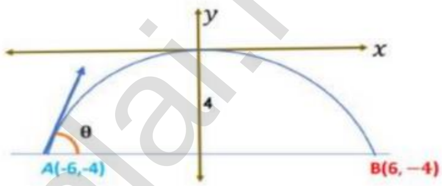
Soln: $x^2 = -4ay ----\longrightarrow (1)$

At $(6,-4)$

$$(1) \Longrightarrow (6)^2 = -4a(-4) \Rightarrow a = \frac{36}{16} = \frac{9}{4}$$

$(1) \Rightarrow x^2 = -4ay \Rightarrow x^2 = -4\left(\frac{9}{4}\right)y \Rightarrow x^2 =$

$-9y ----\longrightarrow (2)$



$(2)\ diff.w.r.t.'x' \Longrightarrow 2x = -9\frac{dy}{dx}$

$\therefore \frac{dy}{dx} = \frac{2x}{-9}$

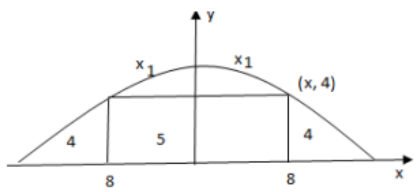
At $(-6,-4) \Rightarrow \frac{dy}{dx} = \frac{2(-6)}{-9}$

$\frac{dy}{dx} = tan\theta = \frac{4}{3}$

$\therefore \boldsymbol{\theta = tan^{-1}\left(\frac{4}{3}\right)}$

11. A tunnel through a mountain for a four lane highway is to have an elliptical opening. The total width of the highway (not the opening) is to be 16*m*, and the height at the edge of the road must be sufficient for a truck 4*m* high to clear if the highest point of the opening is to be 5*m* approximately. How wide must the opening be?

Soln: $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1 ----\longrightarrow (1)$



Given $b = 5$ $(1) \Longrightarrow \frac{x^2}{a^2} + \frac{y^2}{5^2} = 1 ----\longrightarrow (2)$

At $(8,4)$

$$(2) \Longrightarrow \frac{8^2}{a^2} + \frac{4^2}{5^2} = 1$$

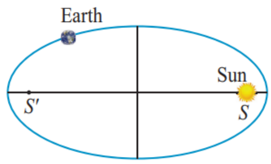
$$\frac{8^2}{a^2} = 1 - \frac{16}{25} = \frac{25-16}{25} = \frac{9}{25} = \left(\frac{3}{5}\right)^2$$

$$\frac{8^2}{a^2} = \left(\frac{3}{5}\right)^2 \Rightarrow \frac{8}{a} = \frac{3}{5} \Longrightarrow a = \frac{40}{3}$$

Required opening is $\boldsymbol{2a = \frac{80}{3} = 26.66}$ **m**

12. The maximum and minimum distances of the Earth from the Sun respectively are 152×10^6 km and 94.5×10^6 km. The Sun is at one focus of the elliptical orbit. Find the distance from the Sun to the other focus.

Soln.



$$AS = 94.5 \times 10^6 \text{ km}, SA' = 152 \times 10^6 \text{ km}$$
$$a + c = 152 \times 10^6$$
$$a - c = 94.5 \times 10^6$$

Subtracting $2c = 57.5 \times 10^6 = 575 \times 10^5$ km

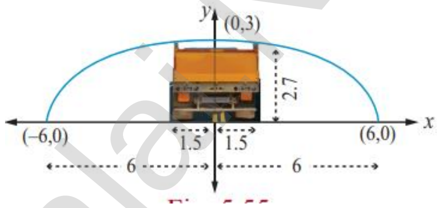
Distance from the Sun to the other focus is $SS' = 575 \times 10^5$ km.

13. **.A semielliptical archway over a one-way road has a height of 3 m and a width of 12 m. The truck has a width of 3 m and a height of 2.7 m. Will the truck clear the opening of the archway?**

Soln.From the diagram $a = 6$ and $b = 3$

Equation of ellipse as $\frac{x^2}{6^2} + \frac{y^2}{3^2} = 1 ---\rightarrow (1)$

Substituting $x = 1.5 = \frac{3}{2}$



$$(1) \Rightarrow \frac{\left(\frac{3}{2}\right)^2}{36} + \frac{y^2}{9} = 1$$

$$y^2 = 9\left(1 - \frac{9}{144}\right)$$

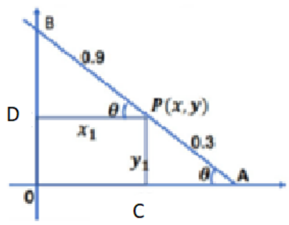
$$y = \frac{135}{16} = \frac{\sqrt{135}}{4} = 2.90m$$

The truck will clear the archway.

14. **A rod of length 1.2 m moves with its ends always touching the coordinate axes. The locus of the point P on the rod 0.3 m from the end in contact with x-axis is an ellipse. Find the eccentricity.**

Soln: Right angle triangle PAC

$$sin\theta = \frac{y_1}{0.3} \Longrightarrow sin^2\theta = \frac{{y_1}^2}{0.09} ---\rightarrow (1)$$



Right angle triangle BPD

$$cos\theta = \frac{x}{0.9} \Longrightarrow cos^2\theta = \frac{{x_1}^2}{0.81} ---\rightarrow (2)$$

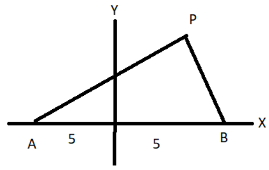
$$(1)^2 + (2)^2 \Rightarrow \frac{{x_1}^2}{0.81} + \frac{{y_1}^2}{0.09} = cos^2\theta + sin^2\theta = 1$$

The locus of $(x_1, y_1) is$ $\frac{x^2}{0.81} + \frac{y^2}{0.09} = 1$.This is ellipse

Eccentricity $e = \sqrt{\frac{a^2-b^2}{a^2}} = \sqrt{\frac{0.81-0.09}{0.81}} = \sqrt{\frac{0.72}{0.81}} = \sqrt{\frac{8}{9}} = \frac{2\sqrt{2}}{3}$ **m**

15. Points A and B are $10km$ apart and it is determined from the sound of an explosion heard at those points at different times that the location of the explosion is 6 km closer to A than B . Show that the location of the explosion is restricted to a particular curve and find an equation of it.

Soln: $2ae = 10 \quad \Rightarrow ae = 5\,; \quad 2a = 6 \quad \Rightarrow a = 3$



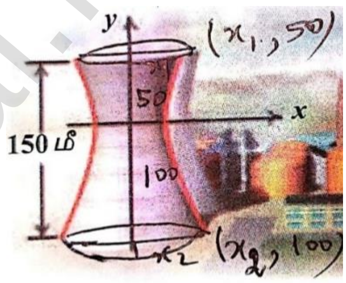
$$3e = 5 \quad \Rightarrow \quad e = \frac{5}{3} > 1, \therefore \text{The curve is an hyperbola.}$$

$$b^2 = a^2(e^2 - 1) \Rightarrow b^2 = 9\left(\frac{25}{9} - 1\right) \Rightarrow b^2 = 9\left(\frac{25-9}{9}\right) \Rightarrow b^2 = 16$$

Equation of hyperbola is $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1 \Rightarrow \frac{x^2}{9} - \frac{y^2}{16} = 1$

16. Cross section of a Nuclear cooling tower is in the shape of a hyperbola with equation $\frac{x^2}{30^2} - \frac{y^2}{44^2} = 1$ The tower is $150m$ tall and the distance from the top of the tower to the centre of the hyperbola is half the distance from the base of the tower to the centre of the hyperbola. Find the diameter of the top and base of the tower.

Soln: Given Equation $\frac{x^2}{30^2} - \frac{y^2}{44^2} = 1 - - - \longrightarrow (1)$



At $(x_1, 50)$

$$(1) \Rightarrow \frac{(x_1)^2}{30^2} - \frac{(50)^2}{44^2} = 1 \Rightarrow \frac{(x_1)^2}{30^2} = 1 + \frac{(50)^2}{44^2}$$

$$x_1 = \frac{30}{44}\sqrt{44^2 + 50^2} = 45.41$$

$\therefore$ **the diameter of the top is** $2x_1 = 2(45.41) = 90.82$

At $(x_2, 100)$

$$(1) \Rightarrow \frac{(x_2)^2}{30^2} - \frac{(100)^2}{44^2} = 1 \Rightarrow \frac{(x_2)^2}{30^2} = 1 + \frac{(100)^2}{44^2}$$

$$x_2 = \frac{30}{44}\sqrt{44^2 + 100^2} = 74.49$$

$\therefore$ **the diameter of the top is** $2x_2 = 2(74.49) = 148.98$ **m**

COMPLEX NUMBERS

Important Hints:

$i = \sqrt{-1}, i^2 = -1,\ i^3 = -i,\ i^4 = 1,\ i^{4n} = 1$

Rectangular form of a complex number is x + iy real part is x, Imaginary part is y.

The conjugate of the complex number z = x + iy is x −iy and is denoted by $\bar{z}$

If $z = x +$ iy then modulus of z is $|z| = \sqrt{x^2 + y^2}$

Triangle inequality: For any two complex number z_1 and z_2, $|z_1 + z_2| \le |z_1| + |z_2|$

$||z_1| - |z_2|| \le |z_1 + z_2| \le |z_1| + |z_2|$

$$\sqrt{a \pm ib} = \pm\left[\sqrt{\frac{|z| + a}{2}} \pm i\sqrt{\frac{|z| - a}{2}}\right]$$

Additive inverse of $\boldsymbol{z}$ **is** $-\boldsymbol{z}$, Multiplicative inverse of $\boldsymbol{z}$ **is** $\frac{1}{z} = \bar{z}$

z is real if any only if z = $\bar{z}$ and **z is purely imaginary if and only if z =** $-\bar{z}$

Distance between two complex numbers, z_1 and z_2 is $|z_1 - z_2|$

$|z - z_0| = r$ is the complex form of the equation of a circle. Centre is z_0 and radius is r.

5 Marks

1. If $z = x + iy$ is a complex number such that $\left|\frac{z-4i}{z+4i}\right| = 1$, S.T. the locus of z is real axis.

Soln:

$$\boldsymbol{z = x + iy}$$

$$\left|\frac{z-4i}{z+4i}\right| = 1 \Rightarrow |z - 4i| = |z + 4i|$$

$$|x + iy - 4i| = |x + iy + 4i|$$

$$|x + i(y - 4)|^2 = |x + i(y + 4)|^2$$

$$x^2 + (y - 4)^2 = x^2 + (y + 4)^2$$

$$y = 0$$

$$\therefore \boldsymbol{z\ is\ real}$$

2. If $z = x + iy$ is a complex number such that $Im\left(\frac{2z+1}{iz+1}\right) = 0$, show that the locus of z is $2x^2 + 2y^2 + x - 2y = 0$.

Soln: Given $Im\left(\frac{2z+1}{iz+1}\right) = 0$ put $z = x + iy$

$$Im\left(\frac{2(x+iy)+1}{i(x+iy)+1}\right) = 0$$

$$Im\left(\frac{2x+i2y+1}{ix+i^2y+1}\right) = 0 \qquad Im\left(\frac{a+ib}{c+id}\right) = \frac{bc-ad}{c^2+d^2}$$

$$Im\left(\frac{(2x+1)+i2y}{(1-y)+ix}\right) = 0$$

$$\left(\frac{2y(1-y)-x(2x+1)}{(1-y)^2+x^2}\right) = 0$$

$$\boldsymbol{2y - 2x^2 - 2y^2 - x = 0 \quad (or)\ \ 2x^2 + 2y^2 + x - 2y = 0}$$

3. If $z = x + iy$ is a complex number such that $Re\left(\frac{z-1}{z+1}\right) = 0$, S.T the locus of z is $x^2 + y^2 = 1$.

Soln: Given $Re\left(\frac{z-1}{z+1}\right) = 0$ put $z = x + iy$

$$Re\left(\frac{x+iy-1}{x+iy+1}\right) = 0$$

$$Re\left(\frac{(x-1)+iy}{(x+1)+iy}\right) = 0 \qquad Re\left(\frac{a+ib}{c+id}\right) = \frac{ac+bd}{c^2+d^2}$$

$$\left(\frac{(x-1)(x+1)+y^2}{(x+1)^2+y^2}\right) = 0$$

$$\boldsymbol{x^2 - 1 + y^2 = 0 \implies x^2 + y^2 = 1}$$

4. If $z = x + iy$ is a complex number such that $arg\left(\frac{z-1}{z+1}\right) = \frac{\pi}{2}$, S.T the locus of z is $x^2 + y^2 = 1$.

Soln: Given $arg\left(\frac{z-1}{z+1}\right) = \frac{\pi}{2}$ put $z = x + iy$

$$arg\left(\frac{x+iy-1}{x+iy+1}\right) = \frac{\pi}{2}$$

$$arg\left(\frac{(x-1)+iy}{(x+1)+iy}\right) = \frac{\pi}{2} \qquad \arg\left(\frac{a+ib}{c+id}\right) = \tan^{-1}\left(\frac{bc-ad}{ac+bd}\right)$$

$$tan^{-1}\left(\frac{y(x+1)-y(x-1)}{(x-1)(x+1)+y^2}\right) = \frac{\pi}{2}$$

$$\left(\frac{y(x+1)-y(x-1)}{(x-1)(x+1)+y^2}\right) = tan\frac{\pi}{2} = \infty = \frac{1}{0}$$

$$(x-1)(x+1) + y^2 = 0$$

$$\boldsymbol{x^2 - 1 + y^2 = 0 \implies x^2 + y^2 = 1}$$

5. If $z = x + iy$ is a complex number such that $arg\left(\frac{z-i}{z+2}\right) = \frac{\pi}{4}$, show that the locus of z is $x^2 + y^2 + 3x - 3y + 2 = 0$.

Soln: Given $arg\left(\frac{z-i}{z+2}\right) = \frac{\pi}{4}$ put $z = x + iy$

$$arg\left(\frac{x+iy-i}{x+iy+2}\right) = \frac{\pi}{4}$$

$$arg\left(\frac{x+i(y-1)}{(x+2)+iy}\right) = \frac{\pi}{4} \qquad \arg\left(\frac{a+ib}{c+id}\right) = \tan^{-1}\left(\frac{bc-ad}{ac+bd}\right)$$

$$tan^{-1}\left(\frac{(x+2)(y-1)-xy}{x(x+2)+y(y-1)}\right) = \frac{\pi}{4}$$

$$\left(\frac{(x+2)(y-1)-xy}{x(x+2)+y(y-1)}\right) = tan\frac{\pi}{4} = 1$$

$$(x+2)(y-1) - xy = x(x+2) + y(y-1)$$

$$\boldsymbol{x^2 + y^2 + 3x - 3y + 2 = 0}$$

Try yourself

- ✓ **If $z = x + iy$ is a complex number such that $arg\left(\frac{z-1}{z+1}\right) = \frac{\pi}{3}$, show that the locus of z is $\sqrt{3}x^2 + \sqrt{3}y^2 - 2y - 3 = 0$.**

6. If $z = 3 + 2i$, represent the complex numbers z, iz, and $z + iz$ in one Argand plane. S.t. these complex numbers form the vertices of an isosceles right triangle.

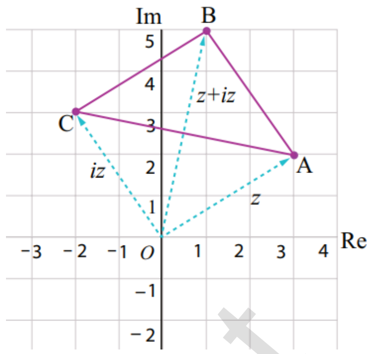
Soln: Given, $z = 3 + 2i$

Then $iz = i(3 + 2i) = 3i - 2 = -2 + 3i$;

$z + iz = 1 + 5i$

Let $z_1 = z = 3 + 2i$, $z_2 = iz = -2 + 3i$,

$z_3 = z + iz = 1 + 5i$



$AB = |z_1 - z_2| = |(3 + 2i) - (-2 + 3i)|$

$= |5 - i| = \sqrt{(5)^2 + (-1)^2} = \sqrt{\mathbf{26}}$

$BC = |z_2 - z_3| = |(-2 + 3i) - (1 + 5i)|$

$= |-3 - 2i| = \sqrt{(-3)^2 + (-2)^2} = \sqrt{\mathbf{13}}$

$CA = |z_3 - z_1| = |(1 + 5i) - (3 + 2i)|$

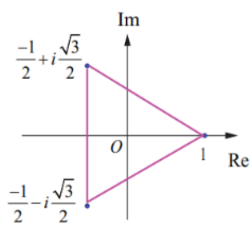
$= |-2 + 3i| = \sqrt{(-2)^2 + (3)^2} = \sqrt{\mathbf{13}}$

$BC^2 + CA^2 = AB^2 \implies (\sqrt{13})^2 + (\sqrt{13})^2 = (\sqrt{26})^2 \implies \mathbf{26 = 26}$

$\therefore$ **Given complex numbers form the vertices of an isosceles right triangle.**

7. Show that the points 1, $\frac{-1}{2} + i\frac{\sqrt{3}}{2}$ and $\frac{1}{2} - i\frac{\sqrt{3}}{2}$ are the vertices of an equilateral triangle.

Soln: Let $z_1 = 1$ $\quad z_2 = \frac{-1}{2} + i\frac{\sqrt{3}}{2}$ $\quad z_3 = \frac{1}{2} - i\frac{\sqrt{3}}{2}$



$AB = |z_1 - z_2| = \left|1 - \left(\frac{-1}{2} + i\frac{\sqrt{3}}{2}\right)\right| = \sqrt{\mathbf{3}}$

$BC = |z_2 - z_3| = \left|\left(\frac{-1}{2} + i\frac{\sqrt{3}}{2}\right) - \left(\frac{-1}{2} - i\frac{\sqrt{3}}{2}\right)\right| = |0 + i\sqrt{3}| = \sqrt{\mathbf{3}}$

$CA = |z_3 - z_1| = \left|\left(\frac{-1}{2} - i\frac{\sqrt{3}}{2}\right) - 1\right| = \left|\frac{-3}{2} - i\frac{\sqrt{3}}{2}\right| = \sqrt{\mathbf{3}}$

AB=BC=CA $\therefore$ **Given points are the vertices of an equilateral triangle.**

8. If z_1, z_2 and z_3 are three complex numbers S.T $|z_1| = 1, |z_2| = 2$, $|z_3| = 3$ and $|z_1 + z_2 + z_3| = 1$, show that $|9z_1z_2 + 4z_1z_3 + z_yz_3| = 6$.

Soln: Given $|z_1| = 1, |z_2| = 2$, $|z_3| = 3$ and $|z_1 + z_2 + z_3| = 1$ $\quad \because |z|^2 = z\bar{z}$

$z_1\overline{z_1} = 1, z_2\overline{z_2} = 4 \quad , z_3\overline{z_3} = 9$

$z_1 = \frac{1}{\overline{z_1}}$, $z_2 = \frac{4}{\overline{z_2}}$, $z_3 = \frac{9}{\overline{z_3}}$

$|\mathbf{z_1 + z_2 + z_3}| = \left|\frac{1}{\overline{z_1}} + \frac{4}{\overline{z_2}} + \frac{9}{\overline{z_3}}\right|$

$\mathbf{1} = \frac{|z_2z_3 + 4z_1z_3 + 9z_1z_2|}{|z_1||z_2||z_3|}$

$|\mathbf{z_2z_3 + 4z_1z_3 + 9z_1z_2}| = |\mathbf{z_1}||\mathbf{z_2}||\mathbf{z_3}|$

$= \mathbf{1 \times 2 \times 3 = 6}$

9. If z_1, z_2 and z_3 are three complex number S.T. $|z_1| = |z_2| = |z_3| = r > 0$ and $z_1 + z_2 + z_3 \neq 0$. Prove that $\left|\frac{z_1z_2+z_2z_3+z_3z_1}{z_1+z_2+z_3}\right| = r$.

Soln: Given $|z_1| = |z_2| = |z_3| = r$ $\quad\because |z|^2 = z\bar{z}$

$$z_1\overline{z_1} = z_2\overline{z_2} = z_3\overline{z_3} = r^2$$

$$z_1 = \frac{r^2}{\overline{z_1}}, z_2 = \frac{r^2}{\overline{z_2}}, z_3 = \frac{r^2}{\overline{z_3}}$$

$$|z_1 + z_2 + z_3| = \left|\frac{r^2}{\overline{z_1}} + \frac{r^2}{\overline{z_2}} + \frac{r^2}{\overline{z_3}}\right|$$

$$= r^2\left|\frac{\overline{z_1}\,\overline{z_2}+\overline{z_2}\,\overline{z_3}+\overline{z_3}\,\overline{z_1}}{\overline{z_1}\,\overline{z_2}\,\overline{z_3}}\right|$$

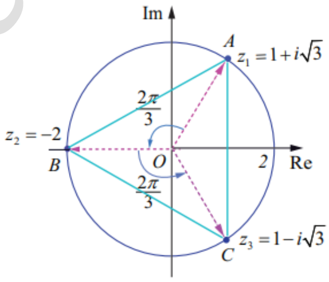
$$|z_1 + z_2 + z_3| = r^2\frac{|z_1z_2+z_2z_3+z_1z_3|}{r^3}$$

$$= \frac{|z_1z_2+z_2z_3+z_1z_3|}{r}$$

$$\therefore \quad \left|\frac{z_1z_2+z_2z_3+z_3z_1}{z_1+z_2+z_3}\right| = r$$

10. Suppose z_1, z_2 and z_3 are the vertices of an equilateral triangle inscribed in the circle $|z| = 2$. If $z_1 = 1 + i\sqrt{3}$, then find z_2 and z_3.

Soln: Given, $|z| = r = 2$ and $z_1 = 1 + i\sqrt{3}$;



$$\theta = \alpha = tan^{-1}\left(\frac{\sqrt{3}}{1}\right) = \frac{\pi}{3}$$

$\therefore$ Euler's form of $z_1 = re^{i\theta} = 2e^{i\frac{\pi}{3}}$

Clearly, z_2 is rotation of z_1 anti-clockwise by $\frac{2\pi}{3}$

$$z_2 = z_1e^{i\frac{2\pi}{3}} = 2e^{i\frac{\pi}{3}}e^{i\frac{2\pi}{3}} = 2e^{i\pi} = -2$$

Clearly, z_3 is rotation of z_1 clockwise by $\frac{2\pi}{3}$

$$\boldsymbol{z_3 = z_1e^{-i\frac{2\pi}{3}} = 2e^{i\frac{\pi}{3}}e^{-i\frac{2\pi}{3}} = 2e^{-i\frac{\pi}{3}} = 2\left(\frac{1}{2} - i\frac{\sqrt{3}}{2}\right) = 1 - i\sqrt{3}}$$

11. Find the fourth roots of unity.

Soln: Given $\quad z^4 = 1$

$$(z^2)^2 = 1$$

$$z^2 = \pm\sqrt{1}$$

$$z^2 = \pm 1$$

$z^2 = 1$ $\qquad\qquad$ $z^2 = -1$

$z = \pm\sqrt{1}$ $\qquad\qquad$ $z = \pm\sqrt{-1}$

$\boldsymbol{z = \pm 1}$ $\qquad\qquad$ $\boldsymbol{z = \pm i}$

12. Find the cube roots of unity.

Soln: Given $\quad z^3 = 1$

$$z^3 - 1 = 0$$

$$(z - 1)(z^2 + z + 1) = 0$$

$z - 1 = 0$ $\qquad$ $z^2 + z + 1 = 0$

$\boldsymbol{z = 1}$ $\qquad$ $\boldsymbol{z = \frac{-1 \pm i\sqrt{3}}{2}}$

Note :

$$\boldsymbol{z = (1)^{\frac{1}{3}} = (1, \omega, \omega^2)}$$

Here $\boldsymbol{\omega = \frac{-1}{2} + i\frac{\sqrt{3}}{2}, \omega^2 = \frac{-1}{2} - i\frac{\sqrt{3}}{2}}$

13. Solve the equation $z^3 + 8i = 0$, where $z \in C$.

Soln: Given $z^3 + 8i = 0$

$$z^3 = -8i$$

$$z^3 = (2i)^3 \times 1$$

$$z = 2i \times (1)^{\frac{1}{3}}$$

$$z = 2i(1, \omega, \omega^2)$$

$$z = 2i, \ 2i\left(\frac{-1}{2} + i\frac{\sqrt{3}}{2}\right), \ 2i\left(\frac{-1}{2} - i\frac{\sqrt{3}}{2}\right)$$

$$\boldsymbol{z = 2i, -i - \sqrt{3}, \quad -i + \sqrt{3}}$$

14. Solve the equation $z^3 + 27 = 0$, where $z \in C$

Soln**:** Given $z^3 + 27 = 0$

$$z^3 = -27 = -3 \times -3 \times -3 = (-3)^3 \times 1$$

$$z = -3 \times (1)^{\frac{1}{3}}$$

$$z = -3(1, \omega, \omega^2)$$

$$\boldsymbol{Z = -3, -3\omega, -3\omega^2}$$

15. If $\omega \neq 1$ is a cube root of unity, show that the roots of the equation $(z-1)^3 + 8 = 0$ are $-1, 1-2\omega, 1-2\omega^2$.

Soln: Given $(z-1)^3 + 8 = 0$

$$(z-1)^3 = -8 = (-2)^3 \times 1$$

$$(z-1) = -2 \times (1)^{\frac{1}{3}}$$

$$z - 1 = -2(1, \omega, \omega^2) = -2, -2\omega, -2\omega^2$$

$$\boldsymbol{Z = -1, 1 - 2\omega, 1 - 2\omega^2}$$

16. Show that (i) $\left(\frac{19-7i}{9+i}\right)^{12} + \left(\frac{20-5i}{7-6i}\right)^{12}$ is real

Soln : $\frac{19-7i}{9+i} = 2 - i$, $\frac{20-5i}{7-6i} = 2 + i$

$$z = \left(\frac{19-7i}{9+i}\right)^{12} + \left(\frac{20-5i}{7-6i}\right)^{12}$$

$$z = (2-i)^{12} + (2+i)^{12}$$

$$\bar{z} = (2+i)^{12} + (2-i)^{12}$$

$\boldsymbol{\bar{z} = z}$ **, z is real**

17. Show that (i) $\left(\frac{19+9i}{5-3i}\right)^{15} - \left(\frac{8+i}{1+2i}\right)^{15}$ is purely imaginary

Soln: $\frac{19+9i}{5-3i} = 2+3i, \frac{8+i}{1+2i} = 2-3i$

$$z = \left(\frac{19+9i}{5-3i}\right)^{15} - \left(\frac{8+i}{1+2i}\right)^{15}$$

$$z = (2+3i)^{15} - (2-3i)^{15}$$

$$\bar{z} = (2-3i)^{15} - (2+3i)^{15}$$

$$\bar{z} = -z$$

$\therefore$ ***z* is purely imaginary.**

18. Find all the cube roots of $\sqrt{3}+i$

Soln: Let $z^3 = re^{i\theta} \Rightarrow z = \left(re^{i\theta}\right)^{\frac{1}{3}}$

$$z = \left(\sqrt{3}+i\right)^{\frac{1}{3}}$$

$$r = \sqrt{\left(\sqrt{3}\right)^2 + (1)^2} = 2, \quad \theta = \alpha = tan^{-1}\left(\frac{1}{\sqrt{3}}\right) = \frac{\pi}{6}$$

$$z = \left(2e^{i\frac{\pi}{6}}\right)^{\frac{1}{3}}$$

$$z = 2^{\frac{1}{3}}\left(e^{i\left(\frac{\pi}{6}+2k\pi\right)}\right)^{\frac{1}{3}} \quad k = 0,1,2$$

$$z = 2^{\frac{1}{3}}\left(e^{i\frac{(1+12k)\pi}{6}}\right)^{\frac{1}{3}} \quad k = 0,1,2$$

$$\boldsymbol{z = 2^{\frac{1}{3}}\, e^{i\frac{(1+12k)\pi}{18}}} \quad k = 0,1,2$$

For $k = 0$, $\quad z = 2^{\frac{1}{3}}\, e^{i\frac{\pi}{18}}$

For $k = 1$, $\quad z = 2^{\frac{1}{3}}\, e^{i\frac{13\pi}{18}}$

For $k = 2$, $\quad z = 2^{\frac{1}{3}}\, e^{i\frac{25\pi}{18}}$

19. If $\cos\alpha + \cos\beta + \cos\gamma = \sin\alpha + \sin\beta + \sin\gamma = 0$, show that

(i) $\cos 3\alpha + \cos 3\beta + \cos 3\gamma = 3\cos(\alpha+\beta+\gamma)$ and

(ii) $\sin 3\alpha + \sin 3\beta + \sin 3\gamma = 3\sin(\alpha+\beta+\gamma)$.

Soln: $a = \cos\alpha + i\sin\alpha$, $\quad b = \cos\beta + i\sin\beta$, $\; c = \cos\gamma + i\sin\gamma$

***if* $\boldsymbol{a+b+c = 0}$ then $\boldsymbol{a^3+b^3+c^3 = 3abc}$**

$$(\cos\alpha + i\sin\alpha)^3 + (\cos\beta + i\sin\beta)^3 + (\cos\gamma + i\sin\gamma)^3$$
$$= 3(\cos\alpha + i\sin\alpha)(\cos\beta + i\sin\beta)(\cos\gamma + i\sin\gamma)$$

$$(\cos 3\alpha + \cos 3\beta + \cos 3\gamma) + i(\sin 3\alpha + \sin 3\beta + \sin 3\gamma)$$
$$= 3[\cos\,(\alpha+\beta+\gamma) + i\sin(\alpha+\beta+\gamma)]$$

(i) $\cos 3\alpha + \cos 3\beta + \cos 3\gamma = 3\cos(\alpha+\beta+\gamma)$

(ii) $\sin 3\alpha + \sin 3\beta + \sin 3\gamma = 3\sin(\alpha+\beta+\gamma)$

20. If $2\cos\alpha = x + \frac{1}{x}$ and $2\cos\beta = y + \frac{1}{y}$, show that (i) $\frac{x}{y} + \frac{y}{x} = 2\cos(\alpha - \beta)$

(ii) $xy - \frac{1}{xy} = 2i\sin(\alpha + \beta)$ (iii) $\frac{x^m}{y^n} - \frac{y^n}{x^m} = 2i\sin(m\alpha - n\beta)$

(iv) $x^m y^n + \frac{1}{x^m y^n} = 2\cos(m\alpha + n\beta)$

Soln: Given $x + \frac{1}{x} = 2\cos\alpha \Rightarrow \frac{x^2+1}{x} = 2\cos\alpha$

$$x^2 - 2\cos\alpha x + 1 = 0$$

$$x = \cos\alpha \pm i\sin\alpha$$

Let $x = \cos\alpha + i\sin\alpha$,

similarly $y = \cos\beta + i\sin\beta$

(i) $\frac{x}{y} = \cos(\alpha-\beta) + i\sin(\alpha-\beta)$ $\frac{y}{x} = \cos(\alpha-\beta) - i\sin(\alpha-\beta)$ $\frac{x}{y} + \frac{y}{x} = \mathbf{2\cos(\alpha-\beta)}$	(ii) $xy = \cos(\alpha+\beta) + i\sin(\alpha+\beta)$ $\frac{1}{xy} = \cos(\alpha+\beta) - i\sin(\alpha+\beta)$ $xy - \frac{1}{xy} = \mathbf{2i\sin(\alpha+\beta)}$
(iii) $\frac{x^m}{y^n} = \cos(m\alpha-n\beta) + i\sin(m\alpha-n\beta)$ $\frac{y^n}{x^m} = \cos(m\alpha-n\beta) - i\sin(m\alpha-n\beta)$ $\frac{x^m}{y^n} - \frac{y^n}{x^m} = \mathbf{2i\sin(m\alpha-n\beta)}$	(iv) $x^m y^n = \cos(m\alpha+n\beta) + i\sin(m\alpha+n\beta)$ $\frac{1}{x^m y^n} = \cos(m\alpha+n\beta) - i\sin(m\alpha+n\beta)$ $x^m y^n + \frac{1}{x^m y^n} = \mathbf{2\cos(m\alpha+n\beta)}$

21. simplify (i) $(1+i)^{18}$ (ii) $\left(-\sqrt{3}+3i\right)^{31}$

soln:

(i) $(1+i)^{18} = ((1+i)^2)^9$ $= (2i)^9$ $= 512i$	(ii) $\left(-\sqrt{3}+3i\right)^{31} = \left[2\sqrt{3}\left(-\frac{1}{2}+\frac{i\sqrt{3}}{2}\right)\right]^{31}$ $= \left(2\sqrt{3}\right)^{31}\omega^{31} = \left(2\sqrt{3}\right)^{31}\omega$ $= \left(\mathbf{2\sqrt{3}}\right)^{\mathbf{31}}\left(\frac{-1}{2} + \mathbf{i}\frac{\sqrt{3}}{2}\right)$

2 ,3 Mark

1.Evaluate

(i) $i^{1729} = i$

(ii) $i^{-1924} + i^{2018} = i^0 + i^2 = 1 - 1 = 0;$

(iii) $i^{59} + \frac{1}{i^{59}} = i^{59} - i^{59} = 0$

(iv) $i i^2 i^3 \ldots\ldots i^{40} = i^{1+2+3+\cdots+40} = i^{\left(\frac{40\times41}{2}\right)} = i^{820} = 1$

2. If $z_1 = 6 + 7i$, $z_2 = 3 - 5i$

$z_1 + z_2 = (6+3) + i(7-5) = 9 + 2i$

$z_1 - z_2 = (6-3) + i(7+5) = 3 + 12i$

$z_1 z_2 = (6+7i)(3-5i) = 18 - 30i + 21i - 35(-1) = 53 - 9i$

$\frac{z_1}{z_2} = \frac{6+7i}{3-5i} = \frac{-17+51i}{34} = \frac{-17}{34} + \frac{51i}{34}$ $\qquad \frac{a+ib}{c+id} = \frac{(ac+bd)+i(bc-ad)}{c^2+d^2}$

3. If $z = 3 + 4i$, then find z^{-1}

$z^{-1} = \frac{1}{z} = \frac{1}{3+4i} = \frac{3-4i}{3^2+4^2} = \frac{3-4i}{25} = \frac{3}{25} + \frac{-4i}{25}$ $\qquad \frac{1}{a+ib} = \frac{(a-ib)}{a^2+b^2}$

4. If $z = (2+3i)(1-i)$, then find z^{-1}

$z = 2 - 2i + 3i + 3i(-i) = 2 + i - 3 = -1 + i$

$z^{-1} = \frac{1}{z} = \frac{1}{-1+i} = \frac{-1-i}{(-1)^2+1^2} = \frac{-1-i}{2} = \frac{-1}{2} - \frac{i}{2}$

5. If $z_1 = 3$, $z_2 = -7i$, $z_3 = 5 + 4i$ show that $z_1(z_2+z_3) = z_1z_2 + z_1z_3$

$z_2+z_3 = -7i + (5+4i) = 5 - 3i$

$z_1(z_2+z_3) = 3(5-3i) = 15 - 9i ----------------------\rightarrow (1)$

$z_1z_2 + z_1z_3 = 3(-7i) + 3(5+4i) = -21i + 15 + 12i = 15 - 9i ---- \rightarrow (2)$

$(1),(2) \Rightarrow z_1(z_2+z_3) = z_1z_2 + z_1z_3$

6. Which one of the point $i, -2+i$ and 3 is farthest and shortest from the origin?

Soln : Let $z_1 = i$, $z_2 = -2 + i$, $z_3 = 3$

$|z_1| = |i| = \sqrt{1^2} = 1$

$|z_2| = |-2+i| = \sqrt{(-2)^2 + 1^2} = \sqrt{5}$

$|z_3| = |3| = 3$

Farthest point is 3 and shortest point is i

7. Which one of the point $10 - 8i$, $11 + 6i$ is closest to $1 + i$.

Soln : Let $z_1 = 10 - 8i$, $z_2 = 11 + 6i$, and $z = 1 + i$

$|z_1 - z| = |(10-8i) - (1+i)| = |9 - 9i| = \sqrt{9^2 + (-9)^2} = \sqrt{162}$

$|z_2 - z| = |(11+6i) - (1+i)| = |10 + 5i| = \sqrt{10^2 + 5^2} = \sqrt{125}$

$11 + 6i$ is closest to $1 + i$.

8. If $(1+i)(1+2i)(1+3i) \ldots \ldots .. (1+ni) = x + iy$ then show that $2.5.10 \ldots .. (1+n^2) = x^2 + y^2$.

$|(1+i)(1+2i)(1+3i) \ldots \ldots .. (1+ni)| = |x+iy|$

$|(1+i)||(1+2i)||(1+3i)| \ldots \ldots .. |(1+ni)| = |x+iy|$

$\left(\sqrt{1^2+1^2}\right)\left(\sqrt{1^2+2^2}\right)\left(\sqrt{1^2+3^2}\right) \ldots \ldots .. \left(\sqrt{1^2+n^2}\right) = \sqrt{x^2+y^2}$

$\left(\sqrt{2}\right)\left(\sqrt{5}\right)\left(\sqrt{10}\right) \ldots \ldots .. \left(\sqrt{1^2+n^2}\right) = \sqrt{x^2+y^2}$

Taking square on both sides

$2.5.10 \ldots .. (1+n^2) = x^2 + y^2$

<u>Square root of a complex number</u>

If $z = x \pm iy$, then $\sqrt{z} = \sqrt{x \pm iy} = \pm\left(\sqrt{\frac{|z|+x}{2}} \pm i\sqrt{\frac{|z|-x}{2}}\right)$

9. Find the square root of a complex number $6 - 8i$, $4 + 3i$.

$\lvert 6-8i\rvert = \sqrt{(6)^2 + (-8)^2} = \sqrt{100}$ $\lvert z\rvert = 10$	$\lvert 4+3i\rvert = \sqrt{4^2+3^2} = \sqrt{25}$ $\lvert z\rvert = 5$

$\sqrt{6-8i} = \pm\left(\sqrt{\frac{10+6}{2}} - i\sqrt{\frac{10-6}{2}}\right)$ $= \pm\left(\sqrt{\frac{16}{2}} - i\sqrt{\frac{4}{2}}\right)$ $= \pm(\sqrt{8} - i\sqrt{2})$ $= \pm(2\sqrt{2} - i\sqrt{2})$	$\sqrt{4+3i} = \pm\left(\sqrt{\frac{5+4}{2}} + i\sqrt{\frac{5-4}{2}}\right)$ $= \pm\left(\sqrt{\frac{9}{2}} + i\sqrt{\frac{1}{2}}\right)$ $= \pm\left(\frac{3}{\sqrt{2}} + i\frac{1}{\sqrt{2}}\right)$

Try yourself :

Find the square root of a complex number $-6+8i$, $-5-12i$

10. If area of triangle formed by z, iz, z+iz is 50 sq.unit. find the value of $|z|$.

Soln : Area of triangle = $\frac{1}{2}|z|^2 = 50$

$$|z|^2 = 100 \quad \Rightarrow |z| = 10$$

11. If $|z| = 2$ show that $8 \le |z+6-8i| \le 12$

Soln: $\big||z| - |6-8i|\big| \le |z+6-8i| \le |z| + |6-8i|$

$$|2-10| \le |z+6-8i| \le 2+10$$

$$|-8| \le |z+6-8i| \le 12$$

$$8 \le |z+6-8i| \le 12$$

12. Find the value of n such that $\left(\frac{1+i}{1-i}\right)^n = 1$

Soln

$$\frac{1+i}{1-i} = i$$

$\left(\frac{1+i}{1-i}\right)^n = i^n = 1$ Possible values of n is 4,8,12,….

13. Show that $\left(\frac{1+i}{1-i}\right)^3 - \left(\frac{1-i}{1+i}\right)^3 = -2i$

Soln

$$\frac{1+i}{1-i} = i \;,\; \frac{1-i}{1+i} = -i$$

$$\left(\frac{1+i}{1-i}\right)^3 - \left(\frac{1-i}{1+i}\right)^3 = i^3 - (-i)^3 = -i - i = -2i$$

14.Simplify $\left[\frac{1+cos2\theta+isin2\theta}{1+cos2\theta-isin2\theta}\right]^{30}$

Soln

$$\left[\frac{1+cos2\theta+isin2\theta}{1+cos2\theta-isin2\theta}\right] = cos2\theta + isin2\theta$$

$$\left[\frac{1+cos2\theta+isin2\theta}{1+cos2\theta-isin2\theta}\right]^{30} = (cos2\theta + isin2\theta)^{30}$$

$$= (cos60\theta + isin60\theta)$$

15. Find the locus of z If $|z+i| = |z-1|$

Soln: $|z+i| = |z-1|$

$$|x+iy+i| = |x+iy-1|$$
$$|x+i(y+1)| = |(x-1)\ +iy|$$
$$\sqrt{x^2+(y+1)^2} = \sqrt{(x-1)^2+y^2}$$
$$x^2+(y+1)^2 = (x-1)^2+y^2$$
$$x^2+y^2+2y+1 = x^2-2x+1+y^2$$
$$2x+2y = 0$$
$$x+y = 0$$

16. write $\overline{3\imath}+\frac{1}{2-i}$ in rectangular form.

Soln: $\overline{3}\imath+\frac{1}{2-i} = -3i+\frac{2+i}{5}$ $\qquad \frac{1}{a+ib} = \frac{(a-ib)}{a^2+b^2}$

$= \frac{2-14i}{5} = \frac{2}{5}-\frac{14i}{5}$

17.Find $\left|\frac{i(2+i)^3}{(1+i)^2}\right|$

Soln: $\left|\frac{i(2+i)^3}{(1+i)^2}\right| = \frac{1\left(\sqrt{2^2+1^2}\right)^3}{\left(\sqrt{1^2+1^2}\right)^2} = \frac{\left(\sqrt{5}\right)^3}{\left(\sqrt{2}\right)^2} = \frac{5\sqrt{5}}{2}$

18.The complex numbers u,v and w are related by $\frac{1}{u}=\frac{1}{v}+\frac{1}{w}$. If $v=3-4i$ and $w=4+3i$, Find u in rectangular form.

Soln: $\frac{1}{v} = \frac{1}{3-4i} = \frac{3+4i}{25}$ $\qquad$ *hint:* $\frac{1}{a+ib} = \frac{(a-ib)}{a^2+b^2}$

$\frac{1}{w} = \frac{1}{4+3i} = \frac{4-3i}{25}$

$\frac{1}{u} = \frac{1}{v}+\frac{1}{w} = \frac{7+i}{25}$

$u = \frac{25}{7+i} = \frac{25(7-i)}{50} = \frac{7}{2}-\frac{i}{2}$

19.Show that $|3z-5+i| = 4$ represents a circle, then find its centre and radius.

Soln: $|3z-(5-i)| = 4$

$\left|z-\left(\frac{5}{3}-\frac{i}{3}\right)\right| = \frac{4}{3}$ centre $\left(\frac{5}{3},-\frac{1}{3}\right)$ radius $\frac{4}{3}$ $\qquad$ **Hint:**$|z-z_0| = r$

Try yourself

(i)$|z+2-i| < 2$,(ii) $|z-2-i| = 3$ (iii)$|2z+2-4i| = 2$ (iv) $|3z-6+12i| = 8$

20. If $\omega \neq 1$ is a cube root of unity , show that $\frac{a+b\omega+c\omega^2}{b+c\omega+a\omega^2}+\frac{a+b\omega+c\omega^2}{c+a\omega+b\omega^2} = -1$

Soln: $\frac{a+b\omega+c\omega^2}{b+c\omega+a\omega^2}\times\frac{\omega}{\omega}+\frac{a+b\omega+c\omega^2}{c+a\omega+b\omega^2}\times\frac{\omega^2}{\omega^2} = \frac{\omega\left(a+b\omega+c\omega^2\right)}{a+b\omega+c\omega^2}+\frac{\omega^2\left(a+b\omega+c\omega^2\right)}{a+b\omega+c\omega^2}$

$$= \omega+\omega^2 = -1$$

21.Show that $\left(\frac{\sqrt{3}}{2}+\frac{i}{2}\right)^5+\left(\frac{\sqrt{3}}{2}-\frac{i}{2}\right)^5 = -\sqrt{3}$

Soln: $\left(\frac{\sqrt{3}}{2}+\frac{i}{2}\right)^5+\left(\frac{\sqrt{3}}{2}-\frac{i}{2}\right)^5 = (-i\omega)^5+(i\omega^2)^5 = -i\omega^2+i\omega$

$$= -i\left(-\frac{1}{2} - i\frac{\sqrt{3}}{2}\right) + i\left(-\frac{1}{2} + i\frac{\sqrt{3}}{2}\right)$$

$$= -\sqrt{3}$$

22. Evaluate $\sum_{k=1}^{8}\left(cos\frac{2k\pi}{9} + isin\frac{2k\pi}{9}\right)$

Soln: $\sum_{k=0}^{8}\left(cos\frac{2k\pi}{9} + isin\frac{2k\pi}{9}\right) = 0$

$1 + \sum_{k=1}^{8}\left(cos\frac{2k\pi}{9} + isin\frac{2k\pi}{9}\right) = 0$

$\sum_{k=1}^{8}\left(cos\frac{2k\pi}{9} + isin\frac{2k\pi}{9}\right) = -1$

23. If $\omega \neq 1$ is a cube root of unity, then show that the following

(i) $(1-\omega+\omega^2)^6 + (1+\omega-\omega^2)^6 = 128$

(ii) $(1+\omega)(1+\omega^2)(1+\omega^4)(1+\omega^8)\ldots..(1+\omega^{2^{11}}) = 1$

Soln: (i) $(1-\omega+\omega^2)^6 + (1+\omega-\omega^2)^6 = (-\omega-\omega)^6 + (-\omega^2-\omega^2)^6$

$$= (-2\omega)^6 + (-2\omega^2)^6$$

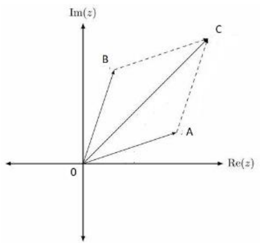
$$= 64 + 64 = 128$$

(ii) $(1+\omega)(1+\omega^2)(1+\omega^4)(1+\omega^8)\ldots..(1+\omega^{2^{11}})$

$$= [(-\omega^2)(-\omega)][(-\omega^2)(-\omega)] \ldots upto\ 6\ times$$

$$= 1 \times 1 \times 1 \times 1 \times 1 \times 1 = 1$$

24.State and prove Triangle inequality



$|z_1 + z_2| \leq |z_1| + |z_2|$

$OA = |z_1|, OB = |z_2|, OC = |z_1 + z_2|$

In Δ OAC, $OC < OA + AC$

$|z_1 + z_2| < |z_1| + |z_2| - - - - - (1)$

Suppose the points are in colinear

$|z_1 + z_2| = |z_1| + |z_2| - - - - \rightarrow (2)$

From (1),(2)

$|z_1 + z_2| \leq |z_1| + |z_2|$

DISCRETE MATHEMATICS

5 MARK

Important hints:

Let * be a binary operation on S

i) Closure property : $\forall a, b \in S \Rightarrow a * b \in S$

ii) Commutative property : $a * b = b * a$, $\forall a, b \in S$

iii) Associative property : $a * (b * c) = (a * b) * c, \forall a, b, c \in S$

iv) Existence of identity : $a * e = e * a = a$, e is the identity element, $e \in S, \forall a \in S$

v) Existence of inverse : a^{-1} is the inverse of a $a * a^{-1} = a^{-1} * a = e, \quad a^{-1} \in S$

1. Verify closure, commutative, associative, existence of identity, and existence of inverse for $m * n = m + n - mn, \quad m, n\epsilon Z$

Soln: **Closure property**: $m, n\epsilon Z$, clearly $m + n - mn\ \epsilon Z$

$\therefore$ closure property true

Associative property: $(l * m) * n = l * (m * n)$

$(l * m) * n = l * (m * n) = l + m + n - lm - mn - nl + lmn$

$\therefore$ associative property true

Identity property: $m * e = e * m = m$

$$m + e - me = m$$

$$e = 0\ \epsilon Z$$

$\therefore$ identity property true

Inverse property: $m * m^{-1} = m^{-1} * m = e = 0$

$$m^{-1} = \frac{-m}{1-m} \notin z$$

$\therefore$ inverse property not true

Commutative property: $m * n = n * m = m + n - mn = n + m - nm$

$\therefore$ commutative property true

2. Verify closure, commutative, associative, existence of identity, and existence of inverse for $x * y = x + y - xy, \ \forall\ x, y\epsilon Q \setminus \{1\}$.

Soln:**Closure property**: $x, y\epsilon Q \setminus \{1\}, \quad x \neq 1, \ y \neq 1$

$\Rightarrow \ x + y - xy \neq 1$

$x * y\epsilon Q \setminus \{1\}$ $\therefore$ closure property true

Associative property: $(x * y) * z = x * (y * z)$

$\therefore$ associative property true

Identity property: $x * e = e * x = x$

$$e = 0\ \epsilon Q \setminus \{1\}$$

$\therefore$ identity property true

Inverse property: $x * x^{-1} = x^{-1} * x = e = 0$

$$x^{-1} = \frac{-x}{1-x} \epsilon Q \setminus \{1\}$$

$\therefore$ inverse property true

Commutative property: $m * n = m + n - mn$

$$= n + m - nm = n * m$$

$\therefore$ commutative property true

3. Verify closure, commutative , associative, existence of identity, and inverse for $M = \left\{ \begin{pmatrix} x & x \\ x & x \end{pmatrix} \in R - \{0\} \right\}$.

Soln:

Let $*$be the matrix multiplication.

Closure property: Let, $A = \begin{pmatrix} x & x \\ x & x \end{pmatrix}, B = \begin{pmatrix} y & y \\ y & y \end{pmatrix}$ $\because\ x, y \neq 0 \Longrightarrow 2xy \neq 0$

$$AB = \begin{pmatrix} 2xy & 2xy \\ 2xy & 2xy \end{pmatrix} \epsilon M$$

$\therefore$ closure property true

Commutative property :

$$AB = \begin{pmatrix} 2xy & 2xy \\ 2xy & 2xy \end{pmatrix}\ , BA = \begin{pmatrix} 2yx & 2yx \\ 2yx & 2yx \end{pmatrix}$$

$AB = BA$

$\therefore$ commutative property true

Associative property:

Matrix multiplication always satisfies associative property

Existence of identity property: $A * E = E * A = A$

$$\begin{pmatrix} x & x \\ x & x \end{pmatrix}\begin{pmatrix} e & e \\ e & e \end{pmatrix} = \begin{pmatrix} x & x \\ x & x \end{pmatrix}$$

$$2ex = x$$

$$e = \frac{1}{2} \qquad \therefore\ E = \begin{pmatrix} \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & \frac{1}{2} \end{pmatrix} \epsilon M$$

$\therefore$ identity property true

Existence of inverse property: $A * A^{-1} = A^{-1} * A = E = \begin{pmatrix} \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & \frac{1}{2} \end{pmatrix}$

$$\begin{pmatrix} x & x \\ x & x \end{pmatrix}\begin{pmatrix} a & a \\ a & a \end{pmatrix} = \begin{pmatrix} \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & \frac{1}{2} \end{pmatrix}$$

$$2ax = \frac{1}{2} \quad \Longrightarrow a = \frac{1}{4x}$$

$$\therefore\ A^{-1} = \begin{pmatrix} \frac{1}{4x} & \frac{1}{4x} \\ \frac{1}{4x} & \frac{1}{4x} \end{pmatrix} \epsilon M$$

$\therefore$ inverse property true

4. Verify closure property, commutative property, associative property ,existence of identity, and existence of inverse for the operation $\times_{11}$ on a subset $A = \{1,3,4,5,9\}$of the set of remainders {0,1,2,3,4,5,6,7,8,9,10}.

Soln:Closure property:

From the table closure property true.

$\times_{11}$	1	3	4	5	9
1	1	3	4	5	9
3	3	9	1	4	5
4	4	1	5	9	3
5	5	4	9	3	1
9	9	5	3	1	4

Commutative property:

From the table commutative property true.

Associative property:

$\times_{11}$ always satisfies associative property .

Identity property:

Identity element $1\epsilon A$

$\therefore$ identity property true

Inverse property:

Inverse element of 1,3,4,5 and 9 are 1,4,3,9 and 5 respectively.

∴ inverse property true

5. Verify closure property, commutative property, associative property,existence of identity, and existence of inverse for the operation $+_5$ on Z_5 using table corresponding to addition modulo 5.

Soln: $Z_5 = \{0,1,2,3,4\}$

Closure property:

From the table closure property true

Commutative property:

From the table commutative property true

$+_5$	0	1	2	3	4
0	0	1	2	3	4
1	1	2	3	4	0
2	2	3	4	0	1
3	3	4	0	1	2
4	4	0	1	2	3

Associative property:

$+_5$ always satisfies associative property.

Identity property:

identity element $0 \epsilon Z_5$

∴ identity property true.

Inverse property:

Inverse element of 0,1,2,3 and 4 are 0,4,3,2 and 1 respectively.

∴ inverse property true

6. Verify closure, commutative, associative, identity, and inverse property for $a * b = \frac{a+b}{2} \forall\ a, b \epsilon Q$

Soln:

Closure property:

Clearly $a, b \epsilon Q \implies \frac{a+b}{2} \epsilon Q$ ∴ closure property true

Associative property:

$(a * b) * c = \frac{a+b+2c}{4}$

$a * (b * c) = \frac{2a+b+c}{4}$

$(a * b) * c \neq a * (b * c)$ ∴ Associative property is not true

Identity property: $a * e = e * a = a$

$a * e = a$

$\frac{a+e}{2} = a$

$e = a$

Uniqueness of identity is not preserved ∴ identity property is not true

Inverse property:

∴ inverse property is not true

Commutative property:

$a * b = b * a = \frac{a+b}{2}$ ∴commutative property true

2,3 Marks

1.In an algebraic structure the identity element must be unique

Soln:

Let e_1 and e_2 be the identity element of S

$$a * e_1 = e_1 * a = a \qquad \forall a \in S$$

$$a * e_2 = e_2 * a = a \qquad \forall a \in S$$

$$a * e_1 = a * e_2$$

$$\therefore e_1 = e_2$$

2.In an algebraic structure the inverse element must be unique

Soln:

Let a_1 and a_2 be the inverse element of a in S

$$a * a_1 = a_1 * a = e \qquad \forall a \in S$$

$$a * a_2 = a_2 * a = e \qquad \forall a \in S$$

$$a * a_1 = a * a_2$$

$$a_1 = a_2$$

3. Let $A = \begin{bmatrix} 0 & 1 \\ 1 & 1 \end{bmatrix}$ and $B = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$ be any two Boolean matrices of the same type. Find $A \vee B$ and $A \wedge B$.

Soln:

$$A \vee B = \begin{bmatrix} 0 & 1 \\ 1 & 1 \end{bmatrix} \vee \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}$$

$$A \wedge B = \begin{bmatrix} 0 & 1 \\ 1 & 1 \end{bmatrix} \wedge \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 0 & 1 \end{bmatrix}$$

Try Yourself: Let $A = \begin{pmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ 1 & 0 & 0 & 1 \end{pmatrix}$, $B = \begin{pmatrix} 0 & 1 & 0 & 1 \\ 1 & 0 & 1 & 0 \\ 1 & 0 & 0 & 1 \end{pmatrix}$, $C = \begin{pmatrix} 1 & 1 & 0 & 1 \\ 0 & 1 & 1 & 0 \\ 1 & 1 & 1 & 1 \end{pmatrix}$ be any three Boolean matrices of the same type. Find (i) $A \vee B$ (ii) $A \wedge B$ (iii) $(A \vee B) \wedge C$ (iv) $(A \wedge B) \vee C$.

4.Show that $p \leftrightarrow q \equiv (p \to q) \wedge (q \to p)$

p	q	$p \leftrightarrow q$	$p \to q$	$q \to p$	$(p \to q) \wedge (q \to p)$
T	T	T	T	T	T
T	F	F	F	T	F
F	T	F	T	F	F
F	F	T	T	T	T

$$p \leftrightarrow q \equiv (p \to q) \wedge (q \to p)$$

5. Show that $p \leftrightarrow q \equiv (p \wedge q) \vee (\neg q \wedge \neg p)$

p	q	$p \leftrightarrow q$	$\neg p$	$\neg q$	$p \wedge q$	$\neg q \wedge \neg p$	$(p \wedge q) \vee (\neg q \to \neg p)$
T	T	T	F	F	T	F	T
T	F	F	F	T	F	F	F
F	T	F	T	F	F	F	F
F	F	T	T	T	F	T	T

$$p \leftrightarrow q \equiv (p \wedge q) \vee (\neg q \wedge \neg p)$$

6. Show that $p \to (\neg q \vee r) \equiv \neg p \vee (\neg q \vee r)$

P	q	r	$\neg p$	$\neg q$	$\neg q \vee r$	$p \to (\neg q \vee r)$	$\neg p \vee (\neg q \vee r)$
T	T	T	F	F	T	T	T
T	T	F	F	F	F	F	F
T	F	T	F	T	T	T	T
T	F	F	F	T	T	T	T
F	T	T	T	F	T	T	T
F	T	F	T	F	F	T	T
F	F	T	T	T	T	T	T
F	F	F	T	T	T	T	T

$$p \to (\neg q \vee r) \equiv \neg p \vee (\neg q \vee r)$$

7. Show that $p \to (q \to r) \equiv (p \wedge q) \to r$

P	q	r	$q \to r$	$p \to (q \to r)$	$p \wedge q$	$(p \wedge q) \to r$
T	T	T	T	T	T	T
T	T	F	F	F	T	F
T	F	T	T	T	F	T
T	F	F	T	T	F	T
F	T	T	T	T	F	T
F	T	F	F	T	F	T
F	F	T	T	T	F	T
F	F	F	T	T	F	T

$p \to (q \to r) \equiv (p \wedge q) \to r$

8.Using truth table whether the statements $\neg(p \vee q) \vee (\neg p \wedge q)$ *and* $\neg p$ are logically equivalent.

P	q	$\neg p$	$p \vee q$	$\neg(p \vee q)$	$\neg p \wedge q$	$\neg(p \vee q) \vee (\neg p \wedge q)$
T	T	F	T	F	F	F
T	F	F	T	F	F	F
F	T	T	T	F	T	T
F	F	T	F	T	F	T

$\neg(p \vee q) \vee (\neg p \wedge q)$ *and* $\neg p$ are logically equivalent.

9.Show that $\neg(p \leftrightarrow q) \equiv p \leftrightarrow \neg q$

P	q	$p \leftrightarrow q$	$\neg(p \leftrightarrow q)$	$\neg q$	$p \leftrightarrow \neg q$
T	T	T	F	F	F
T	F	F	T	T	T
F	T	F	T	F	T
F	F	T	F	T	F

$\neg(p \leftrightarrow q) \equiv p \leftrightarrow \neg q$

10. Show that $p \to q$ *and* $q \to p$ are not equivalent

P	q	$p \to q$	$q \to p$
T	T	T	T
T	F	F	T
F	T	T	F
F	F	T	T

$p \to q$ *and* $q \to p$ are not equivalent

11. Show that q$\to p \equiv \neg p \to \neg q$

P	q	q$\to p$	$\neg p$	$\neg q$	$\neg p \to \neg q$
T	T	T	F	F	T
T	F	T	F	T	T
F	T	F	T	F	F
F	F	T	T	T	T

q$\to p \equiv \neg p \to \neg q$

12.Prove that $\neg(p \wedge q) \equiv \neg p \vee \neg q$

P	q	$p \wedge q$	$\neg(p \wedge q)$	$\neg p$	$\neg q$	$\neg p \vee \neg q$
T	T	T	F	F	F	F
T	F	F	T	F	T	T
F	T	F	T	T	F	T
F	F	F	T	T	T	T

$\neg(p \wedge q) \equiv \neg p \vee \neg q$

13.Verify whether the compound propositions are tautology or contradiction or contingency.
$((p\vee q)\wedge\neg p)\to q$

P	q	$p\vee q$	$\neg p$	$(p\vee q)\wedge\neg p$	$((p\vee q)\wedge\neg p)\to q$
T	T	T	F	F	T
T	F	T	F	F	T
F	T	T	T	T	T
F	F	F	T	F	T

$((p\vee q)\wedge\neg p)\to q$ is a tautology.

14. show that without using truth table $p\to(q\to r)\equiv(p\wedge q)\to r$

Soln:
$$p\to(q\to r)\equiv\neg p\vee(q\to r)$$
$$\equiv\neg p\vee(\neg q\vee r)$$
$$\equiv(\neg p\vee\neg q)\vee r$$
$$\equiv\neg(p\wedge q)\vee r$$
$$\equiv(p\wedge q)\to r$$
$$p\to(q\to r)\equiv(p\wedge q)\to r$$

15.Construct the truth table for $(p\bar{\vee}q)\wedge(p\bar{\vee}\neg q)$

P	q	$p\bar{\vee}q$	$\neg q$	$p\bar{\vee}\neg q$	$(p\bar{\vee}q)\wedge(p\bar{\vee}\neg q)$
T	T	F	F	T	F
T	F	T	T	F	F
F	T	T	F	F	F
F	F	F	T	T	F

16.Show that without using truth table $p\leftrightarrow q\equiv(p\wedge q)\vee(\neg q\wedge\neg p)$

Soln: $p\leftrightarrow q\equiv(p\to q)\wedge(q\to p)$
$$\equiv(\neg p\vee q)\wedge(p\vee\neg q)$$
$$\equiv(\neg p\wedge(p\vee\neg q)q)\vee(q\wedge(p\vee\neg q))$$
$$\equiv(\neg p\wedge p)\vee(\neg p\wedge\neg q)\vee(q\wedge p)\vee(q\wedge\neg q)$$
$$\equiv F\vee(\neg p\wedge\neg q)\vee(q\wedge p)\vee F$$
$$\equiv(q\wedge p)\vee(\neg p\wedge\neg q)$$
$$p\leftrightarrow q\equiv(p\wedge q)\vee(\neg q\wedge\neg p)$$

17. check whether the $p\to(q\to p)$is tautology or contradiction without using truth table.

Soln:
$$p\to(q\to p)\equiv\neg p\vee(q\to p)$$
$$\equiv\neg p\vee(\neg q\vee p)$$
$$\equiv\neg p\vee(p\vee\neg q)$$
$$\equiv(\neg p\vee p)\vee\neg q$$
$$\equiv T\vee\neg q$$
$$\equiv T\ \therefore$$ $p\to(q\to p)$**is tautology**

DIFFERENTIALS AND PARTIAL DERIVATIVES

Important hints:

linear approximation : $L(x)=f(x_0)+f'(x_0)(x-x_0)$

Euler theorem : $x\frac{\partial f}{\partial x}+y\frac{\partial f}{\partial y}=nf$
$Degree=n=N.degree-D.degree$

5 MARKS

1. ***Find the linear approximation for $f(x) = \sqrt{1+x}, x \geq -1$, at $x_0 = 3$. Use the linear approximation to estimate $f(3.2)$***

Soln

$f(x) = \sqrt{1+x}$, $x_0 = 3, \Delta x = 0.2$ and hence $f(3) = \sqrt{1+3} = 2.$

$$f'(x) = \frac{1}{2\sqrt{1+x}} \Rightarrow f'(3) = \frac{1}{2\sqrt{1+3}} = \frac{1}{4}$$

$$L(x) = f(x_0) + f'(x_0)(x - x_0)$$

$$= 2 + \frac{1}{4}(x-3) = \frac{x}{4} + \frac{5}{4}$$

$$f(3.2) = \sqrt{4.2} \cong L(3.2) = \frac{3.2}{4} + \frac{5}{4} = 2.050$$

2. **Use linear approximation to find an approximate value of $\sqrt{9.2}$ without using a calculator**

Soln

$$f(x) = \sqrt{x}, x_0 = 9, \Delta x = 0.2$$

$$f(9) = 3,$$

$$f'(x) = \frac{1}{2\sqrt{x}} \quad , \quad f'(x) = \frac{1}{2\sqrt{9}} = \frac{1}{(2\times3)} = \frac{1}{6}$$

$$L(x) = f(x_0) + f'(x_0)(x - x_0)$$

$$\sqrt{9.2} = f(9) + f'(9)(x-9)$$

$$= 3 + \frac{1}{6}(9.2 - 9) = 3 + \frac{0.2}{6} = 3.0333$$

3. ***If $u = \sin^{-1}\left(\frac{x+y}{\sqrt{x}+\sqrt{y}}\right)$, show that $x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} = \frac{1}{2}\tan u$***

$$u = \sin^{-1}\left(\frac{x+y}{\sqrt{x}+\sqrt{y}}\right)$$

$$\sin u = \left(\frac{x+y}{\sqrt{x}+\sqrt{y}}\right) = f(x,y)$$

$$Degree = n = N.degree - D.degree$$

$$n = 1 - \frac{1}{2} = \frac{1}{2}$$

$$\therefore f(x,y) is\ a\ homgeneous\ function\ of\ degree\ is\ n = \frac{1}{2}$$

By Euler theorem, $\quad x\frac{\partial f}{\partial x} + y\frac{\partial f}{\partial y} = \mathrm{n}f$

$$x\frac{\partial}{\partial x}(\sin u) + y\frac{\partial}{\partial y}(\sin u) = \frac{1}{2}\sin u$$

$$x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} = \frac{1}{2}\tan u$$

4. ***If $u(x,y) = \frac{x^2+y^2}{\sqrt{x+y}}$, prove that $x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} = \frac{3}{2}u$.***

Soln

$$u(x,y) = \frac{x^2+y^2}{\sqrt{x+y}}$$

$$Degree = n = N.degree - D.degree$$

$$n = 2 - \frac{1}{2} = \frac{3}{2}$$

$$\therefore u(x,y) is\ a\ homgeneous\ function\ of\ degree\ is\ n = \frac{3}{2}$$

By Euler theorem, $\quad x\frac{\partial f}{\partial x} + y\frac{\partial f}{\partial y} = \mathrm{n}f$

$$x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} = \frac{3}{2}u$$

5. $If\ v(x,y) = \log\left(\frac{x^2+y^2}{x+y}\right), prove\ that\ x\frac{\partial v}{\partial x} + y\frac{\partial v}{\partial y} = 1$

Soln

$$v(x,y) = \log\left(\frac{x^2+y^2}{x+y}\right)$$

$$f = e^v = \frac{x^2+y^2}{x+y}$$

$$Degree = n = N.degree - D.degree$$

$$n = 2 - 1 = 1$$

$$\therefore f(x,y) is\ a\ homgeneous\ function\ of\ degree\ is\ \ n = 1$$

$By\ Euler\ theorem,$ $$x\frac{\partial f}{\partial x} + y\frac{\partial f}{\partial y} = \mathrm{n}f$$

$$x\frac{\partial e^v}{\partial x} + y\frac{\partial e^v}{\partial y} = (1)e^v$$

$$x\frac{\partial v}{\partial x} + y\frac{\partial v}{\partial y} = 1$$

6. $If\ w(x,y,z) = log\ \left(\frac{5x^3y^4+7y^2xz^4-75y^3z^4}{x^2+y^2}\right), find\ x\frac{\partial w}{\partial x} + y\frac{\partial w}{\partial y} + z\frac{\partial w}{\partial z}$

Soln

$$w(x,y,z) = \log\left(\frac{5x^3y^4+7y^2xz^4-75y^3z^4}{x^2+y^2}\right)$$

$$e^w = \left(\frac{5x^3y^4+7y^2xz^4-75y^3z^4}{x^2+y^2}\right) = f(x,y)$$

$$Degree = n = N.degree - D.degree$$

$$n = 7 - 2 = 5$$

$$\therefore f(x,y,z) is\ a\ homgeneous\ function\ of\ degree\ is\ \ n = 5$$

$By\ Euler\ theorem,$ $$x\frac{\partial f}{\partial x} + y\frac{\partial f}{\partial y} + \mathrm{z}\frac{\partial f}{\partial z} = \mathrm{n}f$$

$$x\frac{\partial e^w}{\partial x} + y\frac{\partial e^w}{\partial y} + \mathrm{z}\frac{\partial e^w}{\partial z} = (5)e^w$$

$$x\frac{\partial w}{\partial x} + y\frac{\partial w}{\partial y} + \mathrm{z}\frac{\partial w}{\partial z} = 5$$

7. Prove that $g(x,y) = xlog\ \left(\frac{y}{x}\right)$ is homogeneous, verify Euler's theorem for g

Soln

$$g(x,y) = xlog\ \left(\frac{y}{x}\right)$$

$$Degree = n = N.degree - D.degree$$

$$n = 2 - 1 = 1$$

$$\therefore f(x,y) is\ a\ homgeneous\ function\ of\ degree\ is\ \ n = 1$$

$$By\ Euler\ theorem, x\frac{\partial g}{\partial x} + y\frac{\partial g}{\partial y} = 1g$$

$$= x\frac{\partial g}{\partial x} + y\frac{\partial g}{\partial y} = x\frac{\partial}{\partial x}\left(\mathrm{x}\log\ \frac{y}{x}\right) + y\frac{\partial}{\partial y}\left(\mathrm{x}\log\ \frac{y}{x}\right)$$

$$= x\log\frac{y}{x} = g$$

Hence $x\frac{\partial g}{\partial x} + y\frac{\partial g}{\partial y} = 1g$

DIFFERENTIAL EQUATIONS
5 MARKS

1. The rate of increase in the number of bacteria in a certain bacteria culture is proportional to the number present. Given that the number triples in 5 hours, find how many bacteria will be present after 10 hours?

Soln: Let A be the no.of bacteria at present

t	A
0	A_0
5	$3A_0$
10	?

$\frac{dA}{dt} \propto A$

$A = Ce^{kt} ----\to (1)$

$\boldsymbol{t=0, A=A_0}$ (1) $\Longrightarrow A_0 = C$

$\therefore A = A_0 e^{kt} ----\to (2)$

$\boldsymbol{t=5, A=3A_0}$ (2) $\Longrightarrow 3A_0 = A_0 e^{5k} \Longrightarrow 3 = e^{5k}$

$\boldsymbol{t=10}$, **A=?** (2) $\Rightarrow A = A_0 e^{10k} \Longrightarrow A = A_0(e^{5k})^2 \Longrightarrow \boldsymbol{A = A_0(3)^2 = 9A_0}$

2. Find the population of a city at any time *t*, given that the rate of increase of population is proportional to the population at that instant and that in a period of 40 years the population increased from 3,00,000 to 4,00,000.

Soln: Let A be the population of a city at present

t	A
0	300000
40	400000
t	?

$\frac{dA}{dt} \propto A$

$A = Ce^{kt} ----\to (1)$

$\boldsymbol{t=0, A=30000}$ (1) $\Longrightarrow 3{,}00{,}000 = C$

$\therefore A = 3{,}00{,}000e^{kt} ----\to (2)$

$\boldsymbol{t=40\ , A=4,00,000}$ (2) $\Longrightarrow 4{,}00{,}000 = 3{,}00{,}000e^{40k}$

$\Longrightarrow k = \frac{1}{40}log\left(\frac{4}{3}\right)$

$\boldsymbol{t=t, A=?\,(2)} \Longrightarrow A = 3{,}00{,}000e^{\frac{1}{40}log\left(\frac{4}{3}\right)^t} \Longrightarrow \boldsymbol{A = 3,00,000(\frac{4}{3})^{\frac{t}{40}}}$

3.The engine of a motor boat moving at 10 *m / s* is shut off. Given that the retardation at any subsequent time (after shutting off the engine) equal to the velocity at that time. Find the velocity after 2 seconds of switching off the engine.

Soln:

t	A
0	10
2	?

$\frac{dA}{dt} \propto A$

$\Longrightarrow \frac{dA}{dt} = -A \Longrightarrow A = Ce^{-t} ----\to (1)$

$\boldsymbol{t=0, A=10}$ (1) $\Rightarrow 10 = C$

$\therefore A = 10e^{-t} ----\to (2)$

$\boldsymbol{t=2, A=?}$ (2) $\Rightarrow A = 10e^{-2} \Longrightarrow \boldsymbol{A = 10/_{e^2}}$

4.Suppose a person deposits 10,000 Indian rupees in a bank account at the rate of 5% per annum compounded continuously. How much money will be in his bank account 18 months later?

Soln:

K=0.05

t	x
0	10,000
1.5	?

$\frac{dA}{dt} \propto A \Longrightarrow \frac{dA}{dt} = 0.05A$

$\Longrightarrow \quad A = Ce^{0.05t} ----\longrightarrow (1)$

$\boldsymbol{t = 0, A = 10000}$ $(1) \Longrightarrow\ 10{,}000 = C$

$\therefore\ A = 10{,}000e^{0.05t} ----\longrightarrow (2)$

$\boldsymbol{t = \frac{3}{2} = 1.5, A = ?}$ $(2) \Rightarrow A = 10{,}000e^{0.05\left(\frac{3}{2}\right)}$

$$\Longrightarrow \boldsymbol{A = 10{,}000e^{0.075}}$$

5.Assume that the rate at which radioactive nuclei decay is proportional to the number of such nuclei that are present in a given sample. In a certain sample 10% of the original number of radioactive nuclei have undergone disintegration in a period of 100 years. What percentage of the original radioactive nuclei will remain after 1000 years?

Soln:

t	A
0	A_0
100	$\frac{9}{10}A_0$
1000	?

$\frac{dA}{dt} \propto A \Longrightarrow \frac{dA}{dt} = -kA$

$\Longrightarrow \quad A = Ce^{-kt} ----\longrightarrow (1)$

$\boldsymbol{t = 0, A = A_0}$ $(1) \Rightarrow\ A_0 = C$

$\therefore\ A = A_0e^{-kt} ----\longrightarrow (2)$

$\boldsymbol{t = 100, A = \frac{9}{10}A_0}$ $(2) \Longrightarrow\ \frac{9}{10}A_0 = A_0e^{-100k}$

$\Longrightarrow\ \frac{9}{10} = e^{-100k} \Longrightarrow k = \frac{-1}{100}log\left(\frac{9}{10}\right)$

$\boldsymbol{t = 1000, A = ?}$ $(2) \Rightarrow A = A_0e^{-1000k} \Longrightarrow A = A_0e^{-1000\left[\frac{-1}{100}log\left(\frac{9}{10}\right)\right]}$

$$\Longrightarrow \frac{A}{A_0} = e^{10log\left(\frac{9}{10}\right)} = \left(\frac{9}{10}\right)^{10}$$

Percentage of radioactive nuclei $\boldsymbol{\frac{A}{A_0} \times 100 = \left(\frac{9}{10}\right)^{10} \times 100\ (or)\ \frac{9^{10}}{10^8}}\ \%$

6.Water at temperature 100°C cools in 10 minutes to 80°C in a room temperature of 25°C .Find(i) The temperature of water after 20 minutes (ii) The time when the temperature is 40°C. $\left[log\frac{11}{15} = -0.3101; log5 = 1.6094\right]$.

Soln:

t	T	S
0	100	25
10	80	
20	?	
?	40	

$\frac{dT}{dt} \propto T - T_m \Longrightarrow T - T_m = Ce^{kt}$

$\Longrightarrow T - 25 = Ce^{kt} ----\longrightarrow (1)$

$\boldsymbol{t = 0, T = 100}$ $(1) \Longrightarrow C = 75$

$(1) \Longrightarrow\ T - 25 = 75e^{kt} ----\longrightarrow (2)$

$\boldsymbol{t = 10, T = 80}$ $(2) \Longrightarrow$ 80-25 $= 75e^{10k}$

$\Longrightarrow k = \frac{1}{10}log\left(\frac{11}{15}\right)$

$\boldsymbol{t = 20, T = ?}$ $(2) \Rightarrow T - 25 = 75e^{20k} \quad \Longrightarrow T = 65.33$

$\boldsymbol{t = ?, T = 40}$

$(2) \Rightarrow 40 - 25 = 75e^{kt} \Longrightarrow \frac{15}{75} = e^{kt} \Longrightarrow log\left(\frac{15}{75}\right) = kt$

$\Longrightarrow log\left(\frac{15}{75}\right) = \frac{1}{10}log\left(\frac{11}{15}\right)t$

$$\boldsymbol{t = \frac{10log\left(\frac{15}{75}\right)}{log\left(\frac{11}{15}\right)} \Longrightarrow t = 53.46\ min}$$

7. At 10.00 A.M. a woman took a cup of hot instant coffee from her microwave oven and placed it on a nearby Kitchen counter to cool. At this instant the temperature of the coffee was 180°F, and 10 minutes later it was 160°F . Assume that constant temperature of the kitchen was 70°F . (i) What was the temperature of the coffee at 10.15A.M.? (ii) The woman likes to drink coffee when its temperature is between 130°F and 140°F.between what times should she have drunk the coffee?

Soln:

t	T	s
0	180	70
10	160	
15	?	
?	130	
?	140	

$$\frac{dT}{dt} \propto T - T_m ----\rightarrow (1)$$
$$\Longrightarrow T - T_m = Ce^{kt}$$
$$\Longrightarrow T - 70 = Ce^{kt} ----\rightarrow (2)$$
$$\boldsymbol{t = 0, T = 180} \ \ (2) \Rightarrow C = 110$$
$$(2) \Rightarrow \ T - 70 = 110e^{kt} ----\rightarrow (3)$$
$$\boldsymbol{t = 10, T = 1600} \ \ (3) \Rightarrow 160 - 70 = 110e^{10k} \Rightarrow e^k = \left(\frac{9}{11}\right)^{\frac{1}{10}}$$
$$\boldsymbol{t = 15, T = ?} \ \ (3) \Rightarrow T - 70 = 110e^{kt} \Rightarrow T = 151.33$$
$$\boldsymbol{T = 130, t = ?} \ (3) \Rightarrow 130 - 70 = 110e^{kt} \Rightarrow t = \frac{10log\left(\frac{6}{11}\right)}{log\left(\frac{9}{11}\right)} \quad \Rightarrow t = 30.20 \, min$$
$$\boldsymbol{t = ? \ T = 140} \ 140 - 70 = 110e^{kt} \Rightarrow t = \frac{10log\left(\frac{7}{11}\right)}{log\left(\frac{9}{11}\right)} \quad \Rightarrow t = 22.52 \, min$$

She drunk coffee between 10.22 min to 10.30 min

8.A pot of boiling water at 100°C is removed from a stove at time $t = 0$ and left to cool in the kitchen. After 5 minutes, the water temperature has decreased to 80°C , and another 5 minutes later it has dropped to 65°C. Determine the temperature of the kitchen.

Soln:

t	T	S
0	100	
5	80	
10	65	

$$T - T_m = Ce^{kt} ----\rightarrow (1)$$
$$\boldsymbol{t = 0, T = 100} \ (1) \Rightarrow 100 - T_m = C$$
$$(1) \Rightarrow \ T - T_m = (100 - T_m)e^{kt} ----\rightarrow (2)$$
$$\boldsymbol{t = 5, T = 80} \ \ (2) \Rightarrow 80 - T_m = (100 - T_m)e^{5k}$$
$$\Rightarrow \ e^{5k} = \frac{80 - T_m}{100 - T_m}$$
$$\boldsymbol{t = 10, T = 65} \ (2) \ \Rightarrow 65 - T_m = (100 - T_m)e^{10k}$$
$$\Rightarrow e^{10k} = \frac{65 - T_m}{100 - T_m}$$
$$\frac{65 - T_m}{100 - T_m} = (e^{5k})^2 = \left(\frac{80 - T_m}{100 - T_m}\right)^2$$
$$\Rightarrow \boldsymbol{T_m = 20}$$

9.A tank initially contains 50 litres of pure water. Starting at time $t = 0$ a brine containing with 2 grams of dissolved salt per litre flows into the tank at the rate of 3 litres per minute. The mixture is kept uniform by stirring and the well-stirred mixture simultaneously flows out of the tank at the same rate. Find the amount of salt present in the tank at any time $t > 0$.

Soln: $\frac{dA}{dt} = IN - OUT$

t	A
0	100

$$\frac{dA}{dt} = 50 - 0.01A$$
$$\frac{dA}{dt} = -0.01(A - 50000)$$
$$\Rightarrow \quad A - 5000 = Ce^{-0.01t} ----\rightarrow (1)$$
$$\boldsymbol{t = 0, A = 100} \quad (1) \Rightarrow \quad 100 - 5000 = C \ \Rightarrow C = -4900$$
$$\therefore \boldsymbol{A - 5000 = -4900e^{-0.01t}}$$

10. A tank contains 1000 litres of water in which 100 grams of salt is dissolved. Brine (Brine is a high-concentration solution of salt usually sodium chloride) in water runs in a rate of 10 litres per minute, and each litre contains 5grams of dissolved salt. The mixture of the tank is kept uniform by stirring. Brine runs out at 10 litres per minute. Find the amount of salt at any time *t* .

Soln: $\frac{dA}{dt} = IN - OUT$

$$\frac{dA}{dt} = 6 - \frac{3}{50}A = \frac{300-3A}{50}$$

$$\frac{dA}{dt} = \frac{-3}{50}(A-100) \implies A - 100 = Ce^{\frac{-3}{50}t} ----\longrightarrow (1)$$

t	A
0	0

$$\boldsymbol{t = 0, A = 0} \quad (1) \implies -100 = C$$

$$\therefore \quad \boldsymbol{A - 100 = -100e^{\frac{-3}{50}t}}$$

11.The growth of a population is proportional to the number present. If the population of a colony doubles in 50 years, in how many years will the population become triple.

Soln:

$$\frac{dA}{dt} \propto A \implies A = Ce^{kt} ----\longrightarrow (1)$$

$$\boldsymbol{t = 0, A = A_0} \implies A_0 = C$$

$$\therefore A = A_0 e^{kt} ----\longrightarrow (2)$$

$$\boldsymbol{t = 50, A = 2A_0} \ (2) \implies 2A_0 = A_0 e^{50k} \implies 2 = e^{50k}$$

$$\implies k = \frac{1}{50}log2$$

$$\boldsymbol{t = ?, A = 3A_0} \ (2) \Rightarrow 3A_0 = A_0 e^{kt} \implies \boldsymbol{t = 50\frac{log3}{log2}}.$$

t	A
0	A_0
50	$2A_0$
?	$3A_0$

12. A radioactive isotope has an initial mass 200mg, which two years later is 50mg . Find the expression for the amount of the isotope remaining at any time. What is its half-life? (half-life means the time taken for the radioactivity of a specified isotope to fall to half its original value).

Soln:

$$\frac{dA}{dt} \propto A \implies \frac{dA}{dt} = -kA$$

$$\implies A = Ce^{-kt} ----\longrightarrow (1)$$

$$\boldsymbol{t = 0, A = 200} \ (1) \implies 200 = C$$

$$\therefore A = 200e^{-kt} ----\longrightarrow (2)$$

$$\boldsymbol{t = 2, A = 150} \ (2) \implies 150 = 200e^{-2k}$$

$$\implies \frac{3}{4} = e^{-2k} \implies log\left(\frac{3}{4}\right) = -2k$$

$$\boldsymbol{t = ?, A = 100} \ (2) \Rightarrow 100 = 200e^{kt} \implies \boldsymbol{t = 2\frac{log\left(\frac{1}{2}\right)}{log\left(\frac{3}{4}\right)}}$$

t	A
0	200
2	150
?	100

13. In a murder investigation, a corpse was found by a detective at exactly 8 p.m. Being alert, the detective also measured the body temperature and found it to be 70°F. Two hours later, the detective measured the body temperature again and found it to be 60°F. If the room temperature is 50°F, and assuming that the body temperature of the person before death was 98.6°F, at what time did the murder occur? [$\log(2.43) = 0.88789; \log(0.5) = -0.69315$].

Soln

$$T - T_m = Ce^{kt}$$

$$\implies T - 50 = Ce^{kt} ----\longrightarrow (1)$$

$$\boldsymbol{t = 0, T = 70} \ (1) \implies C = 20 \ (1) \implies T - 50 = 20e^{kt} ----\longrightarrow (2)$$

$\boldsymbol{t = 2, T = 60}$ 6 (2) $\implies$ 0-50 =20e^{2k}

$$\implies e^{2k} = \frac{1}{2} \implies k = \frac{1}{2}log\left(\frac{1}{2}\right)$$

$$\boldsymbol{t = ?, T = 98.6} \ (2) \Rightarrow 98.6 - 50 = 20e^{kt}$$

t	T	S
0	70	50
2	60	
t_1	98.6	

$\Longrightarrow \frac{48.6}{20} = e^{kt} \Longrightarrow t \cong -2.56$

$\therefore$ the murder time is $8 - 2.56 \cong 5{:}30\ PM$

THEORY OF EQUATION

Important hints :

- $ax^3 + bx^2 + cx + d = 0$
- Sum of co-efficients = 0 $\Rightarrow x = 1$ *is a root*
- Sum of co-efficients a+c=b+d $\Rightarrow x = -1$ *is a root*
- Otherwise try $x = 2$ *or* 3 *is a root*

5 MARKS

1. Solve $6x^4 - 5x^3 - 38x^2 - 5x + 6 = 0$ whose one of the roots is $\frac{1}{3}$ then find the other roots

Soln: $6x^4 - 5x^3 - 38x^2 - 5x + 6 = 0$ given equation is reciprocal type

Given root is $\frac{1}{3}$ then another root is 3

$\frac{1}{3}$	6	-5	-38	-5	6
	0	2	-1	-13	-6
3	6	-3	-39	-18	0
	0	18	45	18	
	6	15	6	0	

Reduced equation $6x^2 + 15x + 6 = 0$

$x = \frac{-12}{6}, \frac{-3}{6} = -2, \frac{-1}{2}$

The solution are $\mathbf{\frac{1}{3}, 3, -2, -\frac{1}{2}}$

2. Solve $6x^4 - 35x^3 + 62x^2 - 35x + 6 = 0$

Soln :

2	6	-35	62	-35	6
	0	12	-46	32	-6
3	6	-23	16	-3	0
	0	18	-15	3	
	6	-5	10	0	

$6x^2 - 5x + 1 = 0$

$\left(x - \frac{1}{3}\right)\left(x - \frac{1}{2}\right) = 0$

The solutions are $\mathbf{x = 2, 3, \frac{1}{2}, \frac{1}{3}}$

3. Solve $x^4 + 3x^3 - 3x - 1 = 0$

Soln

1	1	3	0	-3	-1
	0	1	4	4	1
-1	1	4	4	1	0
	0	-1	-3	-1	
	1	3	1	0	

$x^2 + 3x + 1 = 0$

$x = \frac{-3 + \sqrt{5}}{2}, x = \frac{-3 - \sqrt{5}}{2}$

The solutions are $\mathbf{x = -1, 1, \frac{-3+\sqrt{5}}{2}, \frac{-3-\sqrt{5}}{2}}$

4. Solve $x^4 - 10x^3 + 26x^2 - 10x + 1 = 0$

Soln: Given $x^4 - 10x^3 + 26x^2 - 10x + 1 = 0$

$\left(x^2 + \frac{1}{x^2}\right) - 10\left(x + \frac{1}{x}\right) + 26 = 0$

$y^2 - 2 - 10y + 26 = 0$

$y^2 - 10y + 24 = 0$

$(y - 6)(y - 4) = 0$

$y = 6, \ y = 4$

Case(i)	**Case(ii)**
$x + \frac{1}{x} = 6$	$x + \frac{1}{x} = 4$
$\frac{x^2+1}{x} = 6$	$\frac{x^2+1}{x} = 4$
$x^2 + 1 = 6x$	$x^2 + 1 = 4x$
$x^2 - 6x + 1 = 0$	$x^2 - 4x + 1 = 0$
$\mathbf{x = 3 \pm 2\sqrt{2}}$	$\mathbf{x = 2 \pm \sqrt{3}}$

5. If $2 + i$ and $3 - \sqrt{2}$ are the roots of the equation $x^6 - 13x^5 + 62x^4 - 126x^3 + 65x^2 + 127x - 140 = 0$ find all roots.

Soln: Given roots $2 + i, 3 - \sqrt{2}$

then other roots $2 - i, 3 + \sqrt{2}$

Let assume missing roots a and b.

Sum of roots $2+i\ +3-\sqrt{2}\ +2-i\ +3+\sqrt{2}+a+b=13$

$$10+a+b=13$$
$$a+b=3$$

Product of roots $(2+i)(2-i)(3-\sqrt{2})(3+\sqrt{2})\,ab=-140$

$$5(7)ab=-140$$
$$ab=\frac{-140}{35}=-4$$

Required roots of the equation $x^2-(s.r)x+p.r=0$

$$x^2-3x-4=0$$
$$(x-4)(x+1)=0$$
$$x=4,\quad x=-1$$

6. If $1+2i$ and $\sqrt{3}$ are the roots of the equation $x^6-3x^5-5x^4+22x^3-39x^2-39x+135=0$ find all roots.

Soln: Given roots $1+2i\ ,\sqrt{3}$

then other roots $1-2i\ ,-\sqrt{3}$

Let assume missing roots a and b.

Sum of roots $1+2i\ +\sqrt{3}\ +1-2i\ +(-\sqrt{3})+a+b=3$

$$2+a+b=3$$
$$a+b=1$$

Product of roots $(1+2i)(1-2i)(\sqrt{3})(-\sqrt{3})\,ab=135$

$$5(-3)\,ab=135$$
$$ab=\frac{135}{-15}=-9$$

Required roots of the equation $x^2-(s.r)x+p.r=0$

$$x^2-x-9=0$$

$$\boldsymbol{x=\frac{-b\pm\sqrt{b^2-4ac}}{2a}=\frac{1\pm\sqrt{1^2-4\times1\times-9}}{2\times1}=\frac{1\pm\sqrt{1+36}}{2}=\frac{1\pm\sqrt{37}}{2}}$$

7. Solve $(x-2)(x-7)(x-3)(x+2)+19=0$

Soln:
$$(x-2)(x-3)(x-7)(x+2)+19=0$$
$$(x^2-5x+6)(x^2-5x-14)+19=0$$

Put $x^2-5x=y$
$$(y+6)(y-14)+19=0$$
$$y^2-8y-84+19=0$$
$$y^2-8y-65=0$$
$$y=13,\qquad y=-5$$

Case(i) $\boldsymbol{y=13}$
$$x^2-5x=13$$
$$x^2-5x-13=0$$
$$x=\frac{-b\pm\sqrt{b^2-4ac}}{2a}=\frac{5\pm\sqrt{77}}{2}$$

Case(ii) $\boldsymbol{y=-5}$
$$x^2-5x=-5$$
$$x^2-5x+5=0$$
$$x=\frac{-b\pm\sqrt{b^2-4ac}}{2a}=\frac{5\pm\sqrt{5}}{2}$$

The solutions are $\boldsymbol{x=\frac{5-\sqrt{5}}{2},\frac{5+\sqrt{5}}{2},\frac{5-\sqrt{77}}{2},\frac{5+\sqrt{77}}{2}}$

8. Solve $(2x-3)(6x-1)(3x-2)(x-12)-7=0$

Soln:
$$(2x-3)(3x-2)(6x-1)(x-12)-7=0$$
$$(6x^2-13x+6)(6x^2-13x+12)-7=0$$

Put $6x^2-13x=y$
$$(y+6)(y+12)-7=0$$
$$y^2+18y+72-7=0$$
$$y^2+18y+65=0$$
$$y=-13,\qquad y=-5$$

Case(i) $\boldsymbol{y=-13}$
$$6x^2-13x=-13$$

Case(ii) $\boldsymbol{y=-5}$
$$6x^2-13x=-5$$

$6x^2 - 13x + 13 = 0$ $\qquad$ $6x^2 - 13x + 5 = 0$

$x = \frac{-b \pm \sqrt{b^2-4ac}}{2a} = \frac{13 \pm i\sqrt{143}}{12}$ $\qquad$ $x = \frac{10}{6}, \frac{3}{6} = \frac{5}{3}, \frac{1}{2}$

The solutions are $x = \frac{1}{2}, \frac{5}{3}, \frac{13-i\sqrt{143}}{12}, \frac{13+i\sqrt{143}}{12}$

9. Solve $(2x-1)(x+3)(x-2)(2x+3) + 20 = 0$

Soln: $(2x-1)(2x+3)(x+3)(x-2) + 20 = 0$

$(4x^2 + 4x - 3)(x^2 + x - 6) + 20 = 0$

$(4(x^2 + x) - 3)(x^2 + x - 6) + 20 = 0$

put $x^2 + x = y$ $\qquad$ $(4y - 3)(y - 6) + 20 = 0$

$4y^2 - 27y + 18 + 20 = 0$

$4y^2 - 27y + 38 = 0$

$y = \frac{19}{4}, \quad y = \frac{8}{4} = 2$

Case(i) $\boldsymbol{y = \frac{19}{4}}$

$x^2 + x = \frac{19}{4}$

$4x^2 + 4x - 19 = 0$

$x = \frac{-1 \pm 2\sqrt{5}}{2}$

Case(ii) $\boldsymbol{y = 2}$

$x^2 + x = 2$

$x^2 + x - 2 = 0$

$x = -2, \; x = 1$

The solutions are $x = -2, 1, \frac{-1-2\sqrt{5}}{2}, \frac{-1+2\sqrt{5}}{2}$

10. Find the sum of squares of roots of the equation $2x^4 - 8x^3 + 6x^2 - 3 = 0$

Soln

$ax^4 + bx^3 + cx^2 + dx + e = 0$

a = 2, b = -8, c = 6 , d = 0, e = -3

Let $\alpha, \beta, \gamma, \delta$ be the roots

$$\alpha + \beta + \gamma + \delta = \frac{-b}{a} = \frac{-8}{2} = -4$$

$$\alpha\beta + \alpha\gamma + \alpha\delta + \beta\gamma + \beta\delta + \gamma\delta = \frac{c}{a} = \frac{6}{2} = 3$$

$$\alpha^2 + \beta^2 + \gamma^2 + \delta^2 = (\alpha + \beta + \gamma + \delta)^2 - 2(\alpha\beta + \alpha\gamma + \alpha\delta + \beta\gamma + \beta\delta + \gamma\delta)$$

$$= (-4)^2 - 2(3) = 16 - 6 = 10$$

$$\alpha^2 + \beta^2 + \gamma^2 + \delta^2 = 10$$

2,3 Marks

1.Find the polynomial equation of minimum degree with rational co-efficient, having roots (i) $2 + i\sqrt{3}$ (ii) $2i + 3$ (iii) $\sqrt{5} - \sqrt{3}$ (iv) $\sqrt{\frac{\sqrt{2}}{\sqrt{3}}}$

Soln: (i) $x = 2 + i\sqrt{3}$

$x - 2 = i\sqrt{3}$

$(x-2)^2 = (i\sqrt{3})^2$

$x^2 - 4x + 4 = -3$

$x^2 - 4x + 7 = 0.$

(ii) $x = 2i + 3$

$x - 3 = 2i$

$(x-3)^2 = (2i)^2$

$x^2 - 6x + 9 = -4$

$x^2 - 6x + 13 = 0.$

(iii)
$$x = \sqrt{5} - \sqrt{3}$$
$$x^2 = \left(\sqrt{5} - \sqrt{3}\right)^2$$
$$x^2 = 5 + 3 - 2\sqrt{15}$$
$$x^2 - 8 = -2\sqrt{15}$$
$$(x^2 - 8)^2 = \left(-2\sqrt{15}\right)^2$$
$$x^4 - 16x^2 + 64 = 60.$$
$$x^4 - 16x^2 + 4 = 0$$

(iv)
$$x = \sqrt{\frac{\sqrt{2}}{\sqrt{3}}}$$
$$x^2 = \frac{\sqrt{2}}{\sqrt{3}}$$
$$(x^2)^2 = \left(\frac{\sqrt{2}}{\sqrt{3}}\right)^2$$
$$x^4 = \frac{2}{3}$$
$$3x^4 = 2$$
$$3x^4 - 2 = 0$$

2.Discuss the nature of the roots of equation

(i) $9x^9 + 2x^5 - x^4 - 7x^2 + 2 = 0$

fun	Signs	No.of changes	No.of Real roots
$f(x)$	+ + - - +	2	2 +Ve
$f(-x)$	- - - - +	1	1 -Ve

No. of Imaginary roots = 9 - 3 = 6

(ii) $x^9 + 9x^7 + 7x^5 + 5x^3 + 3x = 0 \Rightarrow x(x^8 + 9x^6 + 7x^4 + 5x^2 + 3) = 0$

$x = 0$ is a root with multiplicity one

fun	Signs	No.of changes	No.of Real roots
$f(x)$	+ + + + +	0	0 +Ve
$f(-x)$	- - - - -	0	0 -Ve

No. of Imaginary roots = 8 - 0 = 8

(iii) $x^9 - 5x^8 - 14x^7 = 0 \Rightarrow x^7(x^2 - 5x - 14) = 0$

$x = 0$ is a root with multiplicity seven

Fun	Signs	No.of changes	No.of Real roots
$f(x)$	+ - -	1	1 +Ve
$f(-x)$	- - +	1	1 -Ve

No. of Imaginary roots = 2 - 2 = 0

3. If the sides of a cubic box are increased by 1, 2, 3 units respectively to form a cuboid, then the volume is increased by 52 cubic units. Find the volume of the cuboid.

Soln: $(x + 1)(x + 2)(x + 3) - x^3 = 52$
$$x^3 + 6x^2 + 11x + 6 - x^3 = 52$$
$$6x^2 + 11x + 6 = 52$$
$$6x^2 + 11x - 46 = 0$$
$$x = \frac{12}{6}, x = \frac{-23}{6} \quad \Rightarrow x = 2, \quad x = \frac{-23}{6} (not\ possible)$$

∴ The volume of the cuboid $(x + 1)(x + 2)(x + 3) = 3 \times 4 \times 5 = 60$

4. Construct a cubic equation with roots

(i) 1, 2 , and 3

$(x - 1)(x - 2)(x - 3) = 0 \Rightarrow x^3 - 6x^2 + 11x - 6 = 0$

(ii) 1,1, and -2

$(x - 1)(x - 1)(x + 2) = 0 \Rightarrow x^3 - 0x^2 - 3x + 2 = 0$

(iii) 2 ,$\frac{1}{2}$, and 1.

$(x - 2)\left(x - \frac{1}{2}\right)(x - 1) = 0 \Rightarrow x^3 - \frac{7}{2}x^2 + \frac{7}{2}x - 1 = 0$
$$2x^3 - 7x^2 + 7x - 2 = 0$$

5. If α , β and γ are the roots of the cubic equation $x^3+2x^2+3x+4=0$, form a cubic equation whose roots are (i) 2α , 2β , 2γ (ii) $\frac{1}{\alpha},\frac{1}{\beta},\frac{1}{\gamma}$ (iii) $-\alpha,-\beta,-\gamma$ (iv) $\frac{\alpha}{2},\frac{\beta}{2},\frac{\gamma}{2}$

Soln: (i) 2α , 2β , 2γ

$2^0x^3+2^1 2x^2+2^2 3x+2^3 4=0$

$x^3+4x^2+12x+32=0$

(ii) $\frac{1}{\alpha},\frac{1}{\beta},\frac{1}{\gamma}$

$4x^3+3x^2+2x+1=0$ (reverse the co eff)

(iii) $-\alpha,-\beta,-\gamma$

$-x^3+2x^2-3x+4=0$ (sign change only in odd deg)

$x^3-2x^2+3x-4=0$

(iv) $\frac{\alpha}{2},\frac{\beta}{2},\frac{\gamma}{2}$

$\left(\frac{1}{2}\right)^0x^3+\left(\frac{1}{2}\right)^1 2x^2+\left(\frac{1}{2}\right)^2 3x+\left(\frac{1}{2}\right)^3 4=0$

$x^3+x^2+\frac{3}{4}x+\frac{4}{8}=0$

$8x^3+8x^2+6x+4=0$

6. Solve the eqn $3x^3-16x^2+23x-6=0$ if the product of two roots is 1.

Soln: Let the roots are a,b,c

Given ab = 1

$3x^3-16x^2+23x-6=0$

$x^3-\frac{16}{3}x^2+\frac{23}{3}x-\frac{6}{3}=0$

Product of roots $abc=\frac{6}{3}=2\Rightarrow c=2$

2	3	-16	23	-6
	0	6	-20	6
	3	-10	3	0

Reduced equation $3x^2-10x+3=0$

$x=\frac{9}{3},\frac{1}{3}\Rightarrow x=3,\frac{1}{3}$

7. Solve the equation $x^3-9x^2+14x+24=0$ if it is given that two of its roots are in the ratio 3:2 .

Soln: Given $x^3-9x^2+14x+24=0$

Sum of odd deg= 15 Sum of even deg= 15

∴ -1 is one of the root

-1	1	-9	14	24
	0	-1	10	-24
	1	-10	24	0

Reduced equation $x^2-10x+24=0$

$x=6,4$

Do it yourself $2x^3+11x^2-9x-18=0$

$7x^3-43x^2=439x-7$

8. If α , β , γ and δ are the roots of polynomial eqn $2x^4+5x^3-7x^2+8=0$, find a quadratic equation whose roots are $\alpha+\beta+\gamma+\delta$ and $\alpha\beta\gamma\delta$.

Soln: Given $2x^4+5x^3-7x^2+8=0 \Rightarrow x^4+\frac{5}{2}x^3-\frac{7}{2}x^2+0x+\frac{8}{2}=0$

Sum of roots $s=\alpha+\beta+\gamma+\delta=\frac{-5}{2}$

Product of roots $p=\alpha\beta\gamma\delta=\frac{8}{2}$

Required quadratic equation is $x^2-(S.R)x+(P.R)=0$

$x^2-(\frac{-5}{2}+\frac{8}{2})x+(\frac{-5}{2}\times\frac{8}{2})=0$

$x^2-\frac{3}{2}x-\frac{40}{4}=0$

$2x^2-3x-20=0$

9. If p and q are roots of eqn $lx^2+nx+n=0$, show that $\sqrt{\frac{p}{q}}+\sqrt{\frac{q}{p}}+\sqrt{\frac{n}{l}}=0$.

Soln: Given, $lx^2+nx+n=0 \implies x^2+\frac{n}{l}x+\frac{n}{l}=0$

Sum of roots $p+q=-\frac{n}{l}$ Product of roots $pq=\frac{n}{l} \implies \sqrt{pq}=\sqrt{\frac{n}{l}}$

$$\frac{p+q}{\sqrt{pq}}=\frac{-\frac{n}{l}}{\sqrt{\frac{n}{l}}}$$

$$\frac{p}{\sqrt{pq}}+\frac{q}{\sqrt{pq}}=-\sqrt{\frac{n}{l}}$$

$$\sqrt{\frac{p}{q}}+\sqrt{\frac{q}{p}}+\sqrt{\frac{n}{l}}=0.$$

10. If the equations $x^2+px+q=0$ and $x^2+p'x+q'=0$ have a common root, show that it must be equal to $\frac{pq'-qp'}{q-q'}$ or $\frac{q-q'}{p-p'}$.

Soln: Let us assume 'a' be the common root

$a^2+pa+q=0$

$a^2+p'a+q'=0$

$$\frac{a^2}{\begin{vmatrix} p & q \\ p' & q' \end{vmatrix}}=\frac{a}{\begin{vmatrix} q & 1 \\ q' & 1 \end{vmatrix}}=\frac{1}{\begin{vmatrix} 1 & p \\ 1 & p' \end{vmatrix}} \implies \frac{a^2}{pq'-qp'}=\frac{a}{q-q'}=\frac{1}{p'-p}$$

$$\frac{a^2}{pq'-qp'}=\frac{a}{q-q'}$$
$$\frac{a^2}{a}=\frac{pq'-qp'}{q-q'}$$
$$a=\frac{pq'-qp'}{q-q'}$$

(or)

$$\frac{a}{q-q'}=\frac{1}{p'-p}$$
$$a=\frac{q-q'}{p-p'}$$

11.Discuss the nature of roots of $4x^2+4px+p+2=0$ in terms of p.

Soln: $\Delta=b^2-4ac=(4p)^2-4(4)(p+2)$
$=16(p^2-p-2)$
$=16(p+1)(p-2)$

Interval	$p+1$	$p-2$	Δ	Nature of roots
$p=-1$	0		0	Real and Equal
$p=2$		0	0	Real and Equal
$-\infty<p<-1$	-	-	+	Real and Unequal
$2<p<\infty$	+	+	+	Real and Unequal
$-1<p<2$	+	-	-	Imaginary

12.If α,β are the roots of $x^2-5x+6=0$ then find $\alpha^2-\beta^2$

Soln: $\alpha+\beta=5, \quad \alpha\beta=6$

$(\alpha-\beta)^2=(\alpha+\beta)^2-4\alpha\beta=5^2-4(6)=1$

$\alpha-\beta=1$

$\alpha^2-\beta^2=(\alpha+\beta)(\alpha-\beta)=5(-1)=-5$

13.If α,β are the roots of $x^2+5x+6=0$ then find $\alpha^2+\beta^2$

Soln: $\alpha+\beta=-5, \quad \alpha\beta=6$

$\alpha^2+\beta^2=(\alpha+\beta)^2-2\alpha\beta=(-5)^2-2(6)=13$

14.Solve $2x^3-9x^2+10x=3$

Soln: $2x^3-9x^2+10x-3=0$

Sum of co-efficients = 0

$\therefore$ 1 is one of the root

$$\begin{array}{r|rrrr} 1 & 2 & -9 & 10 & -3 \\ & 0 & 2 & -7 & 3 \\ \hline & 2 & -7 & 3 & \boxed{0} \end{array}$$

Reduced equation is $2x^2-7x+3=0$

$x=\frac{6}{2}, \quad x=\frac{1}{2}$ $\qquad x=3, \quad x=\frac{1}{2}$

Do it your self $x^3-3x^2-33x+35=0$

15.Solve (i) $x^4-3x^2-4=0$ (ii) $x^4-14x^2+45=0$

Soln:

(i) $x^4-3x^2-4=0$		(ii) $x^4-14x^2+45=0$	
$x^2=-1$	$x^2=4$	$x^2=9$	$x^2=5$
$x=\pm\sqrt{-1}=\pm i$	$x=\pm 2$	$x=\pm 3$	$x=\pm\sqrt{5}$

Do it yourself $x^4-9x^2+20=0$

16.Find the condition of the eqn $x^3+px^2+qx+r=0$ whose roots are $in\ A.P.$

Soln: Let roots are $a-d, a, a+d$

sum of roots $a-d+a+a+d=-p$

$$3a=-p$$
$$a=\frac{-p}{3}$$
$$\left(\frac{-p}{3}\right)^3+p\left(\frac{-p}{3}\right)^2+q\left(\frac{-p}{3}\right)+r=0$$
$$-p^3+3p^3-9pq+r=0$$
$$2p^3+r=9pq$$

Do it yourself: If the roots of $x^3+px^2+qx+r=0$ whose roots are $in\ H.P$.prove that $9pqr=27r^3+2q^3$ (hint: reciprocal of AP is HP)

17.If α, β, and γ are the roots of the equation $x^3+px^2+qx+r=0$ find the value of $\sum\frac{1}{\beta\gamma}$ in terms of the coefficients.

Soln: $\alpha+\beta+\gamma=-p;\quad \alpha\beta\gamma=-r$

$$\sum\frac{1}{\beta\gamma}=\frac{\alpha+\beta+\gamma}{\alpha\beta\gamma}=\frac{-p}{-r}=\frac{p}{r}$$

18.If α and β are the roots of the quadratic equation $2x^2-7x+13=0$ construct a quadratic equation whose roots are $\alpha^2$and β^2 .

Soln: put $x^2=y$ $\qquad 2y-7\sqrt{y}+13=0$

$$2y+13=7\sqrt{y}$$
$$(2y+13)^2=\left(7\sqrt{y}\right)^2$$
$$4y^2+52y+169=49y$$
$$4y^2+3y+169=0$$

19.If α and β are the roots of the quadratic equation$17x^2+43x-73=0$ construct a quadratic equation whose roots are α+2and β+2 .

Soln: roots increase by 2 then equation diminish by

	17	43	-73
-2	0	-34	-18
	17	9	-91
	0	-34	
	17	-25	

Required equation is $17x^2-25x-91=0$

20.Find the sum of squares of roots of $ax^4+bx^3+cx^2+dx+e=0$

Soln: Let $\alpha,\beta,\gamma\ and\ \delta$ be the roots of $x^4+\frac{b}{a}x^3+\frac{c}{a}x^2+\frac{d}{a}x+\frac{e}{a}=0$

$$\alpha+\beta+\gamma+\delta=-\frac{b}{a}$$
$$\alpha\beta+\alpha\gamma+\alpha\delta+\beta\gamma+\beta\delta+\gamma\delta=\frac{c}{a}$$
$$\alpha^2+\beta^2+\gamma^2+\delta^2=(\alpha+\beta+\gamma+\delta)^2-2(\alpha\beta+\alpha\gamma+\alpha\delta+\beta\gamma+\beta\delta+\gamma\delta)$$

$$= \left(-\frac{b}{a}\right)^2 - 2\left(\frac{c}{a}\right)$$

$$= \frac{b^2-2ac}{a^2}$$

21.solve the equation $2x^3 - x^2 - 18x + 9 = 0$ if sum of two of its roots vanishes.

Soln: Let α, β, γ be the roots of $x^3 - \frac{1}{2}x^2 - \frac{18}{2}x + \frac{9}{2} = 0$ Given $\alpha + \beta = 0$

$\alpha + \beta + \gamma = \frac{1}{2}$ $\Longrightarrow$ $\gamma = \frac{1}{2}$

$\frac{1}{2}$	2	-1	-18	9
	0	1	0	-9
	2	0	-18	0

Reduced equation is $2x^2 - 18 = 0$

$x = 3, \ x = -3$

22.solve the equation $9x^3 - 36x^2 + 44x - 16 = 0$ if the roots form an A.P.

Soln: Let $a - b, a, a + b$ be the roots of $x^3 - \frac{36}{9}x^2 + \frac{44}{9}x - \frac{19}{9} = 0$

$a - b + a + a + b = 4$ $\Longrightarrow$ $a = \frac{4}{3}$

$\frac{4}{3}$	9	-36	44	-16
	0	12	-32	16
	9	-24	12	0

Reduced equation is $9x^2 - 24x + 12 = 0$

$x = \frac{18}{9}, \quad x = \frac{6}{9}$

$x = 2, \ x = \frac{2}{3}$

23.solve the equation $3x^3 - 26x^2 + 52x - 24 = 0$ if the roots form an G.P.

Soln: Let $ar, a, \frac{a}{r}$ be the roots of $x^3 - \frac{26}{3}x^2 + \frac{52}{3}x - \frac{24}{3} = 0$

$a^3 = 8$ $\Longrightarrow$ $a = 2$

2	3	-26	52	-24
	0	6	-40	24
	3	-20	12	0

Reduced equation is $3x^2 - 20x + 12 = 0$

$x = \frac{18}{3}, \quad x = \frac{2}{3}$

$x = 6, \quad x = \frac{2}{3}$

24.Determine k and solve the equation $2x^3 - 6x^2 + 3x + k = 0$ if one root is twice the sum of the other two roots.

Soln: Let a, b, c be the roots of $x^3 - \frac{6}{2}x^2 + \frac{3}{2}x + \frac{k}{2} = 0$ Given $a = 2(b + c)$

$a + b + c = 3$ $\Longrightarrow$ $\frac{3}{2}a = 3$ $\Longrightarrow$ $a = 2$

2	2	-6	3	k
	0	4	-4	-2
	2	-2	-1	k-2

clearly $k - 2 = 0$ $\Longrightarrow$ $k = 2$

Reduced equation is $2x^2 - 2x - 1 = 0$

$x = \frac{1 \pm \sqrt{3}}{2}$

PROBABILITY DISTRIBUTIONS

1.A six sided die is marked '1' on one face, '2' on two of its faces, and '3' on remaining three faces. The die is rolled twice. If X denotes the total score in two throws. Find (i) The probability mass function. (ii) The cumulative distribution function.(iii) $P(3 \le X < 6)$ (iv) $P(X \ge 4)$.

+	1	2	2	3	3	3
1	2	3	3	4	4	4
2	3	4	4	5	5	5
2	3	4	4	5	5	5
3	4	5	5	6	6	6
3	4	5	5	6	6	6
3	4	5	5	6	6	6

Soln: The random variable X takes the value 2,3,4,5 and 6.

X	2	3	4	5	6
PMF	$\frac{1}{36}$	$\frac{4}{36}$	$\frac{10}{36}$	$\frac{12}{36}$	$\frac{9}{36}$
CDF	$\frac{1}{36}$	$\frac{5}{36}$	$\frac{15}{36}$	$\frac{27}{36}$	$\frac{36}{36}$

(iii) $P(3 \le X < 6) = P(x = 3) + P(x = 4) + P(x = 5)$

$= \frac{4}{36} + \frac{10}{36} + \frac{12}{36} = \frac{26}{36}$

(iv) $P(X \ge 4) = P(x = 4) + P(x = 5) + P(x = 6)$

$= \frac{10}{36} + \frac{12}{36} + \frac{9}{36} = \frac{31}{36}$

2. A six sided die is marked '1' on one face, '3' on two of its faces ,and '5' on remaining three faces. The die is rolled twice. If X denotes the total score in two throws. Find (i) The probability mass function (ii) The cumulative distribution function.(iii) $P(4 \le X < 10)$ (iv) $P(X \ge 6)$.

+	1	3	3	5	5	5
1	2	4	4	6	6	6
3	4	6	6	8	8	8
3	4	6	6	8	8	8
5	6	8	8	10	10	10
5	6	8	8	10	10	10
5	6	8	8	10	10	10

Soln: The random variable X takes the value 2,4,6,8 and 10.

X	2	4	6	8	10
PMF	$\frac{1}{36}$	$\frac{4}{36}$	$\frac{10}{36}$	$\frac{12}{36}$	$\frac{9}{36}$
CDF	$\frac{1}{36}$	$\frac{5}{36}$	$\frac{15}{36}$	$\frac{27}{36}$	$\frac{36}{36}$

(iii) $P(4 \le X < 10)$

$= P(x = 4) + P(x = 6) + P(x = 8)$

$= \frac{4}{36} + \frac{10}{36} + \frac{12}{36} = \frac{26}{36}$

(iv) $P(X \ge 6) = P(x = 6) + P(x = 8) + P(x = 10)$

$= \frac{10}{36} + \frac{12}{36} + \frac{9}{36} = \frac{31}{36}$

3.A random variable X has the following probability mass function.

x	1	2	3	4	5	6
$f(x)$	k	$2k$	$6k$	$5k$	$6k$	$10k$

Find (i) $P(2 < X < 6)$ (ii) $P(2 \le X < 5)$ (iii) $P(X \le 4)$ (iv) $P(3 < X)$

Soln: Given f is P.M.F

x	1	2	3	4	5	6
$f(x)$	$\frac{1}{30}$	$\frac{2}{30}$	$\frac{6}{30}$	$\frac{5}{30}$	$\frac{6}{30}$	$\frac{10}{30}$

$\therefore \sum f(x) = 1$

$k + 2k + 6k + 5k + 6k + 10k = 1$

$30k = 1 \implies k = \frac{1}{30}$

(i) $P(2 < X < 6) = P(x = 3) + P(x = 4) + P(x = 5) = \frac{6}{30} + \frac{5}{30} + \frac{6}{30} = \frac{17}{30}$

(ii) $P(2 \le X < 5) = P(x = 2) + P(x = 3) + P(x = 4) = \frac{2}{30} + \frac{6}{30} + \frac{5}{30} = \frac{13}{30}$

(iii) $P(X \le 4) = P(x = 1) + P(x = 2) + P(x = 3) + P(x = 4) = \frac{1}{30} + \frac{2}{30} + \frac{6}{30} + \frac{5}{30} = \frac{14}{30}$

(iv) $P(3 < X) = P(x = 4) + P(x = 5) + P(x = 6) = \frac{5}{30} + \frac{6}{30} + \frac{10}{30} = \frac{21}{30}$

4. A random variable X has the following probability mass function.

x	1	2	3	4	5
$f(x)$	k^2	$2k^2$	$3k^2$	$2k$	$3k$

Find (i) $P(2 \le X < 5)$ (ii) $P(3 < X)$

Soln: Given f is P.M.F $\therefore \sum f(x) = 1$

$$k^2 + 2k^2 + 3k^2 + 2k + 3k = 1$$

$$6k^2 + 5k - 1 = 0$$

$$k = -1 \ , \ k = \frac{1}{6}$$

x	1	2	3	4	5
$f(x)$	$\frac{1}{36}$	$\frac{2}{36}$	$\frac{3}{36}$	$\frac{2}{6} = \frac{12}{36}$	$\frac{3}{6} = \frac{18}{36}$

(i) $P(2 \le X < 5) = P(x = 2) + P(x = 3) + P(x = 4) = \frac{2}{36} + \frac{3}{36} + \frac{12}{36} = \frac{17}{36}$

(ii) $P(3 < X) = P(x = 4) + P(x = 5) = \frac{12}{36} + \frac{18}{36} = \frac{30}{36}$

5.The cumulative distribution function of a discrete random variable is given by

$$F(x) = \begin{cases} 0 & ; -\infty < x < 0 \\ \frac{1}{2} & ; 0 \le x < 1 \\ \frac{3}{5} & ; 1 \le x < 2 \\ \frac{4}{5} & ; 2 \le x < 3 \\ \frac{9}{10} & ; 3 \le x < 4 \\ 1 & ; 4 \le x < \infty \end{cases}.$$

find

(i) The Probability mass function $f(x)$

(ii) $P(X < 3)$

(iii) (iii) $P(X \ge 2)$

Soln.

The values of the discrete random variable X are 0,1,2,3,4.

(i) The Probability mass function $f(x)$:

x	0	1	2	3	4
$F(x)$	$\frac{1}{2}$	$\frac{3}{5}$	$\frac{4}{5}$	$\frac{9}{10}$	1
$f(x)$	$\frac{1}{2}$ or $\frac{5}{10}$	$\frac{1}{5}$ or $\frac{2}{10}$	$\frac{1}{10}$	$\frac{1}{10}$	$\frac{1}{10}$

(i) $P(X < 3) = P(X = 0) + P(X = 1) + P(X = 2) = \frac{5}{10} + \frac{1}{10} + \frac{2}{10} = \frac{8}{10} = \frac{4}{5}$

(ii) $P(X \ge 2) = P(X = 2) + P(X = 3) + P(X = 4) = \frac{2}{10} + \frac{1}{10} + \frac{1}{10} = \frac{4}{10} = \frac{2}{5}$

6.The cumulative distribution function of a discrete random variable is given by $F(x) =$

$$\begin{cases} 0 & ; -\infty < x < -1 \\ 0.15 & ; -1 \le x < 0 \\ 0.35 & ; 0 \le x < 1 \\ 0.60 & ; 1 \le x < 2 \\ 0.85 & ; 2 \le x < 3 \\ 1 & ; 3 \le x < \infty \end{cases}$$

Find (i)the probability mass function (ii)$p(X < 1)$ and(iii) $P(X \ge 2)$

The values of the discrete random variable X are -1,0,1,2,3.

(i)The Probability mass function $f(x)$:

x	-1	0	1	2	3
$F(x)$	0.15	0.35	0.60	0.85	1
$f(x)$	0.15	0.20	0.25	0.25	0.15

(i) $P(X < 1) = P(X = -1) + P(X = 0) = 0.15 + 0.20 = 0.35$

(ii) $P(X \geq 2) = P(X = 2) + P(X = 3) = 0.25 + 0.15 = 0.40$

INVERSE TRIGONOMETRY

5 Marks

Important Hints:

$tan^{-1}x + tan^{-1}y = tan^{-1}\left(\frac{x+y}{1-xy}\right)$, $tan^{-1}x - tan^{-1}y = tan^{-1}\left(\frac{x-y}{1+xy}\right)$

$cos^{-1}x + cos^{-1}y = cos^{-1}(xy - \sqrt{(1-x^2)}\ \sqrt{1-y^2}\ )$

1) If $a_1, a_2, a_3 \dots a_n$ is an arithmetic progression with common difference d, prove that

$$\tan\left[tan^{-1}\left(\frac{d}{1+a_1a_2}\right) + tan^{-1}\left(\frac{d}{1+a_2a_3}\right) + \cdots + tan^{-1}\left(\frac{d}{1+a_na_{n-1}}\right)\right] = \frac{a_n - a_1}{1+a_1a_n}$$

soln

$$tan^{-1}\left(\frac{d}{1+a_1a_2}\right) = tan^{-1}\left(\frac{a_2-a_1}{1+a_1a_2}\right) = tan^{-1}a_2 - tan^{-1}a_1$$

$$tan^{-1}\left(\frac{d}{1+a_2a_3}\right) = tan^{-1}\left(\frac{a_3-a_2}{1+a_2a_3}\right) = tan^{-1}a_3 - tan^{-1}a_2$$

$$tan^{-1}\left(\frac{d}{1+a_na_{n-1}}\right) = tan^{-1}\left(\frac{a_n-a_{n-1}}{1+a_{n-1}a_n}\right) = tan^{-1}a_n - tan^{-1}a_{n-1}$$

$$tan^{-1}\left(\frac{d}{1+a_1a_2}\right) + tan^{-1}\left(\frac{d}{1+a_2a_3}\right) + \cdots + tan^{-1}\left(\frac{d}{1+a_na_{n-1}}\right) = tan^{-1}a_n - tan^{-1}a_1$$

$$tan\left[tan^{-1}\left(\frac{d}{1+a_1a_2}\right) + tan^{-1}\left(\frac{d}{1+a_2a_3}\right) + \cdots + tan^{-1}\left(\frac{d}{1+a_na_{n-1}}\right)\right]$$

$$= tan[\,tan^{-1}a_n - tan^{-1}a_1]$$

$$= \tan\left[tan^{-1}\left(\frac{a_n-a_1}{1+a_1a_n}\right)\right]$$

$$= \frac{a_n-a_1}{1+a_1a_n}$$

2) prove that: $tan^{-1}x + tan^{-1}\left(\frac{2x}{1-x^2}\right) = tan^{-1}\left(\frac{3x-x^3}{1-3x^2}\right)$, $|x| < 1/\sqrt{3}$

soln

$$tan^{-1}x + tan^{-1}\left(\frac{2x}{1-x^2}\right) = tan^{-1}\left(\frac{x+\frac{2x}{1-x^2}}{1-x\left(\frac{2x}{1-x^2}\right)}\right)$$

$$= tan^{-1}\left(\frac{x-x^3+2x}{1-x^2-2x^2}\right)$$

$$= tan^{-1}\left(\frac{3x-x^3}{1-3x^2}\right)$$

3) Show that $tan^{-1}x + tan^{-1}y + tan^{-1}z = tan^{-1}\left(\frac{x+y+z-xyz}{1-xy-yz-zx}\right)$

Soln:

$$tan^{-1}x + tan^{-1}y + tan^{-1}z = tan^{-1}\left(\frac{x+y}{1-xy}\right) + tan^{-1}(z)$$

$$= tan^{-1}\left(\frac{\left(\frac{x+y}{1-xy}\right)+z}{1-\left(\frac{x+y}{1-xy}\right)z}\right)$$

$$= tan^{-1}\left(\frac{[x+y+z(1-xy)]/(1-xy)}{[1-xy-(xz+yz)]/(1-xy)}\right)$$
$$= tan^{-1}\left(\frac{x+y+z-xyz}{1-xy-yz-zx}\right)$$

4) IF $tan^{-1}x + tan^{-1}y + tan^{-1}z = \pi$ then show that $x + y + z = xyz$

Soln:

$$tan^{-1}x + tan^{-1}y + tan^{-1}z = tan^{-1}\left(\frac{x+y}{1-xy}\right) + tan^{-1}(z)$$
$$= tan^{-1}\left(\frac{\left(\frac{x+y}{1-xy}\right)+z}{1-\left(\frac{x+y}{1-xy}\right)z}\right)$$
$$= tan^{-1}\left(\frac{[x+y+z(1-xy)]/(1-xy)}{[1-xy-(xz+yz)]/(1-xy)}\right)$$
$$tan^{-1}x + tan^{-1}y + tan^{-1}z = tan^{-1}\left(\frac{x+y+z-xyz}{1-xy-yz-zx}\right) = \pi$$
$$\frac{x+y+z-xyz}{1-xy-yz-zx} = tan\pi = 0$$
$$x + y + z - xyz = 0$$
$$x + y + z = xyz$$

5) Find the number of solutions of the equation
$$tan^{-1}(x-1) + tan^{-1}(x) + tan^{-1}(x+1) = tan^{-1}(3x)$$

Soln: $tan^{-1}(x-1) + tan^{-1}(x+1) = tan^{-1}(3x) - tan^{-1}(x)$

$$tan^{-1}\left(\frac{(x-1)+(x+1)}{1-(x-1)(x+1)}\right) = tan^{-1}\left(\frac{3x-x}{1+3x(x)}\right)$$
$$\frac{2x}{1-(x^2-1)} = \frac{2x}{1+3x^2}$$
$$2x(1+3x^2) = 2x(x^2+2)$$
$$2x + 6x^3 = 2x^3 + 4x$$
$4x^3 - 2x = 0$ $\quad\therefore$ Given equation has 3 solutions

6) Solve $\tan^{-1}\left(\frac{x-1}{x-2}\right) + \tan^{-1}\left(\frac{x+1}{x+2}\right) = \frac{\pi}{4}$.

Soln: Given $\tan^{-1}\left(\frac{x-1}{x-2}\right) + \tan^{-1}\left(\frac{x+1}{x+2}\right) = \frac{\pi}{4}$

$$\tan^{-1}\left(\frac{\frac{x-1}{x-2}+\frac{x+1}{x+2}}{1-\left(\frac{x-1}{x-2}\right)\left(\frac{x+1}{x+2}\right)}\right) = \frac{\pi}{4}$$
$$\frac{(x-1)(x+2)+(x+1)(x-2)}{(x-2)(x+2)-(x-1)(x+1)} = \tan\frac{\pi}{4}$$
$$\frac{x^2-x+2x-2+x^2+x-2x-2}{x^2-4-(x^2-1)} = 1$$

$$\frac{2x^2-4}{x^2-4-x^2+1} = 1$$

$$\frac{2x^2-4}{-3} = 1 \quad \Rightarrow 2x^2 = -3+4$$

$$2x^2 = 1 \Rightarrow x^2 = \frac{1}{2}$$

$$x = \pm\frac{1}{\sqrt{2}}$$

6) If $cos^{-1}x + cos^{-1}y + cos^{-1}z = \pi$ and $0 < x, y, z < 1$, then show that $x^2 + y^2 + z^2 + 2xyz = 1$.

Soln

Let $\cos^{-1}x = \alpha, \cos^{-1}y = \beta$ Then $x = cos\,\alpha, y = \cos\beta$

Given, $cos^{-1}x + cos^{-1}y + cos^{-1}z = \pi$

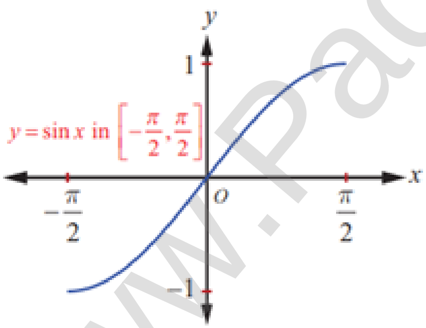
$$cos^{-1}x + cos^{-1}y = \pi - cos^{-1}z$$

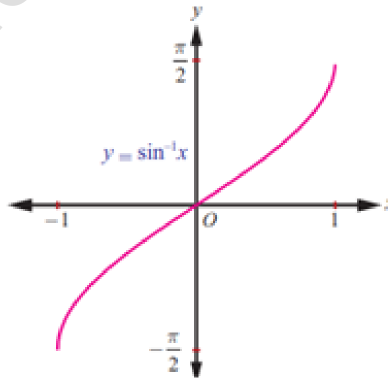
$$cos^{-1}(xy - \sqrt{1-x^2}\sqrt{1-y^2}) = cos^{-1}(-z)$$

$$-z = xy - \sqrt{1-x^2}\sqrt{1-y^2}$$

$$x^2 + y^2 + z^2 + 2xyz = 1.$$

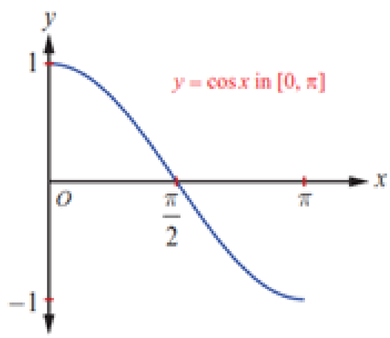
7) Draw the curve $sin\,x$ in the domain $[-\frac{\pi}{2}, \frac{\pi}{2}]$ and $sin^{-1}x$ in [-1,1]

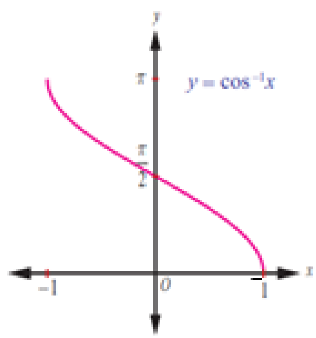




Domain : $\left[-\frac{\pi}{2}, \frac{\pi}{2}\right] \rightarrow [-1, 1]$ **Domain:** $[-1, 1] \rightarrow \left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$

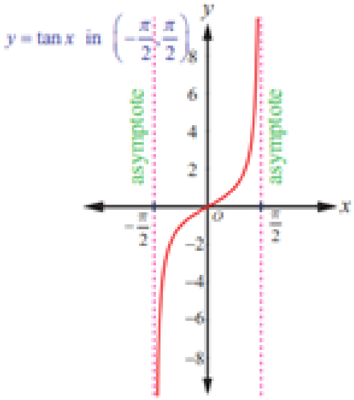
8) Draw the curve $cosx$ in the domain $[0.\pi]$ and $cos^{-1}x$ in $[-1, 1]$

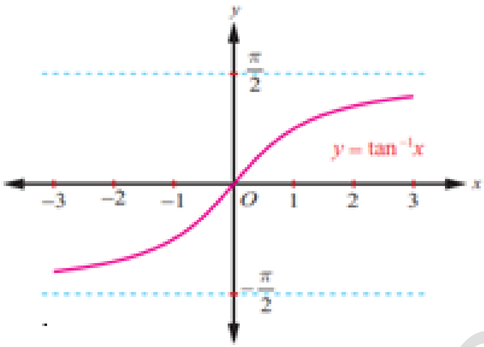




Domain: $[0, \pi] \rightarrow [-1, 1]$ **Domain :** $[-1, 1] \rightarrow [0, \pi]$

9) Draw the curve $tan\ x$ in the domain $(-\frac{\pi}{2}, \frac{\pi}{2})$ and $tan^{-1}x$ in R





Domain: $\left(-\frac{\pi}{2}, \frac{\pi}{2}\right) \rightarrow R$ **Domain:** $R \rightarrow \left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$