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TRB – P.G. – MATHS - UNIT – II

REAL ANALYSIS

CARDINAL NUMBERS

Def: Two sets A and B are called **Similar**, or **equinumerous,** and we write $A \sim B$, iff there exists a **one-to-one** function F whose domain is the set A and whose range is the set B.

Def: A set S is called **finite** and is said to contain n elements if $S \sim \{1, 2, 3, \ldots, n\}$ the integer n is called the **Cardinal Number** of S ie., no. of elements in the set. The **Cardinal No. of** a finite set is well defined. The **empty set** is also considered **finite** Its Cardinal no. is defined to be **0**.

Eg: $A = \{1, 2, 3, 5\}$. The Cardinal no. of the set A is 4 ie., $n(A) = 4$.

COUNTABLE (Enumerable or Denumerable) AND UNCOUNTABLE SETS (Non-Denumerable)

Def: A set S is said to be **countably** infinite if it is **equinumerous** with the set of all positive integers ie., if $S \sim \{1, 2, 3, \ldots\ldots\}$

Def: A set S is called **Countable** if it is **either finite** or **countably infinite.** A set which is **not countable** is called **uncountable**.

Eg: (1) The set of all **integers (Z)** is **Countable**.

(2) $A = \{1, 2, 3, \ldots\ldots\ldots\ldots, 100\}$ is **Countable**

(1) Every subset of a countable set is Countable. **(P.G.05-06)**

(2) The set of all real numbers (R) is uncountable. **(P.G.2001) (P.G.05-06)**

(3) Let Z^+ denote the set of all positive integers, the Cartesian product $Z^+ \times Z^+$ is Countable.

(4) * If $A_1, A_2, \ldots$ are countable sets, then $\bigcup_{n=1}^{\infty} A_n$ is countable (In words, the countable Union of Countable set is countable).

(5) The set of all rational numbers (Q) is a countable set. **[P.G.02-03., 03-04]**

(6) The set S of intervals with rational end points is a countable set.

(7) Let F be a collection of Sets. Then for any set B, we have

$$B - \bigcup_{A\in F} A = \bigcap_{A\in F}(B-A) \text{ and } B - \bigcap_{A\in F} A = \bigcup_{A\in F}(B-A)$$

Results:

* 1. The set of all natural numbers (N) is countable

2. The set N X N is countable

3. Empty set (φ) and Prime Numbers (P) are countable

Coro:

1. The set of all positive rational numbers is countable.
2. The set of all negative rational numbers is countable.
3. The set of all rational numbers in [0, 1] is countable. **[P.G.-05-06, 06-07]**

Theorem(8): The set of irrational number is uncountable. **[P.G.05-06, P.G.12-13]**

(9): The set [0,1] is uncountable. **[P.G.03-04, P.G.06-07]**

Results:

1. The set P_n of Polynomial functions with integer coefficients is countable.
2. The set Q_n of Polynomial functions with rational coefficients is countable.

Def: A real no. is said to be **algebraic** if it is the root-of some polynomial equation with rational coefficients.

Theorem (10): The set of algebraic no. is countable.

Def: A real no. is said to be **transcendental** if it is not algebraic.

Theorem (11): The set of transcendental numbers is uncountable.

Results:

1. The set of all ordered Pairs of integers is countable.
2. If A and B are countable sets, then the Cartesian product A X B is countable.
3. If A is countable set and B is uncountable set, then B – A is uncountable set.
4. Infinite subset of a countable set is countable.
5. Every infinite set contains a countably infinite subset.
6. The intervals (0,1) and [0,1] are equivalent. Also, R is equivalent to [0, 1]
7. If A is an infinite set an $x \in A$, then A and A – {x} are equivalent.

8. The set R is **not equivalent** to the class of all subsets of R.

Def: (Cantor set)

The **cantor set** K is the set of all numbers x in [0,1] which have a ternary expansion without the digit 1.

Note (1): The ternary expansion for a real no. x uses only the digits: 0, 1 and 2.

Eg: $\frac{1}{3} = 0_3.0222\ldots\ldots$, $\frac{2}{3} = 0_3.0222\ldots\ldots$ are in K but, any no. $x \ni \frac{1}{3} < x < \frac{2}{3}$ are not in K. Also, $\frac{1}{3} = 0_3.1000\ldots$ is not in K.

Results:

1. Cantor set K is not countable, compact, closed, perfect, nowhere dense and measure zero.
2. $K \subset [0,1]$ and K is equivalent to [0,1]
3. If f : A → B and the range of f is uncountable, then the domain of f is uncountable.
4. If B is a countable subset of the uncountable set A, then A – B is uncountable.
5. The set of all characteristic functions on I is uncountable.

Def: (Characteristic function)

Let G be an open subset of R. The characteristic function on G defined by

$$\chi_G(x) = \begin{Bmatrix} 1 \text{ if } x \in G \\ 0 \text{ if } x \notin G \end{Bmatrix}$$

Note: (1) Every infinite set is equivalent to a proper subset.

(2) If A, set of all sequences, whose elements are the digits 0 and 1. Then A is uncountable.

Eg: 1, 0, 0, 1, 0, 1, 1, 1, …. **[P.G.05-06]**

(3) Binary Exp. Uses digits: **0, 1** **[P.G.12-13]**

CANTOR'S DIAGONAL PROCESS:

Consider the process:

$$\begin{array}{ccccc}
a_{11} \rightarrow & a_{12} & a_{13} \rightarrow & a_{14} & a_{15} \ldots\ldots\ldots\ldots \\
a_{21} & a_{22} & a_{23} & a_{24} & \ldots\ldots\ldots\ldots \\
a_{31} & a_{32} & a_{33} & a_{34} & \ldots\ldots\ldots\ldots \\
a_{41} & a_{42} & a_{43} & a_{44} & \ldots\ldots\ldots\ldots \\
a_{51} & \ldots\ldots\ldots\ldots & \ldots & \ldots.. & \ldots\ldots\ldots\ldots
\end{array}$$

Let us define the height of the element aij to be i + j. Eg. Height of a_{11} is 1 + 1 = 2 and the height of a_{12} is 3 etc. There are exactly m – 1 elements of height m.

According to this scheme, the elements are to be counted as a_{11}, a_{12}, a_{21}, a_{31}, a_{22}, a_{13}, a_{14}, a_{23}, a_{32}, a_{41}, …. Thus all elements will be counted out through Cantor's Diagonal Process. Eg: N x N , Q

PROPERTIES OF REAL NUMBERS:

The rational and irrational numbers together constitute the real number system. Real Numbers satisfy the following axioms.

The Field Axioms:

Axiom 1: $x + y = y + x$; $xy = yx$ (commutative law)

Axiom 2: $x + (y+z) = (x+y) + z$; $x(yz) = (xy)z$ (Associative)

Axiom 3: $x(y+z) = xy + xz$ (distributive law)

Axiom 4: $x + (-x) = (-x) + x = 0$, (negative of x ie., Additive Inverse)

Axiom 5: $xy = yx = 1 \Rightarrow y = 1/x$, (Multiplicative Inverse of x), $(x \neq 0)$

Theorem (1): There can exist at the most one identity element for addition and multiplication in R. (ie., $x + 0 = 0 + x = x$ & $x.1 = 1.x = x$) (Additive identify 0 & multiplicative identify 1).

(2): To each x in R, there corresponds one and only one real no $y \ni x+y = y+x = 0$ (Uniqueness of inverse)

(3): To each $x \in R$, $x \neq 0$, there corresponds one and only real no. $y \ni xy = yx = 1$ (Uniqueness of Inverse)

* (4): If x, y be real numbers $\ni xy = 0$, then either $x = 0$ or $y = 0$.

The Order Axioms:

We also assume that the existence of a relation < which establishes an ordering among the real numbers and which satisfies the following Axioms.

Axiom (1): Exactly one of the relations $x = y$, $x < y$, $x > y$ holds.

Axiom 2: If $x < y$, then for every z, we have $x + z < y + z$

Axiom 3: If $x > 0$ and $y > 0$, then $xy > 0$

Axiom 4: If $x > y$ and $y > z$, then $x > z$

Theorem: Given real numbers a and b $\ni a \leq b + \varepsilon$ for every $\varepsilon > 0$. Then $a \leq b$.

Note: (1) The set C of complex numbers is an example of a field which is not ordered.

(2) R is ordered field.

(3) If $x < y$, then $xz < yz$ if z is +ve where as $xz > yz$ if z is –ve.

(4) If $x > y$ and $z > w$ where both y and w are positive, then $xz > yw$.

Def: (Inductive Set)

A set of real numbers is called an inductive set if it has the following two properties.

(i) The number 1 is in the set

(ii) For every x in the set, the number x+1 is also in the set.

Note that R is an Inductive set. So is the set R^+ (+ve Real No.s)

Def: A real no. is called a positive integer if it belongs to every inductive set.

Note: 1. The set Z^+ is itself an inductive set. (Principle of Induction)

2. Z^+ is the smallest inductive set.

3. The set of integers Z is the Union of Z^-, Z^+ and 0.

The Unique Factorization theorem for integers

(The fundamental theorem of Arithmetic)

Every integer $n > 1$, can be represented as a product of Prime factors, and this factorization can be done in only one way, apart from the order of the factors.

Theorem (1): Every integer $n > 1$ is either a prime or a product of Primes. **[P.G.06-07]**

Theorem (2): Every pair of integers a and b has a common divisor d of the form $d = ax + by$ where x and y are integers. Moreover, every common divisor of a and b divides this d.

Note: (1) $-d = a(-x) + b(-y)$

(2) GCD is non negative. It is denoted by gcd (a,b) or (a,b). If $(a,b) = 1$, then a and b are relatively prime.

Theorem (3): **(Euclid's Lemma)**

If a/ bc and $(a,b) = 1$, then a / c

Theorem: If a prime p divides ab, then p/a or p/b. More generally , if a prime p divides a product $a_1, \ldots\ldots, a_k$, then p divides at least one of the factors.

RATIONAL NUMBER:

Quotient of integers a/ b (where $b \neq 0$) Eg. ½ , -7/5, 6 etc.

The set of Rational Nos is denoted by Q, contains z as a subset.

Q (set of rational Nos) satisfies all the field Axioms and order Axioms.

IRRATIONAL NUMBERS:

Real Nos that are not rational are called irrational. Eg. $\sqrt{2}$, e, π, e^{π} etc.

Theorem (1):

If n is a positive integer which is not a perfect square, then $\sqrt{n}$ is irrational.

Theorem (2): If $e^x = 1 + \frac{x}{\angle 1} + \frac{x^2}{\angle 2} + \ldots$.., then the number e is irrational.

Note: (1) Between any two rational numbers there are infinitely rational nos.

(2) Another Rational No. between a and b is $\left(\frac{a+b}{2}\right)$

UPPER BOUNDS, MAXIMUM ELEMENT, LEAST UPPER BOUNDS

Def: (**Bounded Above**):

Let S be a set of real nos. If there is a real no b ∍ x ≤ b for every x in S, then b is called an upperbound for S and we say that S is bounded above by b (Every no greater than b will be an upper bound).

Def: (**Max. elt**.)

If an upper bound b is also a member of S, then b is called the largest element or **maximum elt** of S. There can be at most one such b we write **b = max. S.**

Def: Least Upper Bounds (l.u.b):

Let the subset S of R be bounded above the no. b is called the Least Upper Bound (l.u.b) for S if.

(i) b is an upperbound for S, and

(ii) No number smaller than b is an upper bound for S.

LOWER BOUNDS, MINIMUM ELEMENT, GREATEST LOWER BOUND

Def: (Lower Bound)

Let S be a set of real nos. If there is a real no a ∍ x ≥ a for every x ∈ S, then a is called an lower bound for S and we say that S is **bounded below by a** (Every no. less than "a" will also be an lower bound).

Def: (Min. element)

If a lower bound "a" is also a member of S, then "a" is called the **lowest member** or the **minimum element** of S. If it exists we write **a = min.S**.

Def: (Greatest Lower Bound) : g.l.b

Let the subset S of R be bounded below. Then "a" called the Greatest lower bound (g.l.b) for S if

(i) A is a lower bound for S and

(ii) No number less than a is a lower bound for S.

Theorem (1): The greatest lower bound of a given set x bounded below is **unique**. Similarly, l.u.b. is unique.

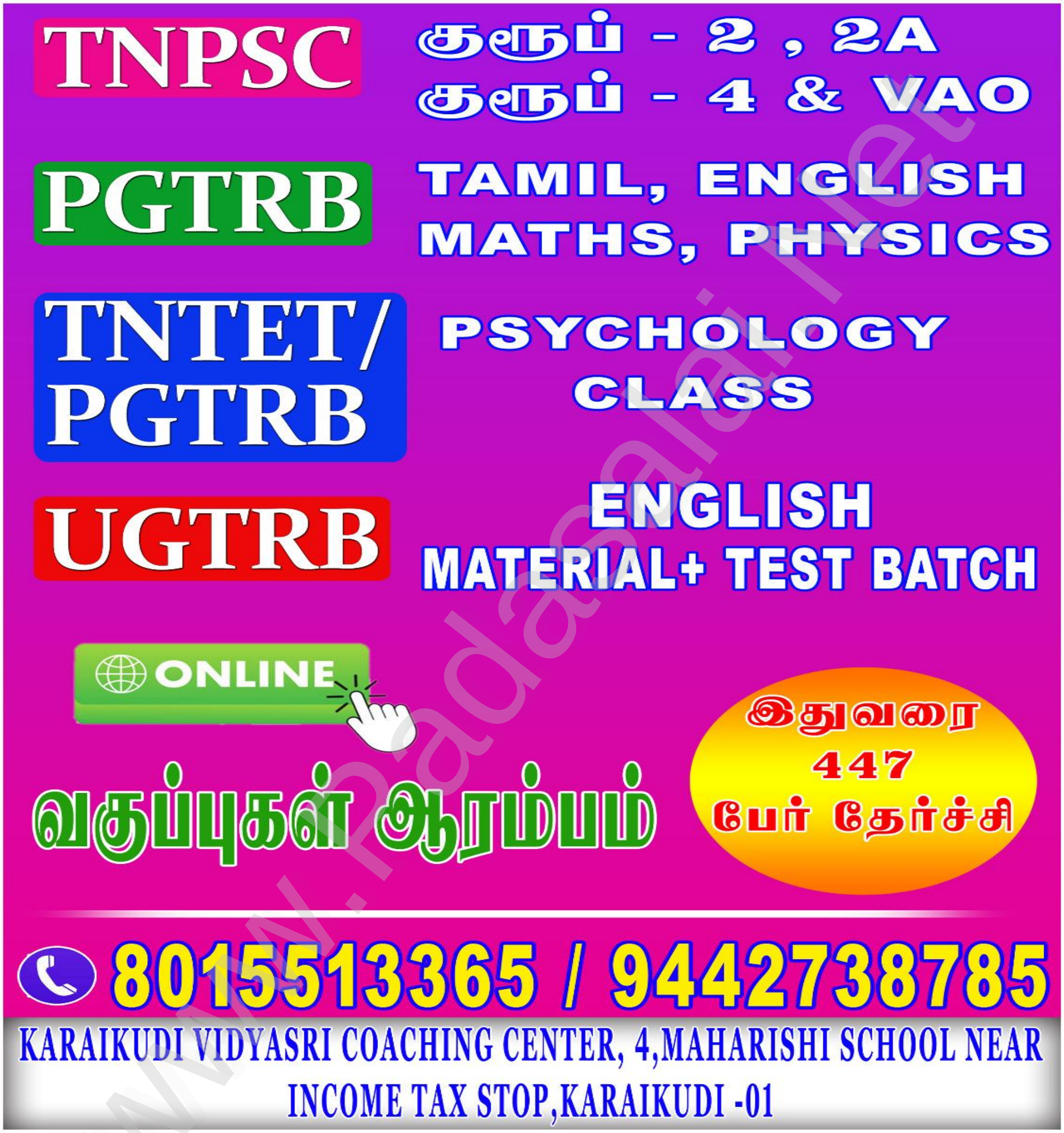
(2). If A is a non – empty subset of R that is bounded below, then A has a glb in R.

Examples:

1. The set $R^+ = (0, \infty)$ is unbounded above. It has no upper bounds and no maximum element. It is bounded below by 0 but has no minimum element.
2. The closed interval S= [0,1] is bounded above by 1 and is bounded below by 0. In fact, max S= 1 and min S= 0. Also, l.u.b = 1 & g.l.b. = 0.
3. The half open interval [0,1) is bounded above by 1 but it has no max. elt. Its min.elt. is 0.
4. **B = {$^1/_2$, ¾, …… , $\frac{2^n-1}{2^n}$, ….. }. Then g.l.b = ½ and l.u.b. = 1 (g.l.b. for B is an element of B but that l.u.b. for B is not an elt. of B)**
5. I = (3,4). Then l.u.b. = 4, g.l.b = 3. But both 3 and 4 are not in I.
6. **S = {1/n, n $\in$ N}. Then l.u.b. = 1 $\in$ S; g.l.b. = 0 $\notin$ S.**
7. S = {2, 3, 5}. The l.u.b. = 5; g.l.b. = 2
8. In N = {1,2,3, …..}, g.l.b. = 1
9. A = {1, 3, 5, 9, 20, π} ; g.l.b. = 1 & l.u.b. = 20.
10. **g.l.b. & l.u.b. for {0} is 0. ($\therefore$ for singleton set, l.u.b. = g.l.b.)**
11. φ = { }. It is bounded ($\phi \subseteq$ [a, b]) Every No. N$\in$R is an upper bound for ϕ and so ϕ does not have a l.u.b.
12. {π+1, π+2, ……} ; g.l.b. = π+1
13. { π+1, π+1/2, ……}; l.u.b. = π+1, g.l.b. = π **[P.G.11-12]**
14. **{1, 1/2, 1/3, …..}; l.u.b. = 1, g.l.b. = 0.**
15. The example of a countable bounded subset S of R whose g.l.b. and l.u.b. are both in R – A is (0,1) (ie., all open intervals form)
16. If A is non-empty bounded subset R, and B is the set of all upper bounds for A. Then $\underset{y \in B}{g.l.b}\; y = \underset{x \in A}{l.u.b}\; x$

Note:

1. There is only one l.u.b. and g.l.b.
2. The l.u.b. and g.l.b. of a set may or may not belong to the set.



3.

Def: (Supremum)

A non-empty subset S of a R when bounded above has least upper bound known as Supremum.

Def: (Infimum)

A non-empty subset S of a R when bounded below has greatest lower bound known as infimum.

The Completeness Axiom: [Poly.05-06]

A non-empty subset S of a real number which is bounded above has a Supremum; ie., there is a real no. b ∍ b = sup S.

Theorem (1): (Approximation property)

Let S be a non-empty set of real nos with a supremum say b = sup S. Then for every $a < b$ there is some $x \in S$ ∍ $a < x \le b$ (ie., a set with a supremum contains nos arbitrarily close to its supremum).

(2): (Additive property)

Given non-empty subsets A and B of R, let C denote the set C ={x+y; x∈ A, y∈B}. It each of A and B has a supremum, then C has a supremum and sup C = sup A + sup B.

(3): (Comparision property)

Given non-empty subset S and T of R ∍ $s \le t$ for every s in S and t in T. If T has asupremum then S has a sup. and sup.S ≤ sup. T.

(4): The set Z^+ of +ve integers 1, 2, 3, …… is unbounded above.

Theorem: For every real x, there is a positive integer n ∍ n > x. **[P.G.06-07]**

The Archimedean property of the Real Number system:

If x > 0 an if y is an arbitrary real numbers, there is a positive integer n such that n x > y.

Theorem (1) For every real x > 0 and every integer n > 0 there is one and only one positive real y such that $y^n = x$.

(2) Assume x ≥ o. Then for every integer n ≥ 1 there is a finite decimal $r_n = a_0, a_1, a_2, \ldots a_n$ ∍ $r_n \le x < r_n + \frac{1}{10^n}$

(3) If a ≥ 0, then we have the inequality $|x| \le a$ iff $-a \le x \le a$.

(4) For arbitrary real x and y, we have $|x + y| \leq |x| + |y|$ (the triangle inequality)

The Cauchy – Schwarz Inequality:

If $a_1, \ldots, a_n$ and $b_1, \ldots, b_n$ are arbitrary real numbers, we have

$$\left(\sum_{k=1}^{n} a_k b_k\right)^2 \leq \left(\sum_{k=1}^{n} a_k^2\right)\left(\sum_{k=1}^{n} b_k^2\right)$$

Moreover, if some $a_i \neq 0$ equality holds iff there is a real x ∍ $a_k x + b_k = 0$ for each k = 1, 2, ….. n.

The Extended Real No. System: (R^*):-

$R^*[-\infty, \infty]$ where as R $(-\infty, \infty)$

i. If $x \in R$, then $x + (+\infty) = +\infty$, $x + (-\infty) = -\infty$, $\frac{x}{\infty} = \frac{x}{-\infty} = 0$.

ii. If $x > 0$, $x(+\infty) = +\infty$, $x(-\infty) = -\infty$

iii. If $x < 0$, $x(+\infty) = -\infty$, $x(-\infty) = +\infty$

iv. If $x \in R$, then we have $-\infty < x < \infty$

v. R* does not form a field.

Def: Every open interval $(a, +\infty)$ is called a neighborhood of $+\infty$ or a ball with centre $+\infty$. Every open interval $(-\infty, a)$ is called a neighbourhood of $-\infty$ or a ball with centre $-\infty$.

Sup E = $+\infty$, inf E = $-\infty$ in R*.

ELEMENTS OF POINT SET TOPOLOTY:

Euclidean Space R^n:

The set of all n-dimensional points is called a n-dimensional Euclidean Space or simply n-space, and is denoted by R^n (n-dim Pt. $\boldsymbol{x} = (\boldsymbol{x_1}, \boldsymbol{x_2}, \ldots, \boldsymbol{x_n})$

Def: **(neighborhood)**

A set S in R is said to be a **neighborhood** of a point $x \in R$ if there exists an open interval (a,b) containing x and contained in S.

Eg: 1. The open interval (a, b) is a neighborhood of each of its points.

2. The closed interval [a, b] is a neighbourhood of each point of (a, b) but is not a neighborhood of the end points a and b.

Def: **(Open balls)**

Let "a" be a given point in R^n and let r be a given positive no. then set of all points x in R^n ∋ $||x - a|| < r$, is called an **open n-ball** of radius r and centre a. We write this ball as B (a ; r)

Def: (Interior points)

Let S be a subset of R^n, and assume that $a \in S$. then a is called an **interior point** of S if there is an open n – ball with centre at a, all of whose points belong to S. ie., every interior points a of S can be surrounded by n-ball.

The set of all interior point of S is called the interior of S and is denoted by **int. S.**

Note: Any set containing a ball with centre a is called a **neighborhood.**

Def: (open set) [P.G.03-04]

A set in R^n is called open if all its points are interior points (or) if it is a neighborhood of each of its points.

Note: 1. In R^1, B (a ; r) is an open interval with centre at a. In R^2, it is a Circular disk and in IR^3, it is a spherical solid with centre at a and radius r.

Theorem: Such open sphere is a **open set** in a metric space.

Eg:
1. Every open interval (a, b) is a open set
2. R is an open set
3. The set $(1, 2) \cup (3, 4)$ is open set
4. The closed interval [a, b] is not an open set.
5. The set Q of rational nos, the set R – Q, of irrational nos, the set Z of integers are not an open set.
6. Empty set ϕ is open (as well as closed)
7. A finite non – empty set is not open set.
8. Every infinite set is not open.
9. Open rays (a, ∞) & $(-\infty, a)$ are open sets
10. The closed rays $[a, \infty]$, $[-\infty, a]$ are not open sets.
11. The interval $(-1/n, 1/n)$ where n = 1, 2, 3, …. is open.
12. The set z of integers is not a neighborhood of any of its points.
13. If M and N are neighborhoods of a point, then $M \cap N$ is also a neighborhood of that point.
14. The empty set is a neighborhood of each of its points.
15. A non-empty finite set is not a neighborhood of each of its points.

Theorem (1) Arbitrary Union of open sets is open set.

(2) Finite Union of Open sets is open.

(3) Finite intersection of open sets is open.

(4) Arbitrary intersection of open sets is **not open**. **[P.G.12-13]**

Eg: 1.The intersection of two open sets is open.

2.Let $I_n = (-1/n, 1/n)$ be open set.

Then $\bigcap_{n \in N} I_n = \{0\}$, which is not an open set . **[P.G.04-05]**

3. Any set with only one point $a \in R_d$ is open in R_d. On the other hand if $a \in R^1$, then $\{a\}$ is not open in R^1

4. Every subset of R_d is open.

Def: (Metric space)

Let M be any set. A metric for M is a function & with domain M x M and range contained in $[0, \infty)$ ie., $\rho : M \times M \to [0, \infty)$ (this may be R also) such that

(i) $\rho(x, x) = 0 \ (x \in M)$

(ii) $\rho(x, y) > 0$, $(x, y \in M, \ x \neq y)$

(iii) $\rho(x, y) = \rho(y, x)$

(iv) $\rho(x, y) \leq \rho(x, z) + \rho(z, y)$, $x, y, z \in M$ (triangle inequality)

If ρ is a metric for M, then the ordered pair (M, ρ) is called a **metric space.**

Def: (Discrete Metric)

Define $d: R \times R \to [0, \infty)$ by

i) $d(x, x) = 0$, $x \in R$ (ii) $d(x, y) = 1$, $x, y \in R$, $x \neq y$.

The metric d is called the **discrete metric.** Denote the metric space <R, d> by R_d.

Def: (Usual Metric)

Let M = R

Define $\rho : M \times M \to R$ by $d(x, y) = |x - y|$, $x, y \in M$

Eg: 1. $d_1(x, y) = \dfrac{d(x,y)}{1+d(x,y)}$, $\sqrt{d}$, nd are Metric

2. d^2, $|x^2 - y^2|$, $|x - 2y|$ are not metric

3. $d(x, y) = \dfrac{|x-y|}{1+|x-y|}$ is metric.

Results:

1. Any open ball in a metric space (M, d) is an open set.
2. Every open ball is a bounded set (A is bounded if there exists a +ve real no. $k \ni d(x, y) \leq k \ \forall \ x, y \in A$)

3. In any metric space (M, d) both M and the empty set ϕ are open sets.

Def: (Component Interval)

Let S be an open subset of R^1. An open interval I (which may be finite or infinite) is called a component interval of S if $I \subseteq S$ and if there is no open interval $J \neq I \ni \; I \subseteq J \subseteq S$.

Theorem (1): Every point of a non – empty open set S belongs to one and only one component interval of S.

(2): (Representation theorem for open sets on the real line).

Every non – empty open set S in R^1 is the Union of a countable collection of disjoint component intervals of S.

Note: 1. Int. $\phi = \phi$, Int.M = M 2. Int. $(A \cap B)$ = Int. $A \cap$ Int. B

3. Int. $(A \cup B) \supseteq$ Int. $A \cup$ Int. B 4. $A \subseteq B \Rightarrow$ Int. $A \subseteq$ Int. B

CLOSED SETS:

Def: A set S in IR^n is called **closed** if its complement $IR^n - S$ is open. (ie., a closed set was defined to be the complement of an open set)

Eg: [a, b] in IR^1 is closed ($[a, b]^c$ is $(-\infty, a) \cup (b, \infty)$ which is open)

Def: (**Adherent Points**)

Let S be a subset of IR^n and $x \in IR^n$. Also, x is not necessarily in S. Then x is said to be **adherent** to S if every n-ball B(x) contains atleast one point of S.

Eg. 1. If $x \in S$, then x adheres to S (every ball B (x) contains x)

2. If S is a subset of R which is bounded above, then sup. S is adherent to S.

Def: (Accumulation Points)

If $S \subseteq IR^n$ and $x \in IR^n$, then x is called an **accumulation point** of S if every n-ball B(x) contains atleast one point of S distinct from x. (ie., x is an accumulation point of S iff. X adheres to $S - \{x\}$)

Eg: 1. The set $\{1/n\}$, n = 1, 2, 3, ….. has 0 as an accumulation point.

2. The set of rational no. (Q) has every real no. as an accumulation point.

3. Every point of the closed interval [a,b] is an accu. Point of the set of no.s in the open interval (a, b)

Theorem (1) If x is an accumulation point of S, then every n – ball B(x) contains infinitely many points of S.

(2) A set S in R^n is closed iff it contains all its adherent points.

(3) Each closed sphere is a **closed set** in a metric space.

Examples (Closed sets):

1. Every closed interval [a, b] is closed.

2. *The Empty set ϕ is both open and closed.*
3. *In Metric Space M, both M and ϕ are open as well as closed.*
4. *The set of integers (z) is closed . [P.G. 03-04]*
5. The set [1, 2] $\cup$ [3, 4] is a closed set.
6. *The closed ray (-∞, a], [a, ∞) are closed. [P.G.03-04]*
7. Every Singleton and finite sets are closed.
8. *Set of Real Nos IR is closed (As well as open)*
9. *The set of Rational Nos (Q). Irrational Nos (IR-Q) and Infinite sets are not closed.*
10. Open rays (-∞, a) and (a, ∞) are not closed.
11. Intervals (a, b) , [a, b), (a, b] are not closed.

Theorem (1) Finite Union of closed sets is closed.

(2) Arbitrary Intersection of closed sets is closed.

(3) Finite Intersection of closed sets is closed.

*(4) Arbitrary Union of closed sets is **not closed**.*

(5) If A is open and B is closed, then A-B is open and B – A is closed.

Eg: 1. If In = [1/n, 1-1/n], then $\bigcup_{n=1}^{\infty} I_n$ is not closed in R^1. **[P.G.12-13]**

Theorem (6) If A and B are open subsets of R^1, then A x B is an open subset of R^2.

(7) If A and B are closed subsets of R^1, then A x B is a closed subset of R^2.

Eg: 1. In M = R with usual Metric, A = [0, 1] is closed.

2. In M = IR with usual Metric, A = (0, 1) is not closed.

3. If M = R^2, infinite lines are closed subsets of IR^2.

4. If M = R^3, planes are closed in R^3.

Theorem (8) If F_1 and F_2 are closed subsets of the Metric space M, the $F_1 \cup F_2$ is also a closed set in M.

(9) Let G be an open subsets of the metric space M then M – G is closed. III^{ly}, if F is closed subset of M, then M – F is open.

Result: 1. Every subset of a discrete metric space is closed.

2. In any metric space every closed ball is a closed set.

3. A set E is open iff its complement is closed.

4. A set F is closed iff its complement is open.

Def: (Closure)

The set of all adherent points of a set S is called the **closure** of S and is denoted by $\overline{S}$ or cl.s.

Note: $A \subseteq M$. $\overline{A}$ is the intersection of all closed sets which contains A.

Def: (Derived set)

The set of all **limit points** (or a **cluster point** or an **accumulation point**) of A is called the **derived set** of A and is denoted by D(A).

Note: If x is a limit point of A iff there exists an open ball B (x, r) $\ni$

$$B(x, r) \cap (A - \{x\}) \neq \phi. \ \forall \ r > 0.$$

Examples: (**Closure**)

Eg: 1. Consider R with usual Metric.

i. Let A = [0, 1]. Then $\overline{A}$ = A = [0, 1]

ii. Let A = (0, 1) . Then $\overline{A}$ = [0, 1]

iii. For Open ray $(-\infty, 0)$, closure is $(-\infty, 0]$

2. In a discrete metric space (M, d) or R_d, any subset A of M is closed and hence $\overline{A}$ = A

Results:

i. A is closed iff A = $\overline{A}$

ii. $\phi = \overline{\phi}$, M = $\overline{M}$

iii. Let A, B $\subseteq$ M. then

(a) $A \subseteq B \Rightarrow \overline{A} \subseteq \overline{B}$

(b) $\overline{A \cup B} = \overline{A} \cup \overline{B}$ c) $\overline{A \cap B} \subseteq \overline{A} \cap \overline{B}$

Examples (**Derived set and closure**)

1. Let A = [0, 1] then D [0, 1] = [0, 1]
   cl. A = A $\cup$ D (A) = [0,1] $\cup$ [0,1] = [0,1]
2. Let A= {1, 1/2, 1/3, …….. 1/n, …} , D(A) = {0}
   cl. A = A $\cup$ D (A) = A $\cup$ {0}
3. D (z) = ϕ but z = $\overline{z}$ ie., clz = z $\cup$ D (z) = z $\cup$ ϕ = Z
4. ***D (Q) = R, cl. Q = R (ie., closure of Q is R)***
5. D (Q x Q) = R x R
6. ***Any subset of a discrete metric space has no limit point.***
7. Consider C with usual metric. Let A= $\{z / |z| < 1\}$ then D(A) = $\{z / |z| \leq 1\}$
8. [0,1] contains all its limit points (ie., All closed intervals)

Results:

Let A, B be two subsets of a metric space. Then (i) A $\subseteq$ B $\Rightarrow$ D(A) $\subseteq$ D (B)

ii) D (A$\cup$B) = D(A) $\cup$ D(B)

iii) D(A$\cap$B) $\subseteq$ D(A) $\cap$ D (B)

Theorem ***(1) A is closed iff A = $\overline{A}$ (ie., cl. A = A)***

(2) A set S is closed iff it contains all its limit points.

(3) The derived set of any set is closed set.

(4) If S be any subset of R, then $\overline{S}$ or cl.s = S$\cup$ S′ or $\overline{S}$ = S $\cup$ D (S)

(5) A set S is open iff int. S = S.

(6) Let (M, d) be a metric space. Let A $\subseteq$ M. Then x is a limit point of A iff each open ball with centre x contains an infinite no. of points of A.

(7): Any finite subset of a Metric Space has no limit points.

Ex: $x \in \overline{A}$ iff $B(x ; r) \cap A \neq \phi \ \forall r > 0$,

Note: (Limit Points – Examples)

1. ***Every point of [0,1] is a limit point of the open interval (0,1)***
2. ***Every real no. (R) is a limit point of the set Q of all rational no.s.***
3. ***The set Z of integers has no limit point.***
4. ***The set S = {1/n, n $\in z^+$} has only one limit point namely , 0.***
5. ***A finite set has no limit points***
6. ***A sub set of R_d has no limit point.*** ***[P.G.05-06]***

Note: If E is any subset of the metric space M, then E $\subseteq \overline{E}$

DENSE SETS:

Def: A subset A of a metric space M is said to be **dense** in M or everywhere dense if

$\overline{A} = M$.

Def: A metric space M is said to be **separable** if there exists a countable dense subsets in M

Examples: (Dense & separable)

1. Let M be a metric space then M is dense in M. Also, any countable Metric space is separable.
2. In R with usual metric, Q is dense in IR. Also, Q is countable and hence R is separable. **[P.G.03-04]** **[P.G.11-12]**
3. Subset of a discrete metric space is not dense in M. Also, any uncountable discrete metric space is not separable.
4. In R x R with usual metric, Q x Q is a dense set. Also, R x R is separable.
5. ***Both Q and Q^c (Irrational) are dense in R***
6. ***R^n with usual metric is separable.***
7. If A is dense in M and $A \subseteq B$, then B is also dense in M.
8. R_d has no dense subsets.

Perfect set:-

The set E is perfect if E is closed and if every point of E is a limit point of E.

Result: (1) A non – empty perfect set in R^k is **uncountable.**

Bounded sets in a Metric Space:

Def: Let (M, d) be a metric space. We say that a subset A of M is bounded if there exists a +ve real number k $\ni d(x, y) \le k \quad \forall\, x, y \in A$

Eg:
1. Any finite subset A of a metric space (M, d) is bounded.
2. [0, 1] is a bounded subset of IR with usual metric.
3. $(0, \infty)$ is an unbounded subset of R. **[P.G.03-04]**
4. In discrete metric R_d, $(0, \infty)$ is bounded (Any subset of a discrete metric space M is a bounded subset of M since $d(x, y) \le 1$)
5. In ℓ_∞ let $e_1 = (1, 0, .)$, $e_2 = (0, 1, \ldots)$ etc. Let A $\{e_1, e_2, \ldots.. e_n, \ldots..\}$. Then A is a bounded subset of ℓ_∞.
6. Let (M, d) be a metric space. Define $d_1(x, y) = \dfrac{d(x,y)}{1+d(x,y)}$. Then (M, d_1) is a bounded Metric space.

7. In a Metric Space, any subset of a bounded set is bounded.

Diameter of the subset A, d (A):

Def: Let (M, d) be a metric space. Let $A \subseteq M$ then the diameter of A, denoted by d(A), is defined by $d(A) = \text{l.u.b.}\{d(x, y) / x, y \in A\}$

Note: 1. A non-empty set A is a bounded set iff d(A) is finite.

2. Let $A, B \subseteq M$. Then $A \subseteq B \Rightarrow d(A) \leq d(B)$

Eg: 1. The diameter of any non-empty subset in a discrete metric space is 1.

2. In R, the diameter of any interval is the length of the interval. ie. $d([0,1]) = 1$.

3. In any metric space, $d(\phi) = -\infty$

4. $A = \{1,3, 5, 7, 9\}$, $d(A) = 8$

5. $d(N) = \infty$, $d(Q) = \infty$

Open Ball (Open Sphere) in a Metric Space:

Def: The open ball or the open sphere with centre a and radius r denoted by $B_d(a, r)$ is the subset of M given by $B_d(a, r) = \{x \in M / d(a, x) < r\}$; $B_d(a, r)$ can be written as B (a, r).

Note: 1. B (a, r) is non-empty since it contains atleast its centre a.

2. B (a, r) is a bounded set.

Eg: 1. In IR with usual metric,

$B(a, r) = \{x \in R / |a - x| < r\} = (a - r, a + r)$ is an Interval.

2. C with usal metric, $B(a, r) = \{Z \in C / |Z - a| < r\}$. This is interior of circle with centre a and radius r.

3. In R^2 with usual metric, B (a, r) is the interior of circle with centre a and radius r.

4. Let d be the discrete metric on M. Then $B(a, r) = \begin{cases} M & \text{if } r > 1 \\ \{a\} & \text{if } r \leq 1 \end{cases}$

Eg: In R with usual metric, find B (1/2, 1)

Sol. $B(1/2, 1) = (1/2 - 1, 1 + ½) = (-1/2, 3/2)$

$(B(a,r) = (a - r, a + r))$

SUBSPACE:

Def: Let (M, d) be a metric space. Let M_1 be a non-empty subset of M. Then M_1 is also a metric space with same metric d. We say that (M_1, d) is a subspace of M, d)

Note: If M_1 is a subspace of M, a set which is open in M_1 need not be open in M.

Eg (1): If M = R with usual metric and M_1 = [0, 1] then [0, ½) is open in M_1 but not open in M.

Theorem: Let M be a metric space and M_1 a subspace of M. Let $A_1 \subseteq M_1$. Then A_1 is open in M_1 iff there exists an open set A in M such that $A_1 = A \cap M_1$ (It is true for closed set. Change as closed instead of open)

Eg (2) : Let M = R and $M_1 = [1, 2] \cup [3, 4]$ Then [1,2] is open in M_1 and also, [3, 4] is open in M_1 **[P.G.: 2011 – 12]**

$A_1 = [1, 2]$, then $A_1 = [1, 2] = (1- ½, 2 + ½) \cap M_1 = (+1/2, 5/2) \cap M_1$

$\Rightarrow$ [1, 2] is open in M_1 $\quad$ III^{ly} [3, 4] is open in M_1)

COMPLETE METRIC SPACE:

Def: The metric space M is **complete** if every Cauchy sequence of points in M converges to a point in M. **[P.G.2002-03]**

Eg:
1. IR^1, IR^2, IR_d, R^n, Z are compelte
2. Q is not complete
3. C with usual metric is complete
4. l_2 is complete.
5. Space of all convergent sequences is a complete metric space.

Result: A subspace of a complete metric space need not be complete.

Theorem: A subset A of a complete metric space M is complete iff A is closed. **[Poly: 06-07]**

Eg:
1. Sub set (0, 1] of R is not complete.
2. [0, 1] with usual metric is complete.
3. Let A, B be subsets of R. Then $\overline{A \times B} = \overline{A} \times \overline{B}$
4. If A and B are closed subsets of R, then A x B is a closed subset in R x R.
5. ℓp is complete metric space for any $p \geq 1$ (ie., $\ell_1, \ell_2, \ldots . \ell^{\infty}$)
6. The set {0, 1, ½, 1/3, …. 1/n, …} is complete
7. $[0, 1] \cup$ [[2, 3] is complete subsets of R.
8. Set of all rationals with absolute value metric is **not complete.**
9. (0, 1) with metric R_d is a complete metric space.
10. Every finite metric space is **complete.**
11. If $T : X \to X$ is defined as $Tx = x^2$, where $x = [0. 1/3]$, then T is a contraction on [0, 1/3].

Generalization of the nested interval theorem:

Let (M, d) be a complete metric space for each $n \in I$ let F_n be a closed bounded subset of M $\ni F_1 \supset F_2 \supset \ldots . . \supset F_n \supset F_{n+1} \ldots \ldots$ and

ii) Diam $F_n \to 0$ as $n \to \infty$ then $\bigcap_{n=1}^{\infty} F_n$ contains **precisely one point**

Def: Let (M, d) be a metric space. If $T : M \to M$ we say that T is a **contraction** on M if there exists $\alpha \in IR$ with $o \leq \alpha < 1 \ni d(Tx, Ty) \leq \alpha \, d(x, y)$, $x, y \in M$.

Theorem (1) Every contraction Mapping is continuous

*** (2) (Picard Fixed Point Theorem): [P.G.2002-03]**

If T is a contraction on the complete metric space M, then T has precisely one fixed point (ie., one and only one point x in M $\ni T_x = x$)

Baire's category theorem:

Def: A subset A of a metric space M is said to be **nowhere dense** in M if int $\overline{A} = \phi$

Def: A subset A of a metric space M is said to be of **first category** in M if A can be expressed as a **countable union of nowhere dense sets.**

Def: A set which is not of first category is said to be of **second category.**

Note: If A is of first category then A $= \bigcup_{n=1}^{\infty} E_n$, where E_n is nowhere dense subsets in M.

Eg: 1. In R with usual metric

A = {1, ½, 1/3, …… , 1/n, ……} is nowhere dense.

2. In any discrete metric space M, any non-empty subset A is not nowhere dense (Int. $\overline{A} \neq \phi$)
3. In R with usual metric any finite subset A is nowhere desne.
4. In R with usual metric, any singleton set {x} is nowhere dense.

Note: 1. Any countable subset of R being a countable union of singleton sets is of first category. In particular Q is of first category.

2. If A and B are sets of first category in a metric space M, then $A \cup B$ is also of first category.

Baire's category Theorem:

Any complete metric space is of second category.

Eg: 1. R is of second category.

2. Any discrete Metric space R_d is of second category.
3. [a, b], [a, b),(a, b],(a, b) in IR are of second category.

Note: A metric space which is of second category **need not be complete.**

* Eg: IR – Q (Irrational No.) is of **second category** (but it is not a closed subspace of IR and hence it is **not complete**) **[P.G.06-07]**

Totally Bounded sets:

The subset A of M is totally bounded iff, for every $\in > 0$, A can be covered by a **finite number** of **subsets** of M whose diameters are all less than $\in$.

Eg: (1) (0,1) [0,1] are totally bounded

(2) R and any infinite set with Rd are **not totally bounded.**

(3) bounded sub set E = { e_1 , e_2, …..e_n, …..}. of ℓ^∞ is not totally bounded where e_1 = (1, 0, 0…, 0, ….) etc.

Theorem (1): Let A be a subset of a metric space M. If A is totally bounded then A is bounded.

Note: The converse of the above theorem is not true in general. But, both are one and same in R and R^2.

(2) A metric Space (M) is totally bounded iff every sequence in M has a Cauchy sequence.

(3) A non- empty subset of a totally bounded set is totally bounded.

Results:

1. Any totally bounded metric space is separable
2. Any bounded sequence in R has a convergent sub sequence.
3. The closure of a totally bounded set is totally bounded.

4. Closure of a totally bounded subset of R is compact.
5. Any Cauchy sequence in a metric space is totally bounded.
6. Any bounded infinite subset of R has a limit point.
7. Every finite subset of a metric space is totally bounded.

Def: (continuous)

Let f: $M_1 \to M_2$ be a function. Let a $\in M_1$ and l $\in M_2$. Then $\lim_{x \to a} f(x) = l$.

(f is said to have limit l as x $\to$ a)

Def: f: $M_1 \to M_2$ is continuous at a if $\lim_{x \to a} f(x) = f(a)$.

Results:

1. Let f: $M_1 \to M_2$ be a constant function. Then constant function is continuous.
2. Let M_1 be a discrete metric space and M_2 be any metric space. Then f : $M_1 \to M_2$ is continuous.
3. A function f : $M_1 \to M_2$ is continuous iff $(x_n) \to a \Rightarrow (f(x_n)) \to f(a)$

❖ Theorem: f : $M_1 \to M_2$ is continuous iff $f^{-1}(G)$ is open in M, whenever G is open in M_2.
(f is continuous iff inverse image of every open set is **open**) **[P.G. 06-07]**

❖ Theorem: f : $M_1 \to M_2$ is continuous iff $f^{-1}(F)$ is closed in M_1 whenever F is closed in M_2.
(ie., f is continuous iff inverse image of every closed set is **closed)** **[P.G.06-07]**

Note: 1. Under a continuous map, the image of an open set **need not be an open set**

2. Under a continuous map, the image of an closed set **need not be closed set.**

3. If f is a continuous bijection, f^{-1} need not be continuous. (If f is **compact**, then f^{-1} is **continuous**) **[P.G.03-04]**

Theorem: f : $M_1 \to M_2$ is continuous iff $f(\overline{A}) \subseteq \overline{f(A)}$ $\forall$ A $\subseteq M_1$

Results:1. Let f be a continuous real valued fn. Defined on a metric space M.
Let A = $\{x \in M / f(x) \geq 0\}$. Then A is **closed.**

*2. f : IR $\to$ IR defined by

$$f(x) = \begin{cases} 0 \text{ if x is irrational} \\ 1 \text{ if x is rational, is not continuous.} \end{cases}$$ [P.G.05-06]

3. If f : $M_1 \to M_2$ and g : $M_2 \to M_3$ are continuous functions,

then gof : $M_1 \to M_3$ is also continuous (composition of two continuous function is conti.)

ie., If f is continuous at P and it g is continuous at f(P), then h = g o f is continuous at P.

4. Let f, g be conti. Real valved fn. On a metric space M. Let $A = \{x / x \in M \text{ and } f(x) < g(x)\}$. Then A is **open**.

5. Let f be a function from IR^2 on to IR defined by $f(x, y) = x \quad \forall (x, y) \in IR^2$. Then f is continuous in IR^2.

6. Let $f : M \to R$ and $g : M \to R$ be any two continuous functions. Define

   i) $(fg)\, x = f(x)\, g(x)$ ii) $(cf)\, x = c\, f(x), \; c \in R$

   iii) $(f/g)\, x = \dfrac{f(x)}{g(x)}$ if $\mathbf{g(x) \neq 0} \quad \forall x \in M$.

   Then fg, cf, f/g & f-g, | f | are also continuous.

Ex:

* 1. The function g defined by

$$g(x) = \begin{cases} \dfrac{\sin x}{x}, & x \neq 0 \\ 1, & x = 0 \end{cases}$$

, is continuous at x = 0 **[Poly:05-06]**

* 2. Let f be defined on IR by

$$g(x) = \begin{cases} \dfrac{\sin x}{x}, & x \neq 0 \\ 0, & x = 0 \end{cases}$$

, is not continuous at x = 0.

$\lim_{x \to 0} f(x) \neq f(0)$) Also it is **Removable Discontinuous**.

* 3. $f(x) = \begin{cases} \sin(1/x), & x \neq 0 \\ 1, & x = 0 \end{cases}$, then f is not continuous at x = 0. **[P.G.03-04]**

4. $f(x) = \begin{cases} x \sin(1/x), & x \neq 0 \\ 0, & x = 0 \end{cases}$, is continuous at x = 0

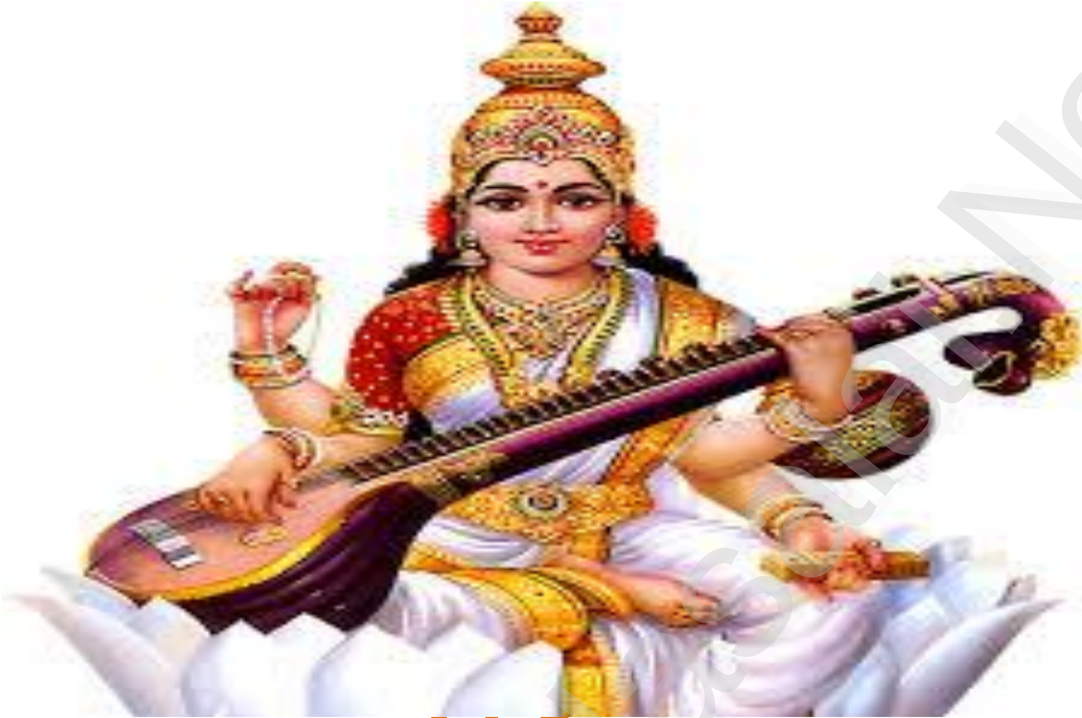
5. $f(x) = \begin{cases} \cos(1/x), & x \neq 0 \\ 0, & x = 0 \end{cases}$, is not continuous at x = 0

6. $f(x) = \begin{cases} x \cos(1/x), & x \neq 0 \\ 0, & x = 0 \end{cases}$, is continuous at x = 0

7. $f : Rd \to Rd^1$ by $f(x) = x$, $-\infty < x < \infty$, then f is continuous, but f^{-1} is not continuous.

8. If g(x) = $\sqrt{x}$, (0 < x < ∞), then g is continuous at each point of (0, ∞)
9. The fn . f(x) = 1/x, 0 < x ≤ 1 is continuous on(0, 1]. **[P.G.11-12]**
10. Every constant fn; exponential fn. Every polynomial fn. of deg. n are also continuous.

KARAIKUDI VIDYA SRI



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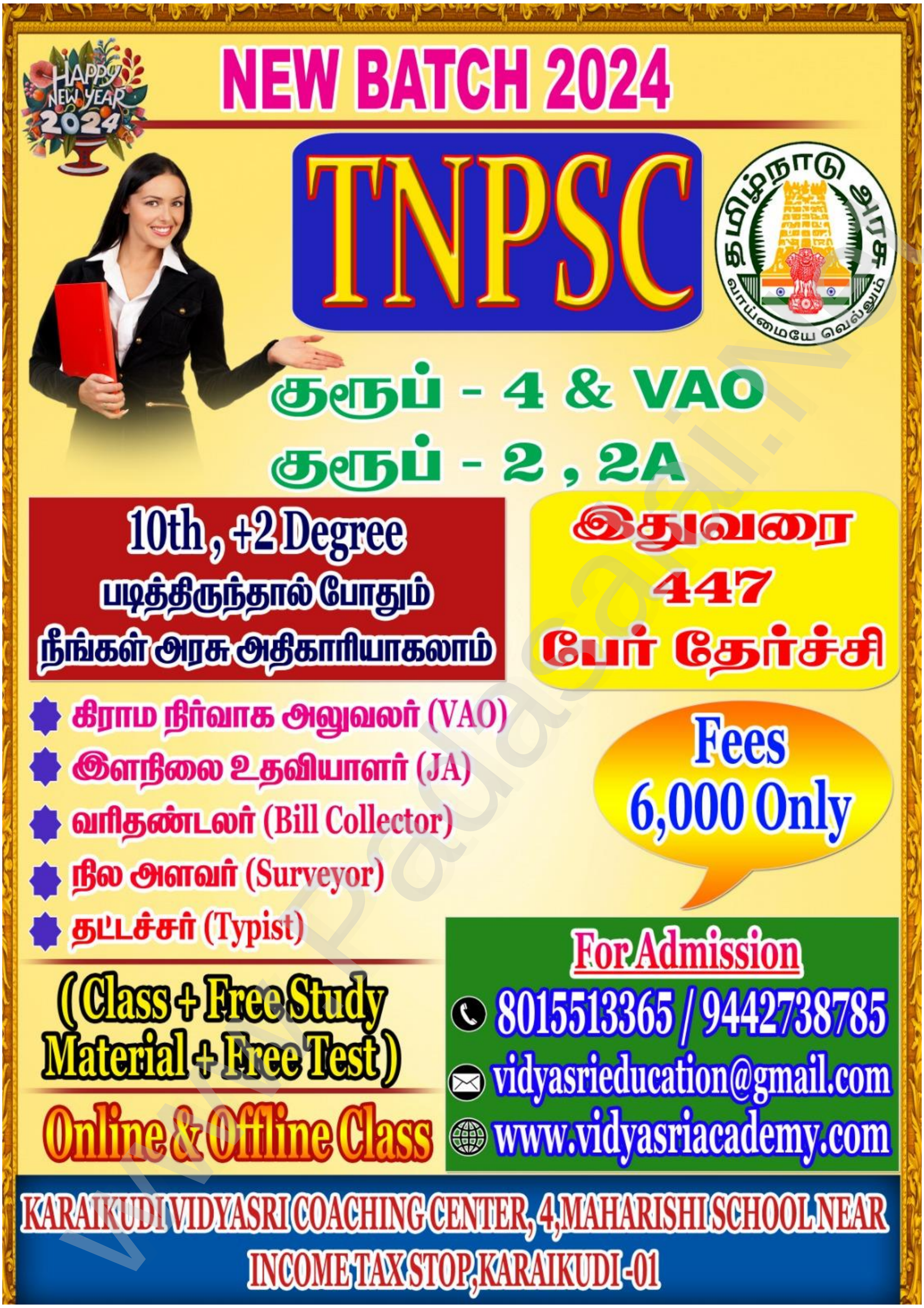
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