+2

Mathematics

MINIMUM LEARNING MATERIAL

பள்ளிக் கல்வித்துறை
விழுப்புரம் மாவட்டம்

(2023-2024)

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முதன்மைக் கல்வி அலுவலர்

விழுப்புரம்.

HIGHER SECONDARY – SECOND YEAR

MATHEMATICS

MINIMUM LEARNING MATERIAL

(2023-2024)

CHIEF EDUCATIONAL OFFICER
VILLUPURAM DISTRICT

+2 கணித பாடத்தில் மெல்ல கற்கும் மாணவர்கள் எளிய முறையில் தேர்ச்சிபெற சிறப்பு பயிற்சி கையேடு

மாணவர்களின் கவனத்திற்கு

- ❖ மாணவர்கள் கணிதத்தில் 100 சதவீதம் தேர்ச்சி பெறும் வகையில் இச்சிறப்பு பயிற்சி கையேடு தயாரிக்கப்பட்டுள்ளது.
- ❖ **Blue Print** இல்லாத காரணத்தினால் அனைத்து அத்தியாயங்களையும் கவனமுடன் படிக்க வேண்டும்
- ❖ இப்பயிற்சி கையேட்டில் உள்ள 5 மதிப்பெண் வினாக்களுக்கான விடைகளை நன்றாக பயிற்சி மேற்கொண்டால் 35/35 மதிப்பெண்கள் எளிதாக பொதுத்தேர்வில் பெறலாம்.
- ❖ பாடப் புத்தகத்தில் ஒவ்வொரு அத்தியாத்திலும் இறுதி பயிற்சியில் உள்ள ஒரு மதிப்பெண் வினாக்கள் மொத்தம் 250க்கும் தீவிர பயிற்சி மேற்கொண்டால் 10 முதல் 15 வரை மதிப்பெண்கள் கண்டிப்பாக பெற்று விடலாம்.
- ❖ தேவையான இடங்களில் படங்கள் வரைய வேண்டும், படத்திற்கு மதிப்பெண்கள் உண்டு.
- ❖ ஒவ்வொரு விடைக்கும் தீர்வு காண உதவும் சூத்திரத்தை அவசியம் எழுத வேண்டும்.
- ❖ தேர்வு நெருங்கும்போது அவசர அவசரமாக கணக்குகளை மனப்பாடம் செய்வதை தவிர்த்து ஆரம்பம் முதலே புரிந்து கணக்குகளை செய்து பார்த்தல் வேண்டும்.
- ❖ தேர்வு எழுதும்போது எவ்வித குழப்பமும் இல்லாமல் எழுதுவது அவசியம்.
- ❖ கேள்வி எண்ணையும், கேள்வியில் உள்ள விவரங்களையும் இரு முறை சரி பார்க்க வேண்டும்.
- ❖ தமிழ்நாடு மாநில பெற்றோர் ஆசிரியர் கழகம் (COME BOOK-PTA) வெளியிட்டுள்ள 6 மாதிரி வினாத்தாள்களுக்கு பயிற்சி மேற்கொண்டு தேர்வு எழுதி பழகினால் கூடுதல் மதிப்பெண் பெறுவது உறுதி.
- ❖ மாணவர்கள் எல்லா வினாக்களுக்கும் முழுமையான பதில் தெரியவில்லை என்றால் கூட விடைத் தெரிந்த அளவிற்கு பதில் அளித்தால் உரிய நிலை மதிப்பெண்களை (Step Mark) எளிதாகப் பெற்று விடலாம்.
- ❖ எனவே நம்பிக்கையோடு தேர்வு எழுதுங்கள் 100 சதவீத தேர்ச்சி பெற்று மாவட்டத்திற்கும் மற்றும் மாநிலத்திற்கும் பெருமை சேர்த்திட வாழ்த்துக்கள்.

இவன்

+2 கணிதவியல் சிறப்பு பயிற்சி கையேடு தொகுப்பாளர் குழு

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1.Applications of Matrices and Determinants

(Important 2 & 3 Marks)

1) If $A = \begin{bmatrix} 8 & -4 \\ -5 & 3 \end{bmatrix}$, verify that $A\,(adjA) = (adjA)\,A = |A|\,I_2$ **(3-Marks) (SEP-2020,JUL-2022)**

Sol:

$A = \begin{bmatrix} 8 & -4 \\ -5 & 3 \end{bmatrix}$, adj A = $\begin{bmatrix} 3 & 4 \\ 5 & 8 \end{bmatrix}$

$|A| = \begin{vmatrix} 8 & -4 \\ -5 & 3 \end{vmatrix} = 24 - 20 = 4$

$A\,(adj\ A) = \begin{bmatrix} 8 & -4 \\ -5 & 3 \end{bmatrix}\begin{bmatrix} 3 & 4 \\ 5 & 8 \end{bmatrix}$

$= \begin{bmatrix} 4 & 0 \\ 0 & 4 \end{bmatrix} = |A|\,I$

$(adj\ A)A = \begin{bmatrix} 3 & 4 \\ 5 & 8 \end{bmatrix}\begin{bmatrix} 8 & -4 \\ -5 & 3 \end{bmatrix}$

$= \begin{bmatrix} 4 & 0 \\ 0 & 4 \end{bmatrix} = |A|\,I$

Hence, $A\,(adj\ A) = (adj\ A)\,A = |A|\,I$

2) If $A = \begin{bmatrix} 1 & 3 \\ 2 & -5 \end{bmatrix}$ verify that $A\,(adjA) = (adjA)\,A = |A|\,I_2$ **(3-Marks)**

Sol:

$A = \begin{bmatrix} 1 & 3 \\ 2 & -5 \end{bmatrix}$, adj A = $\begin{bmatrix} -5 & -3 \\ -2 & 1 \end{bmatrix}$

$|A| = \begin{vmatrix} 1 & 3 \\ 2 & -5 \end{vmatrix} = -5 - 6 = -11$

$A\,(adj\ A) = \begin{bmatrix} 1 & 3 \\ 2 & -5 \end{bmatrix}\begin{bmatrix} -5 & -3 \\ -2 & 1 \end{bmatrix}$

$= \begin{bmatrix} -11 & 0 \\ 0 & -11 \end{bmatrix} = |A|\,I$

$(adj\ A)A = \begin{bmatrix} -5 & -3 \\ -2 & 1 \end{bmatrix}\begin{bmatrix} 1 & 3 \\ 2 & -5 \end{bmatrix}$

$= \begin{bmatrix} -11 & 0 \\ 0 & -11 \end{bmatrix} = |A|\,I$

Hence, $A\,(adj\ A) = (adj\ A)\,A = |A|\,I$

3) If $A = \begin{bmatrix} 2 & 3 & 1 \\ 3 & 4 & 1 \\ 3 & 7 & 2 \end{bmatrix}$ then find adjoint of A. **(2-Marks)**

Sol:

$A_C = \begin{bmatrix} 1 & -3 & 9 \\ 1 & 1 & -5 \\ -1 & 1 & -1 \end{bmatrix}$

$adj\ A = \begin{bmatrix} 1 & 1 & -1 \\ -3 & 1 & 1 \\ 9 & -5 & -1 \end{bmatrix}$

4) If $A = \frac{1}{9}\begin{bmatrix} -8 & 1 & 4 \\ 4 & 4 & 7 \\ 1 & -8 & 4 \end{bmatrix}$, prove that $A^{-1} = A^T$. **(3-Marks)**

Sol: To prove $A^{-1} = A^T$

$AA^{-1} = AA^T$

It is sufficient to prove $AA^T = I$

$= \frac{1}{9}\begin{bmatrix} -8 & 1 & 4 \\ 4 & 4 & 7 \\ 1 & -8 & 4 \end{bmatrix} \frac{1}{9}\begin{bmatrix} -8 & 4 & 1 \\ 1 & 4 & -8 \\ 4 & 7 & 4 \end{bmatrix}$

$= \frac{1}{81}\begin{vmatrix} 81 & 0 & 0 \\ 0 & 81 & 0 \\ 0 & 0 & 81 \end{vmatrix} =$

$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$

$AA^T = I$

5) If adj(A)= $\begin{bmatrix} 0 & -2 & 0 \\ 6 & 2 & -6 \\ -3 & 0 & 6 \end{bmatrix}$ then find A^{-1}. **(2-Marks) (JUN-2023)**

Sol:

6) If adj(A)= $\begin{bmatrix} -1 & 2 & 2 \\ 1 & 1 & 2 \\ 2 & 2 & 1 \end{bmatrix}$ then find A^{-1} **(2-Marks)**

Sol:

$\lvert adjA\rvert=\begin{vmatrix}0 & -2 & 0\\ 6 & 2 & -6\\ -3 & 0 & 6\end{vmatrix}=36,$ $A^{-1}=\pm\frac{1}{\sqrt{\lvert adjA\rvert}}(adjA)$ $A^{-1}=\pm\frac{1}{\sqrt{36}}\begin{bmatrix}0 & -2 & 0\\ 6 & 2 & -6\\ -3 & 0 & 6\end{bmatrix}$ $A^{-1}=\pm\frac{1}{6}\begin{bmatrix}0 & -2 & 0\\ 6 & 2 & -6\\ -3 & 0 & 6\end{bmatrix}$	$\lvert adjA\rvert=9,$ $A^{-1}=\pm\frac{1}{\sqrt{\lvert adjA\rvert}}(adjA)$ $A^{-1}=\pm\frac{1}{3}\begin{bmatrix}-1 & 2 & 2\\ 1 & 1 & 2\\ 2 & 2 & 1\end{bmatrix}$

7) Find the rank of the matrix: $\begin{bmatrix}1 & -2 & -1 & 0\\ 3 & -6 & -3 & 1\end{bmatrix}$ **(2-Marks)**	**8) Find the rank of the matrix:** $\begin{bmatrix}-1 & 3\\ 4 & -7\\ 3 & -4\end{bmatrix}$ **(2-Marks)**
Sol: $A=\begin{bmatrix}1 & -2 & -1 & 0\\ 3 & -6 & -3 & 1\end{bmatrix}, \begin{vmatrix}-1 & 0\\ -3 & 1\end{vmatrix}=-1\neq 0,$ $\rho(A)=2$	**Sol:** $A=\begin{bmatrix}-1 & 3\\ 4 & -7\\ 3 & -4\end{bmatrix}, \begin{vmatrix}-1 & 3\\ 4 & -7\end{vmatrix}=-5\neq 0,$ $\rho(A)=2$

9) Find the rank of the matrix: $\begin{bmatrix}1 & 1 & 1 & 3\\ 2 & -1 & 3 & 4\\ 5 & -1 & 7 & 11\end{bmatrix}$ **(3-Marks)**	**10) Find the rank of the matrix:** $\begin{bmatrix}3 & -8 & 5 & 2\\ 2 & -5 & 1 & 4\\ -1 & 2 & 3 & -2\end{bmatrix}$ **(3-Marks)**
Sol: $A=\begin{bmatrix}1 & 1 & 1 & 3\\ 2 & -1 & 3 & 4\\ 5 & -1 & 7 & 11\end{bmatrix}$ $\sim\begin{bmatrix}1 & 1 & 1 & 3\\ 0 & -3 & 1 & -2\\ 0 & -6 & 2 & -4\end{bmatrix}\begin{matrix}\\ R_2\to R_2-2R_1\\ R_3\to R_3-5R_1\end{matrix}$ $\sim\begin{bmatrix}1 & 1 & 1 & 3\\ 0 & -3 & 1 & -2\\ 0 & 0 & 0 & 0\end{bmatrix} R_3\to R_3-2R_2$ $\rho(A)=2$	**Sol:** $A=\begin{bmatrix}3 & -8 & 5 & 2\\ 2 & -5 & 1 & 4\\ -1 & 2 & 3 & -2\end{bmatrix}$ என்க. $\sim\begin{bmatrix}-1 & 2 & 3 & -2\\ 2 & -5 & 1 & 4\\ 3 & -8 & 5 & 2\end{bmatrix} R_1\leftrightarrow R_3$ $\sim\begin{bmatrix}-1 & 2 & 3 & -2\\ 0 & -1 & 7 & 0\\ 0 & -2 & 14 & -4\end{bmatrix}\begin{matrix}R_2\to R_2+2R_1\\ R_3\to R_3+3R_1\end{matrix}$ $\sim\begin{bmatrix}-1 & 2 & 3 & -2\\ 0 & -1 & 7 & 0\\ 0 & 0 & 0 & -4\end{bmatrix} R_3\to R_3-2R_2$ $\rho(A)=3$
11). *Prove that* $\begin{bmatrix}cos\theta & -sin\theta\\ sin\theta & cos\theta\end{bmatrix}$ *is orthogonal.* **(2-Marks) (MAR-2023)**	**12) If A=**$\begin{bmatrix}2 & 3\\ 1 & 2\end{bmatrix}$**, and B=**$\begin{bmatrix}4 & 0\\ 2 & 5\end{bmatrix}$**, then find** adj(AB) **(2-Marks)**
Sol: $AA^T=\begin{bmatrix}cos\theta & -sin\theta\\ sin\theta & cos\theta\end{bmatrix}\begin{bmatrix}cos\theta & sin\theta\\ -sin\theta & cos\theta\end{bmatrix}$	**Sol:** $adj(AB)=(adjB)(adjA)$ $=\begin{bmatrix}5 & 0\\ -2 & 4\end{bmatrix}\begin{bmatrix}2 & -3\\ -1 & 2\end{bmatrix}$

$= \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$

AA^T=I , A is orthogonal.

$adj(AB) = \begin{bmatrix} 10 & -15 \\ -8 & 14 \end{bmatrix}$

13)Solve by matrix inversion method:

$2x - y = 8, 3x + 2y = -2$ **(3-Marks)**

Sol:

$$\begin{bmatrix} 2 & -1 \\ 3 & 2 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 8 \\ -2 \end{bmatrix}$$

$$AX = B$$

$$|A| = \begin{vmatrix} 2 & -1 \\ 3 & 2 \end{vmatrix} = 4 + 3 = 7 \neq 0$$

$$\text{adj}A = \begin{bmatrix} 2 & 1 \\ -3 & 2 \end{bmatrix}$$

$$A^{-1} = \frac{1}{|A|}\text{adj}A = \frac{1}{7}\begin{bmatrix} 2 & 1 \\ -3 & 2 \end{bmatrix}$$

$$Solution, X = A^{-1}B$$

$$\begin{pmatrix} x \\ y \end{pmatrix} = \frac{1}{7}\begin{bmatrix} 2 & 1 \\ -3 & 2 \end{bmatrix}\begin{bmatrix} 8 \\ -2 \end{bmatrix} = \frac{1}{7}\begin{bmatrix} 14 \\ -28 \end{bmatrix} = \begin{bmatrix} 2 \\ -4 \end{bmatrix}$$

$\therefore x = 2, y = -4$

14) Solve by matrix inversion method:

$5x + 2y = 3, 3x + 2y = 5$**(3-Marks) (Mar-2022)**

Sol:

$$\begin{bmatrix} 5 & 2 \\ 3 & 2 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 3 \\ 5 \end{bmatrix}$$

$$AX = B$$

$$|A| = \begin{vmatrix} 5 & 2 \\ 3 & 2 \end{vmatrix} = 10 - 6 = 4 \neq 0$$

$$\text{adj}A = \begin{bmatrix} 2 & -2 \\ -3 & 5 \end{bmatrix}$$

$$A^{-1} = \frac{1}{|A|}\text{adj}A = \frac{1}{4}\begin{bmatrix} 2 & -2 \\ -3 & 5 \end{bmatrix}$$

$$Solution, X = A^{-1}B$$

$$X = \begin{pmatrix} x \\ y \end{pmatrix} = \frac{1}{4}\begin{bmatrix} 2 & -2 \\ -3 & 5 \end{bmatrix}\begin{bmatrix} 3 \\ 5 \end{bmatrix} = \frac{1}{4}\begin{bmatrix} -4 \\ 16 \end{bmatrix} = \begin{bmatrix} -1 \\ 4 \end{bmatrix}$$

$\therefore x = -1, y = 4$

15) Solve by using cramer's rule: $5x - 2y + 16 = 0, x + 3y - 7 = 0$ **(3-Marks)**

Sol:

$$\Delta = \begin{vmatrix} 5 & -2 \\ 1 & 3 \end{vmatrix} = 15 + 2 = 17$$

$$\Delta_x = \begin{vmatrix} -16 & -2 \\ 7 & 3 \end{vmatrix} = -48 + 14 = -34$$

$$\Delta_y = \begin{vmatrix} 5 & -16 \\ 1 & 7 \end{vmatrix} = 35 + 16 = 51$$

$$x = \frac{\Delta_x}{\Delta} = -\frac{34}{17} = -2$$

$$y = \frac{\Delta_y}{\Delta} = \frac{51}{17} = 3$$

$\therefore x = -2, y = 3$

16) If adj(A)=$\begin{bmatrix} 1 & 0 & 1 \\ 0 & 2 & 0 \\ -1 & 0 & 1 \end{bmatrix}$ **then find adj(adj(A)).(2-Marks)**

Sol:

$$\text{adj(adj(A))} = \begin{bmatrix} 2 & 0 & 2 \\ 0 & 2 & 0 \\ -2 & 0 & 2 \end{bmatrix}^T = \begin{bmatrix} 2 & 0 & -2 \\ 0 & 2 & 0 \\ 2 & 0 & 2 \end{bmatrix}$$

Do it yourself:

Exercise: $1.1 - 3, 7, 11,\ 1.1 - 1(i)$

Example: 1.8,1.3,1.1

17). If A=$\begin{bmatrix} 0 & -3 \\ 1 & 4 \end{bmatrix}$**, B=**$\begin{bmatrix} -2 & -3 \\ 0 & -1 \end{bmatrix}$**, then verify that** $(AB)^{-1} = B^{-1}A^{-1}$**. (3-Marks)(SEP-2020,JUL-2022)**

Sol:$AB = \begin{bmatrix} 0 & 3 \\ -2 & -7 \end{bmatrix}$, $(AB)^{-1} = \frac{1}{6}\begin{bmatrix} -7 & -3 \\ 2 & 0 \end{bmatrix}$, $(A)^{-1} = \frac{1}{3}\begin{bmatrix} 4 & 3 \\ -1 & 0 \end{bmatrix}$, $(B)^{-1} = \frac{1}{2}\begin{bmatrix} -1 & 3 \\ 0 & -2 \end{bmatrix}$,

$B^{-1}A^{-1}=\frac{1}{6}\begin{bmatrix}-7 & -3\\ 2 & 0\end{bmatrix}, \therefore (AB)^{-1}=B^{-1}A^{-1}$

(Important 5-Marks)

1) $If\ A=\begin{bmatrix}-4 & 4 & 4\\ -7 & 1 & 3\\ 5 & -3 & -1\end{bmatrix}$ **and** $B=\begin{bmatrix}1 & -1 & 1\\ 1 & -2 & -2\\ 2 & 1 & 3\end{bmatrix}$**,find the product AB and** BA**. and hence solve the system of equations** $x-y+z=4, x-2y-2z=9, 2x+y+3z=1$

Sol:

$$AB=\begin{bmatrix}-4 & 4 & 4\\ -7 & 1 & 3\\ 5 & -3 & -1\end{bmatrix}\begin{bmatrix}1 & -1 & 1\\ 1 & -2 & -2\\ 2 & 1 & 3\end{bmatrix}$$

$$=\begin{bmatrix}-4+4+8 & 4-8+4 & -4-8+12\\ -7+1+6 & 7-2+3 & -7-2+9\\ 5-3-2 & -5+6-1 & 5+6-3\end{bmatrix}$$

$$AB=\begin{bmatrix}8 & 0 & 0\\ 0 & 8 & 0\\ 0 & 0 & 8\end{bmatrix}=8\begin{bmatrix}1 & 0 & 0\\ 0 & 1 & 0\\ 0 & 0 & 1\end{bmatrix}$$

$$BA=\begin{bmatrix}1 & -1 & 1\\ 1 & -2 & -2\\ 2 & 1 & 3\end{bmatrix}\begin{bmatrix}-4 & 4 & 4\\ -7 & 1 & 3\\ 5 & -3 & -1\end{bmatrix}$$

$$=8\begin{bmatrix}1 & 0 & 0\\ 0 & 1 & 0\\ 0 & 0 & 1\end{bmatrix}$$

$AB=8I$

$\Rightarrow\left(\frac{1}{8}AB\right)=I$

$\therefore B^{-1}=\frac{1}{8}A$

$$\begin{bmatrix}1 & -1 & 1\\ 1 & -2 & -2\\ 2 & 1 & 3\end{bmatrix}\begin{bmatrix}x\\ y\\ z\end{bmatrix}=\begin{bmatrix}4\\ 9\\ 1\end{bmatrix}$$

$$B\begin{bmatrix}x\\ y\\ z\end{bmatrix}=\begin{bmatrix}4\\ 9\\ 1\end{bmatrix}$$

$$\begin{bmatrix}x\\ y\\ z\end{bmatrix}=B^{-1}\begin{bmatrix}4\\ 9\\ 1\end{bmatrix}=\frac{1}{8}A\begin{bmatrix}4\\ 9\\ 1\end{bmatrix}$$

$$=\frac{1}{8}\begin{bmatrix}-4 & 4 & 4\\ -7 & 1 & 3\\ 5 & -3 & -1\end{bmatrix}\begin{bmatrix}4\\ 9\\ 1\end{bmatrix}$$

2) In a T20 match, CSK needed just 6 runs to win with 1 ball left to go in the last over. The last ball was bowled and the batsman at the crease hit it high up. The ball travelled along a path in a vertical plane and the equation of the path is $y=ax^2+bx+c$ **with respect to xy-coordinate system in the vertical plane and the ball traversed through the points** $(10,8),(20,16),(40,22)$ **can you conclude that CSK won the match? Justify your answer. (All the distances are measured in meters and the meeting point of the plane of the path with the farthest boundary line is (70,0))**

Sol:

$y=ax^2+bx+c$

$It\ passes\ (10,8),(20,16),(40,22)$

$8=100a+10b+c$

$16=400a+20b+c$

$22=1600a+40b+c$

$$\Delta=\begin{vmatrix}100 & 10 & 1\\ 400 & 20 & 1\\ 1600 & 40 & 1\end{vmatrix}=1000\begin{vmatrix}1 & 1 & 1\\ 4 & 2 & 1\\ 16 & 4 & 1\end{vmatrix}$$

$=1000[(2-4)-(4-16)+(16-32)]$

$=1000[-2+12-16]=1000(-6)$

$\Delta=-6000$

$$\Delta_a=\begin{vmatrix}8 & 10 & 1\\ 16 & 20 & 1\\ 22 & 40 & 1\end{vmatrix}=20\begin{vmatrix}4 & 1 & 1\\ 8 & 2 & 1\\ 11 & 4 & 1\end{vmatrix}$$

$=20[4(2-4)-(8-11)+(32-22)]$

$\Delta_a=20[(-8+3+10]=20\times 5=100$

$$\Delta_b=\begin{vmatrix}100 & 8 & 1\\ 400 & 16 & 1\\ 1600 & 22 & 1\end{vmatrix}=200\begin{vmatrix}1 & 4 & 1\\ 4 & 8 & 1\\ 16 & 11 & 1\end{vmatrix}$$

$=200[(8-11)-4(4-16)$
$+(44-128)]$

$\Delta_b=200[-3+48-84=200(-39)$
$=-7800$

$$\Delta_c=\begin{vmatrix}100 & 10 & 8\\ 400 & 20 & 16\\ 1600 & 40 & 22\end{vmatrix}$$

$$=100\times 10\times 2\begin{vmatrix}1 & 1 & 4\\ 4 & 2 & 8\\ 16 & 4 & 11\end{vmatrix}$$

$=2000[(22-32)-(44-128)$
$+4(16-32)]$

$= \frac{1}{8}\begin{bmatrix} -16+36+4 \\ -28+9+3 \\ 20-27-1 \end{bmatrix} = \frac{1}{8}\begin{bmatrix} 24 \\ -16 \\ -8 \end{bmatrix}$

$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 3 \\ -2 \\ -1 \end{bmatrix}$

$\therefore x = 3, y = -2, z = -1$

Do it yourself:

Exercise: 1.3 – 2,1(iv),(iii)
Exercise: 1.4 – 1(iii),5
Example: 1.23,1.27,1.12

3) The upward speed v(t) of a rocket at a time is approximated by $v(t) = at^2 + bt + c$, $0 \le t \le 100$ where a, b, and c are constants. It has been found the speed at times t=3,t=6 and t=9 seconds are respectively,64,133and 208 miles per second respectively. Find the speed at time t=15 seconds.

Sol: by $v(t) = c + bt + at^2$, $0 \le t \le 100$

$v(3) = 64 \Rightarrow c + 3b + 9a = 64$
$v(6) = 133 \Rightarrow c + 6b + 36a = 133$
$v(9) = 208 \Rightarrow c + 9b + 81a = 208$

$[A|B] = \left[\begin{array}{ccc|c} 1 & 3 & 9 & 64 \\ 1 & 6 & 36 & 133 \\ 1 & 9 & 81 & 208 \end{array}\right]$

$\xrightarrow[R_2 \to R_2 - R_1]{R_3 \to R_3 - R_1} \left[\begin{array}{ccc|c} 1 & 3 & 9 & 64 \\ 0 & 3 & 27 & 69 \\ 0 & 6 & 72 & 144 \end{array}\right]$

$\xrightarrow[R_2 \to R_2/3]{R_3 \to R_3/6} \left[\begin{array}{ccc|c} 9 & 3 & 1 & 64 \\ 0 & 1 & 9 & 23 \\ 0 & 1 & 12 & 24 \end{array}\right]$

$\sim \left[\begin{array}{ccc|c} 1 & 3 & 9 & 64 \\ 0 & 1 & 9 & 23 \\ 0 & 0 & 3 & 1 \end{array}\right]$

$3a = 1, a = \frac{1}{3}$

$b + 9a = 23$

$= 2000[-10 + 84 - 64]$
$= 2000 \times 10 = 20000$
$\Delta_c = 20000$

By Cramer's rule, $a = \frac{\Delta_a}{a} = \frac{100}{-6000} = -\frac{1}{60}$

$b = \frac{\Delta_b}{b} = \frac{-7800}{-6000} = \frac{78}{60} = \frac{13}{10}$

$c = \frac{\Delta_c}{c} = \frac{20,000}{-6000} = -\frac{20}{6} = -\frac{10}{3}$

The equation of path is

$y = -\frac{1}{60}x^2 + \frac{13}{10}x - \frac{10}{3}$

$x = 70, \; y = -\frac{70^2}{60} + \frac{13}{10}(70) - \frac{10}{3}$

$y = -\frac{4900}{60} + \frac{910}{60} \times 6 - \frac{10}{3}$

$= \frac{-4900 + 5460 - 200}{60}$

$= \frac{5460 - 5100}{60}$

$= \frac{360}{60} = 6$

when $x = 70$, *we get* $y = 6$, *CSK won.*

4). A boy is walking along the path $y = ax^2 + bx + c$ through the points (-6,8),(-2,-12) and (3,8).He wants to meet his friend at P(7,60).Will he meets his friend?

Sol:

$y = c + bx + ax^2$

It passes $(-6, 8), (-2, -12), and\ (3, 8)$

$c - 6b + 36a = 8$
$c - 2b + 4a = -12$
$c + 3b + 9a = 8$

$[A|B] = \left[\begin{array}{ccc|c} 1 & -6 & 36 & 8 \\ 1 & -2 & 4 & -12 \\ 1 & 3 & 9 & 8 \end{array}\right]$

$\xrightarrow[R_2 \to R_2 - R_1]{R_3 \to R_3 - R_1} \left[\begin{array}{ccc|c} 1 & -6 & 36 & 8 \\ 0 & 4 & -32 & -20 \\ 0 & 9 & -27 & 0 \end{array}\right]$

$R_3 \to \frac{1}{3}R_3$

$\xrightarrow{R_2 \to \frac{1}{4}R_2} \left[\begin{array}{ccc|c} 1 & -6 & 36 & 8 \\ 0 & 1 & -8 & -5 \\ 0 & 1 & -3 & 0 \end{array}\right]$

$\xrightarrow{R_3 \to R_3 - R_2} \left[\begin{array}{ccc|c} 1 & -6 & 1 & 8 \\ 0 & 1 & -8 & -5 \\ 0 & 0 & 5 & 5 \end{array}\right]$

$5a = 5 \Rightarrow a = 1$..............(i)
$b - 8a = -5$....................(ii)
$b - 8 = -5, b = 3$
$c - 6b + 36a = 8$.................(iii)
$(iii) \Rightarrow c - 6(3) + 36(1) = 8$
$c = -10$

$b+9\left(\frac{1}{3}\right)=23, b=20$

$c+3b+9a=64 \Rightarrow c+60+9(\frac{1}{3})=64$

$c+63=64 \Rightarrow c=1$

$\therefore v(t)=\frac{1}{3}t^2+20t+1$

$when\ \ t=15$

$v(15)=\frac{225}{3}+20\times 15+1$

$=75+300+1$
$=376$

$b=3$
$a=1$
The equation is
$\therefore y=x^2+3x-10$
$x=7$, $y=7^2+3(7)-10$
$=49+21-10$
$=49+11$
$=60$
The point $(7,60)$ satisfies this equation. Hence the boy will meet his friend.

5) If ax^2+bx+c is divided by x+3, x-5, and x-1, the remainders are 21, 61 and 9 respectively. Find a, b and c (Use Gaussian elimination method)

Sol:

$c+bx+ax^2=p(x)$

p(-3)=21 ⇒c-3b+9a=21

p(5)=61 ⇒ c+5b+25a=61

p(1)=9⇒c+b+a=9

$$[A|B]=\begin{bmatrix}1 & -3 & 9 & 21\\ 1 & 5 & 25 & 61\\ 1 & 1 & 1 & 9\end{bmatrix}$$

$$\sim\begin{bmatrix}1 & -3 & 9 & 21\\ 0 & 8 & 16 & 40\\ 0 & 4 & -8 & -12\end{bmatrix}(R_2\to R_2-R_1,\ R_3\to R_3-R_1)$$

$$\sim\begin{bmatrix}1 & -3 & 9 & 21\\ 0 & 4 & 8 & 20\\ 0 & 4 & -8 & -12\end{bmatrix}\quad (R_2\to R_2/2)$$

$$\sim\begin{bmatrix}1 & -3 & 9 & 21\\ 0 & 4 & 8 & 20\\ 0 & 0 & 16 & 32\end{bmatrix}\quad (R_3\to\ R_2-R_3)$$

$c+bx+ax^2=p(x)$
16a=32 , a=2
4b+8a=20
4b+8(2)=20, 4b=4, b=1
c-3b+9a=21, c-3(1)+9(2)=21, c-3+18=21, c=6
a=2, b=1, c=6

6) Solve the following system of linear equations, by Gaussian elimination method

2x-2y+3z=2, x+2y-z=3, 3x-y+2z=1

Sol:

$$[A|B]=\begin{bmatrix}2 & -2 & 3 & 2\\ 1 & 2 & -1 & 3\\ 3 & -1 & 2 & 1\end{bmatrix}$$

$$\sim\begin{bmatrix}1 & 2 & -1 & 3\\ 2 & -2 & 3 & 2\\ 3 & -1 & 2 & 1\end{bmatrix}\quad R_1\leftrightarrow R_2$$

$(R_2\to R_2-2R_1,\ R_3\to R_3-3R_1)$

$$\sim\begin{bmatrix}1 & 2 & -1 & 3\\ 0 & -6 & 5 & -4\\ 0 & -7 & 5 & -8\end{bmatrix}$$

$$\sim\begin{bmatrix}1 & 2 & -1 & 3\\ 0 & -6 & 5 & -4\\ 0 & 0 & -5 & -20\end{bmatrix}(R_3\to 6R_3-7R_2)$$

$-5z=-20,$
$z=4$

$-6y=-24,$
$y=4$

$x+2y-z=3,$
$x=-1$

$x=-1, y=4, z=4$

7)Solve $\frac{3}{x}-\frac{4}{y}-\frac{2}{z}-1=0$, $\frac{1}{x}+\frac{2}{y}+\frac{1}{z}-2=0$, $\frac{2}{x}-\frac{5}{y}-\frac{4}{z}+1=0$, by using Cramer's rule.

Sol:

$X=\frac{1}{x}, Y=\frac{1}{y}, Z=\frac{1}{z}$
$3X-4Y-2Z=1$
$X+2Y+Z=2$

8). Solve the following system of equations, using matrix inversion method: $2x_1+3x_2+3x_3=5, x_1-2x_2+x_3=-4, 3x_1-x_2-2x_3=3$

Sol:

$AX=B$

$2X - 5Y - 4Z = -1$

$\Delta = \begin{vmatrix} 3 & -4 & -2 \\ 1 & 2 & 1 \\ 2 & -5 & -4 \end{vmatrix} = -15$

$\Delta_X = \begin{vmatrix} 1 & -4 & -2 \\ 2 & 2 & 1 \\ -1 & -5 & -4 \end{vmatrix} = -15$

$\Delta_Y = \begin{vmatrix} 3 & 1 & -2 \\ 1 & 2 & 1 \\ 2 & -1 & -4 \end{vmatrix} = -5$

$\Delta_Z = \begin{vmatrix} 3 & -4 & 1 \\ 1 & 2 & 2 \\ 2 & -5 & -1 \end{vmatrix} = -5$

$X = \frac{\Delta}{\Delta_X} = \frac{-15}{-15} = 1$

$Y = \frac{\Delta}{\Delta_Y} = \frac{-5}{-15} = \frac{1}{3}$

$Z = \frac{\Delta}{\Delta_Z} = \frac{-5}{-15} = \frac{1}{3}$

$\left(X = \frac{1}{x}, Y = \frac{1}{y}, Z = \frac{1}{z}\right)$

$x = 1, y = 3, z = 3$

$\begin{bmatrix} 2 & 3 & 3 \\ 1 & -2 & 1 \\ 3 & -1 & -2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 5 \\ -4 \\ 3 \end{bmatrix}$

$|A| = \begin{vmatrix} 2 & 3 & 3 \\ 1 & -2 & 1 \\ 3 & -1 & -2 \end{vmatrix} = 40 \neq 0$

$adjA = \begin{bmatrix} 5 & 5 & 5 \\ 3 & -13 & 11 \\ 9 & 1 & -7 \end{bmatrix}^T$

$= \begin{bmatrix} 5 & 3 & 9 \\ 5 & -13 & 1 \\ 5 & 11 & -7 \end{bmatrix}$

$A^{-1} = \frac{1}{|A|} \text{adj}A = \frac{1}{40} \begin{bmatrix} 5 & 3 & 9 \\ 5 & -13 & 1 \\ 5 & 11 & -7 \end{bmatrix}$

$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \frac{1}{40} \begin{bmatrix} 5 & 3 & 9 \\ 5 & -13 & 1 \\ 5 & 11 & -7 \end{bmatrix} \begin{bmatrix} 5 \\ -4 \\ 3 \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \\ -1 \end{bmatrix}$

$x_1 = 1, x_2 = 2, x_3 = -1$

9). **By using Gaussian elimination method, balance the chemical reaction equation:** $C_5H_8 + O_2 \rightarrow CO_2 + H_2O$.

Sol: Let positive

x_1, x_2, x_3 **and** x_4 **positive integers**

$x_1C_5H_8 + x_2O_2 \rightarrow x_3CO_2 + x_4H_2O$ -------❶

the number of carbon atoms

$5x_1 = x_3 \Rightarrow 5x_1 - x_3 = 0$----------------------❷

the number of hydrogen atoms

$8x_1 = 2x_4 \Rightarrow 4x_1 - x_4 = 0$----------------------❸

the number of oxygen atoms

$2x_2 = 2x_3 + x_4$

$\Rightarrow 2x_2 - 2x_3 - x_4 = 0$-------------------------❹

$[A|B] = \left[\begin{array}{cccc|c} 5 & 0 & -1 & 0 & 0 \\ 4 & 0 & 0 & -1 & 0 \\ 0 & 2 & -2 & -1 & 0 \end{array}\right]$

$\xrightarrow{R_1 \leftrightarrow R_2} \left[\begin{array}{cccc|c} 4 & 0 & 0 & -1 & 0 \\ 5 & 0 & -1 & 0 & 0 \\ 0 & 2 & -2 & -1 & 0 \end{array}\right]$

$\xrightarrow{R_2 \leftrightarrow R_3} \left[\begin{array}{cccc|c} 4 & 0 & 0 & -1 & 0 \\ 0 & 2 & -2 & -1 & 0 \\ 5 & 0 & -1 & 0 & 0 \end{array}\right]$

$\xrightarrow{R_3 \rightarrow 4R_3 - 5R_1} \left[\begin{array}{cccc|c} 4 & 0 & 0 & -1 & 0 \\ 0 & 2 & -2 & -1 & 0 \\ 0 & 0 & -4 & 5 & 0 \end{array}\right]$

$\therefore \rho(A) = 3, \rho[A|B] = 3$

$\rho(A) = \rho[A|B] = 3 < 4$

$-4x_3 + 5x_4 = 0$---------------(i)

$2x_2 - 2x_3 - x_4 = 0$---------------(ii)

$4x_1 - x_4 = 0$---------------(iii)

10) **By using Gaussian elimination method, balance the chemical reaction equations:** $C_2H_6 + O_2 \rightarrow H_2O + CO_2$

Sol: Let positive integers

x_1, x_2, x_3 **and** x_4 **positive integers**

$x_1C_2H_6 + x_2O_2 \rightarrow x_3H_2O + x_4CO_2$--------❶

the number of carbon atoms

$2x_1 = x_4 \Rightarrow 2x_1 - x_4 = 0$--------------------❷

the number of hydrogen atoms

$6x_1 = 2x_3 \Rightarrow 3x_1 - x_3 = 0$-------------------❸

the number of oxygen atoms

$2x_2 = x_3 + 2x_4 \Rightarrow 2x_2 - x_3 - 2x_4 = 0$-----
------------------------❹

$[A|B] = \left[\begin{array}{cccc|c} 2 & 0 & 0 & -1 & 0 \\ 3 & 0 & -1 & 0 & 0 \\ 0 & 2 & -1 & -2 & 0 \end{array}\right]$

$\xrightarrow{R_2 \leftrightarrow R_3} \left[\begin{array}{cccc|c} 2 & 0 & 0 & -1 & 0 \\ 0 & 2 & -1 & -2 & 0 \\ 3 & 0 & -1 & 0 & 0 \end{array}\right]$

$\xrightarrow{R_3 \rightarrow 2R_3 - 3R_2} \left[\begin{array}{cccc|c} 2 & 0 & 0 & -1 & 0 \\ 0 & 2 & -1 & -2 & 0 \\ 0 & 0 & -2 & 3 & 0 \end{array}\right]$

$-2x_3 + 3x_4 = 0$---------------(i)

$2x_2 - x_3 - 2x_4 = 0$---------------(ii)

$2x_1 - x_4 = 0$---------------(iii)

$Let\ x_4 =$

$(iii) \Rightarrow 2x_1 - t = 0$

$Let\ x_4 = t$
$(iii) \Rightarrow 4x_1 - t = 0$
$4x_1 = t$
$x_1 = \frac{1}{4}t$
$(i) \Rightarrow -4x_3 - 5t = 0$
$-4x_3 = -5t$
$x_3 = \frac{5}{4}t$
$(ii) 2x_2 - \frac{5}{2}t - t = 0$
$2x_2 = \frac{7}{2}t$
$x_2 = \frac{7}{4}t$
$\therefore x_1 = \frac{t}{4}, x_2 = \frac{7}{4}t, x_3 = \frac{5}{4}t, x_4 = t$
$Let\ t = 4$
$\therefore x_1 = 1, x_2 = 7, x_3 = 5, x_4 = 4$
❶ $\Rightarrow C_5H_8 + 7O_2 \rightarrow 5CO_2 + 4H_2O$

$2x_1 = t$
$x_1 = \frac{1}{2}t$
$(i) \Rightarrow -2x_3 + 3t = 0$
$2x_3 = 3t$
$x_3 = \frac{3}{2}t$
$(ii) 2x_2 - \frac{3t}{2} - 2t = 0$
$2x_2 = \frac{3t}{2} + 2t$
$x_2 = \frac{7t}{4}$
$Let\ t = 4$
$\therefore x_1 = 2, x_2 = 7, x_3 = 6, x_4 = 4$
❶ $\Rightarrow 2C_2H_6 + 7O_2 \rightarrow 6H_2O + 4CO_2$

11) Investigate for what values of λ and μ the system of linear equations $x + 2y + z = 7, x + y + \lambda z = \mu, x + 3y - 5z = 5$ has (i) no solution (ii) a unique solution (iii) an infinite number of solutions.

Sol: $\begin{bmatrix} 1 & 2 & 1 \\ 1 & 1 & \lambda \\ 1 & 3 & -5 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 7 \\ \mu \\ 5 \end{bmatrix}$

$AX = B$

$$[A|B] = \left[\begin{array}{ccc|c} 1 & 2 & 1 & 7 \\ 1 & 1 & \lambda & \mu \\ 1 & 3 & -5 & 5 \end{array}\right] \xrightarrow{R_2 \leftrightarrow R_3} \left[\begin{array}{ccc|c} 1 & 2 & 1 & 7 \\ 1 & 3 & -5 & 5 \\ 1 & 1 & \lambda & \mu \end{array}\right]$$

$$\xrightarrow[R_3 \to R_3 - R_1]{R_2 \to R_2 - R_1} \left[\begin{array}{ccc|c} 1 & 2 & 1 & 7 \\ 0 & 1 & -6 & -2 \\ 0 & -1 & \lambda - 1 & \mu - 7 \end{array}\right]$$

$$\xrightarrow{R_3 \to R_3 + R_2} \left[\begin{array}{ccc|c} 1 & 2 & 1 & 7 \\ 0 & 1 & -6 & -2 \\ 0 & 0 & \lambda - 7 & \mu - 9 \end{array}\right]$$

(i) no solution
$when\ \lambda = 7, \mu \neq 9$
$\rho(A) = 2, \rho[A|B] = 3$
$\therefore \rho(A) \neq \rho[A|B]$
(ii) a unique solution
$when\ \lambda \neq 7, \mu \in R$
$\rho(A) = 3, \rho[A|B] = 3$
$\therefore \rho(A) = \rho[A|B] = 3$
(iii) an infinite number of solutions
$when\ \lambda = 7, \mu = 9$
$\rho(A) = 2, \rho[A|B] = 2$
$\therefore \rho(A) = \rho[A|B] = 2 < 3$

12). Investigate the values of λ and μ the system of linear equations $2x + 3y + 5z = 9,\ 7x + 3y - 5z = 8,\ 2x + 3y + \lambda z = \mu$, have (i) no solution (ii) a unique solution (iii) an infinite number of solutions

Sol: $\begin{bmatrix} 2 & 3 & 5 \\ 7 & 3 & -5 \\ 2 & 3 & \lambda \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 9 \\ 8 \\ \mu \end{bmatrix}$

$AX = B$

$$[A|B] = \left[\begin{array}{ccc|c} 2 & 3 & 5 & 9 \\ 7 & 3 & -5 & 8 \\ 2 & 3 & \lambda & \mu \end{array}\right]$$

$$\xrightarrow[R_2 \to 2R_2 - 7R_1]{R_3 \to R_3 - R_1} \left[\begin{array}{ccc|c} 2 & 3 & 5 & 9 \\ 0 & -15 & -45 & -47 \\ 0 & 0 & \lambda - 5 & \mu - 9 \end{array}\right]$$

(i) no solution
$When\ \lambda = 5, \mu \neq 9$
$\rho(A) = 2, \rho[A|B] = 3$
$\therefore \rho(A) \neq \rho[A|B]$
(ii) a unique solution
$when\ \lambda \neq 5, \mu \in R$
$\rho(A) = 3, \rho[A|B] = 3$
$\therefore \rho(A) = \rho[A|B] = 3$
(iii) an infinite number of solutions
$when\ \lambda = 5, \mu = 9$
$\rho(A) = 2, \rho[A|B] = 2$
$\therefore \rho(A) = \rho[A|B] = 2 < 3$

13) Find the value of k for which the equations $kx - 2y + z = 1, x - 2ky + z = -2, x - 2y +$

14). Determine the values of λ for which the following system of equations $x + y + 3z = 0, 4x + 3y + \lambda z = 0, 2x +$

$kz = 1$ have (i)no solution (ii)unique solution (iii)infinitely many solution

Sol:

$$\begin{bmatrix} k & -2 & 1 \\ 1 & -2k & 1 \\ 1 & -2 & k \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$$

$AX = B$

$$[A|B] = \left[\begin{array}{ccc|c} k & -2 & 1 & 1 \\ 1 & -2k & 1 & 1 \\ 1 & -2 & k & 1 \end{array}\right]$$

$$\xrightarrow{R_1 \leftrightarrow R_3} \left[\begin{array}{ccc|c} 1 & -2 & 1 & 1 \\ 1 & -2k & 1 & 1 \\ 1 & -2 & k & 1 \end{array}\right]$$

$$\xrightarrow[R_2 \to R_2 - R_1]{R_3 \to R_3 - kR_1} \left[\begin{array}{ccc|c} 1 & -2 & k & 1 \\ 0 & -2k+2 & 1-k & 0 \\ 0 & -2+2k & 1-k^2 & 1-k \end{array}\right]$$

$$\longrightarrow \left[\begin{array}{ccc|c} 1 & -2 & k & 1 \\ 0 & 2(1-k) & 1-k & 0 \\ 0 & 2(k+1) & 1-k^2 & 1-k \end{array}\right]$$

$$\xrightarrow{R_3 \to R_3 + R_2} \left[\begin{array}{ccc|c} 1 & -2 & 1-k & 1 \\ 0 & 2(1-k) & 1-k & 0 \\ 0 & 0 & 2-k-k^2 & 1-k \end{array}\right]$$

$$\longrightarrow \left[\begin{array}{ccc|c} 1 & -2 & k & 1 \\ 0 & 2(1-k) & 1-k & 0 \\ 0 & 0 & (1-k)(k+2) & 1-k \end{array}\right]$$

(i) no solution :

$when\ k = -2$

$\rho(A) = 2, \rho[A|B] = 3$

$\therefore \rho(A) \neq \rho[A|B]$

(ii)unique Solution

$when\ k \neq 1, k \neq -2$

$\rho(A) = 3, \rho[A|B] = 3$

$\therefore \rho(A) = \rho[A|B] = 3$

(iii) an infinite number of solutions

$when\ k = 1$

$\rho(A) = 1, \rho[A|B] = 1$

$\therefore \rho(A) = \rho[A|B] = 1 < 3$

$y + 2z = 0$ has (i) a unique solution (ii) a non-trivial solution.

Sol:

$$\begin{bmatrix} 1 & 1 & 3 \\ 4 & 3 & \lambda \\ 2 & 1 & 2 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$AX = B$

$$[A|B] = \left[\begin{array}{ccc|c} 1 & 1 & 3 & 0 \\ 4 & 3 & \lambda & 0 \\ 2 & 1 & 2 & 0 \end{array}\right]$$

$$\xrightarrow{R_2 \leftrightarrow R_3} \left[\begin{array}{ccc|c} 1 & 1 & 3 & 0 \\ 2 & 1 & 2 & 0 \\ 4 & 3 & \lambda & 0 \end{array}\right]$$

$$\xrightarrow[R_2 \to R_2 - 2R_1]{R_3 \to R_3 - 4R_1} \left[\begin{array}{ccc|c} 1 & 1 & 3 & 0 \\ 0 & -1 & -4 & 0 \\ 0 & -1 & \lambda - 12 & 0 \end{array}\right]$$

$$\xrightarrow{R_3 \to R_3 - R_2} \left[\begin{array}{ccc|c} 1 & 1 & 3 & 0 \\ 0 & -1 & -4 & 0 \\ 0 & 0 & \lambda - 8 & 0 \end{array}\right]$$

(i) a unique solution: For having trivial solution $\lambda \neq 8$

(ii): a non-trivial solution:

For having non-trivial solution $\lambda = 8$

15). If the system of equations $px + by + cz = 0, ax + qy + cz = 0, ax + by + rz = 0$ has a non-trivial solution and $p \neq a, q \neq b, r \neq c$, prove that $\frac{p}{p-a} + \frac{q}{q-b} + \frac{r}{r-c} = 2$

Sol:

$$\begin{vmatrix} p & b & c \\ a & q & c \\ a & b & r \end{vmatrix} = 0$$

$$\begin{vmatrix} p & b & c \\ a-p & q-b & 0 \\ a-p & 0 & r-c \end{vmatrix} = 0 \qquad \begin{matrix} R_2 \to R_2 - R_1 \\ R_3 \to R_3 - R_1 \end{matrix}$$

$$\begin{vmatrix} a & b & c \\ -(a-p) & q-b & 0 \\ -(a-p) & 0 & r-c \end{vmatrix} = 0$$

$$(p-a)(q-b)(r-c) \begin{vmatrix} \frac{p}{p-a} & \frac{b}{q-b} & \frac{c}{r-c} \\ -1 & 1 & 0 \\ -1 & 0 & 1 \end{vmatrix} = 0$$

$(r-c)(p-a)(q-b)\left[\frac{R}{p-a}(1-0)-\frac{b}{q-b}(-1-0)+\frac{c}{r-c}(0+1)\right]=0$

$(r-c)(p-a)(q-b)\left(\frac{R}{p-a}+\frac{b}{q-b}+\frac{c}{r-c}\right)=0$

$\Rightarrow \frac{p}{p-a}+\frac{b}{q-b}+\frac{c}{r-c}=0$

$\frac{p}{p-a}+\frac{q-(q-b)}{q-b}+\frac{c-(r-c)}{r-c}=0, \Rightarrow \frac{p}{p-a}+\frac{q}{q-b}+\frac{r}{r-c}=2$

2.Complex numbers

(Important 2 & 3 Marks)

1). Find the square root of $-6+8i$. (2-Marks)

Sol:

$$\sqrt{a+ib}=\pm\left[\sqrt{\frac{|z|+a}{2}}+i\sqrt{\frac{|z|-a}{2}}\right]$$

$$|z|=10\,, a=-6$$

$$\sqrt{-6+8i}=\pm\left[\sqrt{\frac{10-6}{2}}+i\sqrt{\frac{10+6}{2}}\right]$$

$$\sqrt{-6+8i}=\pm(\sqrt{2}+i2\sqrt{2})$$

Do it yourself:

Find the square root of $4+3i$

2). Find the square root of $(-5-12i)$ (2-Marks)

Sol:

$$\sqrt{a-ib}=\pm\left[\sqrt{\frac{|z|+a}{2}}-i\sqrt{\frac{|z|-a}{2}}\right]$$

$$|z|=13\,, a=-5$$

$$\sqrt{-5-12i}=\pm\left[\sqrt{\frac{13-5}{2}}-i\sqrt{\frac{13+5}{2}}\right]$$

$$\sqrt{-6+8i}=\pm(2-i3)$$

Do it yourself:

Find the square root of $6-8i$ (JUL-2022)

3)Show that $(2+i\sqrt{3})^{10}+(2-i\sqrt{3})^{10}$ is real.(2-Marks)

Sol:

$$z=(2+i\sqrt{3})^{10}+(2-i\sqrt{3})^{10}$$

$$\bar{z}=\overline{(2+i\sqrt{3})}^{10}+\overline{(2-i\sqrt{3})}^{10}$$

$$\bar{z}=(2-i\sqrt{3})^{10}+(2+i\sqrt{3})^{10}$$

$\bar{z}=z \Rightarrow$ z is real.

4).Show $(2+i\sqrt{3})^{10}-(2-i\sqrt{3})^{10}$ is purely imaginary. (2-Marks)

Sol:

$$z=(2+i\sqrt{3})^{10}-(2-i\sqrt{3})^{10}$$

$$\bar{z}=\overline{(2+i\sqrt{3})}^{10}-\overline{(2-i\sqrt{3})}^{10}$$

$$\bar{z}=(2-i\sqrt{3})^{10}-(2+i\sqrt{3})^{10}$$

$\bar{z}=-z \Rightarrow$ z is purely imaginary.

5)Simplify $\left(\frac{1+i}{1-i}\right)^3-\left(\frac{1-i}{1+i}\right)^3$, into rectangular form. (2-Marks) (MAR-2020)

Sol:

$$\frac{1+i}{1-i}=i, \frac{1-i}{1+i}=-i$$

$$\left(\frac{1+i}{1-i}\right)^3-\left(\frac{1-i}{1+i}\right)^3=(i)^3-(-i)^3$$

$$=-i-i=-2i$$

6)If $z_1=3-2i$ and $z_2=6+4i$, find $\frac{z_1}{z_2}$ in the rectangular form. (2-Marks)

Sol:

$$\frac{z_1}{z_2}=\frac{3-2i}{6+4i}=\frac{3-2i}{6+4i}\times\frac{6-4i}{6-4i}$$

$$=\frac{10-24i}{52}$$

Do it yourself:

Exercise 2.4-1,2

7) Show that the points $1, \frac{-1}{2}+i\frac{\sqrt{3}}{2}$, and $\frac{-1}{2}-i\frac{\sqrt{3}}{2}$ are the vertices of an equilateral triangle. **(3-Marks)(MAR-2021)**

Sol:

$z_1 = 1, z_2 = \frac{-1}{2}+i\frac{\sqrt{3}}{2},\ z_3 = \frac{-1}{2}-i\frac{\sqrt{3}}{2}$

$|z_1 - z_2| = \left|1-\left(\frac{-1}{2}+i\frac{\sqrt{3}}{2}\right)\right| = \sqrt{3}$

$|z_2 - z_3| = \left|\left(\frac{-1}{2}+i\frac{\sqrt{3}}{2}\right)-\left(\frac{-1}{2}-i\frac{\sqrt{3}}{2}\right)\right| = \sqrt{3}$

$|z_3 - z_1| = \left|\left(\frac{-1}{2}-i\frac{\sqrt{3}}{2}\right)-1\right| = \sqrt{3}$

It forms an equilateral triangle.

8) Show that the equation $z^2 = \bar{z}$ has three solutions. **(3-Marks)**

Sol:

$z^2 = \bar{z}$

$|z|^2 = |z|$

$|z|(|z|-1) = 0$

$|z| = 0 \Rightarrow z = 0$ (one solution)

$|z| - 1 = 0 \Rightarrow z\bar{z} = 1$

$z^2 = \bar{z} \Rightarrow z^2 = \frac{1}{z}$

$\Rightarrow z^3 = 1$(3,non-zero solution)

Hence, the equation $z^2 = \bar{z}$ has three solutions.

Do it yourself:

Exercise 2.5-9

9) *If* z_1, z_2 *and* z_3 *are complex numbers such that* $|z_1| = |z_2| = |z_3| = |z_1+z_2+z_3| = 1$ *find the value of* $\left|\frac{1}{z_1}+\frac{1}{z_1}+\frac{1}{z_1}\right| = 1$

(3-Marks)

Sol: $|z_1| = |z_2| = |z_3| = 1$

$|z_1| = 1 \Rightarrow |z_1|^2 = 1; \therefore z_1\ \overline{z_1} = 1; z_1 = \frac{1}{\overline{z_1}}$

$|z_2| = 1 \Rightarrow z_2 = \frac{1}{\overline{z_2}}$

$|z_3| = 1 \Rightarrow z_3 = \frac{1}{\overline{z_3}}$

$\left|\frac{1}{z_1}+\frac{1}{z_2}+\frac{1}{z_3}\right| = |\overline{z_1}+\overline{z_2}+\overline{z_3}|$

$= |\overline{z_1+z_2+z_3}| = |z_1+z_2+z_3| = 1$

10) *Find the centre and radius of the circle* $|z-2-i| = 3$ **(2-Marks)**

Sol:

$|z - z_0| = r,$

$|z-(2+i)| = 3$

$Centre = (2+i) = (2,1)$

$Raduis = 3$

Do it yourself:

Show that following equations represent a circle and find its centre and radius.

$(i)|3z-6+12i| = 8$

$(ii)|3z-5+i| = 4$

$(iii)|2z+2-4i| = 2$

11) *If* $|z| = 2$ *show that* $3 \le |z+3+4i| \le 7$ **(3-Marks) (MAR-2023)**

Sol:$||z_1|-|z_2|| \le |\ z_1+z_2| \le |z_1|+|z_2|$

$z_1 = z \Rightarrow |z_1| = |z| = 2$

$z_2 = 3+4i \Rightarrow |z_2| = |3+4i| = 5$

$|2-5| \le |z+3+4i\ | \le 2+5$

$3 \le |z+3+4i| \le 7$

Do it yourself:

$If|z| = 3\ show\ that\ \ 7 \le |z+6-8i| \le 13$

$If|z| = 1\ show\ that\ \ 2 \le |z^2-3| \le 4$

$If|z| = 2\ show\ that\ \ 7 \le |z+6+8i| \le 12$

12) *Find the modus of* $(1-i)^{10}$

(2-Marks)

Sol:$|(1-i)^{10}| = |(1-i)|^{10} = (\sqrt{2})^{10} = 32$

Do it yourself:

Find the modulus of the following:

$(i)\ \frac{2i}{3+4i}$

$(ii)\ \frac{1-i}{3+i}+\frac{4i}{5}$

$(iii)\ \frac{2+i}{-1+2i}$

13)If $z_1 = 2 - i$ and $z_2 = -4 + 3i$, find the inverse of $z_1 z_2$ and $\frac{z_1}{z_2}$. (3-Marks) Sol: $z_1 z_2 = (2-i)(-4+3i) = -5+10i$ $(z_1 z_2)^{-1} = \frac{1}{z_1 z_2}$ $= \frac{1}{-5+10i} \times \frac{-5-10i}{-5-10i}$ $(z_1 z_2)^{-1} = \frac{-1-2i}{25}$ $(\frac{z_1}{z_2})^{-1} = \frac{z_2}{z_1} = \frac{-4+3i}{2-i}$ $= \frac{-4+3i}{2-i} \times \frac{2+i}{2+i}$ $(\frac{z_1}{z_2})^{-1} = \frac{(-11+i2)}{5}$	14)*If* $z = (2+3i)(1-i)$ then find z^{-1}. (2-Marks) (MAR-2023) Sol: $z = (2+3i)(1-i) = 5+i$ $z^{-1} = \frac{1}{z} = \frac{1}{5+i}$ $z^{-1} = \frac{1}{5+i} \times \frac{5-i}{5-i} = \frac{5-i}{26}$ 15) *Solve*: $(z-1)^3 + 8 = 0$ Sol: $(z-1)^3 = -8$ $z - 1 = (-8)^{\frac{1}{3}} = -2(1)^{\frac{1}{3}} = -2(1, w, w^2)$ $z = -1, 1-2\omega, 1-2\omega^2$ <u>Do it yourself:</u> <u>Solve:</u> $z^3 + 27 = 0$ <u>Do it yourself:</u> Exercise 2.4-5,6 Exercise 2.5-2,8

2.Complex numbers

<u>(Important 5- Marks)</u>

1)*Let.* z_1, z_2 and z_3 be complex numbers such that $\lvert z_1\rvert = \lvert z_2\rvert = \lvert z_3\rvert = r > 0$ and $z_1 + z_2 + z_3$ Prove that $\left\lvert\frac{9z_1z_2+4z_1z_3+z_2z_3}{z_1+z_2+z_3}\right\rvert = r$	4)*let* z_1, z_2 and z_3 be complex numbers such that $\lvert z_1\rvert = 1, \lvert z_2\rvert = 2, \lvert z_3\rvert = 3$ and $\lvert z_1+z_2+z_3\rvert = 1$, Prove that $\lvert 9z_1z_2 + 4z_1z_3 + z_2z_3\rvert = 6$
Sol: $\lvert z_1\rvert = r \Rightarrow \lvert z_1\rvert^2 = r^2;\ \therefore z_1\ \overline{z_1} = r^2;\ \ z_1 = \frac{r^2}{\overline{z_1}}$ $\lvert z_2\rvert = r^2 \Rightarrow z_2 = \frac{r^2}{\overline{z_2}}$ $\lvert z_3\rvert = r^2 \Rightarrow z_3 = \frac{r^2}{\overline{z_3}}$ $z_1 + z_2 + z_3 = \frac{r^2}{\overline{z_1}} + \frac{r^2}{\overline{z_2}} + \frac{r^2}{\overline{z_3}}$ $\lvert z_1 + z_2 + z_3\rvert = r^2\left\lvert\frac{\overline{z_2}\ \overline{z_3} + \overline{z_1 z_3} + \overline{z_1 z_2}}{\overline{z_1 z_2 z_3}}\right\rvert$ $= r^2\left\lvert\frac{\overline{z_2 z_3 + z_1 z_2 + z_1 z_2}}{\overline{z_1 z_2 z_3}}\right\rvert$ $= r^2 \frac{\lvert z_1z_2 + z_1z_2 + z_2z_3\rvert}{\lvert z_1\rvert\lvert z_2\rvert\lvert z_3\rvert}$	Sol: $\lvert z_1\rvert = 1 \Rightarrow \lvert z_1\rvert^2 = 1;\ \therefore z_1\ \overline{z_1} = 1;\ z_1 = \frac{1}{\overline{z_1}}$ $\lvert z_2\rvert = 2 \Rightarrow z_2 = \frac{4}{\overline{z_2}}$ $\lvert z_3\rvert = 3 \Rightarrow z_3 = \frac{9}{\overline{z_3}}$ $\lvert z_1 + z_2 + z_3\rvert = \left\lvert\frac{1}{\overline{z_1}} + \frac{4}{\overline{z_2}} + \frac{9}{\overline{z_3}}\right\rvert$ $= \left\lvert\frac{\overline{z_2}\ \overline{z_3} + 4\overline{z_1 z_3} + 9\overline{z_1 z_2}}{\overline{z_1 z_2 z_3}}\right\rvert$ $= \left\lvert\frac{\overline{z_2 z_3 + 4z_1 z_2 + 9z_1 z_2}}{\overline{z_1 z_2 z_3}}\right\rvert$ $= \frac{\lvert 9z_1z_2 + 4z_1z_2 + z_2z_3\rvert}{\lvert z_1\rvert\lvert z_2\rvert\lvert z_3\rvert}$

$$|z_1+z_2+z_3| = r^2\frac{|z_1z_2+z_1z_2+z_2z_3|}{r^3}$$

$$\left|\frac{z_1z_2+z_1z_3+z_2z_3}{z_1+z_2+z_3}\right| = r$$

$$= \frac{|9z_1z_2+4z_1z_2+z_2z_3|}{1\times2\times3}$$

$$\therefore |9z_1z_2+4z_1z_2+z_2z_3| = 6$$

5) *Show that* $\left(\frac{19+9i}{5-3i}\right)^{15} - \left(\frac{8+i}{1+2i}\right)^{15}$ is purely imaginary.

Sol:

$$\frac{19+9i}{5-3i} = \frac{19+9i}{5-3i}\times\frac{5+3i}{5+3i} = \frac{68+102i}{34}$$

$$\frac{19+9i}{5-3i} = 2+3i$$

$$\frac{8+i}{1+2i} = \frac{8+i}{1+2i}\times\frac{1-2i}{1-2i} = \frac{10-15i}{5}$$

$$\frac{8+i}{1+2i} = 2-3i$$

$$z = \left(\frac{19+9i}{5-3i}\right)^{15} - \left(\frac{8+i}{1+2i}\right)^{15}$$

$$z = (2+3i)^{15} - (2-3i)^{15}$$

$$\bar{z} = -(2+3i)^{15} + (2-3i)^{15}$$

$\bar{z} = -z \Rightarrow z$ purely imaginary.

6) *Show that* $\left(\frac{19-7i}{9+i}\right)^{12} + \left(\frac{20-5i}{7-6i}\right)^{12}$ is real.

Sol:

$$\frac{19-7i}{9+i} = \frac{19-7i}{9+i}\times\frac{9-i}{9-i} = \frac{164-82i}{82}$$

$$\frac{19-7i}{9+i} = 2-i$$

$$\frac{20-5i}{7-6i} = \frac{20-5i}{7-6i}\times\frac{7+6i}{7+6i} = \frac{170+85i}{85}$$

$$\frac{20-5i}{7-6i} = 2+i$$

$$z = \left(\frac{19-7i}{9+i}\right)^{12} + \left(\frac{20-5i}{7-6i}\right)^{12}$$

$$z = (2-i)^{12} + (2+i)^{12}$$

$$\bar{z} = (2+i)^{12} + (2-i)^{12}$$

$\bar{z} = z \Rightarrow z$ *is real.*

5). Find the locus of a complex number z = x + iy satisfying $\left|\frac{z-4i}{z+4i}\right| = 1$

Sol:

$$z = x+iy$$

$$\left|\frac{z-4i}{z+4i}\right| = 1 \Rightarrow |z-4i| = |z+4i|$$

$$|x+iy-4i| = |x+iy+4i|$$

$$|x+i(y-4)|^2 = |x+i(y+4)|^2$$

$$x^2+(y-4)^2 = x^2+(y+4)^2$$

$$y = 0$$

16). *Evaluate*: $\sum_{k=1}^{8}\left(cos\frac{2k\pi}{9} + i\,sin\frac{2k\pi}{9}\right)$

(3-Marks)

Sol:

$$\sum_{k=1}^{k=n}\left(cos\frac{2k\pi}{n+1} \pm i\,sin\frac{2k\pi}{n+1}\right) = \mp1$$

$$\sum_{k=1}^{8}\left(cos\frac{2k\pi}{9} + i\,sin\frac{2k\pi}{9}\right) = -1$$

6) *If* $z = x+iy, \arg\left[\frac{z-1}{z+1}\right] = \frac{\pi}{2}$ *then prove that* $x^2+y^2 = 1$

Sol:

$$z = x+iy$$

$$\frac{z-1}{z+1} = \frac{x+iy-1}{x+iy+1} = \frac{(x-1)+iy}{(x+1)+iy}$$

$$= \frac{(x-1)+iy}{(x+1)+iy}\times\frac{(x+1)-iy}{(x+1)-iy}$$

$$= \frac{(x-1)(x+1)+y^2+i[y(x+1)-y(x-1)]}{(x+1)^2+y^2}$$

$$\arg\left[\frac{z-1}{z+1}\right] = \frac{\pi}{2}$$

$$\Rightarrow \tan^{-1}\left[\frac{y(x+1)-y(x-1)}{(x-1)(x+1)+y^2}\right] = \frac{\pi}{2}$$

$$\frac{xy+y-xy+y}{(x-1)(x+1)+y^2} = \tan\frac{\pi}{2}$$

$$\frac{2y}{x^2-1+y^2} = \infty$$

$$\Rightarrow x^2-1+y^2 = 0$$

$$\therefore x^2+y^2 = 1$$

7) If $z = x + iy, \arg\left[\frac{z-i}{z+2}\right] = \frac{\pi}{4}$ then prove that $x^2 + y^2 + 3x - 3y + 2 = 0$.

Sol:

$\frac{z-i}{z+2} = \frac{x+iy-i}{x+iy+2} = \frac{x+i(y-1)}{x+2+iy}$

$= \frac{x+i(y-1)}{x+2+iy} \times \frac{(x+2)-iy}{(x+2)-iy}$

$= \frac{x(x+2)+y(y-1)+i[(y-1)(x+2)-xy]}{(x+2)^2+y^2}$

$\Rightarrow \tan^{-1}\left[\frac{(y-1)(x+2)-xy}{x(x+2)+y(y-1)}\right] = \frac{\pi}{4}$

$\frac{(y-1)(x+2)-xy}{x(x+2)+y(y-1)} = \tan\frac{\pi}{4}$

$\frac{(y-1)(x+2)-xy}{x(x+2)+y(y-1)} = 1$

$(y-1)(x+2) - xy = x(x+2) + y(y-1)$

$xy + 2y - x - 2 - xy = x^2 + 2x + y^2 - y$

$\therefore x^2 + y^2 + 3x - 3y + 2 = 0$

8) If z=x+iy is a complex number such that $Im\left(\frac{2z+1}{iz+1}\right) = 0$, show that the locus of z is $2x^2 + 2y^2 + x - 2y = 0$

Sol:

$z = x + iy$

$\frac{2z+1}{iz+1} = \frac{2(x+iy)+1}{i(x+iy)+1} = \frac{2x+1+i2y}{1-y+ix}$

$= \frac{(2x+1)+i2y}{1-y+ix} \times \frac{(1-y)-ix}{(1-y)-ix}$

$= \frac{(2x+1)(1-y)+2xy+i(2y(1-y)-x(2x+1)}{(1-y)^2+x^2}$

$Im\left[\frac{2z+1}{iz+1}\right] = 0$

$\Rightarrow \frac{2y(1-y)-x(2x+1)}{(1-y)^2+x^2}$

$2y(1-y) - x(2x+1) = 0$

$2y - 2y^2 - 2x^2 - x = 0$

$\Rightarrow 2x^2 + 2y^2 + x - 2y = 0$

9) If. $(x_1 + iy_1)(x_2 + iy_2)(x_3 + iy_3)\cdots(x_n + iy_n) = a + ib$ then prove that

(i) $(x_1{}^2 + y_1{}^2)(x_2{}^2 + y_2{}^2)(x_3{}^2 + y_3{}^2)\cdots(x_n{}^2 + y_n{}^2) = a^2 + b^2$

(ii) $\sum_{r=1}^{n} tan^{-1}\left(\frac{y_r}{x_r}\right) = tan^{-1}\left(\frac{b}{a}\right) + 2k\pi, k \in \mathbb{Z}$

Sol:

(i) $|(x_1 + iy_1)(x_2 + iy_2)(x_3 + iy_3)\cdots(x_n + iy_n)| = |a + ib|$

$|(x_1 + iy_1)||(x_2 + iy_2)|\cdots|(x_n + iy_n)| = |a + ib|$

$(x_1{}^2 + y_1{}^2)(x_2{}^2 + y_2{}^2)(x_3{}^2 + y_3{}^2)\cdots(x_n{}^2 + y_n{}^2) = a^2 + b^2$

(ii) $(x_1 + iy_1)(x_2 + iy_2)(x_3 + iy_3)\cdots(x_n + iy_n) = a + ib$

$arg((x_1 + iy_1)(x_2 + iy_2)(x_3 + iy_3)\cdots(x_n + iy_n)) = arg((a + ib))$

$\sum_{r=1}^{n} tan^{-1}\left(\frac{y_r}{x_r}\right) = tan^{-1}\left(\frac{b}{a}\right) + 2k\pi, k \in \mathbb{Z}$

10) Find all cube roots of $\sqrt{3} + i$.

Sol:

$z = (\sqrt{3} + i)^{\frac{1}{3}}$

$\sqrt{3} + i = r[\cos\theta + i\sin\theta]$

$r = \sqrt{3+1} = \sqrt{4} = 2$

$\theta = \tan^{-1}\left(\frac{y}{x}\right) = \tan^{-1}\left(\frac{1}{\sqrt{3}}\right) = \frac{\pi}{6}$

$\therefore r = 2, \theta = \frac{\pi}{6}$

$\sqrt{3} + i = 2\left[\cos\frac{\pi}{6} + i\sin\frac{\pi}{6}\right]$

$= 2\left[\cos\left(2k\pi + \frac{\pi}{6}\right) + i\sin\left(2k\pi + \frac{\pi}{6}\right)\right]$

$z = (\sqrt{3} + i)^{\frac{1}{3}} = 2^{\frac{1}{3}}\left[\cos\left(\frac{12k+1}{6}\right)\pi + i\sin\left(\frac{12k+1}{6}\right)\pi\right]^{\frac{1}{3}}$

$= 2^{\frac{1}{3}}\left[\cos\left(\frac{12k+1}{18}\right)\pi + i\sin\left(\frac{12k+1}{18}\right)\pi\right]^{\frac{1}{3}}$

$k = 0$ எனில் $z = 2^{\frac{1}{3}}\left[\cos\frac{\pi}{18} + i\sin\frac{\pi}{8}\right]$

$k = 1$ எனில் $z = 2^{\frac{1}{3}}\left[\cos\frac{13\pi}{18} + i\sin\frac{13\pi}{8}\right]$

$k = 2$ எனில் $z = 2^{\frac{1}{3}}\left[\cos\frac{25\pi}{18} + i\sin\frac{25\pi}{8}\right]$

11) If $\cos\alpha + \cos\beta + \cos\gamma = \sin\alpha + \sin\beta + \sin\gamma = 0$ then prove that (i) $\cos 3\alpha + \cos 3\beta + \cos 3\gamma = 3\cos(\alpha + \beta + \gamma)$

(ii) $\sin 3\alpha + \sin 3\beta + \sin 3\gamma = 3\sin(\alpha + \beta + \gamma)$

Sol:

12). $\frac{1+z}{1-z} = cos\,2\,\theta + i\,sin\,2\,\theta$ எனில், $z = i\,tan\,\theta$ என நிறுவுக.

Sol:

$a = \cos\alpha + i\sin\alpha, \quad b = \cos\beta + i\sin\beta$
$c = \cos\gamma + i\sin\gamma$

$If\ a + b + c = 0\ then\ a^3 + b^3 + c^3 = 3abc$

$(\cos\alpha + i\sin\alpha)^3 + (\cos\beta + i\sin\beta)^3 + (\cos\gamma + i\sin\gamma)^3 = 3(\cos\alpha + i\sin\alpha)(\cos\beta + i\sin\beta)(\cos\gamma + i\sin\gamma)$

$\cos 3\alpha + i\sin 3\alpha + \cos\beta + i\sin 3\beta + \cos 3\gamma + i\sin 3\gamma = 3[\cos(\alpha+\beta+\gamma) + i\sin(\alpha+\beta+\gamma)]$

$(\cos 3\alpha + \cos 3\beta + \cos 3\gamma) + i(\sin 3\alpha + \sin 3\beta + \sin 3\gamma) = 3[\cos(\alpha+\beta+\gamma) + i\sin(\alpha+\beta+\gamma)]$

Equating Real and Imaginary parts

$(i)\ \cos 3\alpha + \cos 3\beta + \cos 3\gamma = 3\cos(\alpha+\beta+\gamma)$

$(ii)\ \sin 3\alpha + \sin 3\beta + \sin 3\gamma = 3\sin(\alpha+\beta+\gamma)$

$$\frac{1+z}{1-z} = \cos 2\theta + i\sin 2\theta = e^{i2\theta}$$
$$\frac{1+z}{1-z} = \frac{e^{i\theta}}{e^{-i\theta}}$$
$$\frac{1+z}{1-z} = \frac{\cos\theta + i\sin\theta}{\cos\theta - i\sin\theta}$$
$$\frac{1+z}{1-z} = \frac{1 + i\tan\theta}{1 - i\tan\theta}$$
$$z = i\tan\theta$$

13)*Prove that* : $\left(\frac{\sqrt{3}}{2} + \frac{i}{2}\right)^5 + \left(\frac{\sqrt{3}}{2} - \frac{i}{2}\right)^5 = -\sqrt{3}$

Sol:

$$\left(\frac{\sqrt{3}}{2} + \frac{i}{2}\right)^5 + \left(\frac{\sqrt{3}}{2} - \frac{i}{2}\right)^5$$
$$= \left(\cos\frac{\pi}{6} + i\sin\frac{\pi}{6}\right)^5 + \left(\cos\frac{\pi}{6} - i\sin\frac{\pi}{6}\right)^5$$
$$= 2\cos\frac{5\pi}{6} = 2\cos\left(\pi - \frac{\pi}{6}\right)$$
$$= -2\cos\frac{\pi}{6} = -2 \times \frac{\sqrt{3}}{2}$$
$$= -\sqrt{3} \quad \therefore \left(\frac{\sqrt{3}}{2} + \frac{i}{2}\right)^5 + \left(\frac{\sqrt{3}}{2} - \frac{i}{2}\right)^5 = -\sqrt{3}$$

14)Solve the equation $z^3 + 8i = 0$, where$z \in \mathbb{C}$.

Sol:

$z^3 + 8i = 0 \Rightarrow z^3 - (2i)^3 = 0$

$(z - 2i)(z^2 + 2iz - 4) = 0$

$(z - 2i) = 0$

$z = 2i$

$(z^2 + 2iz - 4) = 0$

$z = \sqrt{3} - i, -\sqrt{3} - i$

$\therefore z$ −ன் மதிப்புகள் $\sqrt{3} - i, 2i, -\sqrt{3} - i$

15).If $2\cos\alpha = x + \frac{1}{x}$ *and* $2\cos\beta = y + \frac{1}{y}$ *then prove that*

(i) $\frac{x^m}{y^n} - \frac{y^n}{x^m} = 2i\sin(m\alpha - n\beta)$

(ii) $x^m y^n + \frac{1}{x^m y^n} = 2\cos(m\alpha + n\beta)$

Sol:

:$x = \cos\alpha + i\sin\alpha, y = \cos\beta + i\sin\beta$

$(i)\ \frac{x^m}{y^n} = \cos(m\alpha - n\beta) + i\sin(m\alpha - n\beta)$

$\frac{y^n}{x^m} = \cos(m\alpha - n\beta) - i\sin(m\alpha - n\beta)$

$\frac{x^m}{y^n} - \frac{y^n}{x^m} = 2i\sin(m\alpha - n\beta)$

$(ii) x^m y^n = \cos(m\alpha + n\beta) + i\sin(m\alpha + n\beta)$

15).Suppose z_1, z_2, and z_3 are the vertices of an equilateral triangle inscribed in the circle $|z| = 2$. If $z_1 = 1 + i\sqrt{3}$, then find z_2 and z_3.

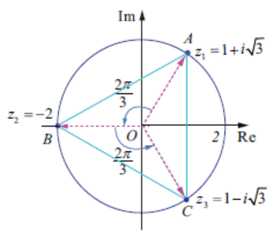
Sol:

Given z_1, z_2, and z_3 are the vertices of an equilateral triangle inscribed in the circle $|z| = 2$.

Vertices of triangle=z, zw, zw^2

$here, z = -2, zw = -2\left(\frac{-1 + i\sqrt{3}}{2}\right) = 1 - i\sqrt{3}$

$zw^2 = -2\left(\frac{-1 - i\sqrt{3}}{2}\right) = 1 + i\sqrt{3}$



$z_1 = 1 + i\sqrt{3}$

$z_2 = -2$

$z_3 = 1 - i\sqrt{3}$

$\frac{1}{x^m y^n} = cos(m\alpha + n\beta) - i\, sin(m\alpha + n\beta)$ $x^m y^n + \frac{1}{x^m y^n} = 2\, cos(m\alpha + n\beta)$		

3.Theory of Equations

(Important 2 & 3 Marks)

1)Find the polynomial equation of minimum degree with rational coefficients, having $2 + i\sqrt{3}$ as a root (2-Marks)	**2)Form a polynomial equation with integer coefficients with $\sqrt{\frac{\sqrt{2}}{\sqrt{3}}}$ as a root. (3-Marks)**
Sol: $\alpha = 2 + i\sqrt{3}, \beta = 2 - i\sqrt{3}$ $\alpha + \beta = 4$ $\alpha\beta = 1$ $x^2 - (\alpha + \beta)x + \alpha\beta = 0$ $x^2 - 4x + 1 = 0$ **Do it yourself:** **Find the polynomial equation of minimum degree with rational coefficients, having $2 - \sqrt{3}$ as a root (2-Marks)**	**Sol:** $x = \sqrt{\frac{\sqrt{2}}{\sqrt{3}}}$ **Squaring,** $x^2 = \frac{\sqrt{2}}{\sqrt{3}}$ **Again squaring,** $x^4 = \frac{2}{3}$ $3x^4 - 2 = 0$

3) Show that the polynomial $9x^9 + 2x^5 - x^4 - 7x^2 + 2$ has at least six imaginary roots. (2-Marks) **Sol:** $f(x) = 9x^9 + 2x^5 - x^4 - 7x^2 + 2$ $f(x) = + + - - +$ Clearly 2 sign changes in f(x), hence number of positive roots of f(x) cannot be more than two. $f(-x) = -9x^9 - 2x^5 - x^4 - 7x^2 + 2$ $f(-x) = - - - - +$ Clearly 1 sign change in f(-x), hence number of negative roots of f(-x) cannot be more than one. It has at least six imaginary roots. **Do it yourself:** **Exercise : 3.6**	**4) If α and β are roots of the equation $17x^2 + 43x - 73 = 0$, construct a quadratic equation whose roots are $\alpha + 2$ and $\beta + 2$. (3-Marks)** **Sol:** A quadratic equation whose roots are $\alpha + 2$ and $\beta + 2$ $\alpha + \beta = -\frac{43}{17}, \alpha\beta = -\frac{73}{17}$ $\alpha + 2 + \beta + 2 = -\frac{43}{17} + 4 = \frac{25}{17}$ $(\alpha + 2)(\beta + 2) = \alpha\beta + 2(\alpha + \beta) + 4$ $= -\frac{73}{17} + 2\left(-\frac{43}{17}\right) + 4$ $= \frac{-91}{17}$ $x^2 - \frac{25}{17}x - \frac{91}{17} = 0$

5) Solve the equation: $x^4 - 14x^2 + 45 = 0$ **(3-Marks)** **Sol:** $x^2 = t, x^4 = t^2$	**6) If α and β are roots of the equation**

$t^2 - 14t^2 + 45 = 0$

$(t-9)(t-5) = 0 \Rightarrow t = 9, 5$

$x^2 = 9 \Rightarrow x = 3, -3$

$x^2 = 5 \Rightarrow x = \sqrt{5}, -\sqrt{5}$

Do it yourself:

Solve the equation: $x^4 - 9x^2 + 20 = 0$

$2x^2 - 7x + 13 = 0$, construct a quadratic equation whose roots are α^2 and β^2 **(3-Marks)**

Sol:

A quadratic equation whose roots are $\alpha + 2$ and $\beta + 2$

$\alpha + \beta = \frac{7}{2}, \alpha\beta = \frac{13}{2}$

$\alpha^2 + \beta^2 = (\alpha+\beta)^2 - 2\alpha\beta = \frac{-3}{4}$

$\alpha^2\beta^2 = \frac{169}{4}$

$x^2 + \frac{3}{4}x + \frac{169}{4} = 0$

7) If p and q are the roots of the equation $lx^2 + nx + n = 0$, show that $\sqrt{\frac{p}{q}} + \sqrt{\frac{q}{p}} + \sqrt{\frac{n}{l}} = 0$ **(3-Marks)**

Sol:

$lx^2 + nx + n = 0$

$p + q = -\frac{n}{l}, pq = \frac{n}{l}$

$\sqrt{\frac{p}{q}} + \sqrt{\frac{q}{p}} + \sqrt{\frac{n}{l}} = \frac{p+q}{\sqrt{pq}} + \sqrt{\frac{n}{l}}$

$\sqrt{\frac{p}{q}} + \sqrt{\frac{q}{p}} + \sqrt{\frac{n}{l}} = \frac{-\frac{n}{l}}{\sqrt{\frac{n}{l}}} + \sqrt{\frac{n}{l}} = 0$

8) If the sides of a cubic box are increased by 1,2,3 units respectively to form a cuboid, the volume is increased by 52 cubic units. Find the volume of the cuboid. **(3-Marks)**

Sol:

The sides of a cubic $= x$

$(x+1)(x+2)(x+3) - x^3 = 52$

$6x^2 + 11x + 6 = 52$

$6x^2 + 11x - 46 = 0$

$x = 2, -\frac{23}{6}$

$x = 2$

The volume of the cuboid $= 60$

9) Construct a cubic equation with roots $1, 2$ *and* 3 **(2-Marks)**

Sol:

A cubic equation $x^3 - S_1x^2 + S_2x - S_3 = 0$

$x^3 - (1+2+3)x^2 + (1.2 + 2.3 + 3.1)x - (1.2.3) = 0$

$x^3 - 6x^2 + 11x - 6 = 0$

Do it yourself:

Exercise 3.1 – 2, 3, *Example*-3.16 3.17, 3.18

(Important 5 Marks)

1) Solve the equation $6x^4 - 35x^3 + 62x^2 - 35x + 6 = 0$

Sol:

$x \neq 0$

$\div x^2 \Rightarrow 6x^2 - 35x + 62 - \frac{35}{x} + \frac{6}{x^2} = 0$

2) Solve the equation $x^4 - 10x^3 + 26x^2 - 10x + 1 = 0$

Sol:

$x \neq 0$

$\div x^2 =>$

$6\left(x^2+\frac{1}{x^2}\right)-35\left(x+\frac{1}{x}\right)+62=0$

$Put, y = x+\frac{1}{x}$

$y^2=\left(x+\frac{1}{x}\right)^2$

$y^2=x^2+\frac{1}{x^2}+2.x.\frac{1}{x}$

$y^2-2=x^2+\frac{1}{x^2}$

$6(y^2-2)-35y+62=0$

$6y^2-12-35y+62=0$

$6y^2-35y+50=0$

$\left(y-\frac{5}{2}\right)\left(y-\frac{10}{3}\right)=0$

$y-\frac{5}{2}=0;\quad y-\frac{10}{3}=0$

$x+\frac{1}{x}=\frac{5}{2}\quad x+\frac{1}{x}=\frac{10}{3}$

$x+\frac{1}{x}=2\frac{1}{2}\quad x+\frac{1}{x}=3\frac{1}{3}$

$x=2,\frac{1}{2}\quad x=3,\frac{1}{3}$

Hence, the roots are $2,\frac{1}{2},3,\frac{1}{3}$

Alternative Method:

	6	-35	62	-35	6
2	0	12	-46	32	-6
	6	-23	16	-3	0
3	0	18	-15	3	
	6	-5	1	0	

$6x^2-5x+1=0$

$(2x-1)(3x-1)=0$

$x=\frac{1}{2},\frac{1}{3}$

: $x=2,\frac{1}{2},3,\frac{1}{3}$

$x^2-10x+26-\frac{10}{x}+\frac{1}{x^2}=0$

$\left(x^2+\frac{1}{x^2}\right)-10\left(x+\frac{1}{x}\right)+26=0$

$Put, y = x+\frac{1}{x}$

$y^2=\left(x+\frac{1}{x}\right)^2$

$y^2-2=x^2+\frac{1}{x^2}$

$(y^2-2)-10(y)+26=0$

$y^2-10y+24=0$

$(y-6)(y-4)=0$

$y-6=0\qquad y-4=0$

$x+\frac{1}{x}-6=0\qquad x+\frac{1}{x}-4=0$

$x^2-6x+1=0\qquad x^2-4x+1$

$(x-3)^2-9+1=0,\qquad (x-2)^2-4+1=0$

$(x-3)^2-8=0\qquad (x-2)^2-3=0$

$(x-3)^2=8\qquad (x-2)^2=3$

$x-3=\pm\sqrt{8}\qquad x-2=\pm\sqrt{3}$

$x=3\pm\sqrt{8}\qquad x=2\pm\sqrt{3}$

Hence, the roots are

$3\pm\sqrt{8}$ and $2\pm\sqrt{3}$

3) Find a polynomial equation of minimum degree with rational coefficients, having $\sqrt{5}-\sqrt{3}$ as a root.

Sol: (Alternative Method)

$x=\sqrt{5}+\sqrt{3}$

Squaring,

$x^2=(\sqrt{5}+\sqrt{3})^2$

$x^2=5+3+2\sqrt{15}$

$x^2=8+2\sqrt{15}$

$x^2-8=2\sqrt{15}$

Again squaring,

$(x^2-8)^2=(2\sqrt{15})^2$

4) $If the equation\ x^2+px+q=0$ and $x^2+p'x+q'=0$ have a common root, show that it must be equal to $\frac{pq'-p'q}{q-q'}$ or $\frac{q-q'}{p'-p}$

Sol:

$x^2+px+q=0$

$x^2+p'x+q'=0$

common root is α

$\alpha^2+p\alpha+q=0$

$\alpha^2+p'\alpha+q'=0$

$\alpha^2\quad \alpha\quad 1$

$p\quad q\quad 1\quad p'$

$p'\quad q'\quad 1\quad p'$

$x^4-16x^2+64=60$
$x^4-16x^2+4=0$

5) *If* $2+i$ and $3-\sqrt{2}$ are the roots of equation $x^6-13x^5+62x^4-126x^4+65x^2+127x-140=0$. Find all roots.

Sol:

Given roots are $2+i, 3-\sqrt{2}$

Other roots will be $2-i, 3+\sqrt{2}, \alpha, \beta$

$\Sigma_1 = \frac{\text{-co-effi of } x^5}{\text{co-effi of } x^6}$

$4+6+\alpha+\beta=13$

$\alpha+\beta=3$……………….❶

$\Sigma_6 = \frac{\text{constant}}{\text{co-effi of } x^6}$

$(2+i)(2-i)(3-\sqrt{5})(3+\sqrt{2})\alpha\beta=-140$

$(4+1)(9-2)\alpha\beta=-140$

$(5)(7)\alpha\beta=-140$

$\alpha\beta=-4$

$x^2-(\alpha+\beta)x+\alpha\beta=0$

$x^2-3x-4=0$

$(x-4)(x+1)=0$

$x=-1,4$

The roots are

$2+i, 2-i, 3-\sqrt{2}, 3+\sqrt{2}, -1, 4.$

7) *Determine k and solve the equation* $2x^3-6x^2+3x+k=0$ if one of its roots is twice the sum of the other two roots.

Sol:

$p(x)=2x^3-6x^2+3x+k=0$

$\alpha, \beta, 2(\alpha+\beta)$

$\Sigma_1 = \frac{\text{co-effi of } x^2}{\text{co-effi of } x^3}$

$\frac{\alpha^2}{pq'-p'q}=\frac{\alpha}{q-q'}=\frac{1}{p'-p}$

$\frac{\alpha^2}{pq'-p'q}=\frac{\alpha}{q-q'} (or) \frac{\alpha}{q-q'}=\frac{1}{p'-p}$

$\alpha=\frac{pq'-qp'}{q-q'} (or) \alpha=\frac{q-q'}{p'-p}$

6) *Find all zeros of polynomial* $x^6-3x^5-5x^4+22x^3-39x^2-39x+135=0$, if it is known that $1+2i$ and $\sqrt{3}$ *are two of its zeros.*

Sol:

Given roots are $1+2i$ and $\sqrt{3}$

Other roots will be

$1+2i, 1-2i, \sqrt{3}, -\sqrt{3}, \alpha, \beta$

$\Sigma_1 = \frac{\text{-co-effi of } x^5}{\text{co-effi of } x^6}$

$1+2i+1-2i+\sqrt{3}-\sqrt{3}+\alpha+\beta=3$

$2+\alpha+\beta=3$

$\alpha+\beta=1$-------------------------❶

$\Sigma_6 = \frac{\text{constant}}{\text{co-effi of } x^6}$

$(1+2i)(1-2i)(\sqrt{3})(-\sqrt{3})\alpha\beta=135$

$(1+4)(-3)\alpha\beta=135$

$-15\alpha\beta=135$

$\alpha\beta=-\frac{135}{15}$

$\alpha\beta=-9$

$x^2-(\alpha+\beta)x+\alpha\beta=0$

$x^2-1x-9=0$

$x=\frac{1\pm\sqrt{37}}{2}$

The roots are

$1+2i, -2i, \sqrt{3}, -\sqrt{3}, \frac{1+\sqrt{37}}{2}, \frac{1-\sqrt{37}}{2}$

8) *Solve the equation* $6x^4-5x^3-38x^2-5x+6=0$ if it is known that $\frac{1}{3}$ is a solution.

Sol:

$6x^4-5x^3-38x^2-5x+6=0,$

Given $\frac{1}{3}$ is a solution for the reciprocal equation. Hence 3 is a solution.

$\alpha+\beta+2(\alpha+\beta)=\frac{6}{2}$
$3\alpha+3\beta=3$
$\alpha+\beta=1$
The roots of p(x) ∵ $\alpha,\beta,2$
$\therefore p(2)=0$
$2(8)-6(4)+3(2)+k=0$
$16-24+6+k=0$
$-2+k=0$
$k=2$
$p(x)=2x^3-6x^2+3x+k=0$

	2	-6	3	2
2	0	4	-4	-2
	2	-2	-1	0

$2x^2-2x-1=0$
$x^2-x-\frac{1}{2}=0$
$\left(x-\frac{1}{2}\right)^2-\frac{1}{4}-\frac{1}{2}=0$
$\left(x-\frac{1}{2}\right)^2=\frac{3}{4}$
$x-\frac{1}{2}=\pm\frac{\sqrt{3}}{2}$
$x=\frac{1}{2}\pm\frac{\sqrt{3}}{2}$
$k=2,\frac{1\pm\sqrt{3}}{2}$

	6	-5	-38	-5	6
1/3	0	2	-1	-13	-6
	6	-3	-39	-18	0
3	0	18	45	18	
	6	15	6	0	

$6x^2+15x+6=0$
$(x+2)\left(x+\frac{1}{2}\right)=0$
$x=-2,x=-\frac{1}{2}$
$x=\frac{1}{3},3,-2,-\frac{1}{2}$
Hence the roots are
$\frac{1}{3},3,-2,-\frac{1}{2}$

9).Solve: $(x-4)(x-7)(x-2)(x+1)=16$

தீர்வு:
$(x-4)(x-2)(x-7)(x+1)=16$
$(x^2-6x+8)(x^2-6x-7)=16$
$x^2-6x=y$ என்க
$(y+8)(y-7)=16$
$y^2-7y+8y-56-16=0$
$y^2+y-72=0$
$(y+9)(y-8)=0$
$(y+9)=0$ $\quad$ $(y-8)=0$
$x^2-6x+9=0$ $\quad$ $x^2-6x-8=0$
$(x-3)\,(x-3)=0$ $\quad$ $(x-3)^2-9-8=0$
$x=3,x=3$ $\quad$ $(x-3)^2=17$
$x-3=\pm\sqrt{17}$
$x=3\pm\sqrt{17}$

Solutions:
$x=3,3\;\;3\pm\sqrt{17}.$

11). Solve: $(x-5)(x-7)(x+6)(x+4)=504$

10).Solve: $(2x-1)(x+3)(x-2)(2x+3)+20=0$

தீர்வு:
$(2x-1)(2x+3)(x+3)(x-2)+20=0$
$(4x^2+4x-3)(x^2+x-6)+20=0$
$(4(x^2+x)-3)(x^2+x-6)+20=0$
$Put\ y=x^2+x$
$(4y-3)(y-6)+20=0$
$4y^2-24y-3y+18+20=0$
$4y^2-27y+38=0$
$(y-2)\left(y-\frac{19}{4}\right)=0$
$y-2=0$ $\quad$ $y-\frac{19}{4}=0$
$x^2+x-2=0$ $\quad$ $x^2+x-\frac{19}{4}=0$
$(x+2)\,(x-1)=0$ $\quad$ $\left(x+\frac{1}{2}\right)^2-\frac{1}{4}-\frac{19}{4}=0$
$x=-2,x=1$ $\quad$ $\left(x+\frac{1}{2}\right)^2=5$
$x+\frac{1}{2}=\pm\sqrt{5}$
$x=\frac{-1\pm2\sqrt{5}}{2}$

Solutions:

Sol: $(x-5)(x+4)(x-7)(x+6)=504$ $(x^2+4x-5x-20)(x^2+6x-7x-42)=504$ $(x^2-x-20)(x^2-x-42)=504$ $x^2-x=y$ எனக $(y-20)(y-42)=504$ $y^2-42y-20y+840-504=0$ $y^2-62y+336=0$ $(y-56)(y-6)=0$ $y-56=0$ $x^2-x-56=0, (x-8)(x+7)=0$ $x=8, x=-7$ $y-6=0, x^2-x-6=0$ $(x-3)(x+2)=0$ $x=3, x=-2$ Solutions: $x=8,-7,3,-2$	$x=1,-2,-\frac{-1\pm2\sqrt{5}}{2}$ **Do it yourself:** Solve the equation $(x-2)(x-7)(x-3)(x+2)+19=0.$ Solve the equation $(2x-3)(6x-1)(3x-2)(x-2)-5=0.$

4.Inverse Trigonometric Functions

(Important 2 & 3 Marks)

1)Find the principal value of $\operatorname{cosec}^{-1}(-\sqrt{2})$ (2-Marks)	2) Find the value of $\tan^{-1}\left(\tan(-\frac{\pi}{6})\right)$ (2-Marks)
Sol: $\operatorname{cosec}^{-1}(-\sqrt{2})=-45°\ (or)\ -\frac{\pi}{4}$ **Do it yourself:** Find the principal value of the following: $\sec^{-1}(\frac{2}{\sqrt{3}})$, $\sin^{-1}(-\frac{1}{2})$, $\cos^{-1}(-\frac{1}{\sqrt{2}})$, $\tan^{-1}(-\sqrt{3})$, $\cos^{-1}(\frac{1}{2})$	Sol: $\tan^{-1}\left(\tan(-\frac{\pi}{6})\right)=-\frac{\pi}{6}\epsilon\left(-\frac{\pi}{2},\frac{\pi}{2}\right)$ **Do it yourself:** Find the value of $\tan^{-1}\left(\tan(\frac{5\pi}{6})\right)$ Find the value of $\tan^{-1}\left(\tan(\frac{3\pi}{5})\right)$

3) Find the value of $\sin^{-1}\left(\sin(\frac{5\pi}{4})\right)$ (2-Marks) Sol: $\frac{5\pi}{4}\notin\left[-\frac{\pi}{2},\frac{\pi}{2}\right]$ $\sin^{-1}\left(\sin(\frac{5\pi}{4})\right)=\sin^{-1}\left(\sin(\pi+\frac{\pi}{4})\right)$ $=\sin^{-1}\left(\sin(-\frac{\pi}{4})\right)=-\frac{\pi}{4}\epsilon\left[-\frac{\pi}{2},\frac{\pi}{2}\right]$ **Do it yourself:** Find the value of $\sin^{-1}\left(\sin(\frac{2\pi}{3})\right)$ Find the value of $\sin^{-1}\left(\sin(\frac{5\pi}{6})\right)$	4) Find the value of $tan(cos^{-1}\left(\frac{1}{2}\right)-\sin^{-1}(-\frac{1}{2}))$ (2-Marks) Sol: $tan(cos^{-1}\left(\frac{1}{2}\right)-\sin^{-1}(-\frac{1}{2}))=$ $\tan\left(cos^{-1}\left(\frac{1}{2}\right)+\sin^{-1}(\frac{1}{2})\right)=tan\frac{\pi}{2}$ $=\infty$ **Do it yourself:** Find the value of $cos^{-1}\left(\frac{1}{2}\right)-sin^{-1}\left(-\frac{1}{2}\right)$ Exercise 4.4 -2

5) Is $cos^{-1}\left[cos\left(-\frac{\pi}{6}\right)\right] \neq -\frac{\pi}{6}$ true? Justify your answer. (2-Marks) Sol: $-\frac{\pi}{6} \notin [-0,\pi]$ $cos^{-1}\left(\cos(-\frac{\pi}{6})\right)$ $= cos^{-1}\left(\cos(\frac{\pi}{6})\right) = \frac{\pi}{6}$ $cos^{-1}\left(\cos(-\frac{\pi}{6})\right) \neq -\frac{\pi}{6}$	6)For what value of x does $\sin x = \sin^{-1} x$? (2-Marks) Sol: $\sin x = \sin^{-1} x$ $\sin 0 = \sin^{-1} 0 = 0$ $x = 0$ Do it yourself: Exercise 4.2 – 7

(Important 5- Marks)

1)Find the value of $cos^{-1}(cos\left(\frac{4\pi}{3}\right)) + cos^{-1}(cos\left(\frac{5\pi}{4}\right))$

Sol:

$cos^{-1}(cos\left(\frac{4\pi}{3}\right)) + cos^{-1}(cos\left(\frac{5\pi}{4}\right))$

$= cos^{-1}(cos\left(\pi+\frac{\pi}{3}\right)) + cos^{-1}(cos\left(\pi+\frac{\pi}{4}\right))$

$(\cos(\pi+\theta) = \cos(\pi-\theta) = -cos\theta)$

$= cos^{-1}(cos\left(\pi-\frac{\pi}{3}\right)) + cos^{-1}(cos\left(\pi-\frac{\pi}{4}\right))$

$= cos^{-1}(cos\left(\frac{2\pi}{3}\right)) + cos^{-1}(cos\left(\frac{3\pi}{4}\right))$

$= \frac{2\pi}{3} + \frac{3\pi}{4} = \frac{17\pi}{12}$

3) Find the value of $tan^{-1}(-1) + cos^{-1}(\frac{1}{2}) + +sin^{-1}(-\frac{1}{2})$

Sol:

$tan^{-1}(-1) + cos^{-1}(\frac{1}{2}) + +sin^{-1}(-\frac{1}{2})$

$= -45° + 60° - 30° = -15°$ (or) $-\frac{\pi}{12}$

$tan^{-1}(-1) + cos^{-1}(\frac{1}{2}) + +sin^{-1}(-\frac{1}{2}) = -\frac{\pi}{12}$

4) Find the value of $cot^{-1}(1) + sin^{-1}(-\frac{\sqrt{3}}{2}) + -sec^{-1}(-\sqrt{2})$

Sol:

$cot^{-1}(1) + sin^{-1}(-\frac{\sqrt{3}}{2}) + -sec^{-1}(-\sqrt{2})$

$= 45° + (-60°) - (180° - 45°) = -150°$ (or) $-\frac{5\pi}{6}$

2) Find the domain of $f(x) = \sin^{-1}\left(\frac{x^2+1}{2x}\right)$

Sol:

$\left|\frac{x^2+1}{2x}\right| \leq 1$, $x \neq 0$

$x^2 + 1 - 2|x| \leq 0$

$(|x| - 1)^2 \leq 0$

$|x| - 1 = 0$

$x = \{-1, 1\}$

5) Find the domain of $f(x) = sin^{-1}\left(\frac{|x|-2}{3}\right) + cos^{-1}\left(\frac{1-|x|}{4}\right)$

Sol:

$-1 \leq \frac{|x|-2}{3} \leq 1$

$-3 \leq |x| - 2 \leq 3$

$-1 \leq |x| \leq 5$

$\therefore |x| \leq 5$

$-5 \leq x \leq 5$ ------------------------------(1)

$-1 \leq \frac{1-|x|}{4} \leq 1$

$-4 \leq 1 - |x| \leq 4$

$-3 \leq |x| \leq 5$------------------------------(2)-

From (1) and (2),

Domain is $-5 \leq x \leq 5$

$cot^{-1}(1) + sin^{-1}(-\frac{\sqrt{3}}{2}) + -sec^{-1}(-\sqrt{2})$
$= -\frac{5\pi}{6}$

Do it yourself:
Exercise 4.3 -4(iii),4(ii)

6).Solve: $\tan^{-1}\left(\frac{x-1}{x-2}\right)+\tan^{-1}\left(\frac{x+1}{x+2}\right)=\frac{\pi}{4}$

Sol: $\tan^{-1}\left(\frac{x-1}{x-2}\right)+\tan^{-1}\left(\frac{x+1}{x+2}\right)=\frac{\pi}{4}$

$$\tan^{-1}\left[\frac{\frac{x-1}{x-2}+\frac{x+1}{x+2}}{1-\left(\frac{x-1}{x-2}\right)\left(\frac{x+1}{x+2}\right)}\right]=\frac{\pi}{4}$$

$$\frac{\frac{x-1}{x-2}+\frac{x+1}{x+2}}{1-\left(\frac{x-1}{x-2}\right)\left(\frac{x+1}{x+2}\right)}=1$$

$$x^2=\frac{1}{2}$$

$$x=\pm\frac{1}{\sqrt{2}}$$

7).Solve: $\cos\left(\sin^{-1}\left(\frac{x}{\sqrt{1+x^2}}\right)\right)=\sin\left\{\cot^{-1}\left(\frac{3}{4}\right)\right\}$

Sol:

$$\cos\left(\sin^{-1}\left(\frac{x}{\sqrt{1+x^2}}\right)\right)=\sin\left\{\cot^{-1}\left(\frac{3}{4}\right)\right\}$$

$$\cos\left(\cos^{-1}\left(\frac{1}{\sqrt{1+x^2}}\right)\right)=\sin\left\{\sin^{-1}\left(\frac{4}{5}\right)\right\}$$

$$\frac{1}{\sqrt{1+x^2}}=\frac{4}{5}$$

$$4\sqrt{1+x^2}=5$$

$$16(1+x^2)=25$$

$$x^2=\frac{9}{16}$$

$$x=\pm\frac{3}{4}$$

8). Solve: $\sin^{-1}\frac{5}{x}+\sin^{-1}\frac{12}{x}=\frac{\pi}{2}$

$$\sin^{-1}\left(\frac{5}{x}\right)+\sin^{-1}\left(\frac{12}{x}\right)=\frac{\pi}{2}$$

$$\sin^{-1}\left(\frac{5}{x}\right)=\frac{\pi}{2}-\sin^{-1}\left(\frac{12}{x}\right)$$

$$\sin^{-1}\left(\frac{5}{x}\right)=\cos^{-1}\left(\frac{12}{x}\right)=\theta$$

$$\therefore \sin\theta=\frac{5}{x};\cos\theta=\frac{12}{x}$$

$$\sin^2\theta+\cos^2\theta=1$$

$$\frac{25}{x^2}+\frac{144}{x^2}=1$$

$$\frac{169}{x^2}=1$$

$$x^2=169\Rightarrow x=\pm 13$$

9). Solve: $2\tan^{-1}(\cos x)=\tan^{-1}(2\operatorname{cosec} x)$

$$2\tan^{-1}(\cos x)=\tan^{-1}(2\operatorname{cosec} x)$$

$$\tan^{-1}\left(\frac{\cos x+\cos x}{1-\cos^2 x}\right)=\tan^{-1}(2\operatorname{cosec} x)$$

$$\tan^{-1}\left(\frac{2\cos x}{\sin^2 x}\right)=\tan^{-1}(2\operatorname{cosec} x)$$

$$\frac{2\cos x}{\sin^2 x}=\frac{2}{\sin x}$$

$$\cos x.\sin x=\sin^2 x$$

$$\cos x.\sin x-\sin^2 x=0$$

$$\sin x\,x(cosx-\sin x)=0$$

$$\sin x=0;\cos x=\sin x$$

$$\therefore x=n\pi;x=n\pi+\frac{\pi}{4},n\in z$$

Do it yourself:
Exercise 4.5 – 10, 9, 5, 6, 8
Exercise 4.2 - 8 (ii), 6(i)

10) If $a_1, a_2, a_3, \dots a_n$ is an arithmetic progression with common difference d, then prove that

$$tan\left[tan^{-1}\left(\frac{d}{1+a_1a_2}\right)+tan^{-1}\left(\frac{d}{1+a_2a_3}\right)+\ldots+tan^{-1}\left(\frac{d}{1+a_na_{n-1}}\right)\right]=\frac{a_n-a_1}{1+a_1a_n}.$$

$$\tan\left[\tan^{-1}\left(\frac{d}{1+a_1a_2}\right)+\tan^{-1}\left(\frac{d}{1+a_2a_3}\right)+\cdots+\tan^{-1}\left(\frac{d}{1+a_na_{n-1}}\right)\right]=\frac{a_n-a_1}{1+a_1a_n}$$

$$d=a_2-a_1=a_3-a_2=\cdots=a_n-a_{n-1}$$

L.H.S:

$$= \tan\left[\tan^{-1}\left(\frac{a_2 - a_1}{1 + a_1 a_2}\right) + \tan^{-1}\left(\frac{a_3 - a_2}{1 + a_2 a_3}\right) + \cdots + \tan^{-1}\left(\frac{a_n - a_{n-1}}{1 + a_n a_{n-1}}\right)\right]$$

$$= \tan[\tan^{-1}(a_2) - \tan^{-1}(a_1) + \tan^{-1}(a_3) - \tan^{-1}(a_2) + \tan^{-1}(a_n) - \tan^{-1}(a_{n-1})]$$

$$= \tan[\tan^{-1}(a_n) - \tan^{-1}(a_1)]$$

$$= \tan\left[\tan^{-1}\left(\frac{a_n - a_1}{1 + a_1 a_n}\right)\right]$$

$$= \left(\frac{a_n - a_1}{1 + a_1 a_n}\right) = R.H.S$$

12) If $\cos^{-1} x + \cos^{-1} y + \cos^{-1} z = \pi$ and $0 < x, y, z < 1$ show that $x^2 + y^2 + z^2 + 2xyz = 1$

$\cos^{-1} x = \alpha; \cos^{-1} y = \beta$

$x = \cos\alpha; y = \cos\beta$

$\therefore \alpha + \beta + \cos^{-1} z = \pi$

$\pi - \cos^{-1} z = \alpha + \beta$

$\cos(\pi - \cos^{-1} y) = \cos(\alpha + \beta)$

$-\cos(\cos^{-1} z) = \cos\alpha\cos\beta - \sin\alpha\sin\beta$

$-z = xy - \sqrt{1 - x^2}\sqrt{1 - y^2}$

$-z - xy = -\sqrt{1 - x^2}\sqrt{1 - y^2}$

$z + xy = \sqrt{1 - x^2}\sqrt{1 - y^2}$

On squaring,

$x^2 + y^2 + z^2 + 2xyz = 1$

13) Solve: $\tan^{-1}(2x) + \tan^{-1}(3x) = \frac{\pi}{4}$, if $6x^2 < 1$

$\tan^{-1}(2x) + \tan^{-1}(3x) = \frac{\pi}{4}$

$\tan^{-1}\left(\frac{2x + 3x}{1 - (2x)(3x)}\right) = \frac{\pi}{4}$

$\frac{5x}{1 - 6x^2} = \tan\frac{\pi}{4}$

$\frac{5x}{1 - 6x^2} = 1$

$6x^2 + 5x - 1 = 0$

$(6x - 1)(x + 1) = -1$

$x = \frac{1}{6}; x = -1$ *not a solution.*

$\therefore x = \frac{1}{6}$

5. Two Dimensional Analytical Geometry-II

(Important 2 & 3 Marks)

1) Find the vertices, foci of the hyperbola $9x^2 - 16y^2 = 144$ (2-Marks)

Sol:

$\frac{x^2}{16} - \frac{y^2}{9} = 1$

Vertices $= (\pm a, 0) = (\pm 4, 0)$

Foci $= (\pm c, 0) = (\pm 5, 0)$

Do it yourself:

Find the equation of directrix and length of latus rectum of the hyperbola $9x^2 - 16y^2 = 144$

2) Find the centre and radius of the circle $2x^2 + 2y^2 - 6x + 4y + 2 = 0$ (2-Marks)

Sol:

$$2x^2 + 2y^2 - 6x - 4y + 2 = 0$$

$\div 2$,

$$x^2 + y^2 - 3x - 2y + 1 = 0$$

$$g = -\frac{3}{2}, f = -1$$

Centre $= (-g, -f) = (\frac{3}{2}, -1)$

Radius $= \sqrt{g^2 + f^2 - c} = \frac{3}{2}$

Do it yourself:

Exercise 5.1 – 11, 12, 8, 2,

Example-5.6, 5.4, 5.12

3)The maximum and minimum distances of the Earth from the Sun respectively are 152×10^6 km and 94.5×10^6 km. The Sun is at one focus of the elliptical orbit. Find the distance from the Sun to the other focus. (3-Marks)

Sol: $AS = 94.5 \text{ x } 10^6$, $SA^1 = 152 \text{ x} 10^6$

$a + c = 152 \text{ x} 10^6$

$a - c = 94.5 \text{ x } 10^6$

Subtracting ,$2c = 57.5 \text{ x} 10^6 = 575 \text{x } 10^5$

Do it yourself:

Example-5.39,5.35,5.34

4)Show that latus rectum of the hyperbola $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$ is $\frac{2b^2}{a}$(3-Marks)

Sol: $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$

$x = ae, \frac{(ae)^2}{a^2} - \frac{y^2}{b^2} = 1 \Rightarrow y = \pm\frac{b^2}{a}$

$L\left(ae, \frac{b^2}{a}\right), L^1\left(ae, -\frac{b^2}{a}\right)$

Length of the latus rectum$(LL^1) = \frac{2b^2}{a}$

Do it yourself:

Example-5.15

(Important 5-Marks)

1)Find the equation of the circle through the points (1,0),(-1,0),and (0,1)

Sol:

The equation of the circle

$x^2 + y^2 + 2gx + 2ty + c = 0$

$(1,0) \Rightarrow 1 + 0 + 2g(1) + 2f(0) + c = 0$

$2g + c = -1 \longrightarrow$ ❶

$(-1,0) \Rightarrow 1 + 0 + 2g(-1) + 2f(0) + c = 0$

$-2g + c = -1 \longrightarrow$ ❷

$(0,1) \Rightarrow 0 + 1 + 2g(0) + 2f(1) + c = 0$

$2f + c = -1 \longrightarrow$ ❸

$(1) + (2) \Rightarrow 2c = -2 \Rightarrow c = -1$

Substitute $c = -1$ in (3)

$2f - 1 = -1$

$2f - 1 = -1 + 1 = 0$

$f = 0$

Substitute $c = -1$ in (1)

$2g - 1 = -1 \Rightarrow 2g = 0$

$\therefore g = 0$

The equation of the circle

$x^2 + y^2 = 1$

Alternative Method:

The equation of the circle passes through the points (1,0),(-1,0),and (0,1) ⇒which form unit circle. That is,

$Centre\ (0,0), Radius = 1$

The equation of the circle is $x^2 + y^2 = 1$

2)Show that the line $x - y + 4 = 0$ is a tangent to the ellipse $x^2 + 3y^2 = 12$. Find the co-ordinates of the point of contact.

Sol:

The ellipse is $x^2 + 3y^2 = 12$

$\frac{x^2}{12} + \frac{y^2}{4} = 1$

$\therefore a^2 = 12 ; b^2 = 4$

Line is $x - y + 4 = 0$

$y = x + 4$

$y = mx + c$

$\therefore m = 1; c = 4$

$Condition\ c^2 = a^2m^2 + b^2$

$16 = 12(1) + 4$

$16 = 16$

Hence, given line is a tangent to the ellipse.

The point of contact $= \left[\frac{-a^2m}{c}, \frac{b^2}{c}\right]$

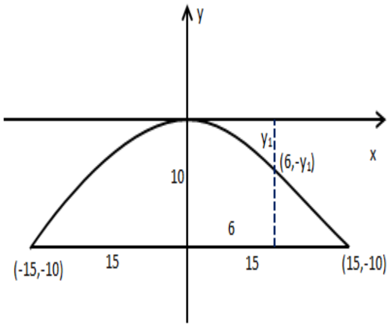
$= \left[-\left(\frac{12(1)}{4}\right), \frac{4}{4}\right]$

$= (-3, 1)$

3)The bridges has a parabolic arch that is 10m high in the centre and 30m wide at the bottom. Find the height of the arch 6m from the centre, on either sides.

Sol:

4)At a water fountain, water attains a maximum height of 4m at horizontal distance of 0.5m from its origin. If the path of water is a parabola, find the height of water at a horizontal distance of 0.75m from the point of origin.



Equation of parabola is

$x^2 = -4ay$

It passes $(15, -10)$

$15^2 = -4a(-10)$

$225 = 40a$

$a = \frac{225}{40}$

$x^2 = -4\left[\frac{225}{40}\right]y$

It passes $(6, -y_1)$

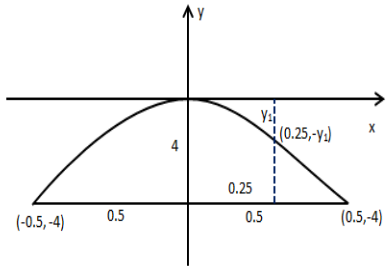
$\frac{36 \times 10}{225} = y_1$

$y_1 = 1.6m$

$\therefore$ *Hieght* $= 10 - y_1$

$\Rightarrow 10 - 1.6 = 8.4m$

Sol:



Equation of parabola is

$x^2 = -4ay$

It passes $(0.5, -4)$

$(0.5)^2 = -4a(-4)$

$\frac{(0.5)^2}{16} = a$

$a = \frac{0.25}{16}$

$x^2 = -4\left[\frac{0.25}{16}\right]y$

It passes $(0.25, -y_1)$

$(0.25)^2 = -4\left[\frac{0.25}{16}\right](-y_1)$

$(0.25)^2 = \frac{0.25}{4}y_1$

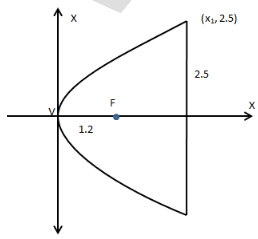
$(0.25)^2 \times \frac{4}{0.25} = y_1$

$y_1 = 1\ m$

Height $= 4 - y_1 \Rightarrow 4 - 1 = 3m$

5) An engineer designs a satellite dish with parabolic cross section. The dish is 5m wide at the opening, and the focus is placed 1.2m from the vertex (a)Position a coordinate system with the origin at the vertex and the x-axis on the parabola's axis of symmetry and find an equation of the parabola. (b)Find the depth of the satellite dish at the vertex.

Sol:

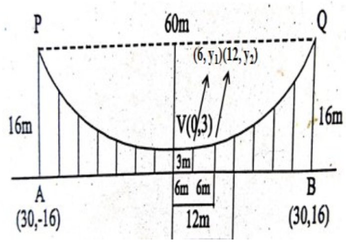


a) $y^2 = 4ax$

$a = 1.2\ m$

6)Parabolic cable of a 60m portion of the roadbed of a suspension bridge are positioned as shown. Vertical Cables are to be spaced every 6m along this portion of the roadbed. Calculate the lengths of first two of these vertical cables from the vertex.

Sol:



Equation of parabola is

$$(x - h)^2 = 4a(y - k)$$

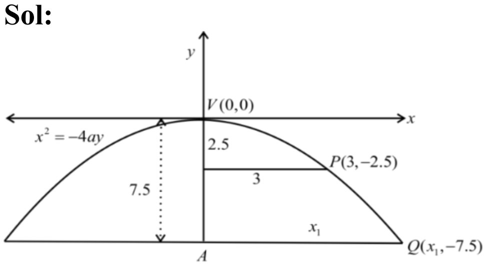
It passes $v\ (0, 3)$

$x^2 = 4a(y - 3)$

$y^2 = 4(1.2)$
$y^2 = 4.8x$
$b)\ (x_1, 2.5)$
$(2.5)^2 = 4(1.2)x_1$
$\frac{(2.5)^2}{4(1.2)} = x_1$
$x_1 = 1.3m$
Depth $= 1.3m.$

7) Assume that water issuing from the end of a horizontal pipe, 7.5m above the ground, describes a parabolic path. The vertex of the parabolic path is at the end of the pipe. At a position 2.5m below the line of the pipe, the flow of water has curved outward 3m beyond the vertical line through the end of the pipe. How far beyond this vertical line will the water strike the ground?

Sol:



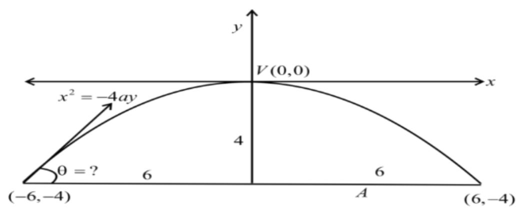
$x^2 = -4ay$
$It\ passes\ (3, -2.5)$
$(3)^2 = -4a(-2.5)$
$9 = 10a$
$a = \frac{9}{10}$
$x^2 = -4\left(\frac{9}{10}\right)y$
$It\ passes (x_1, -7.5)$
$x_1^2 = -4\left(\frac{9}{10}\right)(-7.5)$
$x_1 = \sqrt{3 \times 9}$
$x_1 = 3\sqrt{3}$
∴ water strike the ground=$3\sqrt{3}\ m$

8) A tunnel through a mountain for a highway is to have an semi elliptical opening. The total width of the highway is to be 16m and the height at the edge of the road must be sufficient for a truck 4m high to clear if the highest point of the opening is to be 5m approximately. How wide must the opening be?

$It\ passes (30, 16)$
$(30)^2 = 4a(16 - 3)$
$900 = 13 \times 4a$
$\frac{900}{13 \times 4} = a$
$x^2 = 4\left[\frac{900}{13 \times 4}\right](y - 3)$
$x^2 = 4\left[\frac{900}{13}\right](y - 3)$
$(i)(6, y_1)$
$(6)^2 = \frac{900}{13}(y_1 - 3)$
$\frac{36 \times 13}{900} = y_1 - 3$
$0.52 = y_1 - 3$
$y_1 = 3.52m$
$(ii)(12, y_2)$
$(12)^2 = \frac{900}{13}(y_2 - 3)$
$\frac{144 \times 13}{900} = y_2 - 3$
$2.08 = y_2 - 3$
$y_2 = 5.08m$

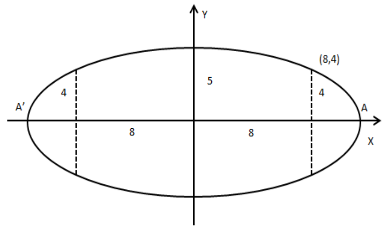
9) On lighting a rocket cracker it gets projected in a parabolic path and reaches a maximum height of 4mts when it is 6 mts away from the point of projection. Finally it reaches the ground 12 mts away from the starting point. Find the angle of projection.

Sol:
Equation of parabola is
$x^2 = -4ay$
$It\ passes\ (6, -4)$



$(6)^2 = 16a$
$\frac{36}{16} = 10a$
$a = \frac{9}{4}$
$x^2 = -4\left(\frac{9}{4}\right)y$
$x^2 = -9y$
$Diff\ wrt\ x$
$2x = -9\frac{dy}{dx}$
$\frac{dy}{dx} = -\frac{2x}{9}$
$\therefore At\ (-6, -4)$

Sol:



The Equation of Ellipse is

$\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$, By given , $b=5$

$\frac{x^2}{a^2}+\frac{y^2}{25}=1,$

It passes $(8,4)$

$\frac{64}{a^2}+\frac{16}{25}=1$

$\frac{64}{a^2}=1-\frac{16}{25}$

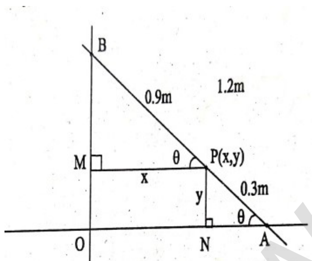
$\Rightarrow a=\frac{40}{3}$

∴The required width, $2a=2\left(\frac{40}{3}\right)=\frac{80}{3}m$

$=26.67m$

10) A rod of length 1.2 m moves with its ends always touching the coordinate axes. The locus of a point P on the rod, which is 0.3m from the end in contact with x-axis an ellipse. Find the eccentricity.

Sol:



$\Delta ANP|||\Delta PMB$

From Δ *ANP*	*From* Δ *PMB*
$\cos\theta=\frac{x_1}{0.9}$	$\sin\theta=\frac{y_1}{0.3}$

Locus of a equation $P(x_1,y_1)$,

$\cos^2\theta+\sin^2\theta=1$

$\frac{(x_1)^2}{(0.9)^2}+\frac{(y_1)^2}{(0.3)^2}=1$

$\frac{x^2}{(0.9)^2}+\frac{y^2}{(0.3)^2}=1$

$a^2=(0.9)^2$

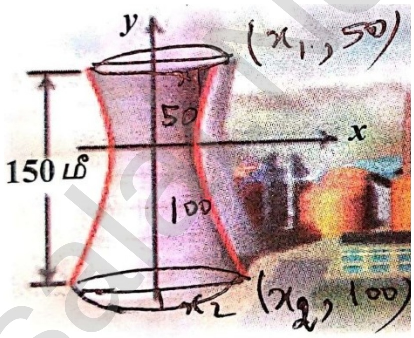
$b^2=(0.3)^2$

The eccentricity $e=\sqrt{\frac{a^2-b^2}{a^2}}=\sqrt{\frac{8}{9}}=\frac{2\sqrt{2}}{3}$

$\tan\theta=-\frac{2(-6)}{9}$

$\tan\theta=\frac{4}{3},\ \theta=\tan^{-1}\left(\frac{4}{3}\right)$

10)Cross section of a Nuclear cooling tower is in the shape of a hyperbola with equation $\frac{x^2}{30^2}-\frac{y^2}{44^2}=1$. The tower is 150$m$ tall and the distance from the top of the tower to the centre of the hyperbola is half the distance from the base of the tower to the centre of the hyperbola. Find the diameter of the top and base of the tower.



Given, hyperbola equation is

$\frac{x^2}{30^2}-\frac{y^2}{44^2}=1$

It passes $(x_1,50)$

$\frac{x_1^2}{30^2}-\frac{50^2}{44^2}=1$

$\frac{x_1^2}{30^2}=1+\frac{50^2}{44^2}$

$x_1=\frac{30}{44}\sqrt{(44)^2+(50)^2}$

$x_1=45.41m$

$2x_1=2(45.41)$

Diameter $=90.82$

It passes $(x_2,100)$

$\frac{x_2^2}{30^2}-\frac{100^2}{44^2}=1$

$x_2=\frac{30}{44}\sqrt{44^2+100^2}$

$x_2=74.49m$

Diameter $2x_2=148.98m$

Do it yourself:

Exercise 5.2-4(iv)(v), 8(v)(vi)

Example:5.19, 5.24, 5.23, 5.10

6.Applications of Vector Algebra

(Important 2 & 3 Marks)

1). Find the angle between the lines 2x=3y=-z and 6x=-y=-4z **(2-Marks)**

Sol:

$$\frac{x}{\frac{1}{2}}=\frac{x}{\frac{1}{3}}=\frac{x}{-1}, \quad \frac{x}{\frac{1}{6}}=\frac{x}{-1}=\frac{x}{-\frac{1}{4}},$$

$$\vec{b}=\left(\frac{1}{2},\frac{1}{3},-1\right), \vec{d}=\left(\frac{1}{6},-1,-\frac{1}{4}\right)$$

$$\vec{b}.\vec{d}=0, \vec{b}\perp\vec{d}$$

The angle between the lines =90°

Do it yourself:

Exercise:$6.4-5$,

Example-6.29,6.30,6.28

2) *Find the angle between the straight line* $\bar{r}=(2\hat{\imath}+3\hat{\jmath}+\hat{k})+t(\hat{\imath}-\hat{\jmath}+\hat{k})$ and the plane. **(2-Marks)**

Sol:

$$\theta=sin^{-1}\left(\frac{\vec{b}.\vec{n}}{|\vec{b}||\vec{n}|}\right)$$

$$\vec{b}=(\hat{\imath}-\hat{\jmath}+\hat{k}), \vec{n}=(2\hat{\imath}-\hat{\jmath}+\hat{k}),$$

$$\theta=sin^{-1}\left(\frac{2\sqrt{2}}{3}\right)$$

Do it yourself:

Exercise:$6.9-4,3$,

Example -6.47

3) Prove that $[\vec{a}-\vec{b}, \vec{b}-\vec{c}, \vec{c}-\vec{a}]=0$. **(2-Marks)**

Sol:

$$[\vec{a}-\vec{b}, \vec{b}-\vec{c}, \vec{c}-\vec{a}]$$

$$=[\vec{a},\vec{b},\vec{c}]\begin{vmatrix}1 & -1 & 0\\ 0 & 1 & -1\\ -1 & 0 & 1\end{vmatrix}$$

$$=[\vec{a},\vec{b},\vec{c}](0)=0$$

$$[\vec{a}-\vec{b}, \vec{b}-\vec{c}, \vec{c}-\vec{a}]=0$$

4) Prove that $[\vec{a}+\vec{b}, \vec{b}+\vec{c}, \vec{c}+\vec{a}]=2\ [\vec{a},\vec{b},\vec{c}]$ **(3-Marks)**

Sol:

$$[\vec{a}+\vec{b}, \vec{b}+\vec{c}, \vec{c}+\vec{a}]$$

$$=[\vec{a},\vec{b},\vec{c}]\begin{vmatrix}1 & 1 & 0\\ 0 & 1 & 1\\ 1 & 0 & 1\end{vmatrix}$$

$$=[\vec{a},\vec{b},\vec{c}](2)$$

$$[\vec{a}+\vec{b}, \vec{b}+\vec{c}, \vec{c}+\vec{a}]=2\ [\vec{a},\vec{b},\vec{c}]$$

Do it yourself:

Example-6.18,6.17

5)Prove that $[\vec{a}\times\vec{b}, \vec{b}\times\vec{c}, \vec{c}\times\vec{a}]=[\vec{a},\vec{b},\vec{c}]^2$ **(3-Marks)**

Sol:

$$[\vec{a}\times\vec{b}, \vec{b}\times\vec{c}, \vec{c}\times\vec{a}]=$$

$$(\vec{a}\times\vec{b}).[(\vec{b}\times\vec{c})\times(\vec{c}\times\vec{a})]$$

$$=(\vec{a}\times\vec{b}).[((\vec{b}\times\vec{c}).\vec{a})\vec{c}-((\vec{b}\times\vec{c}).\vec{c})\vec{a}]$$

$$[\vec{a}\times\vec{b}, \vec{b}\times\vec{c}, \vec{c}\times\vec{a}]=[\vec{a},\vec{b},\vec{c}]^2$$

6)Show that the points$(2,3,4),(-1,4,5)$ *and* $(8,1,4)$are collinear. **(3-Marks)**

Sol:

$$[\vec{a},\vec{b},\vec{c}]=\begin{vmatrix}2 & 3 & 4\\ -1 & 4 & 5\\ 8 & 1 & 4\end{vmatrix}=0$$

Given points are collinear

7) *Show that the vectors* $\hat{\imath}+2\hat{\jmath}-3\hat{k},\ 2\hat{\imath}-\hat{\jmath}+2\hat{k}$ and $3\hat{\imath}+\hat{\jmath}-\hat{k}$ are coplanar.

8) If $2\hat{\imath}-\hat{\jmath}+3\hat{k}, 3\hat{\imath}+2\hat{\jmath}+\hat{k}, \hat{\imath}+m\hat{\jmath}+4\hat{k}$ are coplanar, find the value of m. **(3-Marks)**

(3-Marks)

Sol:

$$[\vec{a},\vec{b},\vec{c}] = \begin{vmatrix} 1 & 2 & -3 \\ 2 & -1 & 2 \\ 3 & 1 & -1 \end{vmatrix} = 0$$

Given vectors are coplanar.

Sol:

$$\begin{vmatrix} 2 & -1 & 3 \\ 3 & 2 & 1 \\ 1 & m & 4 \end{vmatrix} = 0$$

$m = -3$

<u>Do it yourself:</u>

Exercise $6.2 - 6, 7, 8, 9, 10$

9) *Find the volume of the parallepiped whose coterminus edges are given by the vectors* $2\hat{i} - 3\hat{j} + 4\hat{k}$, $\hat{i} + 2\hat{j} - \hat{k}$ and $3\hat{i} - \hat{j} + 2\hat{k}$.

Sol:

$$[\vec{a},\vec{b},\vec{c}] = \begin{vmatrix} 2 & -3 & 4 \\ 1 & 2 & -1 \\ 3 & -1 & 2 \end{vmatrix} = -7$$

Volume of the parallepiped =7

<u>Do it yourself:</u>

<u>EXERCISE $6.3 - 2, 5, 7, 8,$</u>

10) *Find the distance between the parallel palnes* $x + 2y - 2z + 1 = 0$ *and* $2x + 4y - 4z + 5 = 0$

Sol:

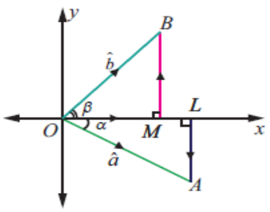
$$Distance = \frac{|d_1 - d_2|}{\sqrt{a^2 + b^2 + c^2}}$$

$a = 1, b = 2, c = -2, d_1 = 1, d_2 = \frac{5}{2}$

$$Distance = \frac{1}{2}$$

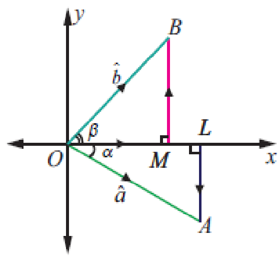
<u>(Important 5 Marks)</u>

1) $\cos(\alpha + \beta) = cos\alpha\ cos\beta - sin\alpha\ sin\beta$



$\hat{a} = cos\alpha\hat{i} - sin\alpha\hat{j}$
$\hat{b} = cos\beta\hat{i} + sin\beta\hat{j}$
$\hat{a}.\hat{b} = \cos(\alpha + \beta) \rightarrow$ ❶
$\hat{a}.\hat{b} = cos\alpha\ cos\beta - sin\alpha\ sin\beta \rightarrow$ ❷
From ❶ and ❷
$\cos(\alpha + \beta) = cos\alpha\ cos\beta - sin\alpha\ sin\beta$

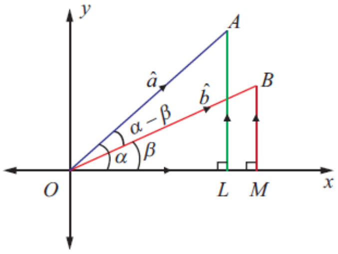
2) $\sin(\alpha + \beta) = sin\alpha\ cos\beta + cos\alpha\ sin$



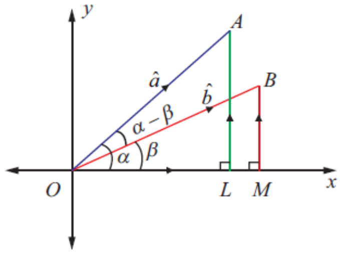
$\hat{a} = cos\alpha\hat{i} - sin\alpha\hat{j}$
$\hat{b} = cos\beta\hat{i} + sin\beta\hat{j}$
$\hat{a} \times \hat{b} = \sin(\alpha + \beta)\hat{k} \rightarrow$ ❶
$\hat{a} \times \hat{b} = (sin\alpha\ cos\beta + cos\alpha\ sin\beta)\hat{k} \rightarrow$ ❷.
From ❶ and ❷
$\sin(\alpha + \beta) = sin\alpha\ cos\beta + cos\alpha\ sin\beta$

3) $\cos(\alpha - \beta) = cos\alpha\ cos\beta + sin\alpha\ sin\beta$

4) $\sin(\alpha - \beta) = sin\alpha\ cos\beta - cos\alpha\ sin\beta$



$\hat{a} = cos\alpha\hat{\imath} + sin\alpha\hat{\jmath}$
$\hat{b} = cos\beta\hat{\imath} + sin\beta\hat{\jmath}$
$\hat{b}.\hat{a} = \cos(\alpha-\beta) \rightarrow$ ❶
$\hat{b}.\hat{a} = cos\alpha\, cos\beta + sin\alpha\, sin\beta \rightarrow$ ❷
From ❶ and ❷
$\cos(\alpha-\beta) = cos\alpha\, cos\beta + sin\alpha\, sin\beta$



$\hat{a} = cos\alpha\hat{\imath} + sin\alpha\hat{\jmath}$
$\hat{b} = cos\beta\hat{\imath} + sin\beta\hat{\jmath}$
$\hat{b}\times\hat{a} = \sin(\alpha-\beta)\,\hat{k} \rightarrow$ ❶.
$\hat{b}\times\hat{a} == (sin\alpha\, cos\beta - cos\alpha\, sin\beta)\hat{k}$
$\rightarrow$ ❷.
From ❶ and ❷
$\sin(\alpha-\beta) = sin\alpha\, cos\beta - cos\alpha\, sin\beta$

Three points:

5) Find the vector and Cartesian form of the equations of the plane passing through the points $(3,6,-2)$, $(-1,-2,6)$ and $(6, 4,-2)$

Sol:

$\vec{a} = 3\hat{\imath} + 6\hat{\jmath} - 2\hat{k}$
$\vec{b} = -\hat{\imath} - 2\hat{\jmath} + 6\hat{k}$
$\vec{c} = 6\hat{\imath} + 4\hat{\jmath} - 2\hat{k}$

Vector equation:

$$\vec{r} = (1-s-t)(3\hat{\imath} + 6\hat{\jmath} - 2\hat{k}) + s(-\hat{\imath} - 2\hat{\jmath} + 6\hat{k}) + t(6\hat{\imath} + 4\hat{\jmath} - 2\hat{k})$$

Cartesian form:

$$\begin{vmatrix} x-x_1 & y-y_1 & z-z_1 \\ x_2-x_1 & y_2-y_1 & z_2-z_1 \\ x_3-x_1 & y_3-y_1 & z_3-z_1 \end{vmatrix} = 0$$

$$\begin{vmatrix} x-3 & y-6 & z+2 \\ -4 & -8 & 8 \\ 3 & -2 & 0 \end{vmatrix} = 0$$

$$2x + 3y + 4z - 16 = 0$$

Two points and one vector:

6)Find the vector parametric,vector non-parametric and cartesian form of the equation of the plane passing through the points $(-1,2,0),(2,2,-1)$ and parallel to the straight line $\frac{x-1}{1} = \frac{2y+1}{2} = \frac{z+1}{-1}$

Sol:

$\vec{a} = -\hat{\imath} + 2\hat{\jmath}$
$\vec{b} = 2\hat{\imath} + 2\hat{\jmath} - \hat{k}$
$\vec{c} = \hat{\imath} + \hat{\jmath} - \hat{k}$
$\vec{b} - \vec{a} = 3\hat{\imath} + 0\hat{\jmath} - \hat{k}$

The parametric vector equation:

$$\vec{r} = \vec{a} + s(\vec{b} - \vec{a}) + t\vec{c}$$

$$\vec{r} = (-\hat{\imath} + 2\hat{\jmath}) + s(3\hat{\imath} - \hat{k}) + t(\hat{\imath} + \hat{\jmath} - \hat{k})$$

The non-parametric form of vector equation:

$$(\vec{r} - \vec{a}).\left((\vec{b} - \vec{a}) \times \vec{c}\right) = 0$$

$$(\vec{b} - \vec{a}) \times \vec{c} = \begin{vmatrix} \hat{\imath} & \hat{\jmath} & \hat{k} \\ 3 & 0 & -1 \\ 1 & 1 & -1 \end{vmatrix}$$

$$= \hat{\imath} + 2\hat{\jmath} + 3\hat{k}$$

$$\vec{r}.(\hat{\imath} + 2\hat{\jmath} + 3\hat{k}) = 3$$

Cartesian equation:

$$\begin{vmatrix} x-x_1 & y-y_1 & z-z_1 \\ x_2-x_1 & y_2-y_1 & z_2-z_1 \\ c_1 & c_2 & c_3 \end{vmatrix} = 0$$

$$x + 2y + 3z = 3$$

Two points and one vector:

7)Find the parametric form of vector equation, and Cartesian equations of the plane passing through the points $(2,2,1),(9,3,6)$ and perpendicular to the plane $2x+6y+6z=9$

Sol:

$\vec{a}=2\hat{\imath}+2\hat{\jmath}+\hat{k}$

$\vec{b}=9\hat{\imath}+3\hat{\jmath}+6\hat{k}$

$\vec{c}=2\hat{\imath}+6\hat{\jmath}+6\hat{k}$

$\vec{b}-\vec{a}=7\hat{\imath}+\hat{\jmath}+5\hat{k}$

The parametric vector equation:

$$\vec{r}=\vec{a}+s(\vec{b}-\vec{a})+t\vec{c}$$

$$\vec{r}=(2\hat{\imath}+2\hat{\jmath}+\hat{k})+s(7\hat{\imath}+\hat{\jmath}+5\hat{k})+t(2\hat{\imath}+6\hat{\jmath}+6\hat{k})$$

Cartesian equation:

$$\begin{vmatrix} x-x_1 & y-y_1 & z-z_1 \\ x_2-x_1 & y_2-y_1 & z_2-z_1 \\ c_1 & c_2 & c_3 \end{vmatrix}=0$$

$$\begin{vmatrix} x-2 & y-2 & z-1 \\ 7 & 1 & 5 \\ 2 & 6 & 6 \end{vmatrix}=0$$

$3x+4y-5z-9=0$

8) Find parametric form of vector equation and Cartesian equations of the plane passing through the points (2, 2, 1), (1, – 2, 3) and parallel to the straight line passing through the points (2, 1, – 3) and (– 1,5, – 8)

Sol:

$\vec{a}=2\hat{\imath}+2\hat{\jmath}+\hat{k}$

$\vec{b}=\hat{\imath}-2\hat{\jmath}+3\hat{k}$

$\vec{c}=-3\hat{\imath}+4\hat{\jmath}-5\hat{k}$

$\vec{b}-\vec{a}=-\hat{\imath}-4\hat{\jmath}+2\hat{k}$

The parametric vector equation:

$$\vec{r}=\vec{a}+s(\vec{b}-\vec{a})+t\vec{c}$$

$$\vec{r}=(2\hat{\imath}+2\hat{\jmath}+\hat{k})+s(-\hat{\imath}-4\hat{\jmath}+2\hat{k})+t(-3\hat{\imath}+4\hat{\jmath}-5\hat{k})$$

Cartesian equation:

$$\begin{vmatrix} x-x_1 & y-y_1 & z-z_1 \\ x_2-x_1 & y_2-y_1 & z_2-z_1 \\ c_1 & c_2 & c_3 \end{vmatrix}=0$$

$$\begin{vmatrix} x-2 & y-2 & z-1 \\ -1 & -4 & 2 \\ -3 & 4 & -5 \end{vmatrix}=0$$

$12x-11y-16z+14=0$

One points and two vectors:

9)Find the non-parametric form of vector equation, and Cartesian equations of the plane passing through the point $(0,1,-5)$ and parallel to the straight lines $\hat{r}=(\hat{\imath}+2\hat{\jmath}-4\hat{k})+s(2\hat{\imath}+3\hat{\jmath}+6\hat{k})$ and $\hat{r}=(\hat{\imath}+3\hat{\jmath}-5\hat{k})+t(\hat{\imath}+\hat{\jmath}-\hat{k})$

Sol:

$\vec{a}=\hat{\jmath}-5\hat{k}$

$\vec{b}=2\hat{\imath}+3\hat{\jmath}+6\hat{k}$

$\vec{c}=\hat{\imath}+\hat{\jmath}-\hat{k}$

The parametric vector equation:

$\vec{r}=\vec{a}+t\vec{b}+s\vec{c}$

$$\vec{r}=(\hat{\jmath}-\hat{5})+t(2\hat{\imath}+3\hat{\jmath}+6\hat{k})+s(\hat{\imath}+\hat{\jmath}-\hat{k})$$

The non-parametric vector equation:

$$(\vec{r}-\vec{a}).(\vec{b}\times\vec{c})=0$$

$$\vec{b}\times\vec{c}=\begin{vmatrix} \hat{\imath} & \hat{\jmath} & \hat{k} \\ 2 & 3 & 6 \\ 1 & 1 & -1 \end{vmatrix}$$

10)Find the non-parametric form of vector equation, and Cartesian equations of the plane passing through the point$(2,3,6)$ and parallel to the straight lines $\frac{x-1}{2}=\frac{y+1}{3}=\frac{z-3}{1}$ and $\frac{x+3}{2}=\frac{y-3}{-5}=\frac{z+1}{-3}$

Sol:

$\vec{a}=2\hat{\imath}+3\hat{\jmath}+6\hat{k}$

$\vec{b}=2\hat{\imath}+3\hat{\jmath}+\hat{k}$

$\vec{c}=2\hat{\imath}-5\hat{\jmath}-3\hat{k}$

The non-parametric vector equation:

$$(\vec{r}-\vec{a}).(\vec{b}\times\vec{c})=0$$

$$\vec{b}\times\vec{c}=\begin{vmatrix} \hat{\imath} & \hat{\jmath} & \hat{k} \\ 2 & 3 & 1 \\ 2 & -5 & -3 \end{vmatrix}$$

$$=-4\hat{\imath}+8\hat{\jmath}-16\hat{k}$$

$$\vec{b}\times\vec{c}=(\hat{\imath}-2\hat{\jmath}+4\hat{k})$$

$$\vec{r}.(\hat{\imath}-2\hat{\jmath}+4\hat{k})-20=0$$

Cartesian equation:

$$\vec{b}\times\vec{c} = -9\hat{\imath} + 8\hat{\jmath} - \hat{k}$$
$$\vec{r}.\left(-9\hat{\imath} + 8\hat{\jmath} - \hat{k}\right) - 13 = 0$$

Cartesian equation:

$$\begin{vmatrix} x-x_1 & y-y_1 & z-z_1 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix} = 0$$
$$9x - 8y + z + 13 = 0$$

11)Find the parametric form of vector equation, and Cartesian equations of the plane containing the line $\vec{r} = \left(\hat{\imath} - \hat{\jmath} + 3\hat{k}\right) + t\left(2\hat{\imath} - \hat{\jmath} + 4\hat{k}\right)$ and perpendicular to the plane $\vec{r}.\left(\hat{\imath} + 2\hat{\jmath} + \hat{k}\right) = 8$

Sol:

$$\vec{a} = \hat{\imath} - \hat{\jmath} + 3\hat{k}$$
$$\vec{b} = 2\hat{\imath} - \hat{\jmath} + 4\hat{k}$$
$$\vec{c} = \hat{\imath} + 2\hat{\jmath} + \hat{k}$$

The parametric vector equation:

$$\vec{r} = \vec{a} + t\vec{b} + s\vec{c}$$
$$\vec{r} = \left(\hat{\imath} - \hat{\jmath} + 3\hat{k}\right) + s\left(2\hat{\imath} - \hat{\jmath} + 4\hat{k}\right) + t\left(\hat{\imath} + 2\hat{\jmath} + \hat{k}\right)$$

Cartesian equation:

$$\begin{vmatrix} x-x_1 & y-y_1 & z-z_1 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix} = 0$$
$$\begin{vmatrix} x-1 & y+1 & z-3 \\ 2 & -1 & 4 \\ 1 & 2 & 1 \end{vmatrix} = 0$$
$$9x - 2y - 5z + 4 = 0$$

$$\begin{vmatrix} x-x_1 & y-y_1 & z-z_1 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix} = 0$$
$$x - 2y + 4z - 20 = 0$$

12)Find the non-parametric form of vector equation, and Cartesian equations of the plane passing through the point $(1,-2,4)$ and perpendicular to the plane $x + 2y - 3z = 11$ and parallel to the line $\frac{x+7}{3} = \frac{y+3}{-1} = \frac{z}{1}$

Sol:

$$\vec{a} = \hat{\imath} - 2\hat{\jmath} + 4\hat{k}$$
$$\vec{b} = \hat{\imath} + 2\hat{\jmath} - 3\hat{k}$$
$$\vec{c} = 3\hat{\imath} - \hat{\jmath} + \hat{k}$$
$$\vec{b}\times\vec{c} = \begin{vmatrix} \hat{\imath} & \hat{\jmath} & \hat{k} \\ 1 & 2 & -3 \\ 3 & -1 & 1 \end{vmatrix}$$
$$= \left(-\hat{\imath} - 10\hat{\jmath} - 7\hat{k}\right)$$

The non-parametric vector equation:

$$(\vec{r} - \vec{a}).\left(\vec{b}\times\vec{c}\right) = 0$$
$$\vec{r}.\left(\hat{\imath} + 10\hat{\jmath} + 7\hat{k}\right) - 9 = 0$$

Cartesian equation:

$$\begin{vmatrix} x-x_1 & y-y_1 & z-z_1 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix} = 0$$
$$x + 10y + 7z - 9 = 0$$

13) Find the non-parametric form of vector equation, and cartesian equation of the plane

$$\vec{r} = \left(6\vec{\imath} - \vec{\jmath} + \vec{k}\right) + s\left(-\vec{\imath} + 2\vec{\jmath} + \vec{k}\right) + t\left(-5\vec{\imath} - 4\vec{\jmath} - 5\vec{k}\right)$$

Sol:

$$\vec{r} = \vec{a} + s\vec{b} + t\vec{c}$$
$$\vec{a} = 6\hat{\imath} - \hat{\jmath} + \hat{k}$$
$$\vec{b} = -\hat{\imath} + 2\hat{\jmath} + \hat{k}$$
$$\vec{c} = -5\hat{\imath} - 4\hat{\jmath} - 5\hat{k}, \vec{b}\times\vec{c} = \begin{vmatrix} \hat{\imath} & \hat{\jmath} & \hat{k} \\ -1 & 2 & 1 \\ -5 & -4 & -5 \end{vmatrix}$$
$$= -2\left(3\hat{\imath} + 5\hat{\jmath} - 7\hat{k}\right)$$
$$\vec{b}\times\vec{c} = 3\hat{\imath} + 5\hat{\jmath} - 7\hat{k}$$

The non-parametric form of vector equation:

$$(\vec{r} - \vec{a}).\left(\vec{b}\times\vec{c}\right) = 0$$
$$\vec{r}.\left(3\hat{\imath} + 5\hat{\jmath} - 7\hat{k}\right) - 6 = 0$$

Cartesian equation: $\begin{vmatrix} x-x_1 & y-y_1 & z-z_1 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix} = 0$

$$3x+5y-7z-6=0$$

14) Prove by vector method that the perpendiculars (altitudes) from the vertices to the opposite sides of a triangle are concurrent.

Sol:

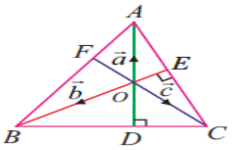
In a ΔABC,

$\overrightarrow{OA} = \vec{a}, \overrightarrow{OB} = \vec{b}, \overrightarrow{OC} = \vec{c}$

To prove $CF \perp AB$

$AD \perp BC$

$\overrightarrow{OA}.\overrightarrow{BC} = 0$



$\vec{a}.\vec{c} - \vec{a}.\vec{b} = 0 \quad \rightarrow$ ❶

$BE \perp CA$

$\overrightarrow{OB}.\overrightarrow{CA} = 0$

$\vec{a}.\vec{b} - \vec{b}.\vec{c} = 0 \quad \rightarrow$ ❷

❶ + ❷ $\Rightarrow \vec{a}.\vec{c} - \vec{b}.\vec{c} = 0 \Rightarrow \vec{c}.(\vec{a} - \vec{b}) = 0$

$\overrightarrow{BA}.\overrightarrow{OC} = 0$

$BA \perp CF$

Thus, the altitude are concurrent.

15) Show that the lines $\frac{x-3}{3} = \frac{y-3}{-1}, z-1 = 0$ and $\frac{x-6}{2} = \frac{z-1}{3}, y-2 = 0$ intersect. Also find the point of intersection.

Sol:

$\frac{x-3}{3} = \frac{y-3}{-1} = \frac{z-1}{0}$

$\frac{x-6}{2} = \frac{y-2}{0} = \frac{z-1}{3}$

$(x_1, y_1, z_1) = (3,3,1)$ $(x_2, y_2, z_2) = (6,2,1)$

$(b_1, b_2, b_3) = (3,-1,0)$ $(d_1, d_2, d_3) = (2,0,3)$

Condition $\begin{vmatrix} x_2-x_1 & y_2-y_1 & z_2-z_1 \\ b_1 & b_2 & b_3 \\ d_1 & d_2 & d_3 \end{vmatrix} = 0$

$\begin{vmatrix} x_2-x_1 & y_2-y_1 & z_2-z_1 \\ b_1 & b_2 & b_3 \\ d_1 & d_2 & d_3 \end{vmatrix}$

$= \begin{vmatrix} 3 & -1 & 0 \\ 3 & -1 & 0 \\ 2 & 0 & 3 \end{vmatrix}$

$= 0$

Given, lines are intersecting.

Point of intersection: $z = 1$ & $y = 2$

$\frac{x-3}{3} = \frac{y-3}{-1}, \frac{x-3}{3} = \frac{2-3}{-1}$

$x - 3 = 3$

$x = 6,$

$\therefore (6,2,1)$

17) If $\vec{a} = \hat{\imath} - \hat{\jmath}$ $\vec{b} = \hat{\imath} - \hat{\jmath} - 4\hat{k}, \vec{c} = 3\hat{\jmath} - \hat{k}$ · $\vec{d} = 2\hat{\imath} + 5\hat{\jmath} + \hat{k}$ verify that

$i)\ (\vec{a} \times \vec{b}) \times (\vec{c} \times \vec{d}) = [\vec{a}, \vec{b}, \vec{d}]\vec{c} - [\vec{a}, \vec{b}, \vec{c}]\vec{d}$ $\quad ii)\ (\vec{a} \times \vec{b}) \times (\vec{c} \times \vec{d}) = [\vec{a}, \vec{b}, \vec{d}]\vec{b} - [\vec{b}, \vec{c}, \vec{d}]\vec{d}$

Sol:

$(\vec{a} \times \vec{b}) = \begin{vmatrix} \hat{\imath} & \hat{\jmath} & \hat{k} \\ 1 & -1 & 0 \\ 1 & -1 & -4 \end{vmatrix}$

$= 4\hat{\imath} + 4\hat{\jmath}$

$(\vec{c} \times \vec{d}) = \begin{vmatrix} \hat{\imath} & \hat{\jmath} & \hat{k} \\ 0 & 3 & -1 \\ 2 & 5 & 1 \end{vmatrix}$

18) If $\vec{a} = 2\hat{\imath} + 3\hat{\jmath} - \vec{b} = 3\hat{\imath} + 5\hat{\jmath} + 2\hat{k}$, $\vec{c} = -\hat{\imath}\ 2\hat{\jmath} + \hat{k}$ verify that

$(i)\ (\vec{a} \times \vec{b}) \times \vec{c} = (\vec{a}.\vec{c})\vec{b} - (\vec{b}.\vec{c})\vec{a}$

$ii)\ \vec{a} \times (\vec{b} \times \vec{c}) = (\vec{a}.\vec{c})\vec{b} - (\vec{a}.\vec{b})\vec{c}$

Sol:

$(\vec{a} \times \vec{b}) = \begin{vmatrix} \hat{\imath} & \hat{\jmath} & \hat{k} \\ 2 & 3 & -1 \\ 3 & 5 & 2 \end{vmatrix}$

$= 11\hat{\imath} - 7\hat{\jmath} + \hat{k}$

$(\vec{a} \times \vec{b}) \times \vec{c} = \begin{vmatrix} \hat{\imath} & \hat{\jmath} & \hat{k} \\ 11 & -7 & 1 \\ -1 & -2 & 3 \end{vmatrix}$

$= -19\hat{\imath} - 34\hat{\jmath} - 29\hat{k} \quad \rightarrow$ ❶

$= 8\hat{i} - 2\hat{j} - 6\hat{k}$

$(\vec{a}\times\vec{b})\times(\vec{c}\times\vec{d}) = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 4 & 4 & 0 \\ 8 & -2 & -6 \end{vmatrix}$

$= -24\hat{i} + 24\hat{j} - 40\hat{k} \quad \rightarrow ❶$

$[\vec{a},\vec{b}\,\vec{d}] = \begin{vmatrix} 1 & -1 & 0 \\ 1 & -1 & -4 \\ 2 & 5 & 1 \end{vmatrix}$

$= 28$

$[\vec{a}\,\vec{b}\,\vec{c}] = \begin{vmatrix} 1 & -1 & 0 \\ 1 & -1 & -4 \\ 0 & 3 & -1 \end{vmatrix}$

$= 12$

$RHS = [\vec{a}\,\vec{b}\,\vec{d}]\vec{c} - [\vec{a}\,\vec{b}\,\vec{c}]\vec{d}$

$= -24\hat{i} + 24\hat{j} - 40\hat{k} \quad \rightarrow ❷$

From ❶ and ❷

$(\vec{a}\times\vec{b})\times(\vec{c}\times\vec{d}) = [\vec{a},\vec{b},\vec{d}]\vec{c} - [\vec{a},\vec{b},\vec{c}]\vec{d}$

Do it yourself:

If $\vec{a} = \hat{i} - \hat{j}$ $\vec{b} = \hat{i} - \hat{j} - 4\hat{k}, \vec{c} = 3\hat{j} - \hat{k}$ $\vec{d} = 2\hat{i} + 5\hat{j} + \hat{k}$ verify that *i*) $(\vec{a}\times\vec{b})\times(\vec{c}\times\vec{d}) = [\vec{a},\vec{b},\vec{d}]\vec{b} - [\vec{b},\vec{c},\vec{d}]\vec{d}$

$(\vec{a}.\vec{c}) = -11 \;\&\; \vec{b}.\vec{c} = -7$

$(\vec{a}.\vec{c})\vec{b} - (\vec{b}.\vec{c})\vec{a} = -19\hat{i} - 34\hat{j} - 29\hat{k} \quad \rightarrow ❷$

From ❶ and ❷

$(\vec{a}\times\vec{b})\times\vec{c} = (\vec{a}.\vec{c}) - (\vec{a}.\vec{b})\vec{c}$

$ii)\vec{a}\times(\vec{b}\times\vec{c}) = (\vec{a}.\vec{c})\vec{b} - (\vec{a}.\vec{b})\vec{c}$

$(\vec{b}\times\vec{c}) = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 3 & 5 & 2 \\ -1 & -2 & 3 \end{vmatrix}$

$= 19\hat{i} - 11\hat{j} - \hat{k}$

$\vec{a}\times(\vec{b}\times\vec{c}) = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 2 & 3 & -1 \\ 19 & -11 & -1 \end{vmatrix}$

$= -14\,\hat{i} - 17\hat{j} - 79\hat{k} \quad ❶$

$(\vec{a}.\vec{c}) = -11 \;\&\; (\vec{a}.\vec{b}) = 19$

$(\vec{a}.\vec{c})\vec{b} - (\vec{a}.\vec{b})\vec{c} = -14\hat{i} - 17\hat{j} - ❷$

From ❶ and ❷

$\vec{a}\times(\vec{b}\times\vec{c}) = (\vec{a}.\vec{c})\vec{b} - (\vec{a}.\vec{b})\vec{c}$

8. Differentials and Partial Derivatives

If $u = \sin^{-1}\left[\frac{x+y}{\sqrt{x}+\sqrt{y}}\right]$ *then prove that* $x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} = \frac{1}{2}\tan u$

Sol: $f(x,y) = \sin u = \frac{x+y}{\sqrt{x}+\sqrt{y}}$

$\therefore$ *f is a homogeneous function of degree* $n = \frac{1}{2}$

$$x\frac{\partial f}{\partial x} + y\frac{\partial f}{\partial y} = \frac{1}{2}f$$

$$x\frac{\partial}{\partial x}\sin u + y\frac{\partial}{\partial x}\sin u = \frac{1}{2}\sin u$$

$$\cos u\left[x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y}\right] = \frac{1}{2}\sin u$$

$$x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} = \frac{1}{2}\tan u$$

Prove that $g(x,y) = x\,log\left(\frac{y}{x}\right)$ is homogeneous; what is the degree? Verify Euler's Theorem for g.

Sol:

$$g(x,y) = x\log\frac{y}{x} = f$$

f is a homogeneous function of degree $n = 1$

If $w(x,y,) = \log\left[\frac{5x^3y^4+7y^2xz^4-75y^3z^4}{x^2+y^2}\right]$ *then prove* that $x\frac{\partial w}{\partial x} + y\frac{\partial w}{\partial y} + z\frac{\partial w}{\partial z}$

Sol:

$$f(x,y,z) = e^w$$

$$= \log\left[\frac{5x^3y^4 + 7y^2xz^4 - 75y^3z^4}{x^2+y^2}\right]$$

$\therefore$ *f is a homogeneous function of degree* $n = 5$

$$x\frac{\partial f}{\partial x} + y\frac{\partial f}{\partial y} + z\frac{\partial f}{\partial z} = nf$$

$$x\frac{\partial}{\partial x}(e^w) + y\frac{\partial}{\partial y}(e^w) + z\frac{\partial}{\partial z}(e^w) = 5e^w$$

$$e^w\left[x\frac{\partial w}{\partial x} + y\frac{\partial w}{\partial y} + z\frac{\partial w}{\partial z}\right] = 5e^w$$

$$x\frac{\partial w}{\partial x} + y\frac{\partial w}{\partial y} + z\frac{\partial w}{\partial z} = 5$$

If $f(x,y) = \tan^{-1}\left(\frac{x}{y}\right)$, *Prove that* $f_{xy} = f_{yx}$

Sol:

$$f = \tan^{-1}\left(\frac{x}{y}\right)$$

$x\frac{\partial f}{\partial x}+y\frac{\partial f}{\partial y}=nf$

$x\frac{\partial g}{\partial x}+y\frac{\partial g}{\partial y}=g$

$g=x\log\left(\frac{y}{x}\right)$

$\frac{\partial g}{\partial x}=x\frac{1}{y/_x}y\left(-\frac{1}{x^2}\right)+\log\left(\frac{y}{x}\right)(1)$

$=x^2\left(-\frac{1}{x^2}\right)+\log\frac{y}{x}$

$=-1+\log\frac{y}{x}$

$x\frac{\partial g}{\partial x}=-x+x\log\left(\frac{y}{x}\right)$----------------------------❶

$\frac{\partial g}{\partial y}=x\frac{1}{y/_x}\left(\frac{1}{x}\right)(1)$

$=\frac{x^2}{yx}=\frac{x}{y}$

$y\frac{\partial g}{\partial y}=y\left(\frac{x}{y}\right)=x$

$y\frac{\partial g}{\partial y}=x$----------------------------------❷

❶ + ❷ $\Rightarrow x\frac{\partial g}{\partial x}+y\frac{\partial g}{\partial y}$

$=-x+x\log\left(\frac{y}{x}\right)+x$

$=x\log\left(\frac{y}{x}\right)$

$x\frac{\partial g}{\partial x}+y\frac{\partial g}{\partial y}=g$

If $f(x,y)=\frac{3x}{y+\sin x}$, ***Prove*** $f_{xy}=f_{yx}$

Sol:

$f_x=\frac{\partial f}{\partial x}=\frac{(y+\sin x)\,(3)-3x(0+\cos x)}{(y+\sin x)^2}$

$f_x=\frac{3y+3\sin x-3x\cos x}{(y+\sin x)^2}$

$f_y=\frac{\partial f}{\partial y}=\frac{(y+\sin x)\,(0)-3x(1+0)}{(y+\sin x)^2}$

$f_y=\frac{-3x}{(y+\sin x)^2}$

$f_{xy}=\frac{\partial}{\partial x}\left(\frac{\partial f}{\partial y}\right)$

$=\frac{\partial}{\partial x}\left[\frac{-3x}{(y+\sin x)^2}\right]$

$=\frac{(y+\sin x)^2(-3)-(-3x)\big(2(y+\sin x)(0+\cos}{(y+\sin x)^4}$

$=\frac{(y+\sin x)(-3y-3\sin x+6x\cos x)}{(y+\sin x)^4}$

$f_{xy}=\frac{6x\cos x-3y-3\sin x}{(y+\sin x)^3}$ ---------→ ❶

$f_x=\frac{\partial f}{\partial x}=\frac{1}{1+\left(\frac{x}{y}\right)^2}\frac{1}{y}$

$f_x=\frac{y}{x^2+y^2}$

$f_y=\frac{1}{1+\left(\frac{x}{y}\right)^2}x\left(-\frac{1}{y^2}\right)$

$f_y=\frac{-x}{x^2+y^2}$

$f_{xy}=\frac{\partial}{\partial x}\left(\frac{\partial f}{\partial y}\right)$

$=\frac{\partial}{\partial x}\left(\frac{-x}{x^2+y^2}\right)$

$=\frac{(x^2+y^2)(-1)-(-x)(2x+0)}{(x^2+y^2)^2}$

$f_{xy}=\frac{x^2-y^2}{(x^2+y^2)^2}$ – –→ ❶

$f_{yx}=\frac{\partial}{\partial y}\left(\frac{\partial f}{\partial x}\right)$

$=\frac{\partial}{\partial y}\left(\frac{y}{x^2+y^2}\right)$

$=\frac{x^2+y^2-2y^2}{(x^2+y^2)^2}$

$f_{yx}=\frac{x^2-y^2}{(x^2+y^2)^2}$ – –→ ❷

From ❶ and ❷

$f_{xy}=f_{yx}$

Let $g(x,y)=\frac{x^2y}{x^4+y^2}$ **for** $(x,y)\neq(0,0)$ **and** $g(0,0)=0$.

(i)Show that $\lim_{(x,y)\to(0,0)} g(x,y)=0$ **along every line** $y=mx, m\in\mathbb{R}$.

(ii)Show that $\lim_{(x,y)\to(0,0)} g(x,y)=\frac{k}{1+k^2}$ **along every parabola** $y=kx^2, k\in\mathbb{R}\backslash\{0\}$.

Sol:

(i) $\lim_{(x,y)\to(0,0)} g(x,y)=$

$\lim_{(x,y)\to(0,0)} \frac{x^2y}{x^2+y^2}$

$y=mx$ என பிரதியிட

$=\lim_{(x,y)\to(0,0)}\frac{x^3m}{x^4+x^2m^2}$

$=\lim_{(x,y)\to(0,0)}\frac{x^3m}{x^2(x^2+m^2)}$

$=\lim_{(x,y)\to(0,0)}\frac{xm}{x^2+m^2}$

$$f_{yx} = \frac{\partial}{\partial y}\left[\frac{3y + 3\sin x - 3x\cos x}{(y+\sin x)^2}\right]$$

$$f_{yx} = \frac{6x\cos x - 3y - 3\sin x}{(y+\sin x)^3} - - - - - - \rightarrow ❷$$

$$\therefore f_{xy} = f_{yx}$$

Let $z(x,y) = x\tan^{-1}(xy), x = t^2, y = se^t, s, t \in \mathbb{R}$**. Find** $\frac{\partial z}{\partial s}$ **and** $\frac{\partial z}{\partial t}$ **at** $s = t = 1$**.**

Sol:

$$\frac{\partial z}{\partial x} = x.\frac{1}{1+x^2y^2}y + \tan^{-1}(xy);$$

$$\frac{\partial z}{\partial x} = \frac{1}{1+x^2y^2}xy + \tan^{-1}(xy); \quad \frac{\partial z}{\partial y} = \frac{x^2}{1+x^2y^2}$$

$$x = t^2 \qquad y = se^t$$

$$\frac{\partial x}{\partial t} = 2t; \quad \frac{\partial x}{\partial s} = 0 \quad \frac{\partial y}{\partial t} = se^t \quad \frac{\partial y}{\partial s} = e^t$$

$$\frac{\partial z}{\partial t} = \frac{\partial z}{\partial x}.\frac{\partial x}{\partial t} + \frac{\partial z}{\partial y}.\frac{\partial y}{\partial t}$$

$$= \left[\frac{1}{1+x^2y^2}xy + \tan^{-1}(xy)\right](2t) + \left[\frac{x^2}{1+x^2y^2}\right]se^t$$

$$= \left[\frac{t^2se^t}{1+t^4s^2e^{2t}} + \tan^{-1}(st^2e^t)\right](2t) + \left[\frac{t^4}{1+t^4s^2e^{2t}}\right]Se^t$$

$$s = t = 1$$

$$\frac{\partial z}{\partial t} = \left[\frac{e}{1+e^2} + \tan^{-1}(e)\right]_2^1 + \left[\frac{1}{1+e^2}\right]e$$

$$\frac{\partial z}{\partial t} = \frac{3e}{1+e^2} + 2\tan^{-1}(e)$$

$$\frac{\partial z}{\partial s} = \frac{\partial z}{\partial x}.\frac{\partial x}{\partial s} + \frac{\partial z}{\partial y}.\frac{\partial y}{\partial s}$$

$$\frac{\partial z}{\partial s} = \left[\frac{t^4e^t}{1+t^4s^2e^{2t}}\right] \Rightarrow s = t = 1 \Rightarrow \frac{\partial z}{\partial s} = \frac{e}{1+e^2}$$

If $w(x,y,z) = \frac{1}{\sqrt{x^2+y^2+z^2}}, (x,y,z) \neq (0,0,0)$ ***then prove that*** $\frac{\partial^2 w}{\partial x^2} + \frac{\partial^2 w}{\partial y^2} + \frac{\partial^2 w}{\partial z^2} = 0$

Sol:

$$w = \frac{1}{\sqrt{x^2+y^2+z^2}} = (x^2+y^2+z^2)^{-\frac{1}{2}}$$

$$\frac{\partial w}{\partial x} = -\frac{1}{2}(x^2+y^2+z^2)^{-\frac{1}{2}-1}(2x)$$

$$= -\frac{1}{2}(x^2+y^2+z^2)^{-\frac{3}{2}}(2x)$$

$$\frac{\partial w}{\partial x} = (x^2+y^2+z^2)^{-\frac{3}{2}}(x)$$

$$= \frac{0}{0+m^2} = 0$$

$$\lim_{(x,y)\to(0,0)} g(x,y) = 0$$

(ii) $\lim_{(x,y)\to(0,0)} g(x,y) = \lim_{(x,y)\to(0,0)} \frac{x^2kx}{x^4+(kx}$

$$= \lim_{(x,y)\to(0,0)} \frac{x^4k}{x^4(1+k^2)}$$

$$\lim_{(x,y)\to(0,0)} g(x,y) = \frac{k}{1+k^2}$$

If $V(x,y) = e^x(x\cos y - y\sin y)$, then prove that $\frac{\partial^2 V}{\partial x^2} + \frac{\partial^2 V}{\partial y^2} = 0$.

Sol:

$$v = e^x(x\cos y - y\sin y)$$

$$\frac{\partial v}{\partial x} = e^x[\cos y - 0] + [x\cos y - y\sin y]e^x$$

$$= e^x[x\cos y - y\sin y + \cos y]$$

$$\frac{\partial^2 v}{\partial x^2} = e^x[2\cos y + x\cos y - y\sin y] - - \rightarrow ❶$$

$$\frac{\partial v}{\partial y} = e^x[x(-\sin y) - (y\cos y + \sin y)]$$

$$= e^x[-x\sin y - y\cos y - \sin y]$$

$$\frac{\partial^2 v}{\partial y^2} = e^x[-2\cos y - x\cos y + y\sin y] - - \rightarrow ❷$$

$$❶ + ❷ \Rightarrow \frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2} = 0$$

If $w(x,y) = xy + \sin(xy)$ ***then prove that*** $\frac{\partial^2 w}{\partial y\partial x} = \frac{\partial^2 w}{\partial x\partial y}$

Sol:

$$w = xy + \sin(xy)$$

$$\frac{\partial w}{\partial x} = y + \cos(xy)(y) = y + y\cos(xy)$$

$$\frac{\partial w}{\partial y} = x + \cos(xy)(x) = x + x\cos(xy)$$

$$\frac{\partial^2 w}{\partial y\partial x} = \frac{\partial}{\partial x}\left(\frac{\partial w}{\partial y}\right)$$

$$\frac{\partial^2 w}{\partial x^2} = x[-\frac{3}{2}(x^2+y^2+z^2)^{-\frac{3}{2}-1}(2x) - (x^2+y^2+z^2)^{-\frac{3}{2}}(1)$$

$$\frac{\partial^2 w}{\partial x^2} = \frac{3x^2}{(x^2+y^2+z^2)^{\frac{5}{2}}} - \frac{1}{(x^2+y^2+z^2)^{\frac{3}{2}}}$$

$|||^{ly}$

$$\frac{\partial^2 w}{\partial y^2} = \frac{3y^2}{(x^2+y^2+z^2)^{\frac{5}{2}}} - \frac{1}{(x^2+y^2+z^2)^{\frac{3}{2}}}$$

$$\frac{\partial^2 w}{\partial z^2} = \frac{3z^2}{(x^2+y^2+z^2)^{\frac{5}{2}}} - \frac{1}{(x^2+y^2+z^2)^{\frac{3}{2}}}$$

$$\frac{\partial^2 w}{\partial x^2} + \frac{\partial^2 w}{\partial y^2} + \frac{\partial^2 w}{\partial z^2} = \frac{3(x^2+y^2+z^2)}{(x^2+y^2+z^2)^{\frac{5}{2}}} - 3\left[\frac{1}{(x^2+y^2+z^2)^{\frac{3}{2}}}\right]$$

$$= 3\left[\frac{1}{(x^2+y^2+z^2)^{\frac{3}{2}}}\right] - 3\left[\frac{1}{(x^2+y^2+z^2)^{\frac{3}{2}}}\right]$$

$$\frac{\partial^2 w}{\partial x^2} + \frac{\partial^2 w}{\partial y^2} + \frac{\partial^2 w}{\partial z^2} = 0$$

$$= \frac{\partial}{\partial x}[x + x\cos(xy)]$$

$$= 1 + x(-\sin(xy))y + \cos(xy)\ (1)$$

$$\frac{\partial^2 w}{\partial y \partial x} = 1 - xy\sin(xy) + \cos(xy) - -\rightarrow ❶$$

$$\frac{\partial^2 w}{\partial x \partial y} = \frac{\partial}{\partial y}\left(\frac{\partial w}{\partial x}\right)$$

$$= \frac{\partial}{\partial y}[y + y\cos(xy)]$$

$$= 1 + y(-\sin xy)x + \cos(xy)$$

$$\frac{\partial^2 w}{\partial y \partial x} = 1 - xy\sin(xy) + \cos(xy) - -\rightarrow ❷$$

From ❶ and ❷

$$\frac{\partial^2 w}{\partial y \partial x} = \frac{\partial^2 w}{\partial x \partial y}$$

12.Discrete Mathematics

(Important 2 & 3 Marks)

1) Let * be defined on R by a*b=a+b+ab-7. Is* binary on R? (2-Marks)	**2) Write the statements in words corresponding to $\neg p, pvq$, where p is 'It is Cold' and q is 'It is raining'.(2-Marks)**
Sol: a+b,ab,-7 are real numbers and their sum is also a real number. * is binary.	**Sol:** $\neg p$: It is not Cold pvq : It is Cold or raining Do it yourself: Exercise 12. 2 − 1, Example:12.12

3)Let $A = \{a + \sqrt{5}\,b : a, b \in Z\}$.Check whether the usual multiplication is a binary operation on A.(2-Marks) **Sol:** Let b$a + \sqrt{5}\,b, c + \sqrt{5}\,d \in A$ Using usual multiplication, $(a+\sqrt{5}\,b)(c+\sqrt{5}\,d) = ac + 5bd + \sqrt{5}(bc+ad) \in A$ Hence, usual multiplication is a binary operation on A.	**4) On Z, define $*$ by $(m * n) = m^n n^m : \forall\ m, n \epsilon Z$. Is binary on Z? (2-Marks)** **Sol:** When $n = -p$, where $p > 0$ then $m^n = m^{-p} = \frac{1}{m^p}$ Now need not be in Z. Hence $*$ is not binary on Z.

5)Verify the associative property under the binary operation $*$ defined by $a * b = a^b, \forall\, a, b \epsilon N$ (2-Marks) **Sol:** $(a * b) * c \neq a * (b * c)$ $*$ **is associative.**	$6) If A = \begin{bmatrix} 0 & 1 \\ 1 & 1 \end{bmatrix}, B = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}, Find$ $AVB, A\Lambda B$ **Sol:** $AVB = \begin{bmatrix} 0 & 1 \\ 1 & 1 \end{bmatrix} V \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}$ $A\Lambda B = \begin{bmatrix} 0 & 1 \\ 1 & 1 \end{bmatrix} \Lambda \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 0 & 1 \end{bmatrix}$ **Do it yourself:** **Exercise 12. 2 – 8**

∧க்கு ஒரு F இருந்தாலும் F வரவேண்டும்.

Vக்கு ஒரு T இருந்தாலும் T வரவேண்டும்.

$p \rightarrow q$க்கு T F க்கு F ……….. மற்றதற்கு T

$p \leftrightarrow q$ க்கு T T க்கு T மேலும் F F க்கு T ……………. மற்றதற்கு F

Example: Construct the truth table (p∨q)∧(~q)

P	Q	$(p \vee p)$	$\sim q$	$(p \vee p) \wedge (\sim p)$
T	T	T	F	F
T	F	T	T	T
F	T	T	F	F
F	F	F	T	F

1) Show that p → q ≡ (~ p) ∨ q

P	q	$P \rightarrow q$
T	T	T
T	F	F
F	T	T
F	F	T

P	q	$\sim p$	$(\sim p) \vee q$
T	T	F	T
T	F	F	F
F	T	T	T
F	F	T	T

$p \to q \equiv (\sim p) \vee q$

2) Show that $p \leftrightarrow q \equiv (p \to q) \wedge (q \to p)$

P	q	$P \leftrightarrow q$
T	T	T
T	F	F
F	T	F
F	F	T

P	q	$P \to q$	$q \to p$	$(P \to q) \wedge (q \to p)$
T	T	T	T	T
T	F	F	T	F
F	T	T	F	F
F	F	T	T	T

$p \leftrightarrow q \equiv (p \to q) \wedge (q \to p)$

3) Show that $p \wedge q \to p \vee q$ **is tautology.**

P	q	$(p \wedge q)$	$p \vee q$	$(p \wedge q) \to p \vee q$
T	T	T	T	T
T	F	F	T	T
F	T	F	T	T

F	F	F	F	T

$p \wedge q \rightarrow p \vee q$ is tautology.

4) Show that $q \rightarrow p \equiv \neg p \rightarrow \neg q$

P	q	$q \rightarrow p$	$\neg p$	$\neg q$	$\neg p \rightarrow \neg q$
T	T	T	F	F	T
T	F	T	F	T	T
F	T	F	T	F	F
F	F	T	T	T	T

5) Verify whether the compound statement $(p \wedge q) \wedge [\neg (p \vee q)]$ is a tautology or contradiction or contingency.

P	q	$p \wedge q$	$(p \vee q)$	$\neg(p \vee q)$	$[(p \wedge q) \vee [\neg(p \vee q)]]$
T	T	T	T	F	F
T	F	F	T	F	F
F	T	F	T	F	F
F	F	F	F	T	F

$(p \wedge q) \wedge [\neg (p \vee q)]$ is contradiction.

6) Show that $\neg(p \wedge q) \equiv (\neg p) \vee (\neg q)$

p	q	$(p \wedge q)$	$\neg(p \wedge q)$
T	T	T	F
T	F	F	T
F	T	F	T

F	F	F	T

p	q	$\neg p$	$\neg q$	$(\neg p)\vee(\neg q)$
T	T	F	F	F
T	F	F	T	T
F	T	T	F	T
F	F	T	T	T

7) Show that : $\neg(p \rightarrow q) \equiv p \wedge \neg q$

P	q	$P \longrightarrow q$	$\neg(p \rightarrow q)$
T	T	T	F
T	F	F	T
F	T	T	F
F	F	T	F

P	q	$\sim q$	$p \wedge \neg q$
T	T	F	T
T	F	T	F
F	T	F	T
F	F	T	T

<u>Do it yourself:</u>

Exercise $12.2 - 6, 7, 12, 14$

Example: $12.18, 12.16$

8) Show that p∧q⟶ p ∨q is a tautology.

P	q	$(p \wedge q)$	$p \vee q$	$(p \wedge q) \longrightarrow p \vee q$
T	T	T	T	T
T	F	F	T	T
F	T	F	T	T
F	F	F	F	T

p∧q⟶ p ∨q is a tautology.

(Important 5- Marks)

1) **Using truth table check whether the statement $\neg(p \vee q) \vee (\neg p \wedge q)$ *and* $\neg p$ are logically equivalent.**

p	q	¬p	(p ∨ q)	¬(p ∨ q)	(¬p ∧ q)	¬(p ∨ q) ∨ (¬p ∧ q)
T	T	F	T	F	F	F
T	F	F	T	F	F	F
F	T	T	T	F	T	T
F	F	T	F	T	F	T
		❶				❷

From ❶ and ❷ ¬(p ∨ q) ∨ (¬p ∧ q) ≡ ¬p

2) **Prove that $p \rightarrow (\neg q \vee r) \equiv \neg p \vee (\neg q \vee r)$ using truth table**

p	q	r	¬q	¬q ∨ r	p → (¬q ∨ r)	¬p ∨ (¬q ∨ r)	¬p
T	T	T	F	T	T	T	F
T	T	F	F	F	F	F	F
T	F	T	T	T	T	T	F
F	T	T	F	T	T	T	T
F	F	T	T	T	T	T	T
F	T	F	F	F	T	T	T
T	F	F	T	T	T	T	F
F	F	F	T	T	T	T	T
					❶	❷	

From ❶ and ❷ $p \to (\neg q \vee r) \equiv \neg p \vee (\neg q \vee r)$

Do it yourself:

Exercise $12.2 - 7(iii)$ **, Exercise** $12.1 - 5, 9, 10$
Example: $12.19, 12.16, 12.9, 12.8, 12.10, 12.4, 12.7$

3) Verify (i) Closure property(ii) commutative property (iii) associative property (iii) existence of identity (iv) existence of inverse for the following operation on the given set. $m * n = m + n - mn \in z$

Sol:

i) $m * n = m + n - mn$ is clearly integer $\therefore m * n \in z, \forall m, n \in z$, $*$ is a binary operation on Z.

ii) $m * n = m + n - mn$
$n * m = n + m - nm$
$\therefore m * n = n * m \in z, \forall m, n \in z$, $*$ is commutative.

iii) $(m * n) * p = (m + n - mn) * p$
$= m + n - mn + p - (m + n - mn)p$
$= m + n + p - mn - mp - np + mnp$
$(m * n) * p = m * (n + p - np)$
$= m + n + p - m(n + p - np)$
$= m + n + p - np - mn - mp + mnp$
$(m * n) * p = m * (n * p) \forall\, m, n, p \in z$,$*$ is associative.

iv) let $e \in z$
$a * e = e * a = a \quad \forall a \in z$
$a * e = a \Rightarrow a + e - ae = a$
$e(1 - a) = 0 \Rightarrow e = 0$
$\therefore 0 \in z$ hence, the existence of identity is assured.

v) Inverse m is n.
$m * n = n * m = e$
$m * n = e \Rightarrow m + n - mn = 0$
$n(1 - m) = -m \Rightarrow n = -\dfrac{m}{1 - m}$
Hence, Inverse does not exist in Z.

4) (i) **Let** A be $\mathbb{Q}\backslash\{1\}$ **Define** $*$ **on A by** $x * y = x + y - xy$. **Is** $*$ $binary$ on A? **If so, examine the commutative and associative properties satisfied by** $*$ **on A .(ii) Let** A be $\mathbb{Q}\backslash\{1\}$ **Define** $*$ **on A by** $x * y = x + y - xy$. **Is** $*$ $binary$ on A? **If so, examine the existence of identity, existence of inverse properties for the operation** $*$ **on A.**

Sol:

(i) $x * y = x + y - xy, A = \mathbb{Q}\backslash\{1\}$
$x, y \in A, x \neq 1, y \neq 1$
$x * y = x + y - xy$ is a rational.
$x * y \neq 1, x * y \in A,$ $*$ $binary$ on A

(ii) $x * y = x + y - xy$
$y * x = y + x - yx$
$\therefore x * y = y * x, \forall x, y \in A$, $*$ is commutative.

(iii) $(x * y) * z = x * (y * z)$

$$x+y+z-xy-xz-yz+xyz = x+y+z-xy-xz-yz+xyz$$

$*$ is associative.

(iv) $\therefore 0 \in A$ is identity element.

(v) Inverse of $x = \frac{x}{x-1}$ (or) $\frac{-x}{1-x}$,The inverse property is also satisfied.

5) Verify (i) Closure property (ii) associative property (iii) existence of identity (iv) existence of inverse and (v) commutative property for the operation $+_5$ on Z_5 using table corresponding to addition modulo 5.

Sol: $Z_5 = \{[0],[1],[2],[3],[4]\}$

$+_5$	[0]	[1]	[2]	[3]	[4]
[0]	[0]	[1]	[2]	[3]	[4]
[1]	[1]	[2]	[3]	[4]	[0]
[2]	[2]	[3]	[4]	[0]	[1]
[3]	[3]	[4]	[0]	[1]	[2]
[4]	[4]	[0]	[1]	[2]	[3]

i) Elements of the table are in the elements of $+_5$, Closure property is true.

ii) From the table, $+_5$ is associative.

iii) From the table, the entries are symmetrical about the main diagonal. $+_5$ is commutative.

iv) The identity element is [0]

v) The inverse of [0] is [0], that of [1] is [4], that of [2] is[3], that of [3] is [2], that of [4] is [1]

6) Verify (i) Closure property (ii) associative property (iii) existence of identity (iv) existence of inverse and (v) commutative property for the operation $\times_{11}$ on a subset $A = \{1,3,4,5,9\}$ of the set of remainders $\{0,1,2,3,4,5,6,7,8,9,10\}$

Sol: $A = \{[1],[3],[4],[5],[9]\}$

X_{11}	[1]	[3]	[4]	[5]	[9]
[1]	[1]	[3]	[4]	[5]	[9]
[3]	[3]	[9]	[1]	[4]	[5]
[4]	[4]	[1]	[5]	[9]	[3]
[5]	[5]	[4]	[9]	[3]	[1]
[9]	[9]	[5]	[3]	[1]	[4]

i) Elements of the table are in the elements of A , Closure property is true.

ii) From the table, X_{11} is associative.

iii) From the table, the entries are symmetrical about the main diagonal. X_{11} is commutative.

iv) The identity element is [1].

v) **The inverse of [1] is [1], that of [3] is [4],that of [4] is [3], that of [5] is [9], that of [9] is [5]**

11.Probability distributions

(Important 5- Marks)

1)**A random variable X has the following probability mass function.**

X	1	2	3	4	5	6
f(x)	k	2k	6k	5k	6k	10k

Find (i) $P(2 < X < 6)$ (ii) $P(2 \le X < 5)$ (iii) $P(X \le 4)$ (iv) $P(3 < X)$

Sol:

$\sum_{x=\infty}^{\infty} f(x) = 1$, $k + 2k + 6k + 5k + 6k + 10k = 1$

$30\,k = 1$,

$k = \frac{1}{30}$

(i) $P(2 < X < 6) = P(X = 3) + P(X = 4) + P(X = 5) = 6k + 5k + 6k = \frac{17}{30}$

(ii) $P(2 \le X < 5) = P(X = 2) + P(X = 3) + P(X = 4) = 2k + 6k + 5k = \frac{13}{30}$

(iii) $P(X \le 4) = P(X = 1) + P(X = 2) + P(X = 3) + P(X = 4) = k + 2k + 6k + 5k = \frac{14}{30}$

(iv) $P(3 < X) = P(X = 4) + P(X = 5) + P(X = 6) = 5k + 6k + 10k = \frac{21}{30}$

2) **A random variable X has the following probability mass function.**

X	1	2	3	4	5
f(x)	k^2	$2k^2$	$3k^2$	2k	3k

Find (i) $k = ?$ (ii) $P(2 \le X < 5)$ (iii) $P(3 < X)$

Sol:

(i) $\sum_{x=\infty}^{\infty} f(x) = 1$, $k^2 + 2k^2 + 3k^2 + 2k + 3k = 1, 6k^2 + 5k = 1$

$6k^2 + 5k - 1 = 0,$

$k = \frac{1}{6}$ (or)-1 ,

$k = \frac{1}{6}$

(ii) $P(2 \le X < 5) = P(X = 2) + P(X = 3) + P(X = 4) = 2k^2 + 3k^2 + 2k = \frac{17}{36}$

(iii) $P(3 < X) = P(X > 3) = P(X = 4) + P(X = 5) = 2k + 3k = \frac{5}{6}$

Do it yourself:

Exercise 11.2 – 4

Exercise 11.3 – 3, 5, 6

Example: 11.7, 11.8, 11.9, 11.11, 11.12, 11.14, 11.15

3) The cumulative distribution function of a discrete random variable is given by

$$F(x) = \begin{cases} 0, & -\infty < x < -1 \\ 0.15, & -1 \le x < 0 \\ 0.35, & 0 \le x < 1 \\ 0.60, & 1 \le x < 2 \\ 0.85, & 2 \le x < 3 \\ 1, & 3 \le x < \infty \end{cases}$$

Find(i) the probability mass function (ii) $P(X < 1)$ (iii)$P(X \ge 2)$

Sol:

(i) The probability mass function

X	-1	0	1	2	3
F(X)	0.15	0.35	0.60	0.85	1
P(X=x)	0.15	0.20	0.25	0.25	0.15

(ii) $P(X < 1) = P(X = -1) + P(X = 0) = 0.35$

(iii)$P(X \ge 2) = P(X = 2) + P(X = 3) = 0.40$

4) The cumulative distribution function of a discrete random variable is given by

$$F(x) = \begin{cases} 0, & -\infty < x < 0 \\ \frac{1}{2}, & 0 \le x < 1 \\ \frac{3}{5}, & 1 \le x < 2 \\ \frac{4}{5}, & 2 \le x < 3 \\ \frac{9}{10}, & 3 \le x < 4 \\ 1, & 4 \le x < \infty \end{cases}$$

Find(i) the probability mass function (ii) $P(X < 3)$ (iii)$P(X \ge 2)$

Sol:

(i) The probability mass function

X	0	1	2	3	4
F(X)	$\frac{1}{2}$	$\frac{3}{5}$	$\frac{4}{5}$	$\frac{9}{10}$	1
P(X=x)	$\frac{5}{10}$	$\frac{1}{10}$	$\frac{2}{10}$	$\frac{1}{10}$	$\frac{1}{10}$

(ii) $P(X < 3) = P(X = 0) + P(X = 1) + P(X = 2) = \frac{5}{10} + \frac{1}{10} + \frac{2}{10} = \frac{4}{5}$

(iii)$P(X \ge 2) = P(X = 2) + P(X = 3) + P(X = 4) = \frac{2}{10} + \frac{1}{10} + \frac{2}{10} = \frac{2}{5}$

5) The probability density function of X is given by $f(x) = \begin{cases} ke^{\frac{-x}{3}} & for\ x > 0 \\ 0 & for\ x \le 0 \end{cases}$

Find (i) the value of k (ii) the distribution function (iii) $P(3 < X)$ (iii)$P(5 \le X)$ (iv)$P(X \le 4)$

Sol:

(i) f(x) is probability density function, $\int_{-\infty}^{\infty} f(x)dx = 1$, $\int_{0}^{\infty} ke^{\frac{-x}{3}} dx = 1$, $k\left[-3e^{\frac{-x}{3}}\right]_0^{\infty}$, $k = \frac{1}{3}$

(ii) The distribution function ,$F(x) = \begin{cases} 0 & ,x \le 0 \\ 1 - e^{\frac{-x}{3}} & ,x > 0 \end{cases}$

(iii) $P(X < 3) = F(3) = 1 - e^{-1}$

(iv) $P(5 \le X) = P(X \ge 5) = 1 - F(5) = 1 - (1 - e^{\frac{-5}{3}}) = e^{\frac{-5}{3}}$

(V) $P(X \le 4) = F(4) = 1 - e^{\frac{-4}{3}}$

10.Ordinary Differential Equations

(Important 5- Marks)

1)The growth of a population is proportional to the number present. If the population of a colony doubles in 50 years, in how many years will the population become triple?

T	0	50	?
x	x_0	$2x_0$	$3x_0$

$\frac{dx}{dt} = kx$

$x = Ce^{kt}$

$\underline{when\ t = 0, x = x_0}$

$x_0 = Ce^0$

$x_0 = C$

$x = x_0 e^{kt}$

$\underline{when\ t = 50, x = 2x_0}$

$2x_0 = x_0 e^{50k}$

$e^{50k} = 2$

$50k = \log 2$

$k = \frac{1}{50}\log 2$

$\underline{when\ \ x = 3x_0}$

$3x_0 = x_0 e^{tk}$

$tk = \log 3$

$t\left(\frac{1}{50}\log 2\right) = \log 3$

$t = 50\left(\frac{\log 3}{\log 2}\right) Years$

2) A radioactive isotope has an initial mass 200mg, which two years later is 50mg. Find the expression for the amount of the isotope remaining at any time. what is its half-life?

t(year)	0	2	?
x(mass)	200	150	100

$\frac{dx}{dt} = k\dot{x}$

$x = ce^{kt}$

$\underline{when\ t = 0, x = 200}$

$200 = Ce^0$

$C = 200$

$x = 200e^{kt}$

$\underline{When\ t = 2, x = 150}$

$150 = 200e^{2k}$

$e^{2k} = \frac{150}{200} = \frac{3}{4}$

$2k = \log\left(\frac{3}{4}\right)$

$k = \log\left(\frac{3}{4}\right)^{\frac{1}{2}}$

$x = 200\ e^{t}\log\left(\frac{3}{4}\right)^{\frac{1}{2}}$

$= 200\ e^{\log\left(\frac{3}{4}\right)^{\frac{t}{2}}}$

$x = 200\left(\frac{3}{4}\right)^{\frac{t}{2}}$

$when\ x = 100$

$100 = 200\left(\frac{3}{4}\right)^{\frac{t}{2}}$

$\frac{t}{2}\log\left(\frac{3}{4}\right) = \log\left(\frac{1}{2}\right)$

$t = \frac{2\log\left(\frac{1}{2}\right)}{\log\left(\frac{3}{4}\right)}$

3) In a murder investigation, a corpse was found by a detective at exactly 8 p.m. Being alert, the detective also measured the body temperature and found it to be 70F. Two hours later, the detective measured the body temperature again and found it to be 60F. If the room temperature is 50F, and assuming that the body temperature of the person before was 98.6F,at what time did the murder occur? $[\log(2.43) = 0.88789; \log(0.5) = -0.69315]$

	?	8.P.M	10P.M
t	?	0	2
T	98.6	70	60

$\frac{dT}{df} = k(T-50);$

$T - 50 = Ce^{kt}$

$\underline{when\ \ t = 0; T = 70}$

4) A tank contains 1000 litres of water in which 100 grams of salt is dissolved. Brine runs in a rate of 10 litres per minute. And each litre contains 5grams of dissolved salt. The mixture of tank is kept uniform by stirring. Brine runs out at 10 litres per minute. Find the amount of salt at any time t.

$\frac{dx}{dt} = 50 - 0.01\,x$

$\frac{dx}{dt} = -0.01(x - 5000)$

$\Rightarrow x - 5000 = ce^{-.01t}$

$70-50=Ce^{0}$

$c=20T-50=20e^{kt}$

$\underline{when\ t=2; T=60}$

$60-50=20e^{2k}$

$10=20e^{2k}$

$e^{2k}=\frac{1}{2}$

$2k=\log\left(\frac{1}{2}\right)$

$k=\frac{1}{2}\log\left(\frac{1}{2}\right)$

$when\ T=98.6$

$98.6-50=20e^{\frac{t}{2}\log\left(\frac{1}{2}\right)}$

$\frac{48.6}{20}=e^{\frac{t}{2}\log\left(\frac{1}{2}\right)}$

$\because\left(\frac{1}{2}\right)^{\frac{t}{2}}=\frac{48.6}{20}$

$\frac{t}{2}\log\left(\frac{1}{2}\right)=\log\left(\frac{48.6}{20}\right)$

$t=2\frac{\log\left(\frac{48.6}{20}\right)}{\log\left(\frac{1}{2}\right)}$

$t\approx-2.56\ (or)-2.30\ hours$

The person was murdered at about $5.30\ P.M$

5) The rate of increase in the number of bacteria in a certain bacteria culture is proportional to the number present. Given that the number triples in 5 hours, find how many bacteria will be present after 10 hours?

t	0	5	10
x	x_0	$3x$	?

$\frac{dx}{dt}=kx$

$x=Ce^{kt}$

$\underline{when\ t=0, x=x_0}$

$x_0=Ce^{0}$

$x_0=c(1)$

$\therefore c=x_0,\ x=x_0e^{kt}$

$\underline{when\ t=5; x=3x_0}$

$3x_0=x_0e^{5k}$

$e^{5k}=3$

$\underline{when\ t=10}$

$x=x_0e^{10k}$

$=x_0\left(e^{5k}\right)^2$

$=x_0(3)^2$

$x=9x_0$

$when\ t=0; x=100$

$\because c=-4900 \because x-5000=-4900\ e^{-0.01t}$

$x=5000-4900\ e^{-0.01t}$

6)A tank initially 50 litres of pure water, Starting at time t=0 a brine containing with 2 grams of dissolved salt per litre flows into the tank at the rate 3 litres per minute . The mixture is kept uniform by stirring and well-stirred mixture simultaneously flows out the tank at the same rate. Find the amount of salt present in the tank at any time $t>0$.

$\frac{dx}{dt}=6-\frac{3x}{50},\quad \frac{dx}{dt}=-\frac{3}{50}(x-100)$

$\therefore x-100=e^{-\frac{3}{50}t}$

$when\ t=0; x=0$

$\therefore C=-100x-100=-100e^{\frac{-3t}{50}}$

$x-100=\left(1-e^{\frac{-3t}{50}}\right)$

7) Find the population of a city at any time t, given that the rate of increase of population is proportional to the population at that instant and that in a period of 40 years the population increased from 300000 to 400000.

t	0	40
x	3,00,000	4,00,000

$\frac{dx}{dt}=kx$

$x=Ce^{kt}$

$\underline{when\ t=0, x=3,00,000}$

$\therefore c=3,00,000$

$x=3,00,000e^{40k}$

$4=3e^{40k}$

$e^{40k}=\frac{4}{3}$

$40k=\log\left(\frac{4}{3}\right)$

$K=\frac{1}{40}\log\left(\frac{4}{3}\right)$

$\therefore x=3,00,000\,e^{\frac{t}{40}\log\left(\frac{4}{3}\right)}$

$=3,00,000\left(\frac{4}{3}\right)^{\frac{t}{40}}$

8) The equation of electromotive force for an electric circuit containing resistance and self-inductance is $E = Ri + L\frac{di}{dt}$ where E is the electromotive force is given to the circuit, R the resistance and L, the coefficient of induction. Find the current i at the time when $E = 0$

$$L\frac{di}{dt} + Ri = E$$
$$\frac{di}{dt} + \frac{Ri}{L} = \frac{E}{L}$$
$$R(t) = \frac{R}{L}; Q(t) = \frac{E}{L}$$
$$I.F = e^{\int P(t)dt} = e^{\frac{Rt}{L}}$$
$$ie^{\frac{Rt}{L}} = \frac{E}{L}\int e^{\frac{Rt}{L}}\,dt + c$$
$$= \frac{E}{L}\frac{e^{\frac{Rt}{L}}}{\frac{R}{L}} + c$$
$$ie^{\frac{Rt}{L}} = \frac{E}{R}e^{\frac{Rt}{L}} + c$$
$$i = \frac{E}{R} + Ce^{\frac{Rt}{L}}$$
$$when\ E = 0$$
$$i = 0 + Ce^{\frac{-Rt}{L}}$$
$$i = Ce^{\frac{-Rt}{L}}$$

9) The engine of a motor boat moving at 10m/s is shut off. Given that the retardation at any subsequent time equal to the velocity at that time, Find the velocity after 2vseconds of switching off the engine.

T	0	2
V	10	?

$$\frac{dv}{dt} = -v$$
$$v = Ce^{-t}$$
$$when\ t = 0; v = 10$$
$$10 = Ce^0$$
$$\therefore C = 10$$
$$V = 10\,e^{-t}$$
$$when\ t = 2$$
$$V = 10\,e^{-2}$$

10) Suppose a person deposits 10000 Indian rupees in a bank account at the rate of 5% per annum compounded continuously. How much money will be in his bank account 18 m0nths later?

T	0	1.5
X	10,000	?

$$\frac{dx}{dt} = 0.05x$$
$$x = Ce^{0.05t}$$
$$when\ t = 0; x = 10,000$$
$$10,000 = Ce^0$$
$$\therefore C = 10,000$$
$$x = 10,000\ e^{0.05t}$$
$$when\ t = 1.5$$
$$x = 10,000\,e^{0.05(1.5)}$$
$$x = 10,000\,e^{0.075}$$

11) Assume that the rate at which radioactive nuclei decay is proportional to the number of such nuclei that are present in a given sample. In a certain sample 10% of the original number of radioactive nuclei have undergone disintegration in a period of 100 years. What percentage of the original radioactive nuclei remain after 1000 years?

T	0	1.5	1000
X	x_0	$90\%x_0$	?

$$\frac{dx}{dt} = kx$$
$$x = ce^{kt}$$
$$\underline{when\ t = 0; x = x_0}$$
$$x_0 = Ce^0$$
$$x_0 = C$$
$$x = x_0e^{kt}$$
$$\underline{when\ t = 100; x = \frac{9}{10}x_0}$$
$$\frac{9}{10}x_0 = x_0e^{100k}$$
$$\therefore e^{100k} = \frac{9}{10}$$
$$\underline{when\ t = 1000}$$
$$\text{Percentage} = \frac{x(1000)}{x(0)} \times 100\%$$
$$= \frac{x_0e^{1000k}}{x_0} \times 100\%$$
$$= \left(e^{100k}\right)^{10} \times 100\%$$
$$= \left(\frac{9}{10}\right)^{10} \times 100\% = \frac{9^{10}}{10^8}\%$$

12) Water at temperature 100C cools in 10 minutes to 80C in a room temperature of 25C.Find (i)The temperature of water

13) At 10.00 A.M a woman took a cup of hot instant coffee from her microwave oven and placed it on a nearby Kitchen counter to cool.

after 20minutes, (ii)The time when the temperature is 40C

$\left[\log_e \frac{11}{15} = -0.3101; \log_e 5 = 1.6094\right]$

T	0	10	20	?
T	100	80	?	40

$\frac{dT}{dt} \propto (T-25)$
$T - 25 = Ce^{kt}$
$\underline{when\ t = 0; T = 100}$
$100 - 25 = Ce^0$
$\therefore C = 75$
$T - 25 = 75e^{kt}$
$\underline{when\ t = 0; T = 80}$
$80 - 25 = 75e^{10k}$
$55 = 75e^{10k}$
$e^{10k} = \frac{11}{15}$
$10k = \log\left(\frac{11}{15}\right)$
$(i) when\ t = 20$
$T - 25 = 75e^{20k}$
$T = 25 + 75\left(e^{-10k}\right)^2$
$= 25 + 75\left(\frac{11}{15}\right)^2$
$T = 65.33℃$
$(ii) When\ t = 40$
$40 - 25 = 75e^{kt}$
$e^{kt} = \frac{15}{75} = \frac{1}{5}$
$kt = \log\frac{1}{5}$
$t = 10\frac{\log\frac{1}{5}}{\log\left(\frac{11}{15}\right)}$
$= 51.91\ Mins$

14)A pot of boiling water at 100C is removed from a stove at time t=0 in the kitchen. After 5 minutes, the water temperature has decreased to 80C,and another 5 minutes later it has dropped to 65C.Determine the temperature of the kitchen.

T	0	5	10
T	100	82	65

$\frac{dT}{dt} \propto (T-A), \quad \frac{dT}{dt} = -k(T-A)$

At this instant the temperature of this coffee was 180F, and 10 minutes later it was 160F.Assume that constant temperature of the kitchen was 70F.(i)What was the temperature of the coffee at 10.15 A.M ?(ii)The women likes to drink coffee when its temperature is between 130F and 140F between what times should she have drunk the coffee?.

t	0	10	15	?
T	130	160	?	130-140

$\frac{dt}{dT} \propto (T-70)$
$\frac{dT}{dt} = -k(T-70)$
$T - 70 = Ce^{kt}$
$\underline{when\ t = 0; T = 180}$
$180 - 70 = ce^0$
$\therefore C = 110$
$\mathrm{T} - 70 = 110\mathrm{e}^{\mathrm{kt}}$
$\underline{when\ t = 10; T = 160}$
$160 - 70 = 110e^{10k}$
$e^{10k} = \frac{9}{11}$
$when (i) t = 15$
$T - 70 = 110e^{15k}$
$= 70 + 110\left(e^{10k}\right)^{\frac{3}{2}}$
$= 70 + 110\left(\frac{9}{11}\right)^{\frac{3}{2}}$
$T = 70 + 81.4$
$T = 151.4$
$T \simeq 151℉$
$(ii) when\ t = 130$
$130 - 70 = 110e^{kt}$
$= e^{kt} = \frac{60}{110} = \frac{6}{11}$

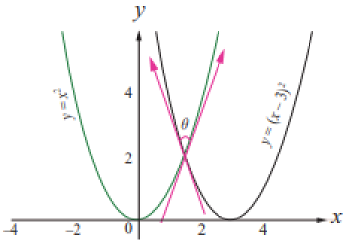
$T = \frac{\log\left(\frac{6}{11}\right)}{\log\left(\frac{11}{9}\right)}$
$= -\frac{10(-0.264)}{0.087}$
$T = 30.34$
$when\ T = 140$
$140 - 70 = 110e^{kt}$
$e^{kt} = \frac{70}{110} = \frac{7}{11}$
$e^{kt} = \frac{11}{7}$

$T-A=Ce^{-kt}, t=0; T=100$ எனில் $100-A=Ce^{\circ}=C(1)=C$ $\therefore C=100-A$ $t=5; T=80$ எனில் $65-A=Ce^{-10k}$ $65-A=C\left(e^{-5k}\right)^2=C\left(\frac{80-A}{C}\right)^2$ $65-A=\left(\frac{80-A}{C}\right)^2=\left(\frac{80-A}{100-A}\right)^2$ $(65-A)(100-A)=(80-A)^2,$ $A=20°C$	$t=10\frac{\log\left(\frac{11}{7}\right)}{\log\left(\frac{11}{9}\right)}=\frac{10(0.197)}{0.087}=22.6$ 10.22 AM between 10.30 AM should she have drunk the coffee

7.Applications of Differential Calculus

(Important 5- Marks)

1)Find the angle between the curves $y=x^2$ and $y=(x-3)^2$



Point of intersection is $\left(\frac{3}{2},\frac{9}{4}\right)$

$y=x^2$

$\frac{dy}{dx}=2x$

$m_1=\left(\frac{dy}{dx}\right)_{\left(\frac{3}{2},\frac{9}{4}\right)}=3$

$y=(x-3)^2$

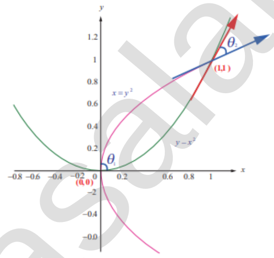
$\frac{dy}{dx}=2(x-3)$

$m_2=\left(\frac{dy}{dx}\right)_{\left(\frac{3}{2},\frac{9}{4}\right)}=-3$

$\tan\theta=\left|\frac{m_1-m_2}{1+m_1m_2}\right|=\frac{3}{4}$

$\therefore \theta=\tan^{-1}\left(\frac{3}{4}\right)$

2)Find the angle between the curves $y=x^2$ and $x=y^2$ at their points of intersection (0,0) and (1,1)



$y=x^2$	$x=y^2$
$\frac{dy}{dx}=2x$	$1=2y\frac{dy}{dx}$
$m_1=2x$	$m_2=\frac{1}{2y}$

The angle between $(0,0)$ and $(1,1)$.

$tan\theta_1=\left|\frac{m_1-m_2}{1+m_1m_2}\right|$

$At(0,0), tan\theta_1=\frac{1}{0}$

$\therefore \theta_1=tan^{-1}(\infty)=\frac{\pi}{2}$

$At\ (1,1)\ \ m_1=2;\ m_2=\frac{1}{2}$

$\therefore tan\theta_2=\left|\frac{2-\frac{1}{2}}{1+(2)(\frac{1}{2})}\right|=\frac{3}{4}$

$\therefore \theta_2=tan^{-1}\left(\frac{3}{4}\right)$

3)If the curves $ax^2+by^2=1$ and $cx^2+dy^2=1$ $\frac{1}{a}-\frac{1}{b}=\frac{1}{c}-\frac{1}{d}$ intersect each other orthogonally.

At (x_0,y_0)

$a{x_0}^2+b{y_0}^2=1\Rightarrow\frac{dy}{dx}=-\frac{ax_0}{by_0}$

$c{x_0}^2+d{y_0}^2=1\Rightarrow\frac{dy}{dx}=-\frac{cx_0}{dy_0}$

4)Prove that the ellipse $x^2+4y^2=8$ and hyperbola $x^2-2y^2=4$ intersect orthogonally.

$x^2+4y^2=8$

$\div 8, \frac{x^2}{8}+\frac{y^2}{2}=1\Rightarrow a=\frac{1}{8}, b=\frac{1}{2}$

$x^2-2y^2=4$

Cuts orthogonally,

$m_1 \times m_2 = -1 \Rightarrow \left(\frac{-ax_0}{by_0}\right) \times \left(\frac{-cx_0}{dy_0}\right) = 1$

$\Rightarrow acx_0{}^2 + bdy_0{}^2 = 0$

$(a-c)x_0{}^2 + (b-d)y_0{}^2 = 0$

$\frac{a-c}{ac} = \frac{b-d}{bd}$

$\Rightarrow \frac{1}{a} - \frac{1}{b} = \frac{1}{c} - \frac{1}{d}$

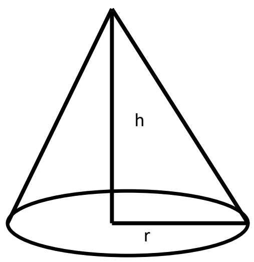
$\div 4, \frac{x^2}{4} - \frac{y^2}{2} = 1 \Rightarrow c = \frac{1}{4}, d = -\frac{1}{2}$

(If the curves $ax^2 + by^2 = 1$ and $x^2 + dy^2 = 1$ $\frac{1}{a} - \frac{1}{b} = \frac{1}{c} - \frac{1}{d}$ intersect each other orthogonally)

$8 - 2 = 4 + 2 \Rightarrow 6 = 6$

Hence, Given curves cut orthogonally.

5) Salt is poured from a conveyer belt at a rate of 30 cubic meter per minute forming a conical pile with a circular base whose height and diameter of base are always equal. How fast is the height of the file increasing when the pile is 10 m high?



At time t, In a Cone Height -h,Raduis-r and Volume-V Given $h = 2r$.

$\frac{dv}{dt} = 30$

$V = \frac{1}{3}\pi r^2 h$

$\frac{dv}{dt} = \frac{1}{4}\pi h^2 \frac{dh}{dt}$

$\frac{dh}{dt} = 4 \times 30 \times \frac{1}{100\pi}$

$\therefore \frac{dh}{dt} = \frac{6}{5\pi}$.

6)A steel plant is capable of producing x tonnes per day of low-grade steel, where $y = \frac{40-5x}{10-x}$. If the fixed market price of low-grade steel is half that of high-grade steel, then what should be optimal productions in low-grade steel and high-grade steel in order to have maximum receipts.

$\text{low} - \text{grade steel} = p$

high-grade steel= $2p$

$R = Px + 2py \Rightarrow px + 2p\left(\frac{40-5x}{10-x}\right)$

$\frac{dR}{dx} = p\left[\frac{x^2 - 20x + 80}{(10-x)^2}\right]$

$\frac{d^2R}{dx^2} = -\frac{40}{(10-x)^3}$

$\frac{dR}{dx} = 0 \Rightarrow x = 10 \pm 2\sqrt{5}$

$x = 10 - 2\sqrt{5}$

$\frac{d^2R}{dx^2} < 0, R$ will be maximum

$x = 10 - 2\sqrt{5}$ and $y = 5 - \sqrt{5}$.

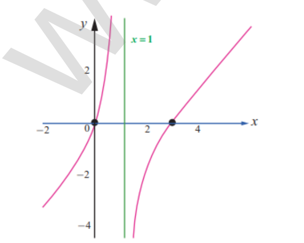
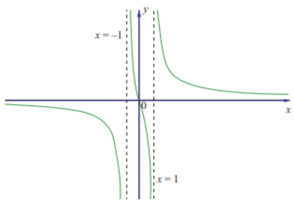
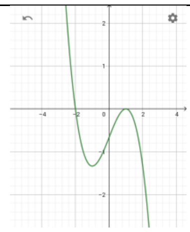
Do it yourself:

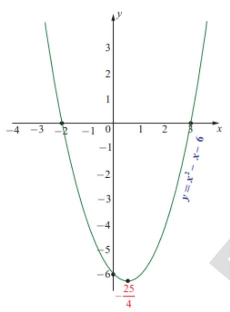
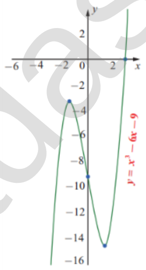
Exercise-10,8,1012,11,6,5

Example:7.62,7.63,7.65

Exercise-7.2-8,9

Example-7.13,7.7

Trace the curve	$y = \frac{x^2 - 3x}{x - 1}$	$y = \frac{3x}{x^2 - 1}$	$y = -\frac{1}{3}(x^3 - 3x + 2)$
Diagram			
Domain, Range	Domain: $R - \{1\}$ Range: R	Domain: $R - \{-1, 1\}$ Range: R	Domain: R Range :R
Intercepts	x – Intercept=0,3 y - Intercepts =0	x - Intercepts =0 y - Intercepts =0	x - Intercepts =2,1 y - Intercepts =$-\frac{2}{3}$

Critical Points	$x = 1$	$x = -1, 1$	$x = -1, 1$
Local extrema	NO	NO	At $x = 1$, local maximum $f(1) = 0$ At $x = -1$,local minimum, $f(-1) = -\frac{4}{3}$
Concavity	For $(-\infty, 1)$, Concave up For $(1, \infty)$, Concave down	For$(-1, 0), (1, \infty), f''(x) > 0$ Concave up For$(-\infty, -1), (0, 1), f''(x) < 0$ Concave down	For$(-\infty, 0), f''(x) > 0$ Concave up For$(0, \infty), f''(x) < 0$ Concave down
Points of inflection	NO	$(0, 0)$	$\left(0, -\frac{2}{3}\right)$
Asymptotes	$x = 1$ Vertical asymptotes	$x = -1, 1$ Vertical asymptotes	NO
Trace the curve	$y = x^2 - x - 6$	$y = x^3 - 6x - 9$	
Diagram			
Domain Range	Domain:$(-\infty, \infty)$ Range: $y \geq -\frac{25}{4}$	Domain:$(-\infty, \infty)$ Range: $(-\infty, \infty)$	
Intercepts	x - Intercept $= -2, 3$ y - Intercept$= -6$	x- Intercept=3 y - Intercept $= -9$	
Critical Points	$f'(x) = 2x - 1$ $x = \frac{1}{2}$	$f'(x) = 3(x^2 - 2)$ $x = \pm\sqrt{2}$	
Local extrema	For $x = 1/2$ Local minimum $f\left(\frac{1}{2}\right) = -\frac{25}{4}$	For $x = \sqrt{2}, f''(x) > 0$ Local minimum $f(\sqrt{2}) = -4\sqrt{2} - 9$ Local maximum $x = -\sqrt{2}, f''(x) < 0$ $f(\sqrt{2}) = 4\sqrt{2} - 9$	