Unit -6 - QUANTUM MECHANICS AND RELATIVITY

Quantum mechanics:

Wave nature of particles deBroglie waves Davison and Germer experiment waves and particle duality photoelectric effect photo electric multiplier Einstein's equation Compton Effect experimental verification of Compton effect wave nature of electron-Heisenberg's uncertainty principle-position and momentum, energy and time uncertainty Schrodinger's wave equation probability amplitude properties of wave function-normalization- potential barriers-tunnelling across barriers-particle in a box (one dimension only)

Relativity:

Relativity postulate of Special theory of Relativity Lorentz transformation of equations and its application - length contraction, time dilation-variation of mass with velocityMass energy equivalence-Physical Significance.

RELATIVITY

Frame of Reference:

A system of co-ordinate axes which defines the position of a particle in two or three dimensional space is called a frame of reference.

Example: Cartesian system of co-ordinates.

- Unaccelerated reference frames in uniform motion of translation relative to one another are called Galilean frames or inertial frames.
- Accelerated frames are called non-inertial frames.

Newtonian Relativity:

The Newtonian principle of relativity may be stated as *"Absolute motion, which is the translation of a body from one absolute place to another absolute place, can never be detected.*

Translatory motion can be perceived only in the form of motion relative to other material bodies".

- The fundamental physical laws and principles are identical in all inertial frames of reference.

According to Newtonian mechanics:

- Space is absolute (i.e., length of object is independent.)
- Time is absolute (i.e., time interval between two events are same)

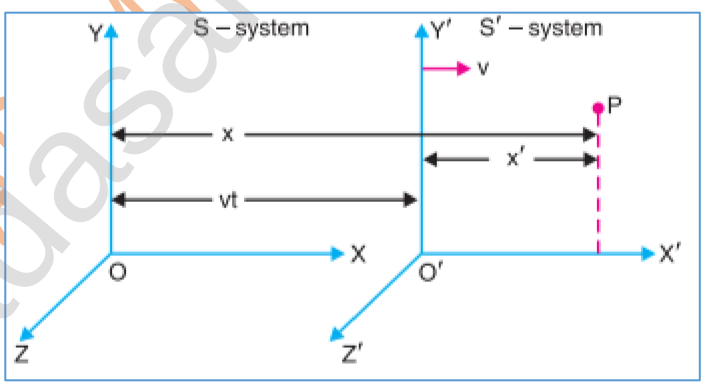
  if two events are simultaneous for an observer they are simultaneous for all observers. i.e., simultaneity is absolute.
- Mass does not depend on the velocity of motion.

Galilean Transformation Equations:

- Let S and S′ be two inertial frames

Let S be at rest and S′ move with uniform velocity v along the positive X direction. We assume that $v << c$.

Let the origins of the two frames coincide at $t = 0$.



Suppose some event occurs at the point P.

The observer O in frame S determines the position of the event by the coordinates x, y, z.

The observer O' in frame S' determines the position of the event by the coordinates x', y', z'.

Galilean Transformation Equations:

$$x' = x - vt,$$

$$y' = y,$$

$$z' = z,$$

$$t' = t$$

$(x, y, z, t) \rightarrow$ *space and time coordinate in S*

$x', y', z', t' \rightarrow$ *space and time coordinate in S′*

- Newton's laws of motion remain unaffected by a Galilian transformation.

Inverse transformations:

$$x = x' + vt$$

$$y = y'$$

$$z = z'$$

$$t = t'$$

$$\frac{dx'}{dt} = \frac{dx}{dt} - \frac{vdt}{dt}$$

$$\boldsymbol{u'} = \boldsymbol{u} - \boldsymbol{v}$$

$$\Longrightarrow \frac{du'}{dt} = \frac{du}{dt} - 0$$

Thus $\boxed{\boldsymbol{a'} = \boldsymbol{a}}$

- **The accelerations, as measured by the two observers in the two frames, are the same.**
- Hence acceleration is invariant under Galilian transformations.

SPECIAL THEORY OF RELATIVITY:

- It deals with inertial or unaccelerated systems.

General theory of relativity:

It deals with non inertial or accelerated systems

Postulates of Special theory of relativity:

1. The Fundamental laws of physics are the same form for all inertial systems.
2. Velocity of light in vacuum is constant for all frames. It is independent of the relative motion of the source and the observer.

Note: The velocity of light is not constant under Galilean transformations. But according to the second postulate, the velocity of light is the same in all inertial frames.

Thus the second postulate is very important and only this postulate is responsible to differentiate the classical theory and Einstein's theory of relativity.

Lorentz transformations:

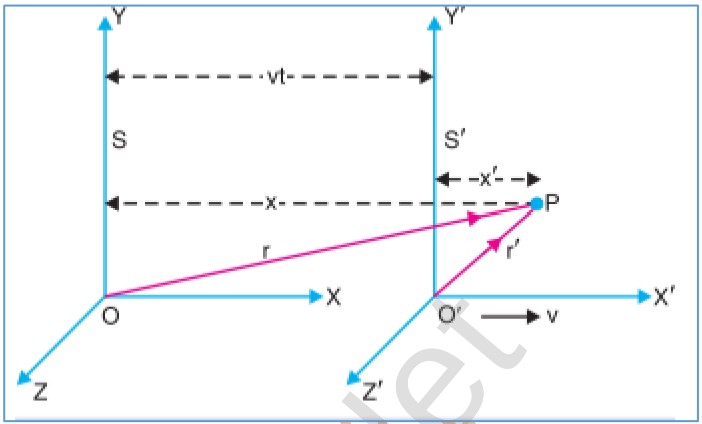
We have to introduce new transformation equations which are consistent with the new concept of the invariance of light velocity in free space. The new transformation equations were discovered by Lorentz, and are known as "Lorentz transformations".

Derivation:

Consider two observers O and O' in two systems S and S′.

System S' is moving with a constant velocity v relative to system S along the positive X −axis.

Suppose we make measurements of time from the instant when the origins of S and S' just coincide, i.e., $t = 0$ when O and O' coincide.



Suppose a light pulse is emitted when O and O′ coincide.

The light pulse produced at $t = 0$ will spread out as a growing sphere.

After a time t, the observer O will note that the light has reached a point $P\ (x, y, z)$ as shown in Fig.

Transformation equations:

$$x' = \frac{x - vt}{\sqrt{1-\frac{v^2}{c^2}}}$$

$$y' = y$$

$$z' = z$$

$$t' = \frac{(t - vx/c^2)}{\sqrt{1-\frac{v^2}{c^2}}}$$

Inverse Lorentz Transformations:

- **Replacing $v \to -v$**

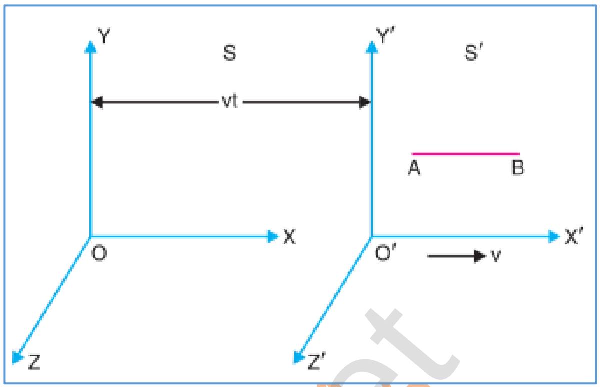
$$x = \frac{x' + vt'}{\sqrt{1-\frac{v^2}{c^2}}}$$

$$y = y'$$

$$z = z'$$

$$t = \frac{t' + \frac{v\,x'}{c^2}}{\sqrt{1-\frac{v^2}{c^2}}}$$

LENGTH CONTRACTION:



Length of the rod in S' (original length or proper length)

$$l_0 = x_2' - x_1'$$

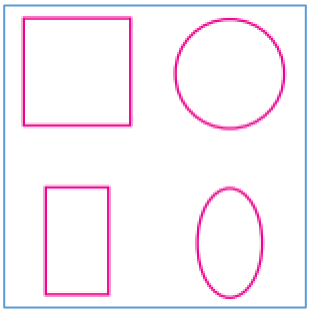
Length of the rod in S' (original length)

Length of the rod w.r.t to S: $l = x_2 - x_1$

- $l = l_0\sqrt{1-\frac{v^2}{c^2}} = l_0\sqrt{1-\beta^2}$
- $l < l_0$
- Where $\beta = \frac{v}{c}$

The length of the object is contracted in the direction of motion by the factor $\sqrt{1-\frac{v^2}{c^2}}$.

This phenomenon is known as ***Lorentz Fitzgerald*** contraction.



- ❖ A body which appears to be **spherical** to an observer at rest relative to it, will appear to be an oblate **spheroid** to a moving observer.
- ❖ Similarly, a **square** appear to be a **rectangle.**
- ❖ a **circle** appear to be a an **ellipse.**
- ❖ The shortening or contraction in the length of an object along its direction of motion is known as the Lorentz-Fitzgerald contraction. There is no contraction in a direction perpendicular to the direction of motion.
- ❖ The contraction becomes appreciable only when $v \approx c$.
- ❖ ***The contraction is reciprocal***, i.e., if two identical rods are at rest—one in S and the other in S', each of the observers finds that the other is shorter than the rod of his own.

EXAMPLE 1. *A rod 1 metre long is moving along its length with a velocity 0.6c. Calculate its length as it appears to an observer (a) on the earth (b) moving with the rod itself.*

SOL. Here, 1 metre is the proper length (l_0) of the rod in its own moving frame of reference. (*a*) Let l be the length of the rod as it appears to an observer in the stationary reference frame of the earth.

Here, $l_0 = 1$ m; $v = 0.6$ c; $l = ?$

$$l = l_0\sqrt{1-\frac{v^2}{c^2}} = 1\sqrt{1-\frac{(0.6c)^2}{c^2}} = 1\sqrt{1-0.36} = 0.8 \text{ m}$$

Hence, the observer on the earth will estimate the length of the rod to be 0.8 metre.

(*b*) For an observer moving with the rod itself, the length of the rod is 1 metre.

EXAMPLE 2. *How fast would a rocket have to go relative to an observer for its length to be contracted to 99% of its length at rest?*

SOL. Here, $l = 0.99 l_0$; $v = ?$

We have, $l = l_0\sqrt{1-(v^2/c^2)}$

$$l^2 = l_0^2\left(1-\frac{v^2}{c^2}\right) \text{ or } \left(1-\frac{v^2}{c^2}\right) = \frac{l^2}{l_0^2} \text{ or } \frac{v^2}{c^2} = 1-\frac{l^2}{l_0^2}.$$

$$\therefore \quad v^2 = c^2\left(1-\frac{l^2}{l_0^2}\right) = c^2\,(1-0.99^2)$$

$$\therefore \quad v = 0.1416\,c = 0.1416 \times (3\times 10^8) = 4.245 \times 10^7 \text{ ms}^{-1}$$

TIME DILATION:

Imagine a gun placed at the position (x′, y′, z′) in S′.

Suppose it fires two shots at times $t_1{}'$ and $t_2{}'$ measured with respect to S′.

In S ′ the clock is at rest relative to the observer.

Proper time interval $\boldsymbol{t_0 = t_2{}' - t_1{}'}$ is the time interval between the two shots for the observer in S′.

Since the gun is fixed in S', it has a velocity v with respect to S in the direction of the positive X −axis.

Let $\boldsymbol{t = t_2 - t_1}$ represent the time interval between the two shots as measured by an observer in S.

- $t = \frac{t_0}{\sqrt{1-\frac{v^2}{c^2}}} = \frac{t_0}{\sqrt{1-\beta^2}}$
- $t > t_0$

- The time interval to be dilated (or lengthened by a factor $\frac{1}{\sqrt{1-\frac{v^2}{c^2}}}$)
- The clock will be found to run slow.
- This phenomenon is known as apparent retardation of clock.

- **Examples of time dilation:** Twin paradox
- **Illustration of time dilation and length contraction:** μ meson decay

EXAMPLE 1. *A clock in a space ship emits signals at intervals of 1 second as observed by an astronaut in the space ship. If the space ship travels with a speed of 3×10^7 ms^{-1}, what is the interval between successive signals as seen by an observer at the control centre on the ground?*

SOL. Here, $t_0 = 1$s; $v = 3 \times 10^7$ ms^{-1}; and $c = 3 \times 10^8$ ms^{-1}; $t = ?$

$$\therefore \quad t = \frac{t_0}{\sqrt{1-\frac{v^2}{c^2}}} = \frac{1}{\sqrt{1-\frac{(3\times10^7)^2}{(3\times10^8)^2}}} = 1.005 \text{ s.}$$

EXAMPLE 2. *A particle with a proper lifetime of 1μs moves through the laboratory at 2.7×10^8 ms^{-1}. (a) What is its lifetime, as measured by observers in the laboratory? (b) What will be the distance traversed by it before disintegrating?*

SOL. Here, $t_0 = 1\mu s = 10^{-6}$s; $v = 2.7 \times 10^8$ ms^{-1}; $t = ?$

$$(a) \quad t = \frac{t_0}{\sqrt{1-\frac{v^2}{c^2}}} = \frac{10^{-6}}{\sqrt{1-\frac{(2.7\times10^8)^2}{(3\times10^8)^2}}} = 2.3 \times 10^{-6} \text{ s.}$$

(b) The average distance moved by the particle before disintegration $= (2.7 \times 10^8) \times (2.3 \times 10^{-6})$
$= 620$ m.

Velocity addition theorem:

Suppose the system S' moves with a uniform velocity v relative to the system S .suppose a particle is moving in the common direction of x and x' axis. Let us velocity as measured by an observer in the system S be u and as measured by an observer in S' be u'.

- $u = \frac{(u'+v)}{1+\frac{u'v}{c^2}}$

- $u' = \frac{(u-v)}{1-\frac{uv}{c^2}}$ (Replace $v \to -v$)

$If\ \ u' = c \quad \& \ \ v = c$

i.e ., in case of photon is moving with a velocity ***c*** in the frame S' and S; is moving with ***c*** relative to S then,

Thus $\quad u = \frac{(c+c)}{1+\frac{c^2}{c^2}} = c$

- Thus the addition of velocity to the velocity of light C merely reproduces the velocity of light.
- Hence, the velocity of light is the maximum attainable velocity.
- Velocity addition theorem only applies when the two velocities are in the same direction.
- If $v \ll c$, we get the classical eqn $(u' = u + v)$

EXAMPLE 1. *An experimenter observes a radioactive atom moving with a velocity of 0.25c. The atom then emits a β particle which has a velocity of 0.9c relative to the atom in the direction of its motion. What is the velocity of the β particle, as observed by the experimenter?*

SOL. Here, $v = 0.25c;\ u' = 0.9c;\ u = ?$

$$u = \frac{u'+v}{1+\frac{v}{c^2}u'} = \frac{0.9c+0.25c}{1+\frac{0.25c}{c^2}\times 0.9c} = 0.94c.$$

EXAMPLE 2. *An electron is moving with a speed of 0.85c in a direction opposite to that of a moving photon. Calculate the relative velocity of the photon with respect to the electron.*

SOL. Let the photon be moving along the positive direction of X-axis and the electron along the negative direction of X-axis. Then, the speed of electron = – 0.85c; the speed of photon = c. Consider that the electron is at rest in system S. Then, we may assume that the system S' (laboratory) is moving with velocity 0.85c relative to system S (electron). *i.e.*, $v = 0.85c;\ u' = c;\ u = ?$

$$u = \frac{u'+v}{1+\frac{u'v}{c^2}} = \frac{c+0.85c}{1+\frac{(c)\times(0.85c)}{c^2}} = c.$$

Relativity of Simultaneity:

Consider two events that occur at the same time to an observer O in a reference frame S. Let the two events occur at different locations x_1 and x_2.

Consider another observer O' in S' moving with a uniform relative speed **v** with respect to S in the positive X −direction.

This indicates that two events which are simultaneous to the observer in S, do not appear so to the observer in S′.

Therefore, the concept of ***simultaneity has only a relative and not an absolute meaning***.

VARIATION OF MASS WITH VELOCITY:

If m denotes the mass of a body when it is moving with a velocity v, then,

- $m = \frac{m_0}{\sqrt{1-\frac{v^2}{c^2}}}$

$i.e\ \ m > m_0$

- This is the relativistic formula for the variation of mass with velocity.
- m *increases when* v *increases*
- m *– mass in motion*
- m_0 *– rest mass*

- If $v \to c., m \to \infty$

- i.e., an object traveling C would have infinite mass.
- Thus no material body can have a velocity equal to or greater than C.

Notes:

(1) The first verification of the increase in mass with velocity came from the work of Kaufmann in 1906 and of Bucherer in 1909.
While studying the β-rays emanating from radioactive materials, they found that their velocities were comparable to the velocity of light and also that their masses were found to be related to their velocities.

(2) The increase of mass with velocity has now been tested in "particle accelerators".
It has been found that the electrons and protons accelerated in these machines to velocities close to the velocity of light acquire increased masses, exactly as predicted.

EXAMPLE. *At what speed is a particle moving if the mass is equal to three times its rest mass?*

SOL. We have, $m = \frac{m_0}{\sqrt{1-v^2/c^2}}$. Here, $m = 3m_0$; $v = ?$

$\therefore$ $3m_0 = \frac{}{\sqrt{1\ \ v^2/c^2}}$ or $\sqrt{1-\frac{v^2}{c^2}} = \frac{1}{3}$ or $1-\frac{v^2}{c^2} = \frac{1}{9}$

Or $\frac{v^2}{c^2} = \frac{8}{9}$ or $v^2 = \frac{8}{9}c^2$ or $v = \sqrt{\frac{8}{9}}c = 0.94c.$

MASS ENERGY EQUIVALENCE:

According to the theory of relativity, both mass and velocity are variable.

- Kinetic Energy $E_k = mc^2 - m_0c^2$
- Rest Energy $= m_0c^2$
- Total Energy is equal to sum of K.E and rest mass energy
- Total Energy = K.E + Rest Mass Energy
- $\therefore Total\ Energy = \boxed{\boldsymbol{E = mc^2}} \rightarrow (\mathbf{1})$
- Eqn $\rightarrow (\mathbf{1})$ is called **Einstein's mass-energy relation.**

Notes:

(1) This relation states a universal equivalence between mass and energy. It means that mass may appear as energy and energy as mass.

Example: Consider the phenomenon of **pair-annihilation or pair production**.

In this phenomenon, an electron and a positron can combine and literally disappear. In their place we find high energy radiation called γ-radiation, whose radiant energy is exactly equal to the rest mass plus kinetic energies of the disappearing particles. The process is reversible, so that a materialization of mass from radiant energy can occur when a high enough energy γ-ray, under proper conditions, disappears. In its place appears a positron-electron pair whose total energy (rest mass + K.E.) is equal to the radiant energy lost.

(2) The relationship ($E = mc^2$) between energy and mass forms the basis of understanding **nuclear reactions such as fission and fusion.**

These reactions take place in nuclear bombs and reactors. When a uranium nucleus is split up, the decrease in its total rest mass appears in the form of an equivalent amount of K.E. of its fragments.

(3) The formula for K.E. reduces to the classical formula for $v << c$.

- $i.e., when\ v << c \Rightarrow K.E = \frac{1}{2} m_0v^2$

Examples of mass energy conversion:

1. Electron – positron pair production & annihilation
2. Nuclear energy
3. Binding energy.

EXAMPLE 1. *If 4 kg of a substance is fully converted into energy, how much energy is produced?*

SOL. Here, $m = 4$ kg, $c = 3 \times 10^8$ ms^{-1}; $E = ?$

$$E = mc^2 = 4 \times (3 \times 10^8)^2 = 3.6 \times 10^{17} \text{ J.}$$

EXAMPLE 2. *Calculate the rest energy of an electron in joules and in electron volts.*

SOL. Here, m_0 = rest mass of the electron = 9.11×10^{-31} kg;

$$c = 3 \times 10^8 \text{ ms}^{-1}$$

$$\therefore \quad E = m_0c^2 = (9.11 \times 10^{-31})(3 \times 10^8)^2 = 8.2 \times 10^{-14} \text{ J.}$$

$$= \frac{8.2 \times 10^{-14}}{1.6 \times 10^{-19}} eV \quad (\text{since } 1eV = 1.6 \times 10^{-19} \ J).$$

$$= 0.51 \text{ MeV [mega electron volt].}$$

Relation between momentum and energy and conservation laws:

$$E^2 = p^2c^2 + m_0^2c^4$$

$$or \quad E = \sqrt{p^2c^2 + m_0^2c^4}$$

if a particle of rest mass m_0 moves with velocity v its momentum p is given by

- $\boldsymbol{p = \frac{m_0 v}{\sqrt{1-\frac{v^2}{c^2}}}}$
- $\boldsymbol{and \ \ E = \frac{m_0 c^2}{\sqrt{1-\frac{v^2}{c^2}}}}$

Particles with zero rest mass:

$$E = (p^2c^2 + m_0^2c^4)^{\frac{1}{2}}$$

$$m_0 = 0 \Longrightarrow E = (p^2c^2)^{1/2} = pc$$

$$p = \frac{E}{c} = \frac{mc^2}{c} = mc$$

$$\Longrightarrow \boldsymbol{v = c}$$

- *Thus a particle of zero rest mass travels with the speed of light. (i.e., photon)*
- **Relativistic mass of photon** $m = \frac{E}{c^2}$
- Momentum of the photon $p = h/\lambda$

Problems:

1. **An experimenter observes a radioactive atom with velocity of 0.25 C . the atom then emits a β particle which has a velocity of 0.9C relative to the atom in the velocity of the β particle , as observed by the experimenter.**

   Data: $V = 0.25C$, $u' = 0.9C$, $u = ?$

   Solution: $u = \frac{(u'+v)}{1+\frac{u'v}{c^2}} = \frac{(0.9c+0.25c)}{1+\frac{0.9c\times0.25c}{c^2}} = \mathbf{0.94C}$

2. **An electron is moving with a speed of 0.85 C in a direction opposite to that of a moving photon. Calculate the relative velocity of the photon with respect to th electron.**

   Solution: $u = \frac{(u'+v)}{1+\frac{u'v}{c^2}} = \frac{(c+0.85c)}{1+\frac{c\times0.85c}{c^2}} = \mathbf{C}$

3. **The mass of the particle travel with a velocity $\frac{C}{\sqrt{2}}$ and its rest mass is m_0.**

   Ans: $\mathbf{1.414\ m_0 = \sqrt{2}\ m_0}$

PARTICLE-WAVE DUALITY:

- The quantum hypothesis of Planck and the subsequent interpretation of the idea by Einstein gave electromagnetic radiation discrete properties somewhat similar to those of a particle.
- These quantized components of light became known as photons.
- According to Einstein, light of frequency ν consists of a photons each of energy $h\nu$.
- It was surprising that such discrete characteristics should be in evidence since light was firmly established as a wave phenomenon.
- The quantum theory of light which treats it strictly as a particle phenomenon, incorporates the light frequency ν, a wave concept.

- The quantum theory made provision for radiation to have both wave and particle aspects in a complementary form of coexistence.

Let us discuss the connection between the wave theory of light and the quantum theory of light.

- The radiation field is a continuous system of propagating electric and magnetic oscillations.
- Consider a monochromatic wave of frequency $\boldsymbol{\nu}$ that is incident on a screen.
- The flow of energy per unit time across unit area is the intensity of the wave

$$\boldsymbol{I = \varepsilon_0 c \bar{E}^2}$$

- Here $\bar{E}^2$ is the average of the square of the instantaneous magnitude of the wave's electric field over a complete cycle.

(ii) **Consider the photon model of the same wave**.

- The intensity of radiation (I) is equal to the number of photons (N) crossing unit area per second multiplied by the energy (hv) of the photon i.e.,

$$I = N\,hv$$

Both descriptions must give the same value for I.

$$N\,hv = \boldsymbol{\varepsilon_0 c \bar{E}^2}$$

This formula is a statement of particle-wave duality since the left side is written in the language of discrete quanta while the right side is written in the language of wave fields.

The rate of arrival of photons at the screen is given by $N = \frac{\boldsymbol{\varepsilon_0 c \bar{E}^2}}{\boldsymbol{hv}}$

If N is large, an observer looking at the screen would find a continuous distribution of light whose pattern corresponds to the distribution of $\bar{E}^2$..

This shows the validity of the wave theory of light.

If N is small, the observer would find a series of apparently random flashes. This points to light being a quantum phenomenon.

Consider an experimental arrangement in which light is diffracted by a double slit and is detected on a screen.

- Suppose the light beam is very weak and there is only one photon at a time in the apparatus.
- We can regard each photon as having a wave associated with it.
- The two slits transmit the photon in the manner of a wave passing through both slits at once.
- When it strikes the screen, light is behaving as a particle does.
- Thus the wave and particle pictures give complementary descriptions of the same system.

Bohr introduced the complementarity principle which states that the wave and particle aspects of matter are complementary rather than being contradictory, both equally essential for a full description of the phenomenon.