மருதம் அகாடமி Youtube channel

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Last minute study questions

MATHS

Example 1.5

Find a matrix A if $\text{adj}(A)=\begin{bmatrix}7&7&-7\\-1&11&7\\11&5&7\end{bmatrix}$.

Solution

First, we find $|\text{adj}(A)|=\begin{vmatrix}7&7&-7\\-1&11&7\\11&5&7\end{vmatrix}=7(77-35)-7(-7-77)-7(-5-121)=1764>0.$

So, we get

$$A=\pm\frac{1}{\sqrt{|\text{adj}\,A|}}\text{adj}(\text{adj}\,A)=\pm\frac{1}{\sqrt{1764}}\begin{bmatrix}+(77-35)&-(-7-77)&+(-5-121)\\-(49+35)&+(49+77)&-(35-77)\\+(49+77)&-(49-7)&+(77+7)\end{bmatrix}^T$$

$$=\pm\frac{1}{42}\begin{bmatrix}42&84&-126\\-84&126&42\\126&-42&84\end{bmatrix}^T=\pm\begin{bmatrix}1&-2&3\\2&3&-1\\-3&1&2\end{bmatrix}.$$

Example 1.11

Prove that $\begin{bmatrix}\cos\theta&-\sin\theta\\\sin\theta&\cos\theta\end{bmatrix}$ is orthogonal.

Solution

Let $A=\begin{bmatrix}\cos\theta&-\sin\theta\\\sin\theta&\cos\theta\end{bmatrix}$. Then, $A^T=\begin{bmatrix}\cos\theta&-\sin\theta\\\sin\theta&\cos\theta\end{bmatrix}^T=\begin{bmatrix}\cos\theta&\sin\theta\\-\sin\theta&\cos\theta\end{bmatrix}$.

So, we get

$$AA^T=\begin{bmatrix}\cos\theta&-\sin\theta\\\sin\theta&\cos\theta\end{bmatrix}\begin{bmatrix}\cos\theta&\sin\theta\\-\sin\theta&\cos\theta\end{bmatrix}$$

$$=\begin{bmatrix}\cos^2\theta+\sin^2\theta&\cos\theta\sin\theta-\sin\theta\cos\theta\\\sin\theta\cos\theta-\cos\theta\sin\theta&\sin^2\theta+\cos^2\theta\end{bmatrix}=\begin{bmatrix}1&0\\0&1\end{bmatrix}=I_2.$$

Similarly, we get $A^TA=I_2$. Hence $AA^T=A^TA=I_2\Rightarrow A$ is orthogonal.

Example 1.9

Verify $(AB)^{-1}=B^{-1}A^{-1}$ with $A=\begin{bmatrix}0&-3\\1&4\end{bmatrix}$, $B=\begin{bmatrix}-2&-3\\0&-1\end{bmatrix}$.

Solution

We get $AB=\begin{bmatrix}0&-3\\1&4\end{bmatrix}\begin{bmatrix}-2&-3\\0&-1\end{bmatrix}=\begin{bmatrix}0+0&0+3\\-2+0&-3-4\end{bmatrix}=\begin{bmatrix}0&3\\-2&-7\end{bmatrix}$

$$(AB)^{-1}=\frac{1}{(0+6)}\begin{bmatrix}-7&-3\\2&0\end{bmatrix}=\frac{1}{6}\begin{bmatrix}-7&-3\\2&0\end{bmatrix}\quad\ldots(1)$$

$$A^{-1}=\frac{1}{(0+3)}\begin{bmatrix}4&3\\-1&0\end{bmatrix}=\frac{1}{3}\begin{bmatrix}4&3\\-1&0\end{bmatrix}$$

$$B^{-1}=\frac{1}{(2-0)}\begin{bmatrix}-1&3\\0&-2\end{bmatrix}=\frac{1}{2}\begin{bmatrix}-1&3\\0&-2\end{bmatrix}$$

$$B^{-1}A^{-1}=\frac{1}{2}\begin{bmatrix}-1&3\\0&-2\end{bmatrix}\frac{1}{3}\begin{bmatrix}4&3\\-1&0\end{bmatrix}=\frac{1}{6}\begin{bmatrix}-7&-3\\2&0\end{bmatrix}.\quad\ldots(2)$$

As the matrices in (1) and (2) are same, $(AB)^{-1}=B^{-1}A^{-1}$ is verified. ■

Example 1.14

Reduce the matrix $\begin{bmatrix}0&3&1&6\\-1&0&2&5\\4&2&0&0\end{bmatrix}$ to a row-echelon form.

Solution

$$\begin{bmatrix}0&3&1&6\\-1&0&2&5\\4&2&0&0\end{bmatrix}\xrightarrow{R_1\leftrightarrow R_2}\begin{bmatrix}-1&0&2&5\\0&3&1&6\\4&2&0&0\end{bmatrix}\xrightarrow{R_3\to R_3+4R_1}\begin{bmatrix}-1&0&2&5\\0&3&1&6\\0&2&8&20\end{bmatrix}$$

$$\xrightarrow{R_3\to R_3-\frac{2}{3}R_2}\begin{bmatrix}-1&0&2&5\\0&3&1&6\\0&0&\frac{22}{3}&16\end{bmatrix}\xrightarrow{R_3\to 3R_3}\begin{bmatrix}-1&0&2&5\\0&3&1&6\\0&0&22&48\end{bmatrix}.$$

Example 1.19

Show that the matrix $\begin{bmatrix}3&1&4\\2&0&-1\\5&2&1\end{bmatrix}$ is non-singular and reduce it to the identity matrix by elementary row transformations.

Solution

Let $A=\begin{bmatrix}3&1&4\\2&0&-1\\5&2&1\end{bmatrix}$. Then, $|A|=3(0+2)-1(2+5)+4(4-0)=6-7+16=15\neq0$. So, A is non-singular. Keeping the identity matrix as our goal, we perform the row operations sequentially on A as follows:

$$\begin{bmatrix}3&1&4\\2&0&-1\\5&2&1\end{bmatrix}\xrightarrow{R_1\to\frac{1}{3}R_1}\begin{bmatrix}1&\frac{1}{3}&\frac{4}{3}\\2&0&-1\\5&2&1\end{bmatrix}\xrightarrow{R_2\to R_2-2R_1,\ R_3\to R_3-5R_1}\begin{bmatrix}1&\frac{1}{3}&\frac{4}{3}\\0&-\frac{2}{3}&-\frac{11}{3}\\0&\frac{1}{3}&-\frac{17}{3}\end{bmatrix}\xrightarrow{R_2\to\left(-\frac{3}{2}\right)R_2}\begin{bmatrix}1&\frac{1}{3}&\frac{4}{3}\\0&1&\frac{11}{2}\\0&\frac{1}{3}&-\frac{17}{3}\end{bmatrix}$$

$$\xrightarrow{R_1\to R_1-\frac{1}{3}R_2,\ R_3\to R_3-\frac{1}{3}R_2}\begin{bmatrix}1&0&-\frac{1}{2}\\0&1&\frac{11}{2}\\0&0&-\frac{15}{2}\end{bmatrix}\xrightarrow{R_3\to\left(-\frac{2}{15}\right)R_3}\begin{bmatrix}1&0&-\frac{1}{2}\\0&1&\frac{11}{2}\\0&0&1\end{bmatrix}\xrightarrow{R_1\to R_1+\frac{1}{2}R_3,\ R_2\to R_2-\frac{11}{2}R_3}\begin{bmatrix}1&0&0\\0&1&0\\0&0&1\end{bmatrix}.$$

■

Example 1.24

If $A=\begin{bmatrix}-4&4&4\\-7&1&3\\5&-3&-1\end{bmatrix}$ and $B=\begin{bmatrix}1&-1&1\\1&-2&-2\\2&1&3\end{bmatrix}$, find the products AB and BA and hence solve the system of equations $x-y+z=4,\ x-2y-2z=9,\ 2x+y+3z=1$.

Solution

We find $AB=\begin{bmatrix}-4&4&4\\-7&1&3\\5&-3&-1\end{bmatrix}\begin{bmatrix}1&-1&1\\1&-2&-2\\2&1&3\end{bmatrix}=\begin{bmatrix}-4+4+8&4-8+4&-4-8+12\\-7+1+6&7-2+3&-7-2+9\\5-3-2&-5+6-1&5+6-3\end{bmatrix}=\begin{bmatrix}8&0&0\\0&8&0\\0&0&8\end{bmatrix}=8I_3$

and $BA=\begin{bmatrix}1&-1&1\\1&-2&-2\\2&1&3\end{bmatrix}\begin{bmatrix}-4&4&4\\-7&1&3\\5&-3&-1\end{bmatrix}=\begin{bmatrix}-4+7+5&4-1-3&4-3-1\\-4+14-10&4-2+6&4-6+2\\-8-7+15&8+1-9&8+3-3\end{bmatrix}=\begin{bmatrix}8&0&0\\0&8&0\\0&0&8\end{bmatrix}=8I_3.$

So, we get $AB=BA=8I_3$. That is, $\left(\frac{1}{8}A\right)B=B\left(\frac{1}{8}A\right)=I_3$. Hence, $B^{-1}=\frac{1}{8}A$.

Writing the given system of equations in matrix form, we get

$\begin{bmatrix}1&-1&1\\1&-2&-2\\2&1&3\end{bmatrix}\begin{bmatrix}x\\y\\z\end{bmatrix}=\begin{bmatrix}4\\9\\1\end{bmatrix}$. That is, $B\begin{bmatrix}x\\y\\z\end{bmatrix}=\begin{bmatrix}4\\9\\1\end{bmatrix}$.

So, $\begin{bmatrix}x\\y\\z\end{bmatrix}=B^{-1}\begin{bmatrix}4\\9\\1\end{bmatrix}=\left(\frac{1}{8}A\right)\begin{bmatrix}4\\9\\1\end{bmatrix}=\frac{1}{8}\begin{bmatrix}-4&4&4\\-7&1&3\\5&-3&-1\end{bmatrix}\begin{bmatrix}4\\9\\1\end{bmatrix}=\frac{1}{8}\begin{bmatrix}-16+36+4\\-28+9+3\\20-27-1\end{bmatrix}=\frac{1}{8}\begin{bmatrix}24\\-16\\-8\end{bmatrix}=\begin{bmatrix}3\\-2\\-1\end{bmatrix}.$

Hence, the solution is $(x=3, y=-2, z=-1)$. ■

Example 1.27

Solve the following system of linear equations, by Gaussian elimination method: $4x+3y+6z=25,\ x+5y+7z=13,\ 2x+9y+z=1$.

Solution

Transforming the augmented matrix to echelon form, we get

$$\left[\begin{array}{ccc|c}4&3&6&25\\1&5&7&13\\2&9&1&1\end{array}\right]\xrightarrow{R_1\leftrightarrow R_2}\left[\begin{array}{ccc|c}1&5&7&13\\4&3&6&25\\2&9&1&1\end{array}\right]\xrightarrow[R_3\to R_3-2R_1]{R_2\to R_2-4R_1}\left[\begin{array}{ccc|c}1&5&7&13\\0&-17&-22&-27\\0&-1&-13&-25\end{array}\right]$$

$$\xrightarrow[R_3\to R_3\times(-1)]{R_2\to R_2\times(-1)}\left[\begin{array}{ccc|c}1&5&7&13\\0&17&22&27\\0&1&13&25\end{array}\right]\xrightarrow{R_3\to 17R_3-R_2}\left[\begin{array}{ccc|c}1&5&7&13\\0&17&22&27\\0&0&199&398\end{array}\right].$$

The equivalent system is written by using the echelon form:

$$x+5y+7z=13, \quad \dots(1)$$
$$17y+22z=27, \quad \dots(2)$$
$$199z=398. \quad \dots(3)$$

From (3), we get $z=\frac{398}{199}=2$.

Substituting $z=2$ in (2), we get $y=\frac{27-22\times2}{17}=\frac{-17}{17}=-1$.

Substituting $z=2, y=-1$ in (1), we get $x=13-5\times(-1)-7\times2=4$.

So, the solution is $(x=4, y=-1, z=2)$.

Note. The above method of going from the last equation to the first equation is called the **method of back substitution.**

4. A boy is walking along the path $y=ax^2+bx+c$ through the points $(-6,8),(-2,-12)$, and $(3,8)$. He wants to meet his friend at $P(7,60)$. Will he meet his friend? (Use Gaussian elimination method.)

Example 1.31

Test for consistency of the following system of linear equations and if possible solve: $x-y+z=-9,\ 2x-2y+2z=-18,\ 3x-3y+3z+27=0$.

Solution

Here the number of unknowns is 3.

The matrix form of the system is $AX=B$, where

$$A=\begin{bmatrix}1&-1&1\\2&-2&2\\3&-3&3\end{bmatrix}, X=\begin{bmatrix}x\\y\\z\end{bmatrix}, B=\begin{bmatrix}-9\\-18\\-27\end{bmatrix}.$$

Applying elementary row operations on the augmented matrix $[A\,|\,B]$, we get

$$[A\,|\,B]=\left[\begin{array}{ccc|c}1&-1&1&-9\\2&-2&2&-18\\3&-3&3&-27\end{array}\right]\xrightarrow[R_3\to R_3-3R_1]{R_2\to R_2-2R_1}\left[\begin{array}{ccc|c}1&-1&1&-9\\0&0&0&0\\0&0&0&0\end{array}\right].$$

So, $\rho(A)=\rho([A\,|\,B])=1<3$.

From the echelon form, we get the equivalent equations $x-y+z=-9,\ 0=0,\ 0=0$.

The equivalent system has one non-trivial equation and three unknowns.

Taking $y=s, z=t$ arbitrarily, we get $x-s+t=-9$; or $x=-9+s-t$.

So, the solution is $(x=-9+s-t, y=s, z=t)$, where s and t are parameters.

The above solution set is a two-parameter family of solutions.

Here, the given system of equations is consistent and has infinitely many solutions which form a two parameter family of solutions.

Example 1.36

Solve the system: $x+3y-2z=0,\ 2x-y+4z=0,\ x-11y+14z=0$.

Solution

Here the number of unknowns is 3.

Transforming into echelon form (Gaussian elimination method), the augmented matrix becomes

$$\left[\begin{array}{ccc|c}1&3&-2&0\\2&-1&4&0\\1&-11&14&0\end{array}\right]\xrightarrow[R_3\to R_3-R_1]{R_2\to R_2-2R_1}\left[\begin{array}{ccc|c}1&3&-2&0\\0&-7&8&0\\0&-14&16&0\end{array}\right]\xrightarrow[R_3\to R_3\times(-2)]{R_2\to R_2\times(-1)}\left[\begin{array}{ccc|c}1&3&-2&0\\0&7&-8&0\\0&7&-8&0\end{array}\right]\xrightarrow{R_3\to R_3-R_2}\left[\begin{array}{ccc|c}1&3&-2&0\\0&7&-8&0\\0&0&0&0\end{array}\right].$$

So, $\rho(A)=\rho([A\,|\,O])=2<3=$ Number of unknowns.

Hence, the system has a one parameter family of solutions.

Writing the equations using the echelon form, we get

$x+3y-2z=0, \quad 7y-8z=0, \quad 0=0.$

Taking $z=t$, where t is an arbitrary real number, we get by back substitution,

Example 1.40

If the system of equations $px+by+cz=0,\ ax+qy+cz=0,\ ax+by+rz=0$ has a non-trivial solution and $p\neq a, q\neq b, r\neq c$, prove that $\frac{p}{p-a}+\frac{q}{q-b}+\frac{r}{r-c}=2$.

Solution

Assume that the system $px+by+cz=0, ax+qy+cz=0, ax+by+rz=0$ has a non-trivial solution.

So, we have $\begin{vmatrix}p&b&c\\a&q&c\\a&b&r\end{vmatrix}=0$. Applying $R_2\to R_2-R_1$ and $R_3\to R_3-R_1$ in the above equation, we get

$$\begin{vmatrix}p&b&c\\a-p&q-b&0\\a-p&0&r-c\end{vmatrix}=0. \text{ That is, } \begin{vmatrix}p&b&c\\-(p-a)&q-b&0\\-(p-a)&0&r-c\end{vmatrix}=0.$$

Since $p\neq a, q\neq b, r\neq c$, we get $(p-a)(q-b)(r-c)\begin{vmatrix}\frac{p}{p-a}&\frac{b}{q-b}&\frac{c}{r-c}\\-1&1&0\\-1&0&1\end{vmatrix}=0$.

So, we have $\begin{vmatrix}\frac{p}{p-a}&\frac{b}{q-b}&\frac{c}{r-c}\\-1&1&0\\-1&0&1\end{vmatrix}=0$.

Expanding the determinant, we get $\frac{p}{p-a}+\frac{b}{q-b}+\frac{c}{r-c}=0$.

That is, $\frac{p}{p-a}+\frac{q-(q-b)}{q-b}+\frac{r-(r-c)}{r-c}=0 \Rightarrow \frac{p}{p-a}+\frac{q}{q-b}+\frac{r}{r-c}=2$.

Example 2.2

Find the value of the real numbers x and y, if the complex number $(2+i)x+(1-i)y+2i-3$ and $x+(-1+2i)y+1+i$ are equal

Solution

Let $z_1=(2+i)x+(1-i)y+2i-3=(2x+y-3)+i(x-y+2)$ and

$z_2=x+(-1+2i)y+1+i=(x-y+1)+i(2y+1)$.

Given that $z_1=z_2$.

Therefore $(2x+y-3)+i(x-y+2)=(x-y+1)+i(2y+1)$.

Equating real and imaginary parts separately, gives

$2x+y-3=x-y+1 \Rightarrow x+2y=4$.

$x-y+2=2y+1 \Rightarrow x-3y=-1$.

Solving the above equations, gives

$x=2$ and $y=1$.

Example 2.5

If $\frac{z+3}{z-5i}=\frac{1+4i}{2}$, find the complex number z in the rectangular form

Solution

We have $\frac{z+3}{z-5i}=\frac{1+4i}{2}$

$\Rightarrow 2(z+3)=(1+4i)(z-5i)$

$\Rightarrow 2z+6=(1+4i)z+20-5i$

$\Rightarrow (2-1-4i)z=20-5i-6$

$\Rightarrow z=\frac{14-5i}{1-4i}=\frac{(14-5i)(1+4i)}{(1-4i)(1+4i)}=\frac{34+51i}{17}=2+3i$.

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Example 2.7

Find z^{-1}, if $z=(2+3i)(1-i)$.

Solution

We have $z=(2+3i)(1-i)=(2+3)+(3-2)i=5+i$

$$\Rightarrow z^{-1}=\frac{1}{z}=\frac{1}{5+i}.$$

Multiplying the numerator and denominator by the conjugate of the denominator, we get

$$z^{-1}=\frac{(5-i)}{(5+i)(5-i)}=\frac{5-i}{5^2+1^2}=\frac{5}{26}-i\frac{1}{26}$$

$$\Rightarrow z^{-1}=\frac{5}{26}-i\frac{1}{26}.$$

Example 2.12

If z_1, z_2, and z_3 are complex numbers such that $|z_1|=|z_2|=|z_3|=|z_1+z_2+z_3|=1$, find the value of $\left|\frac{1}{z_1}+\frac{1}{z_2}+\frac{1}{z_3}\right|$.

Solution

Since, $|z_1|=|z_2|=|z_3|=1$,

$$|z_1|^2=1\Rightarrow z_1\bar{z}_1=1, |z_2|^2=1\Rightarrow z_2\bar{z}_2=1, \text{ and } |z_3|^2=1\Rightarrow z_3\bar{z}_3=1$$

Therefore, $\bar{z}_1=\frac{1}{z_1}, \bar{z}_2=\frac{1}{z_2}$, and $\bar{z}_3=\frac{1}{z_3}$ and hence

$$\left|\frac{1}{z_1}+\frac{1}{z_2}+\frac{1}{z_3}\right|=|\bar{z}_1+\bar{z}_2+\bar{z}_3|$$

$$=|\overline{z_1+z_2+z_3}|=|z_1+z_2+z_3|=1.$$

Example 2.16

Show that the equation $z^2=\bar{z}$ has four solutions.

Solution

We have, $z^2=\bar{z}$.

$$\Rightarrow |z|^2=|z|$$

$$\Rightarrow |z|(|z|-1)=0,$$

$$\Rightarrow |z|=0, \text{ or } |z|=1.$$

$|z|=0 \Rightarrow z=0$ is a solution, $|z|=1\Rightarrow z\bar{z}=1\Rightarrow \bar{z}=\frac{1}{z}$.

Given $z^2=\bar{z}\Rightarrow z^2=\frac{1}{z}\Rightarrow z^3=1$.

It has 3 non-zero solutions. Hence including zero solution, there are four solutions.

Example 2.17

Find the square root of $6-8i$.

Solution

We compute $|6-8i|=\sqrt{6^2+(-8)^2}=10$

and applying the formula for square root, we get

$$\sqrt{6-8i}=\pm\left(\sqrt{\frac{10+6}{2}}-i\sqrt{\frac{10-6}{2}}\right) \quad \left(\because b \text{ is negative}, \frac{b}{|b|}=-1\right)$$

$$=\pm(\sqrt{8}-i\sqrt{2})$$

$$=\pm(2\sqrt{2}-i\sqrt{2}).$$

Example 2.23

Represent the complex number (i) $-1-i$ (ii) $1+i\sqrt{3}$ in polar form.

Solution

(i) Let $-1-i=r(\cos\theta+i\sin\theta)$

We have $r=\sqrt{x^2+y^2}=\sqrt{1^2+1^2}=\sqrt{1+1}=\sqrt{2}$

$$\alpha=\tan^{-1}\left|\frac{y}{x}\right|=\tan^{-1}1=\frac{\pi}{4}.$$

Since the complex number $-1-i$ lies in the third quadrant, it has the principal value,

$$\theta=\alpha-\pi=\frac{\pi}{4}-\pi=-\frac{3\pi}{4}$$

Therefore, $-1-i=\sqrt{2}\left(\cos\left(-\frac{3\pi}{4}\right)+i\sin\left(-\frac{3\pi}{4}\right)\right)$

$$=\sqrt{2}\left(\cos\frac{3\pi}{4}-i\sin\frac{3\pi}{4}\right).$$

$$-1-i=\sqrt{2}\left(\cos\left(\frac{3\pi}{4}+2k\pi\right)-i\sin\left(\frac{3\pi}{4}+2k\pi\right)\right), k\in\mathbb{Z}.$$

Example 2.24

Find the principal argument $\text{Arg}\, z$, when $z=\frac{-2}{1+i\sqrt{3}}$.

Solution

$$\arg z=\arg\frac{-2}{1+i\sqrt{3}}$$

$$=\arg(-2)-\arg(1+i\sqrt{3}) \quad \left(\because \arg\left(\frac{z_1}{z_2}\right)=\arg z_1-\arg z_2\right)$$

$$=\left(\pi-\tan^{-1}\left(\frac{0}{2}\right)\right)-\tan^{-1}\left(\frac{\sqrt{3}}{1}\right)$$

$$=\pi-\frac{\pi}{3}=\frac{2\pi}{3}$$

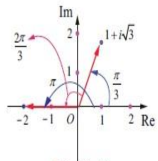


Fig. 2.40

This implies that one of the values of $\arg z$ is $\frac{2\pi}{3}$.

Since $\frac{2\pi}{3}$ lies between $-\pi$ and π, the principal argument $\text{Arg}\, z$ is $\frac{2\pi}{3}$.

Example 2.26

Find the quotient $\frac{2\left(\cos\frac{9\pi}{4}+i\sin\frac{9\pi}{4}\right)}{4\left(\cos\left(\frac{-3\pi}{2}\right)+i\sin\left(\frac{-3\pi}{2}\right)\right)}$ in rectangular form.

Solution

$$\frac{2\left(\cos\frac{9\pi}{4}+i\sin\frac{9\pi}{4}\right)}{4\left(\cos\left(\frac{-3\pi}{2}\right)+i\sin\left(\frac{-3\pi}{2}\right)\right)}$$

$$=\frac{1}{2}\left(\cos\left(\frac{9\pi}{4}-\left(\frac{-3\pi}{2}\right)\right)+i\sin\left(\frac{9\pi}{4}-\left(\frac{-3\pi}{2}\right)\right)\right)$$

$$=\frac{1}{2}\left(\cos\left(\frac{9\pi}{4}+\frac{3\pi}{2}\right)+i\sin\left(\frac{9\pi}{4}+\frac{3\pi}{2}\right)\right)$$

$$=\frac{1}{2}\left(\cos\left(\frac{15\pi}{4}\right)+i\sin\left(\frac{15\pi}{4}\right)\right)=\frac{1}{2}\left(\cos\left(4\pi-\frac{\pi}{4}\right)+i\sin\left(4\pi-\frac{\pi}{4}\right)\right)$$

$$=\frac{1}{2}\left(\cos\left(\frac{\pi}{4}\right)-i\sin\left(\frac{\pi}{4}\right)\right)=\frac{1}{2}\left(\frac{1}{\sqrt{2}}-i\frac{1}{\sqrt{2}}\right)$$

$$\frac{2\left(\cos\frac{9\pi}{4}+i\sin\frac{9\pi}{4}\right)}{4\left(\cos\left(\frac{-3\pi}{2}\right)+i\sin\left(\frac{-3\pi}{2}\right)\right)}=\frac{1}{2\sqrt{2}}-i\frac{1}{2\sqrt{2}}=\frac{\sqrt{2}}{4}-i\frac{\sqrt{2}}{4}.$$ Which is in rectangular form.

Example 2.34

Solve the equation $z^3+8i=0$, where $z\in\mathbb{C}$.

Solution

Let $z^3+8i=0$. Then, we get

$$z^3=-8i$$

$$=8(-i)=8\left(\cos\left(-\frac{\pi}{2}+2k\pi\right)+i\sin\left(-\frac{\pi}{2}+2k\pi\right)\right), k\in\mathbb{Z}.$$

Therefore, $z=\sqrt[3]{8}\left(\cos\left(\frac{-\pi+4k\pi}{6}\right)+i\sin\left(\frac{-\pi+4k\pi}{6}\right)\right)$, $k=0,1,2$.

Taking $k=0,1,2$, we get,

$k=0$, $z=2\left(\cos\left(-\frac{\pi}{6}\right)+i\sin\left(-\frac{\pi}{6}\right)\right)=2\left(\frac{\sqrt{3}}{2}-i\frac{1}{2}\right)=\sqrt{3}-i$

$k=1$, $z=2\left(\cos\left(\frac{\pi}{2}\right)+i\sin\left(\frac{\pi}{2}\right)\right)=2=2(0+i)=0+2i=2i$.

$k=2$, $z=2\left(\cos\left(\frac{7\pi}{6}\right)+i\sin\left(\frac{7\pi}{6}\right)\right)=2\left(\cos\left(\pi+\frac{\pi}{6}\right)+i\sin\left(\pi+\frac{\pi}{6}\right)\right)$

$$=2\left(-\cos\left(\frac{\pi}{6}\right)-i\sin\left(\frac{\pi}{6}\right)\right)=2\left(-\frac{\sqrt{3}}{2}-i\frac{1}{2}\right)=-\sqrt{3}-i.$$

The values of z are $\sqrt{3}-i$, $2i$, and $-\sqrt{3}-i$.

Example 2.35

Find all cube roots of $\sqrt{3}+i$.

Solution

We have to find $(\sqrt{3}+i)^{\frac{1}{3}}$. Let $z=(\sqrt{3}+i)^{\frac{1}{3}}$. Then, $z^3=\sqrt{3}+i=r(\cos\theta+i\sin\theta)$.

Then, $r=\sqrt{3+1}=2$, and $\alpha=\theta=\frac{\pi}{6}$ ($\because \sqrt{3}+i$ lies in the first quadrant)

Therefore, $z^3=\sqrt{3}+i=2\left(\cos\frac{\pi}{6}+i\sin\frac{\pi}{6}\right)$

$\Rightarrow z=\sqrt[3]{2}\left(\cos\left(\frac{\pi+12k\pi}{18}\right)+i\sin\left(\frac{\pi+12k\pi}{18}\right)\right)$, $k=0,1,2$.

Taking $k=0,1,2$, we get

$k=0$, $z=2^{\frac{1}{3}}\left(\cos\frac{\pi}{18}+i\sin\frac{\pi}{18}\right)$;

$k=1$, $z=2^{\frac{1}{3}}\left(\cos\frac{13\pi}{18}+i\sin\frac{13\pi}{18}\right)$;

$k=2$, $z=2^{\frac{1}{3}}\left(\cos\frac{25\pi}{18}+i\sin\frac{25\pi}{18}\right)=2^{\frac{1}{3}}\left(-\cos\frac{7\pi}{18}-i\sin\frac{7\pi}{18}\right)$.

Example 3.5

Find the condition that the roots of cubic equation $x^3+ax^2+bx+c=0$ are in the ratio $p:q:r$.

Solution

Since roots are in the ratio $p:q:r$, we can assume the roots as $p\lambda, q\lambda$ and $r\lambda$.

Then, we get

$\Sigma_1=p\lambda+q\lambda+r\lambda=-a$, ...(1)

$\Sigma_2=(p\lambda)(q\lambda)+(q\lambda)(r\lambda)+(r\lambda)(p\lambda)=b$, ...(2)

$\Sigma_3=(p\lambda)(q\lambda)(r\lambda)=-c$, ...(3)

Now, we get

$(1)\Rightarrow\lambda=-\frac{a}{p+q+r}$ ....(4)

$(3)\Rightarrow\lambda^3=-\frac{c}{pqr}$ .....(5)

Substituting (4) in (5), we get

$$\left(-\frac{a}{p+q+r}\right)^3=-\frac{c}{pqr}\Rightarrow pqra^3=c(p+q+r)^3.$$

Example 3.20

Find the condition that the roots of $ax^3+bx^2+cx+d=0$ are in geometric progression. Assume $a,b,c,d\neq0$

Solution

Let the roots be in G.P.

Then, we can assume them in the form $\frac{\alpha}{\lambda},\alpha,\alpha\lambda$.

Applying the Vieta's formula, we get

$\Sigma_1=\alpha\left(\frac{1}{\lambda}+1+\lambda\right)=-\frac{b}{a}$ ... (1)

$\Sigma_2=\alpha^2\left(\frac{1}{\lambda}+1+\lambda\right)=\frac{c}{a}$ ... (2)

$\Sigma_3=\alpha^3=-\frac{d}{a}$. ... (3)

Dividing (2) by (1), we get

$\alpha=-\frac{c}{b}$ ... (4)

Substituting (4) in (3), we get $\left(-\frac{c}{b}\right)^3=-\frac{d}{a}\Rightarrow ac^3=db^3$.

Example 3.24

Solve the equation $(2x-3)(6x-1)(3x-2)(x-2)-5=0$.

Solution

The given equation is same as

$$(2x-3)(3x-2)(6x-1)(x-2)-5=0.$$

After a computation, the above equation becomes

$$(6x^2-13x+6)(6x^2-13x+2)-5=0.$$

By taking $y=6x^2-13x$, the above equation becomes,

$$(y+6)(y+2)-5=0$$

which is same as

$$y^2+8y+7=0.$$

Solving this equation, we get $y=-1$ and $y=-7$.

Substituting the values of y in $y=6x^2-13x$, we get

$$6x^2-13x+1=0$$
$$6x^2-13x+7=0$$

Solving these two equations, we get

$$x=1, x=\frac{7}{6}, x=\frac{13+\sqrt{145}}{12} \text{ and } x=\frac{13-\sqrt{145}}{12}$$

Example 3.10

Form a polynomial equation with integer coefficients with $\sqrt{\frac{\sqrt{2}}{\sqrt{3}}}$ as a root.

Solution

Since $\sqrt{\frac{\sqrt{2}}{\sqrt{3}}}$ is a root, $x-\sqrt{\frac{\sqrt{2}}{\sqrt{3}}}$ is a factor. To remove the outermost square root, we take $x+\sqrt{\frac{\sqrt{2}}{\sqrt{3}}}$ as another factor and find their product

$$\left(x+\sqrt{\frac{\sqrt{2}}{\sqrt{3}}}\right)\left(x-\sqrt{\frac{\sqrt{2}}{\sqrt{3}}}\right)=x^2-\frac{\sqrt{2}}{\sqrt{3}}.$$

Still we didn't achieve our goal. So we include another factor $x^2+\frac{\sqrt{2}}{\sqrt{3}}$ and get the product

$$\left(x^2-\frac{\sqrt{2}}{\sqrt{3}}\right)\left(x^2+\frac{\sqrt{2}}{\sqrt{3}}\right)=x^4-\frac{2}{3}.$$

So, $3x^4-2=0$ is a required polynomial equation with the integer coefficients.

Now we identify the nature of roots of the given equation without solving the equation. The idea comes from the negativity, equal to 0 and positivity of $\Delta=b^2-4ac$.

Example 4.4

Find the domain of $\sin^{-1}(2-3x^2)$

Solution

We know that the domain of $\sin^{-1}(x)$ is $[-1,1]$.

This leads to $-1 \le 2-3x^2 \le 1$, which implies $-3 \le -3x^2 \le -1$.

Now, $-3 \le -3x^2$, gives $x^2 \le 1$ and ... (1)

$-3x^2 \le -1$, gives $x^2 \ge \frac{1}{3}$ ... (2)

Combining the equations (1) and (2), we get $\frac{1}{3} \le x^2 \le 1$. That is, $\frac{1}{\sqrt{3}} \le |x| \le 1$, which gives

$x \in \left[-1, -\frac{1}{\sqrt{3}}\right] \cup \left[\frac{1}{\sqrt{3}}, 1\right]$, since $a \le |x| \le b$ implies $x \in [-b, -a] \cup [a, b]$.

Example 4.6

Find (i) $\cos^{-1}\left(-\frac{1}{\sqrt{2}}\right)$ (ii) $\cos^{-1}\left(\cos\left(-\frac{\pi}{3}\right)\right)$ (iii) $\cos^{-1}\left(\cos\left(\frac{7\pi}{6}\right)\right)$

Solution

It is known that $\cos^{-1} x : [-1, 1] \to [0, \pi]$ is given by

$\cos^{-1} x = y$ if and only if $x = \cos y$ for $-1 \le x \le 1$ and $0 \le y \le \pi$.

Thus, we have

(i) $\cos^{-1}\left(-\frac{1}{\sqrt{2}}\right) = \frac{3\pi}{4}$, since $\frac{3\pi}{4} \in [0, \pi]$ and $\cos\frac{3\pi}{4} = \cos\left(\pi - \frac{\pi}{4}\right) = -\cos\frac{\pi}{4} = -\frac{1}{\sqrt{2}}$.

(ii) $\cos^{-1}\left(\cos\left(-\frac{\pi}{3}\right)\right) = \cos^{-1}\left(\cos\left(\frac{\pi}{3}\right)\right) = \frac{\pi}{3}$, since $-\frac{\pi}{3} \notin [0, \pi]$, but $\frac{\pi}{3} \in [0, \pi]$.

(iii) $\cos^{-1}\left(\cos\left(\frac{7\pi}{6}\right)\right) = \frac{5\pi}{6}$, since $\cos\left(\frac{7\pi}{6}\right) = \cos\left(\pi + \frac{\pi}{6}\right) = -\frac{\sqrt{3}}{2} = \cos\left(\frac{5\pi}{6}\right)$ and $\frac{5\pi}{6} \in [0, \pi]$.

Example 4.7

Find the domain of $\cos^{-1}\left(\frac{2+\sin x}{3}\right)$.

Solution

By definition, the domain of $y = \cos^{-1} x$ is $-1 \le x \le 1$ or $|x| \le 1$. This leads to

$-1 \le \frac{2+\sin x}{3} \le 1$ which is same as $-3 \le 2 + \sin x \le 3$.

So, $-5 \le \sin x \le 1$ reduces to $-1 \le \sin x \le 1$, which gives

$-\sin^{-1}(1) \le x \le \sin^{-1}(1)$ or $-\frac{\pi}{2} \le x \le \frac{\pi}{2}$.

Thus, the domain of $\cos^{-1}\left(\frac{2+\sin x}{3}\right)$ is $\left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$.

Example 4.10

Find the value of $\tan^{-1}(-1) + \cos^{-1}\left(\frac{1}{2}\right) + \sin^{-1}\left(-\frac{1}{2}\right)$.

Solution

Let $\tan^{-1}(-1) = y$. Then, $\tan y = -1 = -\tan\frac{\pi}{4} = \tan\left(-\frac{\pi}{4}\right)$.

As $-\frac{\pi}{4} \in \left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$, $\tan^{-1}(-1) = -\frac{\pi}{4}$.

Now, $\cos^{-1}\left(\frac{1}{2}\right) = y$ implies $\cos y = \frac{1}{2} = \cos\frac{\pi}{3}$.

As $\frac{\pi}{3} \in [0, \pi]$, $\cos^{-1}\left(\frac{1}{2}\right) = \frac{\pi}{3}$.

Now, $\sin^{-1}\left(-\frac{1}{2}\right) = y$ implies $\sin y = -\frac{1}{2} = \sin\left(-\frac{\pi}{6}\right)$.

As $-\frac{\pi}{6} \in \left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$, $\sin^{-1}\left(-\frac{1}{2}\right) = -\frac{\pi}{6}$.

Therefore, $\tan^{-1}(-1) + \cos^{-1}\left(\frac{1}{2}\right) + \sin^{-1}\left(-\frac{1}{2}\right) = -\frac{\pi}{4} + \frac{\pi}{3} - \frac{\pi}{6} = -\frac{\pi}{12}$.

Example 4.11

Prove that $\tan(\sin^{-1} x) = \frac{x}{\sqrt{1-x^2}}$, $-1 < x < 1$.

Solution

If $x = 0$, then both sides are equal to 0. ... (1)

Assume that $0 < x < 1$.

Let $\theta = \sin^{-1} x$. Then $0 < \theta < \frac{\pi}{2}$. Now, $\sin\theta = \frac{x}{1}$ gives $\tan\theta = \frac{x}{\sqrt{1-x^2}}$.

Hence, $\tan(\sin^{-1} x) = \frac{x}{\sqrt{1-x^2}}$. ... (2)

Assume that $-1 < x < 0$. Then, $\theta = \sin^{-1} x$ gives $-\frac{\pi}{2} < \theta < 0$. Now, $\sin\theta = \frac{x}{1}$ gives $\tan\theta = \frac{x}{\sqrt{1-x^2}}$.

In this case also, $\tan(\sin^{-1} x) = \frac{x}{\sqrt{1-x^2}}$. ... (3)

Equations (1), (2) and (3) establish that $\tan(\sin^{-1} x) = \frac{x}{\sqrt{1-x^2}}$, $-1 < x < 1$.

Example 4.13

Find the value of $\sec^{-1}\left(-\frac{2\sqrt{3}}{3}\right)$.

Solution

Let $\sec^{-1}\left(-\frac{2\sqrt{3}}{3}\right) = \theta$. Then, $\sec\theta = -\frac{2}{\sqrt{3}}$ where $\theta \in [0, \pi] \setminus \left\{\frac{\pi}{2}\right\}$. Thus, $\cos\theta = -\frac{\sqrt{3}}{2}$.

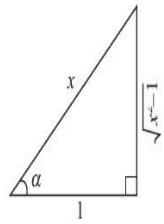
Now, $\cos\frac{5\pi}{6} = \cos\left(\pi - \frac{\pi}{6}\right) = -\cos\left(\frac{\pi}{6}\right) = -\frac{\sqrt{3}}{2}$. Hence, $\sec^{-1}\left(-\frac{2\sqrt{3}}{3}\right) = \frac{5\pi}{6}$.

Example 4.15

Show that $\cot^{-1}\left(\frac{1}{\sqrt{x^2-1}}\right) = \sec^{-1} x$, $|x| > 1$.

Solution

Let $\cot^{-1}\left(\frac{1}{\sqrt{x^2-1}}\right) = \alpha$. Then, $\cot\alpha = \frac{1}{\sqrt{x^2-1}}$ and α is acute.



We construct a right triangle with the given data.

From the triangle, $\sec\alpha = \frac{x}{1} = x$. Thus, $\alpha = \sec^{-1} x$.

Hence, $\cot^{-1}\left(\frac{1}{\sqrt{x^2-1}}\right) = \sec^{-1} x$, $|x| > 1$.

Example 4.16

Prove that $\frac{\pi}{2} \le \sin^{-1} x + 2\cos^{-1} x \le \frac{3\pi}{2}$.

Solution

$\sin^{-1} x + 2\cos^{-1} x = \sin^{-1} x + \cos^{-1} x + \cos^{-1} x = \frac{\pi}{2} + \cos^{-1} x$

We know that $0 \le \cos^{-1} x \le \pi$. Thus, $\frac{\pi}{2} + 0 \le \cos^{-1} x + \frac{\pi}{2} \le \pi + \frac{\pi}{2}$.

Thus, $\frac{\pi}{2} \le \sin^{-1} x + 2\cos^{-1} x \le \frac{3\pi}{2}$.

Example 4.20

Evaluate $\sin\left[\sin^{-1}\left(\frac{3}{5}\right)+\sec^{-1}\left(\frac{5}{4}\right)\right]$

Solution

Let $\sec^{-1}\frac{5}{4}=\theta$. Then, $\sec\theta=\frac{5}{4}$ and hence, $\cos\theta=\frac{4}{5}$.

Also, $\sin\theta=\sqrt{1-\cos^2\theta} = \sqrt{1-\left(\frac{4}{5}\right)^2}=\frac{3}{5}$, which gives $\theta=\sin^{-1}\left(\frac{3}{5}\right)$.

Thus, $\sec^{-1}\left(\frac{5}{4}\right)=\sin^{-1}\left(\frac{3}{5}\right)$ and $\sin^{-1}\frac{3}{5}+\sec^{-1}\left(\frac{5}{4}\right)= 2\sin^{-1}\left(\frac{3}{5}\right)$.

We know that $\sin^{-1}\left(2x\sqrt{1-x^2}\right)=2\sin^{-1}x$, if $|x|\le\frac{1}{\sqrt{2}}$.

Since $\frac{3}{5}<\frac{1}{\sqrt{2}}$, we have $2\sin^{-1}\left(\frac{3}{5}\right)=\sin^{-1}\left(2\times\frac{3}{5}\sqrt{1-\left(\frac{3}{5}\right)^2}\right)=\sin^{-1}\left(\frac{24}{25}\right)$.

Hence, $\sin\left[\sin^{-1}\left(\frac{3}{5}\right)+\sec^{-1}\left(\frac{5}{4}\right)\right] = \sin\left(\sin^{-1}\left(\frac{24}{25}\right)\right)=\frac{24}{25}$, since $\frac{24}{25}\in[-1,1]$.

Example 4.25 Solve $\sin^{-1}x>\cos^{-1}x$

Solution

Given that $\sin^{-1}x>\cos^{-1}x$. Note that $-1\le x\le 1$.

Adding both sides by $\sin^{-1}x$, we get

$\sin^{-1}x+\sin^{-1}x>\cos^{-1}x+\sin^{-1}x$, which reduces to $2\sin^{-1}x>\frac{\pi}{2}$.

As sine function increases in the interval $\left[-\frac{\pi}{2},\frac{\pi}{2}\right]$, we have $x>\sin\frac{\pi}{4}$ or $x>\frac{1}{\sqrt{2}}$.

Thus, the solution set is the interval $\left(\frac{1}{\sqrt{2}},1\right]$.

Example 4.28

Solve $\tan^{-1}\left(\frac{x-1}{x-2}\right)+\tan^{-1}\left(\frac{x+1}{x+2}\right)=\frac{\pi}{4}$.

Solution

Now, $\tan^{-1}\left(\frac{x-1}{x-2}\right)+\tan^{-1}\left(\frac{x+1}{x+2}\right) = \tan^{-1}\left[\frac{\frac{x-1}{x-2}+\frac{x+1}{x+2}}{1-\frac{x-1}{x-2}\left(\frac{x+1}{x+2}\right)}\right]=\frac{\pi}{4}$.

Thus, $\frac{\frac{x-1}{x-2}+\frac{x+1}{x+2}}{1-\frac{x-1}{x-2}\left(\frac{x+1}{x+2}\right)} = 1$, which on simplification gives $2x^2-4=-3$

Thus, $x^2 = \frac{1}{2}$ gives $x=\pm\frac{1}{\sqrt{2}}$.

Example 5.2

Find the equation of the circle described on the chord $3x+y+5=0$ of the circle $x^2+y^2=16$ as diameter.

Solution

Equation of the circle passing through the points of intersection of the chord and circle by Theorem 5.1 is $x^2+y^2-16+\lambda(3x+y+5)=0$.

The chord $3x+y+5=0$ is a diameter of this circle if the centre $\left(\frac{-3\lambda}{2},\frac{-\lambda}{2}\right)$ lies on the chord.

So, we have $3\left(\frac{-3\lambda}{2}\right)-\frac{\lambda}{2}+5=0$,

$\Rightarrow \frac{-9\lambda}{2}-\frac{\lambda}{2}+5=0$,

$\Rightarrow -5\lambda+5=0$,

$\Rightarrow \lambda=1$.

Therefore, the equation of the required circle is $x^2+y^2+3x+y-11=0$. ■

Example 5.20

Find the equation of the ellipse with foci $(\pm2,0)$, vertices $(\pm3,0)$.

Solution

From Fig. 5.36, we get

$SS' = 2c$ and $2c=4$; $A'A=2a=6$

$\Rightarrow c = 2$ and $a=3$,

$\Rightarrow b^2 = a^2-c^2=9-4=5$.

Major axis is along x-axis, since $a>b$.

Centre is (0, 0) and Foci are $(\pm2,0)$.

Therefore, equation of the ellipse is $\frac{x^2}{9}+\frac{y^2}{5}=1$. ■

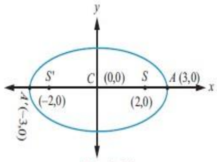


Fig.5.36

Example 5.29

Find the equations of tangent and normal to the parabola $x^2+6x+4y+5=0$ at $(1,-3)$.

Solution

Equation of parabola is $x^2+6x+4y+5=0$.

$x^2+6x+9-9+4y+5 = 0$

$(x+3)^2 = -4(y-1)$ ... (1)

Let $X = x+3, Y=y-1$

Equation (1) takes the standard form

$X^2 = -4Y$

Equation of tangent is $XX_1 = -2(Y+Y_1)$

At $(1,-3)$ $X_1 = 1+3=4; Y_1=-3-1=-4$

Therefore, the equation of tangent at $(1,-3)$ is

$(x+3)4 = -2(y-1-4)$

$2x+6 = -y+5$.

$2x+y+1 = 0$.

Slope of tangent at $(1,-3)$ is -2, so slope of normal at $(1,-3)$ is $\frac{1}{2}$

Therefore, the equation of normal at $(1,-3)$ is given by

$y+3 = \frac{1}{2}(x-1)$

$2y+6 = x-1$

$x-2y-7 = 0$. ■

Example 6.3

By vector method, prove that $\cos(\alpha+\beta)=\cos\alpha\cos\beta-\sin\alpha\sin\beta$.

Solution

Let $\hat{a}=\overline{OA}$ and $\hat{b}=\overline{OB}$ be the unit vectors and which make angles α and β, respectively, with positive x-axis, where A and B are as in the Fig. 6.8. Draw AL and BM perpendicular to the x-axis. Then $|\overline{OL}|=|\overline{OA}|\cos\alpha=\cos\alpha$, $|\overline{LA}|=|\overline{OA}|\sin\alpha=\sin\alpha$.

So, $\overline{OL}=|\overline{OL}|\hat{i} = \cos\alpha\hat{i}$, $\overline{LA}=\sin\alpha(-\hat{j})$.

Therefore, $\hat{a} = \overline{OA}=\overline{OL}+\overline{LA}=\cos\alpha\hat{i}-\sin\alpha\hat{j}$. ... (1)

Similarly, $\hat{b} = \cos\beta\hat{i}+\sin\beta\hat{j}$ ... (2)

The angle between $\hat{a}$ and $\hat{b}$ is $\alpha+\beta$ and so,

$\hat{a}\cdot\hat{b} = |\hat{a}|\,|\hat{b}|\cos(\alpha+\beta)=\cos(\alpha+\beta)$ ... (3)

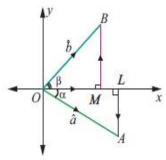


Fig. 6.8

On the other hand, from (1) and (2)

$\hat{a}\cdot\hat{b} = (\cos\alpha\hat{i}-\sin\alpha\hat{j})\cdot(\cos\beta\hat{i}+\sin\beta\hat{j})=\cos\alpha\cos\beta-\sin\alpha\sin\beta$. ... (4)

From (3) and (4), we get $\cos(\alpha+\beta)=\cos\alpha\cos\beta-\sin\alpha\sin\beta$. ■

Example 6.10

A particle is acted upon by the forces $3\hat{i}-2\hat{j}+2\hat{k}$ and $2\hat{i}+\hat{j}-\hat{k}$ is displaced from the point $(1,3,-1)$ to the point $(4,-1,\lambda)$. If the work done by the forces is 16 units, find the value of λ.

Solution

Resultant of the given forces is $\vec{F}=(3\hat{i}-2\hat{j}+2\hat{k})+(2\hat{i}+\hat{j}-\hat{k})=5\hat{i}-\hat{j}+\hat{k}$.

The displacement of the particle is given by

$\vec{d}=(4\hat{i}-\hat{j}+\lambda\hat{k})-(\hat{i}+3\hat{j}-\hat{k})=(3\hat{i}-4\hat{j}+(\lambda+1)\hat{k})$.

As the work done by the forces is 16 units, we have

$\vec{F}\cdot\vec{d}=16$.

That is, $(5\hat{i}-\hat{j}+\hat{k})\cdot(3\hat{i}-4\hat{j}+(\lambda+1)\hat{k}=16\Rightarrow\lambda+20=16$.

So, $\lambda=-4$. ■

Example 6.16

Show that the four points $(6,-7,0)$, $(16,-19,-4)$, $(0,3,-6)$, $(2,-5,10)$ lie on a same plane.

Solution

Let $A=(6,-7,0)$, $B=(16,-19,-4)$, $C=(0,3,-6)$, $D=(2,-5,10)$. To show that the four points A,B,C,D lie on a plane, we have to prove that the three vectors $\overrightarrow{AB},\overrightarrow{AC},\overrightarrow{AD}$ are coplanar.

Now, $$\overrightarrow{AB}=\overrightarrow{OB}-\overrightarrow{OA}=(16\hat{i}-19\hat{j}-4\hat{k})-(6\hat{i}-7\hat{j})=10\hat{i}-12\hat{j}-4\hat{k}$$

$$\overrightarrow{AC}=\overrightarrow{OC}-\overrightarrow{OA}=-6\hat{i}+10\hat{j}-6\hat{k} \text{ and } \overrightarrow{AD}=\overrightarrow{OD}-\overrightarrow{OA}=-4\hat{i}+2\hat{j}+10\hat{k}.$$

We have $$[\overrightarrow{AB},\overrightarrow{AC},\overrightarrow{AD}]=\begin{vmatrix}10&-12&-4\\-6&10&-6\\-4&2&10\end{vmatrix}=0.$$

Therefore, the three vectors $\overrightarrow{AB},\overrightarrow{AC},\overrightarrow{AD}$ are coplanar and hence the four points $A,B,C,$ and D lie on a plane. ■

Example 6.17

If the vectors $\vec{a},\vec{b},\vec{c}$ are coplanar, then prove that the vectors $\vec{a}+\vec{b},\vec{b}+\vec{c},\vec{c}+\vec{a}$ are also coplanar.

Solution

Since the vectors $\vec{a},\vec{b},\vec{c}$ are coplanar, we have $[\vec{a},\vec{b},\vec{c}]=0$. Using the properties of the scalar triple product, we get

$$[\vec{a}+\vec{b},\vec{b}+\vec{c},\vec{c}+\vec{a}]=[\vec{a},\vec{b}+\vec{c},\vec{c}+\vec{a}]+[\vec{b},\vec{b}+\vec{c},\vec{c}+\vec{a}]$$
$$=[\vec{a},\vec{b},\vec{c}+\vec{a}]+[\vec{a},\vec{c},\vec{c}+\vec{a}]+[\vec{b},\vec{b},\vec{c}+\vec{a}]+[\vec{b},\vec{c},\vec{c}+\vec{a}]$$
$$=[\vec{a},\vec{b},\vec{c}]+[\vec{a},\vec{b},\vec{a}]+[\vec{a},\vec{c},\vec{c}]+[\vec{a},\vec{c},\vec{a}]+[\vec{b},\vec{b},\vec{c}]+[\vec{b},\vec{b},\vec{a}]+[\vec{b},\vec{c},\vec{c}]+[\vec{b},\vec{c},\vec{a}]$$
$$=[\vec{a},\vec{b},\vec{c}]+[\vec{a},\vec{b},\vec{c}]=2[a,b,c]=0.$$

Hence the vectors $\vec{a}+\vec{b}$, $\vec{b}+\vec{c}$, $\vec{c}+\vec{a}$ are coplanar. ■

Example 6.19

Prove that $[\vec{a}\times\vec{b},\ \vec{b}\times\vec{c},\ \vec{c}\times\vec{a}]=[\vec{a},\vec{b},\vec{c}]^2$.

Solution

Using the definition of the scalar triple product, we get

$$[\vec{a}\times\vec{b},\vec{b}\times\vec{c},\vec{c}\times\vec{a}]=(\vec{a}\times\vec{b})\cdot[(\vec{b}\times\vec{c})\times(\vec{c}\times\vec{a})]. \quad \ldots(1)$$

By treating $(\vec{b}\times\vec{c})$ as the first vector in the vector triple product, we find

$$(\vec{b}\times\vec{c})\times(\vec{c}\times\vec{a})=((\vec{b}\times\vec{c})\cdot\vec{a})\vec{c}-((\vec{b}\times\vec{c})\cdot\vec{c})\vec{a}=[\vec{a},\vec{b},\vec{c}]\vec{c}.$$

Using this value in (1), we get

$$[\vec{a}\times\vec{b},\vec{b}\times\vec{c},\vec{c}\times\vec{a}]=(\vec{a}\times\vec{b})\cdot([\vec{a},\vec{b},\vec{c}]\vec{c})=[\vec{a},\vec{b},\vec{c}](\vec{a}\times\vec{b})\cdot\vec{c}=[\vec{a},\vec{b},\vec{c}]^2.$$ ■

Example 6.22

If $\vec{a}=-2\hat{i}+3\hat{j}-2\hat{k}$, $\vec{b}=3\hat{i}-\hat{j}+3\hat{k}$, $\vec{c}=2\hat{i}-5\hat{j}+\hat{k}$, find $(\vec{a}\times\vec{b})\times\vec{c}$ and $\vec{a}\times(\vec{b}\times\vec{c})$. State whether they are equal.

Solution

By definition, $$\vec{a}\times\vec{b}=\begin{vmatrix}\hat{i}&\hat{j}&\hat{k}\\-2&3&-2\\3&-1&3\end{vmatrix}=7\hat{i}-7\hat{k}.$$

Then, $$(\vec{a}\times\vec{b})\times\vec{c}=\begin{vmatrix}\hat{i}&\hat{j}&\hat{k}\\7&0&-7\\2&-5&1\end{vmatrix}=-35\hat{i}-21\hat{j}-35\hat{k}. \quad \ldots(1)$$

$$\vec{b}\times\vec{c}=\begin{vmatrix}\hat{i}&\hat{j}&\hat{k}\\3&-1&3\\2&-5&1\end{vmatrix}=14\hat{i}+3\hat{j}-13\hat{k}.$$

Next, $$\vec{a}\times(\vec{b}\times\vec{c})=\begin{vmatrix}\hat{i}&\hat{j}&\hat{k}\\-2&3&-2\\14&3&-13\end{vmatrix}=-33\hat{i}-54\hat{j}-48\hat{k}. \quad \ldots(2)$$

Therefore, equations (1) and (2) lead to $(\vec{a}\times\vec{b})\times\vec{c}\neq\vec{a}\times(\vec{b}\times\vec{c})$. ■

Example 6.32

Show that the lines $\frac{x-1}{4}=\frac{2-y}{6}=\frac{z-4}{12}$ and $\frac{x-3}{-2}=\frac{y-3}{3}=\frac{5-z}{6}$ are parallel.

Solution

We observe that the straight line $\frac{x-1}{4}=\frac{2-y}{6}=\frac{z-4}{12}$ is parallel to the vector $4\hat{i}-6\hat{j}+12\hat{k}$ and the straight line $\frac{x-3}{-2}=\frac{y-3}{3}=\frac{5-z}{6}$ is parallel to the vector $-2\hat{i}+3\hat{j}-6\hat{k}$.

Since $4\hat{i}-6\hat{j}+12\hat{k}=-2(-2\hat{i}+3\hat{j}-6\hat{k})$, the two vectors are parallel, and hence the two straight lines are parallel. ■

Example 6.36

Find the shortest distance between the two given straight lines $\vec{r}=(2\hat{i}+3\hat{j}+4\hat{k})+t(-2\hat{i}+\hat{j}-2\hat{k})$ and $\frac{x-3}{2}=\frac{y}{-1}=\frac{z+2}{2}$.

Solution

The parametric form of vector equations of the given straight lines are

$$\vec{r}=(2\hat{i}+3\hat{j}+4\hat{k})+t(-2\hat{i}+\hat{j}-2\hat{k})$$
and $$\vec{r}=(3\hat{i}-2\hat{k})+t(2\hat{i}-\hat{j}+2\hat{k})$$

Comparing the given two equations with $\vec{r}=\vec{a}+t\vec{b}$, $\vec{r}=\vec{c}+s\vec{d}$

we have $\vec{a}=2\hat{i}+3\hat{j}+4\hat{k}$, $\vec{b}=-2\hat{i}+\hat{j}-2\hat{k}$, $\vec{c}=3\hat{i}-2\hat{k}$, $\vec{d}=2\hat{i}-\hat{j}+2\hat{k}$.

Clearly, $\vec{b}$ is a scalar multiple of $\vec{d}$, and hence the two straight lines are parallel. We know that the shortest distance between two parallel straight lines is given by $d=\frac{|(\vec{c}-\vec{a})\times\vec{b}|}{|\vec{b}|}$.

Now, $$(\vec{c}-\vec{a})\times\vec{b}=\begin{vmatrix}\hat{i}&\hat{j}&\hat{k}\\1&-3&-6\\-2&1&-2\end{vmatrix}=12\hat{i}+14\hat{j}-5\hat{k}$$

Therefore, $$d=\frac{|12\hat{i}+14\hat{j}-5\hat{k}|}{|-2\hat{i}+\hat{j}-2\hat{k}|}=\frac{\sqrt{365}}{3}.$$ ■

Example 6.47

Find the acute angle between the planes $\vec{r}\cdot(2\hat{i}+2\hat{j}+2\hat{k})=11$ and $4x-2y+2z=15$.

Solution

The normal vectors of the two given planes $\vec{r}\cdot(2\hat{i}+2\hat{j}+2\hat{k})=11$ and $4x-2y+2z=15$ are $\vec{n}_1=2\hat{i}+2\hat{j}+2\hat{k}$ and $\vec{n}_2=4\hat{i}-2\hat{j}+2\hat{k}$ respectively.

If θ is the acute angle between the planes, then we have

$$\theta=\cos^{-1}\left(\frac{|\vec{n}_1\cdot\vec{n}_2|}{|\vec{n}_1||\vec{n}_2|}\right)=\cos^{-1}\left(\frac{|(2\hat{i}+2\hat{j}+2\hat{k})\cdot(4\hat{i}-2\hat{j}+2\hat{k})|}{|2\hat{i}+2\hat{j}+2\hat{k}||4\hat{i}-2\hat{j}+2\hat{k}|}\right)=\cos^{-1}\left(\frac{\sqrt{2}}{3}\right)$$ ■

Example 6.48

Find the angle between the straight line $\vec{r}=(2\hat{i}+3\hat{j}+\hat{k})+t(\hat{i}-\hat{j}+\hat{k})$ and the plane $2x-y+z=5$.

Solution

The angle between a line $\vec{r}=\vec{a}+t\vec{b}$ and a plane $\vec{r}\cdot\vec{n}=p$ with normal $\vec{n}$ is $\theta=\sin^{-1}\left(\frac{|\vec{b}\cdot\vec{n}|}{|\vec{b}||\vec{n}|}\right)$.

Here, $\vec{b}=\hat{i}-\hat{j}+\hat{k}$ and $\vec{n}=2\hat{i}-\hat{j}+\hat{k}$.

So, we get $$\theta=\sin^{-1}\left(\frac{|\vec{b}\cdot\vec{n}|}{|\vec{b}||\vec{n}|}\right)=\sin^{-1}\left(\frac{|(\hat{i}-\hat{j}+\hat{k})\cdot(2\hat{i}-\hat{j}+\hat{k})|}{|\hat{i}-\hat{j}+\hat{k}||2\hat{i}-\hat{j}+\hat{k}|}\right)=\sin^{-1}\left(\frac{2\sqrt{2}}{3}\right)$$ ■

Example 6.51

Find the distance between the parallel planes $x+2y-2z+1=0$ and $2x+4y-4z+5=0$.

Solution

We know that the formula for the distance between two parallel planes $ax+by+cz+d_1=0$ and $ax+by+cz+d_2=0$ is $\delta=\frac{|d_1-d_2|}{\sqrt{a^2+b^2+c^2}}$. Rewrite the second equation as $x+2y-2z+\frac{5}{2}=0$.

Comparing the given equations with the general equations, we get $a=1, b=2, c=-2, d_1=1, d_2=\frac{5}{2}$.

Substituting these values in the formula, we get the distance

$$\delta=\frac{|d_1-d_2|}{\sqrt{a^2+b^2+c^2}}=\frac{\left|1-\frac{5}{2}\right|}{\sqrt{1^2+2^2+(-2^2)}}=\frac{1}{2}\text{ units.}$$

Example 6.55

Find the image of the point whose position vector is $\hat{i}+2\hat{j}+3\hat{k}$ in the plane $\vec{r}\cdot(\hat{i}+2\hat{j}+4\hat{k})=38$.

Solution

Here, $\vec{u}=\hat{i}+2\hat{j}+3\hat{k}$, $\vec{n}=\hat{i}+2\hat{j}+4\hat{k}$, $p=38$. Then the position vector of the image $\vec{v}$ of $\vec{u}=\hat{i}+2\hat{j}+3\hat{k}$ is given by $\vec{v}=\vec{u}+\frac{2[p-(\vec{u}\cdot\vec{n})]}{|\vec{n}|^2}\vec{n}$.

$$\vec{v}=(\hat{i}+2\hat{j}+3\hat{k})+\frac{2\left[38-\left((\hat{i}+2\hat{j}+3\hat{k})\cdot(\hat{i}+2\hat{j}+4\hat{k})\right)\right]}{(\hat{i}+2\hat{j}+4\hat{k})\cdot(\hat{i}+2\hat{j}+4\hat{k})}(\hat{i}+2\hat{j}+4\hat{k}).$$

That is, $\vec{v}=(\hat{i}+2\hat{j}+3\hat{k})+2\left(\frac{38-17}{21}\right)(\hat{i}+2\hat{j}+4\hat{k})=3\hat{i}+6\hat{j}+11\hat{k}$.

Therefore, the image of the point with position vector $\hat{i}+2\hat{j}+3\hat{k}$ is $3\hat{i}+6\hat{j}+11\hat{k}$.

Note

The foot of the perpendicular from the point with position vector $\hat{i}+2\hat{j}+3\hat{k}$ in the given plane is

$$\frac{(\hat{i}+2\hat{j}+3\hat{k})+(3\hat{i}+6\hat{j}+11\hat{k})}{2}=2\hat{i}+4\hat{j}+7\hat{k}.$$

Example 7.9

Salt is poured from a conveyer belt at a rate of 30 cubic metre per minute forming a conical pile with a circular base whose height and diameter of base are always equal. How fast is the height of the pile increasing when the pile is 10 metre high?

Solution

Let h and r be the height and the base radius. Therefore $h=2r$. Let V be the volume of the salt cone.

$$V=\frac{1}{3}\pi r^2h=\frac{1}{12}\pi h^3;\quad \frac{dV}{dt}=30\text{ m}^3/\text{min}.$$

Hence, $\frac{dV}{dt}=\frac{1}{4}\pi h^2\frac{dh}{dt}$

Therefore, $\frac{dh}{dt}=4\frac{dV}{dt}\cdot\frac{1}{\pi h^2}$

That is, $\frac{dh}{dt}=4\times30\times\frac{1}{100\pi}$

$=\frac{6}{5\pi}$ m / min.

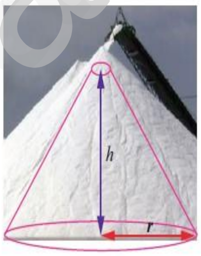


Fig.7.5

Example 7.17

If the curves $ax^2+by^2=1$ and $cx^2+dy^2=1$ intersect each other orthogonally then, show that $\frac{1}{a}-\frac{1}{b}=\frac{1}{c}-\frac{1}{d}$.

Solution

Let the two curves intersect at a point (x_0, y_0). This leads to $(a-c)x_0^2+(b-d)y_0^2=0$.

Let us now find the slope of the curves at the point of intersection (x_0, y_0). The slopes of the curves are as follows :

For the curve $ax^2+by^2=1$, $\frac{dy}{dx}=-\frac{ax}{by}$.

For the curve $cx^2+dy^2=1$, $\frac{dy}{dx}=-\frac{cx}{dy}$.

Now, if two curves cut orthogonally, then the product of their slopes, at the point of intersection (x_0, y_0), is -1. Hence, if the above two curves cut orthogonally at (x_0, y_0) then

$$\left(-\frac{ax_0}{by_0}\right)\times\left(-\frac{cx_0}{dy_0}\right)=-1.$$

That is, $acx_0^2+bdy_0^2=0$,

together with $(a-c)x_0^2+(b-d)y_0^2=0$

gives, $\frac{a-c}{ac}=\frac{b-d}{bd}$.

That is, $\frac{1}{c}-\frac{1}{a}=\frac{1}{d}-\frac{1}{b}$.

Hence, $\frac{1}{a}-\frac{1}{b}=\frac{1}{c}-\frac{1}{d}$.

Remark

Example 7.18

Prove that the ellipse $x^2+4y^2=8$ and the hyperbola $x^2-2y^2=4$ intersect orthogonally.

Solution

Let the point of intersection of the two curves be (a,b). Hence,

$$a^2+4b^2=8 \text{ and } a^2-2b^2=4 \quad \dots(4)$$

It is enough to show that the product of the slopes of the two curves evaluated at (a,b) is -1.

Differentiation of $x^2+4y^2=8$ with respect to x, gives

$$2x+8y\frac{dy}{dx}=0$$

Therefore $\frac{dy}{dx}=-\frac{x}{4y}$.

Then, $\frac{dy}{dx}$ at $(a,b)=m_1=-\frac{a}{4b}$.

Differentiation of $x^2-2y^2=4$ with respect to x, gives

$$2x-4y\frac{dy}{dx}=0$$

Therefore, $\frac{dy}{dx}=\frac{x}{2y}$.

Then $\frac{dy}{dx}$ at $(a,b)=m_2=\frac{a}{2b}$.

Therefore, $m_1\times m_2=\left(-\frac{a}{4b}\right)\times\left(\frac{a}{2b}\right)=-\frac{a^2}{8b^2}$ ...(5)

Applying the ratio of proportions in (4), we get

$$\frac{a^2}{-16-16}=\frac{b^2}{-8+4}=\frac{1}{-2-4}.$$

Therefore $\frac{a^2}{b^2}=\frac{32}{4}=8$. Substituting in (5), we get $m_1\times m_2=-1$. Hence, the curves cut orthogonally.

Example 7.29

A thermometer was taken from a freezer and placed in a boiling water. It took 22 seconds for the thermometer to raise from -10°C to 100°C. Show that the rate of change of temperature at some time t is 5°C per second.

Solution

Let $f(t)$ be the temperature at time t. By the mean value theorem, we have

$$f'(c)=\frac{f(b)-f(a)}{b-a}=\frac{100-(-10)}{22}=\frac{110}{22}=5^\circ\text{C per second.}$$

Hence the instantaneous rate of change of temperature at some time t is 5°C per second.

Example 7.36

Evaluate the limit $\lim_{x\to0}\left(\frac{\sin x}{x^2}\right)$.

Solution

If we directly substitute $x=0$ we get an indeterminate form $\frac{0}{0}$ and hence we apply the l'Hôpital's rule to evaluate the limit as,

$$\lim_{x\to0^+}\left(\frac{\sin x}{x^2}\right) = \lim_{x\to0^+}\left(\frac{\cos x}{2x}\right)=\infty$$

$$\lim_{x\to0^-}\left(\frac{\sin x}{x^2}\right) = \lim_{x\to0^-}\left(\frac{\cos x}{2x}\right)=-\infty$$

As the left limit and the right limit are not the same we conclude that the limit does not exist. ■

Remark

One may be tempted to use the l'Hôpital's rule once again in $\lim_{x\to0^+}\left(\frac{\cos x}{2x}\right)$ to conclude

$$\lim_{x\to0^+}\left(\frac{\cos x}{2x}\right) = \lim_{x\to0^+}\left(\frac{-\sin x}{2}\right)=0.$$

which is not true because it was not an indeterminate form.

Example 7.38

Evaluate : $\lim_{x\to1^-}\left(\frac{\log(1-x)}{\cot(\pi x)}\right)$.

Solution

This is an indeterminate form $\frac{\infty}{\infty}$ and hence we use the l'Hôpital's Rule to evaluate the limit.

$$\text{Thus, } \lim_{x\to1^-}\frac{\log(1-x)}{\cot(\pi x)} = \lim_{x\to1^-}\left(\frac{-\frac{1}{1-x}}{-\pi\operatorname{cosec}^2(\pi x)}\right)\left(\frac{\infty}{\infty}\text{ form}\right)$$

On simplication, we get

$$= \lim_{x\to1^-}\left(\frac{\sin^2(\pi x)}{\pi(1-x)}\right).\quad\left(\frac{0}{0}\text{ form}\right)$$

again applying the l'Hôpital Rule, we get

$$= \lim_{x\to1^-}\left(\frac{2\pi\sin(\pi x)\cdot\cos(\pi x)}{-\pi}\right)$$

$$= \lim_{x\to1^-}\left(-2\sin(\pi x)\cdot\cos(\pi x)\right)$$

$$= 0.$$ ■

Example 7.42

Evaluate : $\lim_{x\to\infty}\left(\frac{e^x}{x^m}\right), m\in N$.

Solution

This is an indeterminate of the form $\left(\frac{\infty}{\infty}\right)$.

To evaluate this limit, we apply l'Hôpital Rule m times.

$$\text{Thus, we have } \lim_{x\to\infty}\frac{e^x}{x^m} = \lim_{x\to\infty}\frac{e^x}{m!}$$

$$= \infty.$$

Example 7.54

Find the intervals of monotonicity and local extrema of the function $f(x)=x\log x+3x$.

Solution

The given function is defined and is differentiable at all $x\in(0,\infty)$.

$$f(x) = x\log x+3x.$$

$$\text{Therefore } f'(x) = \log x+1+3=4+\log x.$$

The stationary numbers are given by $4+\log x = 0$.

That is $x = e^{-4}$.

Hence the intervals of monotonicity are $(0,e^{-4})$ and (e^{-4},∞).

At $x=e^{-5}\in(0,e^{-4}), f'(e^{-5})=-1<0$ and hence in the interval $(0,e^{-4})$ the function is strictly decreasing.

At $x=e^{-3}\in(e^{-4},\infty), f'(e^{-3})=1>0$ and hence strictly increasing in the interval (e^{-4},∞). Since $f'(x)$ changes from negative to positive when passing through $x=e^{-4}$, the first derivative test tells us there is a local minimum at $x=e^{-4}$ and it is $f(e^{-4})=-e^{-4}$. ■

Example 7.61

Find the local maximum and minimum of the function x^2y^2 on the line $x+y=10$.

Solution

Let the given function be written as $f(x)=x^2(10-x)^2$. Now,

$$f(x) = x^2(100-20x+x^2)=x^4-20x^3+100x^2$$

$$\text{Therefore, } f'(x) = 4x^3-60x^2+200x=4x(x^2-15x+50)$$

$$f'(x) = 4x(x^2-15x+50)=0\Rightarrow x=0,5,10$$

$$\text{and } f''(x) = 12x^2-120x+200$$

The stationary numbers of $f(x)$ are $x=0,5,10$ at these points the values of $f''(x)$ are respectively $200,-100$ and 200. At $x=0$, it has local minimum and its value is $f(0)=0$. At $x=5$, it has local maximum and its value is $f(5)=625$. At $x=10$, it has local minimum and its value is $f(10)=0$. ■

Example 8.10

Consider $g(x,y)=\frac{2x^2y}{x^2+y^2}$ if $(x,y)\neq(0,0)$ and $g(0,0)=0$. Show that g is continuous on $\mathbb{R}^2$.

Solution

Observe that the function g is defined for all $(x,y)\in\mathbb{R}^2$. It is easy to check, as in the above examples, that g is continuous at all point $(x,y)\neq(0,0)$. Next we shall check the continuity of g at $(0,0)$. For that we see if g has a limit L at $(0,0)$ and if $L=g(0,0)=0$. So we consider

$$|g(x,y)-g(0,0)|=\left|\frac{2x^2y}{x^2+y^2}-0\right|=\frac{2|x^2y|}{|x^2+y^2|}=\frac{2|xy||x|}{x^2+y^2}\le\frac{(x^2+y^2)|x|}{x^2+y^2}\le|x| \quad \dots(9)$$

Note that in the final step above we have used $2|xy|\le x^2+y^2$ (which follows by considering $0\le(x-y)^2$) for all $x,y\in\mathbb{R}$. Note that $(x,y)\to(0,0)$ implies $|x|\to0$. Then from (9) it follows that $\lim_{(x,y)\to(0,0)}\frac{2x^2y}{x^2+y^2}=0=g(0,0)$; which proves that g is continuous at $(0,0)$. So g is continuous at every point of $\mathbb{R}^2$. ■

Example 8.14

Let $w(x,y)=xy+\frac{e^y}{y^2+1}$ for all $(x,y)\in\mathbb{R}^2$. Calculate $\frac{\partial^2w}{\partial y\partial x}$ and $\frac{\partial^2w}{\partial x\partial y}$.

Solution

First we calculate $\frac{\partial w}{\partial x}(x,y)=\frac{\partial(xy)}{\partial x}+\frac{\partial\left(\frac{e^y}{y^2+1}\right)}{\partial x}$.

This gives $\frac{\partial w}{\partial x}(x,y)=y+0$ and hence $\frac{\partial^2w}{\partial y\partial x}(x,y)=1$. On the other hand,

$$\frac{\partial w}{\partial y}(x,y) = \frac{\partial(xy)}{\partial y}+\frac{\partial\left(\frac{e^y}{y^2+1}\right)}{\partial y}.$$

$$= x+\frac{(y^2+1)e^y-e^y2y}{(y^2+1)^2}.$$

Hence $\frac{\partial^2w}{\partial x\partial y}(x,y)=1$.

Example 8.19

Let $g(x,y)=x^2-yx+\sin(x+y)$, $x(t)=e^{3t}$, $y(t)=t^2$, $t\in\mathbb{R}$. Find $\frac{dg}{dt}$.

Solution

We shall follow the tree diagram to calculate.

So first we need to find $\frac{\partial g}{\partial x}$, $\frac{\partial g}{\partial y}$, $\frac{dx}{dt}$ and $\frac{dy}{dt}$.

Now $\frac{\partial g}{\partial x}=2x-y+\cos(x+y)$, $\frac{\partial g}{\partial y}=-x+\cos(x+y)$, $\frac{dx}{dt}=3e^{3t}$ and $\frac{dy}{dt}=2t$.

Thus

$$\begin{aligned}\frac{dg}{dt} &= \frac{\partial g}{\partial x}\frac{dx}{dt}+\frac{\partial g}{\partial y}\frac{dy}{dt}\\ &=(2x-y+\cos(x+y))3e^{3t}+(-x+\cos(x+y))(2t)\\ &=(2e^{3t}-t^2+\cos(e^{3t}+t^2))3e^{3t}+(-e^{3t}+\cos(e^{3t}+t^2))(2t)\\ &=6e^{6t}-3t^2e^{3t}+3e^{3t}\cos(e^{3t}+t^2)-2te^{3t}+2t\cos(e^{3t}+t^2).\end{aligned}$$

Also, some times our $W(x,y)$ will be such that $x=x(s,t)$, and $y=y(s,t)$ where $s,t\in\mathbb{R}$. Then W can be considered as a function that depends on s and t. If x,y both have partial derivatives with respect to s,t and W has partial derivatives with respect to x and y, then we can calculate the partial derivatives of W with respect to s and t using the following theorem.

Example 9.3

Evaluate $\int_0^1 x^3dx$, as the limit of a sum.

Solution

Here $f(x)=x^3$, $a=0$ and $b=1$. Hence, we get

$$\int_a^b f(x)dx = \lim_{n\to\infty}\frac{1}{n}\sum_{r=1}^{n} f\left(\frac{r}{n}\right) \Rightarrow \int_0^1 x^3dx=\lim_{n\to\infty}\frac{1}{n}\sum_{r=1}^{n}\frac{r^3}{n^3}$$

$$=\lim_{n\to\infty}\frac{1}{n^4}\left[1^3+2^3+\cdots+n^3\right]=\lim_{n\to\infty}\frac{1}{n^4}\frac{n^2(n+1)^2}{4}$$

$$=\lim_{n\to\infty}\frac{1}{4}\left(1+\frac{1}{n}\right)^2=\frac{1}{4}.$$

Example 9.4

Example 9.7

Evaluate : $\int_0^1 [2x]dx$ where $[\cdot]$ is the greatest integer function.

Solution

$$\int_0^1[2x]dx=\int_0^{\frac{1}{2}}[2x]dx+\int_{\frac{1}{2}}^1[2x]dx=\int_0^{\frac{1}{2}}0\,dx+\int_{\frac{1}{2}}^1 1\,dx=0+[x]_{\frac{1}{2}}^1=1-\frac{1}{2}=\frac{1}{2}.$$

Example 9.8

Evaluate : $\int_0^{\frac{\pi}{3}}\frac{\sec x\tan x}{1+\sec^2 x}dx$.

Solution

Let $I=\int_0^{\frac{\pi}{3}}\frac{\sec x\tan x}{1+\sec^2 x}dx$. Put $\sec x=u$. Then, $\sec x\tan x\,dx=du$.

When $x=0$, $u=\sec 0=1$. When $x=\frac{\pi}{3}$, $u=\sec\frac{\pi}{3}=2$.

$$\therefore I=\int_1^2\frac{du}{1+u^2}=[\tan^{-1}u]_1^2=\tan^{-1}(2)-\tan^{-1}1=\tan^{-1}(2)-\frac{\pi}{4}.$$

Example 9.9

Evaluate : $\int_0^9\frac{1}{x+\sqrt{x}}dx$.

Solution

Let $\sqrt{x}=u$. Then $x=u^2$, and so $dx=2u\,du$.

When $x=0$, $u=0$. When $x=9$, $u=3$.

$$\therefore\int_0^9\frac{1}{x+\sqrt{x}}dx=\int_0^3\frac{1}{u^2+u}(2u)du=2\int_0^3\frac{1}{1+u}du=2\Big[\log|1+u|\Big]_0^3=2[\log 4-0]=\log 16.$$

Example 9.11

Evaluate : $\int_0^{\frac{\pi}{2}}\frac{\cos\theta}{(1+\sin\theta)(2+\sin\theta)}d\theta$.

Solution

Let $I=\int_0^{\frac{\pi}{2}}\frac{\cos\theta}{(1+\sin\theta)(2+\sin\theta)}d\theta$. Put $u=1+\sin\theta$. Then, $du=\cos\theta d\theta$.

When $\theta=0$, $u=1$. When $\theta=\frac{\pi}{2}$, $u=2$.

$$\therefore I=\int_1^2\frac{du}{u(1+u)}=\int_1^2\frac{(1+u)-u}{u(1+u)}du=\int_1^2\left(\frac{1}{u}-\frac{1}{1+u}\right)du=\left[\log u-\log(1+u)\right]_1^2$$

$$=(\log 2-\log 3)-(\log 1-\log 2)=2\log 2-\log 3=\log\frac{4}{3}.$$

Example 9.13

Evaluate : $\int_0^{\frac{\pi}{2}}\left(\sqrt{\tan x}+\sqrt{\cot x}\right)dx$.

Solution

Let $I=\int_0^{\frac{\pi}{2}}\left(\sqrt{\tan x}+\sqrt{\cot x}\right)dx$. Then, we get

$$I=\int_0^{\frac{\pi}{2}}\left(\sqrt{\frac{\sin x}{\cos x}}+\sqrt{\frac{\cos x}{\sin x}}\right)dx=\int_0^{\frac{\pi}{2}}\frac{\sin x+\cos x}{\sqrt{\sin x\cos x}}dx=\sqrt{2}\int_0^{\frac{\pi}{2}}\frac{\sin x+\cos x}{\sqrt{2\sin x\cos x}}dx$$

$$=\sqrt{2}\int_0^{\frac{\pi}{2}}\frac{(\sin x+\cos x)dx}{\sqrt{1-(\sin x-\cos x)^2}}.$$

Put $u=\sin x-\cos x$. Then, $du=(\cos x+\sin x)dx$.

When $x=0$, $u=-1$. When $x=\frac{\pi}{2}$, $u=1$.

$$\therefore I=\sqrt{2}\int_{-1}^1\frac{du}{\sqrt{1-u^2}}=\sqrt{2}[\sin^{-1}u]_{-1}^1=\sqrt{2}\left[\sin^{-1}(1)-\sin^{-1}(-1)\right]=\pi\sqrt{2}.$$

Example 9.16

Show that $\int_0^{\frac{\pi}{2}}\frac{dx}{4+5\sin x}=\frac{1}{3}\log_e 2$.

Solution

Put $u=\tan\frac{x}{2}$. Then, $\sin x=\frac{2\tan\frac{x}{2}}{1+\tan^2\frac{x}{2}}=\frac{2u}{1+u^2}$, $du=\frac{1}{2}\sec^2\frac{x}{2}dx\Rightarrow dx=\frac{2du}{1+u^2}$.

When $x=0$, $u=\tan 0=0$. When $x=\frac{\pi}{2}$, $u=\tan\frac{\pi}{4}=1$.

$$\therefore I=\int_0^{\frac{\pi}{2}}\frac{dx}{4+5\sin x}=\int_0^1\frac{\frac{2du}{1+u^2}}{4+5\left(\frac{2u}{1+u^2}\right)}=\int_0^1\frac{du}{2u^2+5u+2}=\frac{1}{2}\int_0^1\frac{du}{u^2+\frac{5}{2}u+1}$$

$$=\frac{1}{2}\int_0^1\frac{du}{\left(u+\frac{5}{4}\right)^2-\left(\frac{3}{4}\right)^2}=\left[\frac{1}{2}\times\frac{1}{2\times\left(\frac{3}{4}\right)}\log\left(\frac{\left(u+\frac{5}{4}\right)-\frac{3}{4}}{\left(u+\frac{5}{4}\right)+\frac{3}{4}}\right)\right]_0^1=\frac{1}{3}\left[\log\left(\frac{u+\frac{1}{2}}{u+2}\right)\right]_0^1=\frac{1}{3}\log 2.$$

Note

To evaluate anti-derivatives of the type $\int\frac{dx}{a\cos x+b\sin x+c}$, we use the substitution method by putting $u=\tan\frac{x}{2}$ so that $\cos x=\frac{1-u^2}{1+u^2}$, $\sin x=\frac{2u}{1+u^2}$, $dx=\frac{2du}{1+u^2}$.

மருதம் அகாடமி Youtube channel

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Example 9.19

Evaluate $\int_0^{\frac{\pi}{4}}\frac{1}{\sin x+\cos x}dx$

Solution

$$I=\int_0^{\frac{\pi}{4}}\frac{1}{\sin x+\cos x}dx=\int_0^{\frac{\pi}{4}}\frac{1}{\sqrt{2}\left(\frac{1}{\sqrt{2}}\sin x+\frac{1}{\sqrt{2}}\cos x\right)}dx$$

$$=\frac{1}{\sqrt{2}}\int_0^{\frac{\pi}{4}}\frac{1}{\left(\cos\frac{\pi}{4}\cos x+\sin\frac{\pi}{4}\sin x\right)}dx=\frac{1}{\sqrt{2}}\int_0^{\frac{\pi}{4}}\frac{1}{\cos\left(\frac{\pi}{4}-x\right)}dx$$

$$=\frac{1}{\sqrt{2}}\int_0^{\frac{\pi}{4}}\frac{1}{\cos x}dx,\ \text{since } \int_0^a f(x)dx=\int_0^a f(a-x)dx$$

$$=\frac{1}{\sqrt{2}}\int_0^{\frac{\pi}{4}}\sec x\,dx=\frac{1}{\sqrt{2}}[\log(\sec x+\tan x)]_0^{\frac{\pi}{4}}$$

$$=\frac{1}{\sqrt{2}}\left[\log(\sqrt{2}+1)-\log(1+0)\right]$$

$$=\frac{1}{\sqrt{2}}\log(\sqrt{2}+1).$$

Example 9.23

If $f(x)=f(a+x)$, then $\int_0^{2a}f(x)dx=2\int_0^a f(x)dx$

Solution

We write $\int_0^{2a}f(x)dx=\int_0^a f(x)dx+\int_a^{2a}f(x)dx$ ... (1)

Consider $\int_a^{2a}f(x)dx$

Substituting $x=a+u$, we have $dx=du$; when $x=a, u=0$ and when $x=2a, u=a$.

$$\therefore\int_a^{2a}f(x)dx=\int_0^a f(a+u)du=\int_0^a f(u)du,\ \text{since } f(x)=f(a+x)$$

$$=\int_0^a f(x)dx. \quad ...(2)$$

Substituting (2) in (1), we get

$$\int_0^{2a}f(x)dx=2\int_0^a f(x)dx.$$

Example 9.26

Evaluate : $\int_0^a\frac{f(x)}{f(x)+f(a-x)}dx.$

Solution

Let $I=\int_0^a\frac{f(x)}{f(x)+f(a-x)}dx$ ... (1)

Applying the formula $\int_0^a f(x)dx=\int_0^a f(a-x)dx$ in equation (1), we get

$$I=\int_0^a\frac{f(a-x)}{f(a-x)+f(a-(a-x))}dx$$

$$=\int_0^a\frac{f(a-x)}{f(x)+f(a-x)}dx. \quad ...(2)$$

Adding equations (1) and (2), we get

$$2I=\int_0^a\frac{f(x)}{f(x)+f(a-x)}dx+\int_0^a\frac{f(a-x)}{f(x)+f(a-x)}dx$$

$$=\int_0^a\frac{f(x)+f(a-x)}{f(x)+f(a-x)}dx$$

$$=\int_0^a dx=a.$$

Hence, we get $I=\frac{a}{2}$.

Example 9.30

Evaluate $\int_{-\pi}^{\pi}\frac{\cos^2 x}{1+a^x}dx$

Solution

Let $I=\int_{-\pi}^{\pi}\frac{\cos^2 x}{1+a^x}dx$ ... (1)

Using $\int_a^b f(x)dx=\int_a^b f(a+b-x)dx$ we get,

$$I=\int_{-\pi}^{\pi}\frac{\cos^2(\pi-\pi-x)}{1+a^{\pi-\pi-x}}dx$$

$$=\int_{-\pi}^{\pi}\frac{\cos^2(-x)}{1+a^{-x}}dx$$

$$=\int_{-\pi}^{\pi}a^x\left(\frac{\cos^2 x}{a^x+1}\right)dx \quad ...(2)$$

Adding (1) and (2) we get

$$2I=\int_{-\pi}^{\pi}\frac{\cos^2 x}{a^x+1}(a^x+1)dx=\int_{-\pi}^{\pi}\cos^2 x\,dx$$

$$=2\int_0^{\pi}\cos^2 x\,dx\ \text{(since } \cos^2 x \text{ is an even function)}$$

Hence $I=\int_0^{\pi}\frac{(1+\cos 2x)}{2}dx=\frac{1}{2}\left[x+\frac{\sin 2x}{2}\right]_0^{\pi}=\frac{1}{2}[\pi]=\frac{\pi}{2}$.

Example 9.33

Evaluate : $\int_0^{2\pi}x^2\sin nx\,dx$, where n is a positive integer.

Solution

Taking $u=x^2$ and $v=\sin nx$, and applying the Bernoulli's formula, we get

$$I=\int_0^{2\pi}x^2\sin nx\,dx=\left[(x^2)\left(-\frac{\cos nx}{n}\right)-(2x)\left(-\frac{\sin nx}{n^2}\right)+(2)\left(\frac{\cos nx}{n^3}\right)\right]_0^{2\pi}$$

$$=\left[(4\pi^2)\left(-\frac{1}{n}\right)-0+(2)\left(\frac{1}{n^3}\right)\right]-\left[0-0+(2)\left(\frac{1}{n^3}\right)\right],\ \text{since } \cos 2n\pi=1 \text{ and } \sin 2n\pi=0$$

$$=-\frac{4\pi^2}{n}+\frac{2}{n^3}-\frac{2}{n^3}=-\frac{4\pi^2}{n}.$$

Example 9.34

Evaluate : $\int_{-1}^{1}e^{-\lambda x}(1-x^2)dx$.

Solution

Taking $u=1-x^2$ and $v=e^{-\lambda x}$, and applying the Bernoulli's formula, we get

$$I=\int_{-1}^{1}e^{-\lambda x}(1-x^2)dx=\left[(1-x^2)\left(\frac{e^{-\lambda x}}{-\lambda}\right)-(-2x)\left(\frac{e^{-\lambda x}}{\lambda^2}\right)+(-2)\left(\frac{e^{-\lambda x}}{-\lambda^3}\right)\right]_{-1}^{1}$$

$$=2\left(\frac{e^{-\lambda}}{\lambda^2}\right)+2\left(\frac{e^{-\lambda}}{\lambda^3}\right)+2\left(\frac{e^{\lambda}}{\lambda^2}\right)-2\left(\frac{e^{\lambda}}{\lambda^3}\right)$$

$$=\frac{2}{\lambda^2}\left(e^{\lambda}+e^{-\lambda}\right)-\frac{2}{\lambda^3}\left(e^{\lambda}-e^{-\lambda}\right).$$

Example 9.42

Evaluate $\int_0^1 x^3(1-x)^4dx$.

Solution

$$\int_0^1 x^m(1-x)^n dx=\frac{m!\times n!}{(m+n+1)!}.$$

$$\therefore\int_0^1 x^3(1-x)^4dx=\frac{3!\times 4!}{(3+4+1)!}=\frac{3!\times 4!}{8!}=\frac{3\times2\times1\times4\times3\times2\times1}{8\times7\times6\times5\times4\times3\times2\times1}=\frac{1}{280}.$$

Example 9.46

Evaluate $\int_0^\infty \frac{x^n}{n^x}dx$, where n is a positive integer ≥ 2.

Solution

Using the formula $n = e^{\log_e n}$, we get

$$I = \int_0^\infty \frac{x^n}{n^x}dx = \int_0^\infty n^{-x}x^n dx = \int_0^\infty \left(e^{\log n}\right)^{-x} x^n dx = \int_0^\infty e^{-x\log n}x^n dx.$$

Using the substitution $u = x\log n$, we get $dx = \frac{du}{\log n}$.

When $x = 0$, we get $u = 0$. When $x = \infty$, we get $u = \infty$.

$$\therefore I = \int_0^\infty e^{-u}\left(\frac{u}{\log n}\right)^n \frac{du}{\log n}$$

$$= \frac{1}{(\log n)^{n+1}}\int_0^\infty e^{-u}u^{(n+1)-1}du = \frac{\Gamma(n+1)}{(\log n)^{n+1}} = \frac{n!}{(\log n)^{n+1}}.$$

Example 9.57

The region enclosed by the circle $x^2 + y^2 = a^2$ is divided into two segments by the line $x = h$. Find the area of the smaller segment.

Solution

The smaller segment is sketched in Fig. 9.29. Here $0 < h < a$. By symmetry about the x-axis, the area of the smaller segment is given by

$$A = 2\int_h^a \sqrt{a^2 - x^2}\,dx = 2\left[\frac{x\sqrt{a^2-x^2}}{2} + \frac{a^2}{2}\sin^{-1}\left(\frac{x}{a}\right)\right]_h^a$$

$$= 2\left[0 + \frac{a^2}{2}\sin^{-1}(1)\right] - 2\left[\frac{h\sqrt{a^2-h^2}}{2} + \frac{a^2}{2}\sin^{-1}\left(\frac{h}{a}\right)\right]$$

$$= a^2\left(\frac{\pi}{2}\right) - h\sqrt{a^2-h^2} - a^2\sin^{-1}\left(\frac{h}{a}\right)$$

$$= a^2\left[\frac{\pi}{2} - \sin^{-1}\left(\frac{h}{a}\right)\right] - h\sqrt{a^2-h^2}$$

$$= a^2\cos^{-1}\left(\frac{h}{a}\right) - h\sqrt{a^2-h^2}.$$

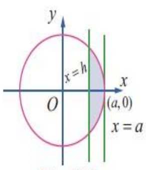

Fig. 9.29

Example 10.4

Find the differential equation of the family of circles passing through the points $(a,0)$ and $(-a,0)$.

Solution

A circle passing through the points $(a,0)$ and $(-a,0)$ has its centre on y-axis.

Let $(0,b)$ be the centre of the circle. So, the radius of the circle is $\sqrt{a^2+b^2}$.

Therefore the equation of the family of circles passing through the points $(a,0)$ and $(-a,0)$ is $x^2 + (y-b)^2 = a^2 + b^2$, b is an arbitrary constant. ... (1)

Differentiating both sides of (1) with respect to x, we get

$$2x + 2(y-b)\frac{dy}{dx} = 0 \Rightarrow y - b = -\frac{x}{\frac{dy}{dx}} \Rightarrow b = \frac{x}{\frac{dy}{dx}} + y.$$

Substituting the value of b in equation (1), we get

$$x^2 + \frac{x^2}{\left(\frac{dy}{dx}\right)^2} = a^2 + \left(\frac{x}{\frac{dy}{dx}} + y\right)^2$$

$$\Rightarrow x^2\left(\frac{dy}{dx}\right)^2 + x^2 = a^2\left(\frac{dy}{dx}\right)^2 + \left[x + y\left(\frac{dy}{dx}\right)\right]^2$$

$$\Rightarrow \left(x^2 - y^2 - a^2\right)\frac{dy}{dx} - 2xy = 0,$$ which is the required differential equation.

Example 10.10

Show that $y = a\cos(\log x) + b\sin(\log x), x > 0$ is a solution of the differential equation $x^2y'' + xy' + y = 0$.

Solution

The given function is $y = a\cos(\log x) + b\sin(\log x)$ ...(1)

where a, b are two arbitrary constants. In order to eliminate the two arbitrary constants, we have to differentiate the given function two times successively.

Differentiating equation (1) with respect to x, we get

$$y' = -a\sin(\log x)\cdot\frac{1}{x} + b\cos(\log x)\cdot\frac{1}{x} \Rightarrow xy' = -a\sin(\log x) + b\cos(\log x).$$

Again differentiating this with respect to x, we get

$$xy'' + y' = -a\cos(\log x)\cdot\frac{1}{x} - b\sin(\log x)\cdot\frac{1}{x} \Rightarrow x^2y'' + xy' + y = 0.$$

Therefore, $y = a\cos(\log x) + b\sin(\log x)$ is a solution of the given differential equation.

Example 10.12

Find the particular solution of $(1+x^3)dy - x^2y\,dx = 0$ satisfying the condition $y(1) = 2$.

Solution

Given that $(1+x^3)dy - x^2y\,dx = 0$.

The above equation is written as $\frac{dy}{y} - \frac{x^2}{1+x^3}dx = 0$.

Integrating both sides gives $\log y - \frac{1}{3}\log(1+x^3) = C_1$, which implies,

$$3\log y - \log(1+x^3) = \log C.$$

Thus, $3\log y = \log(1+x^3) + \log C$,

which reduces to $\log y^3 = \log C(1+x^3)$.

Hence, $y^3 = C(1+x^3)$ gives the general solution of the given differential equation. It is given that when $x = 1, y = 2$. Then $2^3 = C(1+1) \Rightarrow C = 4$ and hence the particular solution is $y^3 = 4(1+x^3)$.

Example 10.17

Solve $\left(x^2 - 3y^2\right)dx + 2xy\,dy = 0$.

Solution

We know that the given equation is homogeneous.

Now, we rewrite the given equation as $\frac{dy}{dx} = \frac{3y}{2x} - \frac{x}{2y}$.

Taking $y = vx$, we have $v + x\frac{dv}{dx} = \frac{3v}{2} - \frac{1}{2v}$ or $x\frac{dv}{dx} = \frac{v^2-1}{2v}$.

Separating the variables, we obtain $\frac{2v\,dv}{v^2-1} = \frac{dx}{x}$.

On integration, we get $\log\left|v^2-1\right| = \log|x| + \log|C|$.

Hence $\left|v^2-1\right| = |Cx|$, where C is an arbitrary constant.

Now, replace v by $\frac{y}{x}$ to get $\left|\frac{y^2}{x^2} - 1\right| = |Cx|$.

Thus, we have $\left|y^2 - x^2\right| = \left|Cx^3\right|$.

Hence, $y^2 - x^2 = \pm Cx^3$ (or) $y^2 - x^2 = kx^3$ gives the general solution.

Example 10.21

Solve $\left(1+2e^{x/y}\right)dx+2e^{x/y}\left(1-\frac{x}{y}\right)dy=0$.

Solution

The given equation can be written as $\frac{dx}{dy}=\frac{\left(\frac{x}{y}-1\right)2e^{x/y}}{1+2e^{x/y}}=g\left(\frac{x}{y}\right)$. ...(1)

The appearance of $\frac{x}{y}$ in equation (1), suggests that the appropriate substitution is $x=vy$.

Put $x=vy$. Then, we have $y\frac{dv}{dy}=-\frac{2e^v+v}{1+2e^v}$.

By separating the variables, we have $\frac{1+2e^v}{v+2e^v}dv=-\frac{dy}{y}$.

On integration, we obtain

$\log\left|2e^v+v\right|=-\log|y|+\log|C|$ or $\log\left|2ye^v+vy\right|=\log|C|$ or $2ye^v+vy=\pm C$.

Replace v by $\frac{x}{y}$ to get, $2ye^{x/y}+x=k$, where $k=\pm C$, which gives the required solution.

Example 10.24

Solve: $\frac{dy}{dx}+2y\cot x=3x^2\operatorname{cosec}^2 x$.

Solution

Given that the equation is $\frac{dy}{dx}+2y\cot x=3x^2\operatorname{cosec}^2 x$.

This is a linear differential equation. Here, $P=2\cot x$; $Q=3x^2\operatorname{cosec}^2 x$.

$$\int Pdx=\int 2\cot x\,dx=2\log|\sin x|=\log|\sin x|^2=\log\sin^2 x.$$

Thus, $\text{I.F}=e^{\int Pdx}=e^{\log\sin^2 x}=\sin^2 x$.

Hence, the solution is $ye^{\int Pdx}=\int Qe^{\int Pdx}dx+C$.

That is, $y\sin^2 x=\int 3x^2\operatorname{cosec}^2 x\cdot\sin^2 x\,dx+C=\int 3x^2dx+C=x^3+C$.

Hence, $y\sin^2 x=x^3+C$ is the required solution.

Example 10.26

Solve $ye^y dx=\left(y^3+2xe^y\right)dy$.

Solution

The given equation can be written as $\frac{dx}{dy}-\frac{2}{y}x=y^2e^{-y}$.

This is a linear differential equation. Here $P=-\frac{2}{y}$; $Q=y^2e^{-y}$.

$$\int pdy=\int-\frac{2}{y}dy=-2\log|y|=\log|y|^{-2}=\log\left(\frac{1}{y^2}\right).$$

Thus, I.F. $=e^{\int Pdy}=e^{\log\left(\frac{1}{y^2}\right)}=\frac{1}{y^2}$.

Hence the solution is $xe^{\int Pdy}=\int Qe^{\int Pdy}dy+C$

That is, $x\left(\frac{1}{y^2}\right)=\int y^2e^{-y}\left(\frac{1}{y^2}\right)dy+C=\int e^{-y}dy+C=-e^{-y}+C$

or $x=-y^2e^{-y}+Cy^2$ is the required solution.

5. A commuter train arrives punctually at a station every half hour. Each morning, a student leaves his house to the train station. Let X denote the amount of time, in minutes, that the student waits for the train from the time he reaches the train station. It is known that the pdf of X is

$$f(x)=\begin{cases}\frac{1}{30} & 0<x<30\\ 0 & \text{elsewhere}\end{cases}$$

Obtain and interpret the expected value of the random variable X.

Example 12.19

Using the equivalence property, show that $p\leftrightarrow q\equiv(p\wedge q)\vee(\neg p\wedge\neg q)$.

Solution

It can be obtained by using examples 12.15 and 12.16 that

$p\leftrightarrow q\equiv(\neg p\vee q)\wedge(\neg q\vee p)$ ... (1)

$\equiv(\neg p\vee q)\wedge(p\vee\neg q)$ (by Commutative Law) ... (2)

$\equiv(\neg p\wedge(p\vee\neg q))\vee(q\wedge(p\vee\neg q))$ (by Distributive Law)

$\equiv(\neg p\wedge p)\vee(\neg p\wedge\neg q)\vee(q\wedge p)\vee(q\wedge\neg q)$ (by Distributive Law)

$\equiv \mathbb{F}\vee(\neg p\wedge\neg q)\vee(q\wedge p)\vee\mathbb{F}$; (by Complement Law)

$\equiv(\neg p\wedge\neg q)\vee(q\wedge p)$; (by Identity Law)

$\equiv(p\wedge q)\vee(\neg p\wedge\neg q)$; (by Commutative Law)

Finally (1) becomes $p\leftrightarrow q\equiv(p\wedge q)\vee(\neg p\wedge\neg q)$.

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