12th STD

PUBLIC EXAMINATION - MARCH 2024

Reg. No.

Part - III

Mathematics (with answers)

TIME ALLOWED : 3.00 Hours] [MAXIMUM MARKS : 90

Instructions :

(1) Check the question paper for fairness of printing. If there is any lack of fairness, inform the Hall Supervisor immediately.

(2) Use **Blue** or **Black** ink to write and underline and pencil to draw diagrams

PART - I

Note : (i) **All** questions are **Compulsory.**

(ii) Choose the most appropriate answer from the given **four** alternatives and write the option code and the corresponding answer. **20 × 1 = 20**

1. The area between $y^2 = 4x$ and its latus rectum is :

(a) $\frac{8}{3}$ (b) $\frac{2}{3}$ (c) $\frac{5}{3}$ (d) $\frac{4}{3}$

2. The value of $\int_0^a \left(\sqrt{a^2-x^2}\right)^3 dx$ is :

(a) $\frac{3\pi a^2}{8}$ (b) $\frac{\pi a^3}{16}$

(c) $\frac{3\pi a^4}{8}$ (d) $\frac{3\pi a^4}{16}$

3. If P(x, y) be any point on $16x^2 + 25y^2 = 400$ with foci F_1 (3, 0) and F_2(–3, 0), then $PF_1 + PF_2$ is :

(a) 10 (b) 8 (c) 12 (d) 6

4. If $|z_1| = 1$, $|z_2| = 2$, $|z_3| = 3$ and $|9z_1z_2 + 4z_1z_3 + z_2z_3| = 12$ then the value of $|z_1 + z_2 + z_3|$ is :

(a) 3 (b) 1 (c) 4 (d) 2

5. The number of rows in the truth table of $(p \vee q) \wedge (p \vee r)$ is :

(a) 6 (b) 9 (c) 3 (d) 8

6. If a vector $\vec{\alpha}$ lies in the plane of $\vec{\beta}$ and $\vec{\gamma}$, then :

(a) $[\vec{\alpha}, \vec{\beta}, \vec{\gamma}] = 0$ (b) $[\vec{\alpha}, \vec{\beta}, \vec{\gamma}] = 1$

(c) $[\vec{\alpha}, \vec{\beta}, \vec{\gamma}] = 2$ (d) $[\vec{\alpha}, \vec{\beta}, \vec{\gamma}] = -1$

7. The differential equation of the family of curves $y = Ae^x + Be^{-x}$, where A and B are arbitrary constants is :

(a) $\frac{dy}{dx} + y = 0$ (b) $\frac{d^2y}{dx^2} + y = 0$

(c) $\frac{dy}{dx} - y = 0$ (d) $\frac{d^2y}{dx^2} - y = 0$

8. The horizontal asymptote of $f(x) = \frac{1}{x}$ is :

(a) $x = c$ (b) $y = 0$

(c) $y = c$ (d) $x = 0$

9. If $f(x)=\dfrac{x}{x+1}$, then its differential is given by :

(a) $\dfrac{1}{x+1}dx$ (b) $\dfrac{-1}{(x+1)^2}dx$

(c) $\dfrac{-1}{x+1}dx$ (d) $\dfrac{1}{(x+1)^2}dx$

10. If $(1+i)(1+2i)(1+3i)\ldots.(1+ni)=x+iy$ then $2.5.10\ldots.(1+n^2)$ is :

(a) x^2+y^2 (b) 1

(c) $1+n^2$ (d) i

11. The number given by the Rolle's theorem for the function $x^3-3x^2, x\in[0, 3]$ is :

(a) $\dfrac{3}{2}$ (b) 1 (c) 2 (d) $\sqrt{2}$

12. The type of conic section for $x^2-3=5x+3y$ is :

(a) hyperbola (b) ellipse

(c) circle (d) parabola

13. If A is a non-singular matrix such that $A^{-1}=\begin{bmatrix}5 & 3\\ -2 & -1\end{bmatrix}$, then $(A^T)^{-1}=$

(a) $\begin{bmatrix}-1 & -3\\ 2 & 5\end{bmatrix}$ (b) $\begin{bmatrix}-5 & 3\\ 2 & 1\end{bmatrix}$

(c) $\begin{bmatrix}5 & -2\\ 3 & -1\end{bmatrix}$ (d) $\begin{bmatrix}5 & 3\\ -2 & -1\end{bmatrix}$

14. The angle between the line $\vec{r}=(\vec{i}+2\vec{j}-3\vec{k})+t(2\vec{i}+\vec{j}-2\vec{k})$ and the plane $\vec{r}.(\vec{i}+\vec{j})+4=0$ is:

(a) 45° (b) 0° (c) 90° (d) 30°

15. If $\sin^{-1}x+\cot^{-1}\left(\dfrac{1}{2}\right)=\dfrac{\pi}{2}$, then x is equal to :

(a) $\dfrac{2}{\sqrt{5}}$ (b) $\dfrac{1}{2}$

(c) $\dfrac{\sqrt{3}}{2}$ (d) $\dfrac{1}{\sqrt{5}}$

16. If α, β and γ are zeros of x^3+px^2+qx+r then $\sum\dfrac{1}{\alpha}$ is :

(a) $\dfrac{q}{r}$ (b) $-\dfrac{q}{r}$

(c) $-\dfrac{q}{p}$ (d) $-\dfrac{p}{r}$

17. The value of Var(3) is :

(a) 0 (b) 3

(c) Var(3) (d) 9

18. The random variable X has binomial distribution with $n = 25$ and $p = 0.8$ then standard deviation of X is :

(a) 3 (b) 6 (c) 2 (d) 4

19. If A, B and C are invertible matrices of some order, then which one of the following is not true?

(a) $\det A^{-1}=(\det A)^{-1}$

(b) $\text{adj } A=|A|A^{-1}$

(c) $(ABC)^{-1}=C^{-1}B^{-1}A^{-1}$

(d) $\text{adj }(AB)=(\text{adj } A)(\text{adj } B)$

20. A zero of x^3+64 is :

(a) $4i$ (b) 0 (c) –4 (d) 4

PART - II

Note : Answer **any seven** questions. Question No. **30** is **Compulsory.** **7 × 2 = 14**

21. Simplify : $\sum_{n=1}^{12} i^n$

22. If α and β are the roots of the quadratic equation $2x^2 - 7x + 13 = 0$, construct a quadratic equation whose roots are α^2 and β^2.

23. Find df for $f(x) = x^2 + 3x$ and evaluate it for $x = 3$ and $dx = 0.02$.

24. Find the differential equation for the family of all straight lines passing through the origin.

25. For the random variable X with the given probability mass function.

$$f(x) = \begin{cases} 2(x-1) & 1 < x < 2 \\ 0 & \text{otherwise} \end{cases}$$

Find the Mean.

26. Find the general equation of a circle with centre (–3, –4) and radius 3 units.

27. Find the rank of the matrix $\begin{bmatrix} -1 & 3 \\ 4 & -7 \\ 3 & -4 \end{bmatrix}$.

28. Evaluate : $\int_0^{\frac{\pi}{2}} \sin^{10} x \, dx$

29. Evaluate : $\lim_{x \to 1} \frac{x^2 - 3x + 2}{x^2 - 4x + 3}$

30. Show that the vectors $2\vec{i} - \vec{j} + 3\vec{k}$, $\vec{i} - \vec{j}$ and $3\vec{i} - \vec{j} + 6\vec{k}$ are coplanar.

PART - III

Note : Answer **any seven** questions. Question No. **40** is **Compulsory.** **7 × 3 = 21**

31. Show that $\cot^{-1}\left(\frac{1}{\sqrt{x^2-1}}\right) = \sec^{-1} x, |x| > 1.$

32. Find the equation of tangent and normal to the parabola $x^2 + 6x + 4y + 5 = 0$ at (1, –3).

33. Prove that $\left[\vec{a} - \vec{b}, \vec{b} - \vec{c}, \vec{c} - \vec{a}\right] = 0.$

34. If $u(x, y) = \frac{x^2 + y^2}{\sqrt{x+y}}$, prove that $x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} = \frac{3}{2}u.$

35. Find two positive numbers whose sum is 12 and their product is maximum.

36. Evaluate : $\int_{\frac{\pi}{8}}^{\frac{3\pi}{8}} \frac{1}{1 + \sqrt{\tan x}} dx$

37. Simplify $\left(\frac{1+i}{1-i}\right)^3 - \left(\frac{1-i}{1+i}\right)^3$ into rectangular form.

38. Solve : $(1 + x^2)\frac{dy}{dx} = 1 + y^2$

39. Three fair coins are tossed simultaneously. Find the probability mass function for number of heads occurred.

40. If $A = \begin{bmatrix} 2 & -1 & 3 \\ -5 & 3 & 1 \\ -3 & 2 & 3 \end{bmatrix}$, then find |adj (adj A)|.

PART - IV

Note : Answer **all** the questions. **7 × 5 = 35**

41. (a) Find the angle between the curves $y = x^2$ and $y = (x-3)^2$.

(OR)

(b) Solve :

$$\tan^{-1}\left(\frac{x-1}{x-2}\right) + \tan^{-1}\left(\frac{x+1}{x+2}\right) = \frac{\pi}{4}.$$

42. (a) A six sided die is marked '1' on one face, '3' on two of its faces, and '5' on remaining three faces. The die is thrown twice. If X denotes the total score in two throws, find :

(i) the probability mass function

(ii) the cumulative distribution function

(iii) $P(4 \le X < 10)$

(OR)

(b) If $z = x + iy$ is a complex number such that $\text{Im}\left(\frac{2z+1}{iz+1}\right) = 0$, show that the locus of z is $2x^2 + 2y^2 + x - 2y = 0$.

43. (a) A conical water tank with vertex down of 12 meters height has a radius of 5 meters at the top. If water flows into the tank at a rate 10 cubic m/min, how fast is the depth of the water increases when the water is 8 metres deep?

(OR)

(b) Prove by vector method that $\sin(\alpha - \beta) = \sin\alpha\cos\beta - \cos\alpha\sin\beta$.

44. (a) Assume that water issuing from the end of horizontal pipe, 7.5m above the ground, describes a parabolic path. The vertex of the parabolic path is at the end of the pipe. At a position 2.5m below the line of the pipe, the flow of water has curved outward 3m beyond the vertical line through the end of the pipe. How far beyond this vertical line will the water strike the ground?

(OR)

(b) Solve the Linear differential equation $\frac{dy}{dx} + \frac{y}{x} = \sin x$.

45. (a) The rate of increase in the number of bacteria in a certain bacteria culture is proportional to the number present. Given that the number triples in 5 hours, find how many bacteria will be present after 10 hours?

(OR)

(b) Find the vector and Cartesian equations of the plane containing $\frac{x-2}{2} = \frac{y-2}{3} = \frac{z-1}{3}$ and parallel to the line $\frac{x+1}{3} = \frac{y-1}{2} = \frac{z+1}{1}$

46. (a) Find the area of the region bounded by the ellipse.

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$$

(OR)

(b) Find the vertex, focus and equation of the directrix of the parabola $y^2 - 4y - 8x + 12 = 0$.

47. (a) Show that $p \leftrightarrow q \equiv ((\sim p) \vee q) \wedge ((\sim q) \vee p)$

(OR)

(b) Solve the system of linear equations by Cramer's Rule.

$$\frac{3}{x}-\frac{4}{y}-\frac{2}{z}-1=0,$$

$$\frac{1}{x}+\frac{2}{y}+\frac{1}{z}-2=0,$$

$$\frac{2}{x}-\frac{5}{y}-\frac{4}{z}+1=0$$

❁❁❁❁❁

Answers

PART - I

1. (a) $\frac{8}{3}$
2. (d) $\frac{3\pi a^4}{16}$
3. (a) 10
4. (d) 2
5. (d) 8
6. (a) $[\vec{\alpha},\vec{\beta},\vec{\gamma}]=0$
7. (d) $\frac{d^2y}{dx^2}-y=0$
8. (b) $y=0$
9. (d) $\frac{1}{(x+1)^2}dx$
10. (a) x^2+y^2
11. (c) 2
12. (d) parabola
13. (c) $\begin{bmatrix}5 & -2\\ 3 & -1\end{bmatrix}$
14. (a) 45^0
15. (d) $\frac{1}{\sqrt{5}}$
16. (b) $-\frac{q}{r}$
17. (a) 0
18. (c) 2
19. (d) adj (AB) = (adj A) (adj B)
20. (c) –4

PART - II

21. $$\sum_{n=1}^{12} i^n = (i^1+i^2+i^3+i^4)+(i^5+i^6+i^7+i^8)+(i^9+i^{10}+i^{11}+i^{12})$$

$$=(i-1-i+1)+(i^{4+1}+i^{4+2}+i^{4+3}+(i^4)^2)+(i^{8+1}+i^{8+2}+i^{8+3}+(i^4)^3)$$

$$=0+(i+i^2+i^3+i^4)+(i^1+i^2+i^3+i^4)$$

$$[\because i^2=-1,\ i^3=-i,\ i^4=1]$$

$$=0+(i-1-i+1)+(i-1-i+1)$$

$$=0+0+0=0$$

22. Since α and β are the roots of the quadratic equation, we have $\alpha+\beta=\frac{7}{2}$ and $\alpha,\beta=\frac{13}{2}$
Thus, to construct a new quadratic equation,

Sum of the roots

$$=\alpha^2+\beta^2=(\alpha+\beta)^2-2\alpha\beta=\frac{-3}{4}$$

Product of the roots

$$=\alpha^2+\beta^2=(\alpha\beta)^2=\frac{169}{4}$$

Thus a required quadratic equation is

$$x^2 + \frac{3}{4}x + \frac{169}{4} = 0$$

$4x^2 + 3x + 169 = 0$ is a quadratic equation with roots α^2 and β^2.

23. $x = 3$ and $dx = 0.02$

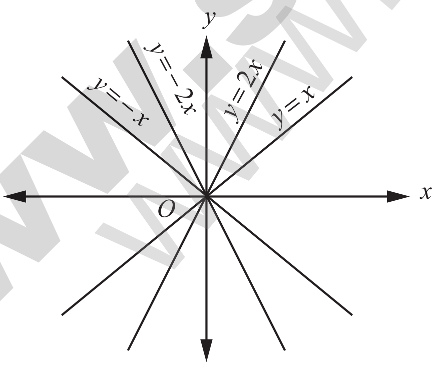
When $x = 3$ and $dx = 0.02$,

$$\begin{aligned} df &= (2x+3)dx \\ &= [2(3)+3]\,(0.02) \\ df &= (6+3)\,(0.02) \\ &= 9(0.02) = 0.18 \end{aligned}$$

24. The family of straight lines passing through the origin is $y = mx$, where m is an arbitrary constant. ... (1)

Differentiating both sides with respect to x, we get $\frac{dy}{dx} = m.$ ... (2)

From (1) and (2), we get $y = x\frac{dy}{dx}$. This is the required differential equation.



Observe that the given equation $y = mx$ contains only one arbitrary constant and thus we get the differential equation of order one.

25. $f(x) = \begin{cases} 2(x-1), & 1 < x < 2 \\ 0, & \text{otherwise} \end{cases}$

$$\text{Mean} = E(X) = \int_1^2 f(x)dx = \int_1^2 2(x-1)dx$$

$$= \left[\frac{2x^2}{3} - x^2\right]_1^2$$

$$= \frac{16}{3} - 4 - \frac{2}{3} + 1 = \frac{5}{3}$$

26. Here, $c\,(h, k) = c\,(-3, -4)$ and $r = 3$

Equation of the circle is

$$\begin{aligned} (x-h)^2 + (y-k)^2 &= r^2 \\ \Rightarrow (x-(-3))^2 + (y-(-4))^2 &= 3^2 \\ \Rightarrow (x+3)^2 + (y+4)^2 &= 9 \\ \Rightarrow x^2 + y^2 + 6x + 8y + 16 &= 0 \end{aligned}$$

27. Let $A = \begin{bmatrix} -1 & 3 \\ 4 & -7 \\ 3 & -4 \end{bmatrix}$

A is a matrix of order 3×2

$\therefore \rho(A) \le = 2$

We find that there is a second order minor,

$$\begin{vmatrix} -1 & 3 \\ 4 & -7 \end{vmatrix} = 7 - 12 = -5 \neq 0$$

$\therefore \rho(A) = 2.$

28. $I_n = \int_0^{\pi/2} \sin^n x = \frac{n-1}{n} I_{n-2},\ n \ge 2$

$$\text{Let } I_{10} = \int_0^{\pi/2} \sin^{10}x\,dx = \frac{9}{10} I_8 = \frac{9}{10} \times \frac{7}{8} \times I_6$$

$$= \frac{9}{10}\times\frac{7}{8}\times\frac{5}{6}\times I_4 = \frac{9}{10}\times\frac{7}{8}\times\frac{5}{6}\times\frac{3}{4}I_2$$

$$= \frac{9}{10}\times\frac{7}{8}\times\frac{5}{6}\times\frac{3}{4}\times\frac{1}{2}\times\frac{\pi}{2}$$

$$= \frac{315}{1280}\times\frac{\pi}{2} = \frac{63\pi}{256(2)} = \frac{63\pi}{512}$$

29. $\lim\limits_{x\to1}\left(\frac{x^2-3x+2}{x^2-4x+3}\right)$ $\left(\frac{0}{0}\text{ form}\right)$

Applying I' Hôpital Rule, we get,

$$= \lim_{x\to1}\left(\frac{2x-3}{2x-4}\right) \qquad \left[\because \frac{x^2-3x+2}{x^2-4x+3} = \frac{(x-1)(x-2)}{(x-1)(x-3)}\right]$$

$$= \frac{2(1)-3}{2(1)-4} = \frac{-1}{-2} = \frac{1}{2}$$

30. Let $\vec{a} = 2\vec{i}-\vec{j}+3\vec{k}$; $\vec{b} = \vec{i}-\vec{j}$, $\vec{c} = 3\vec{i}-\vec{j}+6\vec{k}$

We know that if $\vec{a}, \vec{b}, \vec{c}$ are coplanar

$$= \left[\vec{a}, \vec{b}, \vec{c}\right] = 0$$

$$\left[\vec{a}, \vec{b}, \vec{c}\right] = \begin{vmatrix} \overset{+}{2} & \overset{-}{-1} & \overset{+}{3} \\ 1 & -1 & 0 \\ 3 & -1 & 6 \end{vmatrix}$$

$$= 2(-6-0) + 1(6) + 3(-1+3)$$

$$= -12 + 6 + 6 = -12 + 12 = 0$$

Therefore, the three given vectors are coplanar.

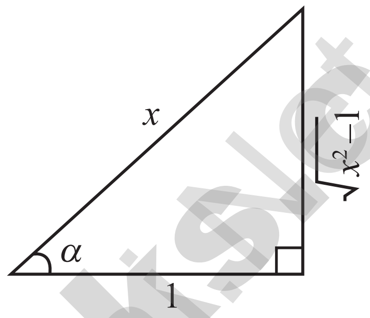
PART - III

31. Let $\cot^{-1}\left(\frac{1}{\sqrt{x^2-1}}\right) = \alpha$.

Then, $\cot\alpha = \frac{1}{\sqrt{x^2-1}}$ and α is acute.

We construct a right triangle with the given data. From the triangle, $\sec\alpha = \frac{x}{1} = x$. Thus, $\alpha = \sec^{-1}x$.

Hence, $\cot^{-1}\left(\frac{1}{\sqrt{x^2-1}}\right) = \sec^{-1}x$, $|x|>1$.



32. Equation of parabola is $x^2 + 6x + 4y + 5 = 0$.

$$x^2 + 6x + 9 - 9 + 4^y + 5 = 0.$$

$$(x+3)^2 = -4(y-1) \quad ...(1)$$

Let $X = x+3$, $Y = y-1$

Equation (1) takes the standard form

$$X^2 = -4Y$$

Equation of tangent is $XX_1 = -2(Y+Y_1)$

At $(1, -3)$ $X_1 = 1+3 = 4$; $Y_1 = -3-1 = -4$

Therefore, the equation of tangent at $(1, -3)$ is

$$(x+3)4 = -2(y-1-4)$$

$$2x+6 = -y+5$$

$$2x+y+1 = 0$$

Slope of tangent at $(1, -3)$ is -2, so slope of normal at $(1, -3)$ is $\frac{1}{2}$

Therefore, the equation of normal at (1, –3) is given by

$$y+3 = \frac{1}{2}(x-1)$$
$$2y+6 = x-1$$
$$x-2y-7 = 0.$$

33. LHS $= [\vec{a}-\vec{b},\ \vec{b}-\vec{c},\ \vec{c}-\vec{a}]$

[∵ cross product is distributive]

$$= (\vec{a}-\vec{b})\cdot[(\vec{b}-\vec{c})\times(\vec{c}-\vec{a})]$$
$$= (\vec{a}-\vec{b})\cdot[\vec{b}\times\vec{c}-\vec{b}\times\vec{a}-\vec{c}\times\vec{c}+\vec{c}\times\vec{a}]$$
$$= (\vec{a}-\vec{b})\cdot[\vec{b}\times\vec{c}-\vec{b}\times\vec{a}-\vec{0}+\vec{c}\times\vec{a}]$$

$[\because \vec{c}\times\vec{c} = \vec{0}]$

$$= [\vec{a}\ \vec{b}\ \vec{c}]-[\vec{a}\ \vec{b}\ \vec{a}]+[\vec{a}\ \vec{c}\ \vec{a}]-[\vec{b}\ \vec{b}\ \vec{c}] +[\vec{b}\ \vec{b}\ \vec{a}]-[\vec{b}\ \vec{c}\ \vec{a}]$$
$$= [\vec{a}\ \vec{b}\ \vec{c}]-0+0-0+0-[\vec{b}\ \vec{c}\ \vec{a}]$$

$[\because [\vec{a}\ \vec{b}\ \vec{a}] = [\vec{b}\ \vec{b}\ \vec{c}] = 0]$

$$= [\vec{a}\ \vec{b}\ \vec{c}]-[\vec{a}\ \vec{b}\ \vec{c}] = 0 = \text{RHS}.$$

34. Given $u(x, y) = \dfrac{x^2+y^2}{\sqrt{x+y}}$

$$u(\lambda x, \lambda y) = \frac{\lambda^2x^2+\lambda^2y^2}{\sqrt{\lambda x+\lambda y}}$$
$$= \frac{\lambda^2(x^2+y^2)}{\sqrt{\lambda}\left(\sqrt{x+y}\right)} = \lambda^{2-\frac{1}{2}}\,u(x, y)$$
$$= \lambda^{\frac{3}{2}}\,u(x, y)$$

∴ $u(x, y)$ is a homogeneous function of degree $\frac{3}{2}$.

∴ By Euler's theorem,

$$x\frac{\partial u}{\partial x}+y\frac{\partial u}{\partial y} = n.u \Rightarrow x\frac{\partial u}{\partial x}+y\frac{\partial u}{\partial y} = \frac{3}{2}u$$

Hence, proved.

35. Let x, y be the two numbers then the sum

$x+y = 12$ gives $\Rightarrow y = 12-x$

Product of the numbers $P = xy = x(12-x)$

$$p = 12x-x^2$$
$$p' = 12-2x$$
$$p'' = -2$$

Substituting $p' = 0$, we get

$$12-2x = 0$$
$$2x = 12 \text{ gives}$$
$$x = 6$$

Since $p'' = -2 < 0$, Product P is maximum at $x = 6$. Then $y = 12-x$, gives $y = 12-6 = 6$.

The required two numbers are 6, 6.

36. $\displaystyle\int_{\frac{\pi}{8}}^{\frac{3\pi}{8}} \frac{1}{1+\sqrt{\tan x}}\,dx$

$$I = \int_{\frac{\pi}{8}}^{\frac{3\pi}{8}} \frac{1}{1+\sqrt{\tan x}}\,dx = \int_{\frac{\pi}{8}}^{\frac{3\pi}{8}} \frac{1}{1+\frac{\sqrt{\sin x}}{\sqrt{\cos x}}}\,dx$$
$$= \int_{\frac{\pi}{8}}^{\frac{3\pi}{8}} \frac{\sqrt{\cos x}}{\sqrt{\cos x}+\sqrt{\sin x}}\,dx \quad ...(1)$$

By the property, $\displaystyle\int_a^b f(x)dx = \int_a^b f(a+b-x)dx$

we get

$$I = \int_{\frac{\pi}{8}}^{\frac{3\pi}{8}} \frac{\sqrt{\cos(\frac{\pi}{8}+\frac{3\pi}{8}-x)}}{\sqrt{\cos(\frac{\pi}{8}+\frac{3\pi}{8}-x)}+\sqrt{\sin(\frac{\pi}{8}+\frac{3\pi}{8}-x)}}\,dx$$

$$= \int_{\frac{\pi}{8}}^{\frac{3\pi}{8}} \frac{\sqrt{\cos(\frac{\pi}{2}-x)}}{\sqrt{\cos(\frac{\pi}{2}-x)}+\sqrt{\sin(\frac{\pi}{2}-x)}}dx$$

$$= \int_{\frac{\pi}{8}}^{\frac{3\pi}{8}} \frac{\sqrt{\sin x}}{\sqrt{\sin x}+\sqrt{\cos x}}dx \qquad ...(2)$$

(1) + (2) →

$$2I = \int_{\frac{\pi}{8}}^{\frac{3\pi}{8}} \frac{\sqrt{\cos x}}{\sqrt{\cos x}+\sqrt{\sin x}}dx + \int_{\frac{\pi}{8}}^{\frac{3\pi}{8}} \frac{\sqrt{\sin x}}{\sqrt{\sin x}+\sqrt{\cos x}}dx$$

$$= \int_{\frac{\pi}{8}}^{\frac{3\pi}{8}} \frac{\sqrt{\cos x}+\sqrt{\sin x}}{\sqrt{\cos x}+\sqrt{\sin x}}dx = \int_{\frac{\pi}{8}}^{\frac{3\pi}{8}} dx = [x]_{\frac{\pi}{8}}^{\frac{3\pi}{8}}$$

$$2I = \frac{3\pi}{8}-\frac{\pi}{8}=\frac{2\pi}{8}=\frac{\pi}{4} \Rightarrow 2I = \frac{\pi}{4} \therefore I = \frac{\pi}{8}$$

37. We consider

$$\frac{1+i}{1-i}=\frac{(1+i)(1+i)}{(1-i)(1+i)}=\frac{1+2i-1}{1+1}=\frac{2i}{2}=i,$$

and $\frac{1-i}{1+i}=\left(\frac{1+i}{1-i}\right)^{-1}=\frac{1}{i}=-i.$

Therefore,

$$\left(\frac{1+i}{1-i}\right)^{-3}-\left(\frac{1-i}{1-i}\right)^{3}=i^3-(-i)^3=-i-i=-2i.$$

38. Given that $(1+x^2)\frac{dy}{dx}=1+y^2.$ ...(1)

The given equation is written in the variables separable form

$$\frac{dy}{1+y^2}=\frac{dx}{1+x^2}. \qquad ...(2)$$

Integrating both sides of (2), we get

$$\tan^{-1} y = \tan^{-1} x + C. \qquad ...(3)$$

But $\tan^{-1} y - \tan^{-1} x = \tan^{-1}\left(\frac{y-x}{1+xy}\right).$ ...(4)

Using (4) in (3) leads to $\left(\frac{y-x}{1+xy}\right). = C,$

which implies $\frac{y-x}{1+xy}$ = tan C = a (say).

Thus, $y - x = a\,(1 + xy)$ gives the required solution.

39. Sample space s = {HHH, HHT, HTH, THH, HTT, THT, TTH, TTT}

Let x denote the number of heads occurred.

$p(x=0)$

p(no head) $= p(\text{TTT}) = \frac{1}{2}\times\frac{1}{2}\times\frac{1}{2} = \frac{1}{8}$

$p(x=1) = p(1\text{ head}) = p(\text{THT, HTT, TTH})$

$=\frac{1}{8}+\frac{1}{8}+\frac{1}{8}=\frac{3}{8} \quad \because p(\text{H}) = \frac{1}{2}$ and $p(\text{T}) = \frac{1}{2}$

$p(x=2) = p(2\text{ heads}) = p(\text{HHT, HTH, THH})$

$=\frac{1}{8}+\frac{1}{8}+\frac{1}{8}=\frac{3}{8}$

$p(x=3) = p(3\text{ heads}) = p(\text{HHH}) = \frac{1}{8}$

∴ X take the value 0,1, 2, 3.

Probability mass function is

x	0	1	2	3
$f(x)$	$\frac{1}{8}$	$\frac{3}{8}$	$\frac{3}{8}$	$\frac{1}{8}$

$$\therefore f(x) = \begin{cases} \frac{1}{8} & \text{for} \quad x = 0,3 \\ \frac{3}{8} & \text{for} \quad x = 1,2 \end{cases}$$

40. Given adj A $= \begin{bmatrix} 2 & -1 & 3 \\ -5 & 3 & 1 \\ -3 & 2 & 3 \end{bmatrix}$.

We know that $|\text{adj (adj A)}| = |A|^{(n-1)^2}$

Here $n = 3$, $|\text{adj (adj)}| = |A|^{(3-1)^2} = |A|^4$

$$|A| = \begin{vmatrix} 2 & -1 & 3 \\ -5 & 3 & 1 \\ -3 & 2 & 3 \end{vmatrix}$$

$= 2(9-2) + 1(-15+3) + 3(-10+9)$

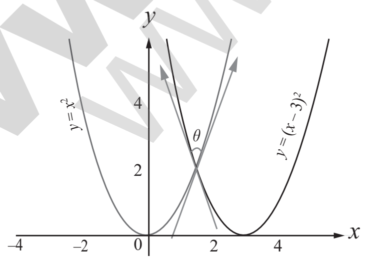
$= 2(7) + 1(-12) + 3(-1) = 14 - 12 - 3$

$= 14 - 15 = -1$

$|\text{adj (adj A)}| = -1$

PART - IV

41. (a) Let us now find the point of intersection of the two given curves. Equating $x^2 = (x-3)^2$ we get, $x = \frac{3}{2}$. Therefore, the point of intersection is $\left(\frac{3}{2}, \frac{9}{4}\right)$. Let θ be the angle between the curves. The slopes of the curves are as follows :



For the curve $y = x^2$, $\frac{dy}{dx} = 2x$.

Let $m_1 = \frac{dy}{dx}$ at $\left(\frac{3}{2}, \frac{9}{4}\right) = 3$.

For the curve $y = (x-3)^2$, $\frac{dy}{dx} = 2(x-3)$.

Let $m_2 = \frac{dy}{dx}$ at $\left(\frac{3}{2}, \frac{9}{4}\right) = -3$.

Using (3), we get $\tan\theta = \left|\frac{3-(-3)}{1-9}\right| = \frac{3}{4}$

Hence, $\theta = \tan^{-1}\left(\frac{3}{4}\right)$.

(OR)

(b) Now, $\tan^{-1}\left(\frac{x-1}{x-2}\right) + \tan^{-1}\left(\frac{x+1}{x+2}\right)$

$$= \tan^{-1}\left[\frac{\frac{x-1}{x-2} + \frac{x+1}{x+2}}{1 - \frac{x-1}{x-2}\left(\frac{x+1}{x+2}\right)}\right] = \frac{\pi}{4}.$$

Thus, $\frac{\frac{x-1}{x-2} + \frac{x+1}{x+2}}{1 - \frac{x-1}{x-2}\left(\frac{x+1}{x+2}\right)} = 1,$

which on simplification gives $2x^2 - 4 = -3$

Thus, $x^2 = \frac{1}{2}$ gives $x = \pm\frac{1}{\sqrt{2}}$.

42.(a) The numbers on the dice are 1, 3, 3, 5, 5, 5 since X denotes the total score in two throws, it take the values 2, 4, 6, 8, 10.

Sample space s

I/II	1	3	3	5	5	5
1	2	4	4	6	6	6
3	4	6	6	8	8	8
3	4	6	6	8	8	8
5	6	8	8	10	10	10
5	6	8	8	10	10	10
5	6	8	8	10	10	10

From the sample space s we have

Values of the random variable	2	4	6	8	10	Total
No. of elements in inverse images	1	4	10	12	9	36

$$p(x=2) = \frac{1}{36}$$

$$\Rightarrow \quad p(x=4) = \frac{4}{36}$$

$$p(x=6) = \frac{10}{36}$$

$$\Rightarrow \quad p\,(x=8) = \frac{12}{36}$$

$$p(x=10) = \frac{9}{36}$$

(i) Probability mass function is

x	2	4	6	8	10	Total
$f(x)$	$\frac{1}{36}$	$\frac{4}{36}$	$\frac{10}{36}$	$\frac{12}{36}$	$\frac{9}{36}$	1

(ii) Cumulative distribution function.

$$F(x) = p(x \le x)$$

$$\Rightarrow \quad F(2) = \frac{1}{36}$$

$$F(4) = \frac{1}{36} + \frac{4}{36} = \frac{5}{36}$$

$$\Rightarrow F(6) = \frac{1}{36} + \frac{4}{36} + \frac{10}{36} = \frac{15}{36}$$

$$F(8) = \frac{15}{36} + \frac{12}{36} = \frac{27}{36}$$

$$\Rightarrow F(10) = \frac{27}{36} + \frac{9}{36} = \frac{36}{36} = 1$$

∵ The cumulative distribution function

$$F(x) = \begin{cases} 0 & \text{for} \quad x < 2 \\ \frac{1}{36} & \text{for} \quad 2 \le x4 \\ \frac{5}{36} & \text{for} \quad 4 \le x < 6 \\ \frac{15}{36} & \text{for} \quad 6 \le x < 8 \\ \frac{27}{36} & \text{for} \quad 8 \le x < 10 \\ 1 & \text{for} \quad 10 \le x < \infty \end{cases}$$

(iii) $p(4 \le x < 10) = p(x=4) + p(x=6) + p(x=8)$

$$= \frac{4}{36} + \frac{10}{36} + \frac{12}{36} = \frac{26}{36} = \frac{13}{18}$$

(OR)

(b) Given $z = x + iy$

$$\text{Im}\left(\frac{2z+1}{iz+1}\right) = 0$$

$$\Rightarrow \quad \text{Im}\left(\frac{2(x+iy)+1}{i(x+iy)+1}\right) = 0$$

$$\Rightarrow \text{Im}\left(\frac{(2x+1)+2i\,y}{ix+i^2\,y+1}\right) = 0$$

$$\Rightarrow \quad \text{Im}\left(\frac{(2x+1)+2iy}{ix-y+1}\right) = 0$$

$$\Rightarrow \quad \text{Im}\left(\frac{(2x+1)+2iy}{(1-y)+ix}\right) = 0$$

Multiply and divide by the conjugate of the denominator

We get $\text{Im}\left(\frac{(2x+1)+2i\,y}{(1-y)+ix} \times \frac{(1-y)-ix}{(1-y)-ix}\right) = 0$

$$\Rightarrow \quad \text{Im}\left(\frac{(2x+1)+2iy\times(1-y)-ix}{(1-y)^2+x^2}\right) = 0$$

Choosing the imaginary part we get,

$$\frac{(2x+1)(-x)+2y(1-y)}{(1-y)^2+x^2} = 0$$

$$\Rightarrow \quad (2x+1)(-x)+2y(1-y) = 0$$

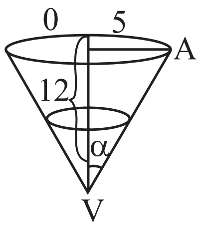
$$\Rightarrow \quad -2x^2-x+2y-2y^2 = 0$$

$$\Rightarrow \quad 2x^2+2y^2+x-2y = 0$$

Hence, locus of z is $2x^2+2y^2+x-2y = 0$

43.(a) Given $h = 12$ m

$r = 5$ m



Let V be the volume of the cone and α be the semi-vertical angle of the cone :

$$\therefore \tan\alpha = \frac{OA}{VO} = \frac{5}{12} = \frac{r}{h}$$

$$\Rightarrow \quad 12r = 5h$$

$$\Rightarrow \quad r = \frac{5h}{12} \qquad \text{... (1)}$$

We know $V = \frac{1}{3}\pi r^2 h$

$$= \frac{1}{3}\pi\left(\frac{5h}{12}\right)^2 . h \qquad \text{[From (1)]}$$

$$V = \frac{1}{3}\pi\left(\frac{25h^3}{144}\right) = \frac{25\pi h^3}{3\times 144}$$

Differentiating with respect to 't' we get,

$$\frac{dv}{dt} = \frac{25\pi}{3\times 144}3h^2\frac{dh}{dt} = \frac{25\pi}{144}h^2\frac{dh}{dt}$$

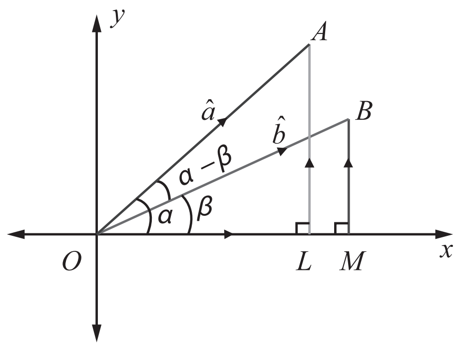
$$\Rightarrow 10 = \frac{25\pi}{144}(8)^2\frac{dh}{dt}$$

$$\left[\because \text{Given } h = 8, \frac{dv}{dt} = 10\text{m}^3/\text{min}\right]$$

$$\Rightarrow \frac{10\times 144}{25\pi\times 64} = \frac{dh}{dt} \quad \Rightarrow \quad \frac{dh}{dt} = \frac{9}{10\pi} \text{ m/ min}$$

(OR)

(b) Let $\hat{a} = \overrightarrow{OA}$ and $\vec{b} = \overrightarrow{OB}$ be the unit vectors making angles α and β respectively, with positive x-axis, where A and B are as shown in the Figure. Then, we get $\hat{a} = \cos\alpha\hat{i} + \sin\alpha\hat{j}$ and $\hat{b} = \cos\beta\hat{i} + \sin\beta\hat{j}$, The angle between $\hat{a}$ and $\hat{b}$ is $\alpha-\beta$ and, the vectors $\hat{a}, \hat{b}, \hat{k}$ bak form a right-handed system.



Hence, we get

$$\hat{b}\times\hat{a} = |\hat{b}||\hat{a}|\sin(\alpha-\beta)\hat{k} = \sin(\alpha-\beta)\hat{k} \qquad \text{... (1)}$$

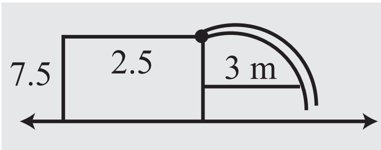
On the other hand,

$$\hat{b}\times\hat{a} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \cos\beta & \sin\beta & 0 \\ \cos\alpha & \sin\alpha & 0 \end{vmatrix} = (\sin\alpha\cos\beta - \cos\alpha\sin\beta)\hat{k} \quad \text{... (2)}$$

Hence, equations (1) and (2), leads to

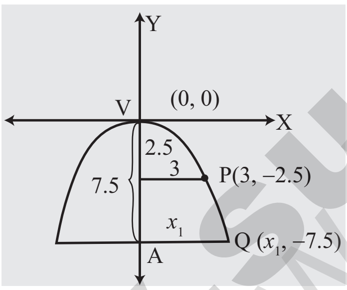
$\sin(\alpha - \beta) = \sin\alpha\ \cos\beta - \cos\alpha\sin\beta$.

44.(a) As per the given information, we can take the parabola as open downward.



$\therefore$ Its equation is $x^2 = -4ay$ ...(1)

Let P be a point on the flow paths, 2.5 m below the line of the pipe and 3m beyond the vertical line through the end of the pipe.



$\therefore$ P is (3, –2.5)

$\therefore$ (1) becomes $3^2 = -4a(-2.5)$

$$\Rightarrow \quad \frac{9}{2.5} = 4a$$

$\therefore$ (1) becomes, $x^2 = \dfrac{-9}{2.5}y$ ...(2)

Let x_1 be the distance between the bottom of the vertical line on the ground from the pipe end and the point on which this water touches the ground. But the height of the pipe from the ground is 7.5 m.

$\therefore$ $(x_1, -7.5)$ lies on

$\therefore$ (2) becomes, $x_1^2 = \dfrac{-9}{2.5}(-7.5)$

$\Rightarrow \quad x_1^2 = 9(3)$

$\Rightarrow x_1 = \sqrt{9\times 3} = 3\sqrt{3}$ m

The water strikes the ground $3\sqrt{3}$ m beyond the vertical line.

(OR)

(b) This is a linear differential equation

$$\therefore P = \frac{1}{x};\ Q = \sin x$$

$$\int p\,dx = \int \frac{1}{x}dx = \log x$$

$$\text{I.F.} = e^{\int p\,dx} = e^{\log x} = x$$

$\therefore$ The solution is

$$ye^{\int p\,dx} = \int Qe^{\int p\,dx}dx + c$$

$$u = x;\ dv = \sin x$$

$$du = dx;\ v = -\cos x$$

$$\because \int u\,dv = uv - \int v\,du$$

$$\Rightarrow \quad yx = \int x\sin x\cdot dx + c$$

$$\Rightarrow \quad xy = -x\cos x + \int \cos x\,dx$$

$$\Rightarrow \quad xy = -x\cos x + \sin x + c$$

$$\Rightarrow \quad xy + x\cos x = \sin x + c$$

$$\Rightarrow \quad x(y + \cos x) = \sin x + c$$

45. (a) Let A be the number of bacteria at any time t.

Given $\frac{dA}{A} \alpha A$

$$\Rightarrow \frac{dA}{dt} = kA$$

$$\Rightarrow \frac{dA}{A} = kdt$$

$$\Rightarrow \int \frac{dA}{A} = k\int dt$$

$$\Rightarrow \log A = kt + \log c$$

$$\Rightarrow \log\left(\frac{A}{c}\right) = kt$$

$$\Rightarrow \frac{A}{c} = e^{kt}$$

$$\Rightarrow A = c.e^{kt} \quad \ldots (1)$$

Initially when $t = 0$, assume that $A = A_0$

(1) becomes, $A_0 = ce^0 \Rightarrow c = A_0$

$$\therefore A = A_0 e^{kt} \quad \ldots (2)$$

Given when $t = 5, A = 3A_0$

$$3A_0 = A_0 e^{5k} \Rightarrow 3 = e^{5k}$$

When $t = 10$, (2) becomes,

$$A = A_0 e^{10k}$$

$$= A_0 (e^{5k})^2 = A_0 (3)^2$$

$$\Rightarrow A = 9 A_0$$

Hence after 10 hours the number of bacteria is 9 times the original number of bacteria.

(OR)

(b) The required plane contains the line $\frac{x-2}{2} = \frac{y-2}{3} = \frac{z-1}{3}$ i.e., It passes through the point (2, 2, 1) and is parallel to the vector $2\vec{i} + 3\vec{j} + 3\vec{k}$.

Also given that the required plane is parallel to the line $\frac{x+1}{3} = \frac{y-1}{2} = \frac{z+1}{1}$. i.e., it is parallel to the vector $3\vec{i} + 2\vec{j} + \vec{k}$.

Therefore $\vec{a} = 2\vec{i} + 2\vec{j} + \vec{k}$; $\vec{u} = 2\vec{i} + 3\vec{j} + 3\vec{k}$

$\vec{v} = 3\vec{i} + 2\vec{j} + \vec{k}$

Vector Equation : The required vector equation is $\vec{r} = \vec{a} + s\vec{u} + t\vec{v}$

$$\vec{r} = \left(2\vec{i} + 2\vec{j} + \vec{k}\right) + s\left(2\vec{i} + 3\vec{j} + 3\vec{k}\right) + t\left(3\vec{i} + 2\vec{j} + \vec{k}\right)$$

Cartesian form : The Cartesian equation of the required plane is =

$$\begin{vmatrix} x - x_1 & y - y_1 & z - z_1 \\ l_1 & m_1 & n_1 \\ l_2 & m_2 & n_2 \end{vmatrix} = 0$$

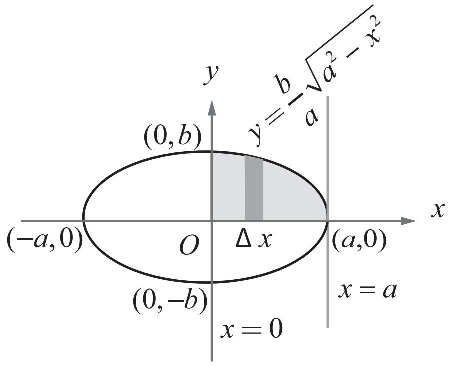
$\Rightarrow (x-2)(3-6) - (y-2)(2-9) + (z-1)(4-0) = 0$

$\Rightarrow -3x + 6 + 7y - 14 - 5z + 5 = 0$

$\Rightarrow -3x + 7y - 5z - 3 = 0$

$\Rightarrow 3x - 7y + 5z + 3 = 0$

46. (a) The ellipse is symmetric about both major and minor axes. It is sketched as in Figure.



So, viewing in the positive direction of y-axis, the required area A is four times the area of the region bounded by the portion of the ellipse in the first quadrant $\left(y=\frac{b}{a}\sqrt{a^2-x^2}, 0<x<a\right)$

x - axis, $x = 0$ and $x = a$.

Hence, by taking vertical strips, we get

$$A = 4\int_0^a y\,dx = 4\int_0^a \frac{b}{a}\sqrt{a^2-x^2}\,dx$$

$$= \frac{4b}{a}\left[\frac{x\sqrt{a^2-x^2}}{2} + \frac{a^2}{2}\sin^{-1}\left(\frac{x}{a}\right)\right]_0^a$$

$$= \frac{4b}{a} \times \frac{\pi a^2}{4} = \pi ab$$

(OR)

(b) $\mathbf{y^2 - 4y - 8x + 12 = 0}$

$y^2 - 4y = 8x - 12$

Adding 4 both sides, we get,

$y^2 - 4y + 4 = 8x - 12 + 4 = 8x - 8$

$\Rightarrow (y-2)^2 = 8(x-1)$

This is a right open parabola and latus rectum is $4a = 8 \Rightarrow a = 2$.

(a) Vertex is $(1, 2) \Rightarrow h = 1, k = 2$

(b) focus is $(h + a, 0 + k)$

$\Rightarrow (1 + 2, 0 + 2) \Rightarrow (3, 2)$

(c) Equation of directrix is $x = h - a$

$\Rightarrow x = 1 - 2 \Rightarrow x = -1$ (or) $x + 1 = 0$

47. (a)

p	q	$p \leftrightarrow q$	$\sim p$	$\sim q$	$(\sim p) \vee q$	$(\sim q) \vee p$	$((\sim p) \vee q) \wedge ((\sim q) \vee p)$
T	T	T	F	F	T	T	T
T	F	F	F	T	F	T	F
F	T	F	T	F	T	F	F
F	F	T	T	T	T	T	T
		(1)					(2)

From (1) and (2) we get $p \leftrightarrow q \equiv ((\sim p) \vee q) \wedge (\sim q) \vee p)$

(OR)

(b) Put $\frac{1}{x} = X, \frac{1}{y} = Y, \frac{1}{z} = Z$

We get $3X - 4Y - 2Z = 1, X + 2Y + Z = 2,$

$$2X - 5Y - 4Z = -1$$

$$\therefore \Delta = \begin{vmatrix} 3 & -4 & -2 \\ 1 & 2 & 1 \\ 2 & -5 & -4 \end{vmatrix} = 3\begin{vmatrix} 2 & 1 \\ -5 & -4 \end{vmatrix} + 4\begin{vmatrix} 1 & 1 \\ 2 & -4 \end{vmatrix} - 2\begin{vmatrix} 1 & 2 \\ 2 & -5 \end{vmatrix}$$

$= 3(-8+5) + 4(-4-2) - 2(-5-4) = 3(-3) + 4(-6) - 2(-9)$

$= -9 - 24 + 18 = -15$

$$\Delta_x = \begin{vmatrix} 1 & -4 & -2 \\ 2 & 2 & 1 \\ -1 & -5 & -4 \end{vmatrix} = 1\begin{vmatrix} 2 & 1 \\ -5 & -4 \end{vmatrix} + 4\begin{vmatrix} 2 & 1 \\ -1 & -4 \end{vmatrix} - 2\begin{vmatrix} 2 & 2 \\ -1 & -5 \end{vmatrix} = 1(-8+5) + 4(-8+1) - 2(-10+2)$$

$= 1(-3) + 4(-7) - 2(-8) = -3 - 28 + 16 = -15$

$$\Delta_y = \begin{vmatrix} 3 & 1 & -2 \\ 1 & 2 & 1 \\ 2 & -1 & -4 \end{vmatrix}$$

$$= 3\begin{vmatrix} 2 & 1 \\ -1 & -4 \end{vmatrix} - 1\begin{vmatrix} 1 & 1 \\ 2 & -4 \end{vmatrix} - 2\begin{vmatrix} 1 & 2 \\ 2 & -1 \end{vmatrix}$$

$= 3(-8+1) - 1(-4-2) - 2(-1-4) = 3(-7) - 1(-6) - 2(-5) = -21 + 6 + 10 = -5$

$$\Delta_z = \begin{vmatrix} 3 & -4 & 1 \\ 1 & 2 & 2 \\ 2 & -5 & -1 \end{vmatrix} = 3\begin{vmatrix} 2 & 2 \\ -5 & -1 \end{vmatrix} + 4\begin{vmatrix} 1 & 2 \\ 2 & -1 \end{vmatrix} + 1\begin{vmatrix} 1 & 2 \\ 2 & -5 \end{vmatrix}$$

$= 3(-2+10) + 4(-1-4) + 1(-5-4) = 3(8) + 4(-5) + 1(-9) = 24 - 20 - 9 = -5$

$$\therefore X = \frac{\Delta_x}{\Delta} = \frac{-15}{-15} = 1 \Rightarrow \frac{1}{x} = 1$$

$$\Rightarrow x = 1$$

$$Y = \frac{\Delta_y}{\Delta} = \frac{-5}{-15} = \frac{1}{3}$$

$$\Rightarrow \frac{1}{y} = \frac{1}{3} \Rightarrow y = 3$$

$$Z = \frac{\Delta_z}{\Delta} = \frac{-5}{-15} = \frac{1}{3} \Rightarrow \frac{1}{z} = \frac{1}{3} \Rightarrow z = 3$$

$$\therefore x = 1, y = 3, z = 3.$$

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