X - * TRIGONOMETRY * UNIT-6

① Prove that: $\tan^2\theta - \sin^2\theta = \tan^2\theta \sin^2\theta$.

Proof: LHS

$$\tan^2\theta - \sin^2\theta = \tan^2\theta - \frac{\sin^2\theta}{\cos^2\theta} \times \cos^2\theta$$

$$= \tan^2\theta - \tan^2\theta \cdot \cos^2\theta$$

$$= \tan^2\theta(1-\cos^2\theta).$$

[$\tan^2\theta$ is common. ∴ Taking out $\tan^2\theta$ we get]

$$= \tan^2\theta \cdot \sin^2\theta.$$

$$\therefore \boxed{\tan^2\theta - \sin^2\theta = \tan^2\theta \cdot \sin^2\theta} \quad (1-\cos^2\theta = \sin^2\theta)$$

② Prove that

$$\frac{\sin A}{1+\cos A} = \frac{1-\cos A}{\sin A}$$

Proof:

$$\frac{\sin A}{1+\cos A} = \frac{\sin A}{1+\cos A} \times \frac{1-\cos A}{1-\cos A}$$

$$= \frac{\sin A(1-\cos A)}{(1)^2 - \cos^2 A}$$

[$a^2-b^2=(a+b)(a-b)$, $1-\cos^2 A = \sin^2 A$]

$$= \frac{\sin A(1-\cos A)}{1-\cos^2 A}$$

$$= \frac{\sin A(1-\cos A)}{\sin^2 A}$$

$$= \frac{\cancel{\sin A}(1-\cos A)}{\cancel{\sin A} \times \sin A} \quad [\because a^2 = a \times a]$$

$$\boxed{\frac{\sin A}{1+\cos A} = \frac{1-\cos A}{\sin A}}$$

③ Prove that:

$$1 + \frac{\cot^2\theta}{1+\operatorname{cosec}\theta} = \operatorname{cosec}\theta.$$

Proof:

$$1 + \frac{\cot^2\theta}{1+\operatorname{cosec}\theta} = 1 + \frac{\operatorname{cosec}^2\theta - 1}{1+\operatorname{cosec}\theta} = 1 + \frac{\operatorname{cosec}^2\theta - (1)^2}{1+\operatorname{cosec}\theta}.$$

$$= 1 + \frac{\cancel{(\operatorname{cosec}\theta+1)}(\operatorname{cosec}\theta - 1)}{\cancel{(\operatorname{cosec}\theta+1)}}$$

$$= \cancel{1} + \operatorname{cosec}\theta - \cancel{1}$$

[$a^2-b^2=(a+b)(a-b)$, $\cot^2\theta = \operatorname{cosec}^2\theta - 1$]

$$= \operatorname{cosec}\theta$$

$$\therefore \boxed{1 + \frac{\cot^2\theta}{1+\operatorname{cosec}\theta} = \operatorname{cosec}\theta}$$

④ Prove that: $\sec\theta - \cos\theta = \tan\theta \cdot \sin\theta$.

[$\sec\theta = \frac{1}{\cos\theta}$, $1-\cos^2\theta = \sin^2\theta$]

$$\sec\theta - \cos\theta = \frac{1}{\cos\theta} - \cos\theta$$

$$= \frac{1-\cos^2\theta}{\cos\theta} = \frac{\sin^2\theta}{\cos\theta} = \left(\frac{\sin\theta}{\cos\theta}\right) \times \sin\theta$$

$$= \tan\theta \cdot \sin\theta. \quad [a^2 = a \times a]$$

$$\boxed{\sec\theta - \cos\theta = \tan\theta \cdot \sin\theta}$$

5. Prove that: $\sqrt{\frac{1+\cos\theta}{1-\cos\theta}} = \operatorname{cosec}\theta + \cot\theta$.

Proof:

$$\sqrt{\frac{1+\cos\theta}{1-\cos\theta}} = \sqrt{\frac{1+\cos\theta}{1-\cos\theta} \times \frac{1+\cos\theta}{1+\cos\theta}}$$

$[(a+b)\times(a+b) = (a+b)^2$

$(a-b)\times(a+b) = a^2-b^2$

$1-\cos^2\theta = \sin^2\theta$.

$\frac{1}{\sin\theta} = \operatorname{cosec}\theta$

$\frac{\cos\theta}{\sin\theta} = \cot\theta$.

$$= \sqrt{\frac{(1+\cos\theta)^2}{1-\cos^2\theta}}$$

$$= \sqrt{\frac{(1+\cos\theta)^2}{\sin^2\theta}}$$

$$= \sqrt{\left[\frac{1+\cos\theta}{\sin\theta}\right]^2}$$

$$= \frac{1+\cos\theta}{\sin\theta} = \frac{1}{\sin\theta} + \frac{\cos\theta}{\sin\theta}$$

$$\boxed{\sqrt{\frac{1+\cos\theta}{1-\cos\theta}} = \operatorname{cosec}\theta + \cot\theta}.$$

6. Prove that.

$$\sqrt{\frac{1+\sin\theta}{1-\sin\theta}} = \sec\theta + \tan\theta.$$

Proof:

$$\sqrt{\frac{1+\sin\theta}{1-\sin\theta}} = \sqrt{\frac{1+\sin\theta}{1-\sin\theta} \times \frac{1+\sin\theta}{1+\sin\theta}}$$

$[1-\sin^2\theta = \cos^2\theta$.

$\sqrt{\left(\frac{1+x}{y}\right)^2} = \frac{1+x}{y}$

$$= \sqrt{\frac{(1+\sin\theta)^2}{1-\sin^2\theta}}$$

$$= \sqrt{\frac{(1+\sin\theta)^2}{\cos^2\theta}} = \sqrt{\left[\frac{1+\sin\theta}{\cos\theta}\right]^2}$$

$$= \frac{1+\sin\theta}{\cos\theta}$$

$\left[\frac{1+x}{y} = \frac{1}{y} + \frac{x}{y}\right.$

$\frac{1}{\cos\theta} = \sec\theta$

$\frac{\sin\theta}{\cos\theta} = \tan\theta$.

$$= \frac{1}{\cos\theta} + \frac{\sin\theta}{\cos\theta}$$

$$\boxed{\sqrt{\frac{1+\sin\theta}{1-\sin\theta}} = \sec\theta + \tan\theta} \text{ ———— } ①.$$

7. Prove that: $\sqrt{\frac{1+\sin\theta}{1-\sin\theta}} + \sqrt{\frac{1-\sin\theta}{1+\sin\theta}} = 2\sec\theta$.

Proof:

$$\sqrt{\frac{1+\sin\theta}{1-\sin\theta}} = \sec\theta + \tan\theta \quad [\text{From } ①] \text{ —— } ①.$$

$$\therefore \sqrt{\frac{1-\sin\theta}{1+\sin\theta}} = \sqrt{\frac{1-\sin\theta}{1+\sin\theta} \times \frac{1-\sin\theta}{1-\sin\theta}}$$

$[(a-b)\times(a-b) = (a-b)^2$

$(a+b)\times(a-b) = a^2-b^2$

$1-\sin^2\theta = \cos^2\theta$.

$$= \sqrt{\frac{(1-\sin\theta)^2}{1-\sin^2\theta}}$$

$$= \sqrt{\frac{(1-\sin\theta)^2}{\cos^2\theta}}.$$

$$= \sqrt{\left[\frac{1-\sin\theta}{\cos\theta}\right]^2} = \frac{1-\sin\theta}{\cos\theta}$$

$$= \frac{1}{\cos\theta} - \frac{\sin\theta}{\cos\theta}$$

$$\sqrt{\frac{1-\sin\theta}{1+\sin\theta}} = \sec\theta - \tan\theta \text{ ———— } ②.$$

Add ① + ② we get:

$$\sqrt{\frac{1+\sin\theta}{1-\sin\theta}} + \sqrt{\frac{1-\sin\theta}{1+\sin\theta}} = \sec\theta + \cancel{\tan\theta} + \sec\theta - \cancel{\tan\theta}$$

$$= 2\sec\theta.$$

(8) Prove that: $\frac{\sec\theta}{\sin\theta} - \frac{\sin\theta}{\cos\theta} = \cot\theta$.

Proof:

$\frac{\sec\theta}{\sin\theta} - \frac{\sin\theta}{\cos\theta} = \frac{\frac{1}{\cos\theta}}{\sin\theta} - \frac{\sin\theta}{\cos\theta}$ $\left[\frac{1/a}{b} = \frac{1}{ab}\right.$

$= \frac{1}{\sin\theta\cos\theta} - \frac{\sin\theta}{\cos\theta}$

$= \frac{1 - \sin\theta \cdot \sin\theta}{\sin\theta \cdot \cos\theta}$ $\left[1-\sin^2\theta = \cos^2\theta,\ \cos^2\theta = \cos\theta \cdot \cos\theta,\ \frac{1}{ab} - \frac{a}{b} = \frac{1-a^2}{ab}\right.$

$= \frac{1-\sin^2\theta}{\sin\theta \cdot \cos\theta}$

$= \frac{\cos^2\theta}{\sin\theta \cdot \cos\theta}$

$= \frac{\cos\theta \cdot \cancel{\cos\theta}}{\sin\theta \cdot \cancel{\cos\theta}}$ $\left[\frac{\cos\theta}{\sin\theta} = \cot\theta\right]$

* $\boxed{\frac{\sec\theta}{\sin\theta} - \frac{\sin\theta}{\cos\theta} = \cot\theta}$ *

(9) $\cot\theta + \tan\theta = \sec\theta \cdot \operatorname{cosec}\theta$.

$\cot\theta + \tan\theta = \frac{\cos\theta}{\sin\theta} + \frac{\sin\theta}{\cos\theta}$

$= \frac{\cos^2\theta + \sin^2\theta}{\sin\theta \cdot \cos\theta}$ $\left[\sin^2\theta + \cos^2\theta = 1,\ \therefore \cos^2\theta + \sin^2\theta = 1,\ \frac{\sin\theta}{\cos\theta} = \tan\theta\right.$

$= \frac{1}{\sin\theta \cdot \cos\theta}$ $\left[\frac{\cos\theta}{\sin\theta} = \cot\theta.\ \frac{1}{\sin\theta} = \operatorname{cosec}\theta,\ \frac{1}{\cos\theta} = \sec\theta\right.$

$= \frac{1}{\sin\theta} \cdot \frac{1}{\cos\theta}$

$= \operatorname{cosec}\theta \cdot \sec\theta$.

* $\boxed{\cot\theta + \tan\theta = \sec\theta \cdot \operatorname{cosec}\theta}$ *

(2)

(10). $\tan^4\theta + \tan^2\theta = \sec^4\theta - \sec^2\theta$.

Proof:

$\tan^4\theta + \tan^2\theta = \tan^2\theta[\tan^2\theta + 1]$ [Taking out $\tan^2\theta$]

[$\tan^2\theta$ is common. $\therefore$

$= (\sec^2\theta - 1) \cdot (\sec^2\theta)$ $\left[\tan^2\theta = \sec^2\theta - 1,\ \tan^2\theta + 1 = \sec^2\theta.\right.$

$= \sec^4\theta - \sec^2\theta$.

* $\boxed{\tan^4\theta + \tan^2\theta = \sec^4\theta - \sec^2\theta}$ *

(11). $\frac{1-\tan^2\theta}{\cot^2\theta - 1} = \tan^2\theta$.

Proof: $\frac{1-\tan^2\theta}{\cot^2\theta - 1} = \frac{1-\tan^2\theta}{\frac{1}{\tan^2\theta} - 1} = \frac{1-\tan^2\theta}{\frac{1-\tan^2\theta}{\tan^2\theta}}$

$= (1-\cancel{\tan^2\theta}) \times \frac{\tan^2\theta}{(1-\cancel{\tan^2\theta})}$

$\boxed{\frac{1-\tan^2\theta}{\cot^2\theta - 1} = \tan^2\theta}$

(12). $\frac{\cos\theta}{1+\sin\theta} = \sec\theta - \tan\theta$ $\left[(a+b)(a-b) = a^2 - b^2\right.$

Proof: $\frac{\cos\theta}{1+\sin\theta} = \frac{\cos\theta}{1+\sin\theta} \times \frac{1-\sin\theta}{1-\sin\theta}$

$= \frac{\cos\theta(1-\sin\theta)}{1-\sin^2\theta} = \frac{\cos\theta(1-\sin\theta)}{\cos^2\theta}$ $\left[\sin^2\theta + \cos^2\theta = 1,\ \cos^2\theta = 1 - \sin^2\theta\right.$

$= \frac{\cancel{\cos\theta}(1-\sin\theta)}{\cancel{\cos\theta} \cdot \cos\theta}$

$= \frac{1-\sin\theta}{\cos\theta}$ $\left[\frac{1-a}{b} = \frac{1}{b} - \frac{a}{b}\right.$

$= \frac{1}{\cos\theta} - \frac{\sin\theta}{\cos\theta}$

$\boxed{\frac{\cos\theta}{1+\sin\theta} = \sec\theta - \tan\theta}$

13. $\frac{\sin A}{1+\cos A} + \frac{\sin A}{1-\cos A} = 2\,\text{cosec}A$.

Proof:

$\frac{\sin A}{1+\cos A} + \frac{\sin A}{1-\cos A} = \frac{\sin A(1-\cos A)+\sin A(1+\cos A)}{(1+\cos A)(1-\cos A)}$

$= \frac{\sin A - \sin A\cancel{\cos A} + \sin A + \cancel{\sin A\cos A}}{1-\cos^2 A}$

$= \frac{2\sin A}{\sin^2 A}$ $\quad [\sin^2 A = \sin A \cdot \sin A$

$= \frac{2\cancel{\sin A}}{\sin A\cdot\cancel{\sin A}} = 2\left(\frac{1}{\sin A}\right)$

$\boxed{\frac{\sin A}{1+\cos A} + \frac{\sin A}{1-\cos A} = 2\,\text{cosec}A}$ $\quad \left[\frac{1}{\sin A} = \text{cosec}A\right]$

14. $\sec^6\theta = \tan^6\theta + 3\tan^2\theta\cdot\sec^2\theta + 1$

L.H.S: $\sec^6\theta = (\sec^2\theta)^3$ $\quad [a^6 = (a^2)^3$

$= [1+\tan^2\theta]^3$ $\quad [1+\tan^2\theta = \sec^2\theta$

$= (1)^3 + (\tan^2\theta)^3 + 3(1)(\tan^2\theta)[1+\tan^2\theta]$

$(a+b)^3 = a^3+b^3+3ab(a+b)$.

$= 1+\tan^6\theta + 3\tan^2\theta(1+\tan^2\theta)$ $\quad [(\tan^2\theta)^3 = \tan^6\theta$, $1+\tan^2\theta = \sec^2\theta$

$= 1+\tan^6\theta + 3\tan^2\theta\cdot\sec^2\theta$

$= \tan^6\theta + 3\tan^2\theta\cdot\sec^2\theta + 1$

15. $\sec^4\theta(1-\sin^4\theta) - 2\tan^2\theta = 1$.

Proof: LHS.

$[a^2-b^2 = (a+b)(a-b)$, $\tan^2\theta = \frac{\sin^2\theta}{\cos^2\theta}$

$= \sec^4\theta\,((1)^4 - \sin^4\theta) - 2\tan^2\theta$.

$= \frac{1}{\cos^4\theta}\cdot[(1^2)^2 - (\sin^2\theta)^2] - 2\tan^2\theta$.

$= \frac{1}{\cos^4\theta}(1^2+\sin^2\theta)(1^2-\sin^2\theta) - 2\frac{\sin^2\theta}{\cos^2\theta}$.

$= \frac{(1+\sin^2\theta)(1-\sin^2\theta)}{\cos^4\theta} - \frac{2\sin^2\theta}{\cos^2\theta}$.

$= \frac{(1+\sin^2\theta)\cancel{(\cos^2\theta)}}{(\cos^2\theta)\cancel{(\cos^2\theta)}} - \frac{2\sin^2\theta}{\cos^2\theta}$.

$[\cos^4\theta = \cos^2\theta\times\cos^2\theta$, $1-\sin^2\theta = \cos^2\theta$.

$= \frac{1+\sin^2\theta}{\cos^2\theta} - \frac{2\sin^2\theta}{\cos^2\theta}$.

$= \frac{1+\sin^2\theta - 2\sin^2\theta}{\cos^2\theta}$. $\quad [\sin^2\theta - 2\sin^2\theta = -\sin^2\theta$.

$= \frac{1-\sin^2\theta}{\cos^2\theta} = \frac{\cancel{\cos^2\theta}}{\cancel{\cos^2\theta}}$

$= 1$

$\therefore \boxed{\sec^4\theta(1-\sin^4\theta) - 2\tan^2\theta = 1}$

U. Karthik 08/10/19

U.KARTHIKKUMAR
M.A.,M.Sc.,M.Ed.,(MATHS),
P.G.TEACHER IN MATHS.,
UNIVERSAL MAT.HR.SEC.SCHOOL,
TIRUPUR - 641664.

(16) $\frac{\cot\theta - \cos\theta}{\cot\theta + \cos\theta} = \frac{\csc\theta - 1}{\csc\theta + 1}$

LHS:

$$\frac{\cot\theta - \cos\theta}{\cot\theta + \cos\theta} = \frac{\frac{\cos\theta}{\sin\theta} - \cos\theta}{\frac{\cos\theta}{\sin\theta} + \cos\theta} \quad \left[\cot\theta = \frac{\cos\theta}{\sin\theta}\right]$$

($\because$ $\cos\theta$ is common. $\therefore$ Taking out $\cos\theta$ we get]

$$= \frac{\cancel{\cos\theta}\left(\frac{1}{\sin\theta} - 1\right)}{\cancel{\cos\theta}\left(\frac{1}{\sin\theta} + 1\right)} \quad \left[\frac{1}{\sin\theta} = \csc\theta\right]$$

$$\boxed{\frac{\cot\theta - \cos\theta}{\cot\theta + \cos\theta} = \frac{\csc\theta - 1}{\csc\theta + 1}}$$

(17) Prove that $\frac{\sin A - \sin B}{\cos A + \cos B} + \frac{\cos A - \cos B}{\sin A + \sin B} = 0$.

LHS:

$$= \frac{\sin A - \sin B}{\cos A + \cos B} + \frac{\cos A - \cos B}{\sin A + \sin B}$$

$$= \frac{(\sin^2 A - \sin^2 B) + (\cos^2 A - \cos^2 B)}{(\cos A + \cos B)\cdot(\sin A + \sin B)}$$

$$= \frac{\sin^2 A + \cos^2 A - \sin^2 B - \cos^2 B}{(\cos A + \cos B)\cdot(\sin A + \sin B)}$$

$$= \frac{(\sin^2 A + \cos^2 A) - (\sin^2 B + \cos^2 B)}{(\cos A + \cos B)\cdot(\sin A + \sin B)}$$

$$= \frac{1 - 1}{(\cos A + \cos B)(\sin A + \sin B)}$$

$$= \frac{0}{(\cos A + \cos B)\cdot(\sin A + \sin B)} \quad \left[\frac{0}{\text{Any Number}} = 0\right]$$

$$= 0.$$

(18) $\frac{\sin^3 A + \cos^3 A}{\sin A + \cos A} + \frac{\sin^3 A - \cos^3 A}{\sin A - \cos A} = 2$

LHS:

$$= \frac{\sin^3 A + \cos^3 A}{\sin A + \cos A} + \frac{\sin^3 A - \cos^3 A}{\sin A - \cos A} \quad \left[\begin{array}{l} a^3+b^3 = (a+b)(a^2-ab+b^2) \\ a^3-b^3 = (a-b)(a^2+ab+b^2)\end{array}\right]$$

$$= \left[\frac{\cancel{(\sin A + \cos A)}\cdot(\sin^2 A - \sin A\cos A + \cos^2 A)}{\cancel{(\sin A + \cos A)}}\right] + \left[\frac{\cancel{(\sin A - \cos A)}\cdot(\sin^2 A + \sin A\cos A + \cos^2 A)}{\cancel{(\sin A - \cos A)}}\right]$$

$$= (\sin^2 A - \sin A\cos A + \cos^2 A) + (\sin^2 A + \sin A\cos A + \cos^2 A)$$

$$= \sin^2 A + \cos^2 A - \cancel{\sin A\cos A} + \sin^2 A + \cos^2 A + \cancel{\sin A\cos A}$$

$$= 1 + 1$$

$$= 2.$$

$$\therefore \boxed{\frac{\sin^3 A + \cos^3 A}{\sin A + \cos A} + \frac{\sin^3 A - \cos^3 A}{\sin A - \cos A} = 2}$$

(19). Prove that $\frac{\sec\theta}{\sin\theta} - \frac{\sin\theta}{\cos\theta} = \cot\theta$.

Proof: L.H.S.

$$\frac{\sec\theta}{\sin\theta} - \frac{\sin\theta}{\cos\theta} = \frac{1/\cos\theta}{\sin\theta} - \frac{\sin\theta}{\cos\theta}$$

$$= \frac{1}{\cos\theta\cdot\sin\theta} - \frac{\sin\theta}{\cos\theta} \qquad \left[\frac{1/a}{b} = \frac{1}{ab}\right.$$

$$= \frac{1}{\cos\theta\cdot\sin\theta} - \frac{\sin\theta}{\cos\theta}\cdot\frac{\sin\theta}{\sin\theta}$$

$$= \frac{1}{\cos\theta\cdot\sin\theta} - \frac{\sin^2\theta}{\cos\theta\cdot\sin\theta}$$

$$= \frac{1-\sin^2\theta}{\cos\theta\cdot\sin\theta} \qquad \left[1-\sin^2\theta = \cos^2\theta\right.$$

$$= \frac{\cos^2\theta}{\cos\theta\cdot\sin\theta} = \frac{\cancel{\cos\theta}\cdot\cos\theta}{\cancel{\cos\theta}\cdot\sin\theta}$$

$$= \frac{\cos\theta}{\sin\theta}$$

$$\boxed{\frac{\sec\theta}{\sin\theta} - \frac{\sin\theta}{\cos\theta} = \cot\theta.}$$

(20). $\sin^2A\cos^2B + \cos^2A\cdot\sin^2B + \cos^2A\cos^2B + \sin^2A\cdot\sin^2B = 1$

Proof: LHS

$$= \sin^2A\cos^2B + \cos^2A\cdot\sin^2B + \cos^2A\cdot\cos^2B + \sin^2A\cdot\sin^2B$$

$$= \sin^2A\cos^2B + \sin^2A\cdot\sin^2B + \cos^2A\cdot\sin^2B + \cos^2A\cdot\cos^2B$$

$$= \sin^2A\left[\cos^2B + \sin^2B\right] + \cos^2A\left[\sin^2B + \cos^2B\right]$$

[$\sin^2A$ is common. Taking out $\sin^2A$ weget] [$\cos^2A$ is common. ∴ Taking out $\cos^2A$]

$$= \sin^2A\,(1) + \cos^2A\,(1) \qquad \left[\begin{array}{l}\sin^2B+\cos^2B=1\\ \sin^2A+\cos^2A=1.\end{array}\right.$$

$$= \sin^2A + \cos^2A.$$

$$= 1.$$

∴ $\boxed{LHS = RHS}$

(21). If $\cos\theta + \sin\theta = \sqrt{2}\cos\theta$, then prove that $\cos\theta - \sin\theta = \sqrt{2}\sin\theta$.

proof:

Given: $\cos\theta + \sin\theta = \sqrt{2}\cos\theta$. ——— ①.

Squaring on both sides weget.

$$(\cos\theta+\sin\theta)^2 = (\sqrt{2}\cos\theta)^2 \qquad \left[(ab)^2 = a^2b^2.\right.$$

$$\cos^2\theta + \sin^2\theta + 2\sin\theta\cdot\cos\theta = (\sqrt{2})^2\cdot\cos^2\theta.$$

$$\cos^2\theta + \sin^2\theta + 2\sin\theta\cdot\cos\theta = 2\cos^2\theta. \qquad \left[(\sqrt{2})^2 = \sqrt{2}\times\sqrt{2} = 2\right.$$

$$2\cos^2\theta - \cos^2\theta - \sin^2\theta = 2\sin\theta\cdot\cos\theta.$$

$$\cos^2\theta - \sin^2\theta = 2\sin\theta\cos\theta.$$

$$(\cos\theta+\sin\theta)(\cos\theta-\sin\theta) = 2\sin\theta\cdot\cos\theta \qquad \left[a^2-b^2=(a+b)(a-b)\right]$$

To prove: $\boxed{\cos\theta - \sin\theta = \sqrt{2}\sin\theta}$

$$\cos\theta - \sin\theta = \frac{2\sin\theta\cdot\cos\theta}{\cos\theta+\sin\theta}$$

$$\cos\theta - \sin\theta = \frac{2\sin\theta\cdot\cancel{\cos\theta}}{\sqrt{2}\cdot\cancel{\cos\theta}} \qquad \left[\because \text{by ①}\right] \quad \left[2 = \sqrt{2}\times\sqrt{2}\right]$$

$$\cos\theta - \sin\theta = \frac{\sqrt{2}\cdot\cancel{\sqrt{2}}\cdot\sin\theta}{\cancel{\sqrt{2}}}$$

$$\boxed{\cos\theta - \sin\theta = \sqrt{2}\cdot\sin\theta}$$

(22) $(\operatorname{cosec}\theta - \sin\theta)(\sec\theta - \cos\theta)(\tan\theta + \cot\theta) = 1.$

Proof:

$(\operatorname{cosec}\theta - \sin\theta)(\sec\theta - \cos\theta)(\tan\theta + \cot\theta)$

$= (\operatorname{cosec}\theta - \sin\theta)(\sec\theta - \cos\theta)(\tan\theta + \cot\theta)$

$= \left(\frac{1}{\sin\theta} - \sin\theta\right)\left(\frac{1}{\cos\theta} - \cos\theta\right)\left(\frac{\sin\theta}{\cos\theta} + \frac{\cos\theta}{\sin\theta}\right)$

$= \left(\frac{1-\sin^2\theta}{\sin\theta}\right) \times \left(\frac{1-\cos^2\theta}{\cos\theta}\right) \times \left(\frac{\sin^2\theta + \cos^2\theta}{\sin\theta \cdot \cos\theta}\right)$

$= \left(\frac{\cos^2\theta}{\sin\theta}\right) \times \left(\frac{\sin^2\theta}{\cos\theta}\right) \times \left[\frac{1}{\sin\theta \cdot \cos\theta}\right]$

$1 - \sin^2\theta = \cos^2\theta$
$1 - \cos^2\theta = \sin^2\theta$
$\sin^2\theta + \cos^2\theta = 1.$

$= \left[\frac{\cos^2\theta \times \sin^2\theta}{\sin\theta \cdot \cos\theta}\right] \times \left[\frac{1}{\sin\theta \cdot \cos\theta}\right]$

$= \frac{\cancel{\cos^2\theta} \times \cancel{\sin^2\theta}}{\cancel{\sin^2\theta} \cdot \cancel{\cos^2\theta}} = 1$

$\therefore \boxed{(\operatorname{cosec}\theta - \sin\theta)(\sec\theta - \cos\theta)(\tan\theta + \cot\theta) = 1}$

(23) $\tan^2 A - \tan^2 B = \frac{\sin^2 A - \sin^2 B}{\cos^2 A \cdot \cos^2 B}$

Proof: LHS

$\tan^2 A - \tan^2 B = \frac{\sin^2 A}{\cos^2 A} - \frac{\sin^2 B}{\cos^2 B}$

$= \frac{\sin^2 A \cos^2 B - \sin^2 B \cdot \cos^2 A}{\cos^2 A \cdot \cos^2 B}$

$= \frac{\sin^2 A(1 - \sin^2 B) - \sin^2 B(1 - \sin^2 A)}{\cos^2 A \cdot \cos^2 B}$

$= \frac{\sin^2 A - \cancel{\sin^2 A \sin^2 B} - \sin^2 B + \cancel{\sin^2 A \sin^2 B}}{\cos^2 A \cdot \cos^2 B}$

$= \frac{\sin^2 A - \sin^2 B}{\cos^2 A \cdot \cos^2 B} = \text{RHS}$

$\therefore \boxed{\tan^2 A - \tan^2 B = \frac{\sin^2 A - \sin^2 B}{\cos^2 A \cdot \cos^2 B}}$

(24) $\left[\frac{\cos^3 A - \sin^3 A}{\cos A - \sin A}\right] - \left[\frac{\cos^3 A + \sin^3 A}{\cos A + \sin A}\right] = 2 \sin A \cos A$

Proof: LHS.

$= \left[\frac{\cos^3 A - \sin^3 A}{\cos A - \sin A}\right] - \left[\frac{\cos^3 A + \sin^3 A}{\cos A + \sin A}\right]$

$a^3 - b^3 = (a-b)(a^2 + b^2 + ab).$
$a^3 + b^3 = (a+b)(a^2 + b^2 - ab).$

$= \left[\frac{\cancel{(\cos A - \sin A)}(\cos^2 A + \sin^2 A + \cos A \sin A)}{\cancel{(\cos A - \sin A)}}\right] - \left[\frac{\cancel{(\cos A + \sin A)}(\cos^2 A + \sin^2 A - \cos A \sin A)}{\cancel{(\cos A + \sin A)}}\right].$

$= (\cos^2 A + \sin^2 A + \cos A \sin A) - (\cos^2 A + \sin^2 A - \cos A \sin A)$

$= (1 + \cos A \sin A) - (1 - \cos A \sin A)$

$= \cancel{1} + \cos A \sin A - \cancel{1} + \cos A \sin A$

$= 2 \cos A \sin A.$

$=$ R.H.S.

(25). Show that $\left[\frac{1+\tan^2 A}{1+\cot^2 A}\right] = \left[\frac{1-\tan A}{1-\cot A}\right]^2$

Solution:

LHS:

$$= \frac{1+\tan^2 A}{1+\cot^2 A}$$

$$= \frac{1+\tan^2 A}{1+\frac{1}{\tan^2 A}}$$

$$= \frac{1+\tan^2 A}{\frac{\tan^2 A+1}{\tan^2 A}}$$

$$= \cancel{(1+\tan^2 A)} \times \frac{\tan^2 A}{\cancel{(1+\tan^2 A)}}$$

$$= \tan^2 A \longrightarrow (1)$$

RHS.

$$= \left[\frac{1-\tan A}{1-\cot A}\right]^2$$

$$= \left[\frac{1-\tan A}{1-\frac{1}{\tan A}}\right]^2$$

$$= \left[\frac{1-\tan A}{\frac{\tan A-1}{\tan A}}\right]^2$$

$$= \left[\frac{(1-\tan A)}{\frac{-(1-\tan A)}{\tan A}}\right]^2$$

$$= \left[\cancel{(1-\tan A)} \times \frac{(-\tan A)}{\cancel{(1-\tan A)}}\right]^2$$

$$= (-\tan A)^2 \quad [(-a)^2 = a^2]$$

$$= \tan^2 A \longrightarrow (2)$$

From (1) and (2) we get

(1) = (2).

$$\therefore \boxed{\frac{1+\tan^2 A}{1+\cot^2 A} = \left[\frac{1-\tan A}{1-\cot A}\right]^2}$$

(26). If $\sin\theta + \cos\theta = \sqrt{3}$; then prove that $\tan\theta + \cot\theta = 1$.

Proof: Given.

$$\sin\theta + \cos\theta = \sqrt{3}$$

$$(\sin\theta + \cos\theta)^2 = (\sqrt{3})^2 \quad (\because \text{Squaring on both sides})$$

$[(\sqrt{3})^2 = \sqrt{3}\times\sqrt{3} = 3$

$$\sin^2\theta + \cos^2\theta + 2\sin\theta \cdot \cos\theta = 3$$

$[\sin^2\theta + \cos^2\theta = 1$.

$$1 + 2\sin\theta \cdot \cos\theta = 3$$

$$2\sin\theta \cdot \cos\theta = 3-1$$

$$2\sin\theta \cdot \cos\theta = 2.$$

$$\sin\theta\cos\theta = \frac{\cancel{2}^1}{\cancel{2}_1}$$

$$\therefore \boxed{\sin\theta \cdot \cos\theta = 1} \longrightarrow (1).$$

To prove: $\tan\theta + \cot\theta = 1$.

L.H.S $\Rightarrow$ $\tan\theta + \cot\theta$

$$= \frac{\sin\theta}{\cos\theta} + \frac{\cos\theta}{\sin\theta} = \frac{\sin^2\theta + \cos^2\theta}{\sin\theta \cdot \cos\theta} = \frac{1}{1} = 1.$$

$$\therefore \boxed{\tan\theta + \cot\theta = 1}$$

(27). If $\sqrt{3}\sin\theta - \cos\theta = 0$ then show that $\tan 3\theta = \frac{3\tan\theta - \tan^3\theta}{1-3\tan^2\theta}$.

Solution: Given:

$$\sqrt{3}\sin\theta - \cos\theta = 0.$$

$$\sqrt{3}\sin\theta = \cos\theta \Rightarrow \sqrt{3}\frac{\sin\theta}{\cos\theta} = 1$$

$\therefore \frac{\sin\theta}{\cos\theta} = \frac{1}{\sqrt{3}} \Rightarrow \boxed{\tan\theta = \frac{1}{\sqrt{3}}}$

$\therefore \tan\theta = \tan 30^\circ$

$\therefore \boxed{\theta = 30^\circ}$

To prove: $\tan 3\theta = \frac{3\tan\theta - \tan^3\theta}{1 - 3\tan^2\theta}$.

LHS $\tan 3\theta = \tan 3(30^\circ) = \tan 90^\circ$

$= \infty$

$=$ undefined ——— ①

R.H.S:

$\frac{3\tan\theta - \tan^3\theta}{1-3\tan^2\theta} = \frac{3\left(\frac{1}{\sqrt{3}}\right) - \left(\frac{1}{\sqrt{3}}\right)^3}{1 - 3\left(\frac{1}{\sqrt{3}}\right)^2}$

$= \frac{\frac{3}{\sqrt{3}} - \frac{1}{3\sqrt{3}}}{1 - \cancel{3}\left(\frac{1}{\cancel{3}}\right)}$

$= \frac{\frac{\sqrt{3}\cdot\cancel{\sqrt{3}}}{\cancel{\sqrt{3}}} - \frac{1}{3\sqrt{3}}}{1-1}$

$= \frac{\sqrt{3} - \frac{1}{3\sqrt{3}}}{1-1}$

$= \frac{\sqrt{3} - \frac{1}{3\sqrt{3}}}{0} = \infty$

$=$ undefined ——— ②

From ① and ② we get ① = ②. $\therefore$ LHS = RHS

5

28. $(\sin\theta + \sec\theta)^2 + (\cos\theta + \mathrm{cosec}\,\theta)^2 = 1 + (\sec\theta + \mathrm{cosec}\,\theta)^2$

Proof: LHS.

$(\sin\theta + \sec\theta)^2 + (\cos\theta + \mathrm{cosec}\,\theta)^2$

$= [\sin^2\theta + \sec^2\theta + 2\sin\theta\cdot\sec\theta] + [\cos^2\theta + \mathrm{cosec}^2\theta + 2\cos\theta\cdot\mathrm{cosec}\,\theta]$

$= \sin^2\theta + \sec^2\theta + \cos^2\theta + \mathrm{cosec}^2\theta + 2\sin\theta\cdot\sec\theta + 2\cdot\cos\theta\cdot\mathrm{cosec}\,\theta$

$= (\sin^2\theta + \cos^2\theta) + \sec^2\theta + \mathrm{cosec}^2\theta + 2\sin\theta\frac{1}{\cos\theta} + 2\cdot\cos\theta\cdot\frac{1}{\sin\theta}$

$= 1 + \sec^2\theta + \mathrm{cosec}^2\theta + 2\cdot\frac{\sin\theta}{\cos\theta} + 2\cdot\frac{\cos\theta}{\sin\theta}$

$= 1 + \sec^2\theta + \mathrm{cosec}^2\theta + 2\left[\frac{\sin\theta}{\cos\theta} + \frac{\cos\theta}{\sin\theta}\right]$

$= 1 + \sec^2\theta + \mathrm{cosec}^2\theta + 2\left[\frac{\sin^2\theta + \cos^2\theta}{\cos\theta\cdot\sin\theta}\right]$

$= 1 + \sec^2\theta + \mathrm{cosec}^2\theta + 2\cdot\left[\frac{1}{\cos\theta\cdot\sin\theta}\right]$

$= 1 + \sec^2\theta + \mathrm{cosec}^2\theta + 2\cdot\left(\frac{1}{\cos\theta}\right)\cdot\left(\frac{1}{\sin\theta}\right)$

$= 1 + \sec^2\theta + \mathrm{cosec}^2\theta + 2\sec\theta\cdot\mathrm{cosec}\,\theta$

$= 1 + (\sec\theta + \mathrm{cosec}\,\theta)^2$

$\therefore \boxed{(\sin\theta + \sec\theta)^2 + (\cos\theta + \mathrm{cosec}\,\theta)^2 = 1 + (\sec\theta + \mathrm{cosec}\,\theta)^2}$

29. If $\frac{\cos\alpha}{\cos\beta} = m$ and $\frac{\cos\alpha}{\sin\beta} = n$.

To prove: $(m^2+n^2)\cos^2\beta = n^2$.

Given: $m = \frac{\cos\alpha}{\cos\beta}$. $n = \frac{\cos\alpha}{\sin\beta}$

$$\therefore\ m^2 = \frac{\cos^2\alpha}{\cos^2\beta}, \quad n^2 = \frac{\cos^2\alpha}{\sin^2\beta}.$$

LHS: $(m^2+n^2)\cos^2\beta$.

$$= \left[\frac{\cos^2\alpha}{\cos^2\beta} + \frac{\cos^2\alpha}{\sin^2\beta}\right]\cdot\cos^2\beta.$$

[$\cos^2\alpha$ is common. $\therefore$ Taking out $\cos^2\alpha$]

$$= \cos^2\alpha\left[\frac{1}{\cos^2\beta} + \frac{1}{\sin^2\beta}\right]\cdot\cos^2\beta.$$

$$= \cos^2\alpha\left[\frac{\sin^2\beta+\cos^2\beta}{\sin^2\beta\cdot\cancel{\cos^2\beta}}\right]\cdot\cancel{\cos^2\beta}$$

$$= \cos^2\alpha\left[\frac{1}{\sin^2\beta}\right] = \frac{\cos^2\alpha}{\sin^2\beta}$$

$$= n^2. \qquad \left[\because n^2 = \frac{\cos^2\alpha}{\sin^2\beta}\right]$$

$$\therefore \boxed{(m^2+n^2)\cos^2\beta = n^2}$$

30. Given: $\cot\theta + \tan\theta = x$

$\sec\theta - \cos\theta = y$.

To prove: $(x^2y)^{2/3} - (xy^2)^{2/3} = 1$

$$\therefore x = \cot\theta + \tan\theta$$

$$x = \frac{\cos\theta}{\sin\theta} + \frac{\sin\theta}{\cos\theta}$$

$$x = \frac{\cos^2\theta+\sin^2\theta}{\sin\theta\cdot\cos\theta}$$

$$y = \sec\theta - \cos\theta.$$

$$y = \frac{1}{\cos\theta} - \cos\theta$$

$$= \frac{1-\cos^2\theta}{\cos\theta}$$

$$\boxed{x = \frac{1}{\sin\theta\cdot\cos\theta}}$$

$$x^2y = \frac{1}{\sin^2\theta\cdot\cos^2\theta} \times \frac{\sin^2\theta}{\cos\theta}$$

$$x^2y = \frac{1}{\cancel{\sin^2\theta}\cdot\cos^2\theta} \times \frac{\cancel{\sin^2\theta}}{\cos\theta}$$

$$x^2y = \frac{1}{\cos^3\theta}$$

$$(x^2y)^{2/3} = \left(\frac{1}{\cos^3\theta}\right)^{2/3}$$

$$\boxed{(x^2y)^{2/3} = \frac{1}{\cos^2\theta}}$$

$$\boxed{y = \frac{\sin^2\theta}{\cos\theta}.}$$

$$xy^2 = \frac{1}{\sin\theta\cdot\cos\theta} \times \left[\frac{\sin^2\theta}{\cos\theta}\right]^2$$

$$xy^2 = \frac{1}{\sin\theta\cdot\cos\theta} \times \frac{\sin^4\theta}{\cos^2\theta}$$

$$= \frac{1}{\cancel{\sin\theta}\cdot\cos\theta} \times \frac{\cancel{\sin\theta}\times\sin^3\theta}{\cos^3\theta}$$

$$xy^2 = \frac{\sin^3\theta}{\cos^3\theta}$$

$$(xy^2)^{2/3} = \left(\frac{\sin^3\theta}{\cos^3\theta}\right)^{2/3}$$

$$\boxed{(xy^2)^{2/3} = \frac{\sin^2\theta}{\cos^2\theta}}$$

$$\left(\frac{\sin^3\theta}{\cos^3\theta}\right)^{2/3} = \frac{\sin^{\cancel{3}\times 2/\cancel{3}}\theta}{\cos^{\cancel{3}\times 2/\cancel{3}}\theta} = \frac{\sin^2\theta}{\cos^2\theta}.$$

$$(a^3)^{2/3} = a^{\cancel{3}\times 2/\cancel{3}} = a^2$$

$$(x^2y)^{2/3} - (xy^2)^{2/3} = \frac{1}{\cos^2\theta} - \frac{\sin^2\theta}{\cos^2\theta}$$

$$= \frac{1-\sin^2\theta}{\cos^2\theta} = \frac{\cancel{\cos^2\theta}}{\cancel{\cos^2\theta}} = 1.$$

$$\boxed{(x^2y)^{2/3} - (xy^2)^{2/3} = 1}$$

U.KARTHIKKUMAR
M.A.,M.Sc.,M.Ed.,(MATHS),
P.G.TEACHER IN MATHS.,
UNIVERSAL MAT.HR.SEC.SCHOOL.,
TIRUPUR - 64166-.

(31) Given: $\frac{1}{a} = \frac{\cos\theta}{1+\sin\theta}$

To prove: $\frac{a^2-1}{a^2+1} = \sin\theta$.

$\therefore \boxed{a = \frac{1+\sin\theta}{\cos\theta}}$

$\therefore a^2 = \left[\frac{1+\sin\theta}{\cos\theta}\right]^2 = \frac{(1+\sin\theta)^2}{\cos^2\theta}$.

$a^2 = \frac{1+\sin^2\theta+2\sin\theta}{\cos^2\theta}$

$\therefore \boxed{a^2 = \frac{\sin^2\theta+2\sin\theta+1}{\cos^2\theta}}$

$a^2-1 \Rightarrow$

$= \frac{\sin^2\theta+2\sin\theta+1}{\cos^2\theta} - 1$

$= \frac{\sin^2\theta+2\sin\theta+1-\cos^2\theta}{\cos^2\theta}$

$= \frac{\sin^2\theta+2\sin\theta+(1-\cos^2\theta)}{\cos^2\theta}$

$= \frac{\sin^2\theta+2\sin\theta+\sin^2\theta}{\cos^2\theta}$

$= \frac{2\sin^2\theta+2\sin\theta}{\cos^2\theta}$

$\therefore \boxed{a^2-1 = \frac{2\sin^2\theta+2\sin\theta}{\cos^2\theta}}$

$a^2+1 \Rightarrow$

$= \frac{\sin^2\theta+2\sin\theta+1}{\cos^2\theta} + 1$

$= \frac{\sin^2\theta+2\sin\theta+1+\cos^2\theta}{\cos^2\theta}$

$= \frac{\sin^2\theta+\cos^2\theta+2\sin\theta+1}{\cos^2\theta}$

$= \frac{1+2\sin\theta+1}{\cos^2\theta}$

$= \frac{2\sin\theta+2}{\cos^2\theta}$

$\boxed{a^2+1 = \frac{2\sin\theta+2}{\cos^2\theta}}$

To find:

$\frac{a^2-1}{a^2+1} = \frac{\frac{2\sin^2\theta+2\sin\theta}{\cancel{\cos^2\theta}}}{\frac{2\sin\theta+2}{\cancel{\cos^2\theta}}} = \frac{2\sin^2\theta+2\sin\theta}{2\sin\theta+2}$

$\frac{a^2-1}{a^2+1} = \frac{\cancel{2}\sin\theta\,\cancel{(\sin\theta+1)}}{\cancel{2}\,\cancel{(\sin\theta+1)}}$

[2 and $\sin\theta$ is common. Taking out 2 and $\sin\theta$ we get.

$\therefore \boxed{\frac{a^2-1}{a^2+1} = \sin\theta}$

Another method:

Given $\frac{1}{a} = \frac{\cos\theta}{1+\sin\theta}$

$\therefore a = \frac{1+\sin\theta}{\cos\theta} = \frac{1}{\cos\theta} + \frac{\sin\theta}{\cos\theta}$.

$\boxed{a = \sec\theta+\tan\theta}$

To find:

$\frac{a^2-1}{a^2+1} = \frac{(\sec\theta+\tan\theta)^2-1}{(\sec\theta+\tan\theta)^2+1}$ [$(a+b)^2 = a^2+b^2+2ab$

$= \frac{\sec^2\theta+\tan^2\theta+2\sec\theta.\tan\theta-1}{\sec^2\theta+\tan^2\theta+2\sec\theta.\tan\theta+1}$

$= \frac{(\sec^2\theta-1)+\tan^2\theta+2\sec\theta.\tan\theta}{\sec^2\theta+(\tan^2\theta+1)+2\sec\theta.\tan\theta}$

6

$$= \frac{\tan^2\theta + \tan^2\theta + 2\sec\theta \cdot \tan\theta}{\sec^2\theta + \sec^2\theta + 2\sec\theta \cdot \tan\theta}$$

$$\left[\begin{array}{l} \sec^2\theta - 1 = \tan^2\theta \\ 1 + \tan^2\theta = \sec^2\theta \end{array}\right]$$

$$= \frac{2\tan^2\theta + 2\sec\theta \cdot \tan\theta}{2\sec^2\theta + 2\sec\theta \cdot \tan\theta}$$

$$= \frac{\cancel{2}\tan\theta\,\cancel{(\tan\theta + \sec\theta)}}{\cancel{2}\sec\theta\,\cancel{(\sec\theta + \tan\theta)}}$$

$$= \frac{\tan\theta}{\sec\theta} = \frac{\frac{\sin\theta}{\cos\theta}}{\frac{1}{\cos\theta}}$$

$$= \frac{\sin\theta}{\cancel{\cos\theta}} \times \frac{\cancel{\cos\theta}}{1}$$

$$\boxed{\frac{a^2 - 1}{a^2 + 1} = \sin\theta}$$

(32). Given:

$p = \sin\theta + \cos\theta$, $q = \sec\theta + \csc\theta$.

To prove: $q(p^2 - 1) = 2p$.

$\boxed{p = \sin\theta + \cos\theta}$ (squaring both sides we get)

$p^2 = (\sin\theta + \cos\theta)^2$

$\therefore p^2 = \sin^2\theta + \cos^2\theta + 2\sin\theta \cdot \cos\theta$.

$p^2 = 1 + 2\sin\theta\cos\theta$

$$\boxed{p^2 - 1 = 2\sin\theta \cdot \cos\theta}$$

$$q = \sec\theta + \csc\theta = \frac{1}{\cos\theta} + \frac{1}{\sin\theta}$$

$$\therefore \boxed{q = \frac{\sin\theta + \cos\theta}{\sin\theta \cdot \cos\theta}}$$

$\therefore$ LHS:

$$q(p^2 - 1) = \frac{\sin\theta + \cos\theta}{\cancel{\sin\theta} \cdot \cancel{\cos\theta}} \times 2\cancel{\sin\theta} \cdot \cancel{\cos\theta}$$

$$= p \times 2 \qquad (\because p = \sin\theta + \cos\theta)$$

$$\boxed{q(p^2 - 1) = 2p}$$

(33). Given:

$$p = \frac{\cos^2\theta}{\sin\theta}, \quad q = \frac{\sin^2\theta}{\cos\theta}$$

To prove: $p^2 q^2 (p^2 + q^2 + 3) = 1$.

LHS: $p^2 q^2 (p^2 + q^2 + 3)$

$$= \left(\frac{\cos^2\theta}{\sin\theta}\right)^2 \left(\frac{\sin^2\theta}{\cos\theta}\right)^2 \left[\left(\frac{\cos^2\theta}{\sin\theta}\right)^2 + \left(\frac{\sin^2\theta}{\cos\theta}\right)^2 + 3\right]$$

$$= \left[\frac{\cos^4\theta}{\sin^2\theta} \times \frac{\sin^4\theta}{\cos^2\theta}\right] \times \left[\frac{\cos^4\theta}{\sin^2\theta} + \frac{\sin^4\theta}{\cos^2\theta} + 3\right]$$

$$= \left[\frac{\cos^2\theta \times \cancel{\cos^2\theta}}{\cancel{\sin^2\theta}} \times \frac{\cancel{\sin^2\theta} \times \sin^2\theta}{\cancel{\cos^2\theta}}\right] \times \left[\frac{\cos^6\theta + \sin^6\theta}{\sin^2\theta \cdot \cos^2\theta} + 3\right]$$

$$= (\cancel{\cos^2\theta} \times \cancel{\sin^2\theta}) \times \left[\frac{\cos^6\theta + \sin^6\theta + 3\sin^2\theta \cdot \cos^2\theta}{\cancel{\sin^2\theta} \cdot \cancel{\cos^2\theta}}\right]$$

$= \cos^6\theta + \sin^6\theta + 3\sin^2\theta\cdot\cos^2\theta$

$= (\cos^2\theta)^3 + (\sin^2\theta)^3 + 3\sin^2\theta\cos^2\theta$

$\left[a^3+b^3 = (a+b)^3 - 3ab(a+b)\right.$ Here $a = \cos^2\theta$, $b = \sin^2\theta$.

$= (\cos^2\theta+\sin^2\theta)^3 - 3\cos^2\theta\sin^2\theta(\cos^2\theta+\sin^2\theta) + 3\sin^2\theta\cdot\cos^2\theta$

$= (1)^3 - 3\cos^2\theta\cdot\sin^2\theta(1) + 3\cos^2\theta\cdot\sin^2\theta$

$= 1 - \cancel{3\cos^2\theta\cdot\sin^2\theta} + \cancel{3\cos^2\theta\sin^2\theta}$

$= 1$

$\therefore \boxed{p^2q^2(p^2+q^2+3) = 1}$

(34) If $\operatorname{cosec}\theta + \cot\theta = p$ then prove that $\cos\theta = \frac{p^2-1}{p^2+1}$

Sol: Given $\boxed{\operatorname{cosec}\theta + \cot\theta = p}$ —— (1).

We know that

$\operatorname{cosec}^2\theta - \cot^2\theta = 1$ $\quad \left[a^2-b^2 = (a+b)(a-b)\right.$

$(\operatorname{cosec}\theta+\cot\theta)(\operatorname{cosec}\theta-\cot\theta) = 1$.

$p \times (\operatorname{cosec}\theta - \cot\theta) = 1$.

$\therefore \boxed{\operatorname{cosec}\theta - \cot\theta = \frac{1}{p}}$ —— (2).

Adding (1)+(2) weget

$\operatorname{cosec}\theta + \cancel{\cot\theta} = p$

$\operatorname{cosec}\theta - \cancel{\cot\theta} = \frac{1}{p}$ (+)

$2\operatorname{cosec}\theta = p + \frac{1}{p}$

$\boxed{2\operatorname{cosec}\theta = \frac{p^2+1}{p}}$ —— (3).

Sub. (1) − (2) weget.

$\cancel{\operatorname{cosec}\theta} + \cot\theta = p$

$\cancel{\operatorname{cosec}\theta} - \cot\theta = \frac{1}{p}$

$(-) \quad (+) \quad (-)$

$2\cot\theta = p - \frac{1}{p}$

$\boxed{2\cot\theta = \frac{p^2-1}{p}}$ —— (4)

Divide (4) ÷ (3) $\Longrightarrow$

$\frac{(4)}{(3)} \Rightarrow \frac{2\cot\theta}{2\operatorname{cosec}\theta} = \frac{\frac{p^2-1}{p}}{\frac{p^2+1}{p}} = \frac{p^2-1}{\cancel{p}} \times \frac{\cancel{p}}{p^2+1}$

$\frac{\cot\theta}{\operatorname{cosec}\theta} = \frac{p^2-1}{p^2+1}$

$\frac{\frac{\cos\theta}{\sin\theta}}{\frac{1}{\sin\theta}} = \frac{p^2-1}{p^2+1} \Rightarrow \frac{\cos\theta}{\cancel{\sin\theta}} \times \frac{\cancel{\sin\theta}}{1} = \frac{p^2-1}{p^2+1}$

$\Rightarrow \boxed{\cos\theta = \frac{p^2-1}{p^2+1}}$

7

(35) Prove that

$$\frac{\sin A}{\sec A+\tan A-1}+\frac{\cos A}{\operatorname{cosec} A+\cot A-1}=1$$

Proof: LHS:-

$$=\frac{\sin A}{\sec A+\tan A-1}+\frac{\cos A}{\operatorname{cosec} A+\cot A-1}$$

$$=\frac{\sin A}{\frac{1}{\cos A}+\frac{\sin A}{\cos A}-1}+\frac{\cos A}{\frac{1}{\sin A}+\frac{\cos A}{\sin A}-1}$$

(Take $\sin A=x$ and $\cos A=y$)

$$=\frac{x}{\frac{1}{y}+\frac{x}{y}-1}+\frac{y}{\frac{1}{x}+\frac{y}{x}-1}$$

$\left[\frac{1}{y}+\frac{x}{y}=\frac{1+x}{y}\right.$, $\left.\frac{1}{x}+\frac{y}{x}=\frac{1+y}{x}\right]$

$$=\frac{x}{\frac{1+x}{y}-1}+\frac{y}{\frac{1+y}{x}-1}$$

$$=\frac{x}{\frac{1+x-y}{y}}+\frac{y}{\frac{1+y-x}{x}}$$

$\left[\frac{a}{b/c}=a\times\frac{c}{b}\right]$

$$=\frac{xy}{1+x-y}+\frac{xy}{1+y-x}=xy\left[\frac{1}{1+x-y}+\frac{1}{1+y-x}\right]$$

($\because xy$ is common. $\therefore$ Taking out xy)

$$=xy\left[\frac{(1+y-x)+(1+x-y)}{(1+x-y)(1+y-x)}\right]$$

$\left[\frac{1}{a}+\frac{1}{b}=\frac{b+a}{ab}\right]$

$$=xy\left[\frac{1+\cancel{y}-\cancel{x}+1+\cancel{x}-\cancel{y}}{1+\cancel{y}-\cancel{x}+\cancel{x}+xy-x^2-\cancel{y}-y^2+xy}\right]$$

$$=xy\left[\frac{2}{1+2xy-x^2-y^2}\right]=\frac{2xy}{1+2xy-(x^2+y^2)}$$

$$=\frac{2xy}{\cancel{1}+2xy-\cancel{1}}$$

$x=\sin A$, $y=\cos A$

$\therefore x^2+y^2=(\sin A)^2+(\cos A)^2$

$=\sin^2A+\cos^2A$

$=1$

$\therefore \boxed{x^2+y^2=1}$

$$=\frac{\cancel{2xy}}{\cancel{2xy}}$$

$$=1$$

$$\boxed{\frac{\sin A}{\sec A+\tan A-1}+\frac{\cos A}{\operatorname{cosec} A+\cot A-1}=1}$$

(36). Prove that:

$$\frac{(1+\cot A+\tan A)(\sin A-\cos A)}{\sec^3A-\operatorname{cosec}^3A}=\sin^2A\cdot\cos^2A.$$

Proof: LHS:

$$=\frac{(1+\cot A+\tan A)(\sin A-\cos A)}{\sec^3A-\operatorname{cosec}^3A}$$

$$=\frac{\left(1+\frac{\cos A}{\sin A}+\frac{\sin A}{\cos A}\right)(\sin A-\cos A)}{\frac{1}{\cos^3A}-\frac{1}{\sin^3A}}$$

Take $\sin A=x$ and $\cos A=y$ we get.

$$=\frac{\left(1+\frac{y}{x}+\frac{x}{y}\right)(x-y)}{\frac{1}{y^3}-\frac{1}{x^3}}$$

$$= \frac{\left(1+\frac{y^2+x^2}{xy}\right)(x-y)}{\frac{x^3-y^3}{x^3y^3}}$$

$$= \frac{\left(\frac{xy+y^2+x^2}{xy}\right)(x-y)}{\frac{x^3-y^3}{x^3y^3}} = \frac{\frac{(xy+y^2+x^2)(x-y)}{xy}}{\frac{x^3-y^3}{x^3y^3}}$$

$$= \frac{(xy+y^2+x^2)(x-y)}{xy} \times \frac{x^3y^3}{x^3-y^3}$$

$$= \frac{\cancel{(xy+y^2+x^2)}\cancel{(x-y)} \times \cancel{(xy)}(x^2y^2)}{\cancel{xy} \times \cancel{(x-y)}\cancel{(x^2+y^2+xy)}}$$

$$= x^2y^2$$

$[\because x = \sin A,\ y = \cos A$
$\therefore x^2y^2 = \sin^2 A \cdot \cos^2 A$

$$= (\sin A)^2 \cdot (\cos A)^2$$

$$= \sin^2 A \cdot \cos^2 A.$$

$\boxed{LHS = RHS}$

(37). Given: $\sin\theta(1+\sin^2\theta) = \cos^2\theta$.

To prove: $\cos^6\theta - 4\cos^4\theta + 8\cos^2\theta = 4$.

LHS: $\Rightarrow$

Take $\sin\theta = x$ and $\cos\theta = y$ weget.

$\therefore \sin\theta(1+\sin^2\theta) = \cos^2\theta$

$= x(1+x^2) = y^2 \Rightarrow x[1+(1-y^2)] = y^2$

Take square on both sides weget

$[\because x^2+y^2 = 1.$
$x^2 = 1-y^2$
$y^2 = 1-x^2$

$$x^2[1+(1-y^2)]^2 = (y^2)^2$$

$$x^2[1+1-y^2]^2 = y^4$$

$$x^2(2-y^2)^2 = y^4$$

$$x^2[2^2+(y^2)^2 - 2(2)(y^2)] = y^4$$

$$x^2[4+y^4-4y^2] = y^4.$$

$$(1-y^2)[4+y^4-4y^2] = y^4$$

$$\cancel{y^4} - 4y^2 + 4 - y^6 + 4y^4 - 4y^2 = \cancel{y^4}$$

$$4 - y^6 + 4y^4 - 8y^2 = 0$$

$$-y^6 + 4y^4 - 8y^2 = -4$$

Divide by (−) on both sides weget.

$$y^6 - 4y^4 + 8y^2 = 4$$

$\therefore \boxed{\cos^6\theta - 4\cos^4\theta + 8\cos^2\theta = 4}$ $[\because y = \cos\theta]$

U. Karthik 08/10/19

U.KARTHIKKUMAR
M.A.,M.Sc.,M.Ed.,(MATHS),
P.G.TEACHER IN MATHS.,
UNIVERSAL MAT.HR.SEC.SCHOOL.,
TIRUPUR - 641664.

38. Example (6.12) Another method:

Prove that: $\tan^2 A - \tan^2 B = \dfrac{\sin^2 A - \sin^2 B}{\cos^2 A \cdot \cos^2 B}$.

Proof: LHS:

$$\tan^2 A - \tan^2 B = (\sec^2 A - 1) - (\sec^2 B - 1)$$
$$= \sec^2 A - \cancel{1} - \sec^2 B + \cancel{1}$$
$$= \sec^2 A - \sec^2 B.$$
$$= \frac{1}{\cos^2 A} - \frac{1}{\cos^2 B} = \frac{\cos^2 B - \cos^2 A}{\cos^2 A \cdot \cos^2 B}$$
$$= \frac{(1-\sin^2 B) - (1-\sin^2 A)}{\cos^2 A \cdot \cos^2 B}.$$
$$= \frac{\cancel{1} - \sin^2 B - \cancel{1} + \sin^2 A}{\cos^2 A \cdot \cos^2 B}$$

$$\boxed{\tan^2 A - \tan^2 B = \frac{\sin^2 A - \sin^2 B}{\cos^2 A \cdot \cos^2 B}}$$

7 ⇔ 39 Another method (Easy way method).

prove that $\sqrt{\dfrac{1+\sin\theta}{1-\sin\theta}} + \sqrt{\dfrac{1-\sin\theta}{1+\sin\theta}} = 2\sec\theta$.

Proof: LHS.

$$\Rightarrow \sqrt{\frac{1+\sin\theta}{1-\sin\theta}} + \sqrt{\frac{1-\sin\theta}{1+\sin\theta}} = \frac{\sqrt{1+\sin\theta}}{\sqrt{1-\sin\theta}} + \frac{\sqrt{1-\sin\theta}}{\sqrt{1+\sin\theta}}$$
$$= \frac{\sqrt{1+\sin\theta}\cdot\sqrt{1+\sin\theta} + \sqrt{1-\sin\theta}\cdot\sqrt{1-\sin\theta}}{\sqrt{1+\sin\theta}\cdot\sqrt{1-\sin\theta}}$$
$$= \frac{\sqrt{(1+\sin\theta)^2} + \sqrt{(1-\sin\theta)^2}}{\sqrt{(1+\sin\theta)(1-\sin\theta)}} = \frac{\sqrt{(1+\sin\theta)^2} + \sqrt{(1-\sin\theta)^2}}{\sqrt{1-\sin^2\theta}}$$
$$= \frac{1+\cancel{\sin\theta} + 1 - \cancel{\sin\theta}}{\sqrt{\cos^2\theta}} = \frac{2}{\cos\theta} = 2 \times \frac{1}{\cos\theta}$$
$$= 2\sec\theta.$$

$$\boxed{\therefore \sqrt{\frac{1+\sin\theta}{1-\sin\theta}} + \sqrt{\frac{1-\sin\theta}{1+\sin\theta}} = 2\sec\theta.}$$

40. Prove that $\sqrt{\dfrac{1+\cos\theta}{1-\cos\theta}} = \operatorname{cosec}\theta + \cot\theta$.

LHS: $\sqrt{\dfrac{1+\cos\theta}{1-\cos\theta}} = \sqrt{\dfrac{1+y}{1-y}} \Rightarrow$ [Take $x = \sin\theta$, $y = \cos\theta$]

$$= \sqrt{\frac{(1+y)}{(1-y)} \times \frac{(1+y)}{(1+y)}} \quad \text{[conjugate]}$$
$$= \sqrt{\frac{(1+y)^2}{1-y^2}} = \sqrt{\frac{(1+y)^2}{x^2}}$$
$$= \sqrt{\left(\frac{1+y}{x}\right)^{\cancel{2}}} = \frac{1+y}{x}$$
$$= \frac{1}{x} + \frac{y}{x} \quad \left[\because x^2+y^2 = 1.\ \therefore x^2 = 1-y^2,\ x = \sin\theta,\ y = \cos\theta\right]$$
$$= \frac{1}{\sin\theta} + \frac{\cos\theta}{\sin\theta}$$

$$\sqrt{\frac{1+\cos\theta}{1-\cos\theta}} = \operatorname{cosec}\theta + \cot\theta$$

U. Karthik 08/10/19
U.KARTHIKKUMAR
M.A.,M.Sc.,M.Ed.,(MATHS),
P.G.TEACHER IN MATHS.,
UNIVERSAL MAT.HR.SEC.SCHOOL.,
TIRUPUR - 641664.