PROFILE

MODEL PAPER - III

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HIGHER SECONDARY SECOND YEAR

MATHEMATICS

MODEL QUESTION PAPER – 3

Time Allowed: 15 Min + 3.00 Hours] **[Maximum Marks:**

Instructions: (a) Check the question paper for fairness of printing. If there is any lack of fairness, inform the Hall Supervisor immediately.

(b) Use **Blue** or **Black** ink to write and underline and pencil to draw diagrams.

PART – I

$20\times1=2$

Note: (i) **All** questions are **compulsory.**

(ii) Choose the correct or most suitable answer from the given **four** alternatives. Write the option code and the corresponding answer.

1. If $A=\begin{bmatrix}\cos\theta & \sin\theta\\ -\sin\theta & \cos\theta\end{bmatrix}$ and $A(\text{adj}\,A)=\begin{bmatrix}k & 0\\ 0 & k\end{bmatrix}$, then $k=$

(1) 0 (2) $\sin\theta$ (3) $\cos\theta$ (4) 1

2. If ρ represents the rank and, A and B are $n\times n$ matrices, then

(1) $\rho(A+B)=\rho(A)+\rho(B)$ (2) $\rho(AB)=\rho(A)\rho(B)$

(3) $\rho(A-B)=\rho(A)-\rho(B)$ (4) $\rho(A+B)\le n$

3. The conjugate of a complex number is $\frac{1}{i-2}$. Then the complex number is

(1) $\frac{1}{i+2}$ (2) $\frac{-1}{i+2}$ (3) $\frac{-1}{i-2}$ (4) $\frac{1}{i-2}$

4. Which of the following are incorrect?

(i) $(\cos\theta+i\sin\theta)^m=\cos m\theta+i\sin m\theta$ if m is a negative integer

(ii) $(\sin\theta+i\cos\theta)^n=\cos n\left(\frac{\pi}{2}-\theta\right)+i\sin n\left(\frac{\pi}{2}-\theta\right)$

(iii) $(\cos\theta-i\sin\theta)^{-m}=\cos m\theta+i\sin m\theta$ if m is a negative integer

(iv) $(\cos\theta-i\sin\theta)^n=\cos n\theta-i\sin n\theta$

(1) none (2) (i) and (iv) (3) (i) and (ii) (4) (iii) and (iv)

5. If $\frac{p}{q}$ (where p and q are co-primes), is a root of a polynomial equation $a_n x^n + a_{n-1}x^{n-1} + \cdots + a_0 = 0$, then identify the correct option.

Statement A : p is a factor of a_0 and q is a factor of a_n.

Statement B : q is a factor of a_0 and p is a factor of a_n.

(1) both are not true (2) both are true

(3) A is correct but B is false (4) A is incorrect but B is correct

6. The domain of the function which is defined by $f(x) = \sin^{-1}\sqrt{x-1}$, is

(1) $[1, 2]$ (2) $[-1, 1]$ (3) $[0, 1]$ (4) $[-1, 0]$

7. The equation of the circle passing through the foci of the ellipse $\frac{x^2}{16} + \frac{y^2}{9} = 1$ and having centre at $(0,3)$ is

(1) $x^2 + y^2 - 6y - 7 = 0$ (2) $x^2 + y^2 - 6y + 7 = 0$

(3) $x^2 + y^2 - 6y - 5 = 0$ (4) $x^2 + y^2 - 6y + 5 = 0$

8. The locus of a point whose distance from $(-2,0)$ is $\frac{2}{3}$ times its distance from the line $x = -\frac{9}{2}$ is

(1) a parabola (2) a hyperbola (3) an ellipse (4) a circle

9. If $\vec{a} \times (\vec{b} \times \vec{c}) = (\vec{a} \times \vec{b}) \times \vec{c}$, where $\vec{a}, \vec{b}, \vec{c}$ are any three vectors such that $\vec{b} \cdot \vec{c} \neq 0$ and $\vec{a} \cdot \vec{b} \neq 0$, then $\vec{a}$ and $\vec{c}$ are

(1) perpendicular (2) parallel

(3) inclinded at an angle $\frac{\pi}{3}$ (4) inclined at an angle $\frac{\pi}{6}$

10. The shortest distance between the two skew lines $\vec{r} = \vec{a} + t\vec{u}$ and $\vec{r} = \vec{b} + t\vec{v}$ is

(1) $\frac{\left|(\vec{b}-\vec{a})\cdot(\vec{u}\times\vec{v})\right|}{|\vec{u}\times\vec{v}|}$ (2) $\frac{\left|(\vec{b}-\vec{a})\cdot(\vec{u}\times\vec{v})\right|}{\vec{u}\times\vec{v}}$

(3) $\frac{\left|(\vec{b}-\vec{a})\cdot(\vec{u}\times\vec{v})\right|}{|\vec{a}\times\vec{b}|}$ (4) $\frac{\left|(\vec{b}-\vec{a})\cdot(\vec{u}\times\vec{v})\right|}{|\vec{a}|}$

11. The number given by the Mean value theorem for the function $\frac{1}{x}, x \in [1,9]$ is

(1) 2 (2) 2.5 (3) 3 (4) 3.5

12. "If $f(x)$ is continuous on $[a,b]$ then f has both absolute maximum and absolute minimum [a b]". This statement is

(1) Extreme value theorem (2) Intermediate value theorem

(3) Lagrange mean value theorem (4) Taylors theorem

13. The approximate change in the volume V of a cube of side x metres caused by increasing the side by 1% is

(1) $0.3xdxm^3$ (2) $0.03xm^3$ (3) $0.03x^2m^3$ (4) $0.03x^3m^3$

14. The volume of solid of revolution of the region bounded by $y^2 = x(a-x)$ about x-axis is

(1) πa^3 (2) $\frac{\pi a^3}{4}$ (3) $\frac{\pi a^3}{5}$ (4) $\frac{\pi a^3}{6}$

15. If $f(2a-x) = -f(x)$ then $\int_0^{2a} f(x)dx =$

(1) $2\int_0^a f(x)dx$ (2) $\int_{-a}^a f(x)dx$ (3) 0 (4) $\int_0^a f(x)dx$

16. If p and q are the order and degree of the differential equation $y\frac{dy}{dx} + x^3\left(\frac{d^2y}{dx^2}\right) + xy = \cos x$ then

(1) $p<q$ (2) $p=q$ (3) $p>q$ (4) none of these

17. The general solution of the differential equation $\frac{dy}{dx} = \frac{y}{x}$ is

(1) $xy=k$ (2) $y = k\log x$ (3) $y = kx$ (4) $\log y = kx$

18. A random variable X has binomial distribution with $n=25$ and $p=0.8$ then standard deviation of X is

(1) 6 (2) 4 (3) 3 (4) 2

19. The probability function of a random variable is defined as :

x	−2	−1	0	1	2
$f(x)$	k	$2k$	$3k$	$4k$	$5k$

Then $E(X)$ is equal to :

(1) $\frac{1}{15}$ (2) $\frac{1}{10}$ (3) $\frac{1}{3}$ (4) $\frac{2}{3}$

20. The operation $*$ is defined by $a*b = \frac{ab}{7}$. It is not a binary operation on

(1) $\mathbb{Q}^+$ (2) $\mathbb{Z}$ (3) $\mathbb{R}$ (4) $\mathbb{C}$

PART – II

Note: (i) Answer any **SEVEN** questions. $7\times2=14$

(ii) Question number **30** is compulsory.

21. If A is a nonsingular matrix of odd order, prove that $|\text{adj } A|$ is positive.

22. If $\frac{z+3}{z-5i}=\frac{1+4i}{2}$, find the complex number z in the rectangular form.

23. State Fermat's theorem.

24. Examine the position of the point $(2,3)$ with respect to the circle $x^2+y^2-6x-8y+12=0$.

25. Find the differential dy if $y=e^{x^2-5x+7}\cos(x^2-1)$.

26. Evaluate $\int_0^{\frac{\pi}{2}}\left(\sin^2 x+\cos^4 x\right)dx$

27. Solve $\frac{dy}{dx}+2y=e^{-x}$.

28. If $E(X)=2$ then find $E(5X+7)$.

29. Let $A=\left\{x-\sqrt{3}y : x,y\in\mathbb{Z}\right\}$. Check whether the usual addition is a binary operation on A.

30. Find the value of $\sin^{-1}\left(-\frac{1}{2}\right)+\sec^{-1}(2)$.

PART– III

Note: (i) Answer any **SEVEN** questions. $7\times3=21$

(ii) Question number **40** is compulsory.

31. If $A=\frac{1}{9}\begin{bmatrix}-8 & 1 & 4\\ 4 & 4 & 7\\ 1 & -8 & 4\end{bmatrix}$, prove that $A^{-1}=A^T$.

32. If $\frac{1+z}{1-z}=\cos 2\theta+i\sin 2\theta$, show that $z=i\tan\theta$.

33. Show that the equation $x^9-5x^5+4x^4+2x^2+1=0$ has atleast 6 imaginary solutions.

34. Solve : $\tan^{-1}2x+\tan^{-1}3x=\frac{\pi}{4}$. Here $6x^2<1$.

35. Find the of vector (parametric) and Cartesian equations of the straight line passing through the point $(-2,3,4)$ and parallel to the straight line $\frac{x-1}{-4}=\frac{y+3}{5}=\frac{8-z}{6}$.

36. Find the equation of a circle whose centre is $(3,2)$ and the radius is the radius of $x^2+y^2+2x+4y-4=0$.

37. Find the volume of the solid generated by revolving the region bounded by the parabola $x=y^2+1$, the y-axis, and the lines $y=1$ and $y=-1$, about y-axis.

38. If α, β and γ are the roots of the cubic equation $x^3+2x^2+3x+4=0$, form a cubic equatio[n] whose roots are $-\alpha, -\beta, -\gamma$

39. If X is the random variable with distribution function $F(x)$ given by,

$$F(x)=\begin{cases}0, & x<0\\ \frac{1}{2}(x^2+x) & 0\le x<1\\ 1, & x\ge 1\end{cases}$$

then find $P[0.3 \le x \le 0.6]$.

40. Evaluate $\int_2^3 \frac{\sqrt{x}}{\sqrt{5-x}+\sqrt{x}}dx$.

PART – IV

$7\times5=3$[5]

Note: Answer **all** the questions.

41. (a) Investigate the values of λ and μ for the system of linear equations $2x+3y+5z=9$, $7x+3y-5z=8$, $2x+3y+\lambda z=\mu$, to have

(i) no solution (ii) a unique solution (iii) infinite number of solutions.

(OR)

(b) Find the area of the region bounded between the curves $y=\sin x$, $y=\cos x$ and $x=0$ a[nd] the lines $x=0$ and $x=\pi$.

42. (a) Solve the equation $z^5+z^3-z^2-1=0$.

(OR)

(b) Write any five points to sketch the function $y=-\frac{1}{3}(x^3-3x+2)$.

43. (a) Find the centre, foci, vertices of the hyperbola $\frac{(x+3)^2}{225}-\frac{(y-4)^2}{64}=1$

(OR)

(b) Prove that $\cos(\alpha-\beta)=\cos\alpha\cos\beta+\sin\alpha\sin\beta$, by using vectors.

44. (a) Draw the curves $\sin x$ and $\sin^{-1}x$ in the respective principal domain.

(OR)

(b) The equation of electromotive force for an electric circuit containing resistance a[nd] self-inductance is $E=Ri+L\frac{di}{dt}$, where E is the electromotive force given to the circuit, [R] the resistance and L, the coefficient of induction. Find the current i at time t when E [=] 0.

45. (a) Show that $\neg(p \leftrightarrow q) \equiv p \leftrightarrow \neg q$.

(OR)

(b) Let $z(x,y) = x^3 - 3x^2y^3$, where $x = se^t, y = se^{-t}, s,t \in \mathbb{R}$. Find $\frac{\partial z}{\partial s}$ and $\frac{\partial z}{\partial t}$.

46. (a) On lighting a rocket cracker it gets projected in a parabolic path and reaches a maximum height of 4m when it is 6m away from the point of projection. Find the angle of projection at the starting point.

(OR)

(b) Find the intervals of increasing, decreasing, concavity and the point of inflection of the curve $y = x^3 - 3x$.

47. (a) Find the non-parametric form of vector equation, and Cartesian equation of the plane $\vec{r} = \left(6\hat{i} - \hat{j} + \hat{k}\right) + s\left(-\hat{i} + 2\hat{j} + \hat{k}\right) + t\left(-5\hat{i} - 4\hat{j} - 5\hat{k}\right)$.

(OR)

(b) The mean and variance of a binomial variate X are respectively 2 and 1.5. Find (i) $P(X = 0)$ (ii) $P(X = 1)$ (iii) $P(X \geq 1)$

MODEL QUESTION PAPER – III

ANSWERS

PART – I

Qn. No.	Option	Answer	Qn. No.	Option	Answer
1.	(4)	1	11.	(3)	3
2.	(4)	$\rho(A+B)\le n$	12.	(1)	Extreme value theorem
3.	(2)	$\frac{-1}{i+2}$	13.	(4)	$0.03x^3\ m^3$
4.	(1)	none	14.	(4)	$\frac{\pi a^3}{6}$
5.	(3)	A is correct but B is false	15.	(3)	0
6.	(1)	$[1,2]$	16.	(3)	$p>q$
7.	(1)	$x^2+y^2-6y-7=0$	17.	(3)	$y=kx$
8.	(3)	an ellipse	18.	(4)	2
9.	(2)	Parallel	19.	(4)	$\frac{2}{3}$
10.	(1)	$\frac{\left\|(\vec{b}-\vec{a})\cdot(\vec{u}\times\vec{v})\right\|}{\left\|\vec{u}\times\vec{v}\right\|}$	20.	(2)	$\mathbb{Z}$

For writing the correct option and the answer – one mark.

PART – II

21. If A is a nonsingular matrix of odd order, prove that $|\text{adj } A|$ is positive.

Solution

We know that property $|adjA| = |A|^{n-1}$. Here, order of A is odd i.e., $2m+1, m \in W$.

$|A| \neq 0$ and we have $|\text{adj } A| = |A|^{(2m+1)-1} = |A|^{2m}$.

Since $|A|^{2m}$ is always positive, we get that $|\text{adj } A|$ is positive.

22. If $\frac{z+3}{z-5i} = \frac{1+4i}{2}$, find the complex number z in the rectangular form.

Solution

$$\text{Given that } \frac{z+3}{z-5i} = \frac{1+4i}{2}$$

$$2(z+3) = (1+4i)(z-5i)$$

$$2z+6 = (1+4i)z+20-5i$$

$$(2-1-4i)z = 20-5i-6$$

$$z = \frac{14-5i}{1-4i} = \frac{(14-5i)(1+4i)}{(1-4i)(1+4i)} = \frac{34+51i}{17} = 2+3i$$

23. State Fermat's theorem.

Solution

If $f(x)$ has a relative or local extrema at $x = c$ then c is a critical number.

24. Examine the position of the point $(2,3)$ with respect to the circle $x^2+y^2-6x-8y+12=0$.

Solution

Substitute the point $(2,3)$ in $x^2+y^2-6x-8y+12$

$$= 2^2+3^2-6\times2-8\times3+12$$

$$= 4+9-12-24+12$$

$$= -11 < 0.$$

Therefore the point $(2,3)$ lies inside the circle.

25. Find the differential dy if $y = e^{x^2-5x+7}\cos(x^2-1)$.

Solution

$$y = e^{x^2-5x+7}\cos(x^2-1)$$

$$dy = e^{x^2-5x+7}\left[(2x-5)\cos(x^2-1)-2x\sin(x^2-1)\right]dx$$

26. Evaluate $\int_0^{\frac{\pi}{2}}(\sin^2 x+\cos^4 x)dx$

Solution

$$I = \int_0^{\frac{\pi}{2}}(\sin^2 x+\cos^4 x)dx = \int_0^{\frac{\pi}{2}}\sin^2 x\,dx + \int_0^{\frac{\pi}{2}}\cos^4 x\,dx = I_2+I_4=\frac{1}{2}\times\frac{\pi}{2}+\frac{3}{4}\times\frac{1}{2}\times\frac{\pi}{2}=\frac{7\pi}{16}.$$

27. Solve $\frac{dy}{dx}+2y=e^{-x}$.

Solution

$$\frac{dy}{dx}+2y = e^{-x} \quad \text{... (1)}$$

This is a linear differential equation in y.

Here $P=2$; $Q=e^{-x}$.

$$\int Pdx = \int 2dx=2x.$$

$$\text{Thus, I.F.} = e^{\int Pdx}=e^{2x}.$$

That is, $ye^{2x}=\int e^{-x}e^{2x}dx+C$ or $ye^{2x}=e^{x}+C$ or $y=e^{-x}+Ce^{-2x}$ is the required solution.

28. If $E(X)=2$ then find $E(5X+7)$.

Solution

$$E(5X+7) = 5E(X)+7=5\times2+7=17.$$

29. Let $A=\{x-\sqrt{3}y : x,y\in\mathbb{Z}\}$. Check whether the usual addition is a binary operation on A.

Solution

Let $x_1-\sqrt{3}y_1, x_2-\sqrt{3}y_2\in A$, where $x_1,x_2,y_1,y_2\in\mathbb{Z}$

$(x_1-\sqrt{3}y_1)+(x_2-\sqrt{3}y_2) = (x_1+x_2)-\sqrt{3}(y_1+y_2)\in A$, since x_1+x_2 and y_1+y_2 are integers.

Hence addition is a binary operation on A.

30. Find the value of $\sin^{-1}\left(-\frac{1}{2}\right)+\sec^{-1}(2)$.

Solution

$$\sin^{-1}\left(-\frac{1}{2}\right)+\sec^{-1}2 = -\sin^{-1}\left(\frac{1}{2}\right)+\sec^{-1}2 \quad \text{since } -\frac{1}{2}\in[-1 \quad 1]$$

$$= -\frac{\pi}{6}+\frac{\pi}{3}=\frac{\pi}{6}$$

PART– III

31. If $A=\frac{1}{9}\begin{bmatrix}-8 & 1 & 4\\ 4 & 4 & 7\\ 1 & -8 & 4\end{bmatrix}$, prove that $A^{-1}=A^T$.

Solution

$$\text{To prove } A^{-1} = A^T$$

$$\text{i.e., It is enough to prove } AA^{-1} = AA^T$$

$$\text{i.e., } AA^T = I$$

$$AA^T = \frac{1}{81}\begin{bmatrix} -8 & 1 & 4 \\ 4 & 4 & 7 \\ 1 & -8 & 4 \end{bmatrix}\begin{bmatrix} -8 & 4 & 1 \\ 1 & 4 & -8 \\ 4 & 7 & 4 \end{bmatrix}$$

$$= \frac{1}{81}\begin{bmatrix} 81 & 0 & 0 \\ 0 & 81 & 0 \\ 0 & 0 & 81 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = I$$

Hence proved.

32. If $\frac{1+z}{1-z} = \cos 2\theta + i\sin 2\theta$, show that $z = i\tan\theta$.

Solution

$$\frac{1+z}{1-z} = \cos 2\theta + i\sin 2\theta$$

$$(1-z)e^{i2\theta} = (1+z)$$

$$z\left(1+e^{i2\theta}\right) = e^{i2\theta} - 1$$

$$z = \frac{e^{i2\theta}-1}{1+e^{i2\theta}} = \frac{e^{i\theta}-e^{-i\theta}}{e^{i\theta}+e^{-i\theta}} = \frac{2i\sin\theta}{2\cos\theta} = i\tan\theta.$$

33. Show that the equation $x^9 - 5x^5 + 4x^4 + 2x^2 + 1 = 0$ has atleast 6 imaginary solutions.

Solution

The signs of the coefficients of $f(x)$ are $+-+++$.

The number of changes in sign is 2.

$\therefore$ The maximum number of positive roots is 2.

The signs of the coefficients of $f(-x)$ are $-++++$.

The number of changes in sign is 1.

The maximum number of negative roots is 1.

$\therefore$ The maximum number of real roots is 3.

Thus the minimum number of imaginary roots is 6.

34. Solve : $\tan^{-1} 2x + \tan^{-1} 3x = \frac{\pi}{4}$, here $6x^2 < 1$.

Solution

Now, $\tan^{-1}2x+\tan^{-1}3x = \tan^{-1}\left(\frac{2x+3x}{1-6x^2}\right).$

So, $\tan^{-1}\left(\frac{5x}{1-6x^2}\right) = \frac{\pi}{4}, \Rightarrow \frac{5x}{1-6x^2} = \tan\frac{\pi}{4} = 1.$

Thus, $1-6x^2 = 5x$, which gives $6x^2+5x-1=0$

Hence, $x = \frac{1}{6},-1.$

$x=-1$ does not satisfy $6x^2<1$. Hence $x=\frac{1}{6}$.

35. Find the parametric form of vector equation and Cartesian equation of the straight line passing through the point $(-2,3,4)$ and parallel to the straight line $\frac{x-1}{-4}=\frac{y+3}{5}=\frac{8-z}{6}$.

Solution

Here, $\vec{a} = -2\hat{i}+3\hat{j}+4\hat{k}$

The required line is parallel to

$$\frac{x-1}{-4} = \frac{y+3}{5} = \frac{z-8}{-6}$$

So the required line is parallel to the vector $-4\hat{i}+5\hat{j}-6\hat{k}$.

The parametric vector equation of the required line is

$$\vec{r} = \vec{a}+t\vec{b}, t\in\mathbb{R}$$

$$\therefore \vec{r} = \left(-2\hat{i}+3\hat{j}+4\hat{k}\right)+t\left(-4\hat{i}+5\hat{j}-6\hat{k}\right)$$

Cartesian equation of the required line is

$$\frac{x-x_1}{b_1} = \frac{y-y_1}{b_2} = \frac{z-z_1}{b_3}$$

Here, $(x_1, y_1, z_1) = (-2,3,4)$ and $(b_1, b_2, b_3)=(-4,5,-6)$.

$\therefore \frac{x+2}{-4} = \frac{y-3}{5} = \frac{z-4}{-6}$ is the required equation.

36. Find the equation of a circle whose centre is $(3,2)$ and the radius is the radius of $x^2+y^2+2x+4y-4=0$.

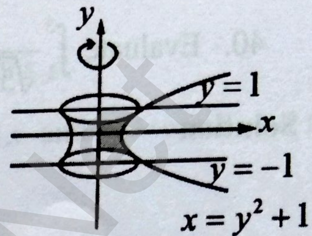
Solution

The radius of $x^2+y^2+2x+4y-4=0$ is $\sqrt{1+4+4}=3$.

The equation of the required circle is $(x-3)^2+(y-2)^2=9$.

37. Find the volume of the solid generated by revolving the region bounded by the parabola $x=y^2+1$, the y-axis, and the lines $y=1$ and $y=-1$, about y-axis.

Solution



The limits are $y=-1$ and $y=1$.

Since it is revolved about y-axis,

$$V = \pi\int_{-1}^{1} x^2\,dy$$

$$= \pi\int_{-1}^{1}\left(y^2+1\right)^2 dy$$

$$= 2\pi\int_{0}^{1}\left(y^4+2y^2+1\right)dy$$

$$= 2\pi\left(\frac{y^5}{5}+2\frac{y^3}{3}+y\right)_0^1 = 2\pi\left(\frac{1}{5}+\frac{2}{3}+1\right) = \frac{56}{15}\pi.$$

38. If α,β amd γ are the roots of the cubic equation $x^3+2x^2+3x+4=0$, form a cubic equation whose roots are $-\alpha,-\beta,-\gamma$.

Solution

$$S_1 = \alpha+\beta+\gamma=-2$$

$$S_2 = \alpha\beta+\beta\gamma+\gamma\alpha=3$$

$$S_3 = \alpha\beta\gamma=-4$$

For the required equation

$$S_1 = -\alpha-\beta-\gamma=2$$

$$S_2 = \alpha\beta+\beta\gamma+\gamma\alpha=3$$

$$S_3 = -(\alpha\beta\gamma)=4$$

The required equation is

$$x^3-2x^2+3x-4=0$$

Note : By putting $x=-x$, we get the answer.

39. If X is the random variable with distribution function $F(x)$ given by,

$$F(x)=\begin{cases}0, & x<0\\ \frac{1}{2}\left(x^2+x\right) & 0\le x<1\\ 1, & x\ge 1\end{cases}$$

then find $P[0.3\le x\le 0.6]$

Solution

$$P[0.3 \le x \le 0.6] = F(0.6) - F(0.3)$$
$$= \frac{1}{2}\left[(0.6)^2 + 0.6\right] - \frac{1}{2}\left[(0.3)^2 + 0.3\right]$$
$$= 0.285.$$

40. Evaluate $\int_2^3 \frac{\sqrt{x}}{\sqrt{5-x}+\sqrt{x}}dx.$

Solution

$$\text{Let } I = \int_2^3 \frac{\sqrt{x}}{\sqrt{5-x}+\sqrt{x}}dx \quad \text{... (1)}$$

$$\text{Again } I = \int_2^3 \frac{\sqrt{(2+3-x)}}{\sqrt{5-(2+3-x)}+\sqrt{(2+3-x)}}dx = \int_2^3 \frac{\sqrt{5-x}}{\sqrt{x}+\sqrt{5-x}}dx \quad \text{... (2)}$$

Adding (1) and (2), we get

$$2I = \int_2^3 \frac{\sqrt{x}+\sqrt{5-x}}{\sqrt{x}+\sqrt{5-x}}dx = \int_2^3 dx = [x]_2^3 = 3-2 = 1.$$

$$\text{Hence, } I = \frac{1}{2}.$$

PART – IV

41. (a) Investigate the values of λ and μ the system of linear equations $2x+3y+5z=9$, $7x+3y-5z=8$, $2x+3y+\lambda z=\mu$, to have
(i) no solution (ii) a unique solution (iii) infinite number of solutions.

Solution

$$[A \quad B] = \begin{bmatrix} 2 & 3 & 5 & 9 \\ 7 & 3 & -5 & 8 \\ 2 & 3 & \lambda & \mu \end{bmatrix}$$

$$\sim \begin{bmatrix} 2 & 3 & 5 & 9 \\ 0 & -15 & -45 & -47 \\ 0 & 0 & \lambda-5 & \mu-9 \end{bmatrix} \begin{matrix} \\ R_2 \to 2R_2 - 7R_1 \\ R_3 \to R_3 - R_1 \end{matrix}$$

(i) **For no solution :** When $\lambda = 5,\ \mu \ne 9$

$$\rho(A) = 2,\ \rho([A \ \ B]) = 3.$$

When $\lambda = 5, \mu \ne 9$, the system has no solution.

(ii) **For unique solution :** When $\lambda \neq 5,\ \mu \in \mathbb{R}$.

$$\rho(A) = \rho([AB]) = 3$$

When $\lambda \neq 5,\ \mu \in \mathbb{R}$ the system has a unique solution.

(iii) **For infinite solutions :** When $\lambda = 5$ and $\mu = 9$,

$$\rho(A) = \rho([AB]) = 2$$

When $\lambda = 5$ and $\mu = 9$, the system has infinite number of solutions.

41. (b) Find the area of the region bounded between the curves $y = \sin x$ and $y = \cos x$ and the lines $x = 0$ and $x = \pi$.

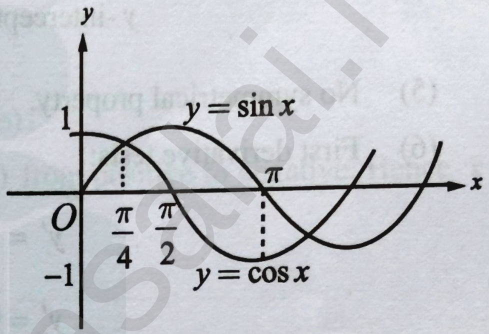
Solution

$$y = \cos x;\ y = \sin x$$

To get points of intersection we have to solve the equations.

i.e., $\sin x = \cos x$

$$\therefore x = \frac{\pi}{4} \in [0\ \pi]$$



From the diagram, the required area

$$= \int_0^{\frac{\pi}{4}} (\cos x - \sin x)dx + \int_{\frac{\pi}{4}}^{\pi} (\sin x - \cos x)dx$$

$$= [\sin x + \cos x]_0^{\frac{\pi}{4}} + [-\cos x - \sin x]_{\frac{\pi}{4}}^{\pi}$$

$$= \left(\frac{1}{\sqrt{2}} + \frac{1}{\sqrt{2}}\right) - (0+1) + \left[(-(-1)-0) - \left(-\frac{1}{\sqrt{2}} - \frac{1}{\sqrt{2}}\right)\right]$$

$$= \frac{2}{\sqrt{2}} - 1 + 1 + \frac{1}{\sqrt{2}} + \frac{1}{\sqrt{2}} = \frac{2}{\sqrt{2}} + \frac{2}{\sqrt{2}} = \sqrt{2} + \sqrt{2} = 2\sqrt{2}.$$

42. (a) Solve the equation $z^5 + z^3 - z^2 - 1 = 0$.

Solution

$$z^5 + z^3 - z^2 - 1 = 0$$

$$z^3(z^2+1) - 1(z^2+1) = 0$$

$$(z^3-1)(z^2+1) = 0$$

$$z^3 = 1,\ z^2 = -1$$

$$z = (1)^{\frac{1}{3}},\ (-1)^{\frac{1}{2}}$$

i.e., $z = 1, \omega, \omega^2, i, -i$

where ω is cube root of unity $(\omega \neq 1)$.

42. (b) Write any five points to sketch the function $y=-\frac{1}{3}(x^3-3x+2)$.

Solution

(1) Domain : $(-\infty,\infty)$

(2) Range : $(-\infty,\infty)$

(3) x-intercept : $y=0 \Rightarrow x=1,1,-2$

The x-intercepts are $1,-2$

(4) y-intercept : Put $x=0 \Rightarrow y=-\frac{2}{3}$

y-intercept is $-\frac{2}{3}$.

(5) No symmetrical property.

(6) First derivative tests:

$$y' = -\frac{1}{3}(3x^2-3)$$

$$y' = 0 \Rightarrow x=\pm 1$$

$(-\infty,-1)$

Let $x=-2$

$y'<0$ for $x=-2$

y is strictly decreasing in $(-\infty,-1)$

$(-1,1)$

Let $x=0$

$y'>0$ for $x=0$

y is strictly increasing in $(-1,1)$.

$(1,\infty)$

Let $x=2$

$y'<0$

y is strictly decreasing in $(1,\infty)$.

(7) Second derivative tests

$y''=-\frac{1}{3}(6x)=-2x$.

(i) **For no solution** : When $\lambda = 5$, $\mu \neq 9$

$$\rho(A) = 2, \rho([A\ B]) = 3.$$

When $\lambda = 5, \mu \neq 9$, the system has no solution.

(ii) **For unique solution :** When $\lambda \neq 5$, $\mu \in \mathbb{R}$.

$$\rho(A) = \rho([AB]) = 3$$

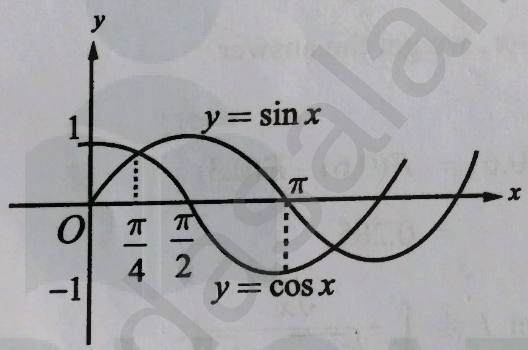
When $\lambda \neq 5$, $\mu \in \mathbb{R}$ the system has a unique solution.

(iii) **For infinite solutions :** When $\lambda = 5$ and $\mu = 9$,

$$\rho(A) = \rho([AB]) = 2$$

When $\lambda = 5$ and $\mu = 9$, the system has infinite number of solutions.

41. (b) Rough diagram



the required area

$$= \int_0^{\frac{\pi}{4}} (\cos x - \sin x)dx + \int_{\frac{\pi}{4}}^{\pi} (\sin x - \cos x)dx$$

$$= 2\sqrt{2}.$$

42. (a)

$$z^3 = 1, z^2 = -1$$

$$z = (1)^{\frac{1}{3}}, (-1)^{\frac{1}{2}}$$

$$z = 1, \omega, \omega^2, i, -i$$

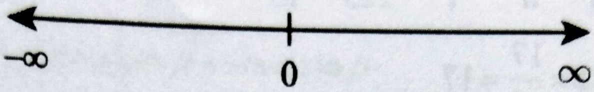
42. (b) Each step 1 mark (or sub heading). Maximum 5 marks

(1) Domain : $(-\infty, \infty)$

(2) Range : $(-\infty, \infty)$

(i) When $x=1, y''<0 \Rightarrow x=1$ gives local maximum.

When $x=-1, y''>0 \Rightarrow x=-1$ gives local minimum.



(ii) $(-\infty,0)$

Let $x=-1$.

$y''>0$

y is concave upward in $(-\infty,0)$

$(0,\infty)$

Let $x=1$

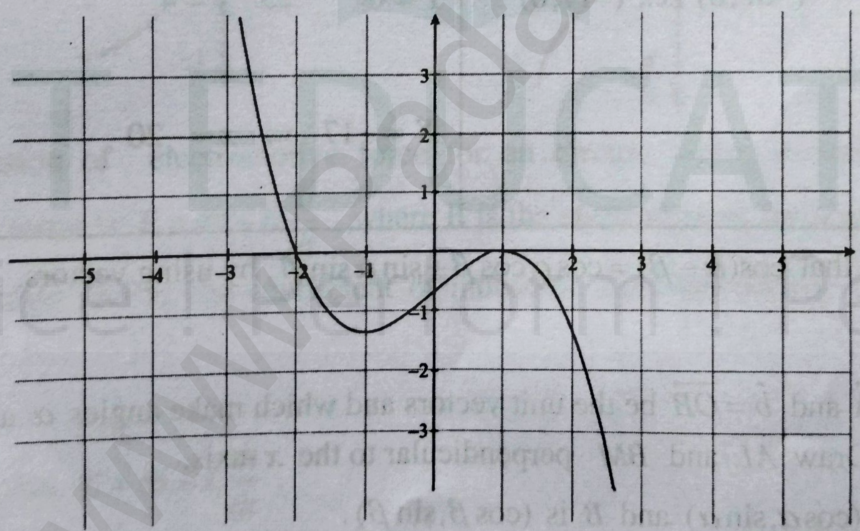
$y''<0$

y is concave downward in $(0,\infty)$.

(iii) $x=0$ changes the sign of $f''(x)$ from positive to negative. Hence $x=0$ gives a point of inflection.

(8) It has no asymptotes.

The curve is



43. (a) Find the centre, foci, vertices of the hyperbola $\frac{(x+3)^2}{225}-\frac{(y-4)^2}{64}=1$.

Solution

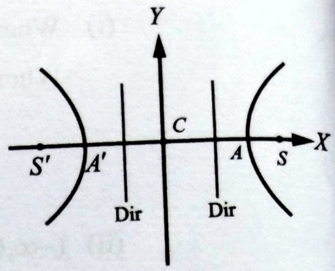
$\frac{(x+3)^2}{225}-\frac{(y-4)^2}{64}=1$ represents a hyperbola with transverse axis parallel to x-axis and the type is I.

$$\frac{X^2}{225}-\frac{Y^2}{64}=1, \text{ where } X=x+3,\ Y=y-4$$

$a^2 = 225,\ b^2 = 64$

$e = \sqrt{1+\frac{b^2}{a^2}} = \sqrt{1+\frac{64}{225}} = \frac{17}{15}$

$ae = 15\times\frac{17}{15} = 17$



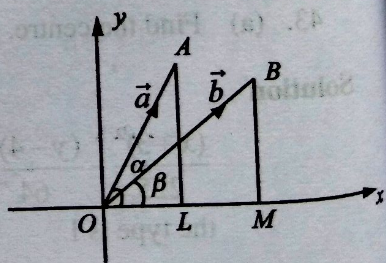
X,Y		x, y
Centre	: $(0,0)$	$C(-3,4)$
Vertices	: $(a,0)$ i.e., $(15,0)$: $(-a,0)$ i.e., $(-15,0)$	$X=15 \Rightarrow x+3=15 \Rightarrow x=12$ $Y=0 \Rightarrow y=4$ $A(12,4)$ $X=-15 \Rightarrow x=-18$ $A'(-18,4)$
Foci	: $(ae,0)$ i.e., $(17,0)$ $(-ae,0)$ i.e., $(-17,0)$	$X=17 \Rightarrow x=14$ $Y=0 \Rightarrow y=4$ $S(14,4)$ $X=-17 \Rightarrow x=-20$ $S'(-20,4)$

43. (b) Prove that $\cos(\alpha-\beta)=\cos\alpha\cos\beta+\sin\alpha\sin\beta$, by using vectors.

Solution :

Let $\hat{a}=\overrightarrow{OA}$ and $\hat{b}=\overrightarrow{OB}$ be the unit vectors and which make angles α and β, respectively with positive x-axis. Draw AL and BM perpendicular to the x-axis.

Here A is $(\cos\alpha,\sin\alpha)$ and B is $(\cos\beta,\sin\beta)$.



Therefore, $\hat{a} = \overrightarrow{OA}=\overrightarrow{OL}+\overrightarrow{LA}$

$= \cos\alpha\hat{i}+\sin\alpha\hat{j}$

Similarly, $\hat{b} = \cos\beta\hat{i}+\sin\beta\hat{j}$

The angle between $\hat{a}$ and $\hat{b}$ is $\alpha-\beta$ and so,

By definition, $\hat{a}\cdot\hat{b} = |\hat{a}|\,|\hat{b}|\cos(\alpha-\beta)=\cos(\alpha-\beta)$ ... (1)

By their values, $\hat{a}\cdot\hat{b} = (\cos\alpha\hat{i}+\sin\alpha\hat{j})\cdot(\cos\beta\hat{i}+\sin\beta\hat{j})$

$= \cos\alpha\cos\beta+\sin\alpha\sin\beta$ ... (2)

From (1) and (2), we get

$$\cos(\alpha-\beta) = \cos\alpha\cos\beta+\sin\alpha\sin\beta.$$

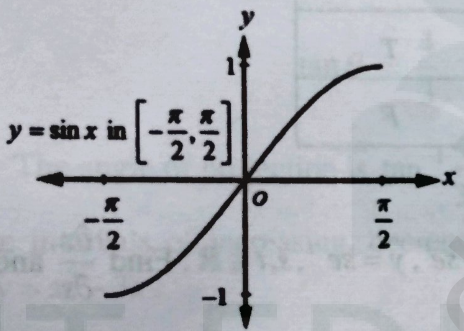
Note : The result can be proved by drawing a unique circle also.

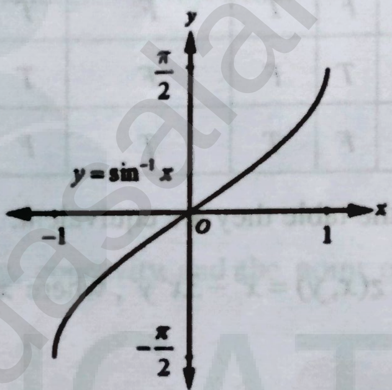
44. (a) Draw the curves $\sin x$ and $\sin^{-1} x$ in the respective principal domain.

Solution

$\sin \; : \left[-\frac{\pi}{2},\frac{\pi}{2}\right] \to [-1, 1]$

$\sin^{-1} \; : [-1, 1] \to \left[-\frac{\pi}{2},\frac{\pi}{2}\right]$





44. (b) The equation of electromotive force for an electric circuit containing resistance and self-inductance is $E = Ri + L\frac{di}{dt}$, where E is the electromotive force given to the circuit, R the resistance and L, the coefficient of induction. Find the current i at time t when E = 0.

Solution

We are given that $E = Ri + L\frac{di}{dt}$.

Dividing by L, we get $\frac{di}{dt}+\frac{R}{L}i=\frac{E}{L}$.

This is a linear differential equation in i, where $P=\frac{R}{L}$ and $Q=\frac{E}{L}$.

$$\text{Now, I.F} = e^{\int P dt} = e^{\int \frac{R}{L}dt} = e^{\frac{R}{L}t}.$$

$$\text{The solution is } i\, e^{\frac{R}{L}t} = \int \frac{E}{L} e^{\frac{R}{L}t}\, dt + c$$

$$i\,e^{\frac{R}{L}t} = \frac{E}{L}\left(\frac{e^{\frac{R}{L}t}}{\frac{R}{L}}\right)+c$$

$$i = \frac{E}{R}+ce^{-\frac{R}{L}t} \quad \ldots(1)$$

When $E=0$, we get $i=ce^{-\frac{R}{L}t}$.

45. (a) Show that $\neg(p\leftrightarrow q)\equiv p\leftrightarrow\neg q$.

Solution

p	q	$\neg q$	$r: p\leftrightarrow q$	$\neg r$	$s: p\leftrightarrow\neg q$
T	T	F	T	F	F
T	F	T	F	T	T
F	T	F	F	T	T
F	F	T	T	F	F

From the table they are equivalent.

45. (b) Let $z(x,y)=x^3-3x^2y^3$, where $x=se^t, y=se^{-t}, s,t\in\mathbb{R}$. Find $\frac{\partial z}{\partial s}$ and $\frac{\partial z}{\partial t}$.

Solution

$$\frac{\partial z}{\partial t} = \frac{\partial z}{\partial x}\cdot\frac{\partial x}{\partial t}+\frac{\partial z}{\partial y}\cdot\frac{\partial y}{\partial t}$$

$$\frac{\partial z}{\partial t} = (3x^2-6xy^3)se^t-(9x^2y^2)(-se^{-t})$$

$$\frac{\partial z}{\partial t} = se^t\left(3s^2e^{2t}-6se^t\cdot s^3e^{-3t}\right)-\left(9s^2e^{2t}\cdot s^2e^{-2t}\right)\left(-se^{-t}\right)$$

$$= 3s^3e^{3t}-6s^5e^{-t}+9s^5e^{-t} = 3s^3e^{3t}+3s^5e^{-t} = 3s^3\left(e^{3t}+s^2e^{-t}\right)$$

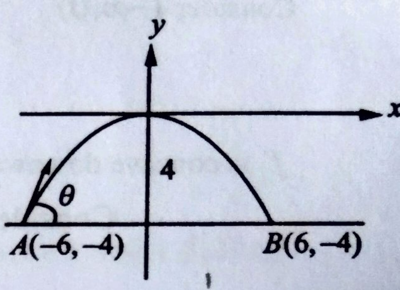
$$\frac{\partial z}{\partial s} = \frac{\partial z}{\partial x}\cdot\frac{\partial x}{\partial s}+\frac{\partial z}{\partial y}\cdot\frac{\partial y}{\partial s} = (3x^2-6xy^3)e^t+(-9x^2y^2)e^{-t}$$

$$= (3s^2e^{2t}-6s^4e^t\cdot e^{-3t})e^t-e^{-t}9s^4e^{2t}\cdot e^{-2t} = 3s^2e^{3t}-6s^4e^{-t}-9s^4e^{-t}$$

$$= 3s^2e^{3t}-15s^4e^{-t}.$$

46. (a) On lighting a rocket cracker it gets projected in a parabolic path and reaches a maximum height of 4m when it is 6m away from the point of projection. Find the angle of projection at the starting point.

Solution



Equation of the parabola is $x^2 = -4ay$.

Let $A(-6,-4)$ be the point of projection.

$$\therefore 36 = -4a(-4)$$

$$4a = 9.$$

$\therefore$ Equation of the parabola is $x^2 = -9y$.

To find the angle of projection, find the slope of tangent at $(-6,-4)$.

Differentiating $x^2 = -9y$ with respect to 'x'.

$$2x = -9\frac{dy}{dx} \Rightarrow \frac{dy}{dx} = \frac{-2x}{9}$$

$\therefore \frac{dy}{dx}$ at $(-6,-4)$ is $\frac{4}{3}$. But $\frac{dy}{dx} = \tan\theta$, θ is the angle of projection.

$$\tan\theta = \frac{4}{3}$$

$\therefore$ The angle of projection is $\tan^{-1}\frac{4}{3}$.

46. (b) Find the intervals of increasing, decreasing, concavity and the point of inflection of the curve $y = x^3 - 3x$.

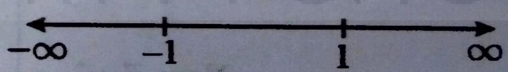
Solution

$$y = x^3 - 3x$$

$$y' = 3x^2 - 3$$

$$y' = 0 \Rightarrow x = \pm 1$$

$$y'' = 6x$$



Consider $(-\infty,-1)$

Take $x = -2$

$y' > 0$

strictly increasing in $(-\infty,-1)$

Consider $(-1,1)$

Take $x = 0$

$y' < 0$

strictly decreasing in $(-1,1)$

Consider $(1,\infty)$

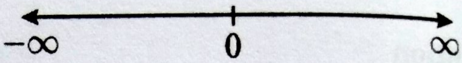
Take $x = 2$

$y' > 0$

strictly increasing in $(1,\infty)$

$$y'' = 0 \Rightarrow x = 0$$

Consider $(-\infty,0)$



$$x = -1$$
$$y'' < 0$$

f is concave downward in $(-\infty,0)$

$$\text{Consider } x = 1$$
$$y'' > 0$$

f is concave upward in $(0,\infty)$

Clearly $x=0$ gives a point of inflection since $x=0$ changes the sign of $f''(x)$ from negative to positive.

The point of inflection is $(0, f(0))$ i.e., $(0,0)$.

47. (a) Find the non-parametric form of vector equation, and Cartesian equations of the plane $\vec{r}=\left(6\hat{i}-\hat{j}+\hat{k}\right)+s\left(-\hat{i}+2\hat{j}+\hat{k}\right)+t\left(-5\hat{i}-4\hat{j}-5\hat{k}\right)$.

Solution

The given plane is passing through a point with position vector $6\hat{i}-\hat{j}+\hat{k}$, and is parallel to the two vectors $-\hat{i}+2\hat{j}+\hat{k}$ and $-5\hat{i}-4\hat{j}-5\hat{k}$.

Therefore, the non-parametric vector equation of the given plane is $[\vec{r}-\vec{a}\ \ \vec{u}\ \ \vec{v}]=0$.

$$(\vec{r}-\vec{a})\cdot(\vec{u}\times\vec{v}) = 0$$

$$\text{where } \vec{a} = 6\hat{i}-\hat{j}+\hat{k}$$
$$\vec{u} = -\hat{i}+2\hat{j}+\hat{k}$$
$$\vec{v} = -5\hat{i}-4\hat{j}-5\hat{k}$$

$$\text{Now, } \vec{u}\times\vec{v} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ -1 & 2 & 1 \\ -5 & -4 & -5 \end{vmatrix}$$

$$= -6\hat{i}-10\hat{j}+14\hat{k}$$

Hence the required equation is

$$\left[\vec{r}-\left(6\hat{i}-\hat{j}+\hat{k}\right)\right]\cdot\left(-6\hat{i}-10\hat{j}+14\hat{k}\right) = 0.$$

Cartesian equation

$$\begin{vmatrix} x-x_1 & y-y_1 & z-z_1 \\ l_1 & m_1 & n_1 \\ l_2 & m_2 & n_2 \end{vmatrix} = 0$$

$$\Rightarrow \begin{vmatrix} x-6 & y+1 & z-1 \\ -1 & 2 & 1 \\ -5 & -4 & -5 \end{vmatrix} = 0$$

$$\Rightarrow \; 3x+5y-7z-6 = 0.$$

47. (b) The mean and variance of a binomial variate X are respectively 2 and 1.5. Find (i) $P(X=0)$ (ii) $P(X=1)$ (iii) $P(X \geq 1)$

Solution

Given that

$$\text{Mean} = np = 2 \text{ and variance } = npq = 1.5$$

$$\text{This gives } \frac{npq}{np} = \frac{1.5}{2} = \frac{3}{4}$$

$$q = \frac{3}{4} \text{ and } p = 1-q = 1-\frac{3}{4} = \frac{1}{4}$$

$$np = 2, \text{ gives } n = \frac{2}{p} = 8. \text{ Therefore } X \sim B\left(8, \frac{1}{4}\right).$$

Therefore probability distribution is

$$P(X=x) = f(x) = \binom{8}{x}\left(\frac{1}{4}\right)^{x}\left(\frac{3}{4}\right)^{8-x} \quad x = 0,1.2,\ldots 8$$

(i) $$P(X=0) = f(0) = \binom{8}{0}\left(\frac{1}{4}\right)^{0}\left(\frac{3}{4}\right)^{8-0} = \left(\frac{3}{4}\right)^{8}$$

(ii) $$P(X=1) = f(1) = \binom{8}{1}\left(\frac{1}{4}\right)^{1}\left(\frac{3}{4}\right)^{8-1} = 2\left(\frac{3}{4}\right)^{7}$$

(iii) $$P(X \geq 1) = 1-P(X<1) = 1-P(X=0) = 1-\left(\frac{3}{4}\right)^{8}$$

MARKING SCHEME - MATHEMATICS

GENERAL INSTRUCTIONS

The marking scheme provides general guidelines to reduce subjectivity in the marking. The answer given in the marking scheme are Text Book, Solution Book and COME book bound. If a student has given any answer which is different from one given in the marking scheme, but carries the prescribed content meaning (rigorous), such answers should be given full credit with suitable distribution.

There is no separate mark allotted for formulae. If a particular stage is wrong and if the candidate writes the appropriate formula then award 1 mark for the formula irrespective of stage marks. This mark is attached with that stage. This is done with the aim that a student who did the problem correctly without writing the formula should not be penalised. That is, mark should not be deducted for not writing the formula.

In the case of Part II, Part III and Part IV, if the solution is correct then award full mark directly. The stage mark is essential only if the part of the solution is incorrect.

Only a Rough sketch of the diagram is expected and full credit must be given for such diagrams that form a part and parcel of the solution of a problem.

In questions on Variable Separable, Homogeneous and Linear differential equations, full marks should be given for an equivalent answer and the students should not be penalised for not getting the final answer as mentioned in the marking scheme.

If the method is not mentioned in the question, then one may adopt any method.

Complicated radicals and combinations need not be simplified (for example $\sqrt{7}$, $\sqrt{82}$, $\frac{3}{\sqrt{2}}$, $10c_6$ etc.)

Important Note : In an answer to a question between any two particular stages of marks (greater than one) if a student starts a stage with correct step but reaches the next stage with a wrong result then suitable credits should be given to the related steps instead of denying the entire marks meant for that stage.

PART – II

21. $|\text{adj } A| = |A|^{(2m+1)-1} = |A|^{2m}$. ... (1)

$|\text{adj } A|$ is positive. ... (1)

22. $z = \dfrac{14-5i}{1-4i}$ ... (1)

$= 2+3i$ ... (1)

23. Fermat's theorem.

If $f(x)$ has a relative or local extrema at $x = c$ then c is a critical number.

To write the statement without affecting inner meaning ... (2)

24. Substitute the point $(2,3)$ in $x^2+y^2-6x-8y+12$

$= -11 < 0$. ... (1)

The point $(2,3)$ lies inside the circle. ... (1)

25. $y = e^{x^2-5x+7}\cos(x^2-1)$

$dy = e^{x^2-5x+7}\left[(2x-5)\cos(x^2-1) - 2x\sin(x^2-1)\right]dx$ ... (2)

26. $\int_0^{\frac{\pi}{2}}(\sin^2 x + \cos^4 x)dx = \int_0^{\frac{\pi}{2}}\sin^2 x\, dx + \int_0^{\frac{\pi}{2}}\cos^4 x\, dx = \frac{1}{2}\times\frac{\pi}{2} + \frac{3}{4}\times\frac{1}{2}\times\frac{\pi}{2}$ ... (1)

$= \dfrac{7\pi}{16}$. ... (1)

27. I.F. $= e^{\int P dx} = e^{2x}$. ... (1)

$ye^{2x} = e^x + C$ or $y = e^{-x} + Ce^{-2x}$ ... (1)

28. $E(5X+7) = 5E(X)+7$ ... (1)

$= 17$ ... (1)

29. $x_1 - \sqrt{3}y_1, x_2 - \sqrt{3}y_2 \in A$, where $x_1, x_2, y_1, y_2 \in \mathbb{Z}$

$(x_1 - \sqrt{3}y_1) + (x_2 - \sqrt{3}y_2) = (x_1+x_2) - \sqrt{3}(y_1+y_2) \in A$, since $x_1+x_2,\ y_1+y_2 \in \mathbb{Z}$. ... (1)

Addition is a binary operation on A ... (1)

30. $\sin^{-1}\left(-\frac{1}{2}\right)+\sec^{-1}2 = -\frac{\pi}{6}+\frac{\pi}{3}$ ... (1)

$= \frac{\pi}{6}$ ... (1)

PART– III

31. To prove $A^{-1} = A^T$

i.e., It is enough to prove $AA^{-1} = AA^T$

(or) $AA^T = I$ ... (1)

Proving $AA^T = I$ ... (2)

32. $z = \frac{e^{i2\theta}-1}{1+e^{i2\theta}}$ ... (1)

$= \frac{e^{i\theta}-e^{-i\theta}}{e^{i\theta}+e^{-i\theta}}$ ... (1)

$= i\tan\theta$ . ... (1)

33. The maximum number of positive roots is 2. ... (1)

The maximum number of negative roots is 1. ... (1)

The minimum number of imaginary roots is 6. ... (1)

34. $\tan^{-1}2x+\tan^{-1}3x = \tan^{-1}\left(\frac{2x+3x}{1-6x^2}\right)$. ... (1)

$6x^2+5x-1=0$ ... (1)

$x=\frac{1}{6}$. ... (1)

35. The required line is parallel to the vector $-4\hat{i}+5\hat{j}-6\hat{k}$. ... (1)

$\vec{r} = \left(-2\hat{i}+3\hat{j}+4\hat{k}\right)+t\left(-4\hat{i}+5\hat{j}-6\hat{k}\right)$ ... (1)

$\frac{x+2}{-4} = \frac{y-3}{5}=\frac{z-4}{-6}$ ... (1)

36. The radius of $x^2+y^2+2x+4y-4=0$ is 3. ... (1)

The equation of the required circle is $(x-3)^2+(y-2)^2=9$. ... (2)

37. $$V = \pi\int_{-1}^{1}\left(y^2+1\right)^2 dy$$ ... (2)

$$= \frac{56}{15}\pi.$$ ... (1)

38. $$S_1 = \alpha+\beta+\gamma=-2$$

$$S_2 = \alpha\beta+\beta\gamma+\gamma\alpha=3$$

$$S_3 = \alpha\beta\gamma=-4$$ ... (1)

For the required equation $S_1 = -\alpha-\beta-\gamma=2$

$$S_2 = \alpha\beta+\beta\gamma+\gamma\alpha=3$$

$$S_3 = -(\alpha\beta\gamma)=4$$ ... (1)

The required equation is

$$x^3-2x^2+3x-4=0$$ ... (1)

Note : By putting $x=-x$, we get the answer.

39. $$P[0.3\le x\le 0.6] = F(0.6)-F(0.3)$$ ... (1)

$$= 0.285.$$ ... (2)

40. Let $I = \int_2^3 \frac{\sqrt{x}}{\sqrt{5-x}+\sqrt{x}}dx$

Again $I = \int_2^3 \frac{\sqrt{5-x}}{\sqrt{x}+\sqrt{5-x}}dx$ ... (1)

$$2I = \int_2^3 dx = [x]_2^3 = 1.$$ ... (1)

$$I = \frac{1}{2}.$$ ... (1)

PART – IV

41. (a) $$[A\quad B] = \begin{bmatrix} 2 & 3 & 5 & 9 \\ 7 & 3 & -5 & 8 \\ 2 & 3 & \lambda & \mu \end{bmatrix}$$

$$\sim \begin{bmatrix} 2 & 3 & 5 & 9 \\ 0 & -15 & -45 & -47 \\ 0 & 0 & \lambda-5 & \mu-9 \end{bmatrix} \begin{matrix} \\ R_2\to 2R_2-7R_1 \\ R_3\to R_3-R_1 \end{matrix}$$ ... (2)

(3) x-intercept : $y=0 \Rightarrow x=1,1,-2$ ... (1)

The x-intercepts are $1,-2$

(4) y-intercept : Put $x=0 \Rightarrow y=-\frac{2}{3}$ ... (1)

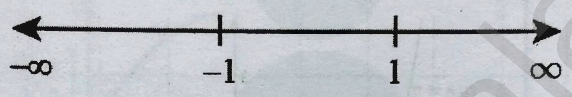
y-intercept is $-\frac{2}{3}$.

(5) No symmetrical property. ... (1)

(6) First derivative tests: .. (1)

$$y' = -\frac{1}{3}(3x^2-3)$$

$$y' = 0 \Rightarrow x = \pm 1$$



$(-\infty,-1)$

Let $x=-2$

$y'<0$ for $x=-2$

y is strictly decreasing in $(-\infty,-1)$

$(-1,1)$

Let $x=0$

$y'>0$ for $x=0$

y is strictly increasing in $(-1,1)$.

$(1,\infty)$

Let $x=2$

$y'<0$

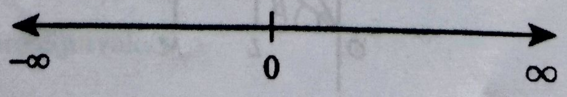
y is strictly decreasing in $(1,\infty)$.

(7) Second derivative tests . .. (1)

$y''=-\frac{1}{3}(6x)=-2x$.

(i) When $x=1, y''<0 \Rightarrow x=1$ gives local maximum.

When $x=-1, y''>0 \Rightarrow x=-1$ gives local minimum.



(ii) $(-\infty,0)$

Let $x=-1$.

$y''>0$

y is concave upward in $(-\infty,0)$

$(0,\infty)$

Let $x=1$

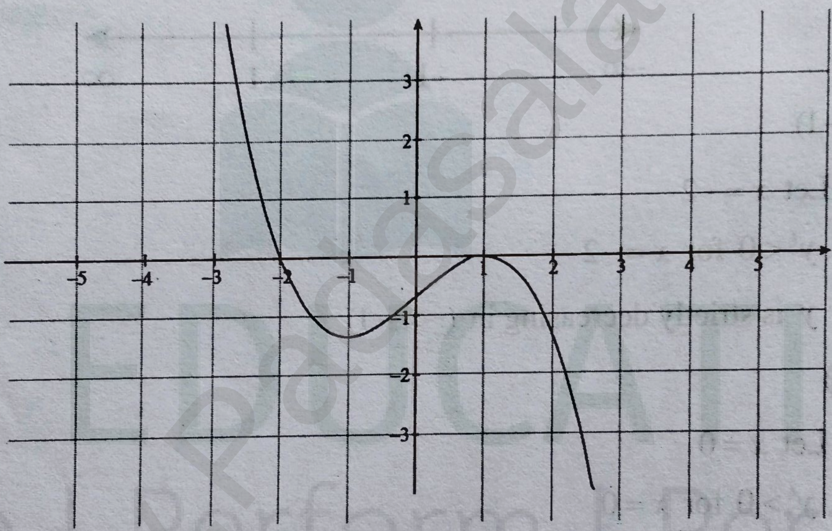
$y''<0$

y is concave downward in $(0,\infty)$.

(iii) $x=0$ changes the sign of $f''(x)$ from positive to negative. Hence $x=0$ gives a point of inflection.

(8) It has no asymptotes. .. (1)

The curve is



43. (a) $a^2 = 225,\ b^2 = 64$

$e = \frac{17}{15}$

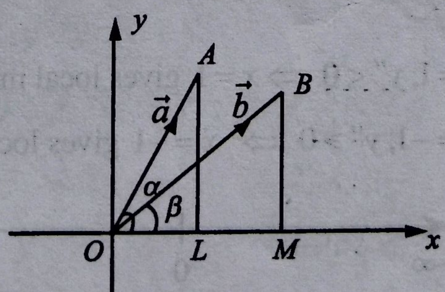
$ae = 17$ .. (1)

Centre $C(-3,4)$ .. (2)

Vertices $A(12,4)$ and $A'(-18,4)$ .. (2)

Foci $S(14,4)$ and $S'(-20,4)$ .. (1)

43. (b) Rough diagram



$$\hat{a} = \cos\alpha\hat{i} + \sin\alpha\hat{j}$$

$$\hat{b} = \cos\beta\hat{i} + \sin\beta\hat{j} \quad \text{.. (1)}$$

$$\hat{a}\cdot\hat{b} = \cos(\alpha-\beta) \quad \text{... (1)}$$

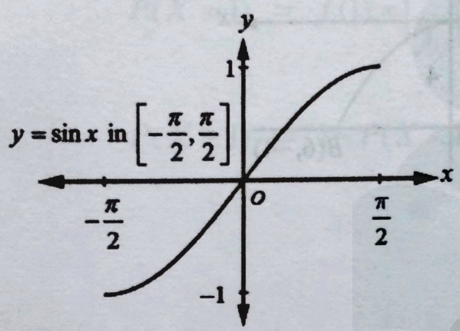
$$\hat{a}\cdot\hat{b} = \cos\alpha\cos\beta + \sin\alpha\sin\beta \quad \text{... (1)}$$

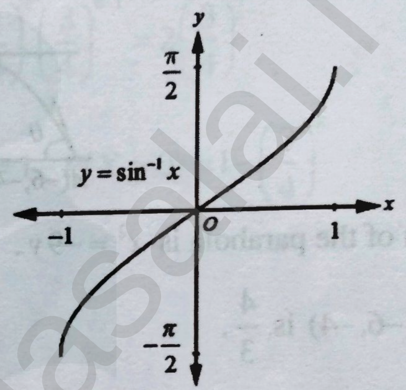
$$\cos(\alpha-\beta) = \cos\alpha\cos\beta + \sin\alpha\sin\beta. \quad \text{... (1)}$$

Note : The result can be proved by drawing a unique circle also.

44. (a) $\sin \quad : \left[-\frac{\pi}{2},\frac{\pi}{2}\right] \to [-1, 1]$ ... (1)

$\sin^{-1} \quad : [-1, 1] \to \left[-\frac{\pi}{2}\ \frac{\pi}{2}\right]$ ... (1)





First diagram ... (1)

Second diagram ... (2)

44. (b) $\frac{di}{dt} + \frac{R}{L}i = \frac{E}{L}.$ ... (1)

$$\text{I.F} = e^{\int P dt} = e^{\int \frac{R}{L} dt} = e^{\frac{R}{L}t}. \quad \text{... (1)}$$

Soution is $i = \frac{E}{R} + ce^{-\frac{R}{L}t}$ ... (2)

When $E = 0$, we get $i = ce^{-\frac{R}{L}t}$. ... (1)

45. (a)

p	q	$\neg q$	$r : p \leftrightarrow q$	$\neg r$	$s : p \leftrightarrow \neg q$
T	T	F	T	F	F
T	F	T	F	T	T
F	T	F	F	T	T
F	F	T	T	F	T

Third, fourth, fifth and sixth column each one mark ... (4)

From the table they are equivalent. ... (1)

PROFILE
MODEL PAPER - IV

Part I		Part II		Part III		Part IV	
1	Exercise 1.8 (17)	21	Example 1.11	31	Exercise 1.1 (4)	41 a	Exercise 1.5 (4)
2	Chapter – 1 Created (2)	22	Example 3.12	32	Exercise 2.5 (5)	41 b	Exercise 2.8 (3)
3	Exercise 2.9 (21)	23	Exercise 4.2 (3)	33	Chapter – 3 Created	42 a	Chapter – 5 Created
4	Chapter – 3 Created	24	Example 6.47	34	Example 6.36	42 b	Chapter – 4 Created
5	Exercise 4.6 (14)	25	Exercise 7.1 (6)	35	Exercise 7.3 (2 – iii)	43 a	Exercise 5.5 (2)
6	Exercise 5.6 (14)	26	Chapter – 8 Created	36	Chapter – 8 Created	43 b	Example 6.2 (i)
7	Exercise 6.10 (2)	27	Chapter – 10 Created	37	Exercise 9.5 (1 - ii)	44 a	Example 7.62
8	Chapter – 6 Created (5)	28	Exercise 11.3 (2 – iii)	38	Example 11.16	44 b	Example 10.25
9	Chapter – 7 Created (15)	29	Example 12.8	39	Exercise 12.2 (9)	45 a	Example 9.56
10	Exercise 7.10 (7)	30	Chapter – 2 Created	40	Chapter – 4 Created	45 b	Chapter – 9 Created
11	Exercise 8.8 (3)					46 a	Example 10.27
12	Chapter – 8 Created (1)					46 b	Exercise 11.5 (7)
13	Exercise 9.10 (14)					47 a	Example 7.45
14	Exercise 9.10 (1)					47 b	Exercise 6.8 (2)
15	Chapter – 10 Created -						
16	Exercise 10.9 (13)						
17	Chapter – 11 Created (1)						
18	Chapter – 11 Created (8)						
19	Exercise 12.3 (3)						
20	Chapter – 12 Created (8)						

HIGHER SECONDARY SECOND YEAR

MATHEMATICS

MODEL QUESTION PAPER – 4

Time Allowed: 15 Min + 3.00 Hours] **[Maximum Marks:90**

Instructions:

(a) Check the question paper for fairness of printing. If there is any lack of fairness, inform the Hall Supervisor immediately.

(b) Use **Blue** or **Black** ink to write and underline and pencil to draw diagrams.

PART – I

Note: (i) **All** questions are **compulsory.** $20 \times 1 = 20$

(ii) Choose the correct or most suitable answer from the given **four** alternatives. Write the option code and the corresponding answer.

1. If $adjA = \begin{bmatrix} 2 & 3 \\ 4 & -1 \end{bmatrix}$ and $adjB = \begin{bmatrix} 1 & -2 \\ -3 & 1 \end{bmatrix}$ then $adj(AB)$ is

(1) $\begin{bmatrix} -7 & -1 \\ 7 & -9 \end{bmatrix}$ (2) $\begin{bmatrix} -6 & 5 \\ -2 & -10 \end{bmatrix}$ (3) $\begin{bmatrix} -7 & 7 \\ -1 & -9 \end{bmatrix}$ (4) $\begin{bmatrix} -6 & -2 \\ 5 & -10 \end{bmatrix}$

2. Which of the following are incorrect?

(i) A is non singular and $AB = AC \Rightarrow B = C$

(ii) A is non singular and $BA = CA \Rightarrow B = C$

(iii) A and B are non singular of same order then $(AB)^{-1} = B^{-1}A^{-1}$

(iv) A is non singular then $A = \left(A^{-1}\right)^{-1}$

(1) none (2) (i) and (ii) (3) (ii) and (iii) (4) (iii) and (iv)

3. If α and β are the roots of $x^2 + x + 1 = 0$, then $\alpha^{2020} + \beta^{2020}$ is

(1) -2 (2) -1 (3) 1 (4) 2

4. If $p + \sqrt{q}$ and $-i\sqrt{q}$ are the roots of a polynomial equation with rational coefficients then the least possible degree of the equation is

(1) 2 (2) 1 (3) 3 (4) 4

5. $\sin^{-1}(2\cos^2 x - 1) + \cos^{-1}(1 - 2\sin^2 x) =$

(1) $\frac{\pi}{2}$ (2) $\frac{\pi}{3}$ (3) $\frac{\pi}{4}$ (4) $\frac{\pi}{6}$

6. Tangents are drawn to the hyperbola $\frac{x^2}{9}-\frac{y^2}{4}=1$ parallel to the straight line $2x-y=1$. One of the points of contact of tangents on the hyperbola is

(1) $\left(\frac{9}{2\sqrt{2}},\frac{-1}{\sqrt{2}}\right)$ (2) $\left(\frac{-9}{2\sqrt{2}},\frac{1}{\sqrt{2}}\right)$ (3) $\left(\frac{9}{2\sqrt{2}},\frac{1}{\sqrt{2}}\right)$ (4) $\left(3\sqrt{3},-2\sqrt{2}\right)$

7. If a vector $\vec{\alpha}$ lies in the plane which contains $\vec{\beta}$ and $\vec{\gamma}$, then

(1) $\left[\vec{\alpha},\vec{\beta},\vec{\gamma}\right]=1$ (2) $\left[\vec{\alpha},\vec{\beta},\vec{\gamma}\right]=-1$ (3) $\left[\vec{\alpha},\vec{\beta},\vec{\gamma}\right]=0$ (4) $\left[\vec{\alpha},\vec{\beta},\vec{\gamma}\right]=2$

8. Which one of the following is insufficient to find the equation of a straight line?

(1) two points on the line

(2) one point on the line and direction ratios of one parallel line

(3) one point on the line and direction ratios of its perpendicular line

(4) a perpendicular line and a parallel line in Cartesian form.

9. The slant asymptote of $f(x)=\frac{x^2-6x+7}{x+5}$ is

(1) $x+y+11=0$ (2) $x+y-11=0$ (3) $x=-5$ (4) $y=x-11$

10. The slope of the normal to the curve $f(x)=2\cos 4x$ at $x=\frac{\pi}{12}$ is

(1) $-4\sqrt{3}$ (2) -4 (3) $\frac{\sqrt{3}}{12}$ (4) $4\sqrt{3}$

11. If $u(x,y)=e^{x^2+y^2}$, then $\frac{\partial u}{\partial x}$ is equal to

(1) $e^{x^2+y^2}$ (2) $2xu$ (3) x^2u (4) y^2u

12. Identify the incorrect statements

(i) absolute error = | Actual value − app. value |

(ii) relative error = $\frac{\text{absolute error}}{\text{actual value}}$

(iii) percentage error = relative error ×100

(iv) absolute error has unit of measurement but relative error and percentage errors are unit free

(1) all (2) (i) and (ii) only (3) (i), (ii), (iii) only (4) none

13. The value of $\int_0^{\infty} e^{-3x}x^2\,dx$ is

(1) $\frac{7}{27}$ (2) $\frac{5}{27}$ (3) $\frac{4}{27}$ (4) $\frac{2}{27}$

14. The value of $\int_0^{\frac{2}{3}} \frac{dx}{\sqrt{4-9x^2}}$ is

(1) $\frac{\pi}{6}$ (2) $\frac{\pi}{2}$ (3) $\frac{\pi}{4}$ (4) π

15. The order and degree of the differential equation

$\left(\frac{d^3y}{dx^3}\right)^{\frac{2}{3}} - 3\frac{d^2y}{dx^2} + 5\frac{dy}{dx} + 4 = 0$ are

(1) $2,3$ (2) $2,2$ (3) $3,2$ (4) $\frac{2}{3},2$

16. The solution of the differential equation $\frac{dy}{dx} + \frac{1}{\sqrt{1-x^2}} = 0$ is

(1) $y + \sin^{-1} x = c$ (2) $x + \sin^{-1} y = 0$

(3) $y^2 + 2\sin^{-1} x = c$ (4) $x^2 + 2\sin^{-1} y = 0$

17. A random variable X is a function from

(1) $S \to \mathbb{R}$ (2) $\mathbb{R} \to S$ (3) $S \to \mathbb{N}$ (4) $\mathbb{N} \to S$

18. With usual notations, which of the following are correct?

(i) $Var(X) = E(X^2) - [E(X)]^2$

(ii) $Var(aX + b) = a^2 Var(X)$

(iii) $E(aX + b) = aE(X) + b$

(iv) $E(X) = \int_{-\infty}^{\infty} f(x)dx$ if X is continuous

(1) all (2) (i), (ii), (iii) only (3) (i), (ii), (iv) only (4) (ii), (iii), (iv) only

19. Which one of the following is a binary operation on $\mathbb{N}$?

(1) Subtraction (2) Multiplication (3) Division (4) All the above

20. The fourth roots of unity under multiplication satisfies the properties

(1) closure only (2) closure and associative only

(3) closure, associative and identity (4) closure, associative identity and inverse

PART – II

$7 \times 2 = 14$

Note: (i) Answer any **SEVEN** questions.

(ii) Question number **30 is compulsory.**

21. Prove that $\begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}$ is orthogonal.

22. If $x^2 + 2(k+2)x + 9k = 0$ has equal roots, find k.

23. Is $\cos^{-1}(-x)=\pi-\cos^{-1}(x)$ true? Justify your answer.

24. Find the acute angle between the planes $\vec{r}\cdot\left(2\hat{i}+2\hat{j}+2\hat{k}\right)=11$ and $4x-2y+2z=15$.

25. A stone is dropped into a pond causing ripples in the form of concentric circles. The radius r of the outer ripple is increasing at a constant rate at 2 cm per second. When the radius is 5 cm find the rate of changing of the total area of the disturbed water?

26. If $y=10^x$, find dy.

27. Obtain the differential equation for $y=mx+c$ where m is the arbitrary constant and c is a ordinary constant.

28. The probability density function of X is $f(x)=\begin{cases} x & 0<x<1 \\ 2-x & 1\le x<2 \\ 0 & \text{otherwise}\end{cases}$

Find $P(0.5\le X<1.5)$

29. Let $A=\begin{bmatrix}0 & 1\\ 1 & 1\end{bmatrix}$, $B=\begin{bmatrix}1 & 1\\ 0 & 1\end{bmatrix}$ be any two boolean matrices of the same type. Find $A\vee B$ and $A\wedge B$.

30. Prove that $\sum_{n=1}^{204}\left(i^{n+1}-i^{n+2}\right)=0$.

PART– III

Note: (i) Answer any **SEVEN** questions. $7\times3=21$

(ii) Question number **40 is compulsory.**

31. If $A=\begin{bmatrix}5 & 3\\ -1 & -2\end{bmatrix}$, show that $A^2-3A-7I_2=0$. Hence find A^{-1}.

32. If $|z|=1$, show that $2\le|z^2-3|\le4$.

33. Discuss the real and imaginary roots of $x^5+x^3+x^2+1=0$.

34. Find the shortest distance between the given straight lines $\vec{r}=(2\hat{i}+3\hat{j}+4\hat{k})+t(-2\hat{i}+\hat{j}-2\hat{k})$ and $\frac{x-3}{2}=\frac{y}{-1}=\frac{z+2}{2}$.

35. Using the Rolle's theorem, determine the values of x at which the tangent is parallel to the x-axis for the function $f(x)=\sqrt{x}-\frac{x}{3}$, $x\in[0,9]$.

36. The edge of a cube was found to be 30*cm* with a possible error in measurement of 0.1*cm*. Use differentials to estimate the maximum possible error in computing

(i) Volume of the cube (ii) Surface area of the cube

37. Evaluate the following: $\int_0^{\frac{\pi}{2}} \frac{dx}{5+4\sin^2 x}$

38. Suppose that $f(x)$ given below represents a probability mass function,

x	1	2	3	4	5	6
$f(x)$	c^2	$2c^2$	$3c^2$	$4c^2$	c	$2c$

Find (i) the value of c (ii) Mean and variance.

39. Prove that $q \to p \equiv \neg p \to \neg q$.

40. Find the equation of the parabola with focus $(-1,-2)$ and the directrix is $x-2y+3=0$.

PART – IV

Note: Answer **all** the questions. $7 \times 5 = 35$

41. (a) A boy is walking along the path $y = ax^2 + bx + c$ $(-6,8), (-2,-12)$ and $(3,8)$. He wants to meet his friend at $P(7,60)$. Will he meet his friend?

(OR)

(b) Find the value of $\left(\frac{1+\sin\frac{\pi}{10}+i\cos\frac{\pi}{10}}{1+\sin\frac{\pi}{10}-i\cos\frac{\pi}{10}} \right)^{10}$.

42. (a) Find the equation of the circle passing through the points $(1,8), (7,2)$ and $(1,-4)$.

(OR)

(b) Find the domain of the function $f(x) = \sin^{-1}\frac{x}{2} + \cos^{-1}\frac{x}{3}$.

43. (a) A tunnel through a mountain for a four lane highway is to have a elliptical opening. The total width of the highway (not the opening) is to be 16m, and the height at the edge of the road must be sufficient for a truck 4m high to clear if the highest point of the opening is to be 5m approximately. How wide must the opening be?

(OR)

(b) With usual notations, in any triangle ABC, prove $a = b\cos C + c\cos B$.

44. (a) A 12 unit square piece of thin material is to be made an open box by cutting small squares from the four corners and folding the sides up. Find the length of the side of the square to be removed, when the volume is maximum and also find the maximum volume.

(OR)

(b) Solve : $(1+x^3)\frac{dy}{dx} + 6x^2 y = 1 + x^2$.

45. (a) Find the area of the region bounded by $y=\cos x, y=\sin x$, the lines $x=\frac{\pi}{4}$ and $x=\frac{5\pi}{4}$.

(OR)

(b) Find the volume of the solid generated when the region enclosed by $y=\sqrt{x}$, $y=2$ an $x=0$ revolved about the y-axis.

46. (a) The growth of a population is proportional to the number present. If the population of colony doubles in 50 years, in how many years will the population become triple?

(OR)

(b) The mean and standard deviation of a binomial variate X are respectively 6 and 2. Find (i) the probability mass function (ii) $P(X=3)$ (iii) $P(X \geq 2)$.

47. (a) Evaluate : $\lim_{x \to 1^+} x^{\frac{1}{1-x}}$.

(OR)

(b) Show that the lines $\frac{x-2}{1}=\frac{y-3}{1}=\frac{z-4}{3}$ and $\frac{x-1}{-3}=\frac{y-4}{2}=\frac{z-5}{1}$ are coplanar. Also, find the cartesian equation of the plane which contains these two lines.

MODEL QUESTION PAPER – IV

ANSWERS

PART – I

Qn. No.	Option	Answer	Qn. No.	Option	Answer
1.	(2)	$\begin{bmatrix} -6 & 5 \\ -2 & -10 \end{bmatrix}$	11.	(2)	$2xu$
2.	(1)	none	12.	(4)	none
3.	(2)	-1	13.	(4)	$\frac{2}{27}$
4.	(4)	4	14.	(1)	$\frac{\pi}{6}$
5.	(1)	$\frac{\pi}{2}$	15.	(3)	3, 2
6.	(3)	$\left(\frac{9}{2\sqrt{2}}, \frac{1}{\sqrt{2}}\right)$	16.	(1)	$y+\sin^{-1}x=c$
7.	(3)	$\left[\vec{\alpha},\vec{\beta},\vec{\gamma}\right]=0$	17.	(1)	$S \to \mathbb{R}$
8.	(4)	a perpendicular line and a parallel line in Cartesian form.	18.	(2)	(i), (ii), (iii) only
9.	(4)	$y=x-11$	19.	(2)	Multiplication
10.	(3)	$\frac{\sqrt{3}}{12}$	20.	(4)	Closure, associative, identity and inverse

For writing the correct option and the answer – one mark.

PART – II

21. Prove that $\begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}$ is orthogonal.

Solution

$$\text{Let } A = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix} \text{ Then, } A^T = \begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix}$$

$$A^{-1} = \frac{1}{1}\begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix}$$

$$A^T = A^{-1}$$

$\therefore$ A is orthogonal.

Note : One can prove by proving $AA^T = A^TA = I$.

22. If $x^2+2(k+2)x+9k=0$ has equal roots, find k.

Solution

Here $\Delta = b^2-4ac=0$ for equal roots.

This implies $4(k+2)^2-4(9)k=0$ i.e., $k=4$ or 1.

23. Is $\cos^{-1}(-x)=\pi-\cos^{-1}(x)$ true? Justify your answer.

Solution

Let $\cos^{-1}(-x)=t \Rightarrow \cos t=-x$

$$x = -\cos t$$
$$x = \cos(\pi-t)$$
$$\cos^{-1}x = \pi-t$$
$$t = \pi-\cos^{-1}x$$
$$\cos^{-1}(-x) = \pi-\cos^{-1}(x)$$

Yes, it is true.

24. Find the acute angle between the planes $\vec{r}\cdot\left(2\hat{i}+2\hat{j}+2\hat{k}\right)=11$ and $4x-2y+2z=15$.

Solution

The normal vectors of the planes $\vec{r}\cdot\left(2\hat{i}+2\hat{j}+2\hat{k}\right)=11$ and $4x-2y+2z=15$ are

$\vec{n}_1=2\hat{i}+2\hat{j}+2\hat{k}$ and $\vec{n}_2=4\hat{i}-2\hat{j}+2\hat{k}$.

If θ is the acute angle between the planes, then

$$\cos\theta = \left(\frac{\left|(2\hat{i}+2\hat{j}+2\hat{k})\cdot(4\hat{i}-2\hat{j}+2\hat{k})\right|}{\left|2\hat{i}+2\hat{j}+2\hat{k}\right|\left|4\hat{i}-2\hat{j}+2\hat{k}\right|}\right) = \left(\frac{\sqrt{2}}{3}\right) \Rightarrow \theta = \cos^{-1}\left(\frac{\sqrt{2}}{3}\right)$$

25. A stone is dropped into a pond causing ripples in the form of concentric circles. The radius r of the outer ripple is increasing at a constant rate at 2 cm per second. When the radius is 5 cm find the rate of changing of the total area of the disturbed water?

Solution

$$A = \pi r^2$$

$$\frac{dA}{dt} = 2\pi r\frac{dr}{dt} = 2\pi\times5\times2 = 20\pi \text{cm}^2/\text{sec}\ .$$

26. If $y = 10^x$, find dy.

Solution

$$y = 10^x$$

$$dy = 10^x \log 10\cdot dx\,.$$

27. Obtain the differential equation for $y = mx + c$ where m is the arbitrary constant and c is an ordinary constant.

Solution

$$y = mx + c \Rightarrow y' = m$$

Replace m with y'.

$$y = y'x + c\,.$$

28. The probability density function of X is $f(x) = \begin{cases} x & 0 < x < 1 \\ 2-x & 1 \le x < 2 \\ 0 & \text{otherwise} \end{cases}$

Find $P(0.5 \le X < 1.5)$

Solution

$$P(0.5 \le X < 1.5) = \int_{0.5}^{1.5} f(x)\,dx = \int_{0.5}^{1.0} x\,dx + \int_{1.0}^{1.5}(2-x)\,dx$$

$$= \left[\frac{x^2}{2}\right]_{0.5}^{1.0} + \left[2x - \frac{x^2}{2}\right]_{1.0}^{1.5} = [0.5 - 0.125] + [3.0 - 1.125 - 2 + 0.5]$$

$$= 0.75\,.$$

29. Let $A=\begin{bmatrix}0 & 1\\1 & 1\end{bmatrix}$, $B=\begin{bmatrix}1 & 1\\0 & 1\end{bmatrix}$ be any two boolean matrices of the same type. Find $A\vee B$ and $A\wedge B$.

Solution

$$\text{Then } A\vee B = \begin{bmatrix}0 & 1\\1 & 1\end{bmatrix}\vee\begin{bmatrix}1 & 1\\0 & 1\end{bmatrix}=\begin{bmatrix}0\vee 1 & 1\vee 1\\1\vee 0 & 1\vee 1\end{bmatrix}=\begin{bmatrix}1 & 1\\1 & 1\end{bmatrix}$$

$$A\wedge B = \begin{bmatrix}0 & 1\\1 & 1\end{bmatrix}\wedge\begin{bmatrix}1 & 1\\0 & 1\end{bmatrix}=\begin{bmatrix}0\wedge 1 & 1\wedge 1\\1\wedge 0 & 1\wedge 1\end{bmatrix}=\begin{bmatrix}0 & 1\\0 & 1\end{bmatrix}$$

30. Prove that $\sum_{n=1}^{204}\left(i^{n+1}-i^{n+2}\right)=0$.

Solution

$$\sum_{n=1}^{204}\left(i^{n+1}-i^{n+2}\right)=\sum_{n=1}^{204}i^{n+1}-\sum_{n=1}^{204}i^{n+2}=0-0=0$$

Since each sum is a multiple of four and sum of four consecutive powers is 0.

PART– III

31. If $A=\begin{bmatrix}5 & 3\\-1 & -2\end{bmatrix}$, show that $A^2-3A-7I_2=O$. Hence find A^{-1}.

Solution

$$A^2 = A\cdot A=\begin{bmatrix}5 & 3\\-1 & -2\end{bmatrix}\begin{bmatrix}5 & 3\\-1 & -2\end{bmatrix} = \begin{bmatrix}22 & 9\\-3 & 1\end{bmatrix}$$

$$A^2-3A-7I_2 = \begin{bmatrix}22 & 9\\-3 & 1\end{bmatrix}+\begin{bmatrix}-15 & -9\\3 & 6\end{bmatrix}+\begin{bmatrix}-7 & 0\\0 & -7\end{bmatrix}=\begin{bmatrix}0 & 0\\0 & 0\end{bmatrix}$$

$$A^2-3A-7I_2 = O$$

$$A^2-3A = 7I_2$$

Pre multiply both sides by A^{-1}, we get

$$A-3I = 7A^{-1}$$

$$A^{-1} = \frac{1}{7}(A-3I)$$

$$A-3I = \begin{bmatrix}5 & 3\\-1 & -2\end{bmatrix}-3\begin{bmatrix}1 & 0\\0 & 1\end{bmatrix} = \begin{bmatrix}2 & 3\\-1 & -5\end{bmatrix}$$

$$A^{-1} = \frac{1}{7}(A-3I)=\frac{1}{7}\begin{bmatrix}2 & 3\\-1 & -5\end{bmatrix}.$$

32. If $|z|=1$, show that $2 \leq |z^2-3| \leq 4$.

Solution

$$|z^2-3| \geq \left||z|^2-|-3|\right| = \left||z|^2-3\right| = |1-3| = 2$$

$$|z^2-3| \leq |z^2|+|-3| = |z|^2+3 = 1+3 = 4$$

$\therefore\ 2 \leq |z^2-3| \leq 4$.

33. Discuss the real and imaginary roots of $x^5+x^3+x^2+1=0$.

Solution

The number of sign changes in $p(x)$ is 0.

There is no positive real root.

The number of sign changes in $p(-x)$ is $(-\ -\ +\ +)$ 1

It has at most one negative root.

Totally it has atmost one real root and hence atleast 4 imaginary roots.

Since the degree is 5, it has one real root and 4 imaginary roots.

34. Find the shortest distance between the given straight lines $\vec{r}=(2\hat{i}+3\hat{j}+4\hat{k})+t(-2\hat{i}+\hat{j}-2\hat{k})$ and $\dfrac{x-3}{2}=\dfrac{y}{-1}=\dfrac{z+2}{2}$.

Solution

The vector equations of the given straight lines are

$$\vec{r} = (2\hat{i}+3\hat{j}+4\hat{k})+t(-2\hat{i}+\hat{j}-2\hat{k})$$

$$\text{and}\quad \vec{r} = (3\hat{i}-2\hat{k})+t(2\hat{i}-\hat{j}+2\hat{k})$$

Comparing the given two equations with $\vec{r}=\vec{a}+t\vec{b},\ \vec{r}=\vec{c}+s\vec{d}$

we have $\vec{a}=2\hat{i}+3\hat{j}+4\hat{k},\ \vec{b}=-2\hat{i}+\hat{j}-2\hat{k},\ \vec{c}=3\hat{i}-2\hat{k},\ \vec{d}=2\hat{i}-\hat{j}+2\hat{k}$.

Clearly, $\vec{b}$ is a scalar multiple of $\vec{d}$, and hence the two straight lines are parallel.

$$\text{Distance} = \frac{|(\vec{c}-\vec{a})\times\vec{b}|}{|\vec{b}|}.$$

$$\text{Now,}\quad (\vec{c}-\vec{a})\times\vec{b} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 1 & -3 & -6 \\ -2 & 1 & -2 \end{vmatrix} = 12\hat{i}+14\hat{j}-5\hat{k}$$

$$\text{Therefore, distance} = \frac{|12\hat{i}+14\hat{j}-5\hat{k}|}{|-2\hat{i}+\hat{j}-2\hat{k}|} = \frac{\sqrt{365}}{3}.$$

35. Using the Rolle's theorem, determine the values of x at which the tangent is parallel to the x-axis for the function $f(x)=\sqrt{x}-\frac{x}{3},\ x\in[0,9]$.

Solution

$f(x)$ is a continuous in $[0,9]$ and differentiable in $(0,9)$.

$$f(0)=0,\ f(9)=3-\frac{9}{3}=0 \Rightarrow f(0)=f(9)$$

All the three conditions are satisfied.

$$f'(x)=\frac{1}{2\sqrt{x}}-\frac{1}{3}$$

$$f'(x)=0 \Rightarrow \frac{1}{2\sqrt{x}}-\frac{1}{3}=0$$

$$\frac{1}{\sqrt{x}}=\frac{2}{3}$$

$$x=\frac{9}{4}\in(0,9) \Rightarrow \therefore c=\frac{9}{4}.$$

36. The edge of a cube was found to be $30cm$ with a possible error in measurement of $0.1cm$. Use differentials to estimate the maximum possible error in computing

(i) Volume of the cube (ii) Surface area of the cube

Solution

(i)
$$V=a^3$$
$$dV=3a^2da$$
$$=3\times900\times0.1=270cm^3$$

(ii)
$$S=6a^2$$
$$dS=12a\cdot da=12\times30\times0.1=36cm^2$$

37. Evaluate : $\int_0^{\frac{\pi}{2}}\frac{dx}{5+4\sin^2 x}$

Solution

$$\text{Let } I=\int_0^{\frac{\pi}{2}}\frac{dx}{5+4\sin^2 x}$$

{Dividing both numerator and denominator by $\cos^2 x$

$$=\int_0^{\frac{\pi}{2}}\frac{\sec^2 x}{5\sec^2 x+4\tan^2 x}dx=\int_0^{\frac{\pi}{2}}\frac{\sec^2 x}{5(1+\tan^2 x)+4\tan^2 x}dx$$

$$=\int_0^{\frac{\pi}{2}}\frac{\sec^2 x}{5+5\tan^2 x+4\tan^2 x}dx$$

$$I = \int_0^{\frac{\pi}{2}} \frac{\sec^2 x}{9\tan^2 x + 5} dx$$

Let $\tan x = t$

when $x = 0,\ t = 0$; when $x = \frac{\pi}{2}, t = \infty$

$$\therefore I = \int_0^{\infty} \frac{dt}{9t^2+5} = \frac{1}{9}\int_0^{\infty} \frac{1}{t^2 + \left(\frac{\sqrt{5}}{3}\right)^2} dt = \frac{1}{9}\cdot\frac{1}{\left(\frac{\sqrt{5}}{3}\right)}\cdot\tan^{-1}\left(\frac{t}{\frac{\sqrt{5}}{3}}\right)$$

$$= \frac{1}{3\sqrt{5}}\left[\tan^{-1}\left(\frac{3t}{\sqrt{5}}\right)\right]_0^{\infty} = \frac{1}{3\sqrt{5}}\left[\frac{\pi}{2} - 0\right]$$

$$= \frac{\pi}{6\sqrt{5}}.$$

38. Suppose that $f(x)$ given below represents a probability mass function,

x	1	2	3	4	5	6
$f(x)$	c^2	$2c^2$	$3c^2$	$4c^2$	c	$2c$

Find (i) the value of c (ii) Mean and variance.

Solution

(i)
$$\sum_x f(x) = 1$$

$$c^2 + 2c^2 + 3c^2 + 4c^2 + c + 2c = 1$$

$$c = \frac{1}{5} \text{ or } -\frac{1}{2}$$

c is $\frac{1}{5}$ since c can't be negative

Hence, the probability mass function is (in the first two rows)

x	1	2	3	4	5	6
$f(x)$	$\frac{1}{25}$	$\frac{2}{25}$	$\frac{3}{25}$	$\frac{4}{25}$	$\frac{1}{5}$	$\frac{2}{5}$
$xf(x)$	$\frac{1}{25}$	$\frac{4}{25}$	$\frac{9}{25}$	$\frac{16}{25}$	$\frac{5}{5}$	$\frac{12}{5}$
$x^2 f(x)$	$\frac{1}{25}$	$\frac{8}{25}$	$\frac{27}{25}$	$\frac{64}{25}$	$\frac{25}{5}$	$\frac{72}{5}$

$\sum x f(x) = \frac{115}{25}$	$\sum x^2 f(x) = \frac{585}{25}$

Mean : $E(X) = \sum x f(x) = \frac{115}{25} = 4.6$

Variance : $V(X) = E(X^2) - (E(X))^2 = \sum x^2 f(x) - \left(\sum x f(x)\right)^2$

$$= \frac{585}{25} - \left(\frac{115}{25}\right)^2 = 23.40 - 21.16 = 2.24$$

Therefore the mean and variance are 4.6 and 2.24 respectively.

39. Prove that $q \to p \equiv \neg p \to \neg q$.

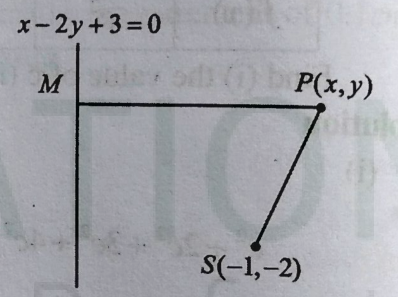
Solution

p	q	$r: q \to p$	$\neg p$	$\neg q$	$s: \neg p \to \neg q$
T	T	T	F	F	T
T	F	T	F	T	T
F	T	F	T	F	F
F	F	T	T	T	T

From the table they are equivalent.

40. Find the equation of the parabola with focus $(-1,-2)$ and the directrix is $x - 2y + 3 = 0$.

Solution



$$\frac{SP}{PM} = e = 1 \Rightarrow SP^2 = PM^2$$

$$\Rightarrow (x+1)^2 + (y+2)^2 = \left(\frac{x-2y+3}{\sqrt{1+4}}\right)^2$$

$$\Rightarrow 5\left[(x+1)^2 + (y+2)^2\right] = (x-2y+3)^2.$$

The equation of the parabola is

$4x^2 + y^2 + 4x + 32y + 16 = 0$.

PART – IV

41. (a) A boy is walking along the path $y = ax^2 + bx + c$ through the points $(-6,8), (-2,-12)$ and $(3,8)$. He wants to meet his friend at $P(7,60)$. Will he meet his friend?

Solution

$$y = ax^2 + bx + c$$

The curve passes through the points $(-6,8), (-2,-12)$ and $(3,8)$.

$$\therefore \quad 36a - 6b + c = 8$$

$$4a - 2b + c = -12$$

$$9a + 3b + c = 8$$

$$[A \quad B] = \left[\begin{array}{ccc|c} 36 & -6 & 1 & 8 \\ 4 & -2 & 1 & -12 \\ 9 & 3 & 1 & 8 \end{array}\right]$$

$$\sim \begin{bmatrix} 4 & -2 & 1 & -12 \\ 36 & -6 & 1 & 8 \\ 9 & 3 & 1 & 8 \end{bmatrix} R_1 \leftrightarrow R_2$$

$$\sim \begin{bmatrix} 4 & -2 & 1 & -12 \\ 0 & 12 & -8 & 116 \\ 0 & 30 & -5 & 140 \end{bmatrix} \begin{matrix} \\ R_2 \to R_2 - 9R_1 \\ R_3 \to 4R_3 - 9R_1 \end{matrix}$$

$$\sim \begin{bmatrix} 4 & -2 & 1 & -12 \\ 0 & 3 & -2 & 29 \\ 0 & 6 & -1 & 28 \end{bmatrix} \begin{matrix} \\ R_2 \to R_2 / 4 \\ R_3 \to R_3 / 5 \end{matrix}$$

$$\sim \begin{bmatrix} 4 & -2 & 1 & -12 \\ 0 & 3 & -2 & 29 \\ 0 & 0 & 3 & -30 \end{bmatrix} \begin{matrix} \\ \\ R_3 \to R_3 - 2R_2 \end{matrix}$$

$$\Rightarrow 3c = -30 \Rightarrow c = -10$$

$$3b - 2c = 29 \Rightarrow b = 3$$

$$4a - 2b + c = -12 \Rightarrow a = 1$$

The equation is $y = x^2 + 3x - 10$.

The point $P(7,60)$ satisfies this equation. Hence the boy will meet his friend.

41. (b) Find the value of $\left(\dfrac{1+\sin\frac{\pi}{10}+i\cos\frac{\pi}{10}}{1+\sin\frac{\pi}{10}-i\cos\frac{\pi}{10}}\right)^{10}$.

Solution

Let $z = \sin\dfrac{\pi}{10} + i\cos\dfrac{\pi}{10}$

$\overline{z} = \sin\dfrac{\pi}{10} - i\cos\dfrac{\pi}{10}$

But here $\dfrac{1}{z} = \overline{z}$ since $|z| = 1$

$$\left[\frac{1+\sin\frac{\pi}{10}+i\cos\frac{\pi}{10}}{1+\sin\frac{\pi}{10}-i\cos\frac{\pi}{10}}\right]^{10} = \left(\frac{1+z}{1+\overline{z}}\right)^{10} = \left(\frac{1+z}{1+\frac{1}{z}}\right)^{10} = z^{10}$$

$$= \left(\sin\frac{\pi}{10} + i\cos\frac{\pi}{10}\right)^{10}$$

$$= \left[\cos\left(\frac{\pi}{2} - \frac{\pi}{10}\right) + i\sin\left(\frac{\pi}{2} - \frac{\pi}{10}\right)\right]^{10}$$

$$= \cos 10\left(\frac{\pi}{2} - \frac{\pi}{10}\right) + i\sin 10\left(\frac{\pi}{2} - \frac{\pi}{10}\right)$$

$$= \cos 4\pi + i\sin 4\pi$$

$$= 1 + 0 = 1.$$

42. (a) Find the equation of the circle passing through the points $(1,8), (7,2)$ and $(1,-4)$.

Solution

The general equation of the circle is $x^2 + y^2 + 2gx + 2fy + c = 0$.

It passes through $(1,8), (7,2)$ and $(1,-4)$.

$$\therefore\ 2g + 16f + c = -65 \quad \ldots (1)$$

$$14g + 4f + c = -53 \quad \ldots (2)$$

$$2g - 8f + c = -17 \quad \ldots (3)$$

$(1)-(2) \Rightarrow \quad -g - f = -1$

$(2)-(3) \Rightarrow \quad g + f = -3$

$$f = -2,\quad g = -1 \text{ and } c = -31$$

The equation is $x^2 + y^2 - 2x - 4y - 31 = 0$ (or) $(x-1)^2 + (y-2)^2 = 6^2$.

42. (b) Find the domain of the function $f(x) = \sin^{-1}\frac{x}{2} + \cos^{-1}\frac{x}{3}$.

Solution

The domain of $\sin^{-1} x$ and $\cos^{-1} x$ is $[-1\ \ 1]$.

$\therefore$ The domain of $\sin^{-1}\frac{x}{2}$ and $\cos^{-1}\frac{x}{3}$ are $-1 \le \frac{x}{2} \le 1$ and $-1 \le \frac{x}{3} \le 1$.

i.e., $-2 \le x \le 2$ and $-3 \le x \le 3$.

Take the common i.e., $-2 \le x \le 2$.

Thus the domain of $f(x)$ is $-2 \le x \le 2$.

43. (a) A tunnel through a mountain for a highway is to have an elliptical opening. The total width of the highway (not the opening) is to be 16m, and the height at the edge of the road must be sufficient for a truck 4m high to clear if the highest point of the opening is to be 5m approximately. How wide must the opening be?

The cross section of the tunnel is in semi-elliptical shape.

Inside the tunnel, we have a roadway of width $16\,m$ at the middle.

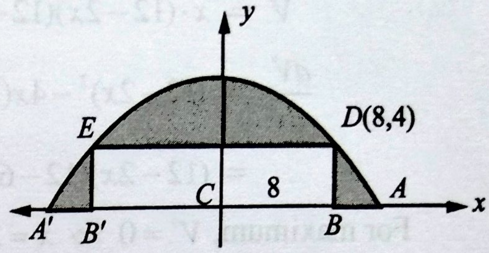
The maximum height of the tunnel is $5\,m$.

To find the width of the tunnel at the ground level, if a truck of $4\,m$ height is to cross the path.

Let the equation of the ellipse be $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$.

Length of semi minor axis $b=5$.

Therefore $\frac{x^2}{a^2}+\frac{y^2}{5^2}=1$



Let BB' be the road width and A, A' be the end points of the opening of the tunnel.

Let $CB = 8$. $BD=4$

$\therefore D$ is $(8,4)$ and lies on the ellipse.

$$\frac{8^2}{a^2}+\frac{4^2}{5^2} = 1 \Rightarrow a^2 = \frac{25}{9}\times 64 \Rightarrow a=\frac{40}{3}.$$

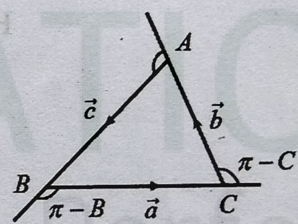
$$\text{The width } AA' = 2a = \frac{80}{3} = 26.66\,\text{m}$$

The required width is 26.66 m

43. (b) With usual notations, in any triangle ABC, prove $a=b\cos C+c\cos B$.

Solution

From the diagram,



$$\vec{a}+\vec{b}+\vec{c} = \vec{0}$$

$$\vec{a} = -\vec{b}-\vec{c}$$

$$\vec{a}\cdot\vec{a} = -\vec{a}\cdot\vec{b}-\vec{a}\cdot\vec{c}$$

$$a^2 = -|\vec{a}|\,|\vec{b}|\cos(\pi-C)-|\vec{a}|\,|\vec{c}|\cos(\pi-B)$$

$$a^2 = ab\cos C+ac\cos B$$

$$a = b\cos C+c\cos B.$$

44. (a) A 12 unit square piece of thin material is to be made an open box by cutting small squares from the four corners and folding the sides up. Find the length of the side of the square to be removed, when the volume is maximum and also find the maximum volume.

Solution

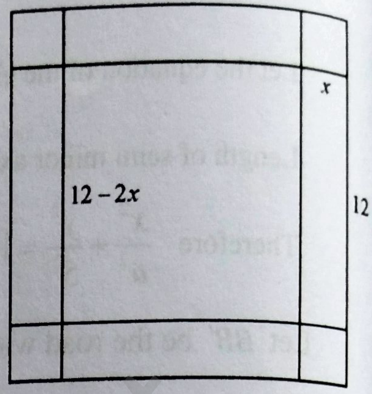
Let x = length of the cut on each side of the little squares.

V = the volume of the folded box.

The length of the base is $12-2x$.

The depth of the box after folding is x.

The problem is to maximize the volume V.



$$V = x\cdot(12-2x)(12-2x) = x(12-2x)^2$$

$$\frac{dV}{dx} = (12-2x)^2 - 4x(12-2x)$$

$$= (12-2x)(12-6x).$$

For maximum, $V' = 0 \Rightarrow x = 2, 6$.

When $x = 6$, $V = 0$. Hence $x \neq 6$

$\therefore$ $x = 2$ is the only value.

When $x = 2$, $V'' = 24x - 96 < 0$

$\therefore V$ attains the maximum volume when $x = 2$.

i.e., the length of the side of the square to be removed is 2.

The maximum volume is $2(12-4)^2 = 128$ cubic units.

44. (b) Solve : $(1+x^3)\frac{dy}{dx} + 6x^2y = 1 + x^2$.

Solution The equation is $\frac{dy}{dx} + \frac{6x^2y}{1+x^3} = \frac{1+x^2}{1+x^3}$.

This is a linear differential equation in y.

Here, $P = \frac{6x^2}{1+x^3}; Q = \frac{1+x^2}{1+x^3}$

$$\int P dx = \int \frac{6x^2}{1+x^3}dx = 2\log|1+x^3| = \log|1+x^3|^2 = \log(1+x^3)^2$$

Thus, I.F. $= e^{\int P dx} = e^{\log(1+x^3)^2} = (1+x^3)^2$

Hence the solution is $ye^{\int P dx} = \int Qe^{\int P dx}dx + C$.

That is, $y(1+x^3)^2 = \int \frac{1+x^2}{1+x^3}(1+x^3)^2\, dx + C$

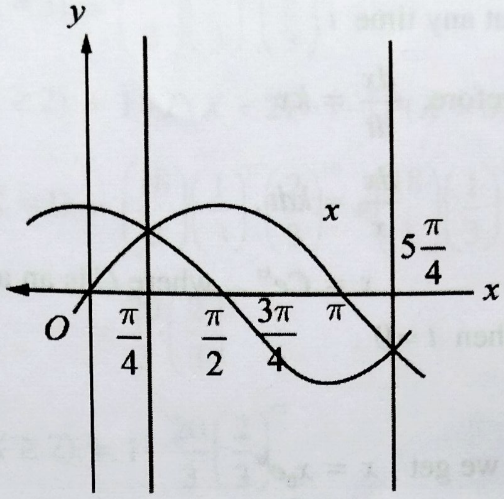
$$y(1+x^3)^2 = \int (1+x^2)(1+x^3)dx + C$$

$$y(1+x^3)^2 = \int (1+x^2+x^3+x^5)dx + C$$

$y(1+x^3)^2 = x + \frac{x^3}{3} + \frac{x^4}{4} + \frac{x^6}{6} + C$ is the required solution.

45. (a) Find the area of the region bounded by $y = \cos x, y = \sin x$, the lines $x = \frac{\pi}{4}$ and $x = \frac{5\pi}{4}$.

Solution



The upper boundary of the region is $y = \sin x$ for $\frac{\pi}{4} \le x \le \frac{5\pi}{4}$ and the lower boundary of the region is $y = \cos x$ for $\frac{\pi}{4} \le x \le \frac{5\pi}{4}$. So, the required area

$$A = \int_{\frac{\pi}{4}}^{\frac{5\pi}{4}} (y_U - y_L)\,dx = \int_{\frac{\pi}{4}}^{\frac{5\pi}{4}} (\sin x - \cos x)\,dx = (-\cos x - \sin x)_{\frac{\pi}{4}}^{\frac{5\pi}{4}}$$

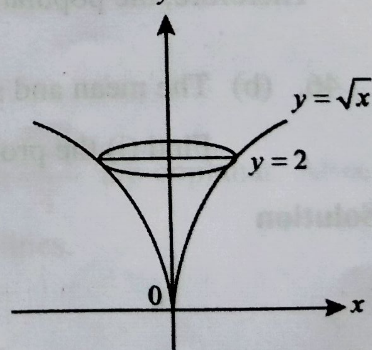
$$= \left(-\sin\frac{5\pi}{4} - \cos\frac{5\pi}{4}\right) - \left(-\sin\frac{\pi}{4} - \cos\frac{\pi}{4}\right)$$

$$= \left(-\left(-\frac{1}{\sqrt{2}}\right) - \left(-\frac{1}{\sqrt{2}}\right)\right) - \left(-\left(\frac{1}{\sqrt{2}}\right) - \left(\frac{1}{\sqrt{2}}\right)\right)$$

$$= \frac{2}{\sqrt{2}} + \frac{2}{\sqrt{2}} = 2\sqrt{2}.$$

45. (b) Find the volume of the solid generated when the region enclosed by $y = \sqrt{x}$, $y = 2$ and $x = 0$ revolved about the y-axis.

Solution



$y = \sqrt{x}$ is the positive branch of $y^2 = x$.

By taking the limits $y = 0$ and $y = 2$

Since it is revolved about y-axis,

$$V = \int_0^2 \pi x^2 dy,$$

$$= \pi \int_0^2 y^4 dy = \pi \left[\frac{y^5}{5}\right]_0^2 = \frac{32}{5}\pi.$$

46. (a) The growth of a population is proportional to the number present. If the population of a colony doubles in 50 years, in how many years will the population become triple?

Solution

Let $x(t)$ be the population at any time t.

$$\text{Therefore, } \frac{dx}{dt} = kx$$

$$\frac{dx}{x} = kdt.$$

$$x = Ce^{kt}, \text{ where } C \text{ is an arbitrary constant.}$$

Let x_0 be the population when $t=0$.

Therefore $C = x_0$.

$$\text{Thus, we get } x = x_0 e^{kt}.$$

$$\text{When } t = 50, x = 2x_0$$

$$\therefore 2x_0 = x_0 e^{50k} \Rightarrow 2 = e^{50k} \Rightarrow 50k = \log 2$$

$$k = \frac{1}{50}\log 2.$$

$$\therefore x = x_o e^{\frac{1}{50}\log 2 \cdot t}$$

$$\text{i.e., } x = x_0 2^{\frac{1}{50}t}$$

To find the time when the population is tripled. That is, $x = 3x_0$

$$3x_0 = x_0 2^{\frac{t}{50}} \Rightarrow 3 = 2^{\frac{t}{50}} \Rightarrow \log 3 = \frac{t}{50}\log 2$$

$$t = 50\frac{\log 3}{\log 2}$$

Therefore, the population is tripled in $50\left(\frac{\log 3}{\log 2}\right)$ years.

46. (b) The mean and standard deviation of a binomial variate X are respectively 6 and 2. Find (i) the probability mass function (ii) $P(X=3)$ (iii) $P(X \geq 2)$.

Solution

$$\text{Given } np = 6 \text{ and } npq = 4.$$

$$\text{Hence } q = \frac{2}{3} \text{ and } p = \frac{1}{3}, n = 18.$$

(i) Probability mass function = $P(X=x) = \binom{18}{x}\left(\frac{1}{3}\right)^x\left(\frac{2}{3}\right)^{18-x}$, $n = 0, 1, \cdots 18$.

(ii) $$P(X=3) = \binom{18}{3}\left(\frac{1}{3}\right)^{3}\left(\frac{2}{3}\right)^{15}$$

(iii) $$P(X\geq 2) = 1-P(X<2)=1-[P(X=0)+P(X=1)]$$

$$P(X=0)+P(X=1) = \binom{18}{0}\left(\frac{1}{3}\right)^{0}\left(\frac{2}{3}\right)^{18} + \binom{18}{1}\left(\frac{1}{3}\right)^{1}\left(\frac{2}{3}\right)^{17} = \left(\frac{2}{3}\right)^{17}\left(\frac{2}{3}+6\right)$$

$$= \frac{20}{3}\left(\frac{2}{3}\right)^{17}$$

$$P(X\geq 2) = 1-\frac{20}{3}\left(\frac{2}{3}\right)^{17}$$

47. (a) Evaluate : $\lim_{x\to 1^+} x^{\frac{1}{1-x}}$.

Solution

Let $g(x)=x^{\frac{1}{1-x}}$.

$$\log g(x) = \log x^{\frac{1}{1-x}}$$

$$\log g(x) = \frac{\log x}{1-x}$$

$$\lim_{x\to 1} \log g(x) = \lim_{x\to 1}\left(\frac{\log x}{1-x}\right)\left(\frac{0}{0}\text{ form}\right)$$

$$\lim_{x\to 1}\log g(x) = \lim_{x\to 1}\left(\frac{\frac{1}{x}}{-1}\right) = -1$$

But, $\lim_{x\to 1}\log g(x) = \log\left(\lim_{x\to 1} g(x)\right) = -1 \Rightarrow \lim_{x\to 1} g(x)=e^{-1}$.

$$\lim_{x\to 1} g(x) = \lim_{x\to 1} x^{\frac{x}{1-x}} = e^{-1}=\frac{1}{e}$$

47. (b) Show that the lines $\frac{x-2}{1}=\frac{y-3}{1}=\frac{z-4}{3}$ and $\frac{x-1}{-3}=\frac{y-4}{2}=\frac{z-5}{1}$ are coplanar. Also, find the cartesian equation of the plane which contains these two lines.

Solution

We have, $(x_1,y_1,z_1) = (2,3,4)$

$$(x_2,y_2,z_2) = (1,4,5)$$

$$(l_1,m_1,n_1) = (1,1,3)$$

$$(l_1,m_2,n_2) = (-3,2,1)$$

We know that the two given lines are coplanar if

$$\begin{vmatrix} x_2-x_1 & y_2-y_1 & z_2-z_1 \\ l_1 & m_1 & n_1 \\ l_2 & m_2 & n_2 \end{vmatrix} = 0$$

$$\text{Now, } \begin{vmatrix} 1-2 & 4-3 & 5-4 \\ 1 & 1 & 3 \\ -3 & 2 & 1 \end{vmatrix} = \begin{vmatrix} -1 & 1 & 1 \\ 1 & 1 & 3 \\ -3 & 2 & 1 \end{vmatrix} = 0$$

$\therefore$ The two given lines are coplanar.

Equation of the plane containing the two given coplanar lines is

$$\begin{vmatrix} x-x_1 & y-y_1 & z-z_1 \\ l_1 & m_1 & n_1 \\ l_2 & m_2 & n_2 \end{vmatrix} = 0$$

$$\Rightarrow \begin{vmatrix} x-2 & y-3 & z-4 \\ 1 & 1 & 3 \\ -3 & 2 & 1 \end{vmatrix} = 0$$

$$-5(x-2)-10(y-3)+5(z-4) = 0 \quad \text{(or)}$$

$$x+2y-z-4 = 0.$$

Note : The equation of a plane can be obtained by taking different points and different parallel vectors

MARKING SCHEME - MATHEMATICS

GENERAL INSTRUCTIONS

The marking scheme provides general guidelines to reduce subjectivity in the marking. The answer given in the marking scheme are Text Book, Solution Book and COME book bound. If a student has given any answer which is different from one given in the marking scheme, but carries the prescribed content meaning (rigorous), such answers should be given full credit with suitable distribution.

There is no separate mark allotted for formulae. If a particular stage is wrong and if the candidate writes the appropriate formula then award 1 mark for the formula irrespective of stage marks. This mark is attached with that stage. This is done with the aim that a student who did the problem correctly without writing the formula should not be penalised. That is, mark should not be deducted for not writing the formula.

In the case of Part II, Part III and Part IV, if the solution is correct then award full mark directly. The stage mark is essential only if the part of the solution is incorrect.

Only a Rough sketch of the diagram is expected and full credit must be given for such diagrams that form a part and parcel of the solution of a problem.

In questions on Variable Separable, Homogeneous and Linear differential equations, full marks should be given for an equivalent answer and the students should not be penalised for not getting the final answer as mentioned in the marking scheme.

If the method is not mentioned in the question, then one may adopt any method.

Complicated radicals and combinations need not be simplified (for example $\sqrt{7}$, $\sqrt{82}$, $\frac{3}{\sqrt{2}}$, $10c_6$ etc.)

Important Note : In an answer to a question between any two particular stages of marks (greater than one) if a student starts a stage with correct step but reaches the next stage with a wrong result then suitable credits should be given to the related steps instead of denying the entire marks meant for that stage.

PART – II

21. $A^T = \begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix}$

$A^{-1} = \frac{1}{1}\begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix}$ ... (

$A^T = A^{-1}$

A is orthogonal. ... (

Note : One can prove by proving $AA^T = A^T A = I$.

22. $4(k+2)^2 - 4(9)k = 0$ ... (

$k = 4$ or 1. ... (

23. To prove $\cos^{-1}(-x) = \pi - \cos^{-1}(x)$ ... (

Yes, it is true. ... (

24. Identifying the normal vectors

$\vec{n}_1 = 2\hat{i} + 2\hat{j} + 2\hat{k}$ and $\vec{n}_2 = 4\hat{i} - 2\hat{j} + 2\hat{k}$. ... (

$\theta = \cos^{-1}\left(\frac{\sqrt{2}}{3}\right)$ ... (

25. $\frac{dA}{dt} = 2\pi r\frac{dr}{dt}$ ... (

$= 20\pi\, cm^2 / \sec$ ... (

26. $y = 10^x$

$dy = 10^x \log 10 \cdot dx$. ... (

27. $y = mx + c \Rightarrow y' = m$ ... (

$y = y'x + c$. ... (

28. $P(0.5 \le X < 1.5) = \int_{0.5}^{1.5} f(x)\, dx = \int_{0.5}^{1.0} x\, dx + \int_{1.0}^{1.5} (2-x)\, dx$ ... (

$= 0.75$. ... (

29. $$A \vee B = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}$$ ... (1)

$$A \wedge B = \begin{bmatrix} 0 & 1 \\ 0 & 1 \end{bmatrix}$$ ... (1)

30. $$\sum_{n=1}^{204}\left(i^{n+1} - i^{n+2}\right) = \sum_{n=1}^{204} i^{n+1} - \sum_{n=1}^{204} i^{n+2}$$ ... (1)

$$= 0 - 0 = 0$$ ... (1)

PART– III

31. $$A^2 - 3A - 7I_2 = O$$ ... (1)

$$A^{-1} = \frac{1}{7}(A - 3I)$$ ... (1)

$$A^{-1} = \frac{1}{7}\begin{bmatrix} 2 & 3 \\ -1 & -5 \end{bmatrix}$$ ... (1)

32. To prove $|z^2 - 3| \geq 2$ ... (1)

To prove $|z^2 - 3| \leq 4$ ... (1)

$$2 \leq |z^2 - 3| \leq 4$$ ... (1)

33. There is no positive real root. ... (1)

It has at most one negative root. ... (1)

Since the degree is 5, it has one real root and 4 imaginary roots. ... (1)

34. To write one vector is scalar multiple of other i.e., the two lines are parallel. ... (1)

$$\text{distance} = \frac{\sqrt{365}}{3}.$$ ... (2)

35. $f(x)$ is a continuous in $[0,9]$ and differentiable in $(0,9)$. $f(0) = f(9)$ ... (1)

$$f'(x) = \frac{1}{2\sqrt{x}} - \frac{1}{3}$$ ... (1)

$$x \text{ or } c = \frac{9}{4}$$ ... (1)

36. (i)

$$V = a^3$$ ... (2)

$$dV = 3a^2 da$$

$$= 3 \times 900 \times 0.1 = 270 cm^3$$ ... (1)

(ii)

$$S = 6a^2$$

$$dS = 12a \cdot da = 12 \times 30 \times 0.1 = 36 cm^2$$ ... (1)

37.

$$I = \int_0^{\frac{\pi}{2}} \frac{dx}{5 + 4\sin^2 x} = \int_0^{\frac{\pi}{2}} \frac{\sec^2 x}{9\tan^2 x + 5} dx$$ ... (1)

$$= \int_0^{\infty} \frac{dt}{9t^2 + 5}$$ ... (1)

$$= \frac{\pi}{6\sqrt{5}}$$ ... (1)

38.

$$c = \frac{1}{5}$$ ... (1)

Mean : $E(X) = 4.6$ ... (1)

Variance : $V(X) = 2.24$

39.

p	q	$r: q \to p$	$\neg p$	$\neg q$	$s: \neg p \to \neg q$
T	T	T	F	F	T
T	F	T	F	T	T
F	T	F	T	F	F
F	F	T	T	T	T

Third column ... (1)

Sixth column ... (1)

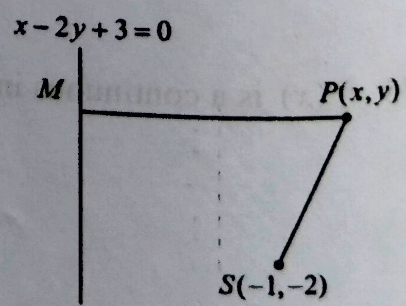
From the table they are equivalent. ... (1)

40.

$$\frac{SP}{PM} = e = 1 \Rightarrow SP^2 = PM^2$$ ... (1)

The equation of the parabola is

$4x^2 + y^2 + 4x + 32y + 16 = 0.$ ... (2)



PART – IV

41. (a)

$$36a-6b+c = 8$$
$$4a-2b+c = -12$$
$$9a+3b+c = 8 \qquad \text{... (1)}$$

$$[A \quad B] = \left[\begin{array}{ccc|c} 36 & -6 & 1 & 8 \\ 4 & -2 & 1 & -12 \\ 9 & 3 & 1 & 8 \end{array}\right]$$

$$\sim \left[\begin{array}{cccc} 4 & -2 & 1 & -12 \\ 36 & -6 & 1 & 8 \\ 9 & 3 & 1 & 8 \end{array}\right] R_1 \leftrightarrow R_2$$

$$\sim \left[\begin{array}{cccc} 4 & -2 & 1 & -12 \\ 0 & 12 & -8 & 116 \\ 0 & 30 & -5 & 140 \end{array}\right] \begin{array}{l} \\ R_2 \to R_2 - 9R_1 \\ R_3 \to 4R_3 - 9R_1 \end{array}$$

$$\sim \left[\begin{array}{cccc} 4 & -2 & 1 & -12 \\ 0 & 3 & -2 & 29 \\ 0 & 6 & -1 & 28 \end{array}\right] \begin{array}{l} \\ R_2 \to R_2 / 4 \\ R_3 \to R_3 / 5 \end{array}$$

$$\sim \left[\begin{array}{cccc} 4 & -2 & 1 & -12 \\ 0 & 3 & -2 & 29 \\ 0 & 0 & 3 & -30 \end{array}\right] \begin{array}{l} \\ \\ R_3 \to R_3 - 2R_2 \end{array}$$

$$\Rightarrow 3c = -30 \Rightarrow c = -10$$
$$3b-2c = 29 \Rightarrow b = 3$$
$$4a-2b+c = -12 \Rightarrow a = 1$$

The equation is $y = x^2+3x-10$. ... (3)

The point $P(7,60)$ satisfies this equation. Hence the boy will meet his friend. ... (1)

Note : The last echelon form may differ from the above echelon form.

41. (b) Let $z = \sin\frac{\pi}{10} + i\cos\frac{\pi}{10}$ ... (1)

$$\left[\frac{1+\sin\frac{\pi}{10}+i\cos\frac{\pi}{10}}{1+\sin\frac{\pi}{10}-i\cos\frac{\pi}{10}}\right]^{10} = z^{10} = \left(\sin\frac{\pi}{10}+i\cos\frac{\pi}{10}\right)^{10} \qquad \text{... (2)}$$

$$= 1. \qquad \text{... (2)}$$

42. (a)
$$2g+16f+c = -65$$
$$14g+4f+c = -53$$
$$2g-8f+c = -17$$
$$f = -2, \quad g=-1 \text{ and } c=-31 \qquad \text{... (1)}$$

The equation is $x^2+y^2-2x-4y-31=0$ (or) $(x-1)^2+(y-2)^2=6^2$. ... (2)

... (1)

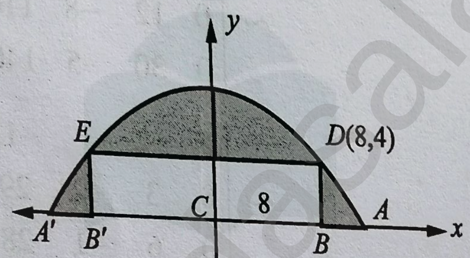
42. (b) The domain of $\sin^{-1}x$ and $\cos^{-1}x$ is $[-1 \quad 1]$. ... (1)

The domain of $\sin^{-1}\frac{x}{2}$ and $\cos^{-1}\frac{x}{3}$ are $-1\le\frac{x}{2}\le1$ and $-1\le\frac{x}{3}\le1$. ... (2)

$-2\le x\le2$ and $-3\le x\le3$. ... (1)

Thus the domain of $f(x)$ is $-2\le x\le2$. ... (1)

43. (a) Rough diagram ... (1)



The equation of the ellipse be $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$.

Length of semi minor axis $b=5$.

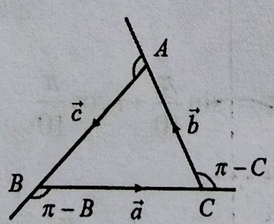
$$\frac{x^2}{a^2}+\frac{y^2}{5^2}=1 \qquad \text{... (1)}$$

D is $(8,4)$ and lies on the ellipse.

$$\frac{8^2}{a^2}+\frac{4^2}{5^2} = 1 \qquad \text{... (1)}$$

The width $AA' = 2a = \frac{80}{3}$ (or) $26.66\,m$ ... (2)

43. (b) Rough diagram ... (1)

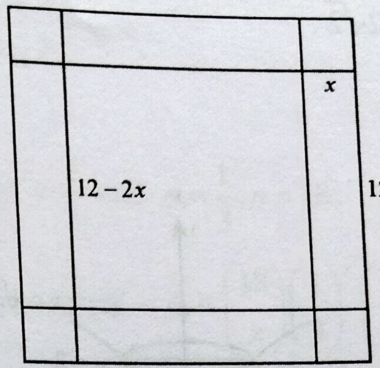


$$\vec{a}+\vec{b}+\vec{c} = \vec{0} \qquad \text{... (1)}$$
$$\vec{a} = -\vec{b}-\vec{c}$$

$$\vec{a}\cdot\vec{a} = -\vec{a}\cdot\vec{b}-\vec{a}\cdot\vec{c}$$ ... (1)

$a = b\cos C + c\cos B$. ... (2)

44. (a) Rough diagram ... (1)



$V = x(12-2x)^2$ (or any other form) ... (1)

For maximum, $V'=0 \Rightarrow x=2,6$.

When $x=6$, $V=0$. Hence $x\neq 6$

$x=2$ is the only value. ... (1)

When $x=2$, $V''=24x-96<0$

V attains the maximum volume when $x=2$. ... (1)

The maximum volume is $=128$. ... (1)

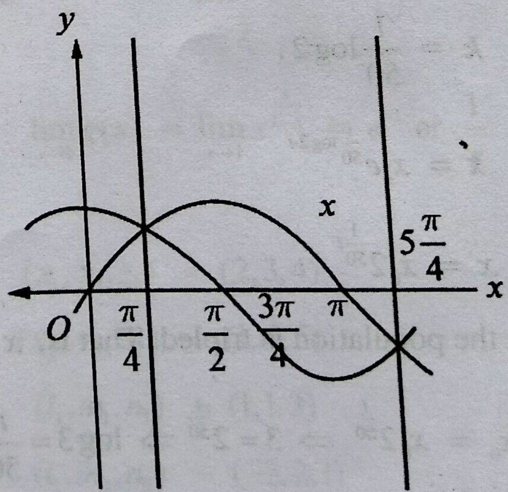
44. (b) $$\frac{dy}{dx}+\frac{6x^2y}{1+x^3}=\frac{1+x^2}{1+x^3}.$$

$$\text{I.F.} = e^{\int Pdx} = e^{\log(1+x^3)^2} = (1+x^3)^2$$ ... (2)

$$y(1+x^3)^2 = \int\frac{1+x^2}{1+x^3}(1+x^3)^2\,dx + C$$ ... (1

$$y(1+x^3)^2 = x+\frac{x^3}{3}+\frac{x^4}{4}+\frac{x^6}{6}+C$$ (or any other form) ... (2

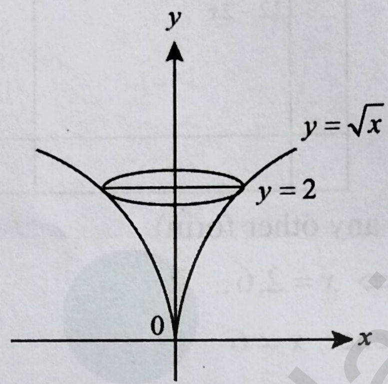
45. (a) Rough diagram ... (



$$A = \int_{\frac{\pi}{4}}^{\frac{5\pi}{4}} (\sin x - \cos x)\,dx \quad \text{... (2)}$$

$$= 2\sqrt{2}. \quad \text{... (2)}$$

45. (b) Rough diagram ... (1)



$$V = \int_0^2 \pi x^2 dy \quad \text{... (2)}$$

$$= \frac{32}{5}\pi. \quad \text{... (2)}$$

46. (a) $x = Ce^{kt}$, where C is an arbitrary constant. ... (1)

Let x_0 be the population when $t=0$.

Therefore $C = x_0$.

$$x = x_0 e^{kt}. \quad \text{... (1)}$$

When $t = 50, x = 2x_0$

$$\therefore 2x_0 = x_0 e^{50k} \Rightarrow 2 = e^{50k} \Rightarrow 50k = \log 2$$

$$k = \frac{1}{50}\log 2.$$

$$x = x_o e^{\frac{1}{50}\log 2 \cdot t}$$

$$x = x_0 2^{\frac{1}{50}t} \quad \text{... (1)}$$

To find the time when the population is tripled. That is, $x = 3x_0$

$$3x_0 = x_0 2^{\frac{t}{50}} \Rightarrow 3 = 2^{\frac{t}{50}} \Rightarrow \log 3 = \frac{t}{50}\log 2$$

$$t = 50\frac{\log 3}{\log 2}$$ ... (2)

The population is tripled in $50\left(\frac{\log 3}{\log 2}\right)$ years.

46. (b) $q = \frac{2}{3}$ and $p = \frac{1}{3}, n = 18$. ... (1)

(i) Probability mass function $= P(X = x) = \binom{18}{x}\left(\frac{1}{3}\right)^x\left(\frac{2}{3}\right)^{18-x}$ ... (1)

(ii) $P(X = 3) = \binom{18}{3}\left(\frac{1}{3}\right)^3\left(\frac{2}{3}\right)^{15}$ ... (1)

(iii) $P(X \geq 2) = 1 - \frac{20}{3}\left(\frac{2}{3}\right)^{17}$ (or any other form) ... (2)

Note : The final values need not be simplified.

47. (a) $g(x) = x^{\frac{1}{1-x}}$.

$$\log g(x) = \log x^{\frac{1}{1-x}}$$ ... (1)

$$\log g(x) = \frac{\log x}{1-x}$$

$$\lim_{x\to 1} \log g(x) = \lim_{x\to 1}\left(\frac{\log x}{1-x}\right) \left(\frac{0}{0} \text{ form}\right)$$ ... (1)

$$\lim_{x\to 1}\log g(x) = \lim_{x\to 1}\left(\frac{\frac{1}{x}}{-1}\right) = -1$$ ... (1)

$$\lim_{x\to 1} g(x) = \lim_{x\to 1} x^{\frac{x}{1-x}} = e^{-1} \text{ or } \frac{1}{e}$$ ... (2)

47. (b)

$$(x_1, y_1, z_1) = (2,3,4)$$

$$(x_2, y_2, z_2) = (1,4,5)$$

$$(l_1, m_1, n_1) = (1,1,3)$$

$$(l_1, m_2, n_2) = (-3,2,1)$$

$$\begin{vmatrix} 1-2 & 4-3 & 5-4 \\ 1 & 1 & 3 \\ -3 & 2 & 1 \end{vmatrix} = \begin{vmatrix} -1 & 1 & 1 \\ 1 & 1 & 3 \\ -3 & 2 & 1 \end{vmatrix} = 0$$

The two given lines are coplanar.

$$\Rightarrow \begin{vmatrix} x-2 & y-3 & z-4 \\ 1 & 1 & 3 \\ -3 & 2 & 1 \end{vmatrix} = 0$$

$$x+2y-z-4 = 0.$$

Note : Since the plane contains two lines, we have two points and two parallel vectors and hence t Cartesian equation can be obtained by taking different points and different parallel vectors.

45. (b)

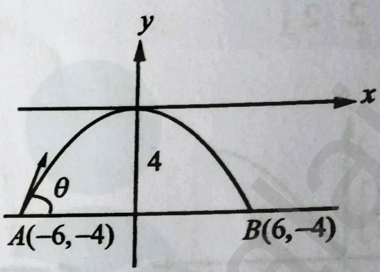
$$\frac{\partial z}{\partial t} = \frac{\partial z}{\partial x}\cdot\frac{\partial x}{\partial t} + \frac{\partial z}{\partial y}\cdot\frac{\partial y}{\partial t}$$

$$\frac{\partial z}{\partial t} = 3s^3\left(e^{3t} + s^2e^{-t}\right)$$ or any other form

$$\frac{\partial z}{\partial s} = \frac{\partial z}{\partial x}\cdot\frac{\partial x}{\partial s} + \frac{\partial z}{\partial y}\cdot\frac{\partial y}{\partial s}$$

$$= 3s^2e^{3t} - 15s^4e^{-t}$$ or any other form.

46. (a) Rough diagram



Equation of the parabola is $x^2 = -9y$.

$\frac{dy}{dx}$ at $(-6,-4)$ is $\frac{4}{3}$.

The angle of projection is $\tan^{-1}\frac{4}{3}$.

46. (b)

strictly increasing in $(-\infty,-1)$ and $(1,\infty)$

strictly decreasing in $(-1,1)$

f is concave downward in $(-\infty,0)$

f is concave upward in $(0,\infty)$

The point of inflection is $(0, f(0))$ i.e., $(0,0)$.

47. (a) Non-parametric vector equation is

$$\left[\vec{r} - \left(6\hat{i} - \hat{j} + \hat{k}\right)\right]\cdot\left(-6\hat{i} - 10\hat{j} + 14\hat{k}\right) = 0.$$

Cartesian equation

$$\Rightarrow \begin{vmatrix} x-6 & y+1 & z-1 \\ -1 & 2 & 1 \\ -5 & -4 & -5 \end{vmatrix} = 0$$

$$\Rightarrow 3x+5y-7z-6=0$$ ... (1)

Note : The Cartesian equation may be obtained from the non-parametric vector equation also.

47. (b) $p=\frac{1}{4}, q=\frac{3}{4}, n=8$ ... (1)

$$P(X=x) = f(x)=\binom{8}{x}\left(\frac{1}{4}\right)^x\left(\frac{3}{4}\right)^{8-x} \quad x=0,1,2,...8$$ ... (1)

(i) $$P(X=0) = f(0)=\binom{8}{0}\left(\frac{1}{4}\right)^0\left(\frac{3}{4}\right)^{8-0}=\left(\frac{3}{4}\right)^8$$ ... (1)

(ii) $$P(X=1) = f(1)=\binom{8}{1}\left(\frac{1}{4}\right)^1\left(\frac{3}{4}\right)^{8-1}=2\left(\frac{3}{4}\right)^7$$ ... (1)

(iii) $$P(X\geq 1) = 1-P(X<1)=1-P(X=0)=1-\left(\frac{3}{4}\right)^8$$ ... (1)