# Relations and Functions

## 2 MARKS-ONLY EXAMPLE SUMS

**Example 1.1** If $A = \{1,3,5\}$ and $B = \{2,3\}$ then (i) find $A \times B$ and $B \times A$.
(ii) Is $A \times B = B \times A$? If not why? (iii) Show that $n(A\times B) = n(B\times A) = n(A)\times n(B)$

**Solution** Given that $A = \{1,3,5\}$ and $B = \{2,3\}$

(i) $A\times B = \{1,3,5\} \times \{2,3\} = \{(1,2), (1,3), (3,2), (3,3), (5,2), (5,3)\}$ ...(1)

$B \times A = \{2,3\} \times \{1,3,5\} = \{(2,1), (2,3), (2,5), (3,1), (3,3), (3,5)\}$ ...(2)

(ii) From (1) and (2) we conclude that $A \times B \neq B \times A$ as $(1,2) \neq (2,1)$ and $(1,3) \neq (3,1)$, etc.

(iii) $n(A)=3$; $n(B) = 2$.

From (1) and (2) we observe that, $n(A\times B) = n(B\times A) = 6$;

we see that, $n(A) \times n(B) = 3 \times 2 = 6$ and $n(B) \times n(A) = 2\times 3 = 6$

Hence, $n(A\times B) = n(B\times A) = n(A) \times n(B) = 6$.

Thus, $n(A\times B) = n(B\times A) = n(A) \times n(B)$.

**Example 1.2** If $A\times B = \{(3,2), (3,4), (5,2), (5,4)\}$ then find $A$ and $B$.

**Solution** $A\times B = \{(3,2), (3,4), (5,2), (5,4)\}$

We have $A = \{$set of all first coordinates of elements of $A \times B\}$. $\therefore A = \{3,5\}$

$B = \{$set of all second coordinates of elements of $A \times B\}$. $\therefore B = \{2,4\}$

Thus $A = \{3,5\}$ and $B = \{2,4\}$.

**Example 1.4** Let $A = \{3,4,7,8\}$ and $B = \{1,7,10\}$. Which of the following sets are relations from $A$ to $B$?

(i) $R_1 = \{(3,7), (4,7), (7,10), (8,1)\}$ (ii) $R_2 = \{(3,1), (4,12)\}$

(iii) $R_3 = \{(3,7), (4,10), (7,7), (7,8), (8,11), (8,7), (8,10)\}$

**Solution** $A \times B = \{(3,1), (3,7), (3,10), (4,1), (4,7), (4,10), (7,1), (7,7), (7,10), (8,1), (8,7), (8,10)\}$

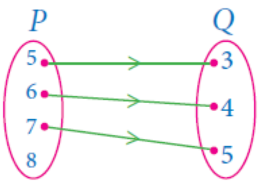
(i) We note that, $R_1 \subseteq A \times B$. Thus, $R_1$ is a relation from $A$ to $B$.

(ii) Here, $(4,12) \in R_2$, but $(4,12) \notin A \times B$. So, $R_2$ is not a relation from $A$ to $B$.

(iii) Here, $(7,8) \in R_3$, but $(7,8) \notin A \times B$. So, $R_3$ is not a relation from $A$ to $B$.

**Example 1.5** The arrow diagram shows (Fig.1.10) a relationship between the sets $P$ and $Q$. Write the relation in (i) Set builder form (ii) Roster form (iii) What is the domain and range of R.

**Solution**

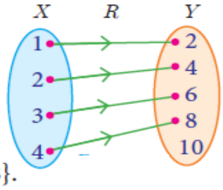



(i) Set builder form of $R = \{(x,y) \mid y = x-2,\ x \in P, y \in Q\}$

(ii) Roster form $R = \{(5,3), (6,4), (7,5)\}$

(iii) Domain of $R = \{5,6,7\}$ and range of $R = \{3,4,5\}$

**Example 1.6** Let $X = \{1,2,3,4\}$ and $Y = \{2,4,6,8,10\}$ and R $= \{(1,2),(2,4),(3,6),(4,8)\}$. Show that R is a function and find its domain, co-domain and range?




**Solution** Pictorial representation of R is given in Fig.1.14. From the diagram, we see that for each $x \in X$, there exists only one $y \in Y$. Thus all elements in $X$ have only one image in $Y$. Therefore R is a function. Domain $X = \{1,2,3,4\}$; Co-domain $Y = \{2,4,6,8,10\}$; Range of $f = \{2,4,6,8\}$.

**Example 1.7** A relation $f: X \to Y$ is defined by $f(x) = x^2 - 2$ where, $X = \{-2,-1,0,3\}$ and $Y =$ R.
(i) List the elements of $f$ (ii) Is $f$ a function?

**Solution** $f(x) = x^2 - 2$ where $X = \{-2,-1,0,3\}$

(i) $f(-2) = (-2)^2 - 2 = 2$; $f(-1) = (-1)^2 - 2 = -1$

$f(0) = (0)^2 - 2 = -2$; $f(3) = (3)^2 - 2 = 7$

$\therefore f = \{(-2,2),(-1,-1),(0,-2),(3,7)\}$

(ii) We note that each element in the domain of $f$ has a unique image. Therefore, $f$ is a function.

**Example 1.9** Given $f(x) = 2x - x^2$,
find (i) $f(1)$ (ii) $f(x+1)$ (iii) $f(x) + f(1)$

**Solution** (i) $x = 1$, we get

$$f(1) = 2(1) - (1)^2 = 2 - 1 = 1$$

(ii) $x = x+1$, we get

$$f(x+1) = 2(x+1) - (x+1)^2 = 2x + 2 - (x^2 + 2x + 1) = -x^2 + 1$$

(iii) $f(x) + f(1) = (2x - x^2) + 1 = -x^2 + 2x + 1$

[Note that $f(x) + f(1) \neq f(x+1)$. In general $f(a+b)$ is not equal to $f(a) + f(b)$]

**Example 1.10** Using vertical line test, determine which of the following curves (Fig.1.18(a), 1.18(b), 1.18(c), 1.18(d)) represent a function?

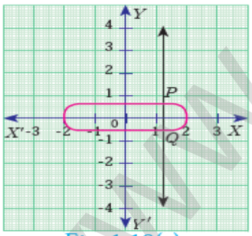


Fig. 1.18(a)

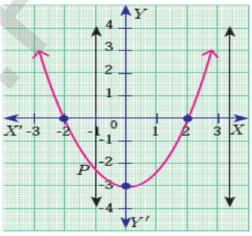


Fig. 1.18(b)

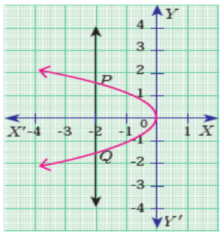


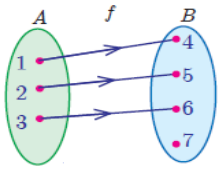
Fig. 1.18(c)

**Solution** The curves in Fig.1.18 ($a$) and Fig.1.18 ($c$) do not represent a function as the vertical lines meet the curves in two points $P$ and $Q$.

The curves in Fig.1.18 (b) and Fig.1.18 (d) represent a function as the vertical lines meet the curve in at most one point.

**Example 1.13** Let $A = \{1,2,3\}$, $B = \{4,5,6,7\}$ and $f = \{(1,4),(2,5),(3,6)\}$ be a function from $A$ to $B$. Show that $f$ is one – one but not onto function.

**Solution** $A = \{1,2,3\}$, $B = \{4,5,6,7\}$; $f = \{(1,4),(2,5),(3,6)\}$




Hence $f$ is one–one function.

Hence $f$ is not onto

$\therefore$ $f$ is one–one but not an onto function.

**Example 1.14** If $A = \{-2,-1,0,1,2\}$ and $f : A \to B$ is an onto function defined by $f(x) = x^2 + x + 1$ then find $B$.

**Solution** Given $A = \{-2,-1,0,1,2\}$ and $f(x) = x^2 + x + 1$.

$f(-2) = (-2)^2 + (-2) + 1 = 3;$

$f(-1) = (-1)^2 + (-1) + 1 = 1$

$f(0) = 0^2 + 0 + 1 = 1;$

$f(1) = 1^2 + 1 + 1 = 3$

$f(2) = 2^2 + 2 + 1 = 7$

$\therefore B = \{1, 3, 7\}$.

**Example 1.15** Let $f$ be a function $f : \mathbb{N} \to \mathbb{N}$ be defined by $f(x) = 3x + 2, x \in \mathbb{N}$

(i) Find the images of 1, 2, 3 (ii) Find the pre-images of 29, 53

(ii) Identify the type of function

**Solution** $f(x) = 3x + 2$

(i) If $x = 1$, $f(1) = 3(1) + 2 = 5$

If $x = 2$, $f(2) = 3(2) + 2 = 8$

If $x = 3$, $f(3) = 3(3) + 2 = 11$

The images of 1, 2, 3 are 5, 8, 11

(ii) If $x$ is the pre-image of 29, then $f(x) = 29$. Hence $3x + 2 = 29$

$3x = 27 \Rightarrow x = 9$.

Similarly, if $x$ is the pre-image of 53, then $f(x) = 53$. Hence $3x + 2 = 53$

$3x = 51 \Rightarrow x = 17$.

Thus the pre-images of 29 and 53 are 9 and 17

(iii)

$f$ is one – one function.

the range of $f$ = {5, 8, 11, 14, 17, ...} is a proper subset of $\mathbb{N}$.

$f$ is an into function.

$f$ is one – one and into function.

**Example 1.17** Let $f$ be a function from $\mathbb{R}$ to $\mathbb{R}$ defined by $f(x) = 3x - 5$. Find the values of $a$ and $b$ given that $(a, 4)$ and $(1, b)$ belong to $f$.

**Solution** $f(x) = 3x - 5$ can be written as $f = \{(x, 3x - 5) \mid x \in R\}$

$(a, 4)$ means the image of $a$ is 4. i.e., $f(a) = 4$

$3a - 5 = 4 \Rightarrow a = 3$

$(1, b)$ means the image of 1 is $b$. i.e., $f(1) = b$

$3(1) - 5 = b \Rightarrow b = -2$

**Example 1.19** Find $f \circ g$ and $g \circ f$ when $f(x) = 2x + 1$ and $g(x) = x^2 - 2$

*Solution* $f(x) = 2x + 1$ , $g(x) = x^2 - 2$

$$f \circ g(x) = f(g(x)) = f(x^2 - 2) = 2(x^2 - 2) + 1 = 2x^2 - 3$$

$$g \circ f(x) = g(f(x)) = g(2x + 1) = (2x + 1)^2 - 2 = 4x^2 + 4x - 1$$

Thus $f \circ g = 2x^2 - 3,\ g \circ f = 4x^2 + 4x - 1$. From the above, we see that $f \circ g \neq g \circ f$.

**Example 1.20** Represent the function $f(x) = \sqrt{2x^2 - 5x + 3}$ as a composition of two functions.

*Solution* $f_2(x) = 2x^2 - 5x + 3$ and $f_1(x) = \sqrt{x}$

$$f(x) = \sqrt{2x^2 - 5x + 3} = \sqrt{f_2(x)}$$

$$= f_1[f_2(x)] \qquad = f_1 f_2(x)$$

**Example 1.22** Find $k$ if $f \circ f(k) = 5$ where $f(k) = 2k - 1$.

*Solution* $f \circ f(k) = f(f(k))$

$$= 2(2k - 1) - 1 = 4k - 3.$$

$$f \circ f(k) = 4k - 3$$

But, $f \circ f(k) = 5$

$$\therefore 4k - 3 = 5 \Rightarrow k = 2.$$

## 5 MARKS-ONLY EXAMPLES

**Example 1.3** Let $A = \{x \in \mathbb{N} \mid 1 < x < 4\}$, $B = \{x \in \mathbb{W} \mid 0 \le x < 2\}$ and $C = \{x \in \mathbb{N} \mid x < 3\}$. Then verify that

(i) $A \times (B \cup C) = (A \times B) \cup (A \times C)$ (ii) $A \times (B \cap C) = (A \times B) \cap (A \times C)$

*Solution* $A = \{x \in \mathbb{N} \mid 1 < x < 4\} = \{2, 3\}$, $B = \{x \in \mathbb{W} \mid 0 \le x < 2\} = \{0, 1\}$,
$C = \{x \in \mathbb{N} \mid x < 3\} = \{1, 2\}$

(i) $A \times (B \cup C) = (A \times B) \cup (A \times C)$

$$B \cup C = \{0,1\} \cup \{1,2\} = \{0,1,2\}$$

$$A \times (B \cup C) = \{2,3\} \times \{0,1,2\} = \{(2,0),(2,1),(2,2),(3,0),(3,1),(3,2)\} \quad \ldots(1)$$

$$A \times B = \{2,3\} \times \{0,1\} = \{(2,0),(2,1),(3,0),(3,1)\}$$

$$A \times C = \{2,3\} \times \{1,2\} = \{(2,1),(2,2),(3,1),(3,2)\}$$

$$(A \times B) \cup (A \times C) = \{(2,0),(2,1),(3,0),(3,1)\} \cup \{(2,1),(2,2),(3,1),(3,2)\}$$

$$= \{(2,0),(2,1),(2,2),(3,0),(3,1),(3,2)\} \quad \ldots(2)$$

From (1) and (2), $A \times (B \cup C) = (A \times B) \cup (A \times C)$ is verified.

(ii) $A \times (B \cap C) = (A \times B) \cap (A \times C)$

$$B \cap C = \{0,1\} \cap \{1,2\} = \{1\}$$

$$A \times (B \cap C) = \{2,3\} \times \{1\} = \{(2,1),(3,1)\} \quad \ldots(3)$$

$$A \times B = \{2,3\} \times \{0,1\} = \{(2,0),(2,1),(3,0),(3,1)\}$$

$$A \times C = \{2,3\} \times \{1,2\} = \{(2,1),(2,2),(3,1),(3,2)\}$$

$$(A \times B) \cap (A \times C) = \{(2,0),(2,1),(3,0),(3,1)\} \cap \{(2,1),(2,2),(3,1),(3,2)\}$$

$$= \{(2,1),(3,1)\} \quad \ldots(4)$$

From (3) and (4), $A \times (B \cap C) = (A \times B) \cap (A \times C)$ is verified.

**Example 1.4** Let $A = \{3,4,7,8\}$ and $B = \{1,7,10\}$. Which of the following sets are relations from $A$ to $B$?

(i) $R_1 = \{(3,7), (4,7), (7,10), (8,1)\}$ (ii) $R_2 = \{(3,1), (4,12)\}$

(iii) $R_3 = \{(3,7), (4,10), (7,7), (7,8), (8,11), (8,7), (8,10)\}$

**Solution** $A \times B = \{(3,1), (3,7), (3,10), (4,1), (4,7), (4,10), (7,1), (7,7), (7,10), (8,1), (8,7), (8,10)\}$

(i) We note that, $R_1 \subseteq A \times B$. Thus, $R_1$ is a relation from $A$ to $B$.

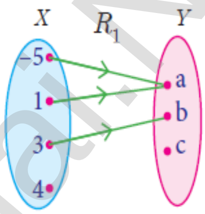
(ii) Here, $(4,12) \in R_2$, but $(4,12) \notin A \times B$. So, $R_2$ is not a relation from $A$ to $B$.

(iii) Here, $(7,8) \in R_3$, but $(7,8) \notin A \times B$. So, $R_3$ is not a relation from $A$ to $B$.

**Example 1.8** If $X = \{-5,1,3,4\}$ and $Y = \{a,b,c\}$, then which of the following relations are functions from $X$ to $Y$?

(i) $R_1 = \{(-5,a), (1,a), (3,b)\}$ (ii) $R_2 = \{(-5,b), (1,b), (3,a),(4,c)\}$

(iii) $R_3 = \{(-5,a), (1,a), (3,b),(4,c),(1,b)\}$



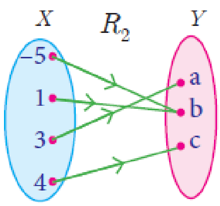

**Solution**

(i) $R_1 = \{(-5,a), (1,a), (3,b)\}$

We may represent the relation $R_1$

$R_1$ is not a function as $4 \in X$ does not have an image in $Y$.

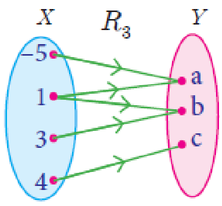
$R_1$ is not a function as $4 \in X$ does not have an image in $Y$.




(ii) $R_2 = \{(-5,b), (1,b), (3,a),(4,c)\}$

Arrow diagram of $R_2$

$R_2$ is a function as each element of $X$ has an unique image in $Y$.




(iii) $R_3 = \{(-5,a), (1,a), (3,b),(4,c),(1,b)\}$

Representing $R_3$ in an arrow diagram

$R_3$ is not a function as $1 \in X$ has two images $a \in Y$ and $b \in Y$.

Note that the image of an element should always be unique.

**Example 1.11** Let $A = \{1,2,3,4\}$ and $B = \{2,5,8,11,14\}$ be two sets. Let $f : A \to B$ be a function given by $f(x) = 3x - 1$. Represent this function

(i) by arrow diagram (ii) in a table form

(iii) as a set of ordered pairs (iv) in a graphical form

**Solution**

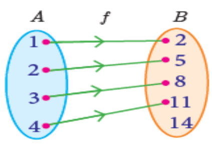
$A = \{1, 2, 3, 4\}$ ; $B = \{2, 5, 8, 11, 14\}$ ; $f(x) = 3x - 1$

$f(1) = 3(1) - 1 = 3 - 1 = 2$ ; $f(2) = 3(2) - 1 = 6 - 1 = 5$

$f(3) = 3(3) - 1 = 9 - 1 = 8$ ; $f(4) = 4(3) - 1 = 12 - 1 = 11$

**(i) Arrow diagram**

Let us represent the function $f : A \to B$ by an arrow diagram



**(ii) Table form**

The given function $f$ can be represented in a tabular form as given below

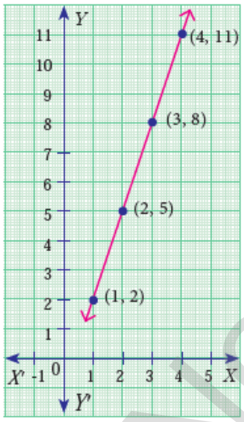
| $x$ | 1 | 2 | 3 | 4 |
|---|---|---|---|---|
| $f(x)$ | 2 | 5 | 8 | 11 |

**(iii) Set of ordered pairs**

The function $f$ can be represented as a set of ordered pairs as

$f = \{(1,2),(2,5),(3,8),(4,11)\}$

**(iv) Graphical form**






**Example 1.15** Let $f$ be a function $f : \mathbb{N} \to \mathbb{N}$ be defined by $f(x) = 3x + 2, x \in \mathbb{N}$

(i) Find the images of 1, 2, 3 (ii) Find the pre-images of 29, 53

(ii) Identify the type of function

*Solution* The function $f : \mathbb{N} \to \mathbb{N}$ is defined by $f(x) = 3x + 2$

(i) If $x = 1$, $f(1) = 3(1) + 2 = 5$

If $x = 2$, $f(2) = 3(2) + 2 = 8$

If $x = 3$, $f(3) = 3(3) + 2 = 11$

The images of 1, 2, 3 are 5, 8, 11

(ii) If $x$ is the pre-image of 29, then $f(x) = 29$. Hence $3x + 2 = 29$

$3x = 27 \Rightarrow x = 9$.

Similarly, if $x$ is the pre-image of 53, then $f(x) = 53$. Hence $3x + 2 = 53$

$3x = 51 \Rightarrow x = 17$.

Thus the pre-images of 29 and 53 are 9 and 17

(iii)

$f$ is one – one function.

But the range of $f = \{5, 8, 11, 14, 17, ...\}$ is a proper subset of $\mathbb{N}$.

$f$ is an into function.

Thus $f$ is one – one and into function.

**Example 1.16** Forensic scientists can determine the height (in cm) of a person based on the length of the thigh bone. They usually do so using the function $h(b) = 2 \cdot 47b + 54 \cdot 10$ where $b$ is the length of the thigh bone.

(i) Verify the function $h$ is one – one or not.

(ii) Also find the height of a person if the length of his thigh bone is 50 cm.

(iii) Find the length of the thigh bone if the height of a person is $147 \cdot 96$ cm.

*Solution* (i) To check if $h$ is one – one, we assume that $h(b_1) = h(b_2)$.

Then we get, $2 \cdot 47b_1 + 54 \cdot 10 = 2 \cdot 47b_2 + 54 \cdot 10$

$$2 \cdot 47b_1 = 2 \cdot 47b_2$$

$$\Rightarrow b_1 = b_2$$

Thus, $h(b_1) = h(b_2) \Rightarrow b_1 = b_2$. So, the function $h$ is one – one.

(ii) If the length of the thigh bone $b = 50$, then the height is $h(50) = (2\cdot 47 \times 50) + 54\cdot 10 = 177\cdot 6$ cm.

(iii) If the height of a person is $147\cdot 96$ cm, then $h(b) = 147\cdot 96$ and so the length of the thigh bone is given by

$$2\cdot 47b + 54\cdot 10 = 147\cdot 96$$
$$\Rightarrow 2\cdot 47b = 147\cdot 96 - 54\cdot 10 = 93\cdot 86$$
$$b = \frac{93\cdot 86}{2\cdot 47} = 38$$

Therefore, the length of the thigh bone is 38 cm.

**Example 1.18** If the function $f:\mathbb{R}\to\mathbb{R}$ is defined by $f(x) = \begin{cases} 2x+7; & x < -2 \\ x^2 - 2; & -2 \le x < 3 \\ 3x - 2; & x \ge 3 \end{cases}$, then find the values of

(i) $f(4)$ (ii) $f(-2)$ (iii) $f(4)+2f(1)$ (iv) $\frac{f(1)-3f(4)}{f(-3)}$

(i) First, we see that, $x = 4$ lie in the third interval.

$\therefore\ f(x) = 3x - 2;\ f(4) = 3(4) - 2 = 10$

(ii) $x = -2$ lies in the second interval.

$\therefore\ f(x) = x^2 - 2;\ f(-2) = (-2)^2 - 2 = 2$

(iii) From (i), $f(4) = 10$.

To find $f(1)$, first we see that $x = 1$ lies in the second interval.

$\therefore\ f(x) = x^2 - 2 \Rightarrow f(1) = 1^2 - 2 = -1$

$$f(4) + 2f(1) = 10 + 2(-1) = 8$$

(iv) We know that $f(1) = -1$ and $f(4) = 10$.

For finding $f(-3)$, we see that $x = -3$, lies in the first interval.

$\therefore\ f(x) = 2x + 7$; thus, $f(-3) = 2(-3) + 7 = 1$

Hence, $\frac{f(1)-3f(4)}{f(-3)} = \frac{-1-3(10)}{1} = -31$

**Example 1.21** If $f(x) = 3x - 2$, $g(x) = 2x + k$ and if $f\circ g = g\circ f$, then find the value of $k$.

*Solution* $f(x) = 3x - 2$, $g(x) = 2x + k$

$f\circ g(x) = f(g(x)) = f(2x+k) = 3(2x+k) - 2 = 6x + 3k - 2$

$f\circ g(x) = 6x + 3k - 2$.

$g\circ f(x) = g(3x-2) = 2(3x-2) + k$

$g\circ f(x) = 6x - 4 + k$.

Given that $f\circ g = g\circ f$

$\therefore 6x + 3k - 2 = 6x - 4 + k$

$6x - 6x + 3k - k = -4 + 2 \Rightarrow k = -1$

**Example 1.23** If $f(x) = 2x+3$, $g(x) = 1-2x$ and $h(x) = 3x$. Prove that $f \circ (g \circ h) = (f \circ g) \circ h$

**Solution** $f(x) = 2x+3$, $g(x) = 1-2x$, $h(x) = 3x$

Now, $(f \circ g)(x) = f(g(x)) = f(1-2x) = 2(1-2x)+3 = 5-4x$

Then, $(f \circ g) \circ h(x) = (f \circ g)(h(x)) = (f \circ g)(3x) = 5-4(3x) = 5-12x$ ....(1)

$(g \circ h)(x) = g(h(x)) = g(3x) = 1-2(3x) = 1-6x$

$\Rightarrow f \circ (g \circ h)(x) = f(1-6x) = 2(1-6x)+3 = 5-12x$ ....(2)

From (1) and (2), we get $(f \circ g) \circ h = f \circ (g \circ h)$

**Example 1.24** Find $x$ if $gff(x) = fgg(x)$, given $f(x) = 3x+1$ and $g(x) = x+3$.

**Solution** $gff(x) = g\,[f\{f(x)\}]$ (This means "$g$ of $f$ of $f$ of $x$")

$= g\,[\,f(3x+1)] = g\,[\,3(3x+1)+1] = g\,(9x+4)$

$g\,(9x+4) = [\,(9x+4)+3] = 9x+7$

$fgg(x) = f\,[g\,\{g\,(x)\}]$ (This means "$f$ of $g$ of $g$ of $x$")

$= f\,[\,g\,(x+3)] = f\,[\,(x+3)+3] = f\,(x+6)$

$f\,(x+6) = [\,3(x+6)+1\,] = 3x+19$

These two quantities being equal, we get $9x+7 = 3x+19$. Solving this equation we obtain $x = 2$.

# Numbers and Sequences

## 2 MARKS

**Example 2.2** Find the quotient and remainder when $a$ is divided by $b$ in the following cases (i) $a = -12$, $b = 5$ (ii) $a = 17$, $b = -3$ (iii) $a = -19$, $b = -4$

**Solutions**

(i) $a = -12$, $b = 5$

By Euclid's division lemma

$a = bq + r$

$-12 = 5 \times (-3) + 3$

Quotient $q = -3$, Remainder $r = 3$

(ii) $a = 17$ $b = -3$

By Euclid's division lemma

$a = bq + r$

$17 = (-3) \times (-5) + 2$,

Quotient $q = -5$,

Remainder $r = 2$

(iii) $a = -19$, $b = -4$

By Euclid's division lemma

$a = bq + r$

$-19 = (-4) \times (5) + 1$

Quotient $q = 5$, Remainder $r = 1$.

**Example 2.3** Show that the square of an odd integer is of the form $4q+1$, for some integer $q$.

**Solution** Let $x$ be any odd integer. Since any odd integer is one more than an even integer, we have $x = 2k+1$, for some integers $k$.

$$\begin{aligned} x^2 &= (2k+1)^2 \\ &= 4k^2 + 4k + 1 \\ &= 4k(k+1)+1 \\ &= 4q+1, \text{ where } q = k(k+1) \text{ is some integer.} \end{aligned}$$

**Example 2.4** If the Highest Common Factor of 210 and 55 is expressible in the form $55x - 325$, find $x$.

**Solution** Using Euclid's Division Algorithm, let us find the HCF of given numbers

$$\begin{aligned} 210 &= 55 \times 3 + 45 \\ 55 &= 45 \times 1 + 10 \\ 45 &= 10 \times 4 + 5 \\ 10 &= 5 \times 2 + 0 \end{aligned}$$

The remainder is zero.

So, the last divisor 5 is the Highest Common Factor (HCF) of 210 and 55.

$\because$ HCF is expressible in the form $55x - 325 = 5$

$$\begin{aligned} \Rightarrow \quad 55x &= 330 \\ x &= 6 \end{aligned}$$

**Example 2.5** Find the greatest number that will divide 445 and 572 leaving remainders 4 and 5 respectively.

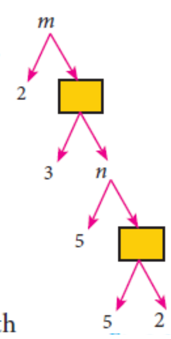
**Solution** Since the remainders are 4, 5 respectively the required number is the HCF of the number $445 - 4 = 441$, $572 - 5 = 567$.

Hence, we will determine the HCF of 441 and 567. Using Euclid's Division Algorithm, we have,

$$\begin{aligned} 567 &= 441 \times 1 + 126 \\ 441 &= 126 \times 3 + 63 \\ 126 &= 63 \times 2 + 0 \end{aligned}$$

Therefore, HCF of $441, 567 = 63$ and so the required number is 63.

**Example 2.7** In the given factorisation, find the numbers $m$ and $n$.




**Solution** Value of the first box from bottom $= 5 \times 2 = 10$

Value of $n$ $= 5 \times 10 = 50$

Value of the second box from bottom $= 3 \times 50 = 150$

Value of $m$ $= 2 \times 150 = 300$

Thus, the required numbers are $m = 300$, $n = 50$

**Example 2.8** Can the number $6^n$, $n$ being a natural number end with the digit 5? Give reason for your answer.

**Solution** Since $6^n = (2\times 3)^n = 2^n \times 3^n$ ,

2 is a factor of $6^n$ . So, $6^n$ is always even.

Hence, $6^n$ cannot end with the digit 5.

**Example 2.9** Is $7\times5\times3\times2+3$ a composite number? Justify your answer.

**Solution**

$$7\times5\times3\times2+3 = 3\times(7\times5\times2+1) = 3\times71$$

it is a composite number.

**Example 2.10** '$a$' and '$b$' are two positive integers such that $a^b \times b^a = 800$. Find '$a$' and '$b$'.

**Solution** The number 800 can be factorized as

$$800 = 2\times2\times2\times2\times2\times5\times5 = 2^5\times5^2$$

Hence, $a^b \times b^a = 2^5 \times 5^2$

This implies that $a = 2$ and $b = 5$ (or) $a = 5$ and $b = 2$.

**Example 2.12** Determine the value of $d$ such that $15 \equiv 3 \pmod{d}$.

**Solution** $15 \equiv 3 \pmod{d}$ means $15 - 3 = kd$, for some integer $k$.

$12 = kd$.

$\Rightarrow$ $d$ divides 12.

The divisors of 12 are 1,2,3,4,6,12. But $d$ should be larger than 3 and so the possible values for $d$ are 4,6,12.

**Example 2.13** Find the least positive value of $x$ such that

(i) $67 + x \equiv 1 \pmod{4}$ (ii) $98 \equiv (x+4) \pmod{5}$

**Solution** (i) $67 + x \equiv 1 \pmod{4}$

$67 + x - 1 = 4n$, for some integer $n$

$66 + x = 4n$

$66 + x$ is a multiple of 4.

Therefore, the least positive value of $x$ must be 2, since 68 is the nearest multiple of 4 more than 66.

(ii) $98 \equiv (x+4) \pmod{5}$

$98 - (x+4) = 5n$, for some integer $n$.

$94 - x = 5n$

$94 - x$ is a multiple of 5.

Therefore, the least positive value of $x$ must be 4

$\because 94 - 4 = 90$ is the nearest multiple of 5 less than 94.

**Example 2.14** Solve $8x \equiv 1 \pmod{11}$

**Solution** $8x \equiv 1 \pmod{11}$ can be written as $8x - 1 = 11k$, for some integer $k$.

$$x = \frac{11k+1}{8}$$

When we put $k = 5, 13, 21, 29, \ldots$ then $11k+1$ is divisible by 8.

$$x = \frac{11\times 5+1}{8} = 7$$

$$x = \frac{11\times 13+1}{8} = 18$$

$\therefore$ The solutions are 7,18,29,40, …

**Example 2.15** Compute $x$, such that $10^4 \equiv x \pmod{19}$

*Solution*

$$10^2 = 100 \equiv 5 \pmod{19}$$
$$10^4 = (10^2)^2 \equiv 5^2 \pmod{19}$$
$$10^4 \equiv 25 \pmod{19}$$
$$10^4 \equiv 6 \pmod{19} \qquad (\because 25 \equiv 6 \pmod{19})$$
$$\therefore x = 6.$$

**Example 2.16** Find the number of integer solutions of $3x \equiv 1 \pmod{15}$.

*Solution* $3x \equiv 1 \pmod{15}$ can be written as

$3x - 1 = 15k$ for some integer $k$

$$3x = 15k + 1$$
$$x = \frac{15k+1}{3}$$
$$x = 5k + \frac{1}{3}$$

$\because$ $5k$ is an integer, $5k + \frac{1}{3}$ cannot be an integer.

So there is no integer solution.

**Example 2.17** A man starts his journey from Chennai to Delhi by train. He starts at 22.30 hours on Wednesday. If it takes 32 hours of travelling time and assuming that the train is not late, when will he reach Delhi?

*Solution* Starting time 22.30, Travelling time 32 hours. Here we use modulo 24.

The reaching time is

22.30 + 32 (mod 24) $\equiv$ 54.30 (mod24)

$\equiv$6.30 (mod24) ($\because$ 32 = (1×24) + 8
Thursday Friday)

Thus, he will reach Delhi on Friday at 6.30 hours.

**Example 2.18** Kala and Vani are friends. Kala says, "Today is my birthday" and she asks Vani, "When will you celebrate your birthday?" Vani replies, "Today is Monday and I celebrated my birthday 75 days ago". Find the day when Vani celebrated her birthday.

*Solution* Let us associate the numbers 0, 1, 2, 3, 4, 5, 6 to represent the weekdays from Sunday to Saturday respectively.

subtract 75 from 1 and a week contain 7 days.

$$-74 \pmod 7 \equiv -4 \pmod 7 \equiv 7-4 \pmod 7 \equiv 3 \pmod 7$$

$$(\because -74 - 3 = -77 \text{ is divisible by } 7)$$

Thus, $1 - 75 \equiv 3 \pmod 7$

3 is Wednesday.

Vani's birthday must be on Wednesday.

**Example 2.19** Find the next three terms of the sequences

(i) $\frac{1}{2},\frac{1}{6},\frac{1}{10},\frac{1}{14},\ldots$ (ii) $5,\ 2,-1,-4,\ldots$ (iii) $1, 0.1, 0.01,\ldots$

***Solution*** (i) $\frac{1}{2},\ \frac{1}{6},\ \frac{1}{10},\ \frac{1}{14},\ \ldots$ (+4, +4, +4)

In the above sequence the numerators are same and the denominator is increased by 4.

So the next three terms are $a_5 = \frac{1}{14+4} = \frac{1}{18}$

$$a_6 = \frac{1}{18+4} = \frac{1}{22}$$

$$a_7 = \frac{1}{22+4} = \frac{1}{26}$$

(ii) $5,\ 2,\ -1,\ -4,\ \ldots$ (−3, −3, −3)

Here each term is decreased by 3. So the next three terms are $-7,\ -10,\ -13$.

(iii) $1,\ 0.1,\ 0.01,\ \ldots$ (÷10, ÷10)

Here each term is divided by 10. Hence, the next three terms are

$$a_4 = \frac{0.01}{10} = 0.001$$

$$a_5 = \frac{0.001}{10} = 0.0001$$

$$a_6 = \frac{0.0001}{10} = 0.00001$$

**Example 2.20** Find the general term for the following sequences

(i) $3, 6, 9, \ldots$ (ii) $\frac{1}{2},\frac{2}{3},\frac{3}{4},\ldots$ (iii) $5, -25, 125, \ldots$

***Solution*** (i) $3, 6, 9, \ldots$

Here the terms are multiples of 3. So the general term is

$a_n = 3n,$

(ii) $\frac{1}{2},\frac{2}{3},\frac{3}{4},\ldots$

$a_1 = \frac{1}{2};\ a_2 = \frac{2}{3};\ a_3 = \frac{3}{4}$

We see that the numerator of $n^{th}$ term is $n$, and the denominator is one more than the numerator. Hence, $a_n = \frac{n}{n+1}, n \in \mathbb{N}$

(iii) $5, -25, 125, \ldots$

The terms of the sequence have + and − sign alternatively and also they are in powers of 5.

So the general term $a_n = (-1)^{n+1}5^n, n \in \mathbb{N}$

**Example 2.21** The general term of a sequence is defined as

$$a_n = \begin{cases} n(n+3);\ n \in \mathbb{N} \text{ is odd} \\ n^2+1 \quad ;\ n \in \mathbb{N} \text{ is even} \end{cases}$$

Find the eleventh and eighteenth terms.

**Solution** To find $a_{11}$, since 11 is odd, we put $n = 11$ in $a_n = n(n+3)$

Thus, the eleventh term $a_{11} = 11(11+3) = 154$.

To find $a_{18}$, since 18 is even, we put $n = 18$ in $a_n = n^2 + 1$

Thus, the eighteenth term $a_{18} = 18^2 + 1 = 325$.

**Example 2.22** Find the first five terms of the following sequence.

$a_1 = 1,\ a_2 = 1, a_n = \dfrac{a_{n-1}}{a_{n-2}+3};\ n \geq 3,\ n \in \mathbb{N}$

**Solution** $a_1 = 1,\ a_2 = 1.$

$$a_3 = \frac{a_{3-1}}{a_{3-2}+3} = \frac{a_2}{a_1+3} = \frac{1}{1+3} = \frac{1}{4}$$

$$a_4 = \frac{a_{4-1}}{a_{4-2}+3} = \frac{a_3}{a_2+3} = \frac{\frac{1}{4}}{1+3} = \frac{\frac{1}{4}}{4} = \frac{1}{4}\times\frac{1}{4} = \frac{1}{16}$$

$$a_5 = \frac{a_{5-1}}{a_{5-2}+3} = \frac{a_4}{a_3+3} = \frac{\frac{1}{16}}{\frac{1}{4}+3} = \frac{1}{16}\times\frac{4}{13} = \frac{1}{52}$$

Therefore, the first five terms of the sequence are 1, 1, $\frac{1}{4}$, $\frac{1}{16}$ and $\frac{1}{52}$

**Example 2.23** Check whether the following sequences are in A.P. or not?

(i) $x+2,\ 2x+3,\ 3x+4,\ldots$ (ii) $2, 4, 8, 16,\ldots.$ (iii) $3\sqrt{2}, 5\sqrt{2}, 7\sqrt{2}, 9\sqrt{2},\ldots$

**Solution**

(i) $t_2 - t_1 = (2x+3)-(x+2) = x+1$

$t_3 - t_2 = (3x+4)-(2x+3) = x+1$

$t_2 - t_1 = t_3 - t_2$

Thus, the differences between consecutive terms are equal.

Hence the sequence $x+2,\ 2x+3,\ 3x+4,\ldots$ is in A.P.

(ii) $t_2 - t_1 = 4-2 = 2$

$t_3 - t_2 = 8-4 = 4$

$t_2 - t_1 \neq t_3 - t_2$

the sequence 2, 4, 8, 16, . . . are not in A.P.

(iii) $t_2 - t_1 = 5\sqrt{2} - 3\sqrt{2} = 2\sqrt{2}$

$t_3 - t_2 = 7\sqrt{2} - 5\sqrt{2} = 2\sqrt{2}$

$t_4 - t_3 = 9\sqrt{2} - 7\sqrt{2} = 2\sqrt{2}$

sequence $3\sqrt{2}, 5\sqrt{2},\ 7\sqrt{2},\ 9\sqrt{2},\ldots$ are in A.P.

**Example 2.24** Write an A.P. whose first term is 20 and common difference is 8.

**Solution** First term $= a = 20$; common difference $= d = 8$

Arithmetic Progression is $a,\ a+d,\ a+2d,\ a+3d,\ldots$

we get $20,\ 20+8,\ 20+2(8),\ 20+3(8),\ldots$

the required A.P. is 20, 28, 36, 44, …

**Example 2.26** Find the number of terms in the A.P. 3, 6, 9, 12,…, 111.

**Solution**

First term $a = 3$; common difference $d = 6 - 3 = 3$; last term $l = 111$

We know that, $n = \left(\frac{l-a}{d}\right) + 1$

$$n = \left(\frac{111-3}{3}\right) + 1 = 37$$

Thus the A.P. contain 37 terms.

**Example 2.27** Determine the general term of an A.P. whose $7^{th}$ term is $-1$ and $16^{th}$ term is 17.

**Solution** Let the A.P. be $t_1, t_2, t_3, t_4, \ldots$

It is given that $t_7 = -1$ and $t_{16} = 17$

$a + (7-1)d = -1$ and $a + (16-1)d = 17$

$a + 6d = -1$ ...(1)

$a + 15d = 17$ ...(2)

Subtracting equation (1) from equation (2), we get $9d = 18 \Rightarrow d = 2$

Putting $d = 2$ in equation (1), we get $a + 12 = -1 \therefore a = -13$

Hence, general term $t_n = a + (n-1)d$

$$= -13 + (n-1) \times 2 = 2n - 15$$

**Example 2.31** Find the sum of first 15 terms of the A. P. $8, 7\frac{1}{4}, 6\frac{1}{2}, 5\frac{3}{4}, \ldots$

**Solution** Here the first term $a = 8$, common difference $d = 7\frac{1}{4} - 8 = -\frac{3}{4}$,

Sum of first $n$ terms of an A.P. $S_n = \frac{n}{2}[2a + (n-1)d]$

$$S_{15} = \frac{15}{2}\left[2 \times 8 + (15-1)(-\frac{3}{4})\right]$$

$$S_{15} = \frac{15}{2}\left[16 - \frac{21}{2}\right] = \frac{165}{4}$$

**Example 2.32** Find the sum of $0.40 + 0.43 + 0.46 + \cdots + 1$.

**Solution** Here the value of $n$ is not given. But the last term is given. From this, we can find the value of $n$.

Given, $a = 0.40$ and $l = 1$, we find $d = 0.43 - 0.40 = 0.03$.

$$\text{Therefore, } n = \left(\frac{l-a}{d}\right) + 1$$

$$= \left(\frac{1-0.40}{0.03}\right) + 1 = 21$$

Sum of first $n$ terms of an A.P. $S_n = \frac{n}{2}[a + l]$

, $n = 21$. Therefore, $S_{21} = \frac{21}{2}[0.40 + 1] = 14.7$

So, the sum of 21 terms of the given series is 14.7.

**Example 2.33** How many terms of the series $1+5+9+\ldots$ must be taken so that their sum is 190?

**Solution** Here we have to find the value of $n$, such that $S_n = 190$.

First term $a = 1$, common difference $d = 5 - 1 = 4$.

Sum of first $n$ terms of an A.P.

$$\begin{aligned} S_n = \frac{n}{2}[2a+(n-1)d] &= 190 \\ \frac{n}{2}[2\times 1+(n-1)\times 4] &= 190 \\ n[4n-2] &= 380 \\ 2n^2 - n - 190 &= 0 \\ (n-10)(2n+19) &= 0 \end{aligned}$$

But, $n = 10$ as $n = -\frac{19}{2}$ is impossible. Therefore, $n = 10$.

**Example 2.35** In an A.P. the sum of first $n$ terms is $\frac{5n^2}{2}+\frac{3n}{2}$. Find the $17^{th}$ term.

**Solution** The $17^{th}$ term can be obtained by subtracting the sum of first 16 terms from the sum of first 17 terms

$$S_{17} = \frac{5\times(17)^2}{2}+\frac{3\times 17}{2} = \frac{1445}{2}+\frac{51}{2} = 748$$

$$S_{16} = \frac{5\times(16)^2}{2}+\frac{3\times 16}{2} = \frac{1280}{2}+\frac{48}{2} = 664$$

Now, $t_{17} = S_{17} - S_{16} = 748 - 664 = 84$

**Example 2.40** Which of the following sequences form a Geometric Progression?

(i) 7, 14, 21, 28, … (ii) $\frac{1}{2}$, 1, 2, 4, … (iii) 5, 25, 50, 75, …

**Solution**

(i) 7, 14, 21, 28, …

$$\frac{t_2}{t_1} = \frac{14}{7} = 2; \quad \frac{t_3}{t_2} = \frac{21}{14} = \frac{3}{2}; \quad \frac{t_4}{t_3} = \frac{28}{21} = \frac{4}{3}$$

the sequence 7, 14, 21, 28, … is not a Geometric Progression.

(ii) $\frac{1}{2}$, 1, 2, 4, …

$$\frac{t_2}{t_1} = \frac{1}{\frac{1}{2}} = 2; \quad \frac{t_3}{t_2} = \frac{2}{1} = 2; \quad \frac{t_4}{t_3} = \frac{4}{2} = 2$$

Therefore the sequence $\frac{1}{2}$, 1, 2, 4, … is a Geometric Progression with common ratio $r = 2$.

(iii) 5, 25, 50, 75,...

$$\frac{t_2}{t_1} = \frac{25}{5} = 5; \quad \frac{t_3}{t_2} = \frac{50}{25} = 2; \quad \frac{t_4}{t_3} = \frac{75}{50} = \frac{3}{2}$$

the sequence 5, 25, 50, 75,... is not a Geometric Progression.

**Example 2.41** Find the geometric progression whose first term and common ratios are given by (i) $a=-7$, $r=6$ (ii) $a=256$, $r=0.5$

**Solution** (i) The general form of Geometric progression is $a, ar, ar^2, ...$

$a=-7$, $ar=-7\times 6=-42$, $ar^2=-7\times 6^2=-252$

Therefore the required Geometric Progression is $-7, -42, -252, ...$

(ii) The general form of Geometric progression is $a, ar, ar^2, ...$

$a=256$, $ar=256\times 0.5=128$, $ar^2=256\times(0.5)^2=64$

Therefore the required Geometric progression is 256, 128, 64,....

**Example 2.42** Find the $8^{th}$ term of the G.P. 9, 3, 1,...

**Solution** To find the $8^{th}$ term we have to use the $n^{th}$ term formula $t_n=ar^{n-1}$

First term $a=9$, Common ratio $r=\frac{t_2}{t_1}=\frac{3}{9}=\frac{1}{3}$

$$t_8=9\times\left(\frac{1}{3}\right)^{8-1}=9\times\left(\frac{1}{3}\right)^7=\frac{1}{243}$$

Therefore the $8^{th}$ term of the G.P. is $\frac{1}{243}$.

**Example 2.46** Find the sum of 8 terms of the G.P. $1,-3,9,-27...$

**Solutions** Here, the first term $a=1$, common ratio $r=\frac{-3}{1}=-3<1$, Here, $n=8$.

Sum to $n$ terms of a G.P. is $S_n=\frac{a(r^n-1)}{r-1}$ if $r\neq 1$

$$\text{Hence,}\quad S_8=\frac{1\left((-3)^8-1\right)}{(-3)-1}=\frac{6561-1}{-4}=-1640$$

**Example 2.47** Find the first term of a G.P. in which $S_6=4095$ and $r=4$.

**Solution** Common ratio $=4>1$, Sum of first 6 terms $S_6=4095$

$$\text{Hence,}\quad S_6=\frac{a(r^n-1)}{r-1}=4095$$

$$\because\quad r=4, \frac{a(4^6-1)}{4-1}=4095 \Rightarrow a\times\frac{4095}{3}=4095$$

First term $a=3$.

**Example 2.48** How many terms of the series $1+4+16+\cdots$ make the sum 1365?

**Solution** Let $n$ be the number of terms to be added to get the sum 1365

$$a=1,\ r=\frac{4}{1}=4>1$$

$$S_n=1365 \Rightarrow \frac{a(r^n-1)}{r-1}=1365$$

$$\frac{1(4^n-1)}{4-1}=1365 \text{ so, } (4^n-1)=4095$$

$$4^n=4096 \text{ then } 4^n=4^6$$

$$n=6$$

**Example 2.49** Find the sum $3+1+\frac{1}{3}+...\infty$

**Solution** Here $a=3$, $r=\frac{t_2}{t_1}=\frac{1}{3}$

Sum of infinite terms $S_\infty=\frac{a}{1-r}=\frac{3}{1-\frac{1}{3}}=\frac{9}{2}$

**Example 2.54** Find the value of (i) $1+2+3+...+50$ (ii) $16+17+18+...+75$

**Solution** (i) $1+2+3+\cdots+50$

$$\text{Using, } 1+2+3+\cdots+n=\frac{n(n+1)}{2}$$

$$1+2+3+\cdots+50=\frac{50\times(50+1)}{2}=1275$$

(ii)

$$16+17+18+\cdots+75=(1+2+3+\cdots+75)-(1+2+3+\cdots+15)$$

$$=\frac{75(75+1)}{2}-\frac{15(15+1)}{2}$$

$$=2850-120=2730$$

**Example 2.55** Find the sum of (i) $1+3+5+\cdots$ to 40 terms

(ii) $2+4+6+\cdots+80$ (iii) $1+3+5+\cdots+55$

**Solution** (i) $1+3+5+\cdots 40$ terms $=40^2=1600$

(ii) $2+4+6+\cdots+80=2(1+2+3+\cdots+40)=2\times\frac{40\times(40+1)}{2}=1640$

(iii) $1+3+5+\cdots+55$

Here the number of terms is not given. Now we have to find the number of terms using the formula, $n=\frac{(l-a)}{d}+1 \Rightarrow n=\frac{(55-1)}{2}+1=28$

Therefore, $1+3+5+\cdots+55=(28)^2=784$

**Example 2.56** Find the sum of (i) $1^2+2^2+\cdots+19^2$

(ii) $5^2+10^2+15^2+\cdots+105^2$ (iii) $15^2+16^2+17^2+\cdots+28^2$

**Solution** (i) $1^2+2^2+\cdots+19^2=\frac{19\times(19+1)(2\times19+1)}{6}=\frac{19\times20\times39}{6}=2470$

(ii) $5^2+10^2+15^2+\cdots+105^2=5^2(1^2+2^2+3^2+\cdots+21^2)$

$=25\times\frac{21\times(21+1)(2\times21+1)}{6}$

$=\frac{25\times21\times22\times43}{6}=82775$

(iii) $15^2+16^2+17^2+\cdots+28^2=(1^2+2^2+3^2+\cdots+28^2)-(1^2+2^2+3^2+\cdots+14^2)$

$=\frac{28\times29\times57}{6}-\frac{14\times15\times29}{6}=7714-1015=6699$

**Example 2.57** Find the sum of (i) $1^3+2^3+3^3+\cdots+16^3$ (ii) $9^3+10^3+\cdots+21^3$

**Solution** (i) $1^3+2^3+3^3+\cdots+16^3=\left[\frac{16\times(16+1)}{2}\right]^2=(136)^2=18496$

(ii) $9^3+10^3+\cdots+21^3=(1^3+2^3+3^3+\cdots+21^3)-(1^3+2^3+3^3+\cdots+8^3)$

$=\left[\frac{21\times(21+1)}{2}\right]^2-\left[\frac{8\times(8+1)}{2}\right]^2=(231)^2-(36)^2=52065$

**Example 2.58** If $1+2+3+\cdots+n=666$ then find $n$.

**Solution** Since, $1+2+3+...+n=\frac{n(n+1)}{2}$, we have $\frac{n(n+1)}{2}=666$

$n^2+n-1332=0 \Rightarrow (n+37)(n-36)=0$

So, $n=-37$ or $n=36$

But $n\neq-37$ ($\because n$ is a natural number); Hence $n=36$.

## 5-MARKS

**Example 2.6** Find the HCF of 396, 504, 636.

**Solution** To find HCF of three given numbers, first we have to find HCF of the first two numbers.

To find HCF of 396 and 504

Using Euclid's division algorithm we get $504=396\times1+108$

The remainder is $108\neq0$

Again applying Euclid's division algorithm $396=108\times3+72$

The remainder is $72\neq0$,

Again applying Euclid's division algorithm $108=72\times1+36$

The remainder is $36\neq0$,

Again applying Euclid's division algorithm $72 = 36 \times 2 + 0$

Here the remainder is zero. Therefore HCF of $396, 504 = 36$.

To find the HCF of 636 and 36.

Using Euclid's division algorithm we get $636 = 36 \times 17 + 24$

The remainder is $24 \neq 0$

Again applying Euclid's division algorithm $36 = 24 \times 1 + 12$

The remainder is $12 \neq 0$

Again applying Euclid's division algorithm $24 = 12 \times 2 + 0$

Here the remainder is zero. Therefore HCF of $636, 36 = 12$

Therefore Highest Common Factor of 396, 504 and 636 is 12.

**Example 2.11** Find the remainders when 70004 and 778 is divided by 7.

***Solution*** Since 70000 is divisible by 7

$$70000 \equiv 0 \pmod 7$$

$$70000 + 4 \equiv 0 + 4 \pmod 7$$

$$70004 \equiv 4 \pmod 7$$

Therefore, the remainder when 70004 is divided by 7 is 4.

$\because$ 777 is divisible by 7

$$777 \equiv 0 \pmod 7$$

$$777 + 1 \equiv 0 + 1 \pmod 7$$

$$778 \equiv 1 \pmod 7$$

Therefore, the remainder when 778 is divided by 7 is 1.

**Example 2.25** Find the $15^{th}$, $24^{th}$ and $n^{th}$ term (general term) of an A.P. given by 3, 15, 27, 39,…

***Solution*** We have, first term $= a = 3$ and common difference $= d = 15 - 3 = 12$.

We know that $n^{th}$ term (general term) of an A.P. with first term $a$ and common difference $d$ is given by $t_n = a + (n-1)d$

$$t_{15} = a + (15-1)d = a + 14d = 3 + 14(12) = 171$$

(Here $a = 3$ and $d = 12$)

$$t_{24} = a + (24-1)d = a + 23d = 3 + 23(12) = 279$$

The $n^{th}$ (general term) term is given by $t_n = a + (n-1)d$

Thus, $t_n = 3 + (n-1)12$

$$t_n = 12n - 9$$

**Example 2.28** If $l^{th}$, $m^{th}$ and $n^{th}$ terms of an A.P. are $x, y, z$ respectively, then show that

(i) $x(m-n) + y(n-l) + z(l-m) = 0$  (ii) $(x-y)n + (y-z)l + (z-x)m = 0$

***Solution*** (i) Let $a$ be the first term and $d$ be the common difference. It is given that

$t_l = x, t_m = y, t_n = z$

Using the general term formula

$a + (l-1)d = x$ …(1)

$a + (m-1)d = y$ …(2)

$a + (n-1)d = z$ …(3)

We have, $x(m-n)+y(n-l)+z(l-m)$

$$= a[(m-n)+(n-l)+(l-m)]+d[(m-n)(l-1)+(n-l)(m-1)+(l-m)(n-1)]$$
$$= a[0]+d[lm-ln-m+n+mn-lm-n+l+ln-mn-l+m]$$
$$= a(0)+d(0)=0$$

(ii) On subtracting equation (2) from equation (1), equation (3) from equation (2) and equation (1) from equation (3), we get

$$x-y=(l-m)d$$
$$y-z=(m-n)d$$
$$z-x=(n-l)d$$
$$(x-y)n+(y-z)l+(z-x)m = [(l-m)n+(m-n)l+(n-l)m]d$$
$$= [ln-mn+lm-nl+nm-lm]d = 0$$

**Example 2.29** In an A.P., sum of four consecutive terms is 28 and the sum of their squares is 276. Find the four numbers.

***Solution*** Let us take the four terms in the form $(a-3d)$, $(a-d)$, $(a+d)$ and $(a+3d)$.

Since, sum of the four terms is 28,

$$a-3d+a-d+a+d+a+3d=28$$
$$4a=28 \Rightarrow a=7$$

Similarly, since sum of their squares is 276,

$$(a-3d)^2+(a-d)^2+(a+d)^2+(a+3d)^2=276.$$
$$a^2-6ad+9d^2+a^2-2ad+d^2+a^2+2ad+d^2+a^2+6ad+9d^2=276$$
$$4a^2+20d^2=276 \Rightarrow 4(7)^2+20d^2=276.$$
$$d^2=4 \Rightarrow d=\pm\sqrt{4} \text{ then, } d=\pm 2$$

If $d=2$ then the four numbers are $7-3(2),\ 7-2,\ 7+2,\ 7+3(2)$

That is the four numbers are 1, 5, 9 and 13.

If $a=7,\ d=-2$ then the four numbers are 13, 9, 5 and 1

Therefore, the four consecutive terms of the A.P. are 1, 5, 9 and 13.

**Example 2.30** A mother divides ₹207 into three parts such that the amount are in A.P. and gives it to her three children. The product of the two least amounts that the children had ₹4623. Find the amount received by each child.

***Solution*** Let the amount received by the three children be in the form of A.P. is given by $a-d$, $a$, $a+d$. Since, sum of the amount is ₹207, we have

$$(a-d)+a+(a+d)=207$$
$$3a=207 \Rightarrow a=69$$

It is given that product of the two least amounts is 4623.

$$(a-d)a=4623$$
$$(69-d)69=4623$$
$$d=2$$

Therefore, amount given by the mother to her three children are

₹(69−2), ₹69, ₹(69+2). That is, ₹67, ₹69 and ₹71.

**Example 2.34** The $13^{th}$ term of an A.P. is 3 and the sum of first 13 terms is 234. Find the common difference and the sum of first 21 terms.

***Solution*** Given, the $13^{th}$ term = 3 so, $t_{13}=a+12d=3$ ...(1)

Sum of first 13 terms = 234 $\Rightarrow S_{13} = \frac{13}{2}[2a + 12d] = 234$

$2a + 12d = 36$ ...(2)

Solving (1) and (2) we get, $a = 33$, $d = \frac{-5}{2}$

Therefore, common difference is $\frac{-5}{2}$.

Sum of first 21 terms $S_{21} = \frac{21}{2}\left[2 \times 33 + (21-1) \times \left(-\frac{5}{2}\right)\right] = \frac{21}{2}[66 - 50] = 168$.

**Example 2.36** Find the sum of all natural numbers between 300 and 600 which are divisible by 7.

**Solution** The natural numbers between 300 and 600 which are divisible by 7 are 301, 308, 315, …, 595.

The sum of all natural numbers between 300 and 600 is $301 + 308 + 315 + \cdots + 595$.

The terms of the above series are in A.P.

First term $a = 301$; common difference $d = 7$; Last term $l = 595$.

$$n = \left(\frac{l-a}{d}\right) + 1 = \left(\frac{595-301}{7}\right) + 1 = 43$$

$\because S_n = \frac{n}{2}[a + l]$, we have $S_{43} = \frac{43}{2}[301 + 595] = 19264$.

**Example 2.39** The sum of first $n$, $2n$ and $3n$ terms of an A.P. are $S_1$, $S_2$ and $S_3$ respectively. Prove that $S_3 = 3(S_2 - S_1)$.

**Solution** If $S_1, S_2$ and $S_3$ are sum of first $n$, $2n$ and $3n$ terms of an A.P. respectively then

$$S_1 = \frac{n}{2}[2a + (n-1)d], \quad S_2 = \frac{2n}{2}[2a + (2n-1)d], \quad S_3 = \frac{3n}{2}[2a + (3n-1)d]$$

Consider,
$$S_2 - S_1 = \frac{2n}{2}[2a + (2n-1)d] - \frac{n}{2}[2a + (n-1)d]$$
$$= \frac{n}{2}[[4a + 2(2n-1)d] - [2a + (n-1)d]]$$
$$S_2 - S_1 = \frac{n}{2} \times [2a + (3n-1)d]$$
$$3(S_2 - S_1) = \frac{3n}{2}[2a + (3n-1)d]$$
$$3(S_2 - S_1) = S_3$$

**Example 2.43** In a Geometric progression, the $4^{th}$ term is $\frac{8}{9}$ and the $7^{th}$ term is $\frac{64}{243}$. Find the Geometric Progression.

**Solution** $4^{th}$ term, $t_4 = \frac{8}{9} \Rightarrow ar^3 = \frac{8}{9}$ ...(1)

$7^{th}$ term, $t_7 = \frac{64}{243} \Rightarrow ar^6 = \frac{64}{243}$ ...(2)

Dividing (2) by (1) we get, $\frac{ar^6}{ar^3} = \frac{\frac{64}{243}}{\frac{8}{9}}$

$$r^3 = \frac{8}{27} \Rightarrow r = \frac{2}{3}$$

Substituting the value of $r$ in (1), we get $a \times \left[\frac{2}{3}\right]^3 = \frac{8}{9} \Rightarrow a = 3$

Therefore the Geometric Progression is $a, ar, ar^2, \ldots$ That is, 3, 2, $\frac{4}{3}$, …

Example 2.44 The product of three consecutive terms of a Geometric Progression is 343 and their sum is $\frac{91}{3}$. Find the three terms.

*Solution* Since the product of 3 consecutive terms is given.

we can take them as $\frac{a}{r}, a, ar$.

Product of the terms = 343

$$\frac{a}{r} \times a \times ar = 343$$
$$a^3 = 7^3 \Rightarrow a = 7$$

Sum of the terms $= \frac{91}{3}$

Hence $a\left(\frac{1}{r} + 1 + r\right) = \frac{91}{3} \Rightarrow 7\left(\frac{1 + r + r^2}{r}\right) = \frac{91}{3}$

$3 + 3r + 3r^2 = 13r \Rightarrow 3r^2 - 10r + 3 = 0$

$(3r - 1)(r - 3) = 0 \Rightarrow r = 3$ or $r = \frac{1}{3}$

If $a = 7, r = 3$ then the three terms are $\frac{7}{3}, 7, 21$.

If $a = 7, r = \frac{1}{3}$ then the three terms are $21, 7, \frac{7}{3}$.

Example 2.51 Find the sum to $n$ terms of the series $5 + 55 + 555 + \cdots$

*Solution* The series is neither Arithmetic nor Geometric series. So it can be split into two series and then find the sum.

$$5 + 55 + 555 + \cdots + n \text{ terms} = 5[1 + 11 + 111 + \cdots + n \text{ terms}]$$
$$= \frac{5}{9}[9 + 99 + 999 + \cdots + n \text{ terms}]$$
$$= \frac{5}{9}[(10 - 1) + (100 - 1) + (1000 - 1) + \cdots + n \text{ terms}]$$
$$= \frac{5}{9}[(10 + 100 + 1000 + \cdots + n \text{ terms}) - n]$$
$$= \frac{5}{9}\left[\frac{10(10^n - 1)}{(10 - 1)} - n\right] = \frac{50(10^n - 1)}{81} - \frac{5n}{9}$$

Example 2.52 Find the least positive integer $n$ such that $1 + 6 + 6^2 + \cdots + 6^n > 5000$

*Solution* We have to find the least number of terms for which the sum must be greater than 5000.

That is, to find the least value of $n$. such that $S_n > 5000$

We have, $S_n = \frac{a(r^n - 1)}{r - 1} = \frac{1(6^n - 1)}{6 - 1} = \frac{6^n - 1}{5}$

$$S_n > 5000 \Rightarrow \frac{6^n - 1}{5} > 5000$$
$$6^n - 1 > 25000 \Rightarrow 6^n > 25001$$
$$\because 6^5 = 7776 \text{ and } 6^6 = 46656$$

The least positive value of $n$ is 6 such that $1 + 6 + 6^2 + \cdots + 6^n > 5000$.

# Algebra

## 2-MARKS (SELECTED SUMS )

Example 3.1 The father's age is six times his son's age. Six years hence the age of father will be four times his son's age. Find the present ages (in years) of the son and father.

**Solution** Let the present age of father be $x$ years and the present age of son be $y$ years

Given, $x = 6y$ ... (1)

$x + 6 = 4(y + 6)$ ... (2)

Substituting (1) in (2), $6y + 6 = 4(y + 6)$

$6y + 6 = 4y + 24 \Rightarrow y = 9$

Therefore, son's age = 9 years and father's age = 54 years.

**Example 3.12** Find the LCM of the following

(i) $8x^4y^2,\ 48x^2y^4$ (ii) $5x - 10,\ 5x^2 - 20$

(iii) $x^4 - 1,\ x^2 - 2x + 1$ (iv) $x^3 - 27,\ (x-3)^2,\ x^2 - 9$

**Solution** (i) $8x^4y^2,\ 48x^2y^4$

$$\text{LCM } (8, 48) = 2 \times 2 \times 2 \times 6 = 48$$

$$\text{LCM } (x^4y^2, x^2y^4) = x^4y^4$$

$$\text{LCM } (8x^4y^2, 48x^2y^4) = 48x^4y^4$$

(ii) $(5x - 10),\ (5x^2 - 20)$

$$5x - 10 = 5(x - 2)$$

$$5x^2 - 20 = 5(x^2 - 4) = 5(x + 2)(x - 2)$$

$$\text{LCM } [(5x - 10), (5x^2 - 20)] = 5(x + 2)(x - 2)$$

(iii) $(x^4 - 1),\ x^2 - 2x + 1$

$$x^4 - 1 = (x^2)^2 - 1 = (x^2 + 1)(x^2 - 1) = (x^2 + 1)(x + 1)(x - 1)$$

$$x^2 - 2x + 1 = (x - 1)^2$$

$$\text{LCM } [(x^4 - 1), (x^2 - 2x + 1)] = (x^2 + 1)(x + 1)(x - 1)^2$$

(iv) $x^3 - 27,\ (x - 3)^2,\ x^2 - 9$

$$x^3 - 27 = (x - 3)(x^2 + 3x + 9)\ ; \quad (x - 3)^2 = (x - 3)^2; \quad (x^2 - 9) = (x + 3)(x - 3)$$

Therefore, $\text{LCM } [(x^3 - 27), (x - 3)^2, (x^2 - 9)] = (x - 3)^2(x + 3)(x^2 + 3x + 9)$

**Example 3.13** Reduce the rational expressions to its lowest form

(i) $\dfrac{x - 3}{x^2 - 9}$ (ii) $\dfrac{x^2 - 16}{x^2 + 8x + 16}$

**Solution** (i) $\dfrac{x - 3}{x^2 - 9} = \dfrac{x - 3}{(x + 3)(x - 3)} = \dfrac{1}{x + 3}$

(ii) $\dfrac{x^2 - 16}{x^2 + 8x + 16} = \dfrac{(x + 4)(x - 4)}{(x + 4)^2} = \dfrac{x - 4}{x + 4}$

**Example 3.14** Find the excluded values of the following expressions (if any).

(i) $\dfrac{x + 10}{8x}$ (ii) $\dfrac{7p + 2}{8p^2 + 13p + 5}$

**Solution**

(i) $\dfrac{x + 10}{8x}$

The expression $\dfrac{x + 10}{8x}$ is undefined when $8x = 0$ or $x = 0$. Hence the excluded value is 0.

(ii) $\frac{7p+2}{8p^2+13p+5}$

The expression $\frac{7p+2}{8p^2+13p+5}$ is undefined when $8p^2+13p+5=0$

that is, $(8p+5)(p+1)=0$

$p=\frac{-5}{8}$, $p=-1$. The excluded values are $\frac{-5}{8}$ and $-1$.

**Example 3.15** (i) Multiply $\frac{x^3}{9y^2}$ by $\frac{27y}{x^5}$ (ii) Multiply $\frac{x^4b^2}{x-1}$ by $\frac{x^2-1}{a^4b^3}$

**Solution** (i) $\frac{x^3}{9y^2}\times\frac{27y}{x^5}=\frac{3}{x^2y}$ (ii) $\frac{x^4b^2}{x-1}\times\frac{x^2-1}{a^4b^3}=\frac{x^4\times b^2}{x-1}\times\frac{(x+1)(x-1)}{a^4\times b^3}=\frac{x^4(x+1)}{a^4b}$

**Example 3.16** Find

(i) $\frac{14x^4}{y}\div\frac{7x}{3y^4}$ (ii) $\frac{x^2-16}{x+4}\div\frac{x-4}{x+4}$

**Solution :** (i) $\frac{14x^4}{y}\div\frac{7x}{3y^4}=\frac{14x^4}{y}\times\frac{3y^4}{7x}=6x^3y^3$

(ii) $\frac{x^2-16}{x+4}\div\frac{x-4}{x+4}=\frac{(x+4)(x-4)}{(x+4)}\times\left(\frac{x+4}{x-4}\right)=x+4$

**Example 3.19** Find the square root of the following expressions

(i) $256(x-a)^8(x-b)^4(x-c)^{16}(x-d)^{20}$ (ii) $\frac{144\ a^8\ b^{12}\ c^{16}}{81\ f^{12}g^4\ h^{14}}$

**Solution** (i) $\sqrt{256(x-a)^8(x-b)^4(x-c)^{16}(x-d)^{20}}=16\left|(x-a)^4(x-b)^2(x-c)^8(x-d)^{10}\right|$

(ii) $\sqrt{\frac{144a^8b^{12}c^{16}}{81f^{12}g^4h^{14}}}=\frac{4}{3}\left|\frac{a^4b^6c^8}{f^6g^2h^7}\right|$

**Example 3.20** Find the square root of the following expressions

$16x^2+9y^2-24xy+24x-18y+9$

**Solution** (i) $\sqrt{16x^2+9y^2-24xy+24x-18y+9}$

$=\sqrt{(4x)^2+(-3y)^2+(3)^2+2(4x)(-3y)+2(-3y)(3)+2(4x)(3)}$

$=\sqrt{(4x-3y+3)^2}=|4x-3y+3|$

**Example 3.23** Find the zeroes of the quadratic expression $x^2+8x+12$.

**Solution** Let $p(x)=x^2+8x+12=(x+2)(x+6)$

$p(-2)=4-16+12=0$

$p(-6)=36-48+12=0$

Therefore $-2$ and $-6$ are zeros of $p(x)=x^2+8x+12$

**Example 3.24** Write down the quadratic equation in general form for which sum and product of the roots are given below.

(i) 9, 14 (ii) $-\frac{7}{2}, \frac{5}{2}$ (iii) $-\frac{3}{5}, -\frac{1}{2}$

**Solution** (i) General form of the quadratic equation when the roots are given is

$$x^2 - (\text{sum of the roots})\,x + \text{product of the roots} = 0$$

$$x^2 - 9x + 14 = 0$$

(ii) $x^2 - \left(-\frac{7}{2}\right)x + \frac{5}{2} = 0 \Rightarrow 2x^2 + 7x + 5 = 0$

(iii) $x^2 - \left(-\frac{3}{5}\right)x + \left(-\frac{1}{2}\right) = 0 \Rightarrow \frac{10x^2 + 6x - 5}{10} = 0$

Therefore, $10x^2 + 6x - 5 = 0$.

**Example 3.25** Find the sum and product of the roots for each of the following quadratic equations : (i) $x^2 + 8x - 65 = 0$ (ii) $2x^2 + 5x + 7 = 0$ (iii) $kx^2 - k^2x - 2k^3 = 0$

**Solution** Let $\alpha$ and $\beta$ be the roots of the given quadratic equation

(i) $x^2 + 8x - 65 = 0$

$a = 1,\ b = 8,\ c = -65$

$$\alpha + \beta = -\frac{b}{a} = -8 \text{ and } \alpha\beta = \frac{c}{a} = -65$$

$$\alpha + \beta = -8;\quad \alpha\beta = -65$$

**Example 3.27** Solve $2m^2 + 19m + 30 = 0$

**Solution**

$$2m^2 + 19m + 30 = 2m^2 + 4m + 15m + 30 = 2m(m+2) + 15(m+2)$$
$$= (m+2)(2m+15)$$

Equating the factors to zero we get,

$(m+2)(2m+15) = 0$

$m + 2 = 0 \Rightarrow m = -2$ or $2m + 15 = 0$ we get, $m = \frac{-15}{2}$

Therefore the roots are $-2, \frac{-15}{2}$.

**Example 3.36** The product of Kumaran's age (in years) two years ago and his age four years from now is one more than twice his present age. What is his present age?

**Solution** Let the present age of Kumaran be $x$ years.

Two years ago, his age $= (x-2)$ years.

Four years from now, his age $= (x+4)$ years.

Given, $(x-2)(x+4) = 1 + 2x$

$x^2 + 2x - 8 = 1 + 2x \Rightarrow (x-3)(x+3) = 0$ then, $x = \pm 3$

Therefore, $x = 3$ (Rejecting $-3$ as age cannot be negative)

Kumaran's present age is 3 years.

**Example 3.40** Determine the nature of roots for the following quadratic equations (i) $x^2 - x - 20 = 0$ (ii) $9x^2 - 24x + 16 = 0$ (iii) $2x^2 - 2x + 9 = 0$

**Solution** (i) $x^2 - x - 20 = 0$

Here, $a = 1,\ b = -1,\ c = -20$

Now, $\Delta = b^2 - 4ac$

$\Delta = (-1)^2 - 4(1)(-20) = 81$

Here, $\Delta = 81 > 0$. So, the equation will have real and unequal roots

(ii) $9x^2 - 24x + 16 = 0$

Here, $a = 9,\ b = -24,\ c = 16$

Now, $\Delta = b^2 - 4ac = (-24)^2 - 4(9)(16) = 0$

Here, $\Delta = 0$. So, the equation will have real and equal roots.

(iii) $2x^2 - 2x + 9 = 0$

Here, $a = 2,\ b = -2,\ c = 9$

Now, $\Delta = b^2 - 4ac = (-2)^2 - 4(2)(9) = -68$

Here, $\Delta = -68 < 0$. So, the equation will have no real roots.

**Example 3.41** (i) Find the values of '$k$', for which the quadratic equation $kx^2 - (8k+4)x + 81 = 0$ has real and equal roots?

(ii) Find the values of '$k$' such that quadratic equation $(k+9)x^2 + (k+1)x + 1 = 0$ has no real roots?

***Solution*** (i) $kx^2 - (8k+4)x + 81 = 0$

Since the equation has real and equal roots, $\Delta = 0$.

That is, $b^2 - 4ac = 0$

Here, $a = k,\ b = -(8k+4),\ c = 81$

That is, $[-(8k+4)]^2 - 4(k)(81) = 0$

$$64k^2 + 64k + 16 - 324k = 0$$

$$64k^2 - 260k + 16 = 0$$

Dividing by 4 we get $16k^2 - 65k + 4 = 0$

$(16k-1)(k-4) = 0$ then, $k = \frac{1}{16}$ or $k = 4$

(ii) $(k+9)x^2 + (k+1)x + 1 = 0$

Since the equation has no real roots, $\Delta < 0$

That is, $b^2 - 4ac < 0$

Here, $a = k+9,\ b = k+1,\ c = 1$

That is, $(k+1)^2 - 4(k+9)(1) < 0$

$$k^2 + 2k + 1 - 4k - 36 < 0$$

$$k^2 - 2k - 35 < 0$$

$$(k+5)(k-7) < 0$$

Therefore, $-5 < k < 7$. {If $\alpha < \beta$ and if $(x-\alpha)(x-\beta) < 0$ then, $\alpha < x < \beta$}.

**Example 3.43** If the difference between the roots of the equation $x^2 - 13x + k = 0$ is 17 find $k$.

***Solution*** $x^2 - 13x + k = 0$ here, $a = 1,\ b = -13,\ c = k$

Let $\alpha, \beta$ be the roots of the equation. Then

$\alpha + \beta = \frac{-b}{a} = \frac{-(-13)}{1} = 13$ ...(1) Also $\alpha - \beta = 17$ ...(2)

(1)+(2) we get, $2\alpha = 30 \Rightarrow \alpha = 15$

Therefore, $15 + \beta = 13$ (from (1)) $\Rightarrow \beta = -2$

But, $\alpha\beta = \frac{c}{a} = \frac{k}{1} \Rightarrow 15 \times (-2) = k$ we get, $k = -30$

## 5-MARKS (SELECTED SUMS)

**Example 3.16** Find

$$\frac{16x^2-2x-3}{3x^2-2x-1} \div \frac{8x^2+11x+3}{3x^2-11x-4}$$

$$\frac{16x^2-2x-3}{3x^2-2x-1} \div \frac{8x^2+11x+3}{3x^2-11x-4} = \frac{16x^2-2x-3}{3x^2-2x-1} \times \frac{3x^2-11x-4}{8x^2+11x+3}$$

$$= \frac{(8x+3)(2x-1)}{(3x+1)(x-1)} \times \frac{(3x+1)(x-4)}{(8x+3)(x+1)} = \frac{(2x-1)(x-4)}{(x-1)(x+1)} = \frac{2x^2-9x+4}{x^2-1}$$

**Example 3.17** Find $\frac{x^2+20x+36}{x^2-3x-28} - \frac{x^2+12x+4}{x^2-3x-28}$

*Solution*

$$\frac{x^2+20x+36}{x^2-3x-28} - \frac{x^2+12x+4}{x^2-3x-28} = \frac{(x^2+20x+36)-(x^2+12x+4)}{x^2-3x-28}$$

$$= \frac{8x+32}{x^2-3x-28} = \frac{8(x+4)}{(x-7)(x+4)} = \frac{8}{x-7}$$

**Example 3.18** Simplify $\frac{1}{x^2-5x+6} + \frac{1}{x^2-3x+2} - \frac{1}{x^2-8x+15}$

*Solution*

$$\frac{1}{x^2-5x+6} + \frac{1}{x^2-3x+2} - \frac{1}{x^2-8x+15}$$

$$= \frac{1}{(x-2)(x-3)} + \frac{1}{(x-2)(x-1)} - \frac{1}{(x-5)(x-3)}$$

$$= \frac{(x-1)(x-5)+(x-3)(x-5)-(x-1)(x-2)}{(x-1)(x-2)(x-3)(x-5)}$$

$$= \frac{(x^2-6x+5)+(x^2-8x+15)-(x^2-3x+2)}{(x-1)(x-2)(x-3)(x-5)}$$

$$= \frac{x^2-11x+18}{(x-1)(x-2)(x-3)(x-5)} = \frac{(x-9)(x-2)}{(x-1)(x-2)(x-3)(x-5)}$$

$$= \frac{x-9}{(x-1)(x-3)(x-5)}$$

**Example 3.20** Find the square root of the following expressions

$$(6x^2+x-1)(3x^2+2x-1)(2x^2+3x+1)$$

$$\sqrt{(6x^2+x-1)(3x^2+2x-1)(2x^2+3x+1)}$$

$$= \sqrt{(3x-1)(2x+1)(3x-1)(x+1)(2x+1)(x+1)} = |(3x-1)(2x+1)(x+1)|$$

**Example 3.21** Find the square root of $64x^4-16x^3+17x^2-2x+1$

Solution

**Example 3.21** Find the square root of $64x^4 - 16x^3 + 17x^2 - 2x + 1$

***Solution***

$$\begin{array}{r|l l}
 & 8x^2 - x + 1 & \\
\hline
8x^2 & 64x^4 - 16x^3 + 17x^2 - 2x + 1 & \\
 & 64x^4 & (-) \\
\hline
16x^2 - x & -16x^3 + 17x^2 & \\
 & -16x^3 + x^2 & (-) \\
\hline
16x^2 - 2x + 1 & 16x^2 - 2x + 1 & \\
 & 16x^2 - 2x + 1 & (-) \\
\hline
 & 0 &
\end{array}$$

Therefore, $\sqrt{64x^4 - 16x^3 + 17x^2 - 2x + 1} = \left|8x^2 - x + 1\right|$

**Example 3.22** If $9x^4 + 12x^3 + 28x^2 + ax + b$ is a perfect square, find the values of $a$ and $b$.

***Solution***

$$\begin{array}{r|l l}
 & 3x^2 + 2x + 4 & \\
\hline
3x^2 & 9x^4 + 12x^3 + 28x^2 + ax + b & \\
 & 9x^4 & (-) \\
\hline
6x^2 + 2x & 12x^3 + 28x^2 & \\
 & 12x^3 + 4x^2 & (-) \\
\hline
6x^2 + 4x + 4 & 24x^2 + ax + b & \\
 & 24x^2 + 16x + 16 & (-) \\
\hline
 & 0 &
\end{array}$$

given polynomial is a perfect square $a - 16 = 0$ , $b - 16 = 0$

Therefore $a = 16$, $b = 16$ .

**Example 3.28** Solve $x^4 - 13x^2 + 42 = 0$

***Solution*** Let $x^2 = a$ . Then, $(x^2)^2 - 13x^2 + 42 = a^2 - 13a + 42 = (a-7)(a-6)$

Given, $(a-7)(a-6) = 0$ we get, $a = 7$ or 6.

Since $a = x^2$ , $x^2 = 7$ then, $x = \pm\sqrt{7}$ or $x^2 = 6$ we get, $x = \pm\sqrt{6}$

Therefore the roots are $x = \pm\sqrt{7}$ , $\pm\sqrt{6}$

**Example 3.29** Solve $\frac{x}{x-1} + \frac{x-1}{x} = 2\frac{1}{2}$

***Solution*** Let $y = \frac{x}{x-1}$ then $\frac{1}{y} = \frac{x-1}{x}$ .

Therefore, $\frac{x}{x-1} + \frac{x-1}{x} = 2\frac{1}{2}$ becomes $y + \frac{1}{y} = \frac{5}{2}$

$2y^2 - 5y + 2 = 0$ then, $y = \frac{1}{2}$, 2

$\frac{x}{x-1} = \frac{1}{2}$ we get, $2x = x - 1$ implies $x = -1$

$\frac{x}{x-1} = 2$ we get, $x = 2x - 2$ implies $x = 2$

Therefore, the roots are $x = -1$, 2.

**Example 3.32** Solve $x^2+2x-2=0$ by formula method

**Solution** Compare $x^2+2x-2=0$ with the standard form $ax^2+bx+c=0$

$a=1,\ b=2,\ c=-2$

$$x=\frac{-b\pm\sqrt{b^2-4ac}}{2a}$$

substituting the values of $a$, $b$ and $c$ in the formula we get,

$$x=\frac{-2\pm\sqrt{(2)^2-4(1)(-2)}}{2(1)}=\frac{-2\pm\sqrt{12}}{2}=-1\pm\sqrt{3}$$

Therefore, $x=-1+\sqrt{3},\ -1-\sqrt{3}$

**Example 3.33** Solve $2x^2-3x-3=0$ by formula method.

**Solution** Compare $2x^2-3x-3=0$ with the standard form $ax^2+bx+c=0$

$a=2,\ b=-3,\ c=-3$

$$x=\frac{-b\pm\sqrt{b^2-4ac}}{2a}$$

substituting the values of $a$, $b$ and $c$ in the formula we get,

$$x=\frac{-(-3)\pm\sqrt{(-3)^2-4(2)(-3)}}{2(2)}=\frac{3\pm\sqrt{33}}{4}$$

Therefore, $x=\frac{3+\sqrt{33}}{4},\ \frac{3-\sqrt{33}}{4}$

**Example 3.37** A ladder 17 feet long is leaning against a wall. If the ladder, vertical wall and the floor from the bottom of the wall to the ladder form a right triangle, find the height of the wall where the top of the ladder meets if the distance between bottom of the wall to bottom of the ladder is 7 feet less than the height of the wall?

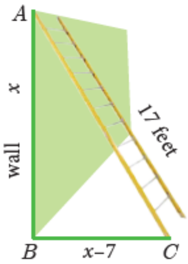


Fig. 3.7

**Solution** Let the height of the wall $AB=x$ feet

As per the given data $BC=(x-7)$ feet

In the right triangle $ABC$, $AC=17$ ft, $BC=(x-7)$ feet

By Pythagoras theorem, $AC^2=AB^2+BC^2$

$(17)^2=x^2+(x-7)^2$; $289=x^2+x^2-14x+49$

$x^2-7x-120=0$ hence, $(x-15)(x+8)=0$ then, $x=15$ (or) $-8$

Therefore, height of the wall AB = 15 ft (Rejecting –8 as height cannot be negative)

**Example 3.38** A flock of swans contained $x^2$ members. As the clouds gathered, $10x$ went to a lake and one-eighth of the members flew away to a garden. The remaining three pairs played about in the water. How many swans were there in total?

**Solution** As given there are $x^2$ swans.

As per the given data $x^2-10x-\frac{1}{8}x^2=6$ we get, $7x^2-80x-48=0$

$$x=\frac{-b\pm\sqrt{b^2-4ac}}{2a}=\frac{80\pm\sqrt{6400-4(7)(-48)}}{14}=\frac{80\pm 88}{14}$$

Therefore, $x = 12, -\frac{4}{7}$.

Here $x = -\frac{4}{7}$ is not possible as the number of swans cannot be negative.

Hence, $x = 12$. Therefore total number of swans is $x^2 = 144$.

**Example 3.39** A passenger train takes 1 hr more than an express train to travel a distance of 240 km from Chennai to Virudhachalam. The speed of passenger train is less than that of an express train by 20 km per hour. Find the average speed of both the trains.

**Solution** Let the average speed of passenger train be $x$ km/hr.

Then the average speed of express train will be $(x+20)$ km/hr

Time taken by the passenger train to cover distance of 240 km $= \frac{240}{x}$ hr

Time taken by express train to cover distance of 240 km $= \frac{240}{x+20}$ hr

Given, $\frac{240}{x} = \frac{240}{x+20} + 1$

$$240\left[\frac{1}{x} - \frac{1}{x+20}\right] = 1 \Rightarrow 240\left[\frac{x+20-x}{x(x+20)}\right] = 1 \text{ we get, } 4800 = (x^2+20x)$$

$$x^2 + 20x - 4800 = 0 \Rightarrow (x+80)(x-60) = 0 \text{ we get, } x = -80 \text{ or } 60.$$

Therefore $x = 60$ (Rejecting $-80$ as speed cannot be negative)

Average speed of the passenger train is 60 km/hr

Average speed of the express train is 80 km/hr.

**Example 3.42** Prove that the equation $x^2(p^2+q^2) + 2x(pr+qs) + r^2 + s^2 = 0$ has no real roots. If $ps = qr$, then show that the roots are real and equal.

**Solution** The given quadratic equation is, $x^2(p^2+q^2) + 2x(pr+qs) + r^2 + s^2 = 0$

Here, $a = p^2+q^2$, $b = 2(pr+qs)$, $c = r^2+s^2$

$$\text{Now, } \Delta = b^2 - 4ac = \left[2(pr+qs)\right]^2 - 4(p^2+q^2)(r^2+s^2)$$

$$= 4\left[p^2r^2 + 2pqrs + q^2s^2 - p^2r^2 - p^2s^2 - q^2r^2 - q^2s^2\right]$$

$$= 4\left[-p^2s^2 + 2pqrs - q^2r^2\right] = -4\left[(ps-qr)^2\right] < 0 \quad ...(1)$$

since, $\Delta = b^2 - 4ac < 0$, the roots are not real.

If $ps = qr$ then $\Delta = -4\left[ps - qr\right]^2 = -4\left[qr - qr\right]^2 = 0$ (using (1))

Thus, $\Delta = 0$ if $ps = qr$ and so the roots will be real and equal.

**Example 3.44** If $\alpha$ and $\beta$ are the roots of $x^2 + 7x + 10 = 0$ find the values of

(i) $(\alpha-\beta)$ (ii) $\alpha^2+\beta^2$ (iii) $\alpha^3-\beta^3$ (iv) $\alpha^4+\beta^4$ (v) $\frac{\alpha}{\beta}+\frac{\beta}{\alpha}$ (vi) $\frac{\alpha^2}{\beta}+\frac{\beta^2}{\alpha}$

**Solution** $x^2 + 7x + 10 = 0$ here, $a = 1$, $b = 7$, $c = 10$

If $\alpha$ and $\beta$ are roots of the equation then,

$$\alpha+\beta = \frac{-b}{a} = \frac{-7}{1} = -7 \text{ ; } \alpha\beta = \frac{c}{a} = \frac{10}{1} = 10$$

(i) $\alpha - \beta = \sqrt{(\alpha+\beta)^2 - 4\alpha\beta} = \sqrt{(-7)^2 - 4\times 10} = \sqrt{9} = 3$

(ii) $\alpha^2 + \beta^2 = (\alpha+\beta)^2 - 2\alpha\beta = (-7)^2 - 2\times 10 = 29$

(iii) $\alpha^3 - \beta^3 = (\alpha-\beta)^3 + 3\alpha\beta(\alpha-\beta) = (3)^3 + 3(10)(3) = 117$

(iv) $\alpha^4+\beta^4 = (\alpha^2+\beta^2)^2-2\alpha^2\beta^2 = 29^2-2\times(10)^2 = 641$ (from (ii), $\alpha^2+\beta^2 = 29$ )

(v) $\dfrac{\alpha}{\beta}+\dfrac{\beta}{\alpha} = \dfrac{\alpha^2+\beta^2}{\alpha\beta} = \dfrac{(\alpha+\beta)^2-2\alpha\beta}{\alpha\beta} = \dfrac{49-20}{10} = \dfrac{29}{10}$

(vi) $\dfrac{\alpha^2}{\beta}+\dfrac{\beta^2}{\alpha} = \dfrac{\alpha^3+\beta^3}{\alpha\beta} = \dfrac{(\alpha+\beta)^3-3\alpha\beta(\alpha+\beta)}{\alpha\beta}$

$= \dfrac{(-343)-3\left(10\times(-7)\right)}{10} = \dfrac{-343+210}{10} = \dfrac{-133}{10}$

**Example 3.45** If $\alpha,\beta$ are the roots of the equation $3x^2+7x-2=0$, find the values of (i) $\dfrac{\alpha}{\beta}+\dfrac{\beta}{\alpha}$ (ii) $\dfrac{\alpha^2}{\beta}+\dfrac{\beta^2}{\alpha}$

***Solution*** $3x^2+7x-2=0$ here, $a=3$, $b=7$, $c=-2$

since, $\alpha,\beta$ are the roots of the equation

(i) $\alpha+\beta = \dfrac{-b}{a} = \dfrac{-7}{3}$; $\alpha\beta = \dfrac{c}{a} = \dfrac{-2}{3}$

$$\frac{\alpha}{\beta}+\frac{\beta}{\alpha} = \frac{\alpha^2+\beta^2}{\alpha\beta} = \frac{(\alpha+\beta)^2-2\alpha\beta}{\alpha\beta} = \frac{\left(\frac{-7}{3}\right)^2-2\left(\frac{-2}{3}\right)}{\frac{-2}{3}} = \frac{-61}{6}$$

(ii) $$\frac{\alpha^2}{\beta}+\frac{\beta^2}{\alpha} = \frac{\alpha^3+\beta^3}{\alpha\beta} = \frac{(\alpha+\beta)^3-3\alpha\beta(\alpha+\beta)}{\alpha\beta} = \frac{\left(-\frac{7}{3}\right)^3-3\left(-\frac{2}{3}\right)\left(-\frac{7}{3}\right)}{-\frac{2}{3}} = \frac{469}{18}$$

**Example 3.46** If $\alpha,\beta$ are the roots of the equation $2x^2-x-1=0$, then form the equation whose roots are (i) $\dfrac{1}{\alpha},\dfrac{1}{\beta}$ (ii) $\alpha^2\beta,\ \beta^2\alpha$ (iii) $2\alpha+\beta,\ 2\beta+\alpha$

***Solution*** $2x^2-x-1=0$ here, $a=2$, $b=-1$, $c=-1$

$\alpha+\beta = \dfrac{-b}{a} = \dfrac{-(-1)}{2} = \dfrac{1}{2}$; $\alpha\beta = \dfrac{c}{a} = -\dfrac{1}{2}$

(i) Given roots are $\dfrac{1}{\alpha},\dfrac{1}{\beta}$

Sum of the roots $= \dfrac{1}{\alpha}+\dfrac{1}{\beta} = \dfrac{\alpha+\beta}{\alpha\beta} = \dfrac{\frac{1}{2}}{-\frac{1}{2}} = -1$

Product of the roots $= \dfrac{1}{\alpha}\times\dfrac{1}{\beta} = \dfrac{1}{\alpha\beta} = \dfrac{1}{-\frac{1}{2}} = -2$

The required equation is $x^2-$(Sum of the roots)$x+$ (Product of the roots)=0

$x^2-(-1)x-2=0 \Rightarrow x^2+x-2=0$

(ii) Given roots are $\alpha^2\beta,\ \beta^2\alpha$

Sum of the roots $\alpha^2\beta+\beta^2\alpha = \alpha\beta(\alpha+\beta) = -\dfrac{1}{2}\left(\dfrac{1}{2}\right) = -\dfrac{1}{4}$

Product of the roots $(\alpha^2\beta)\times(\beta^2\alpha) = \alpha^3\beta^3 = (\alpha\beta)^3 = \left(-\dfrac{1}{2}\right)^3 = -\dfrac{1}{8}$

The required equation is $x^2-$(Sum of the roots)$x+$ (Product of the roots)=0

$x^2-\left(-\dfrac{1}{4}\right)x-\dfrac{1}{8}=0 \Rightarrow 8x^2+2x-1=0$

(iii) $2\alpha+\beta\,,\ 2\beta+\alpha$

Sum of the roots $2\alpha+\beta+2\beta+\alpha=3(\alpha+\beta)=3\left(\frac{1}{2}\right)=\frac{3}{2}$

Product of the roots $=(2\alpha+\beta)(2\beta+\alpha)=4\alpha\beta+2\alpha^2+2\beta^2+\alpha\beta$

$=5\alpha\beta+2(\alpha^2+\beta^2)\ =5\alpha\beta+2\left[(\alpha+\beta)^2-2\alpha\beta\right]$

$=5\left(-\frac{1}{2}\right)+2\left[\frac{1}{4}-2\times-\frac{1}{2}\right]=0$

The required equation is $x^2-(\text{Sum of the roots})x+(\text{Product of the roots})=0$

$x^2-\frac{3}{2}x+0=0\ \Rightarrow\ 2x^2-3x=0$

# Matrices

## 2-MARKS

**Example 3.57** If a matrix has 16 elements, what are the possible orders it can have?

Hence possible orders are $1\times16,\ 16\times1,\ 4\times4,\ 2\times8,\ 8\times2$

**Example 3.58** Construct a $3\times3$ matrix whose elements are $a_{ij}=i^2j^2$

*Solution* The general 3x3 matrix is given by $A=\begin{pmatrix}a_{11}&a_{12}&a_{13}\\a_{21}&a_{22}&a_{23}\\a_{31}&a_{32}&a_{33}\end{pmatrix}$ $\quad a_{ij}=i^2j^2$

$a_{11}=1^2\times1^2=1\times1=1;\quad a_{12}=1^2\times2^2=1\times4=4;\quad a_{13}=1^2\times3^2=1\times9=9;$

$a_{21}=2^2\times1^2=4\times1=4;\quad a_{22}=2^2\times2^2=4\times4=16;\quad a_{23}=2^2\times3^2=4\times9=36$

$a_{31}=3^2\times1^2=9\times1=9;\quad a_{32}=3^2\times2^2=9\times4=36;\quad a_{33}=3^2\times3^2=9\times9=81$

Hence the required matrix is $A=\begin{pmatrix}1&4&9\\4&16&36\\9&36&81\end{pmatrix}$

**Example 3.59** Find the value of $a,\ b,\ c,\ d$ from the equation $\begin{pmatrix}a-b&2a+c\\2a-b&3c+d\end{pmatrix}=\begin{pmatrix}1&5\\0&2\end{pmatrix}$

Put $2a=b$ in equation (1), $a-2a=1\quad\Rightarrow a=-1$

Put $a=-1$ in equation (5), $2(-1)=b\quad\Rightarrow b=-2$

Put $a=-1$ in equation (2), $2(-1)+c=5\quad\Rightarrow c=7$

Put $c=7$ in equation (4), $3(7)+d=2\quad\Rightarrow d=-19$

Therefore, $a=-1,\ b=-2,\ c=7,\ d=-19$

**Example 3.60** If $A=\begin{pmatrix}1&2&3\\4&5&6\\7&8&9\end{pmatrix}$, $B=\begin{pmatrix}1&7&0\\1&3&1\\2&4&0\end{pmatrix}$, find $A+B$.

*Solution* $A+B=\begin{pmatrix}1&2&3\\4&5&6\\7&8&9\end{pmatrix}+\begin{pmatrix}1&7&0\\1&3&1\\2&4&0\end{pmatrix}=\begin{pmatrix}1+1&2+7&3+0\\4+1&5+3&6+1\\7+2&8+4&9+0\end{pmatrix}=\begin{pmatrix}2&9&3\\5&8&7\\9&12&9\end{pmatrix}$

**Example 3.63** If $A=\begin{pmatrix}7 & 8 & 6\\ 1 & 3 & 9\\ -4 & 3 & -1\end{pmatrix}$, $B=\begin{pmatrix}4 & 11 & -3\\ -1 & 2 & 4\\ 7 & 5 & 0\end{pmatrix}$ then Find $2A+B$.

**Solution** Since $A$ and $B$ have same order $3\times 3$, $2A+B$ is defined.

We have $$2A+B = 2\begin{pmatrix}7 & 8 & 6\\ 1 & 3 & 9\\ -4 & 3 & -1\end{pmatrix}+\begin{pmatrix}4 & 11 & -3\\ -1 & 2 & 4\\ 7 & 5 & 0\end{pmatrix}=\begin{pmatrix}14 & 16 & 12\\ 2 & 6 & 18\\ -8 & 6 & -2\end{pmatrix}+\begin{pmatrix}4 & 11 & -3\\ -1 & 2 & 4\\ 7 & 5 & 0\end{pmatrix}$$

$$=\begin{pmatrix}18 & 27 & 9\\ 1 & 8 & 22\\ -1 & 11 & -2\end{pmatrix}$$

**Example 3.64** If $A=\begin{pmatrix}5 & 4 & -2\\ \frac{1}{2} & \frac{3}{4} & \sqrt{2}\\ 1 & 9 & 4\end{pmatrix}$, $B=\begin{pmatrix}-7 & 4 & -3\\ \frac{1}{4} & \frac{7}{2} & 3\\ 5 & -6 & 9\end{pmatrix}$, find $4A-3B$.

**Solution** Since $A$, $B$ are of the same order $3\times 3$, subtraction of $4A$ and $3B$ is defined.

$$4A-3B = 4\begin{pmatrix}5 & 4 & -2\\ \frac{1}{2} & \frac{3}{4} & \sqrt{2}\\ 1 & 9 & 4\end{pmatrix}-3\begin{pmatrix}-7 & 4 & -3\\ \frac{1}{4} & \frac{7}{2} & 3\\ 5 & -6 & 9\end{pmatrix}$$

$$=\begin{pmatrix}20 & 16 & -8\\ 2 & 3 & 4\sqrt{2}\\ 4 & 36 & 16\end{pmatrix}+\begin{pmatrix}21 & -12 & 9\\ -\frac{3}{4} & -\frac{21}{2} & -9\\ -15 & 18 & -27\end{pmatrix}$$

$$=\begin{pmatrix}41 & 4 & 1\\ \frac{5}{4} & -\frac{15}{2} & 4\sqrt{2}-9\\ -11 & 54 & -11\end{pmatrix}$$

**Example 3.65** Find the value of $a$, $b$, $c$, $d$ from the following matrix equation.

$$\begin{pmatrix}d & 8\\ 3b & a\end{pmatrix}+\begin{pmatrix}3 & a\\ -2 & -4\end{pmatrix}=\begin{pmatrix}2 & 2a\\ b & 4c\end{pmatrix}+\begin{pmatrix}0 & 1\\ -5 & 0\end{pmatrix}$$

**Solution**

First, we add the two matrices on both left, right hand sides to get

$$\begin{pmatrix}d+3 & 8+a\\ 3b-2 & a-4\end{pmatrix}=\begin{pmatrix}2 & 2a+1\\ b-5 & 4c\end{pmatrix}$$

Equating the corresponding elements of the two matrices, we have

$d+3=2 \quad \Rightarrow \quad d=-1$

$8+a=2a+1 \quad \Rightarrow \quad a=7$

$3b-2=b-5 \quad \Rightarrow \quad b=\frac{-3}{2}$

Substituting $a=7$ in $a-4=4c \Rightarrow c=\frac{3}{4}$

Therefore, $a=7,\ b=-\frac{3}{2},\ c=\frac{3}{4},\ d=-1$.

**Example 3.67** If $A=\begin{pmatrix}1&2&0\\3&1&5\end{pmatrix}$, $B=\begin{pmatrix}8&3&1\\2&4&1\\5&3&1\end{pmatrix}$, find $AB$.

$$AB=\begin{pmatrix}1&2&0\\3&1&5\end{pmatrix}\times\begin{pmatrix}8&3&1\\2&4&1\\5&3&1\end{pmatrix}$$

$$=\begin{pmatrix}8+4+0&3+8+0&1+2+0\\24+2+25&9+4+15&3+1+5\end{pmatrix}=\begin{pmatrix}12&11&3\\51&28&9\end{pmatrix}$$

**Example 3.68** If $A=\begin{pmatrix}2&1\\1&3\end{pmatrix}$, $B=\begin{pmatrix}2&0\\1&3\end{pmatrix}$ find $AB$ and $BA$. Verify $AB=BA$?

$$AB=\begin{pmatrix}2&1\\1&3\end{pmatrix}\times\begin{pmatrix}2&0\\1&3\end{pmatrix}=\begin{pmatrix}4+1&0+3\\2+3&0+9\end{pmatrix}=\begin{pmatrix}5&3\\5&9\end{pmatrix}$$

$$BA=\begin{pmatrix}2&0\\1&3\end{pmatrix}\times\begin{pmatrix}2&1\\1&3\end{pmatrix}=\begin{pmatrix}4+0&2+0\\2+3&1+9\end{pmatrix}=\begin{pmatrix}4&2\\5&10\end{pmatrix}$$

Therefore, $AB \neq BA$.

## 5-MARKS (SELECTED SUMS)

**Example 3.66** If $A=\begin{pmatrix}1&8&3\\3&5&0\\8&7&6\end{pmatrix}$, $B=\begin{pmatrix}8&-6&-4\\2&11&-3\\0&1&5\end{pmatrix}$, $C=\begin{pmatrix}5&3&0\\-1&-7&2\\1&4&3\end{pmatrix}$

compute the following : (i) $3A+2B-C$ (ii) $\frac{1}{2}A-\frac{3}{2}B$

*Solution* (i) $3A+2B-C=3\begin{pmatrix}1&8&3\\3&5&0\\8&7&6\end{pmatrix}+2\begin{pmatrix}8&-6&-4\\2&11&-3\\0&1&5\end{pmatrix}-\begin{pmatrix}5&3&0\\-1&-7&2\\1&4&3\end{pmatrix}$

$$=\begin{pmatrix}3&24&9\\9&15&0\\24&21&18\end{pmatrix}+\begin{pmatrix}16&-12&-8\\4&22&-6\\0&2&10\end{pmatrix}+\begin{pmatrix}-5&-3&0\\1&7&-2\\-1&-4&-3\end{pmatrix}$$

$$=\begin{pmatrix}14&9&1\\14&44&-8\\23&19&25\end{pmatrix}$$

(ii) $\frac{1}{2}A-\frac{3}{2}B=\frac{1}{2}(A-3B)$

$$=\frac{1}{2}\left(\begin{pmatrix}1&8&3\\3&5&0\\8&7&6\end{pmatrix}-3\begin{pmatrix}8&-6&-4\\2&11&-3\\0&1&5\end{pmatrix}\right)$$

$$=\frac{1}{2}\left(\begin{pmatrix}1&8&3\\3&5&0\\8&7&6\end{pmatrix}+\begin{pmatrix}-24&18&12\\-6&-33&9\\0&-3&-15\end{pmatrix}\right)=\frac{1}{2}\begin{pmatrix}-23&26&15\\-3&-28&9\\8&4&-9\end{pmatrix}$$

$$= \begin{pmatrix} -\frac{23}{2} & 13 & \frac{15}{2} \\ -\frac{3}{2} & -14 & \frac{9}{2} \\ 4 & 2 & -\frac{9}{2} \end{pmatrix}$$

**Example 3.69** If $A = \begin{pmatrix} 2 & -2\sqrt{2} \\ \sqrt{2} & 2 \end{pmatrix}$ and $B = \begin{pmatrix} 2 & 2\sqrt{2} \\ -\sqrt{2} & 2 \end{pmatrix}$

Show that $A$ and $B$ satisfy commutative property with respect to matrix multiplication.

**Solution** We have to show that $AB = BA$

$$\text{LHS} = AB = \begin{pmatrix} 2 & -2\sqrt{2} \\ \sqrt{2} & 2 \end{pmatrix} \times \begin{pmatrix} 2 & 2\sqrt{2} \\ -\sqrt{2} & 2 \end{pmatrix}$$

$$= \begin{pmatrix} 4+4 & 4\sqrt{2}-4\sqrt{2} \\ 2\sqrt{2}-2\sqrt{2} & 4+4 \end{pmatrix}$$

$$= \begin{pmatrix} 8 & 0 \\ 0 & 8 \end{pmatrix}$$

$$\text{RHS} = BA = \begin{pmatrix} 2 & 2\sqrt{2} \\ -\sqrt{2} & 2 \end{pmatrix} \times \begin{pmatrix} 2 & -2\sqrt{2} \\ \sqrt{2} & 2 \end{pmatrix}$$

$$= \begin{pmatrix} 4+4 & -4\sqrt{2}+4\sqrt{2} \\ -2\sqrt{2}+2\sqrt{2} & 4+4 \end{pmatrix}$$

$$= \begin{pmatrix} 8 & 0 \\ 0 & 8 \end{pmatrix}$$

Hence LHS = RHS (ie) $AB = BA$

**Example 3.70** Solve $\begin{pmatrix} 2 & 1 \\ 1 & 2 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 4 \\ 5 \end{pmatrix}$

**Solution** $\begin{pmatrix} 2 & 1 \\ 1 & 2 \end{pmatrix}_{2\times2} \times \begin{pmatrix} x \\ y \end{pmatrix}_{2\times1} = \begin{pmatrix} 4 \\ 5 \end{pmatrix}$

By matrix multiplication $\begin{pmatrix} 2x+y \\ x+2y \end{pmatrix} = \begin{pmatrix} 4 \\ 5 \end{pmatrix}$

Rewriting

$2x + y = 4$ ...(1)

$x + 2y = 5$ ...(2)

(1) $-2 \times$ (2) $\Rightarrow$

$2x + y = 4$

$2x + 4y = 10$ (−)

$-3y = -6 \quad \Rightarrow \quad y = 2$

Substituting $y = 2$ in (1), $2x + 2 = 4 \Rightarrow x = 1$

Therefore, $x = 1$, $y = 2$.

**Example 3.71** If $A = (1 \ \ -1 \ \ 2)$, $B = \begin{pmatrix} 1 & -1 \\ 2 & 1 \\ 1 & 3 \end{pmatrix}$ and $C = \begin{pmatrix} 1 & 2 \\ 2 & -1 \end{pmatrix}$

show that $(AB)C = A(BC)$.

**Solution** LHS $= (AB)C$

$$AB = (1 \ \ -1 \ \ 2)_{1\times3} \times \begin{pmatrix} 1 & -1 \\ 2 & 1 \\ 1 & 3 \end{pmatrix}_{3\times2} = (1-2+2 \ \ -1-1+6) = (1 \ \ 4)$$

$$(AB)C = (1 \ \ 4)_{1\times2} \times \begin{pmatrix} 1 & 2 \\ 2 & -1 \end{pmatrix}_{2\times2} = (1+8 \ \ 2-4) = (9 \ \ -2) \quad ...(1)$$

RHS $= A(BC)$

$$BC = \begin{pmatrix}1 & -1\\ 2 & 1\\ 1 & 3\end{pmatrix}_{3\times 2} \times \begin{pmatrix}1 & 2\\ 2 & -1\end{pmatrix}_{2\times 2} = \begin{pmatrix}1-2 & 2+1\\ 2+2 & 4-1\\ 1+6 & 2-3\end{pmatrix} = \begin{pmatrix}-1 & 3\\ 4 & 3\\ 7 & -1\end{pmatrix}$$

$$A(BC) = (1 \quad -1 \quad 2)_{1\times 3} \times \begin{pmatrix}-1 & 3\\ 4 & 3\\ 7 & -1\end{pmatrix}_{3\times 2}$$

$$A(BC) = (-1-4+14 \quad 3-3-2) = (9 \quad -2) \qquad \ldots(2)$$

From (1) and (2), $(AB)C = A(BC)$.

**Example 3.72** If $A = \begin{pmatrix}1 & 1\\ -1 & 3\end{pmatrix}, B = \begin{pmatrix}1 & 2\\ -4 & 2\end{pmatrix}, C = \begin{pmatrix}-7 & 6\\ 3 & 2\end{pmatrix}$ verify that $A(B+C) = AB + AC$.

***Solution*** LHS $= A(B+C)$

$$B + C = \begin{pmatrix}1 & 2\\ -4 & 2\end{pmatrix} + \begin{pmatrix}-7 & 6\\ 3 & 2\end{pmatrix} = \begin{pmatrix}-6 & 8\\ -1 & 4\end{pmatrix}$$

$$A(B+C) = \begin{pmatrix}1 & 1\\ -1 & 3\end{pmatrix} \times \begin{pmatrix}-6 & 8\\ -1 & 4\end{pmatrix} = \begin{pmatrix}-6-1 & 8+4\\ 6-3 & -8+12\end{pmatrix} = \begin{pmatrix}-7 & 12\\ 3 & 4\end{pmatrix} \qquad \ldots(1)$$

RHS $= AB + AC$

$$AB = \begin{pmatrix}1 & 1\\ -1 & 3\end{pmatrix} \times \begin{pmatrix}1 & 2\\ -4 & 2\end{pmatrix} = \begin{pmatrix}1-4 & 2+2\\ -1-12 & -2+6\end{pmatrix} = \begin{pmatrix}-3 & 4\\ -13 & 4\end{pmatrix}$$

$$AC = \begin{pmatrix}1 & 1\\ -1 & 3\end{pmatrix} \times \begin{pmatrix}-7 & 6\\ 3 & 2\end{pmatrix} = \begin{pmatrix}-7+3 & 6+2\\ 7+9 & -6+6\end{pmatrix} = \begin{pmatrix}-4 & 8\\ 16 & 0\end{pmatrix}$$

Therefore, $AB + AC = \begin{pmatrix}-3 & 4\\ -13 & 4\end{pmatrix} + \begin{pmatrix}-4 & 8\\ 16 & 0\end{pmatrix} = \begin{pmatrix}-7 & 12\\ 3 & 4\end{pmatrix} \qquad \ldots(2)$

From (1) and (2), $A(B+C) = AB + AC$. Hence proved.

**Example 3.73** If $A = \begin{pmatrix}1 & 2 & 1\\ 2 & -1 & 1\end{pmatrix}$ and $B = \begin{pmatrix}2 & -1\\ -1 & 4\\ 0 & 2\end{pmatrix}$ show that $(AB)^T = B^T A^T$

***Solution***

LHS $= (AB)^T$

$$AB = \begin{pmatrix}1 & 2 & 1\\ 2 & -1 & 1\end{pmatrix}_{2\times 3} \times \begin{pmatrix}2 & -1\\ -1 & 4\\ 0 & 2\end{pmatrix}_{3\times 2}$$

$$= \begin{pmatrix}2-2+0 & -1+8+2\\ 4+1+0 & -2-4+2\end{pmatrix} = \begin{pmatrix}0 & 9\\ 5 & -4\end{pmatrix}$$

$$(AB)^T = \begin{pmatrix}0 & 9\\ 5 & -4\end{pmatrix}^T = \begin{pmatrix}0 & 5\\ 9 & -4\end{pmatrix} \qquad \ldots(1)$$

RHS $= (B^T A^T)$

$$B^T = \begin{pmatrix}2 & -1 & 0\\ -1 & 4 & 2\end{pmatrix}, \quad A^T = \begin{pmatrix}1 & 2\\ 2 & -1\\ 1 & 1\end{pmatrix}$$

$$B^T A^T = \begin{pmatrix}2 & -1 & 0\\ -1 & 4 & 2\end{pmatrix}_{2\times 3} \times \begin{pmatrix}1 & 2\\ 2 & -1\\ 1 & 1\end{pmatrix}_{3\times 2}$$

$$= \begin{pmatrix}2-2+0 & 4+1+0\\ -1+8+2 & -2-4+2\end{pmatrix}$$

$$B^T A^T = \begin{pmatrix}0 & 5\\ 9 & -4\end{pmatrix} \qquad \ldots(2)$$

From (1) and (2), $(AB)^T = B^T A^T$.

Hence proved.

# GEOMETRY

## 2-MARKS (SELECTED SUMS)

**Example 4.1** Show that $\Delta PST \sim \Delta PQR$

(i)

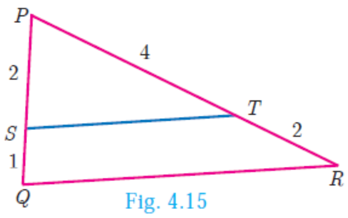



Fig. 4.15

(ii)

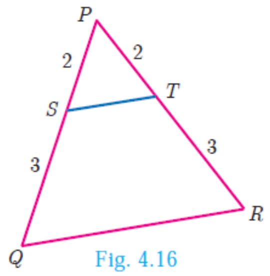



Fig. 4.16

*Solution*

(i) In $\Delta PST$ and $\Delta PQR$,

$$\frac{PS}{PQ}=\frac{2}{2+1}=\frac{2}{3},\ \frac{PT}{PR}=\frac{4}{4+2}=\frac{2}{3}$$

Thus, $\frac{PS}{PQ}=\frac{PT}{PR}$ and $\angle P$ is common

Therefore, by $SAS$ similarity,

$\Delta PST \sim \Delta PQR$

(ii) In $\Delta PST$ and $\Delta PQR$,

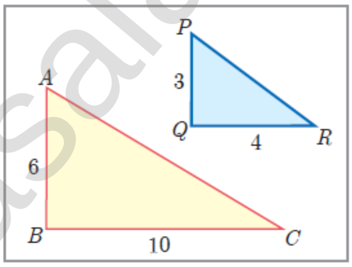
$$\frac{PS}{PQ}=\frac{2}{2+3}=\frac{2}{5},\ \frac{PT}{PR}=\frac{2}{2+3}=\frac{2}{5}$$

Thus, $\frac{PS}{PQ}=\frac{PT}{PR}$ and $\angle P$ is common

Therefore, by SAS similarity,

$\Delta PST \sim \Delta PQR$

**Example 4.2** Is $\Delta ABC \sim \Delta PQR$?




*Solution* In $\Delta ABC$ and $\Delta PQR$,

$$\frac{PQ}{AB}=\frac{3}{6}=\frac{1}{2};\ \frac{QR}{BC}=\frac{4}{10}=\frac{2}{5}$$

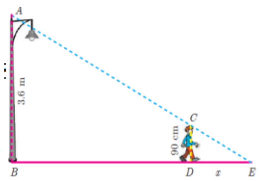
since $\frac{1}{2}\neq\frac{2}{5}$, $\frac{PQ}{AB}\neq\frac{QR}{BC}$.

Therefore $\Delta ABC$ is not similar to $\Delta PQR$.

**Example 4.4** A boy of height 90cm is walking away from the base of a lamp post at a speed of 1.2m/sec. If the lamppost is 3.6m above the ground, find the length of his shadow cast after 4 seconds.

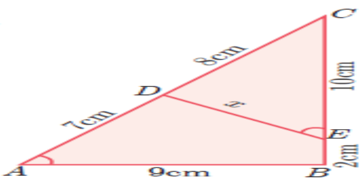
$$)E\ ,\ \frac{BE}{DE}=\frac{AB}{CD}\text{ gives }\frac{4.8+x}{x}=\frac{3.6}{0.9}=4$$

$$4.8+x=4x\text{ gives }3x=4.8\text{ so, }x=1.6\text{ m}$$



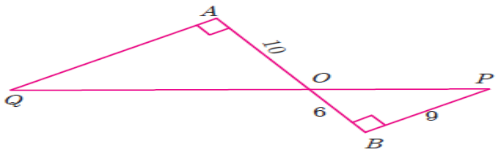
**Example 4.5** In Fig.4.20 $\angle A=\angle CED$ prove that $\Delta CAB \sim \Delta CED$. Also find the value of $x$.

$$\frac{9}{x}=\frac{10+2}{8}\text{ so, }x=\frac{8\times 9}{12}=6\text{ cm.}$$




**Example 4.6** In Fig.4.21, $QA$ and $PB$ are perpendiculars to $AB$. If $AO = 10$ cm, $BO=6$ cm and $PB=9$ cm. Find $AQ$.

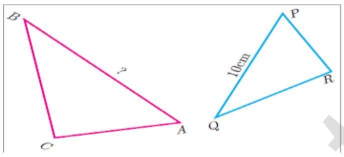
$$\frac{10}{6}=\frac{AQ}{9} \Rightarrow AQ=\frac{10\times 9}{6}=15\text{ cm}$$




**Example 4.7** The perimeters of two similar triangles $ABC$ and $PQR$ are respectively 36 cm and 24 cm. If $PQ = 10$ cm, find $AB$.

$$\frac{AB}{PQ}=\frac{36}{24} \Rightarrow \frac{AB}{10}=\frac{36}{24}$$

$$AB=\frac{36\times 10}{24}=15\text{ cm}$$



**Example 4.8** If $\Delta ABC$ is similar to $\Delta DEF$ such that $BC=3$ cm, $EF=4$ cm and area of $\Delta ABC = 54\text{ cm}^2$. Find the area of $\Delta DEF$.

$$\text{Area}(\Delta DEF)=\frac{16\times 54}{9}=96\text{ cm}^2$$

**Example 4.13** $D$ and $E$ are respectively the points on the sides $AB$ and $AC$ of a $\Delta ABC$ such that $AB = 5.6$ cm, AD = 1.4 cm, $AC = 7.2$ cm and $AE = 1.8$ cm, show that $DE \parallel BC$.

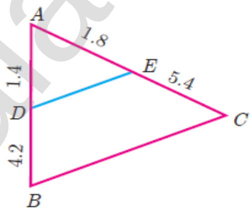
**Solution** We have $AB = 5.6$ cm, $AD = 1.4$ cm, $AC = 7.2$ cm and $AE = 1.8$ cm.

$BD = AB - AD = 5.6 - 1.4 = 4.2$ cm

and $EC = AC - AE = 7.2 - 1.8 = 5.4$ cm.

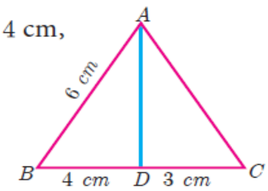
$$\frac{AD}{DB}=\frac{1.4}{4.2}=\frac{1}{3} \text{ and } \frac{AE}{EC}=\frac{1.8}{5.4}=\frac{1}{3}$$

$$\frac{AD}{DB}=\frac{AE}{EC}$$




Therefore, by converse of Basic Proportionality Theorem, we have $DE$ is parallel to $BC$. Hence proved.

**Example 4.15** In the Fig.4.39, $AD$ is the bisector of $\angle A$. If $BD = 4$ cm, $DC = 3$ cm and $AB = 6$ cm, find $AC$.




**Solution** In $\Delta ABC$, $AD$ is the bisector of $\angle A$

By Angle Bisector Theorem

$$\frac{BD}{DC}=\frac{AB}{AC}$$

$$\frac{4}{3}=\frac{6}{AC} \text{ gives } 4AC = 18. \text{ Hence, } AC=\frac{9}{2}=4.5\text{ cm}$$

**Example 4.16** In the Fig. 4.40, $AD$ is the bisector of $\angle BAC$, if $AB = 10$ cm, $AC = 14$ cm and BC = 6 cm. Find $BD$ and $DC$.

**Solution** Let $BD = x$ cm, then $DC = (6-x)$cm

AD is the bisector of $\angle A$

By Angle Bisector Theorem

$$\frac{AB}{AC}=\frac{BD}{DC}$$

$$\frac{10}{14}=\frac{x}{6-x} \text{ gives } \frac{5}{7}=\frac{x}{6-x}$$

$$12x = 30 \quad \text{we get, } x=\frac{30}{12}=2.5\text{ cm}$$

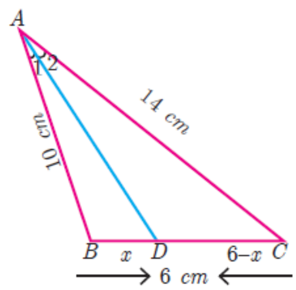
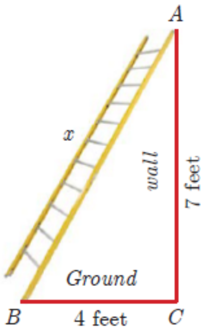


Fig. 4.40

Therefore, $BD = 2.5$ cm, $DC = 6 - x = 6 - 2.5 = 3.5$ cm

**Example 4.22** What length of ladder is needed to reach a height of 7 ft along the wall when the base of the ladder is 4 ft from the wall? Round off your answer to the next tenth place.




**Solution** Let $x$ be the length of the ladder. $BC=4$ ft, $AC=7$ ft.

By Pythagoras theorem we have, $AB^2 = AC^2 + BC^2$

$$x^2 = 7^2 + 4^2 \Rightarrow x^2 = 49 + 16$$

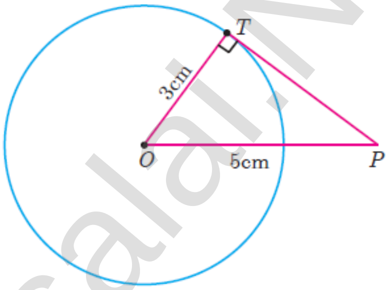
$$x^2 = 65. \quad \text{Hence, } x = \sqrt{65}$$

The number $\sqrt{65}$ is between 8 and 8.1.

$$8^2 = 64 < 65 < 65.61 = 8.1^2$$

Therefore, the length of the ladder is approximately 8.1 ft.

**Example 4.24** Find the length of the tangent drawn from a point whose distance from the centre of a circle is 5 cm and radius of the circle is 3 cm.




**Solution** Given OP = 5 cm, radius $r$ = 3 cm

To find the length of tangent $PT$.

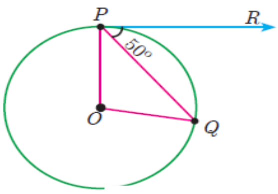
In right angled $\Delta OTP$,

$OP^2 = OT^2 + PT^2$ (by Pythagoras theorem)

$5^2 = 3^2 + PT^2$ gives $PT^2 = 25 - 9 = 16$

Length of the tangent $PT = 4$ cm

**Example 4.26** In Fig.4.64, $O$ is the centre of a circle. $PQ$ is a chord and the tangent $PR$ at $P$ makes an angle of 50° with $PQ$. Find $\angle POQ$.




**Solution** $\angle OPQ = 90° - 50° = 40°$ (angle between the radius and tangent is 90°)

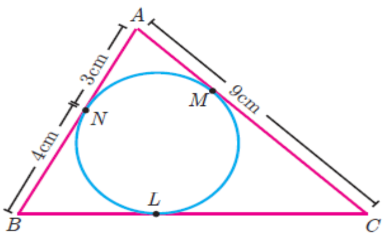
$OP = OQ$ (Radii of a circle are equal)

$\angle OPQ = \angle OQP = 40°$ ($\Delta OPQ$ is isosceles)

$\angle POQ = 180° - \angle OPQ - \angle OQP$

$\angle POQ = 180° - 40° - 40° = 100°$

**Example 4.27** In Fig.4.65, $\Delta ABC$ is circumscribing a circle. Find the length of $BC$.



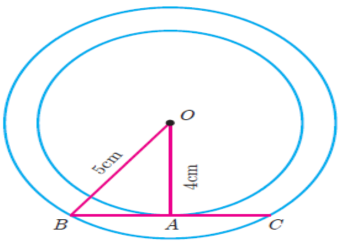

**Solution** $AN = AM = 3$ cm (Tangents drawn from same external point are equal)

$BN = BL = 4$ cm

$CL = CM = AC - AM = 9 - 3 = 6$ cm

$BC = BL + CL = 4 + 6 = 10$ cm

**Example 4.28** If radii of two concentric circles are 4 cm and 5 cm then find the length of the chord of one circle which is a tangent to the other circle.




**Solution** $OA = 4$ cm, OB = 5 cm; also $OA \perp BC$.

$OB^2 = OA^2 + AB^2$

$5^2 = 4^2 + AB^2$ gives $AB^2 = 9$

Therefore $AB = 3$ cm

$BC$ = 2AB hence $BC = 2 \times 3 = 6$ cm

**4.5.1 Construction**

**Construction of tangents to a circle**

**State Pythagoras Theorem.**
In a right angle triange, the square on the hypotenuse is equal to the sum of the squares on the other two sides.

**State Converse of Phytagoras Theorem.**
If the square of the longest side of a triangle is equal to the sum of squares of other two sides, then the triangle is a right angle triangle.

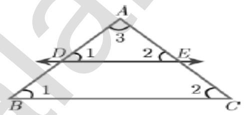
**State Menelaus Theorem.**
A necessary and sufficient condition for points P, Q, R on the respective sides BC, CA, AB (or their extension) of a triangle ABC to be collinear is that $\frac{BP}{PC} \times \frac{CQ}{QA} \times \frac{AR}{RB} = -1$ where all segments in the formula are directed segments.

**State Ceva's Theorem.**
Let ABC be a triangle and let D, E, F be points on the lines BC, CA, AB respectively. Then the cevains AD, BE, CF are concurrent if and only if $\frac{BD}{DC} \times \frac{CE}{EA} \times \frac{AF}{FB} = 1$ where the lengths are directed.

## 5-MARKS (SELECTED SUMS)

**State and Prove Basic Proportionality Theorem (BPT) or Thales Theorem.**

**Statement:**




A straight line drawn parallel to a side of triangle intersecting the other two sides, divides the sides in the same ratio.

**Proof:**

Given: In ΔABC, D is a point on AB and E is a point on AC.

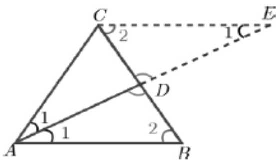
To Prove: $\frac{AD}{DB} = \frac{AE}{EC}$. Construction: Draw a line DE || BC.

| No. | Statement | Reason |
|---|---|---|
| 1. | $\angle ABC = \angle ADE = \angle 1$ | Corresponding angles are equal because DE \|\| BC. |
| 2. | $\angle ACB = \angle AED = \angle 2$ | Corresponding angles are equal because DE \|\| BC. |
| 3. | $\angle DAE = \angle BAC = \angle 3$ | Both triangles have a common angle. |
| 4. | $\Delta ABC \sim \Delta ADE$<br>$\frac{AB}{AD} = \frac{AC}{AE}$<br>$\frac{AD+DB}{AD} = \frac{AE+EC}{AE}$<br>$1+\frac{DB}{AD} = 1+\frac{EC}{AE}$<br>$\frac{DB}{AD} = \frac{EC}{AE}$<br>$\frac{AD}{DB} = \frac{AE}{EC}$ | By AAA similarity.<br>Corresponding sides are proportional.<br>Split AB and AC using the points D and E.<br>On Simplification.<br>Cancelling 1 on both sides.<br>Taking reciprocals. |
| Hence Proved. | | |

**State and Prove Angle Bisector Theorem.**

**Statement:**

The internal bisector of an angle of a triangle divides the opposite side internally in the ratio of the corresponding sides containing the angle.




**Proof:**

Given: In $\Delta ABC$,

AD is the internal bisector.

To Prove: $\frac{AB}{AC} = \frac{BD}{CD}$.

Construction: Draw a line through C parallel to AB.

Extend AD to meet line through C at E.

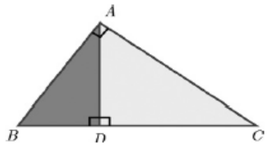
| No. | Statement | Reason |
|---|---|---|
| 1. | $\angle AEC = \angle BAE = \angle 1$ | Two parallel lines cut by a transversal make alternate angles equal. |
| 2. | $\Delta ACE$ is isosceles<br>$AC = CE$ …. (1) | In $\Delta ACE$, $\angle CAE = \angle CEA$. |
| 3. | $\Delta ABD \sim \Delta ECD$<br>$\frac{AB}{CE} = \frac{BD}{CD}$ | By AA Similarity. |
| 4. | $\frac{AB}{AC} = \frac{BD}{CD}$ | From (1) AC = CE.<br>Hence Proved. |

3. **State and Prove Pythagoras Theorem.**

**Statement:**

In a right angle triangle, the square on the hypotenuse is equal to the sum of the squares on the other two sides.

**Proof:**




Given: In $\Delta ABC$, $\angle A = 90°$

To Prove: $AB^2 + AC^2 = BC^2$

Construction: Draw $AD \perp BC$

| No. | Statement | Reason |
|---|---|---|
| 1. | Compare $\Delta ABC$ and $\Delta ABD$.<br>$\angle B$ is common<br>$\angle BAC = \angle BDA = 90°$<br>Therefore, $\Delta ABC \sim \Delta ABD$<br>$\frac{AB}{BD} = \frac{BC}{AB}$<br>$AB^2 = BC \times BD$ ……. (1) | Given $\angle BAC = 90°$ and by construction $\angle BDA = 90°$<br><br>By AA similarity. |

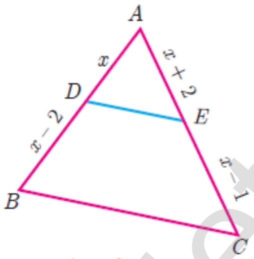
| | | |
|---|---|---|
| 2. | Compare $\Delta ABC$ and $\Delta ADC$.<br>$\angle C$ is common.<br>$\angle BAC = \angle ADC = 90°$<br>Therefore, $\Delta ABC \sim \Delta ADC$<br>$\frac{BC}{AC} = \frac{AC}{DC}$<br>$AC^2 = BC \times DC$ ……. (2) | Given $\angle BAC = 90°$ and by construction $\angle CDA = 90°$<br><br>By AA similarity. |

Adding (1) and (2) we get,

$$AB^2 + AC^2 = BC \times BD + BC \times DC$$
$$= BC(BD + DC)$$
$$AB^2 + AC^2 = BC \times BC = BC^2.$$

Hence the theorem is proved.

**Example 4.12** In $\Delta ABC$, if $DE \parallel BC$, $AD = x$, $DB = x-2$, $AE = x+2$ and $EC = x-1$ then find the lengths of the sides $AB$ and $AC$.




**Solution** In $\Delta ABC$ we have $DE \parallel BC$.

By Thales theorem, we have $\frac{AD}{DB} = \frac{AE}{EC}$

$\frac{x}{x-2} = \frac{x+2}{x-1}$ gives $x(x-1) = (x-2)(x+2)$

Hence, $x^2 - x = x^2 - 4$ so, $x = 4$

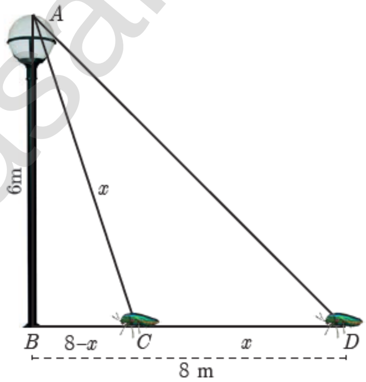
When $x = 4$, $AD = 4$, $DB = x-2 = 2$, $AE = x+2 = 6$, $EC = x-1 = 3$.

Hence, $AB = AD + DB = 4 + 2 = 6$, $AC = AE + EC = 6 + 3 = 9$.

Therefore, $AB = 6$, $AC = 9$.

**Example 4.20** An insect 8 m away initially from the foot of a lamp post which is 6 m tall, crawls towards it moving through a distance. If its distance from the top of the lamp post is equal to the distance it has moved, how far is the insect away from the foot of the lamp post?

**Solution** Distance between the insect and the foot of the lamp post $BD = 8$ m




The height of the lamp post, $AB = 6$ m

After moving a distance of $x$ m, let the insect be at $C$

Let, $AC = CD = x$. Then $BC = BD - CD = 8 - x$

In $\Delta ABC$, $\angle B = 90°$

$AC^2 = AB^2 + BC^2$ gives $x^2 = 6^2 + (8-x)^2$

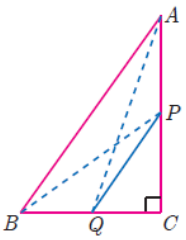
$x^2 = 36 + 64 - 16x + x^2$

$16x = 100$ then $x = 6.25$

Then, $BC = 8 - x = 8 - 6.25 = 1.75$ m

Therefore the insect is 1.75 m away from the foot of the lamp post.

**Example 4.21** $P$ and $Q$ are the mid-points of the sides $CA$ and $CB$ respectively of a $\Delta ABC$, right angled at $C$. Prove that $4(AQ^2 + BP^2) = 5AB^2$.




**Solution** $\Delta AQC$ is a right triangle at C, $AQ^2 = AC^2 + QC^2$ ... (1)
$\Delta BPC$ is a right triangle at C, $BP^2 = BC^2 + CP^2$ ... (2)
$\Delta ABC$ is a rignt triangle at C, $AB^2 = AC^2 + BC^2$ ... (3)

From (1) and (2), $AQ^2 + BP^2 = AC^2 + QC^2 + BC^2 + CP^2$

$4(AQ^2 + BP^2) = 4AC^2 + 4QC^2 + 4BC^2 + 4CP^2$

$= 4AC^2 + (2QC)^2 + 4BC^2 + (2CP)^2$

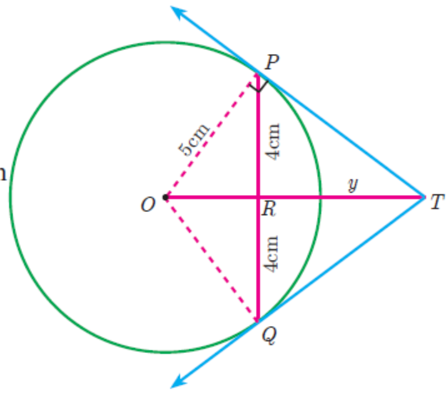
$= 4AC^2 + BC^2 + 4BC^2 + AC^2$ (Since $P$ and $Q$ are mid points)

$= 5(AC^2 + BC^2)$ (From equation (3))

$4(AQ^2 + BP^2) = 5AB^2$

**Example 4.25** PQ is a chord of length 8 cm to a circle of radius 5 cm. The tangents at P and Q intersect at a point T. Find the length of the tangent TP.

**Solution** Let $TR = y$. Since, OT is perpendicular bisector of PQ.




$PR = QR = 4$ cm

In $\Delta ORP$, $OP^2 = OR^2 + PR^2$

$OR^2 = OP^2 - PR^2$

$OR^2 = 5^2 - 4^2 = 25 - 16 = 9 \Rightarrow OR = 3$ cm

$OT = OR + RT = 3 + y$ ... (1)

In $\Delta PRT$, $TP^2 = TR^2 + PR^2$ ... (2)

and $\Delta\, OPT$ we have, $OT^2 = TP^2 + OP^2$

$OT^2 = (TR^2 + PR^2) + OP^2$ (substitute for $TP^2$ from (2))

$(3 + y)^2 = y^2 + 4^2 + 5^2$ (substitute for $OT$ from (1))

$9 + 6y + y^2 = y^2 + 16 + 25$

$6y = 41 - 9$ we get $y = \dfrac{16}{3}$

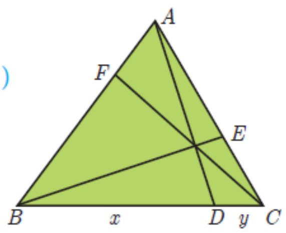
From (2), $TP^2 = TR^2 + PR^2$

$TP^2 = \left(\dfrac{16}{3}\right)^2 + 4^2 = \dfrac{256}{9} + 16 = \dfrac{400}{9}$ so, $TP = \dfrac{20}{3}$ cm

**Example 4.33** In $\Delta ABC$, points $D, E, F$ lies on $BC$, $CA$, $AB$ respectively. Suppose $AB$, $AC$ and $BC$ have lengths 13, 14 and 15 respectively. If $\dfrac{AF}{FB} = \dfrac{2}{5}$ and $\dfrac{CE}{EA} = \dfrac{5}{8}$. Find $BD$ and $DC$.

***Solution*** Given that $AB = 13$, $AC = 14$ and $BC = 15$.

Let $BD = x$ and $DC = y$




Using Ceva's theorem, we have, $\dfrac{BD}{DC} \times \dfrac{CE}{EA} \times \dfrac{AF}{FB} = 1$ ...(1)

Substitute the values of $\dfrac{AF}{FB}$ and $\dfrac{CE}{EA}$ in (1),

we have $\dfrac{BD}{DC} \times \dfrac{5}{8} \times \dfrac{2}{5} = 1$

$\dfrac{x}{y} \times \dfrac{10}{40} = 1$ we get, $\dfrac{x}{y} \times \dfrac{1}{4} = 1$. Hence, $x = 4y$ ...(2)

$BC = BD + DC = 15$ so, $x + y = 15$ ... (3)

From (2), using $x = 4y$ in (3) we get, $4y + y = 15$ gives $5y = 15$ then $y = 3$

Substitute $y = 3$ in (3) we get, $x = 12$. Hence $BD = 12$, $DC = 3$.

# COORDINATE GEOMETRY

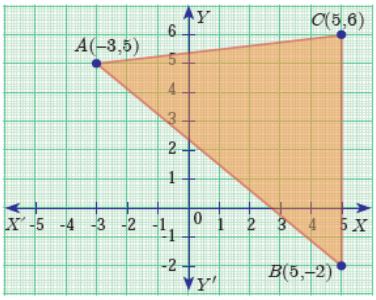
## 2-MARKS (SELECTED SUMS)

**Example 5.1** Find the area of the triangle whose vertices are $(-3,5)$, $(5,6)$ and $(5,-2)$

***Solution*** Plot the points in a rough diagram and take them in counter-clockwise order.

Let the vertices be $A(-3,5)$, $B(5,-2)$, $C(5,6)$

$(x_1,y_1)$ $(x_2,y_2)$ $(x_3,y_3)$




The area of $\Delta ABC$ is

$$= \frac{1}{2}\{(x_1y_2 + x_2y_3 + x_3y_1) - (x_2y_1 + x_3y_2 + x_1y_3)\}$$

$$= \frac{1}{2}\{(6+30+25) - (25-10-18)\}$$

$$= \frac{1}{2}\{61+3\}$$

$$= \frac{1}{2}(64) = 32 \text{ sq.units}$$

**Example 5.2** Show that the points $P(-1.5,3)$, $Q(6,-2)$, $R(-3,4)$ are collinear.

***Solution*** The points are $P(-1.5,3)$, $Q(6,-2)$, $R(-3,4)$

Area of $\Delta PQR$ $= \frac{1}{2}\{(x_1y_2 + x_2y_3 + x_3y_1) - (x_2y_1 + x_3y_2 + x_1y_3)\}$

$$= \frac{1}{2}\{(3+24-9) - (18+6-6)\} = \frac{1}{2}\{18-18\} = 0$$

Therefore, the given points are collinear.

**Example 5.3** If the area of the triangle formed by the vertices $A(-1,2)$, $B(k,-2)$ and $C(7,4)$ (taken in order) is 22 sq. units, find the value of $k$.

***Solution*** The vertices are $A(-1,2)$, $B(k,-2)$ and $C(7,4)$

Area of triangle $ABC$ is 22 sq.units

$$\frac{1}{2}\{(x_1y_2 + x_2y_3 + x_3y_1) - (x_2y_1 + x_3y_2 + x_1y_3)\} = 22$$

$$\frac{1}{2}\{(2+4k+14) - (2k-14-4)\} = 22$$

$$2k+34 = 44 \text{ gives } 2k = 10 \text{ so } k = 5$$

**Example 5.4** If the points $P(-1,-4)$, $Q(b,c)$ and $R(5,-1)$ are collinear and if $2b+c=4$, then find the values of $b$ and $c$.

***Solution*** Since the three points $P(-1,-4)$, $Q(b,c)$ and $R(5,-1)$ are collinear,

Area of triangle $PQR = 0$

$$\frac{1}{2}\{(x_1y_2 + x_2y_3 + x_3y_1) - (x_2y_1 + x_3y_2 + x_1y_3)\} = 0$$

$$\frac{1}{2}\{(-c-b-20) - (-4b+5c+1)\} = 0$$

$$-c-b-20+4b-5c-1 = 0$$

$b - 2c = 7$ ...(1)

Also, $2b + c = 4$ ...(2) (from given information)

Solving (1) and (2) we get $b = 3$, $c = -2$

**Example 5.8** (i) What is the slope of a line whose inclination is $30°$?
(ii) What is the inclination of a line whose slope is $\sqrt{3}$?

***Solution*** (i) Here $\theta = 30°$

Slope $m = \tan\theta$

Therefore, slope $m = \tan 30° = \frac{1}{\sqrt{3}}$

(ii) Given $m = \sqrt{3}$, let $\theta$ be the inclination of the line

$\tan\theta = \sqrt{3}$

We get, $\theta = 60°$

**Example 5.9** Find the slope of a line joining the given points

(i) $(-6,1)$ and $(-3,2)$ (ii) $\left(-\frac{1}{3},\frac{1}{2}\right)$ and $\left(\frac{2}{7},\frac{3}{7}\right)$ (iii) $(14,10)$ and $(14,-6)$

**Solution**

(i) $(-6,1)$ and $(-3,2)$

The slope $= \frac{y_2 - y_1}{x_2 - x_1} = \frac{2-1}{-3+6} = \frac{1}{3}$.

(ii) $\left(-\frac{1}{3},\frac{1}{2}\right)$ and $\left(\frac{2}{7},\frac{3}{7}\right)$

The slope $= \frac{\frac{3}{7}-\frac{1}{2}}{\frac{2}{7}+\frac{1}{3}} = \frac{\frac{6-7}{14}}{\frac{6+7}{21}}$

$= -\frac{1}{14} \times \frac{21}{13} = -\frac{3}{26}$.

(iii) $(14,10)$ and $(14,-6)$

The slope $= \frac{-6-10}{14-14} = \frac{-16}{0}$.

The slope is undefined.

**Example 5.11** The line $p$ passes through the points $(3,-2)$, $(12,4)$ and the line $q$ passes through the points $(6,-2)$ and $(12,2)$. Is $p$ parallel to $q$ ?

**Solution** The slope of line $p$ is $m_1 = \frac{4+2}{12-3} = \frac{6}{9} = \frac{2}{3}$

The slope of line $q$ is $m_2 = \frac{2+2}{12-6} = \frac{4}{6} = \frac{2}{3}$

Thus, slope of line $p$ = slope of line $q$.

Therefore, line $p$ is parallel to the line $q$.

**Example 5.12** Show that the points $(-2,5)$, $(6,-1)$ and $(2,2)$ are collinear.

**Solution** The vertices are $A(-2,5)$, $B(6,-1)$ and $C(2,2)$.

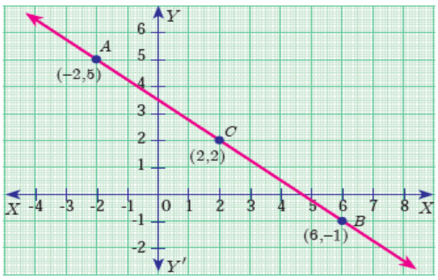
Slope of $AB = \frac{-1-5}{6+2} = \frac{-6}{8} = \frac{-3}{4}$

Slope of $BC = \frac{2+1}{2-6} = \frac{3}{-4} = \frac{-3}{4}$

We get, Slope of $AB$ = Slope of $BC$

Therefore, the points $A$, $B$, $C$ all lie in a same straight line.

Hence the points $A$, $B$ and $C$ are collinear.




**Example 5.18** Find the equation of a straight line whose

(i) Slope is 5 and $y$ intercept is $-9$ (ii) Inclination is $45°$ and $y$ intercept is 11

**Solution** (i) Given, Slope $= 5$, $y$ intercept, $c = -9$

Therefore, equation of a straight line is $y = mx + c$

$y = 5x - 9 \Rightarrow 5x - y - 9 = 0$

(ii) Given, $\theta = 45°$, $y$ intercept, $c = 11$

Slope $m = \tan\theta = \tan 45° = 1$

Therefore, equation of a straight line is of the form $y = mx + c$

Hence we get, $y = x + 11 \Rightarrow x - y + 11 = 0$

**Example 5.19** Calculate the slope and $y$ intercept of the straight line $8x - 7y + 6 = 0$

**Solution** Equation of the given straight line is $8x - 7y + 6 = 0$

$7y = 8x + 6$ (bringing it to the form $y = mx + c$)

$$y = \frac{8}{7}x + \frac{6}{7} \quad \ldots(1)$$

Comparing (1) with $y = mx + c$

Slope $m = \frac{8}{7}$ and $y$ intercept $c = \frac{6}{7}$

**Example 5.21** Find the equation of a line passing through the point $(3, -4)$ and having slope $\frac{-5}{7}$

**Solution** Given, $(x_1, y_1) = (3, -4)$ and $m = \frac{-5}{7}$

The equation of the point-slope form of the straight line is $y - y_1 = m(x - x_1)$

we write it as $y + 4 = -\frac{5}{7}(x - 3)$

$\Rightarrow \quad 5x + 7y + 13 = 0$

**Example 5.23** Find the equation of a straight line passing through $(5, -3)$ and $(7, -4)$.

$$\frac{y - y_1}{y_2 - y_1} = \frac{x - x_1}{x_2 - x_1}$$

Substituting the points we get,

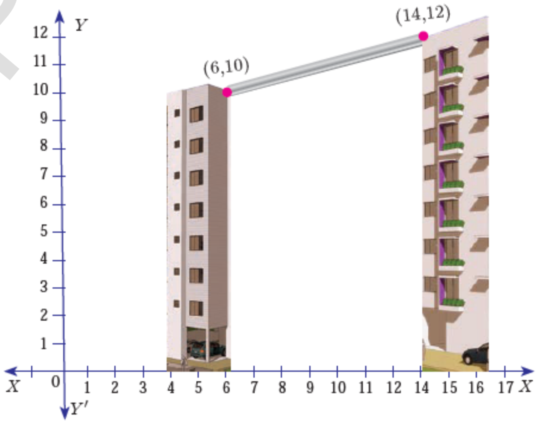
$$\frac{y + 3}{-4 + 3} = \frac{x - 5}{7 - 5}$$

$\Rightarrow \quad 2y + 6 = -x + 5$

Therefore, $x + 2y + 1 = 0$

**Example 5.24** Two buildings of different heights are located at opposite sides of each other. If a heavy rod is attached joining the terrace of the buildings from $(6,10)$ to $(14,12)$, find the equation of the rod joining the buildings ?

**Solution**




The equation of the rod is the equation of the straight line passing through $A(6,10)$ and $B(14,12)$

$$\frac{y - y_1}{y_2 - y_1} = \frac{x - x_1}{x_2 - x_1} \text{ gives } \frac{y - 10}{12 - 10} = \frac{x - 6}{14 - 6}$$

$$\frac{y - 10}{2} = \frac{x - 6}{8}$$

Therefore, $x - 4y + 34 = 0$

Hence, equation of the rod is $x - 4y + 34 = 0$

**Example 5.25** Find the equation of a line which passes through (5,7) and makes intercepts on the axes equal in magnitude but opposite in sign.

*Solution* Let the $x$ intercept be '$a$' and $y$ intercept be '$-a$'.

The equation of the line in intercept form is $\frac{x}{a}+\frac{y}{b}=1$

$$\Rightarrow \frac{x}{a}+\frac{y}{-a}=1 \text{ (Here } b=-a)$$

$$\therefore\; x-y=a\,. \qquad ..(1)$$

Since (1) passes through (5,7)

Therefore, $5-7=a \Rightarrow a=-2$

Thus the required equation of the straight line is $x-y=-2$; or $x-y+2=0$

**Example 5.26** Find the intercepts made by the line $4x-9y+36=0$ on the coordinate axes.

*Solution* Equation of the given line is $4x-9y+36=0$

we write it as $4x-9y=-36$ (bringing it to the normal form)

Dividing by $-36$ we get, $\frac{x}{-9}+\frac{y}{4}=1 \qquad ...(1)$

Comparing (1) with intercept form, we get $x$ intercept $a=-9$; $y$ intercept $b=4$

**Example 5.30** Find the slope of the straight line $6x+8y+7=0$.

*Solution* Given $6x+8y+7=0$

$$\text{slope } m = \frac{-\text{coefficient of } x}{\text{coefficient of } y} = -\frac{6}{8} = -\frac{3}{4}$$

Therefore, the slope of the straight line is $-\frac{3}{4}$.

**Example 5.31** Find the slope of the line which is

(i) parallel to $3x-7y=11$ (ii) perpendicular to $2x-3y+8=0$

*Solution* (i) Given straight line is $3x-7y=11$

$$\Rightarrow 3x-7y-11=0$$

$$\text{Slope } m=\frac{-3}{-7}=\frac{3}{7}$$

Since parallel lines have same slopes, slope of any line parallel to

$3x-7y=11$ is $\frac{3}{7}$.

(ii) Given straight line is $2x-3y+8=0$

$$\text{Slope } m=\frac{-2}{-3}=\frac{2}{3}$$

Since product of slopes is $-1$ for perpendicular lines, slope of any line perpendicular to $2x-3y+8=0$ is $\frac{-1}{\frac{2}{3}}=\frac{-3}{2}$

**Example 5.32** Show that the straight lines $2x+3y-8=0$ and $4x+6y+18=0$ are parallel.

*Solution* Slope of the straight line $2x+3y-8=0$ is

$$m_1 = \frac{-\text{coefficient of } x}{\text{coefficient of } y}$$

$$m_1 = \frac{-2}{3}$$

Slope of the straight line $4x+6y+18=0$ is

$$m_2 = \frac{-4}{6}=\frac{-2}{3}$$

Here, $m_1=m_2$

That is, slopes are equal. Hence, the two straight lines are parallel.

**Aliter**

$a_1=2,\; b_1=3$

$a_2=4,\; b_2=6$

$\frac{a_1}{a_2}=\frac{2}{4}=\frac{1}{2}$

$\frac{b_1}{b_2}=\frac{3}{6}=\frac{1}{2}$

Therefore, $\frac{a_1}{a_2}=\frac{b_1}{b_2}$

Hence the lines are parallel.

**Example 5.33** Show that the straight lines $x-2y+3=0$ and $6x+3y+8=0$ are perpendicular.

**Solution** Slope of the straight line $x-2y+3=0$ is

$$m_1 = \frac{-1}{-2} = \frac{1}{2}$$

Slope of the straight line $6x+3y+8=0$ is

$$m_2 = \frac{-6}{3} = -2$$

Now, $m_1 \times m_2 = \frac{1}{2}\times(-2) = -1$

Hence, the two straight lines are perpendicular.

**Aliter**

$a_1 = 1,\ b_1 = -2;$

$a_2 = 6,\ b_2 = 3$

$a_1a_2 + b_1b_2 = 6-6 = 0$

The lines are perpendicular.

**Example 5.34** Find the equation of a straight line which is parallel to the line $3x-7y=12$ and passing through the point $(6,4)$.

**Solution** Equation of the straight line, parallel to $3x-7y-12=0$ is $3x-7y+k=0$

Since it passes through the point $(6,4)$

$$3(6)-7(4)+k = 0$$
$$k = 28-18 = 10$$

Therefore, equation of the required straight line is $3x-7y+10=0$.

**Example 5.35** Find the equation of a straight line perpendicular to the line $y=\frac{4}{3}x-7$ and passing through the point $(7,\ -1)$.

**Solution** The equation $y=\frac{4x}{3}-7$ can be written as $4x-3y-21=0$.

Equation of a straight line perpendicular to $4x-3y-21=0$ is $3x+4y+k=0$

Since it is passes through the point $(7,\ -1)$,

$21-4+k=0$ we get, $k=-17$

Therefore, equation of the required straight line is $3x+4y-17=0$.

## 5-MARKS (SELECTED SUMS)

**Example 5.5** The floor of a hall is covered with identical tiles which are in the shapes of triangles. One such triangle has the vertices at $(-3,2)$, $(-1,-1)$ and $(1,2)$. If the floor of the hall is completely covered by 110 tiles, find the area of the floor.

**Solution** Vertices of one triangular tile are at

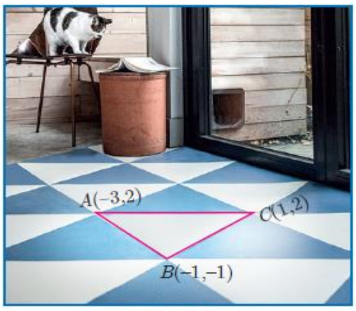
$(-3,2)$, $(-1,-1)$ and $(1,2)$

Area of this tile $= \frac{1}{2}\{(3-2+2)-(-2-1-6)\}$ sq.units

$= \frac{1}{2}(12) = 6$ sq.units

Since the floor is covered by 110 triangle shaped identical tiles,

Area of floor $= 110\times 6 = 660$ sq.units




**Example 5.6** Find the area of the quadrilateral formed by the points $(8,6)$, $(5,11)$, $(-5,12)$ and $(-4,3)$.

**Solution** Before determining the area of quadrilateral, plot the vertices in a graph.

Let the vertices be $A(8,6)$, $B(5,11)$, $C(-5,12)$ and $D(-4,3)$

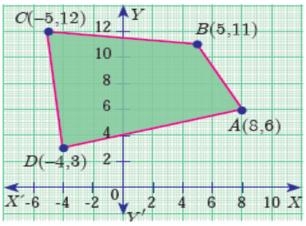
Therefore, area of the quadrilateral $ABCD$

$$= \frac{1}{2}\left\{(x_1y_2 + x_2y_3 + x_3y_4 + x_4y_1) - (x_2y_1 + x_3y_2 + x_4y_3 + x_1y_4)\right\}$$

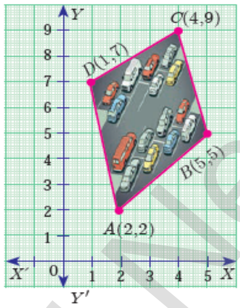
$$= \frac{1}{2}\left\{(88 + 60 - 15 - 24) - (30 - 55 - 48 + 24)\right\}$$

$$= \frac{1}{2}\{109 + 49\}$$

$$= \frac{1}{2}\{158\} = 79 \text{ sq.units}$$




**Example 5.7** The given diagram shows a plan for constructing a new parking lot at a campus. It is estimated that such construction would cost ₹1300 per square feet. What will be the total cost for making the parking lot?




**Solution** The parking lot is a quadrilateral whose vertices are at $A(2,2)$, $B(5,5)$, $C(4,9)$ and $D(1,7)$.

$$\text{Area of parking lot} = \frac{1}{2}\begin{vmatrix} 2 & 5 & 4 & 1 & 2 \\ 2 & 5 & 9 & 7 & 2 \end{vmatrix} \text{ sq.units}$$

$$= \frac{1}{2}\{(10 + 45 + 28 + 2) - (10 + 20 + 9 + 14)\}$$

$$= \frac{1}{2}\{85 - 53\}$$

$$= \frac{1}{2}(32) = 16 \text{ sq.units.}$$

Area of parking lot $= 16$ sq.feets

Construction rate per square feet $=$ ₹1300

Total cost for constructing the parking lot $= 16 \times 1300 =$ ₹20800

**Example 5.10** The line $r$ passes through the points $(-2, 2)$ and $(5, 8)$ and the line $s$ passes through the points $(-8, 7)$ and $(-2, 0)$. Is the line $r$ perpendicular to $s$?

**Solution** The slope of line $r$ is $m_1 = \frac{8-2}{5+2} = \frac{6}{7}$

The slope of line $s$ is $m_2 = \frac{0-7}{-2+8} = \frac{-7}{6}$

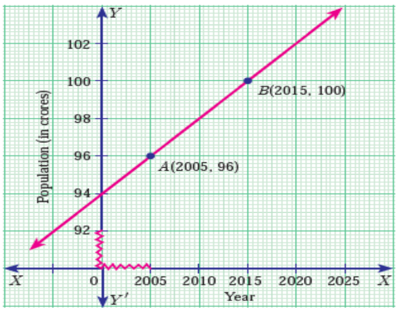
The product of slopes $= \frac{6}{7} \times \frac{-7}{6} = -1$

That is, $m_1 m_2 = -1$

Therefore, the line $r$ is perpendicular to line $s$.

**Example 5.14** Consider the graph representing growth of population (in crores). Find the slope of the line $AB$ and hence estimate the population in the year 2030?

**Solution** The points $A(2005, 96)$ and $B(2015, 100)$ are on the line $AB$.




Slope of AB $= \frac{100-96}{2015-2005} = \frac{4}{10} = \frac{2}{5}$

Let the growth of population in 2030 be $k$ crores.

Assuming that the point $C(2030, k)$ is on $AB$,

we have, slope of $AC$ = slope of $AB$

$$\frac{k-96}{2030-2005} = \frac{2}{5} \quad \text{gives} \quad \frac{k-96}{25} = \frac{2}{5}$$

$$k - 96 = 10$$

$$k = 106$$

Hence the estimated population in 2030 = 106 Crores.

**Example 5.15** Without using Pythagoras theorem, show that the points $(1,-4)$, $(2,-3)$ and $(4,-7)$ form a right angled triangle.

**Solution** Let the given points be $A(1,-4)$, $B(2,-3)$ and $C(4,-7)$.

$$\text{The slope of AB} = \frac{-3+4}{2-1} = \frac{1}{1} = 1$$

$$\text{The slope of BC} = \frac{-7+3}{4-2} = \frac{-4}{2} = -2$$

$$\text{The slope of AC} = \frac{-7+4}{4-1} = \frac{-3}{3} = -1$$

Slope of $AB \times$ slope of $AC = (1)(-1) = -1$

AB is perpendicular to AC. $\angle A = 90^\circ$

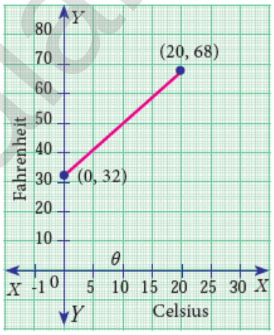
Therefore, $\Delta ABC$ is a right angled triangle.

**Example 5.20** The graph relates temperatures $y$ (in Fahrenheit degree) to temperatures $x$ (in Celsius degree) (a) Find the slope and $y$ intercept (b) Write an equation of the line (c) What is the mean temperature of the earth in Fahrenheit degree if its mean temperature is $25^\circ$ Celsius?

**Solution** (a) From the figure, slope $= \dfrac{\text{change in } y \text{ coordinate}}{\text{change is } x \text{ coordinate}} = \dfrac{68-32}{20-0} = \dfrac{36}{20} = \dfrac{9}{5} = 1.8$

The line crosses the $Y$ axis at $(0, 32)$

So the slope is $\frac{9}{5}$ and $y$ intercept is 32.




(b) Use the slope and $y$ intercept to write an equation

The equation is $y = \frac{9}{5}x + 32$

(c) In Celsius, the mean temperature of the earth is $25^\circ$. To find the mean temperature in Fahrenheit, we find the value of $y$ when $x = 25$

$$y = \frac{9}{5}x + 32$$

$$y = \frac{9}{5}(25) + 32$$

$$y = 77$$

Therefore, the mean temperature of the earth is $77^\circ$ F.

**Example 5.22** Find the equation of a line passing through the point $A(1,4)$ and perpendicular to the line joining points $(2,5)$ and $(4,7)$.

**Solution**

Let the given points be $A(1,4)$, $B(2,5)$ and $C(4,7)$.

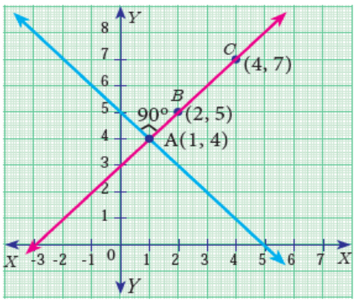
Slope of line $BC = \dfrac{7-5}{4-2} = \dfrac{2}{2} = 1$

Let $m$ be the slope of the required line.

Since the required line is perpendicular to $BC$,

$$m \times 1 = -1$$

$$m = -1$$




The required line also pass through the point $A(1,4)$.

The equation of the required straight line is $y - y_1 = m(x - x_1)$

$$y - 4 = -1(x-1)$$

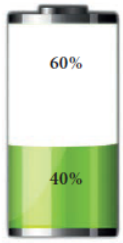
$$y - 4 = -x + 1$$

we get, $x + y - 5 = 0$

**Example 5.27** A mobile phone is put to use when the battery power is 100%. The percent of battery power '$y$' (in decimal) remaining after using the mobile phone for $x$ hours is assumed as $y = -0.25x + 1$

(i) Find the number of hours elapsed if the battery power is 40%.

(ii) How much time does it take so that the battery has no power?




**Solution**

(i) To find the time when the battery power is 40%,
we have to take $y = 0.40$

$$0.40 = -0.25x + 1 \Rightarrow 0.25x = 0.60$$

we get, $$x = \frac{0.60}{0.25} = 2.4 \text{ hours.}$$

(ii) If the battery power is 0 then $y = 0$

Therefore, $0 = -0.25x + 1$ gives $0.25x = 1$ hence $x = 4$ hours.

Thus, after 4 hours, the battery of the mobile phone will have no power.

**Example 5.28** A line makes positive intercepts on coordinate axes whose sum is 7 and it passes through $(-3, 8)$. Find its equation.

**Solution** If $a$ and $b$ are the intercepts then $a + b = 7$ or $b = 7 - a$

By intercept form $$\frac{x}{a} + \frac{y}{b} = 1$$

We have $$\frac{x}{a} + \frac{y}{7-a} = 1$$

As this line pass through the point $(-3, 8)$, we have

$$\frac{-3}{a} + \frac{8}{7-a} = 1 \Rightarrow -3(7-a) + 8a = a(7-a)$$

$$-21 + 3a + 8a = 7a - a^2$$

So, $$a^2 + 4a - 21 = 0$$

Solving this equation $(a-3)(a+7) = 0$

$$a = 3 \text{ or } a = -7$$

Since $a$ is positive, we have $a = 3$ and $b = 7 - a = 7 - 3 = 4$.

Hence $$\frac{x}{3} + \frac{y}{4} = 1$$

Therefore, $4x + 3y - 12 = 0$ is the required equation.

**Example 5.29** A circular garden is bounded by East Avenue and Cross Road. Cross Road intersects North Street at $D$ and East Avenue at $E$. $AD$ is tangential to the circular garden at $A(3, 10)$. Using the figure.


(a) Find the equation of

(i) East Avenue.

(ii) North Street

(iii) Cross Road

(b) Where does the Cross Road intersect?

(i) North Street (ii) East Avenue

**Solution** (a) (i) East Avenue is the straight line joining $C(0,2)$ and $B(7,2)$. Thus the equation of East Avenue is obtained by using two-point form which is

$$\frac{y-2}{2-2} = \frac{x-0}{7-0}$$

$$\frac{y-2}{0} = \frac{x}{7} \Rightarrow y = 2$$

(ii) Since the point $D$ lie vertically above $C(0,2)$. The $x$ coordinate of $D$ is 0.
Since any point on North Street has $x$ coordinate value 0.
The equation of North Street is $x = 0$

(iii) To find equation of Cross Road.
Center of circular garden $M$ is at $(7,\ 7)$, $A$ is $(3,\ 10)$
We first find slope of $MA$, which we call $m_1$

Thus $m_1 = \frac{10-7}{3-7} = \frac{-3}{4}$.

Since the Cross Road is perpendicular to MA, if $m_2$ is the slope of the Cross Road then, $m_1 m_2 = -1$ gives $\frac{-3}{4} m_2 = -1$ so $m_2 = \frac{4}{3}$.

Now, the cross road has slope $\frac{4}{3}$ and it passes through the point $A(3,10)$.

The equation of the Cross Road is $y - 10 = \frac{4}{3}(x-3)$

$$3y - 30 = 4x - 12$$

Hence, $4x - 3y + 18 = 0$

(b) (i) If $D$ is $(0,k)$ then $D$ is a point on the Cross Road.
Therefore, substituting $x = 0$, $y = k$ in the equation of Cross Road,
we get, $0 - 3k + 18 = 0$
Value of $k = 6$
Therefore, $D$ is $(0,\ 6)$

(ii) To find $E$, let $E$ be $(q,2)$
Put $y = 2$ in the equation of the Cross Road,
we get, $4q - 6 + 18 = 0$

$4q = -12$ gives $q = -3$

Therefore, The point $E$ is $(-3,2)$

Thus the Cross Road meets the North Street at $D(0,\ 6)$ and East Avenue at E $(-3,2)$.

**Example 5.37** The line joining the points $A(0,5)$ and $B(4,1)$ is a tangent to a circle whose centre $C$ is at the point $(4,4)$ find

(i) the equation of the line $AB$.
(ii) the equation of the line through $C$ which is perpendicular to the line $AB$.
(iii) the coordinates of the point of contact of tangent line $AB$ with the circle.

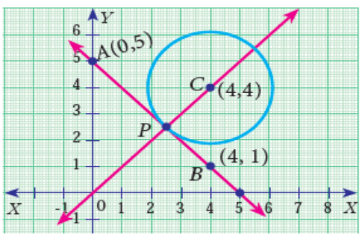
**Solution** (i) Equation of line AB, $A(0,5)$ and $B(4,1)$

$$\frac{y-y_1}{y_2-y_1} = \frac{x-x_1}{x_2-x_1}$$

$$\frac{y-5}{1-5} = \frac{x-0}{4-0}$$

$$4(y-5) = -4x \Rightarrow y - 5 = -x$$

$$x + y - 5 = 0$$



(ii) The equation of a line which is perpendicular to the line $AB: x+y-5=0$ is $x-y+k=0$

Since it is passing through the point $(4,4)$, we have

$4-4+k=0 \Rightarrow k=0$

The equation of a line which is perpendicular to $AB$ and through $C$ is $x-y=0$

(iii) The coordinate of the point of contact $P$ of the tangent line $AB$ with the circle is point of intersection of lines.

$x+y-5=0$ and $x-y=0$

solving, we get $x=\frac{5}{2}$ and $y=\frac{5}{2}$

Therefore, the coordinate of the point of contact is $P\left(\frac{5}{2},\frac{5}{2}\right)$.

# TRIGONOMETRY

## 2-MARKS (SELECTED SUMS)

**Example 6.1** Prove that $\tan^2\theta-\sin^2\theta=\tan^2\theta\sin^2\theta$

***Solution*** $\tan^2\theta-\sin^2\theta=\tan^2\theta-\frac{\sin^2\theta}{\cos^2\theta}.\cos^2\theta$

$=\tan^2\theta(1-\cos^2\theta)=\tan^2\theta\sin^2\theta$

**Example 6.2** Prove that $\frac{\sin A}{1+\cos A}=\frac{1-\cos A}{\sin A}$

***Solution*** $\frac{\sin A}{1+\cos A}=\frac{\sin A}{1+\cos A}\times\frac{1-\cos A}{1-\cos A}$ [multiply numerator and denominator by the conjugate of $1+\cos A$]

$=\frac{\sin A(1-\cos A)}{(1+\cos A)(1-\cos A)}=\frac{\sin A(1-\cos A)}{1-\cos^2 A}$

$=\frac{\sin A(1-\cos A)}{\sin^2 A}=\frac{1-\cos A}{\sin A}$

**Example 6.3** Prove that $1+\frac{\cot^2\theta}{1+\operatorname{cosec}\theta}=\operatorname{cosec}\theta$

***Solution*** $1+\frac{\cot^2\theta}{1+\operatorname{cosec}\theta}=1+\frac{\operatorname{cosec}^2\theta-1}{\operatorname{cosec}\theta+1}$ $\left[\because \operatorname{cosec}^2\theta-1=\cot^2\theta\right]$

$=1+\frac{(\operatorname{cosec}\theta+1)(\operatorname{cosec}\theta-1)}{\operatorname{cosec}\theta+1}$

$=1+(\operatorname{cosec}\theta-1)=\operatorname{cosec}\theta$

**Example 6.4** Prove that $\sec\theta-\cos\theta=\tan\theta\sin\theta$

***Solution*** $\sec\theta-\cos\theta=\frac{1}{\cos\theta}-\cos\theta=\frac{1-\cos^2\theta}{\cos\theta}$

$=\frac{\sin^2\theta}{\cos\theta}$ $\left[\because 1-\cos^2\theta=\sin^2\theta\right]$

$=\frac{\sin\theta}{\cos\theta}\times\sin\theta=\tan\theta\sin\theta$

**Example 6.5** Prove that $\sqrt{\frac{1+\cos\theta}{1-\cos\theta}} = \operatorname{cosec}\theta + \cot\theta$

**Solution** $\sqrt{\frac{1+\cos\theta}{1-\cos\theta}} = \sqrt{\frac{1+\cos\theta}{1-\cos\theta}\times\frac{1+\cos\theta}{1+\cos\theta}}$ [multiply numerator and denominator by the conjugate of $1-\cos\theta$ ]

$= \sqrt{\frac{(1+\cos\theta)^2}{1-\cos^2\theta}} = \frac{1+\cos\theta}{\sqrt{\sin^2\theta}}$ $\quad [\because \sin^2\theta+\cos^2\theta=1]$

$= \frac{1+\cos\theta}{\sin\theta} = \operatorname{cosec}\theta + \cot\theta$

**Example 6.6** Prove that $\frac{\sec\theta}{\sin\theta} - \frac{\sin\theta}{\cos\theta} = \cot\theta$

**Solution** $\frac{\sec\theta}{\sin\theta} - \frac{\sin\theta}{\cos\theta} = \frac{\frac{1}{\cos\theta}}{\sin\theta} - \frac{\sin\theta}{\cos\theta} = \frac{1}{\sin\theta\cos\theta} - \frac{\sin\theta}{\cos\theta}$

$= \frac{1-\sin^2\theta}{\sin\theta\cos\theta} = \frac{\cos^2\theta}{\sin\theta\cos\theta} = \cot\theta$

**Example 6.7** Prove that $\sin^2 A\cos^2 B + \cos^2 A\sin^2 B + \cos^2 A\cos^2 B + \sin^2 A\sin^2 B = 1$

**Solution** $\sin^2 A\cos^2 B + \cos^2 A\sin^2 B + \cos^2 A\cos^2 B + \sin^2 A\sin^2 B$

$= \sin^2 A\cos^2 B + \sin^2 A\sin^2 B + \cos^2 A\sin^2 B + \cos^2 A\cos^2 B$

$= \sin^2 A(\cos^2 B + \sin^2 B) + \cos^2 A(\sin^2 B + \cos^2 B)$

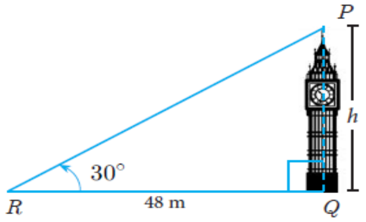
$= \sin^2 A(1) + \cos^2 A(1)$ $\quad (\because \sin^2 B + \cos^2 B = 1)$

$= \sin^2 A + \cos^2 A = 1$

**Example 6.19** A tower stands vertically on the ground. From a point on the ground, which is 48 m away from the foot of the tower, the angle of elevation of the top of the tower is $30^\circ$. Find the height of the tower.

**Solution** Let $PQ$ be the height of the tower.

Take $PQ = h$ and $QR$ is the distance between the tower and the point $R$. In the right angled $\Delta PQR$, $\angle PRQ = 30^\circ$




$\tan\theta = \frac{PQ}{QR}$

$\tan 30^\circ = \frac{h}{48} \Rightarrow \frac{1}{\sqrt{3}} = \frac{h}{48} \Rightarrow h = 16\sqrt{3}$

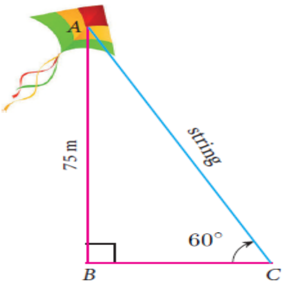
Therefore, the height of the tower is $16\sqrt{3}$ m

**Example 6.20** A kite is flying at a height of 75 m above the ground. The string attached to the kite is temporarily tied to a point on the ground. The inclination of the string with the ground is $60^\circ$. Find the length of the string, assuming that there is no slack in the string.

**Solution** Let $AB$ be the height of the kite above the ground. Then, $AB = 75$.

Let $AC$ be the length of the string.

In the right angled $\Delta ABC$, $\angle ACB = 60^\circ$




$\sin\theta = \frac{AB}{AC}$

$\sin 60^\circ = \frac{75}{AC}$

$\Rightarrow \frac{\sqrt{3}}{2} = \frac{75}{AC} \Rightarrow AC = \frac{150}{\sqrt{3}} = 50\sqrt{3}$

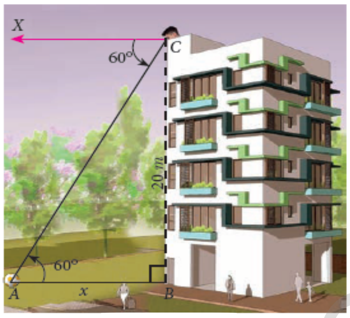
Hence, the length of the string is $50\sqrt{3}$ m.

**Example 6.26** A player sitting on the top of a tower of height 20 m observes the angle of depression of a ball lying on the ground as 60°. Find the distance between the foot of the tower and the ball. ($\sqrt{3} = 1.732$)

**Solution** Let $BC$ be the height of the tower and $A$ be the position of the ball lying on the ground. Then,

$BC = 20$ m and $\angle XCA = 60° = \angle CAB$

Let $AB = x$ metres.




In the right angled $\Delta ABC$,

$$\tan 60° = \frac{BC}{AB}$$

$$\sqrt{3} = \frac{20}{x}$$

$$x = \frac{20 \times \sqrt{3}}{\sqrt{3} \times \sqrt{3}} = \frac{20 \times 1.732}{3} = 11.55 \text{ m.}$$

Hence, the distance between the foot of the tower and the ball is 11.55 m.

## 5-MARKS (SELECTED SUMS)

**Example 6.8** If $\cos\theta + \sin\theta = \sqrt{2}\cos\theta$, then prove that $\cos\theta - \sin\theta = \sqrt{2}\sin\theta$

**Solution** Now, $\cos\theta + \sin\theta = \sqrt{2}\cos\theta$.

Squaring both sides,

$$(\cos\theta + \sin\theta)^2 = (\sqrt{2}\cos\theta)^2$$

$$\cos^2\theta + \sin^2\theta + 2\sin\theta\cos\theta = 2\cos^2\theta$$

$$2\cos^2\theta - \cos^2\theta - \sin^2\theta = 2\sin\theta\cos\theta$$

$$\cos^2\theta - \sin^2\theta = 2\sin\theta\cos\theta$$

$$(\cos\theta + \sin\theta)(\cos\theta - \sin\theta) = 2\sin\theta\cos\theta$$

$$\cos\theta - \sin\theta = \frac{2\sin\theta\cos\theta}{\cos\theta + \sin\theta} = \frac{2\sin\theta\cos\theta}{\sqrt{2}\cos\theta} \quad [\because \cos\theta + \sin\theta = \sqrt{2}\cos\theta]$$

$$= \sqrt{2}\sin\theta$$

Therefore, $\cos\theta - \sin\theta = \sqrt{2}\sin\theta$.

**Example 6.9** Prove that $(\operatorname{cosec}\theta - \sin\theta)(\sec\theta - \cos\theta)(\tan\theta + \cot\theta) = 1$

**Solution** $(\operatorname{cosec}\theta - \sin\theta)(\sec\theta - \cos\theta)(\tan\theta + \cot\theta)$

$$= \left(\frac{1}{\sin\theta} - \sin\theta\right)\left(\frac{1}{\cos\theta} - \cos\theta\right)\left(\frac{\sin\theta}{\cos\theta} + \frac{\cos\theta}{\sin\theta}\right)$$

$$= \frac{1-\sin^2\theta}{\sin\theta} \times \frac{1-\cos^2\theta}{\cos\theta} \times \frac{\sin^2\theta + \cos^2\theta}{\sin\theta\cos\theta}$$

$$= \frac{\cos^2\theta\sin^2\theta \times 1}{\sin^2\theta\cos^2\theta} = 1$$

**Example 6.10** Prove that $\frac{\sin A}{1+\cos A} + \frac{\sin A}{1-\cos A} = 2\operatorname{cosec} A$.

**Solution** $\frac{\sin A}{1+\cos A} + \frac{\sin A}{1-\cos A}$

$$= \frac{\sin A(1-\cos A) + \sin A(1+\cos A)}{(1+\cos A)(1-\cos A)}$$

$$= \frac{\sin A - \sin A\cos A + \sin A + \sin A\cos A}{1-\cos^2 A}$$

$$= \frac{2\sin A}{1-\cos^2 A} = \frac{2\sin A}{\sin^2 A} = 2\operatorname{cosec} A$$

**Example 6.11** If $\operatorname{cosec}\theta+\cot\theta=P$, then prove that $\cos\theta=\frac{P^2-1}{P^2+1}$

**Solution** Given $\operatorname{cosec}\theta+\cot\theta=P$ ...(1)

$$\operatorname{cosec}^2\theta-\cot^2\theta=1 \text{ (identity)}$$

$$\operatorname{cosec}\theta-\cot\theta=\frac{1}{\operatorname{cosec}\theta+\cot\theta}$$

$$\operatorname{cosec}\theta-\cot\theta=\frac{1}{P} \quad ...(2)$$

Adding (1) and (2) we get, $2\operatorname{cosec}\theta=P+\frac{1}{P}$

$$2\operatorname{cosec}\theta=\frac{P^2+1}{P} \quad ...(3)$$

Subtracting (2) from (1), we get, $2\cot\theta=P-\frac{1}{P}$

$$2\cot\theta=\frac{P^2-1}{P} \quad ...(4)$$

Dividing (4) by (3) we get, $\frac{2\cot\theta}{2\operatorname{cosec}\theta}=\frac{P^2-1}{P}\times\frac{P}{P^2+1} \Rightarrow \cos\theta=\frac{P^2-1}{P^2+1}$

**Example 6.12** Prove that $\tan^2 A-\tan^2 B=\frac{\sin^2 A-\sin^2 B}{\cos^2 A\cos^2 B}$

**Solution** $\tan^2 A-\tan^2 B=\frac{\sin^2 A}{\cos^2 A}-\frac{\sin^2 B}{\cos^2 B}$

$$=\frac{\sin^2 A\cos^2 B-\sin^2 B\cos^2 A}{\cos^2 A\cos^2 B}$$

$$=\frac{\sin^2 A(1-\sin^2 B)-\sin^2 B(1-\sin^2 A)}{\cos^2 A\cos^2 B}$$

$$=\frac{\sin^2 A-\sin^2 A\sin^2 B-\sin^2 B+\sin^2 A\sin^2 B}{\cos^2 A\cos^2 B}=\frac{\sin^2 A-\sin^2 B}{\cos^2 A\cos^2 B}$$

**Example 6.17** If $\frac{\cos^2\theta}{\sin\theta}=p$ and $\frac{\sin^2\theta}{\cos\theta}=q$, then prove that $p^2q^2(p^2+q^2+3)=1$

**Solution** We have $\frac{\cos^2\theta}{\sin\theta}=p$ ...(1) and $\frac{\sin^2\theta}{\cos\theta}=q$ ...(2)

$$p^2q^2(p^2+q^2+3)=\left(\frac{\cos^2\theta}{\sin\theta}\right)^2\left(\frac{\sin^2\theta}{\cos\theta}\right)^2\times\left[\left(\frac{\cos^2\theta}{\sin\theta}\right)^2+\left(\frac{\sin^2\theta}{\cos\theta}\right)^2+3\right] \text{ [from (1) and (2)]}$$

$$=\left(\frac{\cos^4\theta}{\sin^2\theta}\right)\left(\frac{\sin^4\theta}{\cos^2\theta}\right)\times\left[\frac{\cos^4\theta}{\sin^2\theta}+\frac{\sin^4\theta}{\cos^2\theta}+3\right]$$

$$=(\cos^2\theta\times\sin^2\theta)\times\left[\left(\frac{\cos^6\theta+\sin^6\theta+3\sin^2\theta\cos^2\theta}{\sin^2\theta\cos^2\theta}\right)\right]$$

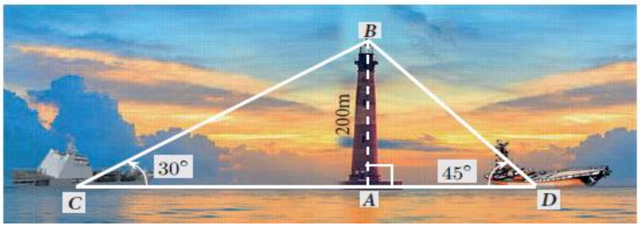
$$=\cos^6\theta+\sin^6\theta+3\sin^2\theta\cos^2\theta$$

$$=(\cos^2\theta)^3+(\sin^2\theta)^3+3\sin^2\theta\cos^2\theta$$

$$=[(\cos^2\theta+\sin^2\theta)^3-3\cos^2\theta\sin^2\theta(\cos^2\theta+\sin^2\theta)]+3\sin^2\theta\cos^2\theta$$

$$=1-3\cos^2\theta\sin^2\theta(1)+3\cos^2\theta\sin^2\theta=1$$

**Example 6.21** Two ships are sailing in the sea on either sides of a lighthouse. The angle of elevation of the top of the lighthouse as observed from the ships are 30° and 45° respectively. If the lighthouse is 200 m high, find the distance between the two ships. ($\sqrt{3}=1.732$)




**Solution** Let $AB$ be the lighthouse. Let $C$ and $D$ be the positions of the two ships.

Then, $AB=200$ m.

$\angle ACB=30°$, $\angle ADB=45°$

In the right angled $\Delta BAC$, $\tan 30^\circ = \frac{AB}{AC}$

$$\frac{1}{\sqrt{3}} = \frac{200}{AC} \Rightarrow AC = 200\sqrt{3} \quad ...(1)$$

In the right angled $\Delta BAD$, $\tan 45^\circ = \frac{AB}{AD}$

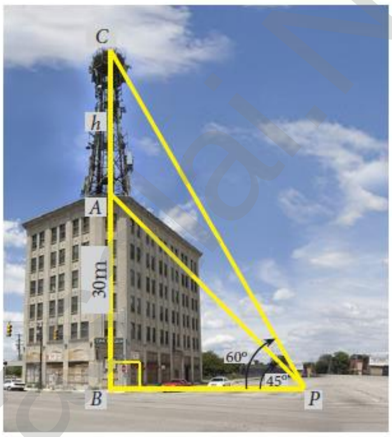
$$1 = \frac{200}{AD} \Rightarrow AD = 200 \quad ...(2)$$

Now, $CD = AC + AD = 200\sqrt{3} + 200$ [by (1) and (2)]

$$CD = 200(\sqrt{3}+1) = 200 \times 2.732 = 546.4$$

Distance between two ships is 546.4 m.

**Example 6.22** From a point on the ground, the angles of elevation of the bottom and top of a tower fixed at the top of a 30 m high building are $45^\circ$ and $60^\circ$ respectively. Find the height of the tower. ($\sqrt{3} = 1.732$)




*Solution* Let $AC$ be the height of the tower.

Let $AB$ be the height of the building.

Then, $AC = h$ metres, $AB = 30$ m

In the right angled $\Delta CBP$, $\angle CPB = 60^\circ$

$$\tan\theta = \frac{BC}{BP}$$

$$\tan 60^\circ = \frac{AB+AC}{BP} \Rightarrow \sqrt{3} = \frac{30+h}{BP} \quad ...(1)$$

In the right angled $\Delta ABP$, $\angle APB = 45^\circ$

$$\tan\theta = \frac{AB}{BP}$$

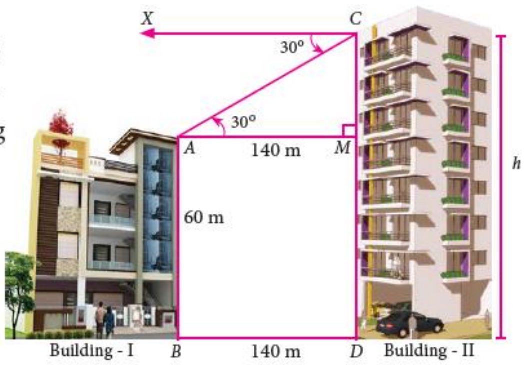
$$\tan 45^\circ = \frac{30}{BP} \Rightarrow BP = 30 \quad ...(2)$$

Substituting (2) in (1), we get $\sqrt{3} = \frac{30+h}{30}$

$$h = 30(\sqrt{3}-1) = 30(1.732-1) = 30(0.732) = 21.96$$

Hence, the height of the tower is 21.96 m.

**Example 6.27** The horizontal distance between two buildings is 140 m. The angle of depression of the top of the first building when seen from the top of the second building is 30°. If the height of the first building is 60 m, find the height of the second building. ($\sqrt{3} = 1.732$)




*Solution* The height of the first building $AB = 60$ m. Now, $AB = MD = 60$ m

Let the height of the second building $CD = h$. Distance $BD = 140$ m

Now, $AM = BD = 140$ m

From the diagram,

$$\angle XCA = 30^\circ = \angle CAM$$

In the right angled $\Delta AMC$, $\tan 30° = \frac{CM}{AM}$

$$\frac{1}{\sqrt{3}} = \frac{CM}{140}$$

$$CM = \frac{140}{\sqrt{3}} = \frac{140\sqrt{3}}{3}$$

$$= \frac{140 \times 1.732}{3}$$

$$CM = 80.83$$

Now, $h = CD = CM + MD = 80.83 + 60 = 140.83$

Therefore, the height of the second building is 140.83 m

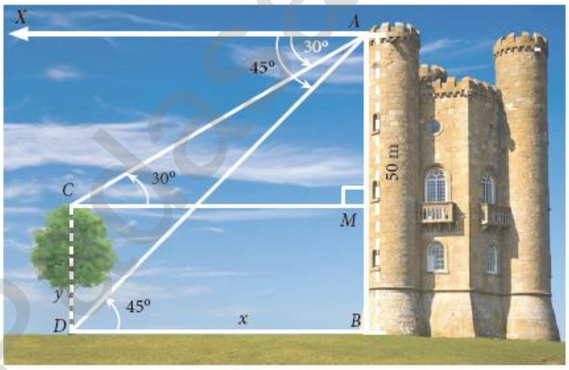
**Example 6.28** From the top of a tower 50 m high, the angles of depression of the top and bottom of a tree are observed to be 30° and 45° respectively. Find the height of the tree. $(\sqrt{3} = 1.732)$

**Solution** The height of the tower $AB$ = 50 m

Let the height of the tree $CD = y$ and $BD = x$

From the diagram, $\angle XAC = 30° = \angle ACM$ and $\angle XAD = 45° = \angle ADB$

In the right angled $\Delta ABD$,




$$\tan 45° = \frac{AB}{BD}$$

$$1 = \frac{50}{x} \Rightarrow x = 50 \text{ m}$$

In the right angled $\Delta AMC$,

$$\tan 30° = \frac{AM}{CM}$$

$$\frac{1}{\sqrt{3}} = \frac{AM}{50} \quad [\because DB = CM]$$

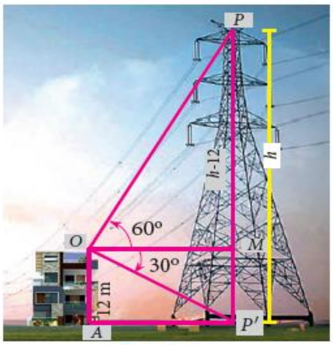
$$AM = \frac{50}{\sqrt{3}} = \frac{50\sqrt{3}}{3} = \frac{50 \times 1.732}{3} = 28.87 \text{ m.}$$

Therefore, height of the tree $= CD = MB = AB - AM = 50 - 28.87 = 21.13$ m

**Example 6.31** From the top of a 12 m high building, the angle of elevation of the top of a cable tower is 60° and the angle of depression of its foot is 30°. Determine the height of the tower.

**Solution** As shown in Fig.6.27, $OA$ is the building, $O$ is the point of observation on the top of the building $OA$. Then, $OA = 12$ m.

$PP'$ is the cable tower with $P$ as the top and $P'$ as the bottom.




Then the angle of elevation of $P$, $\angle MOP = 60°$.

And the angle of depression of $P'$, $\angle MOP' = 30°$.

Suppose, height of the cable tower $PP' = h$ metres.

Through $O$, draw $OM \perp PP'$

$$MP = PP' - MP' = h - OA = h - 12$$

In the right angled $\Delta OMP$, $\frac{MP}{OM} = \tan 60°$

$$\Rightarrow \quad \frac{h-12}{OM} = \sqrt{3}$$

$$OM = \frac{h-12}{\sqrt{3}} \qquad ...(1)$$

In the right angled $\Delta OMP'$, $\frac{MP'}{OM} = \tan 30°$

$$\Rightarrow \quad \frac{12}{OM} = \frac{1}{\sqrt{3}}$$

$$OM = 12\sqrt{3} \qquad ...(2)$$

From (1) and (2) we have, $\frac{h-12}{\sqrt{3}} = 12\sqrt{3}$

$$\Rightarrow \quad h - 12 = 12\sqrt{3} \times \sqrt{3} \quad \text{we get, } h = 48$$

Hence, the required height of the cable tower is 48 m.

**Example 6.32** A pole 5 m high is fixed on the top of a tower. The angle of elevation of the top of the pole observed from a point '$A$' on the ground is 60° and the angle of depression to the point '$A$' from the top of the tower is 45°. Find the height of the tower. ($\sqrt{3} = 1.732$)

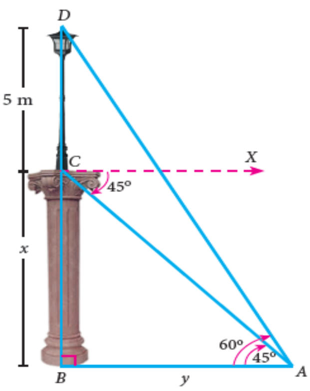
***Solution*** Let $BC$ be the height of the tower and $CD$ be the height of the pole.

Let '$A$' be the point of observation.

Let $BC = x$ and $AB = y$.

From the diagram,

$\angle BAD = 60°$ and $\angle XCA = 45° = \angle BAC$




In the right angled $\Delta ABC$, $\tan 45° = \frac{BC}{AB}$

$$\Rightarrow 1 = \frac{x}{y} \quad \Rightarrow x = y \qquad ...(1)$$

In the right angled $\Delta ABD$, $\tan 60° = \frac{BD}{AB} = \frac{BC + CD}{AB}$

$$\Rightarrow \sqrt{3} = \frac{x+5}{y} \quad \Rightarrow \sqrt{3}\,y = x + 5$$

we get, $\sqrt{3}\,x = x + 5$ [From (1)]

$$x = \frac{5}{\sqrt{3}-1} = \frac{5}{\sqrt{3}-1} \times \frac{\sqrt{3}+1}{\sqrt{3}+1} = \frac{5(1.732+1)}{2} = 6.83$$

Hence, height of the tower is 6.83 m.

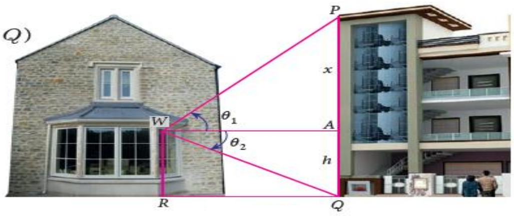
**Example 6.33** From a window ($h$ metres high above the ground) of a house in a street, the angles of elevation and depression of the top and the foot of another house on the opposite side of the street are $\theta_1$ and $\theta_2$ respectively. Show that the height of the opposite house is $h\left(1 + \frac{\cot\theta_2}{\cot\theta_1}\right)$.

***Solution*** Let $W$ be the point on the window where the angles of elevation and depression are measured. Let $PQ$ be the house on the opposite side.

Then $WA$ is the width of the street.

Height of the window $= h$ metres
$= AQ \; (WR = AQ)$

Let $PA = x$ metres.




In the right angled $\Delta PAW$, $\tan\theta_1 = \frac{AP}{AW}$

$\Rightarrow \quad \tan\theta_1 = \frac{x}{AW}$

$AW = \frac{x}{\tan\theta_1}$

we get, $AW = x\cot\theta_1 \quad \ldots(1)$

In the right angled $\Delta QAW$, $\tan\theta_2 = \frac{AQ}{AW}$

$\Rightarrow \quad \tan\theta_2 = \frac{h}{AW}$

we get, $AW = h\cot\theta_2 \quad \ldots(2)$

From (1) and (2) we get, $x\cot\theta_1 = h\cot\theta_2$

$\Rightarrow \quad x = h\frac{\cot\theta_2}{\cot\theta_1}$

Therefore, height of the opposite house $= PA + AQ = x + h = h\frac{\cot\theta_2}{\cot\theta_1} + h = h\left(1 + \frac{\cot\theta_2}{\cot\theta_1}\right)$

Hence Proved.

# MENSURATION

## 2-MARKS (SELECTED SUMS)

**Example 7.1** A cylindrical drum has a height of 20 cm and base radius of 14 cm. Find its curved surface area and the total surface area.

**Solution** Given that, height of the cylinder $h = 20$ cm ; radius $r = 14$ cm

Now, C.S.A. of the cylinder $= 2\pi rh$ sq. units

C.S.A. of the cylinder $= 2 \times \frac{22}{7} \times 14 \times 20 = 2 \times 22 \times 2 \times 20 = 1760 \text{ cm}^2$

T.S.A. of the cylinder $= 2\pi r(h + r)$ sq. units

$= 2 \times \frac{22}{7} \times 14 \times (20 + 14) = 2 \times \frac{22}{7} \times 14 \times 34$

$= 2992 \text{ cm}^2$

Therefore, C.S.A. $= 1760 \text{ cm}^2$ and T.S.A. $= 2992 \text{ cm}^2$

**Example 7.2** The curved surface area of a right circular cylinder of height 14 cm is $88 \text{ cm}^2$. Find the diameter of the cylinder.

**Solution** Given that, C.S.A. of the cylinder = 88 sq. cm

$2\pi rh = 88$

$2 \times \frac{22}{7} \times r \times 14 = 88 \quad (h = 14 \text{ cm})$

$2r = \frac{88 \times 7}{22 \times 14} = 2$

Therefore, diameter $= 2$ cm

**Example 7.6** If the total surface area of a cone of radius 7cm is $704 \text{ cm}^2$, then find its slant height.

**Solution** Given that, radius $r = 7$ cm

Now, total surface area of the cone $= \pi r(l + r)$ sq. units

T.S.A. $= 704 \text{ cm}^2$

$704 = \frac{22}{7} \times 7(l + 7)$

$32 = l + 7 \Rightarrow l = 25$ cm

Therefore, slant height of the cone is 25 cm.

**Example 7.8** Find the diameter of a sphere whose surface area is 154 $m^2$.

**Solution** Let $r$ be the radius of the sphere.

Given that, surface area of sphere = 154 $m^2$

$$4\pi r^2 = 154$$

$$4 \times \frac{22}{7} \times r^2 = 154$$

$$\Rightarrow \quad r^2 = 154 \times \frac{1}{4} \times \frac{7}{22}$$

$$r^2 = \frac{49}{4} \text{ we get } r = \frac{7}{2}$$

Therefore, diameter is 7 m

**Example 7.9** The radius of a spherical balloon increases from 12 cm to 16 cm as air being pumped into it. Find the ratio of the surface area of the balloons in the two cases.

**Solution** Let $r_1$ and $r_2$ be the radii of the balloons.

Given that, $\frac{r_1}{r_2} = \frac{12}{16} = \frac{3}{4}$

Now, ratio of C.S.A. of balloons $= \frac{4\pi r_1^2}{4\pi r_2^2} = \frac{r_1^2}{r_2^2} = \left(\frac{r_1}{r_2}\right)^2 = \left(\frac{3}{4}\right)^2 = \frac{9}{16}$

Therefore, ratio of C.S.A. of balloons is 9:16.

**Example 7.10** If the base area of a hemispherical solid is 1386 sq. metres, then find its total surface area?

**Solution** Let $r$ be the radius of the hemisphere.

Given that, base area $= \pi r^2 = 1386$ sq. m

$$\text{T.S.A.} = 3\pi r^2 \text{ sq.m}$$

$$= 3 \times 1386 = 4158$$

Therefore, T.S.A. of the hemispherical solid is 4158 $m^2$.

**Example 7.13** The slant height of a frustum of a cone is 5 cm and the radii of its ends are 4 cm and 1 cm. Find its curved surface area.

**Solution** Let $l$, $R$ and $r$ be the slant height, top radius and bottom radius of the frustum.

Given that, $l = 5$ cm, $R = 4$ cm, $r = 1$ cm

Now, C.S.A. of the frustum $= \pi(R + r)l$ sq. units

$$= \frac{22}{7} \times (4 + 1) \times 5$$

$$= \frac{550}{7}$$

Therefore, C.S.A. = 78.57 $cm^2$

**Example 7.15** Find the volume of a cylinder whose height is 2 m and whose base area is 250 $m^2$.

**Solution** Let $r$ and $h$ be the radius and height of the cylinder respectively.

Given that, height $h = 2$ m, base area = 250 $m^2$

Now, volume of a cylinder $= \pi r^2 h$ cu. units

$$= \textit{base area} \times h$$

$$= 250 \times 2 = 500 \text{ m}^3$$

Therefore, volume of the cylinder = 500 $m^3$

**Example 7.17** Find the volume of the iron used to make a hollow cylinder of height 9 cm and whose internal and external radii are 21 cm and 28 cm respectively.

**Solution** Let $r$, $R$ and $h$ be the internal radius, external radius and height of the hollow cylinder respectively.

Given that, $r = 21$cm, $R = 28$ cm, $h = 9$ cm

$$\text{Now, volume of hollow cylinder} = \pi(R^2 - r^2)h \text{ cu. units}$$

$$= \frac{22}{7}(28^2 - 21^2) \times 9$$

$$= \frac{22}{7}(784 - 441) \times 9 = 9702$$

Therefore, volume of iron used $= 9702 \text{ cm}^3$

**Example 7.19** The volume of a solid right circular cone is 11088 cm$^3$. If its height is 24 cm then find the radius of the cone.

**Solution** Let $r$ and $h$ be the radius and height of the cone respectively.

Given that, volume of the cone $= 11088 \text{ cm}^3$

$$\frac{1}{3}\pi r^2 h = 11088$$

$$\frac{1}{3} \times \frac{22}{7} \times r^2 \times 24 = 11088$$

$$r^2 = 441$$

Therefore, radius of the cone $r = 21$ cm

**Example 7.20** The ratio of the volumes of two cones is 2:3. Find the ratio of their radii if the height of second cone is double the height of the first.

**Solution** Let $r_1$ and $h_1$ be the radius and height of the cone-I and let $r_2$ and $h_2$ be the radius and height of the cone-II.

Given that, $h_2 = 2h_1$ and $\frac{\text{Volume of the cone I}}{\text{Volume of the cone II}} = \frac{2}{3}$

$$\frac{\frac{1}{3}\pi r_1^2 h_1}{\frac{1}{3}\pi r_2^2 h_2} = \frac{2}{3} \Rightarrow \frac{r_1^2}{r_2^2} \times \frac{h_1}{2h_1} = \frac{2}{3}$$

$$\frac{r_1^2}{r_2^2} = \frac{4}{3} \Rightarrow \frac{r_1}{r_2} = \frac{2}{\sqrt{3}}$$

Therefore, ratio of their radii $= 2 : \sqrt{3}$

**Example 7.29** A metallic sphere of radius 16 cm is melted and recast into small spheres each of radius 2 cm. How many small spheres can be obtained?

**Solution** Let the number of small spheres obtained be $n$.

Let $r$ be the radius of each small sphere and $R$ be the radius of metallic sphere.

Here, $R = 16$ cm, $r = 2$ cm

Now, $n \times$ (Volume of a small sphere) = Volume of big metallic sphere

$$n\left(\frac{4}{3}\pi r^3\right) = \frac{4}{3}\pi R^3$$

$$n\left(\frac{4}{3}\pi \times 2^3\right) = \frac{4}{3}\pi \times 16^3$$

$$8n = 4096 \Rightarrow n = 512$$

**Example 7.30** A cone of height 24 cm is made up of modeling clay. A child reshapes it in the form of a cylinder of same radius as cone. Find the height of the cylinder.

**Solution** Let $h_1$ and $h_2$ be the heights of a cone and cylinder respectively.

Also, let $r$ be the raius of the cone.

Given that, height of the cone $h_1 = 24$ cm; radius of the cone and cylinder $r = 6$ cm

Since, Volume of cylinder = Volume of cone

$$\pi r^2 h_2 = \frac{1}{3}\pi r^2 h_1$$

$$h_2 = \frac{1}{3} \times h_1 \Rightarrow h_2 = \frac{1}{3} \times 24 = 8$$

Therefore, height of cylinder is 8 cm

## 5-MARKS (SELECTED SUMS)

**Example 7.3** A garden roller whose length is 3 m long and whose diameter is 2.8 m is rolled to level a garden. How much area will it cover in 8 revolutions?

**Solution** Given that, diameter $d = 2.8$ m and height $= 3$ m

radius $r = 1.4$ m

Area covered in one revolution = curved surface area of the cylinder

$$= 2\pi rh \text{ sq. units}$$

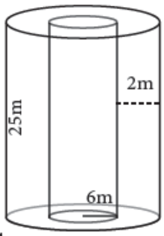
$$= 2 \times \frac{22}{7} \times 1.4 \times 3 = 26.4$$

Area covered in 1 revolution = 26.4 $m^2$

Area covered in 8 revolutions = $8 \times 26.4 = 211.2$

Therefore, area covered is 211.2 $m^2$

**Example 7.4** If one litre of paint covers 10 $m^2$, how many litres of paint is required to paint the internal and external surface areas of a cylindrical tunnel whose thickness is 2 m, internal radius is 6 m and height is 25 m.




**Solution** Given that, height $h = 25$ m; thickness $= 2$ m

internal radius $r = 6$ m

Now, external radius $R = 6 + 2 = 8$ m

C.S.A. of the cylindrical tunnel = C.S.A. of the hollow cylinder

C.S.A. of the hollow cylinder $= 2\pi(R + r)h$ sq. units

$$= 2 \times \frac{22}{7}(8 + 6) \times 25$$

Hence, C.S.A. of the cylindrical tunnel = 2200 $m^2$

Area covered by one litre of paint = 10 $m^2$

Number of litres required to paint the tunnel $= \frac{2200}{10} = 220$

Therefore, 220 litres of paint is needed to paint the tunnel.

**Example 7.5** The radius of a conical tent is 7 m and the height is 24 m. Calculate the length of the canvas used to make the tent if the width of the rectangular canvas is 4 m?

**Solution** Let $r$ and $h$ be the radius and height of the cone respectively.

Given that, radius $r = 7$ m and height $h = 24$ m

$$\text{Hence, } l = \sqrt{r^2 + h^2} = \sqrt{49 + 576}$$

$$l = \sqrt{625} = 25 \text{ m}$$

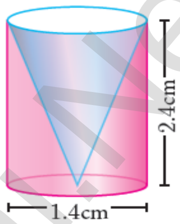
$$\text{C.S.A. of the conical tent} = \pi r l \text{ sq. units}$$

$$\text{Area of the canvas} = \frac{22}{7} \times 7 \times 25 = 550 \text{ m}^2$$

$$\text{Now, length of the canvas} = \frac{\text{Area of the canvas}}{\text{width}} = \frac{550}{4} = 137.5 \text{ m}$$

Therefore, the length of the canvas is137.5 m

**Example 7.7** From a solid cylinder whose height is 2.4 cm and diameter 1.4 cm, a conical cavity of the same height and base is hollowed out (Fig.7.13). Find the total surface area of the remaining solid.




***Solution*** Let $h$ and $r$ be the height and radius of the cone and cylinder.

Let $l$ be the slant height of the cone.

Given that, $h = 2.4$ cm and $d = 1.4$ cm ; $r = 0.7$ cm

$$\left.\begin{array}{r}\text{Total surface area of the}\\ \text{remaining solid}\end{array}\right\} = \text{C.S.A. of the cylinder + C.S.A. of the cone} + \text{area of the bottom}$$

$$= 2\pi rh + \pi rl + \pi r^2 \text{ sq. units}$$

$$= \pi r(2h + l + r) \text{ sq. units}$$

$$\text{Now, } l = \sqrt{r^2 + h^2} = \sqrt{0.49 + 5.76} = \sqrt{6.25} = 2.5 \text{ cm}$$

$$l = 2.5 \text{ cm}$$

$$\text{Area of the remaining solid} = \pi r(2h + l + r) \text{ sq. units}$$

$$= \frac{22}{7} \times 0.7 \times [(2 \times 2.4) + 2.5 + 0.7]$$

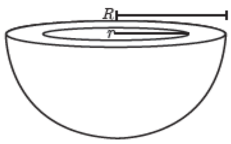
$$= 17.6$$

Therefore, total surface area of the remaining solid is 17.6 cm²

**Example 7.11** The internal and external radii of a hollow hemispherical shell are 3 m and 5 m respectively. Find the T.S.A. and C.S.A. of the shell.

***Solution*** Let the internal and external radii of the hemispherical shell be $r$ and $R$ respectively.

Given that, $R = 5$ m, $r = 3$ m




$$\text{C.S.A. of the shell} = 2\pi(R^2 + r^2) \text{ sq. units}$$

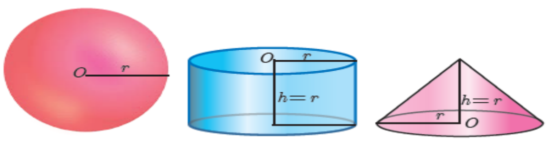
$$= 2 \times \frac{22}{7} \times (25 + 9) = 213.71$$

$$\text{T.S.A. of the shell} = \pi(3R^2 + r^2) \text{ sq. units}$$

$$= \frac{22}{7}(75 + 9) = 264$$

Therefore, C.S.A. $= 213.71$ $m^2$ and T.S.A. $= 264$ m².

**Example 7.12** A sphere, a cylinder and a cone are of the same height which is equal to its radius, where as cone and cylinder are of same height. Find the ratio of their curved surface areas.



**Solution** Required Ratio = C.S.A. of the sphere: C.S.A. of the cylinder : C.S.A. of the cone

$$= 4\pi r^2 : 2\pi rh : \pi rl \qquad (l = \sqrt{r^2 + h^2} = \sqrt{2r^2} = \sqrt{2}r)$$

$$= 4\pi r^2 : 2\pi r^2 : \pi r\sqrt{2}\, r$$

$$= 4\pi r^2 : 2\pi r^2 : \sqrt{2}\pi r^2$$

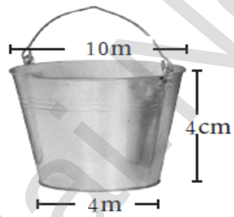
$$= 4 : 2 : \sqrt{2} = 2\sqrt{2} : \sqrt{2} : 1$$

**Example 7.14** An industrial metallic bucket is in the shape of the frustum of a right circular cone whose top and bottom diameters are 10 m and 4 m and whose height is 4 m. Find the curved and total surface area of the bucket.

**Solution** Let $h,\ l,\ R$ and $r$ be the height, slant height, top radius and bottom radius of the frustum.

Given that, diameter of the top =10 m; radius of the top $R = 5$ m.

diameter of the bottom = 4 m; radius of the bottom $r = 2$ m, height $h = 4$ m




Now, $l = \sqrt{h^2 + (R - r)^2}$

$= \sqrt{4^2 + (5-2)^2}$

$l = \sqrt{16 + 9} = \sqrt{25} = 5\text{m}$

C.S.A. $= \pi(R + r)l$ sq. units

$= \frac{22}{7}(5+2) \times 5 = 110\, m^2$

T.S.A. $= \pi(R + r)l + \pi R^2 + \pi r^2$ sq. units

$= \frac{22}{7}\left[(5+2)5 + 25 + 4\right] = \frac{1408}{7} = 201.14$

Therefore, C.S.A. = 110 $m^2$ and T.S.A. = 201.14 $m^2$

**Example 7.16** The volume of a cylindrical water tank is $1.078 \times 10^6$ litres. If the diameter of the tank is 7 m, find its height.

**Solution** Let $r$ and $h$ be the radius and height of the cylinder respectively.

Given that, volume of the tank $= 1.078 \times 10^6 = 1078000$ litre

$= 1078\ m^3 \qquad (\because 1l = \frac{1}{1000} m^3)$

diameter $= 7m \Rightarrow$ radius $= \frac{7}{2}$ m

volume of the tank $= \pi r^2 h$ cu. units

$1078 = \frac{22}{7} \times \frac{7}{2} \times \frac{7}{2} \times h$

Therefore, height of the tank is 28 m

**Example 7.18** For the cylinders $A$ and $B$ (Fig. 7.27),

(i) find out the cylinder whose volume is greater.

(ii) verify whether the cylinder with greater volume has greater total surface area.

(iii) find the ratios of the volumes of the cylinders $A$ and $B$.

**Solution**

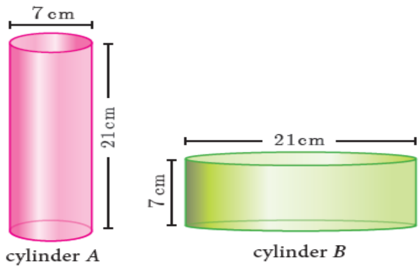
(i) Volume of cylinder $= \pi r^2 h$ cu. units

Volume of cylinder $A = \frac{22}{7} \times \frac{7}{2} \times \frac{7}{2} \times 21$

$= 808.5\, cm^3$

Volume of cylinder $B = \frac{22}{7} \times \frac{21}{2} \times \frac{21}{2} \times 7$

$= 2425.5\, cm^3$




Therefore, volume of cylinder $B$ is greater than volume of cylinder $A$.

(ii) T.S.A. of cylinder $= 2\pi r(h+r)$ sq. units

$$\text{T.S.A. of cylinder } A = 2\times\frac{22}{7}\times\frac{7}{2}\times(21+3.5) = 539 \text{ cm}^2$$

$$\text{T.S.A. of cylinder } B = 2\times\frac{22}{7}\times\frac{21}{2}\times(7+10.5) = 1155 \text{ cm}^2$$

Hence verified that cylinder $B$ with greater volume has a greater surface area.

(iii) $\dfrac{\text{Volume of cylinder A}}{\text{Volume of cylinder B}} = \dfrac{808.5}{2425.5} = \dfrac{1}{3}$

Therefore, ratio of the volumes of cylinders $A$ and $B$ is 1:3.

**Example 7.21** The volume of a solid hemisphere is 29106 cm³. Another hemisphere whose volume is two-third of the above is carved out. Find the radius of the new hemisphere.

**Solution** Let $r$ be the radius of the hemisphere.

Given that, volume of the hemisphere = 29106 cm³

$$\text{Now, volume of new hemisphere} = \frac{2}{3}(\text{Volume of original sphere})$$

$$= \frac{2}{3}\times 29106$$

$$\text{Volume of new hemisphere} = 19404 \text{ cm}^3$$

$$\frac{2}{3}\pi r^3 = 19404$$

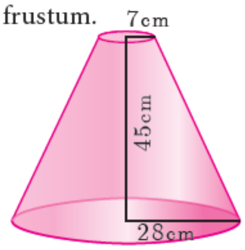
$$r^3 = \frac{19404\times3\times7}{2\times22} = 9261$$

$$r = \sqrt[3]{9261} = 21 \text{ cm}$$

Therefore, $r = 21$ cm

**Example 7.23** If the radii of the circular ends of a frustum which is 45 cm high are 28 cm and 7 cm, find the volume of the frustum.

**Solution** Let $h$, $r$ and $R$ be the height, top and bottom radii of the frustum.



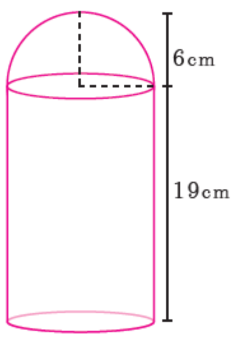

Given that, $h = 45$ cm, $R = 28$ cm, $r = 7$ cm

$$\text{Volume} = \frac{1}{3}\pi\left[R^2 + Rr + r^2\right]h \text{ cu. units}$$

$$= \frac{1}{3}\times\frac{22}{7}\times\left[28^2 + (28\times7) + 7^2\right]\times 45$$

$$= \frac{1}{3}\times\frac{22}{7}\times 1029\times 45 = 48510$$

Therefore, volume of the frustum is 48510 cm³




**Example 7.24** A toy is in the shape of a cylinder surmounted by a hemisphere. The height of the toy is 25 cm. Find the total surface area of the toy if its common diameter is 12 cm.

**Solution** Let $r$ and $h$ be the radius and height of the cylinder respectively.

Given that, diameter $d = 12$ cm, radius $r = 6$ cm

Total height of the toy is 25 cm

Therefore, height of the cylindrical portion $= 25 - 6 = 19$ cm

T.S.A. of the toy = C.S.A. of the cylinder + C.S.A. of the hemisphere + Base Area of the cylinder

$$= 2\pi rh + 2\pi r^2 + \pi r^2$$

$$= \pi r(2h + 3r) \text{ sq. units}$$

$$= \frac{22}{7} \times 6 \times (38 + 18)$$

$$= \frac{22}{7} \times 6 \times 56 = 1056$$

Therefore, T.S.A. of the toy is 1056 $cm^2$

**Example 7.25** A jewel box (Fig. 7.39) is in the shape of a cuboid of dimensions 30 cm × 15 cm × 10 cm surmounted by a half part of a cylinder as shown in the figure. Find the volume of the box.


***Solution*** Let $l$, $b$ and $h_1$ be the length, breadth and height of the cuboid. Also let us take $r$ and $h_2$ be the radius and height of the cylinder.

Now, Volume of the box = Volume of the cuboid + $\frac{1}{2}$(Volume of cylinder)

$$= (l \times b \times h_1) + \frac{1}{2}(\pi r^2 h_2) \text{ cu. units}$$

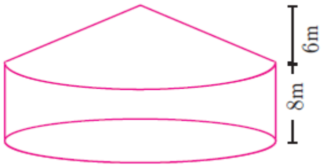
$$= (30 \times 15 \times 10) + \frac{1}{2}\left(\frac{22}{7} \times \frac{15}{2} \times \frac{15}{2} \times 30\right)$$

$$= 4500 + 2651.79 = 7151.79$$

Therefore, Volume of the box = 7151.79 $cm^3$

**Example 7.26** Arul has to make arrangements for the accommodation of 150 persons for his family function. For this purpose, he plans to build a tent which is in the shape of cylinder surmounted by a cone. Each person occupies 4 sq. m of the space on ground and 40 cu. meter of air to breathe. What should be the height of the conical part of the tent if the height of cylindrical part is 8 m?

***Solution*** Let $h_1$ and $h_2$ be the height of cylinder and cone respectively.




Area for one person = 4 sq. m

Total number of persons = 150

Therefore, total base area = 150 × 4

$$\pi r^2 = 600 \quad \ldots (1)$$

Volume of air required for 1 person = 40 $m^3$

Total Volume of air required for 150 persons = 150 × 40 = 6000 $m^3$

$$\pi r^2 h_1 + \frac{1}{3}\pi r^2 h_2 = 6000$$

$$\pi r^2\left(h_1 + \frac{1}{3}h_2\right) = 6000$$

$$600\left(8 + \frac{1}{3}h_2\right) = 6000 \quad \text{[using (1)]}$$

$$8 + \frac{1}{3}h_2 = \frac{6000}{600}$$

$$\frac{1}{3}h_2 = 10 - 8 = 2$$

$$h_2 = 6 \text{ m}$$

Therefore, the height of the conical tent $h_2$ is 6 m

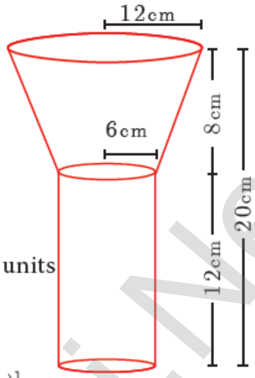
**Example 7.27** A funnel consists of a frustum of a cone attached to a cylindrical portion 12 cm long attached at the bottom. If the total height be 20 cm, diameter of the cylindrical portion be 12 cm and the diameter of the top of the funnel be 24 cm. Find the outer surface area of the funnel.

**Solution** Let $R$, $r$ be the top and bottom radii of the frustum.

Let $h_1$, $h_2$ be the heights of the frustum and cylinder respectively.

Given that, $R = 12$ cm, $r = 6$ cm, $h_2 = 12$ cm

Now, $h_1 = 20 - 12 = 8$ cm




Here, Slant height of the frustum $l = \sqrt{(R-r)^2 + h_1^2}$ units

$= \sqrt{36+64}$

$l = 10$ cm

Outer surface area $= 2\pi r h_2 + \pi(R+r)l$ sq. units

$= \pi[2rh_2 + (R+r)l]$

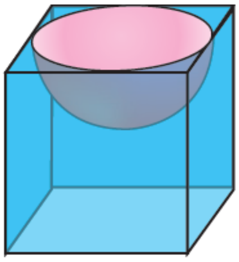
$= \pi[(2\times 6\times 12) + (18\times 10)]$

$= \pi[144+180]$

$= \frac{22}{7}\times 324 = 1018.28$

Therefore, outer surface area of the funnel is 1018.28 cm$^2$

**Example 7.28** A hemispherical section is cut out from one face of a cubical block (Fig.7.42) such that the diameter $l$ of the hemisphere is equal to side length of the cube. Determine the surface area of the remaining solid.



**Solution** Let $r$ be the radius of the hemisphere.

Given that, diameter of the hemisphere = side of the cube = $l$

Radius of the hemisphere = $\frac{l}{2}$

TSA of the remaining solid = Surface area of the cubical part
+ C.S.A. of the hemispherical part
− Area of the base of the hemispherical part

$= 6\times(Edge)^2 + 2\pi r^2 - \pi r^2$

$= 6\times(Edge)^2 + \pi r^2$

$= 6\times(l)^2 + \pi\left(\frac{l}{2}\right)^2 = \frac{1}{4}(24+\pi)l^2$

Total surface area of the remaining solid $= \frac{1}{4}(24+\pi)l^2$ sq. units

**Example 7.31** A right circular cylindrical container of base radius 6 cm and height 15 cm is full of ice cream. The ice cream is to be filled in cones of height 9 cm and base radius 3 cm, having a hemispherical cap. Find the number of cones needed to empty the container.

**Solution** Let $h$ and $r$ be the height and radius of the cylinder respectively.

Given that, $h = 15$ cm, $r = 6$ cm

Volume of the container $V = \pi r^2 h$ cubic units.

$$= \frac{22}{7} \times 6 \times 6 \times 15$$

Let, $r_1 = 3$ cm, $h_1 = 9$ cm be the radius and height of the cone.

Also, $r_1 = 3$ cm is the radius of the hemispherical cap.

Volume of one ice cream cone = (Volume of the cone + Volume of the hemispherical cap)

$$= \frac{1}{3}\pi r_1^2 h_1 + \frac{2}{3}\pi r_1^3$$

$$= \frac{1}{3} \times \frac{22}{7} \times 3 \times 3 \times 9 + \frac{2}{3} \times \frac{22}{7} \times 3 \times 3 \times 3$$

$$= \frac{22}{7} \times 9(3+2) = \frac{22}{7} \times 45$$

$$\text{Number of cones} = \frac{\text{volume of the cylinder}}{\text{volume of one ice cream cone}}$$

$$\text{Number of ice cream cones needed} = \frac{\frac{22}{7} \times 6 \times 6 \times 15}{\frac{22}{7} \times 45} = 12$$

Thus 12 ice cream cones are required to empty the cylindrical container.

# Statistics and Probability

## 2-MARKS (SELECTED SUMS)

**Example 8.1** Find the range and coefficient of range of the following data: 25, 67, 48, 53, 18, 39, 44.

*Solution* Largest value $L = 67$; Smallest value $S = 18$

$$\text{Range R} = L - S = 67 - 18 = 49$$

$$\text{Coefficient of range} = \frac{L-S}{L+S}$$

$$\text{Coefficient of range} = \frac{67-18}{67+18} = \frac{49}{85} = 0.576$$

**Example 8.2** Find the range of the following distribution.

| Age (in years) | 16-18 | 18-20 | 20-22 | 22-24 | 24-26 | 26-28 |
|---|---|---|---|---|---|---|
| Number of students | 0 | 4 | 6 | 8 | 2 | 2 |

*Solution* Here Largest value $L = 28$

Smallest value $S = 18$

Range $R = L - S$

$R = 28 - 18 = 10$ Years

**Example 8.3** The range of a set of data is 13.67 and the largest value is 70.08. Find the smallest value.

*Solution* Range $R = 13.67$

Largest value $L = 70.08$

Range $R = L - S$

$13.67 = 70.08 - S$

$S = 70.08 - 13.67 = 56.41$

Therefore, the smallest value is 56.41.

**Example 8.15** The mean of a data is 25.6 and its coefficient of variation is 18.75. Find the standard deviation.

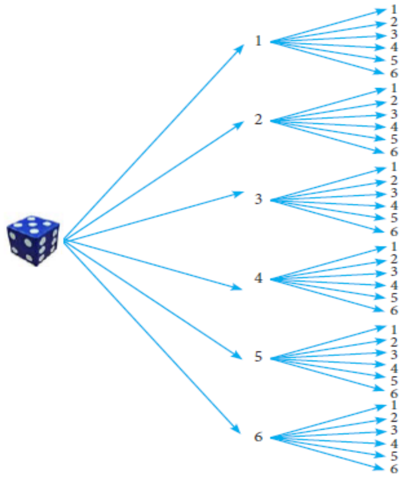
**Solution** Mean $\bar{x} = 25.6$ , Coefficient of variation, C.V. = 18.75

Coefficient of variation, C.V. $= \frac{\sigma}{\bar{x}} \times 100\%$

$$18.75 = \frac{\sigma}{25.6} \times 100 \Rightarrow \sigma = 4.8$$

**Example 8.17** Express the sample space for rolling two dice using tree diagram.

**Solution**




Hence, the sample space can be written as

S= {(1,1),(1,2),(1,3),(1,4),(1,5),(1,6)

(2,1),(2,2),(2,3),(2,4),(2,5),(2,6)

(3,1),(3,2),(3,3),(3,4),(3,5),(3,6)

(4,1),(4,2),(4,3),(4,4),(4,5),(4,6)

(5,1),(5,2),(5,3),(5,4),(5,5),(5,6)

(6,1),(6,2),(6,3),(6,4),(6,5),(6,6)}

**Example 8.18** A bag contains 5 blue balls and 4 green balls. A ball is drawn at random from the bag. Find the probability that the ball drawn is (i) blue (ii) not blue.

**Solution** Total number of possible outcomes $n(S) = 5 + 4 = 9$

(i) Let $A$ be the event of getting a blue ball.

$$n(A) = 5$$

$$P(A) = \frac{n(A)}{n(S)} = \frac{5}{9}$$

(ii) $\bar{A}$ will be the event of not getting a blue ball. So $P(\bar{A}) = 1 - P(A) = 1 - \frac{5}{9} = \frac{4}{9}$

**Example 8.20** Two coins are tossed together. What is the probability of getting different faces on the coins?

**Solution**

$S = \{HH, HT, TH, TT\}$; $n(S) = 4$

$A = \{HT, TH\}$; $n(A) = 2$

$$P(A) = \frac{n(A)}{n(S)} = \frac{2}{4} = \frac{1}{2}$$

**Example 8.22** What is the probability that a leap year selected at random will contain 53 saturdays.

**Solution** A leap year has 366 days. So it has 52 full weeks and 2 days.

$S =$ {Sun-Mon, Mon-Tue, Tue-Wed, Wed-Thu, Thu-Fri, Fri-Sat, Sat-Sun}

$n(S) = 7$

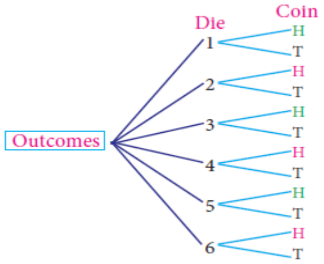
$A =$ {Fri-Sat, Sat-Sun}; $n(A) = 2$

$$P(A) = \frac{n(A)}{n(S)} = \frac{2}{7}$$

**Example 8.23** A die is rolled and a coin is tossed simultaneously. Find the probability that the die shows an odd number and the coin shows a head.

**Solution**

$$S = \{1H, 1T, 2H, 2T, 3H, 3T, 4H, 4T, 5H, 5T, 6H, 6T\};$$
$$n(S) = 12$$




$$A = \{1H, 3H, 5H\};\ n(A) = 3$$
$$P(A) = \frac{n(A)}{n(S)} = \frac{3}{12} = \frac{1}{4}$$

**Example 8.26** If $P(A) = 0.37$, $P(B) = 0.42$, $P(A \cap B) = 0.09$ then find $P(A \cup B)$.

**Solution** $P(A) = 0.37$, $P(B) = 0.42$, $P(A \cap B) = 0.09$

$$P(A \cup B) = P(A) + P(B) - P(A \cap B)$$
$$P(A \cup B) = 0.37 + 0.42 - 0.09 = 0.7$$

**Example 8.27** What is the probability of drawing either a king or a queen in a single draw from a well shuffled pack of 52 cards?

**Solution** Total number of cards $= 52$

Number of king cards $= 4$

Probability of drawing a king card $= \frac{4}{52}$

Number of queen cards $= 4$

Probability of drawing a queen card $= \frac{4}{52}$

Both the events of drawing a king and a queen are mutually exclusive

$$\Rightarrow P(A \cup B) = P(A) + P(B)$$

$\therefore$ Probability of drawing either a king or a queen $= \frac{4}{52} + \frac{4}{52} = \frac{2}{13}$

**Example 8.29** If $A$ and $B$ are two events such that $P(A) = \frac{1}{4}$, $P(B) = \frac{1}{2}$ and $P(A \text{ and } B) = \frac{1}{8}$, find (i) $P(A \text{ or } B)$ (ii) $P(\text{not } A \text{ and not } B)$.

**Solution** (i)

$$P(A \text{ or } B) = P(A \cup B)$$
$$= P(A) + P(B) - P(A \cap B)$$
$$P(A \text{ or } B) = \frac{1}{4} + \frac{1}{2} - \frac{1}{8} = \frac{5}{8}$$

(ii)

$$P(\text{not } A \text{ and not } B) = P(\bar{A} \cap \bar{B})$$
$$= P\left(\overline{A \cup B}\right)$$
$$= 1 - P\left(A \cup B\right)$$
$$P(\text{not } A \text{ and not } B) = 1 - \frac{5}{8} = \frac{3}{8}$$

**Example 8.32** $A$ and $B$ are two candidates seeking admission to IIT. The probability that $A$ getting selected is 0.5 and the probability that both $A$ and $B$ getting selected is 0.3. Prove that the probability of $B$ being selected is atmost 0.8.

**Solution** $P(A) = 0.5$, $P(A \cap B) = 0.3$

We have

$$P(A \cup B) \leq 1$$
$$P(A) + P(B) - P(A \cap B) \leq 1$$
$$0.5 + P(B) - 0.3 \leq 1$$
$$P(B) \leq 1 - 0.2$$
$$P(B) \leq 0.8$$

Therefore, probability of $B$ getting selected is atmost 0.8.

## 5-MARKS (SELECTED SUMS)

**Example 8.4** The number of televisions sold in each day of a week are 13, 8, 4, 9, 7, 12, 10. Find its standard deviation.

*Solution*

| $x_i$ | $x_i^2$ |
|---|---|
| 13 | 169 |
| 8 | 64 |
| 4 | 16 |
| 9 | 81 |
| 7 | 49 |
| 12 | 144 |
| 10 | 100 |
| $\Sigma x_i = 63$ | $\Sigma x_i^2 = 623$ |

Standard deviation

$$\sigma = \sqrt{\frac{\Sigma x_i^2}{n} - \left(\frac{\Sigma x_i}{n}\right)^2}$$

$$= \sqrt{\frac{623}{7} - \left(\frac{63}{7}\right)^2}$$

$$= \sqrt{89 - 81} = \sqrt{8}$$

Hence, $\sigma \simeq 2.83$

**Example 8.5** The amount of rainfall in a particular season for 6 days are given as 17.8 cm, 19.2 cm, 16.3 cm, 12.5 cm, 12.8 cm and 11.4 cm. Find its standard deviation.

*Solution* Arranging the numbers in ascending order we get, 11.4, 12.5, 12.8, 16.3, 17.8, 19.2. Number of observations $n = 6$

$$\text{Mean} = \frac{11.4 + 12.5 + 12.8 + 16.3 + 17.8 + 19.2}{6} = \frac{90}{6} = 15$$

| $x_i$ | $d_i = x_i - \bar{x}$ $= x - 15$ | $d_i^2$ |
|---|---|---|
| 11.4 | −3.6 | 12.96 |
| 12.5 | −2.5 | 6.25 |
| 12.8 | −2.2 | 4.84 |
| 16.3 | 1.3 | 1.69 |
| 17.8 | 2.8 | 7.84 |
| 19.2 | 4.2 | 17.64 |
| | | $\Sigma d_i^2 = 51.22$ |

Standard deviation $\sigma = \sqrt{\frac{\Sigma d_i^2}{n}}$

$$= \sqrt{\frac{51.22}{6}} = \sqrt{8.53}$$

Hence, $\sigma \simeq 2.9$

**Example 8.6** The marks scored by 10 students in a class test are 25, 29, 30, 33, 35, 37, 38, 40, 44, 48. Find the standard deviation.

*Solution* The mean of marks is 35.9 which is not an integer. Hence we take assumed mean, $A = 35$, $n = 10$.

| $x_i$ | $d_i = x_i - A$ $d_i = x_i - 35$ | $d_i^2$ |
|---|---|---|
| 25 | −10 | 100 |
| 29 | −6 | 36 |
| 30 | −5 | 25 |
| 33 | −2 | 4 |
| 35 | 0 | 0 |
| 37 | 2 | 4 |
| 38 | 3 | 9 |
| 40 | 5 | 25 |
| 44 | 9 | 81 |
| 48 | 13 | 169 |
| | $\Sigma d_i = 9$ | $\Sigma d_i^2 = 453$ |

Standard deviation

$$\sigma = \sqrt{\frac{\Sigma d_i^2}{n} - \left(\frac{\Sigma d_i}{n}\right)^2}$$

$$= \sqrt{\frac{453}{10} - \left(\frac{9}{10}\right)^2}$$

$$= \sqrt{45.3 - 0.81}$$

$$= \sqrt{44.49}$$

$$\sigma \simeq 6.67$$

**Example 8.7** The amount that the children have spent for purchasing some eatables in one day trip of a school are 5, 10, 15, 20, 25, 30, 35, 40. Using step deviation method, find the standard deviation of the amount they have spent.

**Solution** We note that all the observations are divisible by 5. Hence we can use the step deviation method. Let the Assumed mean $A = 20$, $n = 8$.

| $x_i$ | $d_i = x_i - A$<br>$d_i = x_i - 20$ | $d_i = \frac{x_i - A}{c}$<br>$c = 5$ | $d_i^2$ |
|---|---|---|---|
| 5 | −15 | −3 | 9 |
| 10 | −10 | −2 | 4 |
| 15 | −5 | −1 | 1 |
| 20 | 0 | 0 | 0 |
| 25 | 5 | 1 | 1 |
| 30 | 10 | 2 | 4 |
| 35 | 15 | 3 | 9 |
| 40 | 20 | 4 | 16 |
| | | $\Sigma d_i = 4$ | $\Sigma d_i^2 = 44$ |

Standard deviation

$$\sigma = \sqrt{\frac{\Sigma d_i^2}{n} - \left(\frac{\Sigma d_i}{n}\right)^2} \times c$$

$$= \sqrt{\frac{44}{8} - \left(\frac{4}{8}\right)^2} \times 5 = \sqrt{\frac{11}{2} - \frac{1}{4}} \times 5$$

$$= \sqrt{5.5 - 0.25} \times 5 = 2.29 \times 5$$

$$\sigma \simeq 11.45$$

**Example 8.8** Find the standard deviation of the following data 7, 4, 8, 10, 11. Add 3 to all the values then find the standard deviation for the new values.

**Solution** Arranging the values in ascending order we get, 4, 7, 8, 10, 11 and $n = 5$

| $x_i$ | $x_i^2$ |
|---|---|
| 4 | 16 |
| 7 | 49 |
| 8 | 64 |
| 10 | 100 |
| 11 | 121 |
| $\Sigma x_i = 40$ | $\Sigma x_i^2 = 350$ |

Standard deviation

$$\sigma = \sqrt{\frac{\Sigma x_i^2}{n} - \left(\frac{\Sigma x_i}{n}\right)^2}$$

$$= \sqrt{\frac{350}{5} - \left(\frac{40}{5}\right)^2}$$

$$\sigma = \sqrt{6} \simeq 2.45$$

When we add 3 to all the values, we get the new values as 7,10,11,13,14.

| $x_i$ | $x_i^2$ |
|---|---|
| 7 | 49 |
| 10 | 100 |
| 11 | 121 |
| 13 | 169 |
| 14 | 196 |
| $\Sigma x_i = 55$ | $\Sigma x_i^2 = 635$ |

Standard deviation

$$\sigma = \sqrt{\frac{\Sigma x_i^2}{n} - \left(\frac{\Sigma x_i}{n}\right)^2}$$

$$= \sqrt{\frac{635}{5} - \left(\frac{55}{5}\right)^2}$$

$$\sigma = \sqrt{6} \simeq 2.45$$

**Example 8.9** Find the standard deviation of the data 2, 3, 5, 7, 8. Multiply each data by 4. Find the standard deviation of the new values.

**Solution** Given, $n = 5$

| $x_i$ | $x_i^2$ |
|---|---|
| 2 | 4 |
| 3 | 9 |
| 5 | 25 |
| 7 | 49 |
| 8 | 64 |
| $\Sigma x_i = 25$ | $\Sigma x_i^2 = 151$ |

Standard deviation $\sigma = \sqrt{\frac{\Sigma x_i^2}{n} - \left(\frac{\Sigma x_i}{n}\right)^2}$

$$\sigma = \sqrt{\frac{151}{5} - \left(\frac{25}{5}\right)^2} = \sqrt{30.2 - 25} = \sqrt{5.2} \simeq 2.28$$

When we multiply each data by 4, we get the new values as 8, 12, 20, 28, 32.

| $x_i$ | $x_i^2$ |
|---|---|
| 8 | 64 |
| 12 | 144 |
| 20 | 400 |
| 28 | 784 |
| 32 | 1024 |
| $\Sigma x_i = 100$ | $\Sigma x_i^2 = 2416$ |

Standard deviation $\sigma = \sqrt{\frac{\Sigma x_i^2}{n} - \left(\frac{\Sigma x_i}{n}\right)^2}$

$$= \sqrt{\frac{2416}{5} - \left(\frac{100}{5}\right)^2} = \sqrt{483.2 - 400} = \sqrt{83.2}$$

$$\sigma = \sqrt{16 \times 5.2} = 4\sqrt{5.2} \simeq 9.12$$

**Example 8.10** Find the mean and variance of the first $n$ natural numbers.

***Solution***

$$\text{Mean } \bar{x} = \frac{\text{Sum of all the observations}}{\text{Number of observations}}$$

$$= \frac{\Sigma x_i}{n} = \frac{1+2+3+...+n}{n} = \frac{n(n+1)}{2 \times n}$$

$$\text{Mean } \bar{x} = \frac{n+1}{2}$$

$$\text{Variance } \sigma^2 = \frac{\Sigma x_i^2}{n} - \left(\frac{\Sigma x_i}{n}\right)^2 \left[\begin{array}{l}\Sigma x_i^2 = 1^2+2^2+3^2+...+n^2 \\ (\Sigma x_i)^2 = (1+2+3+...+n)^2\end{array}\right]$$

$$= \frac{n(n+1)(2n+1)}{6 \times n} - \left[\frac{n(n+1)}{2 \times n}\right]^2$$

$$= \frac{2n^2+3n+1}{6} - \frac{n^2+2n+1}{4}$$

$$\text{Variance } \sigma^2 = \frac{4n^2+6n+2-3n^2-6n-3}{12} = \frac{n^2-1}{12}.$$

**Example 8.11** 48 students were asked to write the total number of hours per week they spent on watching television. With this information find the standard deviation of hours spent for watching television.

| $x$ | 6 | 7 | 8 | 9 | 10 | 11 | 12 |
|---|---|---|---|---|---|---|---|
| $f$ | 3 | 6 | 9 | 13 | 8 | 5 | 4 |

***Solution***

| $x_i$ | $f_i$ | $x_i f_i$ | $d_i = x_i - \bar{x}$ | $d_i^2$ | $f_i d_i^2$ |
|---|---|---|---|---|---|
| 6 | 3 | 18 | −3 | 9 | 27 |
| 7 | 6 | 42 | −2 | 4 | 24 |
| 8 | 9 | 72 | −1 | 1 | 9 |
| 9 | 13 | 117 | 0 | 0 | 0 |
| 10 | 8 | 80 | 1 | 1 | 8 |
| 11 | 5 | 55 | 2 | 4 | 20 |
| 12 | 4 | 48 | 3 | 9 | 36 |
| | $N = 48$ | $\Sigma x_i f_i = 432$ | $\Sigma d_i = 0$ | | $\Sigma f_i d_i^2 = 124$ |

Mean

$$\bar{x} = \frac{\Sigma x_i f_i}{N} = \frac{432}{48} = 9 \quad (\because N = \Sigma f_i)$$

Standard deviation

$$\sigma = \sqrt{\frac{\Sigma f_i d_i^2}{N}} = \sqrt{\frac{124}{48}} = \sqrt{2.58}$$

$$\sigma \simeq 1.6$$

**Example 8.12** The marks scored by the students in a slip test are given below. Find the standard deviation of their marks.

| $x$ | 4 | 6 | 8 | 10 | 12 |
|---|---|---|---|---|---|
| $f$ | 7 | 3 | 5 | 9 | 5 |

***Solution*** Let the assumed mean, $A = 8$

| $x_i$ | $f_i$ | $d_i = x_i - A$ | $f_i d_i$ | $f_i d_i^2$ |
|---|---|---|---|---|
| 4 | 7 | −4 | −28 | 112 |
| 6 | 3 | −2 | −6 | 12 |
| 8 | 5 | 0 | 0 | 0 |
| 10 | 9 | 2 | 18 | 36 |
| 12 | 5 | 4 | 20 | 80 |
| | $N = 29$ | | $\Sigma f_i d_i = 4$ | $\Sigma f_i d_i^2 = 240$ |

Standard deviation

$$\sigma = \sqrt{\frac{\Sigma f_i d_i^2}{N} - \left(\frac{\Sigma f_i d_i}{N}\right)^2}$$

$$= \sqrt{\frac{240}{29} - \left(\frac{4}{29}\right)^2} = \sqrt{\frac{240 \times 29 - 16}{29 \times 29}}$$

$$\sigma = \sqrt{\frac{6944}{29 \times 29}} \Rightarrow \quad \sigma \simeq 2.87$$

**Example 8.13** Marks of the students in a particular subject of a class are given below. Find its standard deviation.

| Marks | 0-10 | 10-20 | 20-30 | 30-40 | 40-50 | 50-60 | 60-70 |
|---|---|---|---|---|---|---|---|
| Number of students | 8 | 12 | 17 | 14 | 9 | 7 | 4 |

***Solution*** Let the assumed mean, $A = 35$, $c = 10$

| Marks | Mid value ($x_i$) | $f_i$ | $d_i = x_i - A$ | $d_i = \frac{x_i - A}{c}$ | $f_i d_i$ | $f_i d_i^2$ |
|---|---|---|---|---|---|---|
| 0-10 | 5 | 8 | -30 | −3 | −24 | 72 |
| 10-20 | 15 | 12 | -20 | −2 | −24 | 48 |
| 20-30 | 25 | 17 | -10 | −1 | −17 | 17 |
| 30-40 | 35 | 14 | 0 | 0 | 0 | 0 |
| 40-50 | 45 | 9 | 10 | 1 | 9 | 9 |
| 50-60 | 55 | 7 | 20 | 2 | 14 | 28 |
| 60-70 | 65 | 4 | 30 | 3 | 12 | 36 |
| | | $N = 71$ | | | $\Sigma f_i d_i = -30$ | $\Sigma f_i d_i^2 = 210$ |

Standard deviation $\sigma = c \times \sqrt{\frac{\Sigma f_i d_i^2}{N} - \left(\frac{\Sigma f_i d_i}{N}\right)^2}$

$$\sigma = 10 \times \sqrt{\frac{210}{71} - \left(-\frac{30}{71}\right)^2} = 10 \times \sqrt{\frac{210}{71} - \frac{900}{5041}}$$

$$= 10 \times \sqrt{2.779}\ ; \quad \sigma \simeq 16.67$$

**Example 8.14** The mean and standard deviation of 15 observations are found to be 10 and 5 respectively. On rechecking it was found that one of the observation with value 8 was incorrect. Calculate the correct mean and standard deviation if the correct observation value was 23?

**Solution** $n = 15,\ \overline{x} = 10,\ \sigma = 5;\quad \overline{x} = \frac{\Sigma x}{n};\quad \Sigma x = 15 \times 10 = 150$

Wrong observation value $= 8$, Correct observation value $= 23$.

Correct total $= 150 - 8 + 23 = 165$

Correct mean $\overline{x} = \frac{165}{15} = 11$

Standard deviation $\sigma = \sqrt{\frac{\Sigma x^2}{n} - \left(\frac{\Sigma x}{n}\right)^2}$

Incorrect value of $\sigma = 5 = \sqrt{\frac{\Sigma x^2}{15} - (10)^2}$

$25 = \frac{\Sigma x^2}{15} - 100 \Rightarrow \frac{\Sigma x^2}{15} = 125$

Incorrect value of $\Sigma x^2 = 1875$

Correct value of $\Sigma x^2 = 1875 - 8^2 + 23^2 = 2340$

Correct standard deviation $\sigma = \sqrt{\frac{2340}{15} - (11)^2}$

$\sigma = \sqrt{156 - 121} = \sqrt{35} \quad \sigma \simeq 5.9$

**Example 8.16** The following table gives the values of mean and variance of heights and weights of the 10th standard students of a school.

| | Height | Weight |
|---|---|---|
| Mean | 155 cm | 46.50 kg |
| Variance | 72.25 cm$^2$ | 28.09 kg |

Which is more varying than the other?

**Solution** For comparing two data, first we have to find their coefficient of variations

Mean $\overline{x}_1 = 155$cm, variance $\sigma_1^2 = 72.25$ cm$^2$

Therefore standard deviation $\sigma_1 = 8.5$

Coefficient of variation $C.V_1 = \frac{\sigma_1}{\overline{x}_1} \times 100\%$

$C.V_1 = \frac{8.5}{155} \times 100\% = 5.48\%$ (for heights)

Mean $\overline{x}_2 = 46.50$ kg, Variance $\sigma_2^2 = 28.09$ kg$^2$

Standard deviation $\sigma_2 = 5.3$ kg

Coefficient of variation $C.V_2 = \frac{\sigma_2}{\bar{x}_2} \times 100\ \%$

$$C.V_2 = \frac{5.3}{46.50} \times 100\% = 11.40\% \text{ (for weights)}$$

$$C.V_1 = 5.48\% \text{ and } C.V_2 = 11.40\%$$

Height is more consistent.

**Example 8.19** Two dice are rolled. Find the probability that the sum of outcomes is (i) equal to 4 (ii) greater than 10 (iii) less than 13

**Solution** When we roll two dice, the sample space is given by

$$S = \{ (1,1),(1,2),(1,3),(1,4),(1,5),(1,6)$$
$$(2,1),(2,2),(2,3),(2,4),(2,5),(2,6)$$
$$(3,1),(3,2),(3,3),(3,4),(3,5),(3,6)$$
$$(4,1),(4,2),(4,3),(4,4),(4,5),(4,6)$$
$$(5,1),(5,2),(5,3),(5,4),(5,5),(5,6)$$
$$(6,1),(6,2),(6,3),(6,4),(6,5),(6,6) \};\ n(S) = 36$$

(i) Let $A$ be the event of getting the sum of outcome values equal to 4.

Then $A = \{(1,3),(2,2),(3,1)\};\ n(A) = 3$.

Probability of getting the sum of outcomes equal to 4 is $P(A) = \frac{n(A)}{n(S)} = \frac{3}{36} = \frac{1}{12}$

(ii) Let B be the event of getting the sum of outcome values greater than 10.

Then $B = \{(5,6),(6,5),(6,6)\};\ n(B) = 3$

Probability of getting the sum of outcomes greater than 10 is $P(B) = \frac{n(B)}{n(S)} = \frac{3}{36} = \frac{1}{12}$

(iii) Let C be the event of getting the sum of outcomes less than 13. Here all the outcomes have the sum value less than 13. Hence $C = S$.

Therefore, $n(C) = n(S) = 36$

Probability of getting the total value less than 13 is $P(C) = \frac{n(C)}{n(S)} = \frac{36}{36} = 1$.

**Example 8.21** From a well shuffled pack of 52 cards, one card is drawn at random. Find the probability of getting (i) red card (ii) heart card (iii) red king (iv) face card (v) number card

| Suits of playing cards | Spade ♠ | Heart ♥ | Clavor ♣ | Diamond ♦ |
|---|---|---|---|---|
| Cards of each suit | A | A | A | A |
| | 2 | 2 | 2 | 2 |
| | 3 | 3 | 3 | 3 |
| | 4 | 4 | 4 | 4 |
| | 5 | 5 | 5 | 5 |
| | 6 | 6 | 6 | 6 |
| | 7 | 7 | 7 | 7 |
| | 8 | 8 | 8 | 8 |
| | 9 | 9 | 9 | 9 |
| | 10 | 10 | 10 | 10 |
| | J | J | J | J |
| | Q | Q | Q | Q |
| | K | K | K | K |
| Set of playing cards in each suit | 13 | 13 | 13 | 13 |

**Solution** $n(S) = 52$

(i) Let $A$ be the event of getting a red card.

$n(A) = 26$

Probability of getting a red card is

$$P(A) = \frac{26}{52} = \frac{1}{2}$$

(ii) Let $B$ be the event of getting a heart card.

$n(B) = 13$

Probability of getting a heart card is

$$P(B) = \frac{n(B)}{n(S)} = \frac{13}{52} = \frac{1}{4}$$

(iii) Let $C$ be the event of getting a red king card. A red king card can be either a diamond king or a heart king.

$n(C) = 2$

Probability of getting a red king card is

$$P(C) = \frac{n(C)}{n(S)} = \frac{2}{52} = \frac{1}{26}$$

(iv) Let $D$ be the event of getting a face card. The face cards are Jack ($J$), Queen ($Q$), and King ($K$).

$$n(D) = 4 \times 3 = 12$$

Probability of getting a face card is

$$P(D) = \frac{n(D)}{n(S)} = \frac{12}{52} = \frac{3}{13}$$

(v) Let $E$ be the event of getting a number card. The number cards are 2, 3, 4, 5, 6, 7, 8, 9 and 10.

$$n(E) = 4 \times 9 = 36$$

Probability of getting a number card is

$$P(E) = \frac{n(E)}{n(S)} = \frac{36}{52} = \frac{9}{13}$$

**Example 8.24** A bag contains 6 green balls, some black and red balls. Number of black balls is as twice as the number of red balls. Probability of getting a green ball is thrice the probability of getting a red ball. Find (i) number of black balls (ii) total number of balls.

***Solution*** Number of green balls is $n(G) = 6$

Let number of red balls is $n(R) = x$

Therefore, number of black balls is $n(B) = 2x$

Total number of balls $n(S) = 6 + x + 2x = 6 + 3x$

It is given that, $P(G) = 3 \times P(R)$

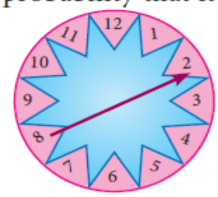
$$\frac{6}{6+3x} = 3 \times \frac{x}{6+3x}$$

$3x = 6$ gives, $x = 2$

(i) Number of black balls $= 2 \times 2 = 4$

(ii) Total number of balls $= 6 + (3 \times 2) = 12$

**Example 8.25** A game of chance consists of spinning an arrow which is equally likely to come to rest pointing to one of the numbers 1, 2, 3, …12. What is the probability that it will point to (i) 7 (ii) a prime number (iii) a composite number?




***Solution*** Sample space $S = \{1,2,3,4,5,6,7,8,9,10,11,12\}$; $n(S) = 12$

(i) Let $A$ be the event of resting in 7. $n(A)=1$

$$P(A) = \frac{n(A)}{n(S)} = \frac{1}{12}$$

(ii) Let $B$ be the event that the arrow will come to rest in a prime number.

$B = \{2,3,5,7,11\}$; $n(B) = 5$

$$P(B) = \frac{n(B)}{n(S)} = \frac{5}{12}$$

(iii) Let $C$ be the event that arrow will come to rest in a composite number.

$C = \{4,6,8,9,10,12\}$; $n(C) = 6$

$$P(C) = \frac{n(C)}{n(S)} = \frac{6}{12} = \frac{1}{2}$$

**Example 8.28** Two dice are rolled together. Find the probability of getting a doublet or sum of faces as 4.

***Solution*** When two dice are rolled together, there will be $6 \times 6 = 36$ outcomes. Let $S$ be the sample space. Then $n(S) = 36$

Let $A$ be the event of getting a doublet and $B$ be the event of getting face sum 4.

Then $A = \{(1,1),(2,2),(3,3),(4,4),(5,5),(6,6)\}$

$B = \{(1,3),(2,2),(3,1)\}$

$\therefore \quad A \cap B = \{(2,2)\}$

Then, $n(A) = 6$, $n(B) = 3$, $n(A \cap B) = 1$.

$$P(A) = \frac{n(A)}{n(S)} = \frac{6}{36}$$

$$P(B) = \frac{n(B)}{n(S)} = \frac{3}{36}$$

$$P(A \cap B) = \frac{n(A \cap B)}{n(S)} = \frac{1}{36}$$

$\therefore P$ (getting a doublet or a total of 4) $= P(A \cup B)$

$$P(A \cup B) = P(A) + P(B) - P(A \cap B)$$

$$= \frac{6}{36} + \frac{3}{36} - \frac{1}{36} = \frac{8}{36} = \frac{2}{9}$$

Hence, the required probability is $\frac{2}{9}$.

**Example 8.30** $A$ card is drawn from a pack of 52 cards. Find the probability of getting a king or a heart or a red card.

***Solution*** Total number of cards $= 52$; $n(S) = 52$

| Suits of playing cards | Spade | Heart | Clavor | Diamond |
|---|---|---|---|---|
| Cards of each suit | A | A | A | A |
| | 2 | 2 | 2 | 2 |
| | 3 | 3 | 3 | 3 |
| | 4 | 4 | 4 | 4 |
| | 5 | 5 | 5 | 5 |
| | 6 | 6 | 6 | 6 |
| | 7 | 7 | 7 | 7 |
| | 8 | 8 | 8 | 8 |
| | 9 | 9 | 9 | 9 |
| | 10 | 10 | 10 | 10 |
| | J | J | J | J |
| | Q | Q | Q | Q |
| | K | K | K | K |
| Set of playing cards in each suit | 13 | 13 | 13 | 13 |

Let $A$ be the event of getting a king card. $n(A) = 4$

$$P(A) = \frac{n(A)}{n(S)} = \frac{4}{52}$$

Let $B$ be the event of getting a heart card. $n(B) = 13$

$$P(B) = \frac{n(B)}{n(S)} = \frac{13}{52}$$

Let $C$ be the event of getting a red card. $n(C) = 26$

$$P(C) = \frac{n(C)}{n(S)} = \frac{26}{52}$$

$$P(A \cap B) = P(\text{getting heart king}) = \frac{1}{52}$$

$$P(B \cap C) = P(\text{getting red and heart})) = \frac{13}{52}$$

$$P(A \cap C) = P(\text{getting red king}) = \frac{2}{52}$$

$$P(A \cap B \cap C) = P(\text{getting heart, king which is red}) = \frac{1}{52}$$

Therefore, required probability is

$$P(A \cup B \cup C) = P(A) + P(B) + P(C) - P(A \cap B) - P(B \cap C) - P(C \cap A) + P(A \cap B \cap C)$$

$$= \frac{4}{52} + \frac{13}{52} + \frac{26}{52} - \frac{1}{52} - \frac{13}{52} - \frac{2}{52} + \frac{1}{52} = \frac{28}{52} = \frac{7}{13}$$

**Example 8.31** In a class of 50 students, 28 opted for NCC, 30 opted for NSS and 18 opted both NCC and NSS. One of the students is selected at random. Find the probability that

(i) The student opted for NCC but not NSS.

(ii) The student opted for NSS but not NCC.
(iii) The student opted for exactly one of them.

*Solution* Total number of students $n(S) = 50$.

Let $A$ and $B$ be the events of students opted for NCC and NSS respectively.

$n(A) = 28$, $n(B) = 30$, $n(A \cap B) = 18$

$$P(A) = \frac{n(A)}{n(S)} = \frac{28}{50}$$

$$P(B) = \frac{n(B)}{n(S)} = \frac{30}{50}$$

$$P(A \cap B) = \frac{n(A \cap B)}{n(S)} = \frac{18}{50}$$

(i) Probability of the students opted for NCC but not NSS

$$P(A \cap \bar{B}) = P(A) - P(A \cap B) = \frac{28}{50} - \frac{18}{50} = \frac{1}{5}$$

(ii) Probability of the students opted for NSS but not NCC.

$$P(\bar{A} \cap B) = P(B) - P(A \cap B) = \frac{30}{50} - \frac{18}{50} = \frac{6}{25}$$

(iii) Probability of the students opted for exactly one of them

$$= P[(A \cap \bar{B}) \cup (\bar{A} \cap B)]$$

$$= P(A \cap \bar{B}) + P(\bar{A} \cap B) = \frac{1}{5} + \frac{6}{25} = \frac{11}{25}$$

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