SRIMAAN

TNPSC CTSE

Combined Technical Services Examination (Non-Interview Posts)

MATHEMATICS

UNIT-2- CALCULUS

TNPSC-CTSE STUDY MATERIALS

TNPSC-CTSE

Combined Technical Services Examination (Non - Interview Posts)

ஒருங்கிணைந்த தொழில் நுட்ப பணிகள் தேர்வு

MATHEMATICS

UNIT-2: CALCULUS

Differential Calculus: nth derivative

Introduction :

Successive Differentiation is the process of differentiating a given function successively n times and the results of such differentiation are called successive derivatives. The higher order differential coefficients are of utmost importance in scientific and engineering applications.

Let $f(x)$ be a differentiable function and let its successive derivatives be denoted by $f'(x), f''(x), \dots, f^{(n)}(x)$.

Common notations of higher order Derivatives of $y = f(x)$

1^{st} **Derivative:** $f'(x)$ or y' or y_1 or $\frac{dy}{dx}$ or Dy

2^{nd} **Derivative:** $f''(x)$ or y'' or y_2 or $\frac{d^2y}{dx^2}$ or D^2y

⋮

n^{th} **Derivative:** $f^{(n)}(x)$ or $y^{(n)}$ or y_n or $\frac{d^ny}{dx^n}$ or D^ny

Calculation of n^{th} derivative

i. n^{th} Derivative of e^{ax+b}

Let $y = e^{ax+b}$

$y_2 = a^2 e^{ax+b}$

⋮

$y_n = a^n e^{ax+b}$

ii. n^{th} Derivative of $y = \log(ax+b)$

Let $y = \log(ax+b)$

$y_1 = \frac{a}{(ax+b)}$

$y_2 = \frac{-a^2}{(ax+b)^2}$

$y_3 = \frac{2!\, a^3}{(ax+b)^3}$

⋮

$y_n = (-1)^{n-1} \frac{(n-1)!\, a^n}{(ax+b)^n}$

iii. n^{th} Derivative of $(ax+b)^m$

Case 1: When m is a positive integer i.e., when $m > 0, m \geq n$

Let $y = (ax+b)^m$

$y_1 = m\,a(ax+b)^{m-1}$

$y_2 = m(m-1)a^2(ax+b)^{m-2}$

$$y_3 = a^3 m(m-1)(m-2)(ax+b)^{m-3}$$

$\vdots$

$$y_n = m(m-1)\ldots(m-n+1)a^n(ax+b)^{m-n}$$

$$y_n = \frac{m!}{(m-n)!}a^n(ax+b)^{m-n}$$

Case 2: When m is a negative integer i.e.,when $m > 0, m < n$Put $m = -p$ where p is a positive integer in the above result.

$D^n(ax+b) \quad = a^n m(m-1)(m-2)\ldots\big(m-(n-1)\big)(ax+b)^{m-n}$

$D^n(ax+b) \quad = a^n(-p)(-p-1)(-p-2)\ldots\big(-p-(n-1)\big)(ax+b)^{-p-n}$ D^n

$(ax+b) \quad = a^n(-1)^n(p)(p+1)(p+2)\ldots(p+n-1)(ax+b)^{-p-n}$

Multiply and divide by $1\,.\,2\,.\,3\,.\ldots\,(p-1)$ to the rhs of above equation

Then $D^n(ax+b)^{-p} = (-1)^n\frac{(p-n+1)!}{(p-1)!}\;a^n(ax+b)^{-p-n}$

Change p to m we get

$$\boldsymbol{D^n(ax+b)^{-m} = (-1)^n\frac{(m-n+1)!}{(m-1)!}\;a^n(ax+b)^{-m-n}}$$

iv. n^{th} Derivative of $y = \sin(ax+b)$

Let $y = \sin(ax+b)$

$$y_1 = a\cos(ax+b) = a\sin\left(ax+b+\frac{\pi}{2}\right)$$

$$y_2 = a^2\cos\left(ax+b+\frac{\pi}{2}\right) = a^2\sin\left(ax+b+\frac{2\pi}{2}\right)$$

$\vdots$

$$y_n = a^n\sin\left(ax+b+\frac{n\pi}{2}\right)$$

Similarly if $y = \cos(ax+b)$

$$y_n = a^n\cos\left(ax+b+\frac{n\pi}{2}\right)$$

v. n^{th} Derivative of $y = e^{ax}\sin(ax+b)$

Let $y = e^{ax}\sin(bx+c)$

$$y_1 = a\,e^{ax}\sin(bx+c) + e^{ax}b\cos(bx+c)$$

$$= e^{ax}\,[a\sin(bx+c) + b\cos(bx+c)]$$

$$= e^{ax}\,[\,r\,cos\alpha\sin(bx+c) + r\,sin\alpha\cos(bx+c)]$$

Putting $a = r\,cos\alpha,\; b = r\,sin\alpha$

$$= e^{ax}\;r\;\sin(bx+c+\alpha)$$

Similarly $y_2 = e^{ax}\;r^2\;\sin(bx+c+2\alpha)$

$\vdots$

$$y_n = e^{ax}\;r^n\;\sin(bx+c+n\alpha)$$

where $r^2 = a^2+b^2$ and $\tan\alpha = \frac{b}{a}$

Similarly if $y = e^{ax}\cos(ax+b)$

$$y_n = e^{ax}\ r^n\ \cos(bx+c+n\alpha)$$
$$= e^{ax}\ (a^2+b^2)^{\frac{n}{2}}\ \cos\left(bx+c+n\tan^{-1}\frac{b}{a}\right)$$

Summary of Results

Function	n[th] derivative
e^{ax+b}	$a^n e^{ax+b}$
$\log(ax+b)$	$(-1)^{n-1}\frac{(n-1)!\,a^n}{(ax+b)^n}$
$(ax+b)^m$	$\frac{m!}{(m-n)!}a^n(ax+b)^{m-n}$ when $m>0$ *and* $m>n$ 0 when $0<m<n$
$(ax+b)^{-m}$	$(-1)^n\frac{(m-n+1)!}{(m-1)!}\quad a^n(ax+b)^{-m-n}$
$\sin(ax+b)$	$a^n\sin\left(ax+b+\frac{n\pi}{2}\right)$
$\cos(ax+b)$	$a^n\cos\left(ax+b+\frac{n\pi}{2}\right)$
$e^{ax}\sin(ax+b)$	$e^{ax}\ (a^2+b^2)^{\frac{n}{2}}\sin\left(bx+c+n\tan^{-1}\frac{b}{a}\right)$
$e^{ax}\cos(ax+b)$	$e^{ax}\ (a^2+b^2)^{\frac{n}{2}}\cos\left(bx+c+n\tan^{-1}\frac{b}{a}\right)$

Example 1 Find the n^{th} derivative of $\frac{1}{1-5x+6x^2}$

Solution: Let $y = \frac{1}{1-5x+6x^2}$

Resolving into partial fractions

$$y = \frac{1}{1-5x+6x^2} = \frac{1}{(1-3x)(1-2x)} = \frac{3}{1-3x} - \frac{2}{1-2x}$$

$$\therefore y_n = \frac{3\,(-3)^n(-1)^n n!}{(1-3x)^{n+1}} - \frac{2(-2)^n(-1)^n n!}{(1-2x)^{n+1}}$$

$$\Rightarrow\ y_n = (-1)^{n+1}\,n!\left[\left(\frac{3}{1-3x}\right)^{n+1} - \left(\frac{2}{1-2x}\right)^{n+1}\right]$$

Example 2 Find the n^{th} derivative of $\sin 6x\cos 4x$

Solution: Let y $= \sin 6x\cos 4x$

$$= \frac{1}{2}(\sin 10\,x + \cos 2\,x)$$

$$\therefore y_n = \frac{1}{2}\left[10^n\sin\left(10x+\frac{n\pi}{2}\right) + 2^n\cos\left(2x+\frac{n\pi}{2}\right)\right]$$

Example 3 Find n^{th} derivative of $sin^2x cos^3x$

Solution: Let y = $sin^2x cos^3x$

$$= sin^2x cos^2x \; cosx$$
$$= \frac{1}{4} sin^2 2x \; cosx \; = \frac{1}{8}(1 - cos4x)cosx$$
$$= \frac{1}{8} cosx - \frac{1}{8} cos4x \; cosx$$
$$= \frac{1}{8} cosx - \frac{1}{16}(cos3x + cos5x)$$
$$= \frac{1}{16}(2cosx - cos3x - cos5x)$$
$$\therefore y_n = \frac{1}{16}\left[2cos\left(x + \frac{n\pi}{2}\right) - 3^n cos\left(3x + \frac{n\pi}{2}\right) - 5^n cos\left(5x + \frac{n\pi}{2}\right)\right]$$

Example 4 Find the n^{th} derivative of sin^4x

Solution: Let $y = sin^4x = (sin^2x)^2$

$$= \left(\frac{1}{2} 2 \; sin^2x\right)^2$$
$$= \frac{1}{4}((1 - \cos 2x)^2$$
$$= \frac{1}{4}\left[1 - 2cos2x + \frac{1}{2}(2cos^2 2x)\right]$$
$$= \frac{1}{4}\left[1 - 2cos2x + \frac{1}{2}(1 + cos4x)\right]$$
$$= \frac{3}{8} - \frac{1}{2} cos2x + \frac{1}{8} cos4x$$
$$\therefore y_n = -\frac{1}{2} 2^n cos\left(2x + \frac{n\pi}{2}\right) + \frac{1}{8} 4^n cos\left(4x + \frac{n\pi}{2}\right)$$

Example 5 Find the n^{th} derivative of $e^{3x} cosx \sin^2 2x$

Solution: Let $y = e^{3x} cosx \sin^2 2x$

Now $\cos x \sin^2 2x = \frac{1}{2}(\cos x - \cos x \cos 4x)$

$\because \sin^2 2x = \frac{1}{2}(1 - \cos 4x)$

$$= \frac{1}{2}\left(\cos x - \frac{1}{2}(\cos 5x + \cos 3x\right)$$
$$\Rightarrow y = e^{3x} cosx \sin^2 2x = \frac{1}{2} e^{3x} \cos x - \frac{1}{4} e^{3x} \cos 5x - \frac{1}{4} e^{3x} \cos 3x$$
$$\therefore y_n = \frac{1}{2} e^{3x}(9+1)^{\frac{n}{2}} \cos\left(x + n\tan^{-1}\frac{1}{3}\right) - \frac{1}{4} e^{3x}(9+25)^{\frac{n}{2}} \cos\left(5x + n\tan^{-1}\frac{5}{3}\right)$$
$$- \frac{1}{4} e^{3x}(9+9)^{\frac{n}{2}} \cos\left(3x + n\tan^{-1}\frac{3}{3}\right)$$
$$= \frac{1}{2} e^{3x} 10^{\frac{n}{2}} \cos\left(x + n\tan^{-1}\frac{1}{3}\right) - \frac{1}{4} e^{3x} 34^{\frac{n}{2}} \cos\left(5x + n\tan^{-1}\frac{5}{3}\right)$$
$$- \frac{1}{4} e^{3x} 18^{\frac{n}{2}} \cos(3x + n\tan^{-1} 1)$$

Example 6 If $y = \sin ax + \cos ax$, prove that $y_n = a^n[1 + (-1)^n \sin 2ax]^{\frac{1}{2}}$

Solution: $y = \sin ax + \cos ax$

$$\therefore y_n = a^n\left[\sin\left(ax + \frac{n\pi}{2}\right) + \cos\left(ax + \frac{n\pi}{2}\right)\right]$$
$$= a^n\left[\left\{\sin\left(ax + \frac{n\pi}{2}\right) + \cos\left(ax + \frac{n\pi}{2}\right)\right\}^2\right]^{\frac{1}{2}}$$
$$= a^n\left[\sin^2\left(ax + \frac{n\pi}{2}\right) + \cos^2\left(ax + \frac{n\pi}{2}\right) + 2\sin\left(ax + \frac{n\pi}{2}\right).\cos\left(ax + \frac{n\pi}{2}\right)\right]^{\frac{1}{2}}$$
$$= a^n[1 + \sin(2ax + n\pi)]^{\frac{1}{2}}$$
$$= a^n[1 + \sin 2ax \cos n\pi + \cos 2ax \sin n\pi]^{\frac{1}{2}}$$
$$= a^n[1 + (-1)^n \sin 2ax]^{\frac{1}{2}} \quad \because \cos n\pi = (-1)^n \text{ and } \sin n\pi = 0$$

Example 7 Find the n^{th} derivative of $\tan^{-1}\frac{x}{a}$

Solution: Let $y = \tan^{-1}\frac{x}{a}$

$$\Rightarrow y_1 = \frac{dy}{dx} = \frac{1}{a\left(1+\frac{x^2}{a^2}\right)} = \frac{a}{x^2+a^2} = \frac{a}{x^2-(ai)^2}$$

$$= \frac{a}{(x+ai)(x-ai)} = \frac{a}{2ai}\left(\frac{1}{x-ai} - \frac{1}{x+ai}\right)$$

$$= \frac{1}{2i}\left(\frac{1}{x-ai} - \frac{1}{x+ai}\right)$$

Differentiating above $(n-1)$ times w.r.t. x, we get

$$y_n = \frac{1}{2i}\left[\frac{(-1)^{n-1}(n-1)!}{(x-ai)^n} - \frac{(-1)^{n-1}(n-1)!}{(x+ai)^n}\right]$$

Substituting $x = r\cos\theta,\ a = r\sin\theta$ such that $\theta = \tan^{-1}\frac{x}{a}$

$$\Rightarrow y_n = \frac{(-1)^{n-1}(n-1)!}{2i}\left[\frac{1}{r^n(\cos\theta - i\sin\theta)^n} - \frac{1}{r^n(\cos\theta + i\sin\theta)^n}\right]$$

$$= \frac{(-1)^{n-1}(n-1)!}{2ir^n}\left[(\cos\theta - i\sin\theta)^{-n} - (\cos\theta + i\sin\theta)^{-n}\right]$$

Using De Moivre's theorem, we get

$$y_n = \frac{(-1)^{n-1}(n-1)!}{2ir^n}\left[\cos n\theta + i\sin n\theta - \cos n\theta + i\sin n\theta\right]$$

$$= \frac{(-1)^{n-1}(n-1)!}{r^n}\sin n\theta$$

$$= \frac{(-1)^{n-1}(n-1)!}{\left(\frac{a}{\sin\theta}\right)^n}\sin n\theta \qquad \because a = r\sin\theta$$

$$= \frac{(-1)^{n-1}(n-1)!}{a^n}\sin n\theta\ \sin^n\theta \qquad \text{where } \theta = \tan^{-1}\frac{a}{x}$$

Example 8 Find the n^{th} derivative of $\frac{1}{1+x+x^2}$

Solution: Let $y = \frac{1}{1+x+x^2}$

$$= \frac{1}{(x-w)(x-w^2)} \quad \text{where } w = \frac{-1+i\sqrt{3}}{2} \text{ and } w^2 = \frac{-1-i\sqrt{3}}{2}$$

Resolving into partial fractions

$$y = \frac{1}{w-w^2}\left(\frac{1}{x-w} - \frac{1}{x-w^2}\right)$$

$$= \frac{1}{i\sqrt{3}}\left(\frac{1}{x-w} - \frac{1}{x-w^2}\right) = \frac{-i}{\sqrt{3}}\left(\frac{1}{x-w} - \frac{1}{x-w^2}\right)$$

Differentiating n times w.r.t. , we get

$$y_n = \frac{-i}{\sqrt{3}}\left[\frac{(-1)^n n!}{(x-w)^{n+1}} - \frac{(-1)^n n!}{(x-w^2)^{n+1}}\right]$$

$$= \frac{-i\,(-1)^n n!}{\sqrt{3}}\left[\frac{1}{(x-w)^{n+1}} - \frac{1}{(x-w^2)^{n+1}}\right]$$

$$= \frac{i\,(-1)^{n+1} n!}{\sqrt{3}}\left[\frac{1}{\left(x+\frac{1}{2}-\frac{i\sqrt{3}}{2}\right)^{n+1}} - \frac{1}{\left(x+\frac{1}{2}+\frac{i\sqrt{3}}{2}\right)^{n+1}}\right]$$

$$= \frac{i\,2^{n+1}\,(-1)^{n+1} n!}{\sqrt{3}}\left[\frac{1}{(2x+1-i\sqrt{3})^{n+1}} - \frac{1}{(2x+1+i\sqrt{3})^{n+1}}\right]$$

Substituting $2x+1 = r\,cos\theta$, $\sqrt{3} = r\,sin\theta$ such that $\theta = \tan^{-1}\frac{\sqrt{3}}{2x+1}$

$$y_n = \frac{i\,2^{n+1}(-1)^{n+1}n!}{\sqrt{3}\,r^{n+1}}\left[(cos\theta - isin\theta)^{-(n+1)} - (cos\theta + isin\theta)^{-(n+1)}\right]$$

Using De Moivre's theorem, we get

$$y_n = \frac{i\,2^{n+1}(-1)^{n+1}n!}{\sqrt{3}\left(\frac{\sqrt{3}}{sin\theta}\right)^{n+1}}\left[cos(n+1)\theta + i\,sin(n+1)\theta - cos(n+1)\theta + isin(n+1)\theta\right]$$

$$\because \sqrt{3} = r\,sin\theta$$

$$= \frac{i\,2^{n+1}(-1)^{n+1}n!}{(\sqrt{3})^{n+2}}\,2i\,sin(n+1)\theta\,sin^{n+1}\theta$$

$$= \frac{(-2)^{n+2}n!}{\sqrt{3}^{n+2}}\,sin(n+1)\theta\,sin^{n+1}\theta \quad \text{where } \theta = \tan^{-1}\frac{\sqrt{3}}{2x+1}$$

Example 9 If $y = x + \tan x$, show that $\cos^2 x\frac{d^2y}{dx^2} - 2y + 2x = 0$

Solution: $y = x + \tan x$

$$\Rightarrow \frac{dy}{dx} = 1 + sec^2 x$$

$$\frac{d^2y}{dx^2} = 2\sec x(\sec x\tan x) = 2\,sec^2 x\tan x$$

$$\therefore \cos^2 x\frac{d^2y}{dx^2} - 2y + 2x = 2\cos^2 x\,sec^2 x\tan x - 2(x+\tan x) + 2x$$

$$= 2\tan x - 2x - 2\tan x + 2x$$

$$= 0$$

Example 10 If $y = \log\left(x + \sqrt{x^2+1}\right)$, show that $(1+x^2)\frac{d^2y}{dx^2} + x\frac{dy}{dx} = 0$

Solution: $y = \log\left(x + \sqrt{x^2+1}\right)$

$$\Rightarrow \frac{dy}{dx} = \frac{1+\frac{x}{\sqrt{1+x^2}}}{x+\sqrt{1+x^2}} = \frac{1}{\sqrt{1+x^2}}$$

$$\Rightarrow (\sqrt{1+x^2})\frac{dy}{dx} = 1$$

Differentiating both sides w.r.t. x , we get

$$(\sqrt{1+x^2})\frac{d^2y}{dx^2} + \frac{x}{\sqrt{1+x^2}}\frac{dy}{dx} = 0$$

$$\Rightarrow (1+x^2)\frac{d^2y}{dx^2} + x\frac{dy}{dx} = 0$$

Leibnitz's theorem and its applications

LEIBNITZ'S THEOREM

If u and v are functions of x such that their n^{th} derivatives exist, then the n^{th} derivative of their product is given by

$$(u\,v)_n = u_n v + n_{C_1}u_{n-1}v_1 + n_{C_2}u_{n-2}v_2 + \cdots + n_{C_r}u_{n-r}v_r + \cdots + uv_n$$

where u_r and v_r represent r^{th} derivatives of u and v respectively.

Example11 Find the n^{th} derivative of $x\log x$

Solution: Let $u = \log x$ and $v = x$

Then $u_n = (-1)^{n-1}\frac{(n-1)!}{x^n}$ and $u_{n-1} = (-1)^{n-2}\frac{(n-2)!}{x^{n-1}}$

By Leibnitz's theorem, we have

$$(u\,v)_n = u_n v + n_{C_1} u_{n-1} v_1 + n_{C_2} u_{n-2} v_2 + \cdots + n_{C_r} u_{n-r} v_r + \cdots + u v_n$$

$$\Rightarrow (x\log x)_n = (-1)^{n-1}\frac{(n-1)!}{x^n}x + n(-1)^{n-2}\frac{(n-2)!}{x^{n-1}} + 0$$

$$= (-1)^{n-1}\frac{(n-1)!}{x^{n-1}} + n(-1)^{n-2}\frac{(n-2)!}{x^{n-1}}$$

$$= (-1)^{n-2}\frac{(n-2)!}{x^{n-1}}[-(n-1)+n]$$

$$= (-1)^{n-2}\frac{(n-2)!}{x^{n-1}}$$

Example 12 Find the n^{th} derivative of $x^2e^{3x}\sin 4x$

Solution: Let $u = e^{3x}\sin 4x$ and $v = x^2$

Then $u_n = e^{3x}\,25^{\frac{n}{2}}\sin\left(4x + n\tan^{-1}\frac{4}{3}\right)$

$= e^{3x}\,5^n\sin\left(4x + n\tan^{-1}\frac{4}{3}\right)$

By Leibnitz's theorem, we have

$$(u\,v)_n = u_n v + n_{C_1} u_{n-1} v_1 + n_{C_2} u_{n-2} v_2 + \cdots + n_{C_r} u_{n-r} v_r + \cdots + u v_n$$

$$\Rightarrow \left(x^2e^{3x}\sin 4x\right)_n = x^2e^{3x}5^n\sin\left(4x + n\tan^{-1}\frac{4}{3}\right) +$$

$$2nxe^{3x}5^{n-1}\sin\left(4x + (n-1)\tan^{-1}\frac{4}{3}\right) +$$

$$n(n-1)e^{3x}5^{n-2}\sin\left(4x + (n-2)\tan^{-1}\frac{4}{3}\right) + 0$$

$$= e^{3x}5^n\left[x^2\sin\left(4x + n\tan^{-1}\frac{4}{3}\right) + \frac{2nx}{5}\sin\left(4x + (n-1)\tan^{-1}\frac{4}{3}\right) + \frac{n(n-1)}{25}\sin\left(4x + (n-2)\tan^{-1}\frac{4}{3}\right)\right]$$

Example 13 If $y = a\cos(\log x) + b\,sin(\log x)$, show that

$$x^2y_{n+2} + (2n+1)xy_{n+1} + n(n+1)y_n = 0$$

Solution: Here $y = a\cos(\log x) + b\sin(\log x)$

$$\Rightarrow y_1 = \frac{-a}{x}\sin(\log x) + \frac{b}{x}\cos(logx)$$

$$\Rightarrow xy_1 = -a\sin(logx) + b\,cos(logx)$$

Differentiating both sides w.r.t. x, we get

$$xy_2 + y_1 = -\frac{a}{x}\cos(logx) + \frac{-b}{x}\,sin(logx)$$

$$\Rightarrow x^2y_2 + xy_1 = -\{acos(logx) + b\,sin(logx)\}$$

$$= -y$$

$\Rightarrow x^2y_2 + xy_1 + y = 0$

Using Leibnitz's theorem, we get

$(y_{n+2}x^2 + n_{C_1}y_{n+1}2x + n_{C_2}y_n.2) + (y_{n+1}x + n_{C_1}y_n.1) + y_n = 0$

$\Rightarrow y_{n+2}x^2 + y_{n+1}2nx + n(n-1)y_n + y_{n+1}x + ny_n + y_n = 0$

$\Rightarrow x^2y_{n+2} + (2n+1)xy_{n+1} + (n^2+1)y_n = 0$

Example 14 If $y = \log(x + \sqrt{1+x^2})$

Prove that $(1+x^2)y_{n+2} + (2n+1)xy_{n+1} + n^2y_n = 0$

Solution: $y = \log(x + \sqrt{1+x^2}$

$\Rightarrow y_1 = \frac{1}{x+\sqrt{1+x^2}}\left(1 + \frac{1}{2\sqrt{1+x^2}}2x\right) = \frac{1}{\sqrt{1+x^2}}$

$\Rightarrow (1+x^2){y_1}^2 = 1$

Differentiating both sides w.r.t. x, we get

$(1+x^2)2y_1y_2 + 2x{y_1}^2 = 0$

$\Rightarrow (1+x^2)y_2 + xy_1 = 0$

Using Leibnitz's theorem

$[y_{n+2}(1+x^2) + n_{C_1}y_{n+1}2x + n_{C_2}y_n.2] + (y_{n+1}x + n_{C_1}y_n.1) = 0$

$\Rightarrow y_{n+2}(1+x^2) + y_{n+1}2nx + n(n-1)y_n + y_{n+1}x + ny_n = 0$

$\Rightarrow (1+x^2)y_{n+2} + (2n+1)xy_{n+1} + n^2y_n = 0$

Example 15 If $y = \sin(m\, sin^{-1}x)$, show that
$(1-x^2)y_{n+2} = (2n+1)xy_{n+1} + (n^2-m^2)y_n$. Also find $y_n(0)$

Solution: Here $y = \sin(m\, sin^{-1}x)$ ......①

$\Rightarrow y_1 = \frac{m}{\sqrt{1-x^2}}\cos(m\, sin^{-1}x)$ ......②

$\Rightarrow (1-x^2){y_1}^2 = \mathrm{m}^2\cos^2(m\, sin^{-1}x)$

$\Rightarrow (1-x^2){y_1}^2 = \mathrm{m}^2[1 - \sin^2(m\, sin^{-1}x)]$

$\Rightarrow (1-x^2){y_1}^2 = m^2(1-y^2)$......③

$\Rightarrow (1\text{-}x^2){y_1}^2 + m^2y^2 = m^2$

Differentiating w.r.t. x, we get

$(1-x^2)2y_1y_2 + {y_1}^2(-2x) + m^22yy_1 = 0$

$\Rightarrow (1-x^2)y_2 - xy_1 + m^2y = 0$

Using Leibnitz's theorem, we get

$[y_{n+2}(1-x^2) + n_{C_1}y_{n+1}(-2x) + n_{C_2}y_n(-2)] - (y_{n+1}x + n_{C_1}y_n1) + m^2y_n = 0$

$$[y_{n+2}(1-x^2)+n_{C_1}y_{n+1}(-2x)+n_{C_2}y_n(-2)]-\left(y_{n+1}x+n_{C_1}y_n1\right)+m^2y_n=0$$

$$\Rightarrow y_{n+2}(1-x^2)-y_{n+1}2nx-n(n-1)y_n-(y_{n+1}x+ny_n)+m^2y_n=0$$

$$\Rightarrow (1-x^2)y_{n+2}=(2n+1)xy_{n+1}+(n^2-m^2)y_n \ldots\ldots④$$

Putting $x=0$ in ①, ②and ③

$y(0)=0, y_1(0)=m$ and $y_2(0)=0$

Putting $x=0$ in ④

$y_{n+2}(0)=(n^2-m^2)y_n(0)$

Putting $n=1,2,3 \ldots\ldots$ in the above equation, we get

$$y_3(0)=(1^2-m^2)y_1(0)$$

$$=(1^2-m^2)m \quad \because y_1(0)=m$$

$$y_4(0)=(2^2-m^2)y_2(0)$$

$$=0 \quad \because y_2(0)=0$$

$$y_5(0)=\left(3^2-m^2\right)y_3(0)$$

$$=m(1^2-m^2)(3^2-m^2)$$

$$\vdots$$

$$\Rightarrow \quad y_n(0)=\begin{cases}0, & if\ n\ is\ even\\ m(1^2-m^2)(3^2-m^2)\ldots[((n-2)^2)-m^2], & if\ n\ is\ odd\end{cases}$$

Example 16 If $y=e^{m\sin^{-1}x}$, show that
$(1-x^2)y_{n+2}-(2n+1)xy_{n+1}-(n^2+m^2)y_n=0$. Also find $y_n(0)$.

Solution: Here $y=e^{m\sin^{-1}x}$ …①

$$\Rightarrow y_1=\frac{m}{\sqrt{1-x^2}}e^{m\sin^{-1}x}$$

$$=\frac{my}{\sqrt{1-x^2}} \ldots\ldots②$$

$$\Rightarrow (1-x^2){y_1}^2=m^2y^2$$

Differentiating above equation w.r.t. x , we get

$$(1-x^2)2y_1y_2+{y_1}^2(-2x)=m^2 2yy_1$$

$$\Rightarrow (1-x^2)y_2-xy_1-m^2y=0 \quad \ldots\ldots③$$

Differentiating above equation n times w.r.t. x using Leibnitz's theorem, we get

$$[y_{n+2}(1-x^2) + n_{C_1} y_{n+1}(-2x) + n_{C_2} y_n(-2)] - \left(y_{n+1}x + n_{C_1} y_n 1\right) - m^2 y_n = 0$$

$$\Rightarrow y_{n+2}(1-x^2) - y_{n+1}2nx - n(n-1)y_n - (y_{n+1}x + ny_n) - m^2 y_n = 0$$

$$\Rightarrow (1-x^2)y_{n+2} - (2n+1)xy_{n+1} - (n^2+m^2)y_n = 0 \ldots\ldots ④$$

To find $y_n(0)$: Putting $x = 0$ in ①, ②and ③

$y(0) = 1,\ y_1(0) = m$ and $y_2(0) = m^2$

Also putting $x = 0$ in , we get

$y_{n+2}(0) = (n^2 + m^2)y_n(0)$

Putting $n = 1,2,3 \ldots$ in the above equation, we get

$$y_3(0) = (1^2 + m^2)y_1(0)$$

$$=(1^2 + m^2)m \qquad \because y_1(0) = m$$

$$y_4(0) = (2^2 + m^2)y_2(0)$$

$$= m^2(2^2 + m^2) \qquad \because y_2(0) = m^2$$

$$y_5(0) = \left(3^2 + m^2\right)y_3(0)$$

$$= m(1^2 + m^2)(3^2 + m^2)$$

$$\vdots$$

$$\Rightarrow \quad y_n(0) = \begin{cases} m^2(2^2+m^2) \ldots [(n-2)^2 + m^2], & if\ n\ is\ even \\ m(1^2+m^2)(3^2+m^2) \ldots [(n-2)^2 + m^2], & if\ n\ is\ odd \end{cases}$$

Example 17 If $y = tan^{-1}x$, show that

$(1-x^2)y_{n+2} + 2(n+1)xy_{n+1} + n(n+1)y_n = 0$. Also find $y_n(0)$

Solution: Here $y = tan^{-1}x \ldots..①$

$\Rightarrow y_1 = \frac{1}{1+x^2} \ldots\ldots ②$

$y_2 = \frac{-2x}{1+x^2}$

$\Rightarrow (1+x^2)y_2 + 2xy_1 = 0 \ldots\ldots ③$

Differentiating equation ③ n times w.r.t. x using Leibnitz's theorem

$$[y_{n+2}(1+x^2) + n_{C_1} y_{n+1}(2x) + n_{C_2} y_n(2)] + 2\left(y_{n+1}x + n_{C_1} y_n 1\right) = 0$$

$$\Rightarrow y_{n+2}(1+x^2) + y_{n+1}2nx + n(n-1)y_n + 2(y_{n+1}x + ny_n) = 0$$

$$\Rightarrow (1+x^2)y_{n+2} + 2(n+1)xy_{n+1} + n(n+1)y_n = 0 \ldots\ldots ④$$

To find $y_n(0)$: Putting $x = 0$ in ①, ②and ③, we get

$y(0) = 0 , y_1(0) = 1$ and $y_2(0) = 0$

Also putting $x = 0$ in ④, we get

$y_{n+2}(0) = -n(n+1)y_n(0)$

Putting $n = 1,2,3 \ldots$ in the above equation, we get

$y_3(0) = -1(2)y_1(0)$

$= -2 \qquad \because y_1(0) = 1$

$y_4(0) = -2(3)y_2(0)$

$= 0 \qquad \because y_2(0) = 0$

$y_5(0) = -3(4)y_3(0)$

$= -3(4)(-2) = 4!$

$y_6(0) = -4(5)y_4(0) = 0$

$y_7(0) = -5(6)y_5(0) = -5(6)4! = -(6!)$

$\vdots$

$\Rightarrow \quad y_{2n+1}(0) = (-1)^n(2n)!$ and $\quad y_{2n}(0) = 0$

Example18 If $y = (sin^{-1}x)^2$, show that $(1-x^2)y_{n+2} - (2n+1)xy_{n+1} - n^2y_n = 0$. Also find $y_n(0)$

Solution: Here $y = (sin^{-1}x)^2$.....①

$\Rightarrow y_1 = 2sin^{-1}x.\frac{1}{\sqrt{1-x^2}}$ ......②

Squaring both the sides, we get

$(1-x^2){y_1}^2 = 4\,(sin^{-1}x)^2$

$\Rightarrow (1-x^2){y_1}^2 = 4\,(y)^2$

Differentiating the above equation w.r.t. x, we get

$(1-x^2)2y_1y_2 + {y_1}^2(-2x) - 4y_1 = 0$

$\Rightarrow (1-x^2)y_2 + y_1(-x) - 2 = 0$ ......③

Differentiating the above equation n times w.r.t. x using Leibnitz's theorem, we get

$[y_{n+2}(1-x^2) + n_{C_1}y_{n+1}(-2x) + n_{C_2}y_n(-2)] - (y_{n+1}x + n_{C_1}y_n1) = 0$

$\Rightarrow y_{n+2}(1-x^2) - y_{n+1}2nx - n(n-1)y_n - (y_{n+1}x + ny_n) = 0$

$\Rightarrow (1-x^2)y_{n+2} - (2n+1)xy_{n+1} - y_nn^2 = 0$......④

To find $y_n(0)$: Putting $x = 0$ in ①, ②and ③, we get

$y(0) = 0\,, y_1(0) = 0$ and $y_2(0) = 2$

Also putting $x = 0$ in ④, we get

$$y_{n+2}(0) = n^2 y_n(0)$$

Putting $n = 1,2,3 \dots$ in the above equation, we get

$$y_3(0) = 1^2 y_1(0)$$

$$= 0 \qquad \because y_1(0) = 0$$

$$y_4(0) = 2^2 y_2(0)$$

$$= 2^2 2 \qquad \because y_2(0) = 2$$

$$y_5(0) = 3^2 y_3(0) = 0$$

$$y_6(0) = 4^2 y_4(0) = 4^2 2^2 2$$

$$\vdots$$

$$\Rightarrow \quad y_n(0) = \begin{cases} 0, & if\ n\ is\ odd \\ 2.2^2.4^2 \dots\dots\dots (n-2)^2, & if\ n\ is\ even \end{cases}$$

CURVATURE -RADIUS OF CURVATURE IN CARTESIAN COORDINATES

Introduction

To characterize a curve completely we have seen various aspects of the curve such as increasing and decreasing nature, maximum and minimum points, concavity and convexity, symmetry and special points such as points of inflexion etc.Another aspect to characterize the shape of a curve is the degree of its bending or curvature.

In many practical problems we are concerned with the bending of a curve at different points or the bending of two curves such as rail tracks. The concept of curvature is considered while laying rail tracks and designing highways. The curvature at a point is a numerical measure of the rate of bending of a curve.

Measure of Curvature

Definition Let Γ be a curve that does not intersect itself and having tangents at each point. Let A be a fixed point on the curve from which arc length is mea
sured. Let P and Q be neighbouring points on the curve so that AP = s and AQ = $s + \Delta s$.

∴ length of arc PQ = Δs

Let the tangents at P and Q make angles ψ and $\psi + \Delta\psi$ respectively with the positive direction of x-axis.

∴ $\Delta\psi$ is the angle between the tangents at P and Q.

P isely, $\Delta\psi$ is the angle through which the tangent turns from P to Q as P moves along the arc through the distance Δs.

1. The angle $\Delta\psi$ is called the **angle of contigence** of the arc PQ or the **total curvature** of the arc PQ.
2. The ratio $\frac{\Delta\psi}{\Delta s}$ is called the average curvature of the arc PQ.
3. The curvature of the curve at P is defined as $\lim_{\Delta s \to 0} \frac{\Delta\psi}{\Delta s} = \frac{d\psi}{ds}$ and it is denoted by the greek letter κ (kappa). Thus, $\kappa = \frac{d\psi}{ds}$.

Definition **Radius of Curvature:** If the curvature at a point P on a curve is k, then $\frac{1}{k}$ is called the radius of curvature at P (if $k \neq 0$). Radius of curvature is denoted by Greek letter ρ.

$$= \frac{1}{k} = \frac{ds}{d}.$$

Radius of Curvature for Cartesian Equation of a Given Curve

Let $y = f(x)$ be the equation of a curve, then we know that at the point (x, y), $\frac{dy}{dx} = \tan\psi$, where ψ is the angle made by the tangent at (x, y) with the positive direction of the x-axis.

∴ $$\frac{d^2y}{dx^2} = \sec^2\psi \frac{d\psi}{dx}$$

$$= (1 + \tan^2\psi) \frac{d\psi}{ds} \frac{ds}{dx}$$

$$= \left[1+\left(\frac{dy}{dx}\right)^2\right]\frac{1}{\boldsymbol{\rho}}\cdot\frac{ds}{dx}$$

$\therefore$ $$\boldsymbol{\rho} = \frac{\left[1+\left(\frac{dy}{dx}\right)^2\right]}{\frac{d^2y}{dx^2}}\cdot\frac{ds}{d\boldsymbol{x}}$$

But we know that

$$(ds)^2 = (dx)^2 + (dy)^2$$

$\therefore$ $$\left(\frac{ds}{dx}\right)^2 = 1+\left(\frac{dy}{dx}\right)^2$$

$\Rightarrow$ $$\frac{ds}{dx} = \left[1+\left(\frac{dy}{dx}\right)^2\right]^{1/2}$$

$\therefore$ $$\boldsymbol{\rho} = \frac{\left[1+\left(\frac{dy}{dx}\right)^2\right]^{3/2}}{\frac{d^2y}{dx^2}}$$

$\Rightarrow$ $$\boldsymbol{\rho} = \frac{(1+\ {}_1^2)^{3/2}}{{}_2}$$

where $y_1 = \frac{dy}{dx}$ and $y_2 = \frac{d^2y}{dx^2}$

Note

1. When calculating $\boldsymbol{\rho}$ only positive value should be taken i.e., numerical value of $\boldsymbol{\rho}$ is taken as radius of curvature, since it cannot be negative. If $y_2 > 0$, the curve is concave up and if $y_2 < 0$ then it is concave down or convex up at the point.
2. At a point of inflexion i.e., when $y_2 = 0$, the curvature is defined as zero.
3. If the equation of curve is $x = f(y)$, then $\boldsymbol{\rho} = \frac{\left[1+\left(\frac{dx}{dy}\right)^2\right]^{3/2}}{\frac{d^2x}{dy^2}} = \frac{(1+x_1^2)^{3/2}}{x_2}$ if $x_2 \neq 0$

   where $x_1 = \frac{dx}{dy}$ and $x_2 = \frac{d^2x}{dy^2}$
4. If at a point $\frac{dy}{dx} = \infty$ formula cannot be used. i.e., if the tangent is parallel to y-axis, then $\frac{dx}{dy} = 0$.

Radius of Curvature for Parametric Equations

If the equation of curve is given by parametric equations $x = f(t)$, $y = g(t)$, then we find $\frac{dx}{dt}, \frac{dy}{dt}$

$$\text{Let } \frac{dx}{dt} = x', \ \frac{dy}{dt} = y'$$

$$\therefore \quad \frac{dy}{dx} = \frac{\frac{dy}{dt}}{\frac{dx}{dt}} = \frac{y'}{x'}$$

$$\therefore \quad \frac{d^2y}{dx^2} = \frac{d}{dx}\left(\frac{dy}{dx}\right) = \frac{d}{dt}\left(\frac{y'}{x'}\right)\cdot\frac{dt}{dx}$$

$$= \frac{x'y'' - y'x''}{(x')^2}\cdot\frac{1}{x'} = \frac{x'y'' - y'x''}{(x')^3}$$

$$\therefore \quad \text{the radius of curvature } \boldsymbol{\rho} = \frac{\left[1+\left(\frac{y'}{x'}\right)^2\right]^{3/2}}{\frac{x'y''-y'x''}{(x')^3}} = \frac{(x'^2+y'^2)^{3/2}}{x'y''-y'x''}$$

EXAMPLE 1

Find the radius of curvature at the point $\left(\frac{1}{4}, \frac{1}{4}\right)$ on $\sqrt{x} + \sqrt{y} = 1$.

Solution.

The given curve is $\sqrt{x} + \sqrt{y} = 1$

Differentiating w.r.to x, we get

$$\frac{1}{2\sqrt{x}} + \frac{1}{2\sqrt{y}}\cdot\frac{dy}{dx} = 0$$

$$\Rightarrow \quad \frac{dy}{dx} = -\frac{\sqrt{y}}{\sqrt{x}} = \frac{\sqrt{x}-1}{\sqrt{x}}$$

$$= 1 - \frac{1}{\sqrt{x}} = 1 - x^{-1/2}$$

$$\frac{d^2y}{dx^2} = -\left(-\frac{1}{2}\cdot x^{-3/2}\right) = \frac{1}{2\boldsymbol{x}^{3/2}}$$

At the point $\left(\frac{1}{4}, \frac{1}{4}\right)$, $\quad \frac{dy}{dx} = 1 - \frac{1}{\sqrt{1/4}} = 1 - 2 = -1$

and $\quad \frac{d^2y}{dx^2} = \frac{1}{2\cdot\left(\frac{1}{4}\right)^{3/2}} = \frac{4^{3/2}}{2} = \frac{4\cdot 2}{2} = 4$

$\therefore$ $y_1 = -1$ and $y_2 = 4$

$\therefore$ the radius of curvature $\boldsymbol{\rho} = \dfrac{(1+y_1^2)^{3/2}}{y_2} = \dfrac{(1+1)^{3/2}}{y_2} = \dfrac{2\sqrt{2}}{4} = \dfrac{1}{\sqrt{2}}$

EXAMPLE 2

Find the radius of curvature at the point $\left(\frac{3a}{2}, \frac{3a}{2}\right)$ on the curve $x^3 + y^3 = 3axy$.

Solution.

The given curve is $x^3 + y^3 = 3axy$.

Differentiating w.r.to x, we get

$$3x^2 + 3y^2\frac{dy}{dx} = 3a\left[x\frac{dy}{dx} + y\cdot 1\right]$$

$$\Rightarrow \quad \frac{dy}{dx}[y^2 - ax] = ay - x^2 \Rightarrow \frac{dy}{dx} = \frac{ay - x^2}{y^2 - ax}$$

Differentiating (2) w.r.to x, we get

$$\frac{d^2y}{dx^2} = \frac{(y^2 - ax)\left(a\frac{dy}{dx} - 2x\right) - (ay - x^2)\left(2y\frac{dy}{dx} - a\right)}{(y^2 - ax)^2}$$

At the point $\left(\frac{3a}{2}, \frac{3a}{2}\right)$,

$$\frac{dy}{dx} = \frac{a\cdot\frac{3a}{2} - \frac{9a^2}{4}}{\frac{9a^2}{4} - a\cdot\frac{3a}{2}} = -1$$

and

$$\frac{d^2y}{dx^2} = \frac{\left(\frac{9a^2}{4} - a\cdot\frac{3a}{2}\right)(-a-3a) - \left(\frac{3a^2}{2} - \frac{9a^2}{4}\right)(-3a-a)}{\left(\frac{9a^2}{4} - \frac{3a^2}{2}\right)^2}$$

$$= \frac{(-4a)\left[\frac{3a^2}{4} - \left(-\frac{3a^2}{4}\right)\right]}{\left(\frac{3a^2}{4}\right)^2} = \frac{-4a\cdot 2\cdot\left(\frac{3a^2}{4}\right)}{\left(\frac{3a^2}{4}\right)^2} = \frac{-8a}{\frac{3a^2}{4}} = -\frac{32}{3a}$$

$\therefore$ $y_1 = -1$ and $y_2 = -\dfrac{32}{3a}$

$\therefore$ the radius of curvature $\boldsymbol{\rho} = \dfrac{(1+y_1^2)^{3/2}}{y_2} = \dfrac{(1+1)^{3/2}}{-\frac{32}{3a}} = \dfrac{-2\sqrt{2}\times 3a}{32} = -\dfrac{3a}{8\sqrt{2}}$

Since $\boldsymbol{\rho}$ is positive, $\boldsymbol{\rho} = \dfrac{3a}{8\sqrt{2}}$

EXAMPLE 3

Find the radius of curvature of the curve $xy^2 = a^3 - x^3$ at $(a, 0)$.

Solution.

The given curve is $xy^2 = a^3 - x^3$

Differentiating w.r.to x, we get

$$x \cdot 2y \frac{dy}{dx} + y^2 \cdot 1 = -3x^2 \quad \Rightarrow \quad \frac{dy}{dx} = \frac{-(3x^2 + y^2)}{2xy}$$

At the point $(a, 0)$, $\frac{dy}{dx} = \infty \qquad \therefore \frac{dx}{dy} = 0 \Rightarrow x_1 = 0$

So, we use the formula, $\rho = \frac{(1 + x_1^2)^{3/2}}{x_2}$

Now, $\frac{dx}{dy} = -\frac{2xy}{3x^2 + y^2}$

TO BE CONTINUED……

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