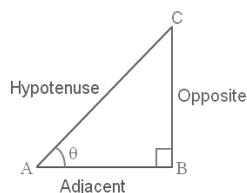


Trigonometric Identities

Six Trigonometric Functions

Right triangle definitions, where $0 < \theta < \pi/2$

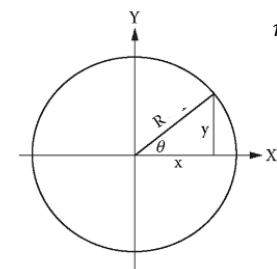


$$\sin \theta = \frac{\text{opp}}{\text{hyp}} \quad \csc \theta = \frac{\text{hyp}}{\text{opp}}$$

$$\cos \theta = \frac{\text{adj}}{\text{hyp}} \quad \sec \theta = \frac{\text{hyp}}{\text{adj}}$$

$$\tan \theta = \frac{\text{opp}}{\text{adj}} \quad \cot \theta = \frac{\text{adj}}{\text{opp}}$$

Circular function definitions, where θ is any angle.

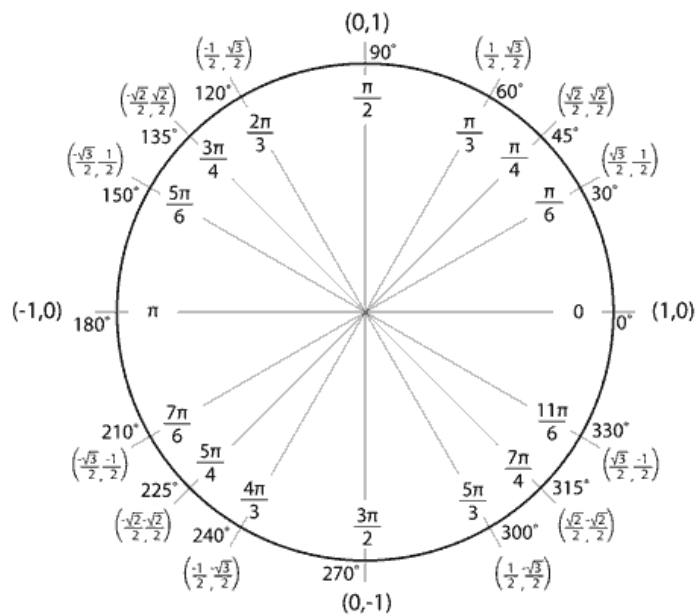


$$r = \sqrt{x^2 + y^2}$$

$$\sin \theta = \frac{y}{r} \quad \csc \theta = \frac{r}{y}$$

$$\cos \theta = \frac{x}{r} \quad \sec \theta = \frac{r}{x}$$

$$\tan \theta = \frac{y}{x} \quad \cot \theta = \frac{x}{y}$$



Negative Angle Identities

$$\sin(-\theta) = -\sin \theta \quad \cos(-\theta) = \cos \theta \quad \tan(-\theta) = -\tan \theta$$

$$\csc(-\theta) = -\csc \theta \quad \sec(-\theta) = \sec \theta \quad \cot(-\theta) = -\cot \theta$$

Tangent and Cotangent Identities

$$\tan \theta = \frac{\sin \theta}{\cos \theta} \quad \cot \theta = \frac{\cos \theta}{\sin \theta}$$

Pythagorean Identities

$$\sin^2 \theta + \cos^2 \theta = 1 \quad \tan^2 \theta + 1 = \sec^2 \theta$$

$$\cot^2 \theta + 1 = \csc^2 \theta$$

Cofunction Identities

$$\sin\left(\frac{\pi}{2} - \theta\right) = \cos \theta \quad \cos\left(\frac{\pi}{2} - \theta\right) = \sin \theta$$

$$\csc\left(\frac{\pi}{2} - \theta\right) = \sec \theta \quad \tan\left(\frac{\pi}{2} - \theta\right) = \cot \theta$$

$$\sec\left(\frac{\pi}{2} - \theta\right) = \csc \theta \quad \cot\left(\frac{\pi}{2} - \theta\right) = \tan \theta$$

Sum and Difference Formulas

$$\sin A + \sin B = 2 \sin\left(\frac{A+B}{2}\right) \cos\left(\frac{A-B}{2}\right)$$

$$\sin A - \sin B = 2 \cos\left(\frac{A+B}{2}\right) \sin\left(\frac{A-B}{2}\right)$$

$$\cos A + \cos B = 2 \cos\left(\frac{A+B}{2}\right) \cos\left(\frac{A-B}{2}\right)$$

$$\cos A - \cos B = -2 \sin\left(\frac{A+B}{2}\right) \sin\left(\frac{A-B}{2}\right)$$

Reciprocal Identities

$$\sin \theta = \frac{1}{\csc \theta} \quad \cos \theta = \frac{1}{\sec \theta} \quad \tan \theta = \frac{1}{\cot \theta}$$

$$\csc \theta = \frac{1}{\sin \theta} \quad \sec \theta = \frac{1}{\cos \theta} \quad \cot \theta = \frac{1}{\tan \theta}$$

Double Angle Identities

$$\sin 2\theta = 2 \sin \theta \cos \theta$$

$$\cos 2\theta = \cos^2 \theta - \sin^2 \theta = 2 \cos^2 \theta - 1 = 1 - 2 \sin^2 \theta$$

$$\tan 2\theta = \frac{2 \tan \theta}{1 - \tan^2 \theta}$$

Half Angle Identities

$$\sin \frac{\theta}{2} = \pm \sqrt{\frac{1 - \cos \theta}{2}} \quad \cos \frac{\theta}{2} = \pm \sqrt{\frac{1 + \cos \theta}{2}}$$

$$\tan \frac{\theta}{2} = \pm \sqrt{\frac{1 - \cos \theta}{1 + \cos \theta}}$$

Addition and Subtraction Formulas

$$\sin(A \pm B) = \sin A \cos B \pm \cos A \sin B$$

$$\cos(A \pm B) = \cos A \cos B \mp \sin A \sin B$$

$$\tan(A \pm B) = \frac{\tan A \pm \tan B}{1 \mp \tan A \tan B}$$

Product Formulas

$$\sin A \sin B = \frac{1}{2} [\cos(A - B) - \cos(A + B)]$$

$$\cos A \cos B = \frac{1}{2} [\cos(A + B) + \cos(A - B)]$$

$$\sin A \cos B = \frac{1}{2} [\sin(A + B) + \sin(A - B)]$$

$$\cos A \sin B = \frac{1}{2} [\sin(A + B) - \sin(A - B)]$$

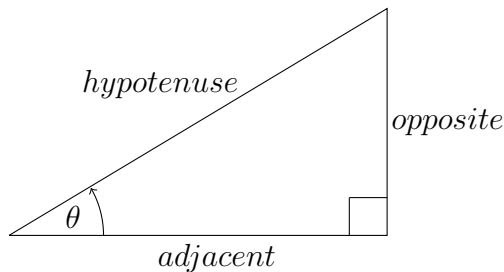
Trigonometric Formula Sheet

Definition of the Trig Functions

Right Triangle Definition

Assume that:

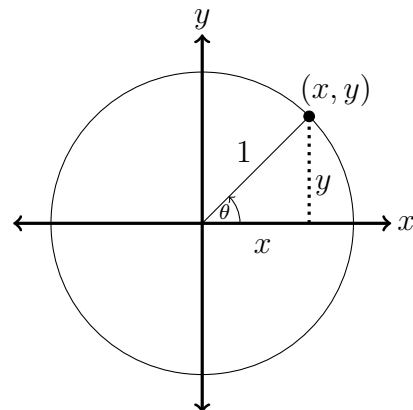
$$0 < \theta < \frac{\pi}{2} \quad \text{or} \quad 0^\circ < \theta < 90^\circ$$



$$\begin{aligned} \sin \theta &= \frac{\text{opp}}{\text{hyp}} & \csc \theta &= \frac{\text{hyp}}{\text{opp}} \\ \cos \theta &= \frac{\text{adj}}{\text{hyp}} & \sec \theta &= \frac{\text{hyp}}{\text{adj}} \\ \tan \theta &= \frac{\text{opp}}{\text{adj}} & \cot \theta &= \frac{\text{adj}}{\text{opp}} \end{aligned}$$

Unit Circle Definition

Assume θ can be any angle.



$$\begin{aligned} \sin \theta &= \frac{y}{1} & \csc \theta &= \frac{1}{y} \\ \cos \theta &= \frac{x}{1} & \sec \theta &= \frac{1}{x} \\ \tan \theta &= \frac{y}{x} & \cot \theta &= \frac{x}{y} \end{aligned}$$

Domains of the Trig Functions

$$\sin \theta, \quad \forall \theta \in (-\infty, \infty)$$

$$\csc \theta, \quad \forall \theta \neq n\pi, \text{ where } n \in \mathbb{Z}$$

$$\cos \theta, \quad \forall \theta \in (-\infty, \infty)$$

$$\sec \theta, \quad \forall \theta \neq \left(n + \frac{1}{2}\right)\pi, \text{ where } n \in \mathbb{Z}$$

$$\tan \theta, \quad \forall \theta \neq \left(n + \frac{1}{2}\right)\pi, \text{ where } n \in \mathbb{Z}$$

$$\cot \theta, \quad \forall \theta \neq n\pi, \text{ where } n \in \mathbb{Z}$$

Ranges of the Trig Functions

$$-1 \leq \sin \theta \leq 1$$

$$\csc \theta \geq 1 \text{ and } \csc \theta \leq -1$$

$$-1 \leq \cos \theta \leq 1$$

$$\sec \theta \geq 1 \text{ and } \sec \theta \leq -1$$

$$-\infty \leq \tan \theta \leq \infty$$

$$-\infty \leq \cot \theta \leq \infty$$

Periods of the Trig Functions

The period of a function is the number, T , such that $f(\theta + T) = f(\theta)$.

So, if ω is a fixed number and θ is any angle we have the following periods.

$$\sin(\omega\theta) \Rightarrow T = \frac{2\pi}{\omega}$$

$$\csc(\omega\theta) \Rightarrow T = \frac{2\pi}{\omega}$$

$$\cos(\omega\theta) \Rightarrow T = \frac{2\pi}{\omega}$$

$$\sec(\omega\theta) \Rightarrow T = \frac{2\pi}{\omega}$$

$$\tan(\omega\theta) \Rightarrow T = \frac{\pi}{\omega}$$

$$\cot(\omega\theta) \Rightarrow T = \frac{\pi}{\omega}$$

Identities and Formulas

Tangent and Cotangent Identities

$$\tan \theta = \frac{\sin \theta}{\cos \theta} \quad \cot \theta = \frac{\cos \theta}{\sin \theta}$$

Reciprocal Identities

$$\begin{aligned} \sin \theta &= \frac{1}{\csc \theta} & \csc \theta &= \frac{1}{\sin \theta} \\ \cos \theta &= \frac{1}{\sec \theta} & \sec \theta &= \frac{1}{\cos \theta} \\ \tan \theta &= \frac{1}{\cot \theta} & \cot \theta &= \frac{1}{\tan \theta} \end{aligned}$$

Pythagorean Identities

$$\begin{aligned} \sin^2 \theta + \cos^2 \theta &= 1 \\ \tan^2 \theta + 1 &= \sec^2 \theta \\ 1 + \cot^2 \theta &= \csc^2 \theta \end{aligned}$$

Even and Odd Formulas

$$\begin{aligned} \sin(-\theta) &= -\sin \theta & \csc(-\theta) &= -\csc \theta \\ \cos(-\theta) &= \cos \theta & \sec(-\theta) &= \sec \theta \\ \tan(-\theta) &= -\tan \theta & \cot(-\theta) &= -\cot \theta \end{aligned}$$

Periodic Formulas

If n is an integer

$$\begin{aligned} \sin(\theta + 2\pi n) &= \sin \theta & \csc(\theta + 2\pi n) &= \csc \theta \\ \cos(\theta + 2\pi n) &= \cos \theta & \sec(\theta + 2\pi n) &= \sec \theta \\ \tan(\theta + \pi n) &= \tan \theta & \cot(\theta + \pi n) &= \cot \theta \end{aligned}$$

Double Angle Formulas

$$\sin(2\theta) = 2 \sin \theta \cos \theta$$

$$\begin{aligned} \cos(2\theta) &= \cos^2 \theta - \sin^2 \theta \\ &= 2 \cos^2 \theta - 1 \\ &= 1 - 2 \sin^2 \theta \end{aligned}$$

$$\tan(2\theta) = \frac{2 \tan \theta}{1 - \tan^2 \theta}$$

Degrees to Radians Formulas

If x is an angle in degrees and t is an angle in radians then:

$$\frac{\pi}{180^\circ} = \frac{t}{x} \quad \Rightarrow \quad t = \frac{\pi x}{180^\circ} \quad \text{and} \quad x = \frac{180^\circ t}{\pi}$$

Half Angle Formulas

$$\sin \theta = \pm \sqrt{\frac{1 - \cos(2\theta)}{2}}$$

$$\cos \theta = \pm \sqrt{\frac{1 + \cos(2\theta)}{2}}$$

$$\tan \theta = \pm \sqrt{\frac{1 - \cos(2\theta)}{1 + \cos(2\theta)}}$$

Sum and Difference Formulas

$$\sin(\alpha \pm \beta) = \sin \alpha \cos \beta \pm \cos \alpha \sin \beta$$

$$\cos(\alpha \pm \beta) = \cos \alpha \cos \beta \mp \sin \alpha \sin \beta$$

$$\tan(\alpha \pm \beta) = \frac{\tan \alpha \pm \tan \beta}{1 \mp \tan \alpha \tan \beta}$$

Product to Sum Formulas

$$\sin \alpha \sin \beta = \frac{1}{2} [\cos(\alpha - \beta) - \cos(\alpha + \beta)]$$

$$\cos \alpha \cos \beta = \frac{1}{2} [\cos(\alpha - \beta) + \cos(\alpha + \beta)]$$

$$\sin \alpha \cos \beta = \frac{1}{2} [\sin(\alpha + \beta) + \sin(\alpha - \beta)]$$

$$\cos \alpha \sin \beta = \frac{1}{2} [\sin(\alpha + \beta) - \sin(\alpha - \beta)]$$

Sum to Product Formulas

$$\sin \alpha + \sin \beta = 2 \sin \left(\frac{\alpha + \beta}{2} \right) \cos \left(\frac{\alpha - \beta}{2} \right)$$

$$\sin \alpha - \sin \beta = 2 \cos \left(\frac{\alpha + \beta}{2} \right) \sin \left(\frac{\alpha - \beta}{2} \right)$$

$$\cos \alpha + \cos \beta = 2 \cos \left(\frac{\alpha + \beta}{2} \right) \cos \left(\frac{\alpha - \beta}{2} \right)$$

$$\cos \alpha - \cos \beta = -2 \sin \left(\frac{\alpha + \beta}{2} \right) \sin \left(\frac{\alpha - \beta}{2} \right)$$

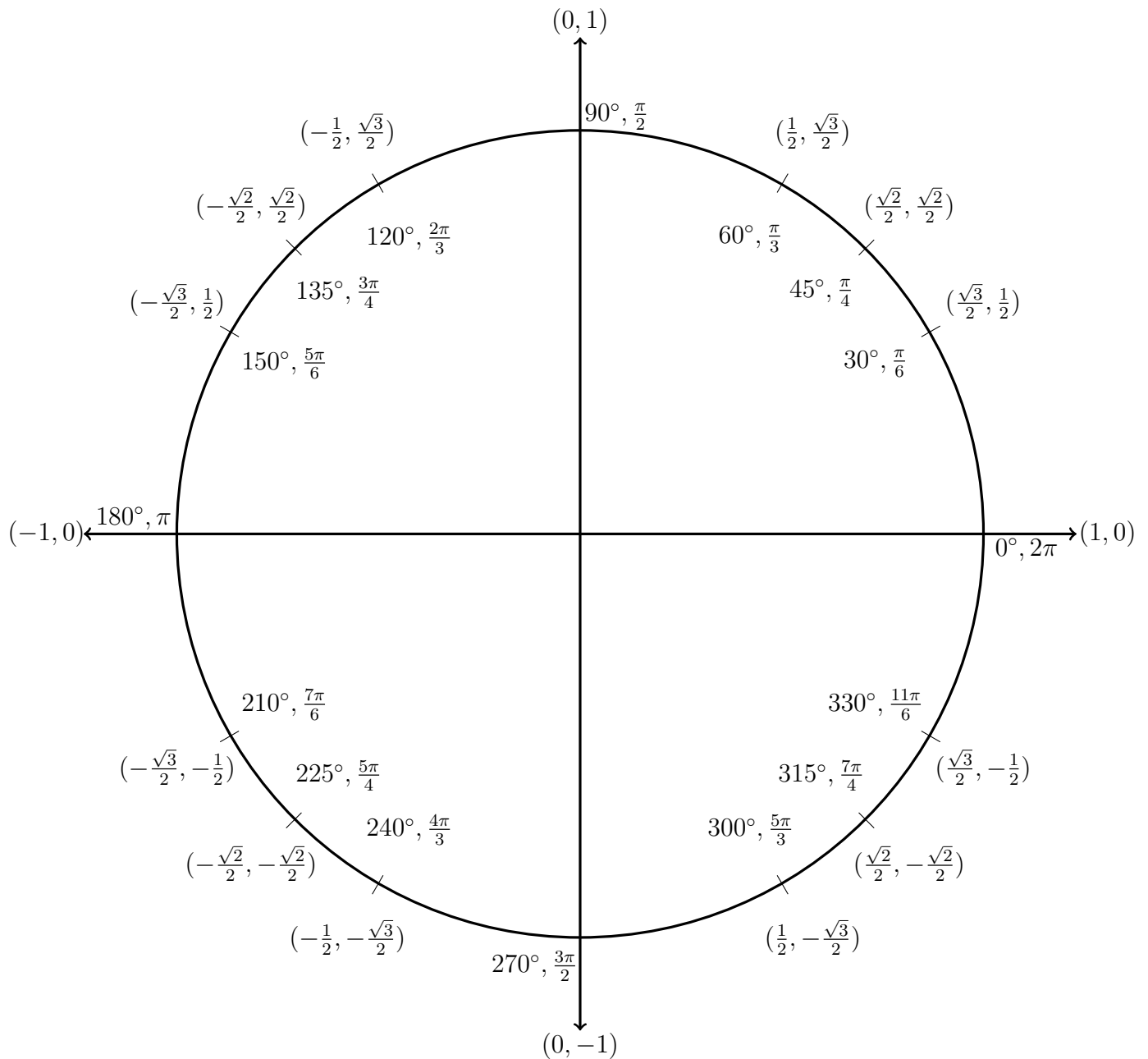
Cofunction Formulas

$$\sin \left(\frac{\pi}{2} - \theta \right) = \cos \theta \quad \cos \left(\frac{\pi}{2} - \theta \right) = \sin \theta$$

$$\csc \left(\frac{\pi}{2} - \theta \right) = \sec \theta \quad \sec \left(\frac{\pi}{2} - \theta \right) = \csc \theta$$

$$\tan \left(\frac{\pi}{2} - \theta \right) = \cot \theta \quad \cot \left(\frac{\pi}{2} - \theta \right) = \tan \theta$$

Unit Circle



For any ordered pair on the unit circle (x, y) : $\cos \theta = x$ and $\sin \theta = y$

Example

$$\cos \left(\frac{7\pi}{6} \right) = -\frac{\sqrt{3}}{2} \quad \sin \left(\frac{7\pi}{6} \right) = -\frac{1}{2}$$

Inverse Trig Functions

Definition

$\theta = \sin^{-1}(x)$ is equivalent to $x = \sin \theta$

$\theta = \cos^{-1}(x)$ is equivalent to $x = \cos \theta$

$\theta = \tan^{-1}(x)$ is equivalent to $x = \tan \theta$

Domain and Range

Function

Domain

Range

$$\theta = \sin^{-1}(x) \quad -1 \leq x \leq 1 \quad -\frac{\pi}{2} \leq \theta \leq \frac{\pi}{2}$$

$$\theta = \cos^{-1}(x) \quad -1 \leq x \leq 1 \quad 0 \leq \theta \leq \pi$$

$$\theta = \tan^{-1}(x) \quad -\infty \leq x \leq \infty \quad -\frac{\pi}{2} < \theta < \frac{\pi}{2}$$

Inverse Properties

These properties hold for x in the domain and θ in the range

$$\sin(\sin^{-1}(x)) = x \quad \sin^{-1}(\sin(\theta)) = \theta$$

$$\cos(\cos^{-1}(x)) = x \quad \cos^{-1}(\cos(\theta)) = \theta$$

$$\tan(\tan^{-1}(x)) = x \quad \tan^{-1}(\tan(\theta)) = \theta$$

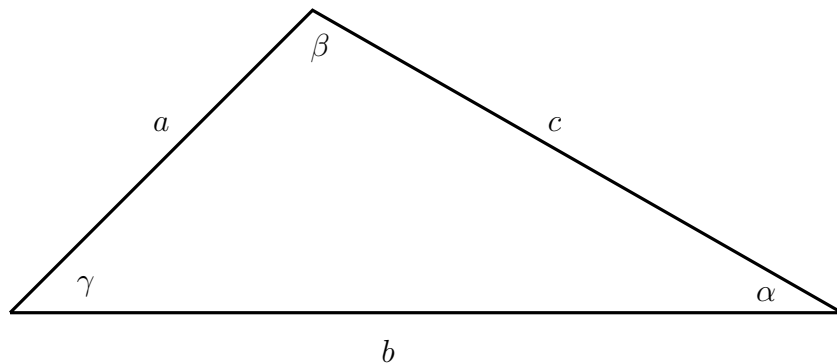
Other Notations

$$\sin^{-1}(x) = \arcsin(x)$$

$$\cos^{-1}(x) = \arccos(x)$$

$$\tan^{-1}(x) = \arctan(x)$$

Law of Sines, Cosines, and Tangents



Law of Sines

$$\frac{\sin \alpha}{a} = \frac{\sin \beta}{b} = \frac{\sin \gamma}{c}$$

Law of Cosines

$$a^2 = b^2 + c^2 - 2bc \cos \alpha$$

$$b^2 = a^2 + c^2 - 2ac \cos \beta$$

$$c^2 = a^2 + b^2 - 2ab \cos \gamma$$

Law of Tangents

$$\frac{a-b}{a+b} = \frac{\tan \frac{1}{2}(\alpha - \beta)}{\tan \frac{1}{2}(\alpha + \beta)}$$

$$\frac{b-c}{b+c} = \frac{\tan \frac{1}{2}(\beta - \gamma)}{\tan \frac{1}{2}(\beta + \gamma)}$$

$$\frac{a-c}{a+c} = \frac{\tan \frac{1}{2}(\alpha - \gamma)}{\tan \frac{1}{2}(\alpha + \gamma)}$$

Complex Numbers

$$i = \sqrt{-1} \quad i^2 = -1 \quad i^3 = -i \quad i^4 = 1$$

$$\sqrt{-a} = i\sqrt{a}, \quad a \geq 0$$

$$(a + bi)(a - bi) = a^2 + b^2$$

$$(a + bi) + (c + di) = a + c + (b + d)i$$

$$|a + bi| = \sqrt{a^2 + b^2} \quad \text{Complex Modulus}$$

$$(a + bi) - (c + di) = a - c + (b - d)i$$

$$\overline{(a + bi)} = a - bi \quad \text{Complex Conjugate}$$

$$(a + bi)(c + di) = ac - bd + (ad + bc)i$$

$$\overline{(a + bi)}(a + bi) = |a + bi|^2$$

DeMoivre's Theorem

Let $z = r(\cos \theta + i \sin \theta)$, and let n be a positive integer.

Then:

$$z^n = r^n(\cos n\theta + i \sin n\theta).$$

Example: Let $z = 1 - i$, find z^6 .

Solution: First write z in polar form.

$$r = \sqrt{(1)^2 + (-1)^2} = \sqrt{2}$$

$$\theta = \arg(z) = \tan^{-1} \left(\frac{-1}{1} \right) = -\frac{\pi}{4}$$

$$\text{Polar Form: } z = \sqrt{2} \left(\cos \left(-\frac{\pi}{4} \right) + i \sin \left(-\frac{\pi}{4} \right) \right)$$

Applying DeMoivre's Theorem gives :

$$\begin{aligned} z^6 &= (\sqrt{2})^6 \left(\cos \left(6 \cdot -\frac{\pi}{4} \right) + i \sin \left(6 \cdot -\frac{\pi}{4} \right) \right) \\ &= 2^3 \left(\cos \left(-\frac{3\pi}{2} \right) + i \sin \left(-\frac{3\pi}{2} \right) \right) \\ &= 8(0 + i(1)) \\ &= 8i \end{aligned}$$

Finding the n th roots of a number using DeMoivre's Theorem

Example: Find all the complex fourth roots of 4. That is, find all the complex solutions of $x^4 = 4$.

We are asked to find all complex fourth roots of 4.

These are all the solutions (including the complex values) of the equation $x^4 = 4$.

For any positive integer n , a nonzero complex number z has exactly n distinct n th roots. More specifically, if z is written in the trigonometric form $r(\cos \theta + i \sin \theta)$, the n th roots of z are given by the following formula.

$$(*) \quad r^{\frac{1}{n}} \left(\cos \left(\frac{\theta}{n} + \frac{360^\circ k}{n} \right) + i \sin \left(\frac{\theta}{n} + \frac{360^\circ k}{n} \right) \right), \quad \text{for } k = 0, 1, 2, \dots, n - 1.$$

Remember from the previous example we need to write 4 in trigonometric form by using:

$$r = \sqrt{(a)^2 + (b)^2} \quad \text{and} \quad \theta = \arg(z) = \tan^{-1} \left(\frac{b}{a} \right).$$

So we have the complex number $a + ib = 4 + i0$.

Therefore $a = 4$ and $b = 0$

So $r = \sqrt{(4)^2 + (0)^2} = 4$ and

$$\theta = \arg(z) = \tan^{-1} \left(\frac{0}{4} \right) = 0$$

Finally our trigonometric form is $4 = 4(\cos 0^\circ + i \sin 0^\circ)$

Using the formula (*) above with $n = 4$, we can find the fourth roots of $4(\cos 0^\circ + i \sin 0^\circ)$

- For $k = 0$, $4^{\frac{1}{4}} \left(\cos \left(\frac{0^\circ}{4} + \frac{360^\circ * 0}{4} \right) + i \sin \left(\frac{0^\circ}{4} + \frac{360^\circ * 0}{4} \right) \right) = \sqrt{2} (\cos(0^\circ) + i \sin(0^\circ)) = \sqrt{2}$
- For $k = 1$, $4^{\frac{1}{4}} \left(\cos \left(\frac{0^\circ}{4} + \frac{360^\circ * 1}{4} \right) + i \sin \left(\frac{0^\circ}{4} + \frac{360^\circ * 1}{4} \right) \right) = \sqrt{2} (\cos(90^\circ) + i \sin(90^\circ)) = \sqrt{2}i$
- For $k = 2$, $4^{\frac{1}{4}} \left(\cos \left(\frac{0^\circ}{4} + \frac{360^\circ * 2}{4} \right) + i \sin \left(\frac{0^\circ}{4} + \frac{360^\circ * 2}{4} \right) \right) = \sqrt{2} (\cos(180^\circ) + i \sin(180^\circ)) = -\sqrt{2}$
- For $k = 3$, $4^{\frac{1}{4}} \left(\cos \left(\frac{0^\circ}{4} + \frac{360^\circ * 3}{4} \right) + i \sin \left(\frac{0^\circ}{4} + \frac{360^\circ * 3}{4} \right) \right) = \sqrt{2} (\cos(270^\circ) + i \sin(270^\circ)) = -\sqrt{2}i$

Thus all of the complex roots of $x^4 = 4$ are:

$$\sqrt{2}, \sqrt{2}i, -\sqrt{2}, -\sqrt{2}i .$$

Formulas for the Conic Sections

Circle

$$\text{Standard Form : } (x - h)^2 + (y - k)^2 = r^2$$

Where (h, k) = center and r = radius

Ellipse

Standard Form for Horizontal Major Axis :

$$\frac{(x - h)^2}{a^2} + \frac{(y - k)^2}{b^2} = 1$$

Standard Form for Vertical Major Axis :

$$\frac{(x - h)^2}{b^2} + \frac{(y - k)^2}{a^2} = 1$$

Where (h, k) = center

$2a$ = length of major axis

$2b$ = length of minor axis

$$(0 < b < a)$$

Foci can be found by using $c^2 = a^2 - b^2$

Where c = foci length

More Conic Sections

Hyperbola

Standard Form for Horizontal Transverse Axis :

$$\frac{(x - h)^2}{a^2} - \frac{(y - k)^2}{b^2} = 1$$

Standard Form for Vertical Transverse Axis :

$$\frac{(y - k)^2}{a^2} - \frac{(x - h)^2}{b^2} = 1$$

Where (h, k) = center

a = distance between center and either vertex

Foci can be found by using $b^2 = c^2 - a^2$

Where c is the distance between center and either focus. ($b > 0$)

Parabola

Vertical axis: $y = a(x - h)^2 + k$

Horizontal axis: $x = a(y - k)^2 + h$

Where (h, k) = vertex

a = scaling factor

