SIDDHIKSHA EDUCATION CARE 2024-25

MATHEMATICS

Chapter 3

XI STD

1. State and prove Napier's Formula.

Soln:-

Statement: In ΔABC, we have

(i) $\tan\frac{A-B}{2} = \frac{a-b}{a+b}\cot\frac{C}{2}$.

(ii) $\tan\frac{B-C}{2} = \frac{b-c}{b+c}\cot\frac{A}{2}$

(iii) $\tan\frac{C-A}{2} = \frac{c-a}{c+a}\cot\frac{B}{2}$.

Proof:-

W.k.T sine formula,

$$\frac{a}{\sin A} = \frac{b}{\sin B} = \frac{c}{\sin C} = 2R$$

Now,

$$\frac{a-b}{a+b}\cot\frac{C}{2} = \frac{2R\sin A - 2R\sin B}{2R\sin A + 2R\sin B}\cot\frac{C}{2}$$

$$= \frac{\cancel{2R}(\sin A - \sin B)}{\cancel{2R}(\sin A + \sin B)}\cot\frac{C}{2}$$

$$= \frac{\cancel{2}\cos\frac{A+B}{2}\cdot\sin\frac{A-B}{2}}{\cancel{2}\sin\frac{A+B}{2}\cdot\cos\frac{A-B}{2}}\cdot\cot\frac{C}{2}$$

$$= \cot\frac{A+B}{2}\cdot\tan\frac{A-B}{2}\cdot\cot\frac{C}{2}$$

$$= \cot\left(90^\circ - \frac{C}{2}\right)\cdot\tan\frac{A-B}{2}\cot\frac{C}{2}$$

$$= \tan\cancel{\frac{C}{2}} \cdot \tan\frac{A-B}{2} \cdot \cancel{\cot\frac{C}{2}}$$

$$= \tan\frac{A-B}{2}$$

Similarly we can prove the other two results.

1. In any triangles. the lengths of the sides are proportional to the sines of the opposite angles. That is, in ΔABC, $\frac{a}{\sin A} = \frac{b}{\sin B} = \frac{c}{\sin C} = 2R$

where R is the circumradius of the triangle.

2. If $A+B+C = 180°$ prove that $\sin 2A + \sin 2B + \sin 2C = 4\sin A \sin B \sin C$.

Soln:-

$LHS \Rightarrow \sin 2A + \sin 2B + \sin 2C$

$= (\sin 2A + \sin 2B) + \sin 2C$

$= 2\sin(A+B)\cos(A-B) + 2\sin C\cos C$

$= 2\sin C\cos(A-B) + 2\sin C\cos C$

$[\because \sin(A+B) = \sin(180-C)$
$= \sin C]$

$= 2\sin C\,[\cos(A-B) + \cos C]$

$= 2\sin C\,[\cos(A-B) - \cos(A+B)]$

$[\because \cos C = \cos(180-(A+B))$
$= -\cos(A+B)]$

$= 2\sin C\left\{2\sin\frac{2A}{2}\sin\frac{2B}{2}\right\}$

$= 4\sin A\sin B\sin C$

$\Rightarrow RHS$

2. If $A+B+C = 180°$, prove that $\cos A + \cos B - \cos C = -1 + 4 \cos \frac{A}{2} . \cos \frac{B}{2} . \sin \frac{C}{2}$.

3. If $A+B+C = \frac{\pi}{2}$, prove the following

$\sin 2A + \sin 2B + \sin 2C = 4 \cos A \cos B \cos C$.

soln:-

LHS $\Rightarrow$ $\sin 2A + \sin 2B + \sin 2C$

$= (\sin 2A + \sin 2B) + \sin 2C$

$= 2 \sin (A+B) \cos (A-B) + 2 \sin C \cos C$

$= 2 \sin (90-C) \cos (A-B) + 2 \sin C \cos C$

$[\because A+B = 90-C]$

$= 2 \cos C \cos (A-B) + 2 \sin C \cos C$

$= 2 \cos C [\cos (A-B) + \sin C]$

$= 2 \cos C [\cos (A-B) + \cos (A+B)]$

$[\because \sin C = \cos (A+B)]$

$= 2 \cos C [2 \cos A \cos B]$

$= 4 \cos A \cos B \cos C$

$\Rightarrow$ RHS.

4. If $A+B+C = \frac{\pi}{2}$, prove the following

$\cos 2A + \cos 2B + \cos 2C = 1 + 4 \sin A \sin B \sin C$.

3. If $x+y+z = xyz$, then prove that

$$\frac{2x}{1-x^2} + \frac{2y}{1-y^2} + \frac{2z}{1-z^2} = \frac{2x}{1-x^2} . \frac{2y}{1-y^2} . \frac{2z}{1-z^2}$$

5. Show that $\cos\frac{\pi}{15}\cdot\cos\frac{2\pi}{15}\cdot\cos\frac{3\pi}{15}\cdot\cos\frac{4\pi}{15}\cdot\cos\frac{5\pi}{15}\cdot\cos\frac{6\pi}{15}\cdot\cos\frac{7\pi}{15}=\frac{1}{128}$

soln:-

Let $\frac{\pi}{15}=12^\circ$

LHS ⇒

$\cos 12^\circ\cdot\cos 24^\circ\cdot\cos 36^\circ\cdot\cos 48^\circ\cdot\cos 60^\circ\cdot\cos 72^\circ\cdot\cos 84^\circ$ — (A)

W.K.T

$\cos A\cdot\cos(60+A)\cdot\cos(60-A)$

$=\cos A\left[\cos^2 60^\circ-\sin^2 A\right]$

$=\cos A\left(\frac{1}{4}-(1-\cos^2 A)\right)$

$=\cos A\left(\frac{1}{4}-1+\cos^2 A\right)$

$=\cos A\left(\cos^2 A-\frac{3}{4}\right)$

$=\frac{4\cos^3 A-3\cos A}{4}=\frac{1}{4}\cos 3A.$

$\therefore \cos 12^\circ\cdot\cos 72^\circ\cdot\cos 48^\circ=\frac{1}{4}\cos 3(12^\circ)$

$=\frac{1}{4}\cos 36^\circ$

$=\frac{1}{4}\left[\frac{\sqrt{5}+1}{4}\right]$ — (1)

$\therefore \cos 24^\circ\cdot\cos 84^\circ\cdot\cos 36^\circ=\frac{1}{4}\cos 3(24^\circ)$

$=\frac{1}{4}\cos 72^\circ$

$\because \cos 72^\circ=\cos(90-18^\circ)$ $=\frac{1}{4}\left[\frac{\sqrt{5}-1}{4}\right]$ — (2)

$=\sin 18^\circ$

From (A)

$$= \frac{1}{4}\left[\frac{\sqrt{5}+1}{4}\right] \cdot \frac{1}{4}\left[\frac{\sqrt{5}-1}{4}\right] \cdot \frac{1}{2}$$

$$= \frac{1}{4\times4\times2} \times \frac{(\sqrt{5})^2-(1)^2}{16}$$

$$= \frac{1}{32} \times \frac{(5-1)}{16}$$

$$= \frac{1}{32} \times \frac{\cancel{4}}{\cancel{16}\,4}$$

$$= \frac{1}{128}$$ Hence proved.

5. Prove that $1+\cos 2x+\cos 4x+\cos 6x = 4\cos x \cos 2x \cos 3x$.

6. Prove that $(1+\tan 1^\circ)(1+\tan 2^\circ)(1+\tan 3^\circ)\ldots(1+\tan 44^\circ)$ is a multiple of 4.

Soln:-

$$1+\tan 44^\circ = 1+\tan(45^\circ-1)$$

$$= 1+\frac{\tan 45^\circ-\tan 1^\circ}{1+\tan 45^\circ \tan 1^\circ}$$

$$= 1+\frac{1-\tan 1^\circ}{1+\tan 1^\circ}$$

$$= \frac{1+\cancel{\tan 1^\circ}+1-\cancel{\tan 1^\circ}}{1+\tan 1^\circ}$$

$$= \frac{2}{1+\tan 1^\circ}$$

$\therefore$ ~~tha~~ $(1+\tan 1^\circ)(1+\tan 44^\circ) = (\cancel{1+\tan 1^\circ})\left(\frac{2}{\cancel{1+\tan 1^\circ}}\right)$

$$= 2.$$

Similarly $(1+\tan 2^\circ)(1+\tan 43^\circ) = 2$

$(1+\tan 3^\circ)(1+\tan 42^\circ) = 2$

$\vdots$

$(1+\tan 22^\circ)(1+\tan 23^\circ) = 2$

$\therefore (1+\tan 1^\circ)(1+\tan 2^\circ)(1+\tan 3^\circ) \ldots (1+\tan 44^\circ)$

$= 2 \times 2 \times 2 \times \ldots$ (22 times)

It is multiple of 4.

6. Prove that $32(\sqrt{3}) \cdot \sin\frac{\pi}{48} \cdot \cos\frac{\pi}{48} \cdot \cos\frac{\pi}{24} \cdot \cos\frac{\pi}{12} \cos\frac{\pi}{6} = 3$.

7. If $\frac{\cos^4\alpha}{\cos^2\beta} + \frac{\sin^4\alpha}{\sin^2\beta} = 1$, prove that $\sin^4\alpha + \sin^4\beta = 2\sin^2\alpha \sin^2\beta$.

Soln:-

Given $\frac{\cos^4\alpha}{\cos^2\beta} + \frac{\sin^4\alpha}{\sin^2\beta} = 1$

$$\frac{\cos^4\alpha \sin^2\beta + \sin^4\alpha\cos^2\beta}{\cos^2\beta\sin^2\beta} = 1$$

$$\cos^4\alpha\sin^2\beta + \sin^4\alpha\cos^2\beta = \cos^2\beta\sin^2\beta$$

$$(1-\sin^2\alpha)^2\sin^2\beta + \sin^4\alpha(1-\sin^2\beta) = \sin^2\beta\cos^2\beta$$

$$(1+\sin^4\alpha - 2\sin^2\alpha)\sin^2\beta + \sin^4\alpha(1-\sin^2\beta) = \sin^2\beta\cos^2\beta.$$

$\sin^2\beta + \cancel{\sin^4\alpha\sin^2\beta} - 2\sin^2\alpha\sin^2\beta + \sin^4\alpha - \cancel{\sin^4\alpha\sin^2\beta} = \sin^2\beta\,(1-\sin^2\beta)$

$\Rightarrow \sin^2\beta - 2\sin^2\alpha\sin^2\beta + \sin^4\alpha = \sin^2\beta - \sin^4\beta$

$\cancel{\sin^2\beta} - 2\sin^2\alpha\sin^2\beta + \sin^4\alpha - \cancel{\sin^2\beta} + \sin^4\beta = 0$

$\Rightarrow \sin^4\alpha + \sin^4\beta - 2\sin^2\alpha\sin^2\beta = 0$

$\sin^4\alpha + \sin^4\beta = 2\sin^2\alpha\sin^2\beta$

Hence proved.

7. If $\dfrac{\cos^4\alpha}{\cos^2\beta} + \dfrac{\sin^4\alpha}{\sin^2\beta} = 1$ prove that $\dfrac{\cos^4\beta}{\cos^2\alpha} + \dfrac{\sin^4\beta}{\sin^2\alpha} = 1.$

8. If $x = \sum_{n=0}^{\infty}\cos^{2n}\theta$, $y = \sum_{n=0}^{\infty}\sin^{2n}\theta$ and $z = \sum_{n=0}^{\infty}\cos^{2n}\theta\,.\,\sin^{2n}\theta$, $0<\theta<\frac{\pi}{2}$ then show that $xyz = x+y+z$. [Hint: Use the formula $1+x+x^2+\dots = \dfrac{1}{1-x}$ where $|x|<1$].

Soln:-

$x = 1+\cos^2\theta+\cos^4\theta+\cos^6\theta+\dots = \dfrac{1}{1-\cos^2\theta} = \dfrac{1}{\sin^2\theta}$

$y = 1+\sin^2\theta+\sin^4\theta+\sin^6\theta+\dots = \dfrac{1}{1-\sin^2\theta} = \dfrac{1}{\cos^2\theta}$

$z = 1+\cos^2\theta\sin^2\theta+\cos^4\theta\sin^4\theta+\dots = \dfrac{1}{1-\cos^2\theta\sin^2\theta}$

LHS $\Rightarrow$ xyz

$$= \frac{1}{\sin^2\theta} \cdot \frac{1}{\cos^2\theta} \cdot \frac{1}{1-\sin^2\theta\cos^2\theta}$$

$$= \frac{1}{\sin^2\theta\cos^2\theta\,(1-\sin^2\theta\cos^2\theta)} \quad \text{--- (1)}$$

RHS $\Rightarrow$ $x+y+z$

$$= \frac{1}{\sin^2\theta} + \frac{1}{\cos^2\theta} + \frac{1}{1-\sin^2\theta\cos^2\theta}$$

$$= \frac{\cos^2\theta+\sin^2\theta}{\sin^2\theta\cos^2\theta} + \frac{1}{1-\sin^2\theta\cos^2\theta}$$

$$= \frac{1}{\sin^2\theta\cos^2\theta} + \frac{1}{1-\sin^2\theta\cos^2\theta}$$

$$= \frac{1-\cancel{\sin^2\theta\cos^2\theta}+1\,(\cancel{\sin^2\theta\cos^2\theta})}{\sin^2\theta\cos^2\theta\,(1-\sin^2\theta\cos^2\theta)}$$

$$= \frac{1}{\sin^2\theta\cos^2\theta\,(1-\sin^2\theta\cos^2\theta)} \quad \text{--- (2)}$$

from (1) & (2)

LHS = RHS

$xyz = x+y+z$ Hence proof.

8. If $\tan^2\theta = 1-k^2$ show that $\sec\theta + \tan^3\theta \operatorname{cosec}\theta = (2-k^2)^{3/2}$. Also find the values of k for which this result holds.