SIDDHIKSHA EDUCATION CARE 2024-25

MATHEMATICS

Chapter 5

XI STD

1. Prove that $\sqrt[3]{x^3+7} - \sqrt[3]{x^3+4}$ is approximately equal to $\frac{1}{x^2}$ when x is large.

Soln:-

$$\sqrt[3]{x^3+7} = (x^3+7)^{1/3}$$

$$= \left[x^3\left(1+\frac{7}{x^3}\right)\right]^{1/3}$$

$$= x\left(1+\frac{7}{x^3}\right)^{1/3}$$

$$= x\left(1+\frac{1}{3}\times\frac{7}{x^3}+\frac{\frac{1}{3}\left(\frac{1}{3}-1\right)}{2!}\left(\frac{7}{x^3}\right)^2+\cdots\right)$$

$$= x\left(1+\frac{7}{3}\times\frac{1}{x^3}-\frac{49}{9}\times\frac{1}{x^6}+\cdots\right)$$

$$= x+\frac{7}{3x^2}-\frac{49}{9x^5}+\cdots$$

$$\sqrt[3]{x^3+4} = (x^3+4)^{1/3}$$

$$= \left[x^3\left(1+\frac{4}{x^3}\right)\right]^{1/3}$$

$$= x\left(1+\frac{4}{x^3}\right)^{1/3}$$

$$= x\left(1+\frac{1}{3}\times\frac{4}{x^3}+\frac{\frac{1}{3}\left(\frac{1}{3}-1\right)}{2!}\left(\frac{4}{x^3}\right)^2+\cdots\right)$$

$$= x\left(1+\frac{4}{3}\times\frac{1}{x^3}-\frac{16}{9}\times\frac{1}{x^6}+\cdots\right)$$

$$= x+\frac{4}{3x^2}-\frac{16}{9x^5}+\cdots$$

Since x is large, $\frac{1}{x}$ is very small and hence the higher powers of $\frac{1}{x}$ are negligible.

Thus $\sqrt[3]{x^3+7} = x + \frac{7}{3x^2}$

and $\sqrt[3]{x^3+4} = x + \frac{4}{3x^2}$

$\therefore \sqrt[3]{x^3+7} - \sqrt[3]{x^3+4} = \cancel{x} + \frac{7}{3x^2} - \cancel{x} - \frac{4}{3x^2}$

$= \frac{7-4}{3x^2}$

$= \frac{3}{3x^2}$

$= \frac{1}{x^2}$

1. Prove that $\sqrt[3]{x^3+6} - \sqrt[3]{x^3+3}$ is approximately equal to $\frac{1}{x^2}$ when x is sufficiently large.

2. Prove that $\sqrt{\frac{1-x}{1+x}}$ is approximately equal to $1 - x + \frac{x^2}{2}$ when x is very small.

Soln:-

LHS $\Rightarrow \sqrt{\frac{1-x}{1+x}} = \frac{(1-x)^{1/2}}{(1+x)^{1/2}}$

$= (1-x)^{1/2}(1+x)^{-1/2}$

$$= \left(1 - \frac{1}{2}x + \frac{\frac{1}{2}\left(-\frac{1}{2}\right)}{2\cdot 1}x^2 \dots\right)\left(1 - \frac{1}{2}x + \frac{\left(\frac{1}{2}\right)\left(\frac{3}{2}\right)}{2\cdot 1}x^2 \dots\right)$$

$$= \left(1 - \frac{x}{2} - \frac{x^2}{8} \dots\right)\left(1 - \frac{x}{2} + \frac{3x^2}{8} \dots\right)$$

$$= 1 - \frac{x}{2} + \frac{3x^2}{8} - \frac{x}{2} + \frac{x^2}{4} - \frac{x^2}{8} \dots$$

$$= 1 - \frac{2x}{2} + \frac{3x^2 + 2x^2 - x^2}{8} \dots$$

$$= 1 - x + \frac{4x^2}{8}$$

$$= 1 - x + \frac{x^2}{2} \Rightarrow \text{RHS} \quad \text{Hence proved.}$$

2. Find the value of $\sum_{n=1}^{\infty} \frac{1}{2n-1}\left(\frac{1}{9^{n-1}} + \frac{1}{9^{2n-1}}\right)$.

3. Compute the sum of first n terms of
$8 + 88 + 888 + 8888 + \dots$

Soln:-

$$S_n = 8 + 88 + 888 + 8888 + \dots n \text{ terms}$$

$$= 8\left[1 + 11 + 111 + 1111 + \dots n \text{ terms}\right]$$

$$= \frac{8}{9}\left[9 + 99 + 999 + 9999 + \dots n \text{ terms}\right]$$

$$= \frac{8}{9}\left[(10-1) + (10^2-1) + (10^3-1) + \dots n \text{ terms}\right]$$

$$= \frac{8}{9}\left[(10 + 10^2 + 10^3 + \dots n \text{ terms}) - (1+1+1 \dots n \text{ terms})\right]$$

$$= \frac{8}{9}\left[\frac{10(10^n - 1)}{10-1} - n\right]$$

$$= \frac{8}{9}\left[\frac{10(10^n-1)}{9} - n\right]$$

$$= \frac{80}{81}(10^n-1) - \frac{8n}{9}.$$

3. Compute the sum of first n terms of $6+66+666+6666+\ldots$

4. The AM of two numbers exceeds their GM by 10 and HM by 16. Find the numbers.

Soln:-

Let the two numbers be a and b.

$AM = \frac{a+b}{2}$, $GM = \sqrt{ab}$ and $HM = \frac{2ab}{a+b}$

Given $AM = GM + 10$

$AM = HM + 16$

$$\frac{a+b}{2} = \sqrt{ab} + 10 \quad \text{--- (1)}$$

$$\frac{a+b}{2} = \frac{2ab}{a+b} + 16 \quad \text{--- (2)}$$

from (2), $\frac{a+b}{2} - \frac{2ab}{a+b} = 16$

$$\frac{(a+b)^2 - 4ab}{2(a+b)} = 16$$

$$(a-b)^2 = 32(a+b) \quad \text{--- (3)}$$

from (1), $\frac{a+b}{2} = \sqrt{ab} + 10$

$$a+b = 2\sqrt{ab} + 20$$

$$a+b-20 = 2\sqrt{ab}$$

Squaring,

So, $(a+b-20)^2 = 4ab$

$$(a+b)^2 - 40(a+b) + 400 = 4ab$$

$$(a+b)^2 - 4ab = 40(a+b) - 400 \quad -(4)$$

from (3) & (4),

$$32(a+b) = 40(a+b) - 400$$

$$40(a+b) - 32(a+b) = 400$$

$$8(a+b) = 400$$

$$a+b = \frac{400}{8}$$

$$a+b = 50$$

$$a = 50-b$$

sub $a = 50-b$ in (3),

$$(50-b-b)^2 = 32(50)$$

$$(50-2b)^2 = 32 \times 50$$

$$50-2b = \sqrt{32 \times 50}$$

$$= \sqrt{16 \times 2 \times 25 \times 2}$$

$$= 4 \times 5 \times 2$$

$$= 40$$

$$-2b = 40-50$$

$$-2b = -10$$

$$b = \frac{-10}{-2} = 5.$$

$$\Rightarrow a = 50-5$$

$$= 45.$$

So the two numbers are 5 and 45.

4. If a, b, c are respectively the p^{th}, q^{th} and r^{th} terms of a GP, show that $(q-r)\log a + (r-p)\log b + (p-q)\log c = 0$.

5\. If the product of the 4^{th}, 5^{th} and 6^{th} terms of a geometric progression is 4096 and if the product of the 5^{th}, 6^{th} and 7^{th} terms of it is 32768, find the sum of first 8 terms of the G.P.

Soln:-

Let $a, ar, ar^2 \ldots$ be the geometric Series.

Given 4^{th}, 5^{th} and 6^{th} terms of G.P = 4096

$$ar^3 \times ar^4 \times ar^5 = 4096$$

$$a^3 r^{12} = 4096 \quad \text{---} (1)$$

III'ly 5^{th}, 6^{th} and 7^{th} terms of G.P = 32768

$$ar^4 \times ar^5 \times ar^6 = 32768$$

$$a^3 r^{15} = 32768 \quad \text{---} (2)$$

Therefore,

$$\frac{\cancel{a^3} r^{15}}{\cancel{a^3} r^{12}} = \frac{32768}{4096}$$

$$r^3 = 8$$

$$r = 2$$

$$\Rightarrow \quad a^3 r^{12} = 4096$$

$$a^3 (2)^{12} = 4096$$

$$a^3 = \frac{4096}{4096}$$

$$a^3 = 1$$

$$a = 1.$$

∴ Sum of the first 8 terms is $\frac{a(1-r^8)}{1-r}$

$$= \frac{1-2^8}{1-2}$$

$$= \frac{1-256}{-1} = \frac{-255}{-1}$$

$$= 255.$$

5. The product of three increasing numbers in GP is 5832. If we add 6 to the second number and 9 to the third number, then resulting numbers form an A.P. Find the numbers in GP.

6. The 2nd, 3rd and 4th terms in the binomial expansion of $(x+a)^n$ are 240, 720 and 1080 for a suitable value of x. Find x, a and n.

Soln:-

Given $T_2 = 240$, $T_3 = 720$ and $T_4 = 1080$

$$T_2 = nC_1 \cdot x^{n-1} \cdot a = 240 \quad \text{--- (1)}$$

$$T_3 = nC_2 \cdot x^{n-2} \cdot a^2 = 720 \quad \text{--- (2)}$$

$$T_4 = nC_3 \cdot x^{n-3} \cdot a^3 = 1080 \quad \text{--- (3)}$$

$$\frac{(2)}{(1)}, \quad \frac{a}{x} = \frac{6}{n-1} \quad \text{--- (4)}$$

$$\frac{(3)}{(2)}, \quad \frac{a}{x} = \frac{9}{2(n-2)} \quad \text{--- (5)}$$

From (4) & (5),

$$\frac{6}{n-1} = \frac{9}{2(n-2)}$$

$$6(2(n-2)) = 9(n-1)$$

$$6(2n-4) = 9n-9$$

$$12n-24 = 9n-9$$

$$12n-9n = 24-9$$

$$3n = 15$$

$$n = \frac{15}{3} \Rightarrow n = 5.$$

When $n=5$ in (1),

$$5x^4 a = 240 \quad \text{--- (6)}$$

When $n = 5$ in (4),

$$\frac{a}{x} = \frac{6}{5-1}$$

$$\frac{a}{x} = \frac{6}{4} \quad \text{--- (7)}$$

$\frac{(6)}{(7)}$,

$$\frac{5x^4 a}{\frac{a}{x}} = \frac{240}{\frac{6}{4}}$$

$$5x^4 \cancel{a} \times \frac{x}{\cancel{a}} = \frac{\overset{40}{\cancel{240}} \times 4}{\cancel{6}}$$

$$5x^5 = 160$$

$$x^5 = \frac{160}{5}$$

$$x^5 = 32$$

$$x^5 = 2^5$$

$$\therefore x = 2$$

When $x = 2$ in (7),

$$\frac{a}{2} = \frac{6}{4}$$

$$a = \frac{\cancel{2} \times \cancel{6}\,3}{\cancel{4}\,\cancel{2}}$$

$$a = 3.$$

6. Expand $\left(x^2 + \sqrt{1-x^2}\right)^5 + \left(x^2 - \sqrt{1-x^2}\right)^5$.

7. Expand $\left(2x^2 - 3\sqrt{1-x^2}\right)^4 + \left(2x^2 + 3\sqrt{1-x^2}\right)^4$

Soln: -

Let $a = 2x^2$

$b = 3\sqrt{1-x^2}$

We have $(a-b)^4 + (a+b)^4$

Now,

$(a-b)^4 = 4C_0\, a^4 (-b)^0 + 4C_1\, a^3(-b)^1 + 4C_2\, a^2(-b)^2 + 4C_3\, a^1(-b)^3 + 4C_4\, a^0(-b)^4$

$\because 4C_0 = 1 = 4C_4$

$4C_1 = 4 = 4C_3$

$4C_2 = \frac{4\times 3}{2\times 1} = 6.$

$(a-b)^4 = a^4 - 4a^3b + 6a^2b^2 - 4ab^3 + b^4$

$(a+b)^4 = a^4 + 4a^3b + 6a^2b^2 + 4ab^3 + b^4$

$(a-b)^4 + (a+b)^4 = 2a^4 + 12a^2b^2 + 2b^4$

$= 2\left[a^4 + 6a^2b^2 + b^4\right]$

sub a and b we get

$= 2\left[(2x^2)^4 + 6(2x^2)^2(3\sqrt{1-x^2})^2 + (3\sqrt{1-x^2})^4\right]$

$= 2\left[16x^8 + 6(4x^4)(9(1-x^2)) + 81(1-x^2)^2\right]$

$= 2\left[16x^8 + 24x^4(9-9x^2) + 81(1+x^4-2x^2)\right]$

$= 2\left[16x^8 + 216x^4 - 216x^6 + 81 + 81x^4 - 162x^2\right]$

$= 2\left[16x^8 - 216x^6 + 297x^4 - 162x^2 + 81\right]$

$= 32x^8 - 432x^6 + 594x^4 - 324x^2 + 162.$

7. If n is a positive integer, using Binomial Theorem show that $9^{n+1} - 8n - 9$ is always divisible by 64.