தமிழ்நாடு பள்ளிக்கல்வித் துறை
மதுரை மாவட்டம்
12 – Mathematics
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Question Paper Design

Questions asked	Question to be answered	Marks
One Mark - 20	20	20 x 1 = 20
2 Mark - 10	7	7 x 2 = 14
3 Mark - 10	7	7 x 3 = 21
5 Mark - 14	7	7 x 5 = 35
Total		90
Internal Marks		10
Total Marks		100

- Write the Exam with focus and confidence
- Write the question number correctly
- One Mark to write the correct option and the corresponding answer
- Give more importance to one mark questions
- Suitable marks will be given to formula and Diagrams
- Answer the answers to the well-known questions first
- Try the prove that or show that questions, and finally prove that or show that.
- If you don't know the complete answer, you can easily get step mark if you write as much you know.
- After reading the questions carefully in the exam room, you should them without any anxiety.
- A lot of Model Exams have to be written

CHAPTER 1 – Application of Matrices and Determinants

1	If $\lvert\operatorname{adj}(\operatorname{adj} A)\rvert = \lvert A\rvert^9$, then the order of the square matrix A is (1) 3 (2) 4 (3) 2 (4) 5	**Ans: 4**
2	If A is a 3 x 3 non-singular matrix such that $AA^T = A^TA$ and $B = A^{-1}A^T$, then $BB^T =$ (1) A (2) B (3) I_3 (4) B^T	**Ans: I_3**
3	If $A = \begin{bmatrix} 3 & 5 \\ 1 & 2 \end{bmatrix}$, $B = \operatorname{adj} A$ and $C = 3A$, then $\frac{\lvert\operatorname{adj} B\rvert}{\lvert C\rvert} =$ (1) $\frac{1}{3}$ (2) $\frac{1}{9}$ (3) $\frac{1}{4}$ (4) 1	**Ans: $\frac{1}{9}$**
4	If $A\begin{bmatrix} 1 & -2 \\ 1 & 4 \end{bmatrix} = \begin{bmatrix} 6 & 0 \\ 0 & 6 \end{bmatrix}$, then $A =$ (1) $\begin{bmatrix} 1 & -2 \\ 1 & 4 \end{bmatrix}$ (2) $\begin{bmatrix} 1 & 2 \\ -1 & 4 \end{bmatrix}$ (3) $\begin{bmatrix} 4 & 2 \\ -1 & 1 \end{bmatrix}$ (4) $\begin{bmatrix} 4 & -1 \\ 2 & 1 \end{bmatrix}$	**Ans: $\begin{bmatrix} 4 & 2 \\ -1 & 1 \end{bmatrix}$**
5	If $A = \begin{bmatrix} 7 & 3 \\ 4 & 2 \end{bmatrix}$, then $9I - A =$ (1) A^{-1} (2) $\frac{A^{-1}}{2}$ (3) $3A^{-1}$ (4) $2A^{-1}$	**Ans: $2A^{-1}$**
6	If $A = \begin{bmatrix} 2 & 0 \\ 1 & 5 \end{bmatrix}$ and $B = \begin{bmatrix} 1 & 4 \\ 2 & 0 \end{bmatrix}$ then $\lvert\operatorname{adj}(AB)\rvert =$ (1) -40 (2) -80 (3) -60 (4) -20	**Ans: -80**
7	If $P = \begin{bmatrix} 1 & x & 0 \\ 1 & 3 & 0 \\ 2 & 4 & -2 \end{bmatrix}$ is the adjoint of 3×3 matrix A and $\lvert A\rvert = 4$, then x is (1) 15 (2) 12 (3) 14 (4) 11	**Ans: 11**
8	If $A = \begin{bmatrix} 3 & 1 & -1 \\ 2 & -2 & 0 \\ 1 & 2 & -1 \end{bmatrix}$ and $A^{-1} = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$ then the value of a_{23} is (1) 0 (2) -2 (3) -3 (4) -1	**Ans: -1**
9	If A, B and C are invertible matrices of some order, then which one of the following is not true? (1) $\operatorname{adj} A = \lvert A\rvert A^{-1}$ (2) $\operatorname{adj}(AB) = (\operatorname{adj} A)(\operatorname{adj} B)$ (3) $\det A^{-1} = (\det A)^{-1}$ (4) $(ABC)^{-1} = C^{-1}B^{-1}A^{-1}$	**Ans: $\operatorname{adj}(AB) = (\operatorname{adj} A)(\operatorname{adj} B)$**

10	If $(AB)^{-1} = \begin{bmatrix} 12 & -17 \\ -19 & 27 \end{bmatrix}$ and $A^{-1} = \begin{bmatrix} 1 & -1 \\ -2 & 3 \end{bmatrix}$, then $B^{-1} =$ (1) $\begin{bmatrix} 2 & -5 \\ -3 & 8 \end{bmatrix}$ (2) $\begin{bmatrix} 8 & 5 \\ 3 & 2 \end{bmatrix}$ (3) $\begin{bmatrix} 3 & 1 \\ 2 & 1 \end{bmatrix}$ (4) $\begin{bmatrix} 8 & -5 \\ -3 & 2 \end{bmatrix}$	**Ans:** $\begin{bmatrix} 2 & -5 \\ -3 & 8 \end{bmatrix}$
11	If $A^T A^{-1}$ is symmetric, then $A^2 =$ (1) A^{-1} (2) $(A^T)^2$ (3) A^T (4) $(A^{-1})^2$	**Ans:** $(A^T)^2$
12	If A is a non-singular matrix such that $A^{-1} = \begin{bmatrix} 5 & 3 \\ -2 & -1 \end{bmatrix}$, then $(A^T)^{-1} =$ (1) $\begin{bmatrix} -5 & 3 \\ 2 & 1 \end{bmatrix}$ (2) $\begin{bmatrix} 5 & 3 \\ -2 & -1 \end{bmatrix}$ (3) $\begin{bmatrix} -1 & -3 \\ 2 & 5 \end{bmatrix}$ (4) $\begin{bmatrix} 5 & -2 \\ 3 & -1 \end{bmatrix}$	**Ans:** $\begin{bmatrix} 5 & -2 \\ 3 & -1 \end{bmatrix}$
13	If $A = \begin{bmatrix} \frac{3}{5} & \frac{4}{5} \\ x & \frac{3}{5} \end{bmatrix}$ and $A^T = A^{-1}$, then the value of x is (1) $\frac{-4}{5}$ (2) $\frac{-3}{5}$ (3) $\frac{3}{5}$ (4) $\frac{4}{5}$	**Ans:** $\frac{-4}{5}$
14	If $A = \begin{bmatrix} 1 & \tan\frac{\theta}{2} \\ -\tan\frac{\theta}{2} & 1 \end{bmatrix}$ and $AB = I$, then $B =$ (1) $\left(\cos^2\frac{\theta}{2}\right)A$ (2) $\left(\cos^2\frac{\theta}{2}\right)A^T$ (3) $(\cos^2\theta)I$ (4) $\left(\sin^2\frac{\theta}{2}\right)A$	**Ans:** $\left(\cos^2\frac{\theta}{2}\right)A^T$
15	If $A = \begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix}$ and $A(\text{adj } A) = \begin{bmatrix} k & 0 \\ 0 & k \end{bmatrix}$, then $k =$ (1) 0 (2) $\sin\theta$ (3) $\cos\theta$ (4) 1	**Ans: 1**
16	If $A = \begin{bmatrix} 2 & 3 \\ 5 & -2 \end{bmatrix}$ be such that $\lambda A^{-1} = A$, then λ is (1) 17 (2) 14 (3) 19 (4) 21	**Ans: 19**
17	If adj $A = \begin{bmatrix} 2 & 3 \\ 4 & -1 \end{bmatrix}$ and adj $B = \begin{bmatrix} 1 & -2 \\ -3 & 1 \end{bmatrix}$ then adj(AB) is (1) $\begin{bmatrix} -7 & -1 \\ 7 & -9 \end{bmatrix}$ (2) $\begin{bmatrix} -6 & 5 \\ -2 & -10 \end{bmatrix}$ (3) $\begin{bmatrix} -7 & 7 \\ -1 & -9 \end{bmatrix}$ (4) $\begin{bmatrix} -6 & -2 \\ 5 & -10 \end{bmatrix}$	**Ans:** $\begin{bmatrix} -6 & 5 \\ -2 & -10 \end{bmatrix}$
18	The rank of the matrix $\begin{bmatrix} 1 & 2 & 3 & 4 \\ 2 & 4 & 6 & 8 \\ -1 & -2 & -3 & -4 \end{bmatrix}$ is (1) 1 (2) 2 (3) 4 (4) 3	**Ans: 1**
19	If $x^a y^b = e^m, x^c y^d = e^n, \Delta_1 = \begin{vmatrix} m & b \\ n & d \end{vmatrix}, \Delta_2 = \begin{vmatrix} a & m \\ c & n \end{vmatrix}, \Delta_3 = \begin{vmatrix} a & b \\ c & d \end{vmatrix}$, then the values of x and y are respectively, (1) $e^{(\Delta_2/\Delta_1)}, e^{(\Delta_3/\Delta_1)}$ (2) $\log(\Delta_1/\Delta_3), \log(\Delta_2/\Delta_3)$ (3) $\log(\Delta_2/\Delta_1), \log(\Delta_3/\Delta_1)$ (4) $e^{(\Delta_1/\Delta_3)}, e^{(\Delta_2/\Delta_3)}$	**Ans:** $e^{(\Delta_1/\Delta_3)}, e^{(\Delta_2/\Delta_3)}$
20	Which of the following is/are correct? (i) Adjoint of a symmetric matrix is also a symmetric matrix. (ii) Adjoint of a diagonal matrix is also a diagonal matrix. (iii) If A is a square matrix of order n and λ is a scalar, then adj$(\lambda A) = \lambda^n$adj(A). (iv) $A(\text{adj } A) = (\text{adj } A)A = \|A\|I$ (1) Only (i) (2) (ii) and (iii) (3) (iii) and (iv) (4) (i), (ii) and (iv)	**Ans:** (i), (ii) and (iv)
21	If $\rho(A) = \rho([A \mid B])$, then the system $AX = B$ of linear equations is (1) consistent and has a unique solution (2) consistent (3) consistent and has infinitely many solution (4) inconsistent	**Ans:** consistent
22	If $0 \le \theta \le \pi$ and the system of equations $x + (\sin\theta)y - (\cos\theta)z = 0$, $(\cos\theta)x - y + z = 0$, $(\sin\theta)x + y - z = 0$ has a non-trivial solution then θ is (1) $\frac{2\pi}{3}$ (2) $\frac{3\pi}{4}$ (3) $\frac{5\pi}{6}$ (4) $\frac{\pi}{4}$	**Ans:** $\frac{\pi}{4}$

23	The augmented matrix of a system of linear equations is $\begin{bmatrix}1 & 2 & 7 & 3\\0 & 1 & 4 & 6\\0 & 0 & \lambda-7 & \mu+5\end{bmatrix}$. The system has infinitely many solutions if (1) $\lambda=7,\mu\neq-5$ (2) $\lambda=-7,\mu=5$ (3) $\lambda\neq7,\mu\neq-5$ (4) $\lambda=7,\mu=-5$	**Ans:** $\boldsymbol{\lambda=7,\mu=-5}$
24	Let $A=\begin{bmatrix}2 & -1 & 1\\-1 & 2 & -1\\1 & -1 & 2\end{bmatrix}$ and $4B=\begin{bmatrix}3 & 1 & -1\\1 & 3 & x\\-1 & 1 & 3\end{bmatrix}$. If B is the inverse of A, then the value of x is (1) 2 (2) 4 (3) 3 (4) 1	**Ans: 1**
25	If $A=\begin{bmatrix}3 & -3 & 4\\2 & -3 & 4\\0 & -1 & 1\end{bmatrix}$, then adj(adj A) is (1) $\begin{bmatrix}3 & -3 & 4\\2 & -3 & 4\\0 & -1 & 1\end{bmatrix}$ (2) $\begin{bmatrix}6 & -6 & 8\\4 & -6 & 8\\0 & -2 & 2\end{bmatrix}$ (3) $\begin{bmatrix}-3 & 3 & -4\\-2 & 3 & -4\\0 & 1 & -1\end{bmatrix}$ (4) $\begin{bmatrix}3 & -3 & 4\\0 & -1 & 1\\2 & -3 & 4\end{bmatrix}$	**Ans:** $\begin{bmatrix}3 & -3 & 4\\2 & -3 & 4\\0 & -1 & 1\end{bmatrix}$

CHAPTER 2 – Complex Numbers

1	$i^n+i^{n+1}+i^{n+2}+i^{n+3}$ is (1) 0 (2) 1 (3) -1 (4) i	**Ans: 0**
2	The value of $\sum_{i=1}^{13}\left(i^n+i^{n-1}\right)$ is (1) $1+i$ (2) i (3) 1 (4) 0	**Ans:** $\boldsymbol{1+i}$
3	The area of the triangle formed by the complex numbers z, iz and $z+iz$ in the Argand's diagram is (1) $\frac{1}{2}\lvert z\rvert^2$ (2) $\lvert z\rvert^2$ (3) $\frac{3}{2}\lvert z\rvert^2$ (4) $2\lvert z\rvert^2$	**Ans:** $\frac{1}{2}\lvert z\rvert^2$
4	The conjugate of a complex number is $\frac{1}{i-2}$. Then, the complex number is (1) $\frac{1}{i+2}$ (2) $\frac{-1}{i+2}$ (3) $\frac{-1}{i-2}$ (4) $\frac{1}{i-2}$	**Ans:** $\boldsymbol{\frac{-1}{i+2}}$
5	If $z=\frac{(\sqrt{3}+i)^3(3i+4)^2}{(8+6i)^2}$, then $\lvert z\rvert$ is equal to (1) 0 (2) 1 (3) 2 (4) 3	**Ans: 2**
6	If z is a non zero complex number, such that $2iz^2=\bar{z}$ then $\lvert z\rvert$ is (1) $\frac{1}{2}$ (2) 1 (3) 2 (4) 3	**Ans:** $\boldsymbol{\frac{1}{2}}$
7	If $\lvert z-2+i\rvert\leq 2$, then the greatest value of $\lvert z\rvert$ is (1) $\sqrt{3}-2$ (2) $\sqrt{3}+2$ (3) $\sqrt{5}-2$ (4) $\sqrt{5}+2$	**Ans:** $\sqrt{5}+2$
8	If $\left\lvert z-\frac{3}{z}\right\rvert=2$, then the least value of $\lvert z\rvert$ is (1) 1 (2) 2 (3) 3 (4) 5	**Ans: 1**
9	If $\lvert z\rvert=1$, then the value of $\frac{1+z}{1+\bar{z}}$ is (1) z (2) $\bar{z}$ (3) $\frac{1}{z}$ (4) 1	**Ans:** z
10	The solution of the equation $\lvert z\rvert-z=1+2i$ is (1) $\frac{3}{2}-2i$ (2) $-\frac{3}{2}+2i$ (3) $2-\frac{3}{2}i$ (4) $2+\frac{3}{2}i$	**Ans:** $\frac{3}{2}-2i$
11	If $\lvert z_1\rvert=1,\lvert z_2\rvert=2,\lvert z_3\rvert=3$ and $\lvert 9z_1z_2+4z_1z_3+z_2z_3\rvert=12$, then the value of $\lvert z_1+z_2+z_3\rvert$ is (1) 1 (2) 2 (3) 3 (4) 4	**Ans: 2**
12	If z is a complex number such that $z\in C\setminus R$ and $z+\frac{1}{z}\in R$, then $\lvert z\rvert$ is (1) 0 (2) 1 (3) 2 (4) 3	**Ans: 1**

13	z_1, z_2 and z_3 are complex numbers such that $z_1+z_2+z_3=0$ and $\|z_1\|=\|z_2\|=\|z_3\|=1$ then $z_1^2+z_2^2+z_3^2$ is (1) 3 (2) 2 (3) 1 (4) 0	**Ans: 0**
14	If $\frac{z-1}{z+1}$ is purely imaginary, then $\|z\|$ is (1) $\frac{1}{2}$ (2) 1 (3) 2 (4) 3	**Ans: 1**
15	If $z=x+iy$ is a complex number such that $\|z+2\|=\|z-2\|$, then the locus of z is (1) real axis (2) imaginary axis (3) ellipse (4) circle	**Ans:** imaginary axis
16	The principal argument of $\frac{3}{-1+i}$ is (1) $\frac{-5\pi}{6}$ (2) $\frac{-2\pi}{3}$ (3) $\frac{-3\pi}{4}$ (4) $\frac{-\pi}{2}$	**Ans:** $\frac{-3\pi}{4}$
17	The principal argument of $(\sin 40^\circ + i\cos 40^\circ)^5$ is (1) -110° (2) -70° (3) 70° (4) 110°	**Ans:** -110°
18	If $(1+i)(1+2i)(1+3i)\cdots(1+ni)=x+iy$, then $2\cdot 5\cdot 10\cdots(1+n^2)$ is (1) 1 (2) i (3) x^2+y^2 (4) $1+n^2$	**Ans:** x^2+y^2
19	If $\omega \neq 1$ is a cubic root of unity and $(1+\omega)^7 = A+B\omega$, then (A,B) equals (1) (1,0) (2) $(-1,1)$ (3) (0,1) (4) (1,1)	**Ans:** (1,1)
20	The principal argument of the complex number $\frac{(1+i\sqrt{3})^2}{4i(1-i\sqrt{3})}$ is (1) $\frac{2\pi}{3}$ (2) $\frac{\pi}{6}$ (3) $\frac{5\pi}{6}$ (4) $\frac{\pi}{2}$	**Ans:** $\frac{\pi}{2}$
21	If α and β are the roots of $x^2+x+1=0$, then $\alpha^{2020}+\beta^{2020}$ is (1) -2 (2) -1 (3) 1 (4) 2	**Ans: -1**
22	The product of all four values of $\left(\cos\frac{\pi}{3}+i\sin\frac{\pi}{3}\right)^{\frac{3}{4}}$ is (1) -2 (2) -1 (3) 1 (4) 2	**Ans: 1**
23	If $\omega \neq 1$ is a cubic root of unity and $\begin{vmatrix} 1 & 1 & 1 \\ 1 & -\omega^2-1 & \omega^2 \\ 1 & \omega^2 & \omega^7 \end{vmatrix} = 3k$, then k is equal to (1) 1 (2) -1 (3) $\sqrt{3}i$ (4) $-\sqrt{3}i$	**Ans:** $-\sqrt{3}i$
24	The value of $\left(\frac{1+\sqrt{3}i}{1-\sqrt{3}i}\right)^{10}$ is (1) $cis\frac{2\pi}{3}$ (2) $cis\frac{4\pi}{3}$ (3) $-cis\frac{2\pi}{3}$ (4) $-cis\frac{4\pi}{3}$	**Ans:** $cis\frac{2\pi}{3}$
25	If $\omega = \text{cis}\frac{2\pi}{3}$, then the number of distinct roots of $\begin{vmatrix} z+1 & \omega & \omega^2 \\ \omega & z+\omega^2 & 1 \\ \omega^2 & 1 & z+\omega \end{vmatrix} = 0$ (1) 1 (2) 2 (3) 3 (4) 4	**Ans: 1**

CHAPTER 3 – Theory of Equations

1	A zero of x^3+64 is (1) 0 (2) 4 (3) $4i$ (4) -4	**Ans: -4**
2	If f and g are polynomials of degrees m and n respectively, and if $h(x)=(f\circ g)(x)$, then the degree of h is (1) mn (2) $m+n$ (3) m^n (4) n^m	**Ans:** mn
3	A polynomial equation in x of degree n always has (1) n distinct roots (2) n real roots (3) n imaginary roots (4) at most one root	Ans: n imaginary roots
4	If α,β and γ are the roots of x^3+px^2+qx+r, then $\sum\frac{1}{\alpha}$ is (1) $-\frac{q}{r}$ (2) $-\frac{p}{r}$ (3) $\frac{q}{r}$ (4) $-\frac{q}{p}$	**Ans:** $-\frac{q}{r}$

5	According to the rational root theorem, which number is not possible rational root of $4x^7+2x^4-10x^3-5$? (1) -1 (2) $\frac{5}{4}$ (3) $\frac{4}{5}$ (4) 5	**Ans:** $\frac{4}{5}$
6	The polynomial x^3-kx^2+9x has three real roots if and only if, k satisfies (1) $\lvert k\rvert\le 6$ (2) $k=0$ (3) $\lvert k\rvert>6$ (4) $\lvert k\rvert\ge 6$	**Ans:** $\lvert k\rvert\ge 6$
7	The number of real numbers in $[0,2\pi]$ satisfying $\sin^4 x-2\sin^2 x+1$ is (1) 2 (2) 4 (3) 1 (4) ∞	**Ans:** **2**
8	If $x^3+12x^2+10ax+1999$ definitely has a positive root, if and only if (1) $a\ge 0$ (2) $a>0$ (3) $a<0$ (4) $a\le 0$	**Ans:** $a<0$
9	The polynomial x^3+2x+3 has (1) one negative and two real roots (2) one positive and two imaginary roots (3) three real roots (4) no solution	Ans: one negative and two real roots
10	The number of positive roots of the polynomial $\sum_{r=0}^{n} n_{c_r}(-1)^r x^r$ is (1) 0 (2) n (3) $<n$ (4) r	**Ans:** n

CHAPTER 4 – Inverse Trigonometric Functions

1	The value of $\sin^{-1}(\cos x), 0\le x\le\pi$ is (1) $\pi-x$ (2) $x-\frac{\pi}{2}$ (3) $\frac{\pi}{2}-x$ (4) $x-\pi$	**Ans:** $\frac{\pi}{2}-x$
2	If $\sin^{-1}x+\sin^{-1}y=\frac{2\pi}{3}$; then $\cos^{-1}x+\cos^{-1}y$ is equal to (1) $\frac{2\pi}{3}$ (2) $\frac{\pi}{3}$ (3) $\frac{\pi}{6}$ (4) π	**Ans:** $\frac{\pi}{3}$
3	$sin^{-1}\frac{3}{5}-\cos^{-1}\frac{12}{13}+\sec^{-1}\frac{5}{3}-\text{cosec}^{-1}\frac{13}{12}$ is equal to (1) 2π (2) π (3) 0 (4) $\tan^{-1}\frac{12}{65}$	**Ans:** **0**
4	If $\sin^{-1}x=2\sin^{-1}\alpha$ has a solution, then (1) $\lvert\alpha\rvert\le\frac{1}{\sqrt{2}}$ (2) $\lvert\alpha\rvert\ge\frac{1}{\sqrt{2}}$ (3) $\lvert\alpha\rvert<\frac{1}{\sqrt{2}}$ (4) $\lvert\alpha\rvert>\frac{1}{\sqrt{2}}$	**Ans:** $\lvert\alpha\rvert<\frac{1}{\sqrt{2}}$
5	$\sin^{-1}(\cos x)=\frac{\pi}{2}-x$ is valid for (1) $-\pi\le x\le 0$ (2) $0\le x\le\pi$ (3) $-\frac{\pi}{2}\le x\le\frac{\pi}{2}$ (4) $-\frac{\pi}{4}\le x\le\frac{3\pi}{4}$	**Ans:** $0\le x\le\pi$
6	If $\sin^{-1}x+\sin^{-1}y+\sin^{-1}z=\frac{3\pi}{2}$, the value of $x^{2017}+y^{2018}+z^{2019}-\frac{9}{x^{101}+y^{101}+z^{101}}$ is (1) 0 (2) 1 (3) 2 (4) 3	**Ans: 0**
7	If $\cot^{-1}x=\frac{2\pi}{5}$ for some $x\in R$, the value of $\tan^{-1}x$ is (1) $-\frac{\pi}{10}$ (2) $\frac{\pi}{5}$ (3) $\frac{\pi}{10}$ (4) $-\frac{\pi}{5}$	**Ans:** $\frac{\pi}{10}$
8	The domain of the function defined by $f(x)=\sin^{-1}\sqrt{x-1}$ is (1) $[1,2]$ (2) $[-1,1]$ (3) $[0,1]$ (4) $[-1,0]$	**Ans:** $[1,2]$
9	If $x=\frac{1}{5}$, the value of $\cos(\cos^{-1}x+2\sin^{-1}x)$ is (1) $-\sqrt{\frac{24}{25}}$ (2) $\sqrt{\frac{24}{25}}$ (3) $\frac{1}{5}$ (4) $-\frac{1}{5}$	**Ans:** $-\frac{1}{5}$
10	$\tan^{-1}\left(\frac{1}{4}\right)+\tan^{-1}\left(\frac{2}{9}\right)$ is equal to (1) $\frac{1}{2}\cos^{-1}\left(\frac{3}{5}\right)$ (2) $\frac{1}{2}\sin^{-1}\left(\frac{3}{5}\right)$ (3) $\frac{1}{2}\tan^{-1}\left(\frac{3}{5}\right)$ (4) $\tan^{-1}\left(\frac{1}{2}\right)$	**Ans:** $\tan^{-1}\left(\frac{1}{2}\right)$

11	If the function $f(x)=\sin^{-1}(x^2-3)$, then x belongs to (1) $[-1,1]$ (2) $[\sqrt{2},2]$ (3) $[-2,-\sqrt{2}]\cup[\sqrt{2},2]$ (4) $[-2,-\sqrt{2}]\cap[\sqrt{2},2]$	**Ans:** $[-2,-\sqrt{2}]$ $\cup[\sqrt{2},2]$
12	If $\cot^{-1}2$ and $\cot^{-1}3$ are two angles of a triangle, then the third angle is (1) $\frac{\pi}{4}$ (2) $\frac{3\pi}{4}$ (3) $\frac{\pi}{6}$ (4) $\frac{\pi}{3}$	**Ans:** $\frac{3\pi}{4}$
13	$\sin^{-1}\left(\tan\frac{\pi}{4}\right)-\sin^{-1}\left(\sqrt{\frac{3}{x}}\right)=\frac{\pi}{6}$. Then x is a root of the equation (1) $x^2-x-6=0$ (2) $x^2-x-12=0$ (3) $x^2+x-12=0$ (4) $x^2+x-6=0$	**Ans:** $x^2-x-12=0$
14	$\sin^{-1}(2\cos^2x-1)+\cos^{-1}(1-2\sin^2x)=$ (1) $\frac{\pi}{2}$ (2) $\frac{\pi}{3}$ (3) $\frac{\pi}{4}$ (4) $\frac{\pi}{6}$	**Ans:** $\frac{\pi}{2}$
15	If $\cot^{-1}(\sqrt{\sin\alpha})+\tan^{-1}(\sqrt{\sin\alpha})=u$, then $\cos 2u$ is equal to (1) $\tan^2\alpha$ (2) 0 (3) -1 (4) $\tan 2\alpha$	**Ans:** -1
16	If $\|x\|\le 1$, then $2\tan^{-1}x-\sin^{-1}\frac{2x}{1+x^2}$ is equal to (1) $\tan^{-1}x$ (2) $\sin^{-1}x$ (3) 0 (4) π	**Ans:** 0
17	The equation $\tan^{-1}x-\cot^{-1}x=\tan^{-1}\left(\frac{1}{\sqrt{3}}\right)$ has (1) no solution (2) unique solution (3) two solutions (4) infinite number of solutions	**Ans:** unique solution
18	If $\sin^{-1}x+\cot^{-1}\left(\frac{1}{2}\right)=\frac{\pi}{2}$, then x is equal to (1) $\frac{1}{2}$ (2) $\frac{1}{\sqrt{5}}$ (3) $\frac{2}{\sqrt{5}}$ (4) $\frac{\sqrt{3}}{2}$	**Ans:** $\frac{1}{\sqrt{5}}$
19	If $\sin^{-1}\frac{x}{5}+\operatorname{cosec}^{-1}\frac{5}{4}=\frac{\pi}{2}$, then the value of x is (1) 4 (2) 5 (3) 2 (4) 3	**Ans:** **3**
20	$\sin(\tan^{-1}x), \|x\|<1$ is equal to (1) $\frac{x}{\sqrt{1-x^2}}$ (2) $\frac{1}{\sqrt{1-x^2}}$ (3) $\frac{1}{\sqrt{1+x^2}}$ (4) $\frac{x}{\sqrt{1+x^2}}$	**Ans:** $\frac{x}{\sqrt{1+x^2}}$

CHAPTER 5 – Two Dimensional Analytical Geometry-II

1	The equation of the circle passing through (1,5) and (4,1) and touching y-axis is $x^2+y^2-5x-6y+9+\lambda(4x+3y-19)=0$ where λ is equal to (1) $0,-\frac{40}{9}$ (2) 0 (3) $\frac{40}{9}$ (4) $\frac{-40}{9}$	**Ans:** $0,-\frac{40}{9}$
2	The eccentricity of the hyperbola whose latus rectum is 8 and conjugate axis is equal to half the distance between the foci is (1) $\frac{4}{3}$ (2) $\frac{4}{\sqrt{3}}$ (3) $\frac{2}{\sqrt{3}}$ (4) $\frac{3}{2}$	**Ans:** $\frac{2}{\sqrt{3}}$
3	The circle $x^2+y^2=4x+8y+5$ intersects the line $3x-4y=m$ at two distinct points if (1) $15<m<65$ (2) $35<m<85$ (3) $-85<m<-35$ (4) $-35<m<15$	**Ans:** $-35<m<15$
4	The length of the diameter of the circle which touches the x-axis at the point (1,0) and passes through the point (2,3) is (1) $\frac{6}{5}$ (2) $\frac{5}{3}$ (3) $\frac{10}{3}$ (4) $\frac{3}{5}$	**Ans:** $\frac{10}{3}$
5	The radius of the circle $3x^2+by^2+4bx-6by+b^2=0$ is (1) 1 (2) 3 (3) $\sqrt{10}$ (4) $\sqrt{11}$	**Ans:** $\sqrt{10}$

6	The centre of the circle inscribed in a square formed by the lines $x^2-8x-12=0$ and $y^2-14y+45=0$ is (1) (4,7) (2) (7,4) (3) (9,4) (4) (4,9)	**Ans:** (4,7)
7	The equation of the normal to the circle $x^2+y^2-2x-2y+1=0$ which is parallel to the line $2x+4y=3$ is (1) $x+2y=3$ (2) $x+2y+3=0$ (3) $2x+4y+3=0$ (4) $x-2y+3=0$	**Ans:** $x+2y=3$
8	If $P(x,y)$ be any point on $16x^2+25y^2=400$ with foci $F_1(3,0)$ and $F_2(-3,0)$ then PF_1+PF_2 is (1) 8 (2) 6 (3) 10 (4) 12	**Ans:** 10
9	The radius of the circle passing through the point (6,2) two of whose diameters are $x+y=6$ and $x+2y=4$ is (1) 10 (2) $2\sqrt{5}$ (3) 6 (4) 4	**Ans:** $2\sqrt{5}$
10	The area of quadrilateral formed with foci of the hyperbolas $\frac{x^2}{a^2}-\frac{y^2}{b^2}=1$ and $\frac{x^2}{a^2}-\frac{y^2}{b^2}=-1$ is (1) $4(a^2+b^2)$ (2) $2(a^2+b^2)$ (3) a^2+b^2 (4) $\frac{1}{2}(a^2+b^2)$	**Ans:** $2(a^2+b^2)$
11	If the normals of the parabola $y^2=4x$ drawn at the end points of its latus rectum are tangents to the circle $(x-3)^2+(y+2)^2=r^2$, then the value of r^2 is (1) 2 (2) 3 (3) 1 (4) 4	**Ans:** **2**
12	If $x+y=k$ is a normal to the parabola $y^2=12x$, then the value of k is (1) 3 (2) −1 (3) 1 (4) 9	**Ans:** **9**
13	The ellipse $E_1:\frac{x^2}{9}+\frac{y^2}{4}=1$ is inscribed in a rectangle R whose sides are parallel to the coordinate axes. Another ellipse E_2 passing through the point (0,4) circumscribes the rectangle R. The eccentricity of the ellipse is (1) $\frac{\sqrt{2}}{2}$ (2) $\frac{\sqrt{3}}{2}$ (3) $\frac{1}{2}$ (4) $\frac{3}{4}$	**ns:** $\frac{1}{2}$
14	Tangents are drawn to the hyperbola $\frac{x^2}{9}-\frac{y^2}{4}=1$ parallel to the straight line $2x-y=1$. One of the points of contact of tangents on the hyperbola is (1) $\left(\frac{9}{2\sqrt{2}},\frac{-1}{\sqrt{2}}\right)$ (2) $\left(\frac{-9}{2\sqrt{2}},\frac{1}{\sqrt{2}}\right)$ (3) $\left(\frac{9}{2\sqrt{2}},\frac{1}{\sqrt{2}}\right)$ (4) $(3\sqrt{3},-2\sqrt{2})$	**Ans:** $\left(\frac{9}{2\sqrt{2}},\frac{1}{\sqrt{2}}\right)$
15	The equation of the circle passing through the foci of the ellipse $\frac{x^2}{16}+\frac{y^2}{9}=1$ having centre at (0,3) is (1) $x^2+y^2-6y-7=0$ (2) $x^2+y^2-6y+7=0$ (3) $x^2+y^2-6y-5=0$ (4) $x^2+y^2-6y+5=0$	**Ans:** $x^2+y^2-6y-7=0$
16	Let C be the circle with centre at (1,1) and radius = 1. If T is the circle centered at $(0,y)$ passing through the origin and touching the circle C externally, then the radius of T is equal to (1) $\frac{\sqrt{3}}{\sqrt{2}}$ (2) $\frac{\sqrt{3}}{2}$ (3) $\frac{1}{2}$ (4) $\frac{1}{4}$	**Ans:** $\frac{1}{4}$
17	Consider an ellipse whose center is of the origin and its major axis is along x-axis. If its eccentrcity is $\frac{3}{5}$ and the distance between its foci is 6 , then the area of the quadrilateral inscribed in the ellipse with diagonals as major and minor axis of the ellipse is (1) 8 (2) 32 (3) 80 (4) 40	**Ans:** 40
18	Area of the greatest rectangle inscribed in the ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$ is (1) $2ab$ (2) ab (3) $\sqrt{ab}$ (4) $\frac{a}{b}$	**Ans:** $2ab$

19	An ellipse has OB as semi minor axes, F and F' its foci and the angle FBF' is a right angle. Then the eccentricity of the ellipse is (1) $\frac{1}{\sqrt{2}}$ (2) $\frac{1}{2}$ (3) $\frac{1}{4}$ (4) $\frac{1}{\sqrt{3}}$	**Ans:** $\frac{1}{\sqrt{2}}$
20	The eccentricity of the ellipse $(x-3)^2+(y-4)^2=\frac{y^2}{9}$ is (1) $\frac{\sqrt{3}}{2}$ (2) $\frac{1}{3}$ (3) $\frac{1}{3\sqrt{2}}$ (4) $\frac{1}{\sqrt{3}}$	**Ans:** $\frac{1}{3}$
21	If the two tangents drawn from a point P to the parabola $y^2=4x$ are at right angles then the locus of P is (1) $2x+1=0$ (2) $x=-1$ (3) $2x-1=0$ (4) $x=1$	**Ans:** $x=-1$
22	The circle passing through $(1,-2)$ and touching the axis of x at $(3,0)$ passing through the point (1) $(-5,2)$ (2) $(2,-5)$ (3) $(5,-2)$ (4) $(-2,5)$	**Ans:** $(5,-2)$
23	The locus of a point whose distance from $(-2,0)$ is $\frac{2}{3}$ times its distance from the line $x=\frac{-9}{2}$ is (1) a parabola (2) a hyperbola (3) an ellipse (4) a circle	**Ans:** an ellipse
24	The values of m for which the line $y=mx+2\sqrt{5}$ touches the hyperbola $16x^2-9y^2=144$ are the roots of $x^2-(a+b)x-4=0$, then the value of $(a+b)$ is (1) 2 (2) 4 (3) 0 (4) -2	**Ans:** **0**
25	If the coordinates at one end of a diameter of the circle $x^2+y^2-8x-4y+c=0$ are $(11,2)$, the coordinates of the other end are (1) $(-5,2)$ (2) $(2,-5)$ (3) $(5,-2)$ (4) $(-2,5)$	**Ans:** $(2,-5)$

CHAPTER 6 – Applications of Vector Algebra

1	If $\vec{a}$ and $\vec{b}$ are parallel vectors, then $[\vec{a},\vec{c},\vec{b}]$ is equal to (1) 2 (2) -1 (3) 1 (4) 0	**Ans: 0**
2	If a vector $\vec{\alpha}$ lies in the plane of $\vec{\beta}$ and $\vec{\gamma}$, then (1) $[\vec{\alpha},\vec{\beta},\vec{\gamma}]=1$ (2) $[\vec{\alpha},\vec{\beta},\vec{\gamma}]=-1$ (3) $[\vec{\alpha},\vec{\beta},\vec{\gamma}]=0$ (4) $[\vec{\alpha},\vec{\beta},\vec{\gamma}]=2$	**Ans:** (3) $[\vec{\alpha},\vec{\beta},\vec{\gamma}]=0$
3	If $\vec{a}\cdot\vec{b}=\vec{b}\cdot\vec{c}=\vec{c}\cdot\vec{a}=0$, then the value of $[\vec{a},\vec{b},\vec{c}]$ is (1) $\lvert\vec{a}\rvert\lvert\vec{b}\rvert\lvert\vec{c}\rvert$ (2) $\frac{1}{3}\lvert\vec{a}\rvert\lvert\vec{b}\rvert\lvert\vec{c}\rvert$ (3) 1 (4) -1	**Ans:** $\lvert\vec{a}\rvert\lvert\vec{b}\rvert\lvert\vec{c}\rvert$
4	If $\vec{a},\vec{b},\vec{c}$ are three unit vectors such that $\vec{a}$ is perpendicular to $\vec{b}$, and is parallel to $\vec{c}$ then $\vec{a}\times(\vec{b}\times\vec{c})$ is equal to (1) $\vec{a}$ (2) $\vec{b}$ (3) $\vec{c}$ (4) $\vec{0}$	**Ans:** $\vec{b}$
5	If $[\vec{a},\vec{b},\vec{c}]=1$, then the value of $\frac{\vec{a}\cdot(\vec{b}\times\vec{c})}{(\vec{c}\times\vec{a})\cdot\vec{b}}+\frac{\vec{b}\cdot(\vec{c}\times\vec{a})}{(\vec{a}\times\vec{b})\cdot\vec{c}}+\frac{\vec{c}\cdot(\vec{a}\times\vec{b})}{(\vec{c}\times\vec{b})\cdot\vec{a}}$ is (1) 1 (2) -1 (3) 2 (4) 3	**Ans: 1**
6	The volume of the parallelepiped with its edges represented by the vectors $\hat{\imath}+\hat{\jmath},\hat{\imath}+2\hat{\jmath},\hat{\imath}+\hat{\jmath}+\pi\hat{k}$ is (1) $\frac{\pi}{2}$ (2) $\frac{\pi}{3}$ (3) π (4) $\frac{\pi}{4}$	**Ans:** π
7	If $\vec{a}$ and $\vec{b}$ are unit vectors such that $[\vec{a},\vec{b},\vec{a}\times\vec{b}]=\frac{\pi}{4}$, then the angle between $\vec{a}$ and $\vec{b}$ is (1) $\frac{\pi}{6}$ (2) $\frac{\pi}{4}$ (3) $\frac{\pi}{3}$ (4) $\frac{\pi}{2}$	**Ans:** $\frac{\pi}{6}$
8	If $\vec{a}=\hat{\imath}+\hat{\jmath}+\hat{k},\vec{b}=\hat{\imath}+\hat{\jmath},\vec{c}=\hat{\imath}$ and $(\vec{a}\times\vec{b})\times\vec{c}=\lambda\vec{a}+\mu\vec{b}$, then the value of $\lambda+\mu$ is (1) 0 (2) 1 (3) 6 (4) 3	**Ans: 0**

9	If $\vec{a},\vec{b},\vec{c}$ are non-coplanar, non-zero vectors such that $[\vec{a},\vec{b},\vec{c}]=3$, then $\{[\vec{a}\times\vec{b},\vec{b}\times\vec{c},\vec{c}\times\vec{a}]\}^2$ is equal to (1) 81 (2) 9 (3) 27 (4) 18	**Ans: 81**
10	If $\vec{a},\vec{b},\vec{c}$ are three non-coplanar vectors such that $\vec{a}\times(\vec{b}\times\vec{c})=\frac{\vec{b}+\vec{c}}{\sqrt{2}}$, then the angle between $\vec{a}$ and $\vec{b}$ is (1) $\frac{\pi}{2}$ (2) $\frac{3\pi}{4}$ (3) $\frac{\pi}{4}$ (4) π	**Ans:** $\frac{3\pi}{4}$
11	If the volume of the parallelepiped with $\vec{a}\times\vec{b},\vec{b}\times\vec{c},\vec{c}\times\vec{a}$ as coterminous edges is 8 cubic units, then the volume of the parallelepiped with $(\vec{a}\times\vec{b})\times(\vec{b}\times\vec{c})$, $(\vec{b}\times\vec{c})\times(\vec{c}\times\vec{a})$ and $(\vec{c}\times\vec{a})\times(\vec{a}\times\vec{b})$ as coterminous edges is, (1) 8 cubic units (2) 512 cubic units (3) 64 cubic units (4) 24 cubic units	**Ans:** 64 cubic untis
12	Consider the vectors $\vec{a},\vec{b},\vec{c},\vec{d}$ such that $(\vec{a}\times\vec{b})\times(\vec{c}\times\vec{d})=\vec{0}$. Let P_1 and P_2 be the planes determined by the pairs of vectors $\vec{a},\vec{b}$ and $\vec{c},\vec{d}$ respectively. Then the angle between P_1 and P_2 is (1) 0° (2) 45° (3) 60° (4) 90°	**Ans:** 0°
13	If $\vec{a}\times(\vec{b}\times\vec{c})=(\vec{a}\times\vec{b})\times\vec{c}$, where $\vec{a},\vec{b},\vec{c}$ are any three vectors such that $\vec{b}\cdot\vec{c}\neq 0$ and $\vec{a}\cdot\vec{b}\neq 0$, then $\vec{a}$ and $\vec{c}$ are (1) perpendicular (2) parallel (3) inclined at an angle $\frac{\pi}{3}$ (4) inclined at an angle $\frac{\pi}{6}$	**Ans:** parallel
14	If $\vec{a}=2\hat{\imath}+3\hat{\jmath}-\hat{k},\vec{b}=\hat{\imath}+2\hat{\jmath}-5\hat{k},\vec{c}=3\hat{\imath}+5\hat{\jmath}-\hat{k}$, then a vector perpendicular to $\vec{a}$ and lies in the plane containing $\vec{b}$ and $\vec{c}$ is (1) $-17\hat{\imath}+21\hat{\jmath}-97\hat{k}$ (2) $17\hat{\imath}+21\hat{\jmath}-123\hat{k}$ (3) $-17\hat{\imath}-21\hat{\jmath}+97\hat{k}$ (4) $-17\hat{\imath}-21\hat{\jmath}-97\hat{k}$	**Ans:** $-17\hat{\imath}-21\hat{\jmath}-97\hat{k}$
15	The angle between the lines $\frac{x-2}{3}=\frac{y+1}{-2}, z=2$ and $\frac{x-1}{1}=\frac{2y+3}{3}=\frac{z+5}{2}$ is (1) $\frac{\pi}{6}$ (2) $\frac{\pi}{4}$ (3) $\frac{\pi}{3}$ (4) $\frac{\pi}{2}$	**Ans:** $\frac{\pi}{2}$
16	If the line $\frac{x-2}{3}=\frac{y-1}{-5}=\frac{z+2}{2}$ lies in the plane $x+3y-\alpha z+\beta=0$, then (α,β) is (1) $(-5,5)$ (2) $(-6,7)$ (3) $(5,-5)$ (4) $(6,-7)$	**Ans:** $(-6,7)$
17	The angle between the line $\vec{r}=(\hat{\imath}+2\hat{\jmath}-3\hat{k})+t(2\hat{\imath}+\hat{\jmath}-2\hat{k})$ and the plane $\vec{r}\cdot(\hat{\imath}+\hat{\jmath})+4=0$ is (1) 0° (2) 30° (3) 45° (4) 90°	**Ans:** 45°
18	The coordinates of the point where the line $\vec{r}=(6\hat{\imath}-\hat{\jmath}-3\hat{k})+t(-\hat{\imath}+4\hat{k})$ meets the plane $\vec{r}\cdot(\hat{\imath}+\hat{\jmath}-\hat{k})=3$ are (1) $(2,1,0)$ (2) $(7,-1,-7)$ (3) $(1,2,-6)$ (4) $(5,-1,1)$	**Ans:** $(5,-1,1)$
19	Distance from the origin to the plane $3x-6y+2z+7=0$ is (1) 0 (2) 1 (3) 2 (4) 3	**Ans: 1**
20	The distance between the planes $x+2y+3z+7=0$ and $2x+4y+6z+7=0$ is (1) $\frac{\sqrt{7}}{2\sqrt{2}}$ (2) $\frac{7}{2}$ (3) $\frac{\sqrt{7}}{2}$ (4) $\frac{7}{2\sqrt{2}}$	**Ans:** $\frac{\sqrt{7}}{2\sqrt{2}}$
21	If the direction cosines of a line are $\frac{1}{c},\frac{1}{c},\frac{1}{c}$, then (1) $c=\pm3$ (2) $c=\pm\sqrt{3}$ (3) $c>0$ (4) $0<c<1$	**Ans:** $c=\pm\sqrt{3}$
22	The vector equation $\vec{r}=(\hat{\imath}-2\hat{\jmath}-\hat{k})+t(6\hat{\imath}-\hat{k})$ represents a straight line passing through the points (1) $(0,6,-1)$ and $(1,-2,-1)$ (2) $(0,6,-1)$ and $(-1,-4,-2)$ (3) $(1,-2,-1)$ and $(1,4,-2)$ (4) $(1,-2,-1)$ and $(0,-6,1)$	**Ans:** $(1,-2,-1)$ and $(1,4,-2)$

23	If the distance of the point (1,1,1) from the origin is half of its distance from the plane $x+y+z+k=0$, then the values of k are (1) ± 3 (2) ± 6 (3) $-3,9$ (4) $3,-9$	**Ans:** $3,-9$
24	If the planes $\vec{r}\cdot(2\hat{\imath}-\lambda\hat{\jmath}+\hat{k})=3$ and $\vec{r}\cdot(4\hat{\imath}+\hat{\jmath}-\mu\hat{k})=5$ are parallel, then the value of λ and μ are (1) $\frac{1}{2},-2$ (2) $-\frac{1}{2},2$ (3) $-\frac{1}{2},-2$ (4) $\frac{1}{2},2$	**Ans:** $-\frac{1}{2},-2$
25	If the length of the perpendicular from the origin to the plane $2x+3y+\lambda z=1$, $\lambda>0$ is $\frac{1}{5}$, then the value of λ is (1) $2\sqrt{3}$ (2) $3\sqrt{2}$ (3) 0 (4) 1	**Ans:** $2\sqrt{3}$

CHAPTER 7 – Application of Differential Calculus

1	The volume of a sphere is increasing in volume at the rate of $3\pi\text{cm}^3/\text{sec}$. The rate of change of its radius when radius is $\frac{1}{2}$ cm (1) 3 cm/s (2) 2 cm/s (3) 1 cm/s (4) $\frac{1}{2}$ cm/s	**Ans:** 3 cm/s
2	A balloon rises straight up at 10 m/s. An observer is 40 m away from the spot where the balloon left the ground. Find the rate of change of the balloon's angle of elevation in radian per second when the balloon is 30 metres above the ground. (1) $\frac{3}{25}$ radians /sec (2) $\frac{4}{25}$ radians /sec (3) $\frac{1}{5}$ radians /sec (4) $\frac{1}{3}$ radians /sec	Ans: $\frac{4}{25}$ radians /sec
3	The position of a particle moving along a horizontal line of any time t is given by $s(t)=3t^2-2t-8$. The time at which the particle is at rest is (1) $t=0$ (2) $t=\frac{1}{3}$ (3) $t=1$ (4) $t=3$	**Ans:** $t=\frac{1}{3}$
4	A stone is thrown up vertically. The height it reaches at time t seconds is given by $x=80t-16t^2$. The stone reaches the maximum height in time t seconds is given by (1) 2 (2) 2.5 (3) 3 (4) 3.5	**Ans:** 2.5
5	Find the point on the curve $6y=x^3+2$ at which y-coordinate changes 8 times as fast as x-coordinate is (1) (4,11) (2) $(4,-11)$ (3) $(-4,11)$ (4) $(-4,-11)$	**Ans:** (4,11)
6	The abscissa of the point on the curve $f(x)=\sqrt{8-2x}$ at which the slope of the tangent is -0.25 ? (1) -8 (2) -4 (3) -2 (4) 0	**Ans:** -4
7	The slope of the line normal to the curve $f(x)=2\cos 4x$ at $x=\frac{\pi}{12}$ is (1) $-4\sqrt{3}$ (2) -4 (3) $\frac{\sqrt{3}}{12}$ (4) $4\sqrt{3}$	**Ans:** $\frac{\sqrt{3}}{12}$
8	The tangent to the curve $y^2-xy+9=0$ is vertical when (1) $y=0$ (2) $y=\pm\sqrt{3}$ (3) $y=\frac{1}{2}$ (4) $y=\pm 3$	**Ans:** $y=\pm 3$
9	Angle between $y^2=x$ and $x^2=y$ at the origin is (1) $\tan^{-1}\left(\frac{3}{4}\right)$ (2) $\tan^{-1}\left(\frac{4}{3}\right)$ (3) $\frac{\pi}{2}$ (4) $\frac{\pi}{4}$	**Ans:** $\frac{\pi}{2}$
10	What is the value of the $\lim_{x\to 0}\left(\cot x-\frac{1}{x}\right)$? (1) 0 (2) 1 (3) 2 (4) ∞	**Ans: 0**
11	The function $\sin^4 x+\cos^4 x$ is increasing in the interval (1) $\left[\frac{5\pi}{8},\frac{3\pi}{4}\right]$ (2) $\left[\frac{\pi}{2},\frac{5\pi}{8}\right]$ (3) $\left[\frac{\pi}{4},\frac{\pi}{2}\right]$ (4) $\left[0,\frac{\pi}{4}\right]$	**Ans:** $\left[\frac{\pi}{4},\frac{\pi}{2}\right]$

12	The number given by the Rolle's theorem for the function $x^3 - 3x^2, x \in [0,3]$ is (1) 1 (2) $\sqrt{2}$ (3) $\frac{3}{2}$ (4) 2	**Ans: 2**
13	The number given by the Mean value theorem for the function $\frac{1}{x}, x \in [1,9]$ is (1) 2 (2) 2.5 (3) 3 (4) 3.5	**Ans: 3**
14	The minimum value of the function $\|3 - x\| + 9$ is (1) 0 (2) 3 (3) 6 (4) 9	**Ans: 9**
15	The maximum slope of the tangent to the curve $y = e^x \sin x, x \in [0,2\pi]$ is at (1) $x = \frac{\pi}{4}$ (2) $x = \frac{\pi}{2}$ (3) $x = \pi$ (4) $x = \frac{3\pi}{2}$	**Ans:** $x = \frac{\pi}{2}$
16	The maximum value of the function $x^2 e^{-2x}, x > 0$ is (1) $\frac{1}{e}$ (2) $\frac{1}{2e}$ (3) $\frac{1}{e^2}$ (4) $\frac{4}{e^4}$	**Ans:** $\frac{1}{e^2}$
17	One of the closest points on the curve $x^2 - y^2 = 4$ to the point (6,0) is (1) (2,0) (2) $(\sqrt{5}, 1)$ (3) $(3, \sqrt{5})$ (4) $(\sqrt{13}, -\sqrt{3})$	**Ans:** $(3, \sqrt{5})$
18	The maximum product of two positive numbers, when their sum of the squares is 200, is (1) 100 (2) $25\sqrt{7}$ (3) 28 (4) $24\sqrt{14}$	**Ans: 100**
19	The curve $y = ax^4 + bx^2$ with $ab > 0$ (1) has no horizontal tangent (2) is concave up (3) is concave down (4) has no points of inflection	Ans: has no points of inflection
20	The point of inflection of the curve $y = (x - 1)^3$ is (1) (0,0) (2) (0,1) (3) (1,0) (4) (1,1)	**Ans: (1, 0)**

CHAPTER 8 – Differentials and Partial Derivatives

1	A circular template has a radius of 10 cm. The measurement of radius has an approximate error of 0.02 cm. Then the percentage error in calculating area of this template is (1) 0.2% (2) 0.4% (3) 0.04% (4) 0.08%	**Ans: 0.4%**
2	The percentage error of fifth root of 31 is approximately how many times the percentage error in 31? (1) $\frac{1}{31}$ (2) $\frac{1}{5}$ (3) 5 (4) 31	**Ans:** $\frac{1}{5}$
3	If $u(x, y) = e^{x^2+y^2}$, then $\frac{\partial u}{\partial x}$ is equal to (1) $e^{x^2+y^2}$ (2) $2xu$ (3) x^2u (4) y^2u	**Ans:** $2xu$
4	If $v(x, y) = \log(e^x + e^y)$, then $\frac{\partial v}{\partial x} + \frac{\partial v}{\partial y}$ is equal to (1) $e^x + e^y$ (2) $\frac{1}{e^x+e^y}$ (3) 2 (4) 1	**Ans: 1**
5	If $w(x, y) = x^y, x > 0$, then $\frac{\partial w}{\partial x}$ is equal to (1) $x^y \log x$ (2) $y \log x$ (3) yx^{y-1} (4) $x \log y$	**Ans:** yx^{y-1}
6	If $f(x, y) = e^{xy}$, then $\frac{\partial^2 f}{\partial x \partial y}$ is equal to (1) xye^{xy} (2) $(1 + xy)e^{xy}$ (3) $(1 + y)e^{xy}$ (4) $(1 + x)e^{xy}$	**Ans:** $(1 + xy)e^{xy}$
7	If we measure the side of a cube to be 4 cm with an error of 0.1 cm, then the error in our calculation of the volume is (1) 0.4 cu.cm (2) 0.45 cu.cm (3) 2 cu.cm (4) 4.8 cu.cm	**Ans:** 4.8 cu.cm
8	The change in the surface area $S = 6x^2$ of a cube when the edge length varies from x_0 to $x_0 + dx$ is (1) $12x_0 + dx$ (2) $12x_0 dx$ (3) $6x_0 dx$ (4) $6x_0 + dx$	**Ans:** $12x_0 dx$

9	The approximate change in the volume V of a cube of side x metres caused by increasing the side by 1% is (1) $0.3xdx\ m^3$ (2) $0.03x\ m^3$ (3) $0.03x^2\ m^3$ (4) $0.03x^3\ m^3$	**Ans:** $0.03x^2\ m^3$
10	If $g(x,y) = 3x^2 - 5y + 2y^2, x(t) = e^t$ and $y(t) = \cos t$, then $\frac{dg}{dt}$ is equal to (1) $6e^{2t} + 5\sin t - 4\cos t \sin t$ (2) $6e^{2t} - 5\sin t + 4\cos t \sin t$ (3) $3e^{2t} + 5\sin t + 4\cos t \sin t$ (4) $3e^{2t} - 5\sin t + 4\cos t \sin t$	**Ans:** $6e^{2t} + 5\sin t - 4\cos t \sin t$
11	If $f(x) = \frac{x}{x+1}$, then its differential is given by (1) $\frac{-1}{(x+1)^2}dx$ (2) $\frac{1}{(x+1)^2}dx$ (3) $\frac{1}{x+1}dx$ (4) $\frac{-1}{x+1}dx$	**Ans:** $\frac{1}{(x+1)^2}dx$
12	If $u(x,y) = x^2 + 3xy + y - 2019$, then $\left.\frac{\partial u}{\partial x}\right\|_{(4,-5)}$ is equal to (1) -4 (2) -3 (3) -7 (4) 13	**Ans: -7**
13	Linear approximation for $g(x) = \cos x$ at $x = \frac{\pi}{2}$ is (1) $x + \frac{\pi}{2}$ (2) $-x + \frac{\pi}{2}$ (3) $x - \frac{\pi}{2}$ (4) $-x - \frac{\pi}{2}$	**Ans:** $-x + \frac{\pi}{2}$
14	If $w(x,y,z) = x^2(y-z) + y^2(z-x) + z^2(x-y)$, then $\frac{\partial w}{\partial x} + \frac{\partial w}{\partial y} + \frac{\partial w}{\partial z}$ is (1) $xy + yz + zx$ (2) $x(y+z)$ (3) $y(z+x)$ (4) 0	**Ans: 0**
15	If $f(x,y,z) = xy + yz + zx$, then $f_x - f_z$ is equal to (1) $z - x$ (2) $y - z$ (3) $x - z$ (4) $y - x$	**Ans:** $z - x$

CHAPTER 9 – Applications of Integration

1	The value of $\int_0^{2/3} \frac{dx}{\sqrt{4-9x^2}}$ is (1) $\frac{\pi}{6}$ (2) $\frac{\pi}{2}$ (3) $\frac{\pi}{4}$ (4) π	**Ans:** $\frac{\pi}{6}$
2	The value of $\int_{-1}^{2} \|x\| dx$ (1) $\frac{1}{2}$ (2) $\frac{3}{2}$ (3) $\frac{5}{2}$ (4) $\frac{7}{2}$	**Ans:** $\frac{5}{2}$
3	For any value of $n \in \mathbb{Z}$, $\int_0^{\pi} e^{\cos^2 x} \cos^3[(2n+1)x]dx$ is (1) $\frac{\pi}{2}$ (2) π (3) 0 (4) 2	**Ans: 0**
4	The value of $\int_{-\pi/2}^{\pi/2} \sin^2 x \cos x\, dx$ is (1) $\frac{3}{2}$ (2) $\frac{1}{2}$ (3) 0 (4) $\frac{2}{3}$	**Ans:** $\frac{2}{3}$
5	The value of $\int_{-4}^{4} \left[\tan^{-1}\left(\frac{x^2}{x^4+1}\right) + \tan^{-1}\left(\frac{x^4+1}{x^2}\right)\right] dx$ is (1) π (2) 2π (3) 3π (4) 4π	**Ans:** 4π
6	The value of $\int_{-\pi/4}^{\pi/4} \left(\frac{2x^7 - 3x^5 + 7x^3 - x + 1}{\cos^2 x}\right) dx$ is (1) 4 (2) 3 (3) 2 (4) 0	**Ans: 2**
7	If $f(x) = \int_0^x t\cos t\, dt$, then $\frac{df}{dx} =$ (1) $\cos x - x\sin x$ (2) $\sin x + x\cos x$ (3) $x\cos x$ (4) $x\sin x$	**Ans:** $x\cos x$

8	The area between $y^2 = 4x$ and its latus rectum is (1) $\frac{2}{3}$ (2) $\frac{4}{3}$ (3) $\frac{8}{3}$ (4) $\frac{5}{3}$	**Ans:** $\frac{8}{3}$
9	The value of $\int_0^1 x(1-x)^{99}dx$ is (1) $\frac{1}{11000}$ (2) $\frac{1}{10100}$ (3) $\frac{1}{10010}$ (4) $\frac{1}{10001}$	**Ans:** $\frac{1}{10100}$
10	The value of $\int_0^{\pi} \frac{dx}{1+5^{\cos x}}$ is (1) $\frac{\pi}{2}$ (2) π (3) $\frac{3\pi}{2}$ (4) 2π	**Ans:** $\frac{\pi}{2}$
11	If $\frac{\Gamma(n+2)}{\Gamma(n)} = 90$ then n is (1) 10 (2) 5 (3) 8 (4) 9	**Ans: 9**
12	The value of $\int_0^{\pi/6} \cos^3 3x\, dx$ is (1) $\frac{2}{3}$ (2) $\frac{2}{9}$ (3) $\frac{1}{9}$ (4) $\frac{1}{3}$	**Ans:** $\frac{2}{9}$
13	The value of $\int_0^{\pi} \sin^4 x\, dx$ is (1) $\frac{3\pi}{10}$ (2) $\frac{3\pi}{8}$ (3) $\frac{3\pi}{4}$ (4) $\frac{3\pi}{2}$	**Ans:** $\frac{3\pi}{8}$
14	The value of $\int_0^{\infty} e^{-3x}x^2 dx$ is (1) $\frac{7}{27}$ (2) $\frac{5}{27}$ (3) $\frac{4}{27}$ (4) $\frac{2}{27}$	**Ans:** $\frac{2}{27}$
15	If $\int_0^a \frac{1}{4+x^2}dx = \frac{\pi}{8}$ then a is (1) 4 (2) 1 (3) 3 (4) 2	**Ans: 2**
16	The volume of solid of revolution of the region bounded by $y^2 = x(a-x)$ about x-axis is (1) πa^3 (2) $\frac{\pi a^3}{4}$ (3) $\frac{\pi a^3}{5}$ (4) $\frac{\pi a^3}{6}$	**Ans:** $\frac{\pi a^3}{6}$
17	If $f(x) = \int_1^x \frac{e^{\sin u}}{u}du, x > 1$ and $\int_1^3 \frac{e^{\sin x^2}}{x}dx = \frac{1}{2}[f(a) - f(1)]$, then one of the possible value of a is (1) 3 (2) 6 (3) 9 (4) 5	**Ans: 9**
18	The value of $\int_0^1 (\sin^{-1} x)^2\, dx$ is (1) $\frac{\pi^2}{4} - 1$ (2) $\frac{\pi^2}{4} + 2$ (3) $\frac{\pi^2}{4} + 1$ (4) $\frac{\pi^2}{4} - 2$	**Ans:** $\frac{\pi^2}{4} - 2$
19	The value of $\int_0^a \left(\sqrt{a^2 - x^2}\right)^3 dx$ is (1) $\frac{\pi a^3}{16}$ (2) $\frac{3\pi a^4}{16}$ (3) $\frac{3\pi a^2}{8}$ (4) $\frac{3\pi a^4}{8}$	**Ans:** $\frac{3\pi a^4}{16}$
20	If $\int_0^x f(t)dt = x + \int_x^1 t\, f(t)dt$, then the value of $f(1)$ is (1) $\frac{1}{2}$ (2) 2 (3) 1 (4) $\frac{3}{4}$	**Ans:** $\frac{1}{2}$

CHAPTER 10 – Ordinary Differential Equations

1	The order and degree of the differential equation $\frac{d^2y}{dx^2}+\left(\frac{dy}{dx}\right)^{1/3}+x^{1/4}=0$ are respectively (1) 2,3 (2) 3,3 (3) 2,6 (4) 2,4	**Ans: 2, 3**
2	The differential equation representing the family of curves $y=A\cos(x+B)$, where A and B are parameters, is (1) $\frac{d^2y}{dx^2}-y=0$ (2) $\frac{d^2y}{dx^2}+y=0$ (3) $\frac{d^2y}{dx^2}=0$ (4) $\frac{d^2x}{dy^2}=0$	**Ans:** $\frac{d^2y}{dx^2}+y=0$
3	The order and degree of the differential equation $\sqrt{\sin x}(dx+dy)=\sqrt{\cos x}(dx-dy)$ is (1) 1,2 (2) 2,2 (3) 1,1 (4) 2,1	**Ans: 1, 1**
4	The order of the differential equation of all circles with centre at (h,k) and radius 'a' is (1) 2 (2) 3 (3) 4 (4) 1	**Ans: 3**
5	The differential equation of the family of curves $y=\mathrm{A}e^x+\mathrm{B}e^{-x}$, where A and B are arbitrary constants is (1) $\frac{d^2y}{dx^2}+y=0$ (2) $\frac{d^2y}{dx^2}-y=0$ (3) $\frac{dy}{dx}+y=0$ (4) $\frac{dy}{dx}-y=0$	**Ans:** $\frac{d^2y}{dx^2}-y=0$
6	The general solution of the differential equation $\frac{dy}{dx}=\frac{y}{x}$ is (1) $xy=k$ (2) $y=k\log x$ (3) $y=kx$ (4) $\log y=kx$	**Ans:** $y=kx$
7	The solution of the differential equation $2x\frac{dy}{dx}-y=3$ represents (1) straight lines (2) circles (3) parabola (4) ellipse	**Ans:** parabola
8	The solution of $\frac{dy}{dx}+p(x)y=0$ is (1) $y=ce^{\int pdx}$ (2) $y=ce^{-\int pdx}$ (3) $x=ce^{-\int pdy}$ (4) $x=ce^{\int pdy}$	**Ans:** $y=ce^{-\int pdx}$
9	The integrating factor of the differential equation $\frac{dy}{dx}+y=\frac{1+y}{x}$ is (1) $\frac{x}{e^x}$ (2) $\frac{e^x}{x}$ (3) λe^x (4) e^x	**Ans:** $\frac{e^x}{x}$
10	The integrating factor of the differential equation $\frac{dy}{dx}+P(x)y=Q(x)$ is x, then $P(x)$ is (1) x (2) $\frac{x^2}{2}$ (3) $\frac{1}{x}$ (4) $\frac{1}{x^2}$	**Ans:** $\frac{1}{x}$
11	The degree of the differential equation $y(x)=1+\frac{dy}{dx}+\frac{1}{1\cdot 2}\left(\frac{dy}{dx}\right)^2+\frac{1}{1\cdot 2\cdot 3}\left(\frac{dy}{dx}\right)^3+\cdots$ is (1) 2 (2) 3 (3) 1 (4) 4	**Ans: 1**
12	If p and q are the order and degree of the differential equation $y\frac{dy}{dx}+x^3\left(\frac{d^2y}{dx^2}\right)+xy=\cos x$, when, (1) $p<q$ (2) $p=q$ (3) $p>q$ (4) p exists and q does not exist	**Ans:** $p>q$
13	The solution of the differential equation $\frac{dy}{dx}+\frac{1}{\sqrt{1-x^2}}=0$ is (1) $y+\sin^{-1}x=c$ (2) $x+\sin^{-1}y=0$ (3) $y^2+2\sin^{-1}x=C$ (4) $x^2+2\sin^{-1}y=0$	**Ans:** $y+\sin^{-1}x=c$
14	The solution of the differential equation $\frac{dy}{dx}=2xy$ is (1) $y=Ce^{x^2}$ (2) $y=2x^2+C$ (3) $y=Ce^{-x^2}+C$ (4) $y=x^2+C$	**Ans:** $y=Ce^{x^2}$
15	The general solution of the differential equation $\log\left(\frac{dy}{dx}\right)=x+y$ is (1) $e^x+e^y=C$ (2) $e^x+e^{-y}=C$ (3) $e^{-x}+e^y=C$ (4) $e^{-x}+e^{-y}=C$	**Ans:** $e^x+e^{-y}=C$

16	The solution of $\frac{dy}{dx} = 2^{y-x}$ is (1) $2^x + 2^y = C$ (2) $2^x - 2^y = C$ (3) $\frac{1}{2^x} - \frac{1}{2^y} = C$ (4) $x + y = C$	**Ans:** $\frac{1}{2^x} - \frac{1}{2^y} = C$
17	The solution of the differential equation $\frac{dy}{dx} = \frac{y}{x} + \frac{\phi\left(\frac{y}{x}\right)}{\phi'\left(\frac{y}{x}\right)}$ is (1) $x\phi\left(\frac{y}{x}\right) = k$ (2) $\phi\left(\frac{y}{x}\right) = kx$ (3) $y\phi\left(\frac{y}{x}\right) = k$ (4) $\phi\left(\frac{y}{x}\right) = ky$	**Ans:** $\phi\left(\frac{y}{x}\right) = kx$
18	If $\sin x$ is the integrating factor of the linear differential equation $\frac{dy}{dx} + Py = Q$, then P is (1) $\log \sin x$ (2) $\cos x$ (3) $\tan x$ (4) $\cot x$	**Ans:** $\cot x$
19	The number of arbitrary constants in the general solutions of order n and $n+1$ are respectively (1) $n-1, n$ (2) $n, n+1$ (3) $n+1, n+2$ (4) $n+1, n$	**Ans:** $n, n+1$
20	The number of arbitrary constants in the particular solution of a differential equation of third order is (1) 3 (2) 2 (3) 1 (4) 0	**Ans: 0**
21	Integrating factor of the differential equation $\frac{dy}{dx} = \frac{x+y+1}{x+1}$ is (1) $\frac{1}{x+1}$ (2) $x+1$ (3) $\frac{1}{\sqrt{x+1}}$ (4) $\sqrt{x+1}$	**Ans:** $\frac{1}{x+1}$
22	The population P in any year t is such that the rate of increase in the population is proportional to the population. Then (1) $P = Ce^{kt}$ (2) $P = Ce^{-kt}$ (3) $P = Ckt$ (4) $P = C$	**Ans:** $P = Ce^{kt}$
23	P is the amount of certain substance left in after time t. If the rate of evaporation of the substance is proportional to the amount remaining, then (1) $P = Ce^{kt}$ (2) $P = Ce^{-kt}$ (3) $P = Ckt$ (4) $Pt = C$	**Ans:** $P = Ce^{-kt}$
24	If the solution of the differential equation $\frac{dy}{dx} = \frac{ax+3}{2y+f}$ represents a circle, then the value of a is (1) 2 (2) -2 (3) 1 (4) -1	**Ans: -2**
25	The slope at any point of a curve $y = f(x)$ is given by $\frac{dy}{dx} = 3x^2$ and it passes through $(-1,1)$. Then the equation of the curve is (1) $y = x^3 + 2$ (2) $y = 3x^2 + 4$ (3) $y = 3x^3 + 4$ (4) $y = x^3 + 5$	**Ans:** $y = x^3 + 2$

CHAPTER 11 – Probability Distributions

1	Let X be random variable with probability density function $f(x) = \begin{cases} \frac{2}{x^3} & x \geq 1 \\ 0 & x < 1 \end{cases}$ Which of the following statement is correct? (1) both mean and variance exist (2) mean exists but variance does not exist (3) both mean and variance do not exist (4) variance exists but mean does not exist	**Ans:** mean exists but variance does not exist
2	A rod of length $2l$ is broken into two pieces at random. The probability density function of the shorter of the two pieces is $f(x) = \begin{cases} \frac{1}{l} & 0 < x < l \\ 0 & l \leq x < 2l \end{cases}$ The mean and variance of the shorter of the two pieces are respectively (1) $\frac{l}{2}, \frac{l^2}{3}$ (2) $\frac{l}{2}, \frac{l^2}{6}$ (3) $l, \frac{l^2}{12}$ (4) $\frac{l}{2}, \frac{l^2}{12}$	**Ans:** $\frac{l}{2}, \frac{l^2}{12}$

3	Consider a game where the player tosses a six-sided fair die. If the face that comes up is 6, the player wins ₹36, otherwise he loses ₹k^2, where k is the face that comes up $k=\{1,2,3,4,5\}$. The expected amount to win at this game in ₹ is (1) $\frac{19}{6}$ (2) $-\frac{19}{6}$ (3) $\frac{3}{2}$ (4) $-\frac{3}{2}$	**Ans:** $-\frac{19}{6}$
4	A pair of dice numbered 1,2,3,4,5,6 of a six-sided die and 1,2,3,4 of a four-sided die is rolled and the sum is determined. Let the random variable X denote this sum. Then the number of elements in the inverse image of 7 is (1) 1 (2) 2 (3) 3 (4) 4	**Ans: 4**
5	A random variable X has binomial distribution with $n=25$ and $p=0.8$ then standard deviation of X is (1) 6 (2) 4 (3) 3 (4) 2	**Ans: 2**
6	Let X represent the difference between the number of heads and the number of tails obtained when a coin is tossed n times. Then the possible values of X are (1) $i+2n, i=0,1,2\ldots n$ (2) $2i-n, i=0,1,2\ldots n$ (3) $n-i, i=0,1,2\ldots n$ (4) $2i+2n, i=0,1,2\ldots n$	**Ans:** $2i-n, i=0,1,2\ldots n$
7	If the function $f(x)=\frac{1}{12}$ for $a<x<b$, represents a probability density function of a continuous random variable X, then which of the following cannot be the value of a and b? (1) 0 and 12 (2) 5 and 17 (3) 7 and 19 (4) 16 and 24	**Ans:** 16 and 24
8	Four buses carrying 160 students from the same school arrive at a football stadium. The buses carry, respectively, 42,36,34 and 48 students. One of the students is randomly selected. Let X denote the number of students that were on the bus carrying the randomly selected student. One of the 4 bus drivers is also randomly selected. Let Y denote the number of students on that bus. Then $E[X]$ and $E[Y]$ respectively are (1) 50,40 (2) 40,50 (3) 40.75,40 (4) 41,41	**Ans:** 40.75,40
9	Two coins are to be flipped. The first coin will land on heads with probability 0.6, the second with probability 0.5. Assume that the results of the flips are independent, and let X equal the total number of heads that result. The value of $E[X]$ is (1) 0.11 (2) 1.1 (3) 11 (4) 1	**Ans: 1.1**
10	On a multiple-choice exam with 3 possible destructives for each of the 5 questions, the probability that a student will get 4 or more correct answers just by guessing is (1) $\frac{11}{243}$ (2) $\frac{3}{8}$ (3) $\frac{1}{243}$ (4) $\frac{5}{243}$	**Ans:** $\frac{11}{243}$
11	If $P\{X=0\}=1-P\{X=1\}$. If $E[X]=3\text{Var}(X)$, then $P\{X=0\}$. (1) $\frac{2}{3}$ (2) $\frac{2}{5}$ (3) $\frac{1}{5}$ (4) $\frac{1}{3}$	**Ans:** $\frac{1}{3}$
12	If X is a binomial random variable with expected value 6 and variance 2.4, Then $P\{X=5\}$ is (1) $\binom{10}{5}\left(\frac{3}{5}\right)^6\left(\frac{2}{5}\right)^4$ (2) $\binom{10}{5}\left(\frac{3}{5}\right)^{10}$ (3) $\binom{10}{5}\left(\frac{3}{5}\right)^4\left(\frac{2}{5}\right)^6$ (4) $\binom{10}{5}\left(\frac{3}{5}\right)^5\left(\frac{2}{5}\right)^5$	**Ans:** $\binom{10}{5}\left(\frac{3}{5}\right)^5\left(\frac{2}{5}\right)^5$
13	The random variable X has the probability density function $f(x)=\begin{cases}ax+b & 0<x<1\\ 0 & \text{otherwise}\end{cases}$ and $E(X)=\frac{7}{12}$, then a and b are respectively (1) 1 and $\frac{1}{2}$ (2) $\frac{1}{2}$ and 1 (3) 2 and 1 (4) 1 and 2	**Ans:** 1 and $\frac{1}{2}$

14	Suppose that X takes on one of the values 0,1 and 2. If for some constant k, $P(X=i)=kP(X=i-1)$ for $i=1,2$ and $P(X=0)=\frac{1}{7}$. Then the value of k is (1) 1 (2) 2 (3) 3 (4) 4	**Ans: 2**
15	Which of the following is a discrete random variable? I. The number of cars crossing a particular signal in a day. II. The number of customers in a queue to buy train tickets at a moment. III. The time taken to complete a telephone call. (1) I and II (2) II only (3) III only (4) II and III	**Ans:** I and II
16	If $f(x)=\begin{cases}2x & 0\le x\le a\\ 0 & \text{otherwise}\end{cases}$ is a probability density function of a random variable, then the value of a is (1) 1 (2) 2 (3) 3 (4) 4	**Ans: 1**
17	The probability function of a random variable is defined as: <table><tr><td>x</td><td>-2</td><td>-1</td><td>0</td><td>1</td><td>2</td></tr><tr><td>$f(x)$</td><td>k</td><td>$2k$</td><td>$3k$</td><td>$4k$</td><td>$5k$</td></tr></table>Then $E(X)$ is equal to: (1) $\frac{1}{15}$ (2) $\frac{1}{10}$ (3) $\frac{1}{3}$ (4) $\frac{2}{3}$	**Ans:** $\frac{2}{3}$
18	Let X have a Bernoulli distribution with mean 0.4, then the variance of $(2X-3)$ is (1) 0.24 (2) 0.48 (3) 0.6 (4) 0.96	**Ans: 0.96**
19	If in 6 trials, X is a binomial variate which follows the relation $9P(X=4)=P(X=2)$, then the probability of success is (1) 0.125 (2) 0.25 (3) 0.375 (4) 0.75	**Ans: 0.25**
20	A computer salesperson knows from his past experience that he sells computers to one in every twenty customers who enter the showroom. What is the probability that he will sell a computer to exactly two of the next three customers? (1) $\frac{57}{20^3}$ (2) $\frac{57}{20^2}$ (3) $\frac{19^3}{20^3}$ (4) $\frac{57}{20}$	**Ans:** $\frac{57}{20^3}$

CHAPTER 12 – Discrete Mathematics

1	A binary operation on a set S is a function from (1) $S\to S$ (2) $(S\times S)\to S$ (3) $S\to(S\times S)$ (4) $(S\times S)\to(S\times S)$	**Ans:** $(S\times S)\to S$
2	Subtraction is not a binary operation in (1) $\mathbb{R}$ (2) $\mathbb{Z}$ (3) $\mathbb{N}$ (4) $\mathbb{Q}$	**Ans:** $\mathbb{N}$
3	Which one of the following is a binary operation on $\mathbb{N}$? (1) Subtraction (2) Multiplication (3) Division (4) All the above	**Ans:** Multiplication
4	In the set $\mathbb{R}$ of real numbers '*' is defined as follows. Which one of the following is not a binary operation on $\mathbb{R}$? (1) $a*b=\min(a\cdot b)$ (2) $a*b=\max(a,b)$ (3) $a*b=a$ (4) $a*b=a^b$	**Ans:** $a*b=a^b$
5	The operation $*$ defined by $a*b=\frac{ab}{7}$ is not a binary operation on (1) $\mathbb{Q}^+$ (2) $\mathbb{Z}$ (3) $\mathbb{R}$ (4) $\mathbb{C}$	**Ans:** $\mathbb{Z}$
6	In the set $\mathbb{Q}$ define $a\odot b=a+b+ab$. For what value of y, $3\odot(y\odot 5)=7$? (1) $y=\frac{2}{3}$ (2) $y=\frac{-2}{3}$ (3) $y=\frac{-3}{2}$ (4) $y=4$	**Ans:** $y=\frac{-2}{3}$

<table>
<tr><td>7</td><td>If $a * b = \sqrt{a^2 + b^2}$ on the real numbers then $*$ is
(1) commutative but not associative (2) associative but not commutative
(3) both commutative and associative (4) neither commutative nor associative</td><td>Ans:
both commutative and associative</td></tr>
<tr><td>8</td><td>Which one of the following statements has the truth value T ?
(1) $\sin x$ is an even function. (2) Every square matrix is non-singular
(3) The product of complex number and its conjugate is purely imaginary
(4) $\sqrt{5}$ is an irrational number</td><td>Ans:
$\sqrt{5}$ is an irrational number</td></tr>
<tr><td>9</td><td>Which one of the following statements has truth value F ?
(1) Chennai is in India or $\sqrt{2}$ is an integer
(2) Chennai is in India or $\sqrt{2}$ is an irrational number
(3) Chennai is in China or $\sqrt{2}$ is an integer
(4) Chennai is in China or $\sqrt{2}$ is an irrational number</td><td>Ans:
Chennai is in China or $\sqrt{2}$ is an integer</td></tr>
<tr><td>10</td><td>If a compound statement involves 3 simple statements, then the number of rows in the truth table is
(1) 9 (2) 8 (3) 6 (4) 3</td><td>Ans: 8</td></tr>
<tr><td>11</td><td>Which one is the inverse of the statement $(p \vee q) \rightarrow (p \wedge q)$?
(1) $(p \wedge q) \rightarrow (p \vee q)$ (2) $\neg(p \vee q) \rightarrow (p \wedge q)$
(3) $(\neg p \vee \neg q) \rightarrow (\neg p \wedge \neg q)$ (4) $(\neg p \wedge \neg q) \rightarrow (\neg p \vee \neg q)$</td><td>Ans:
$(\neg p \wedge \neg q)$
$\rightarrow (\neg p \vee \neg q)$</td></tr>
<tr><td>12</td><td>Which one is the contrapositive of the statement $(p \vee q) \rightarrow r$?
(1) $\neg r \rightarrow (\neg p \wedge \neg q)$ (2) $\neg r \rightarrow (p \vee q)$ (3) $r \rightarrow (p \wedge q)$ (4) $p \rightarrow (q \vee r)$</td><td>Ans:
$\neg r \rightarrow (\neg p \wedge \neg q)$</td></tr>
<tr><td>13</td><td>The truth table for $(p \wedge q) \vee \neg q$ is given below
<table>
<tr><th>p</th><th>q</th><th>$(p \wedge q) \vee (\neg q)$</th></tr>
<tr><td>T</td><td>T</td><td>(a)</td></tr>
<tr><td>T</td><td>F</td><td>(b)</td></tr>
<tr><td>F</td><td>T</td><td>(c)</td></tr>
<tr><td>F</td><td>F</td><td>(d)</td></tr>
</table>
Which one of the following is true?
<table>
<tr><th></th><th>(a)</th><th>(b)</th><th>(c)</th><th>(d)</th></tr>
<tr><td>(1)</td><td>T</td><td>T</td><td>T</td><td>T</td></tr>
<tr><td>(2)</td><td>T</td><td>F</td><td>T</td><td>T</td></tr>
<tr><td>(3)</td><td>T</td><td>T</td><td>F</td><td>T</td></tr>
<tr><td>(4)</td><td>T</td><td>F</td><td>F</td><td>F</td></tr>
</table></td><td>Ans:
a b c d
T T F T</td></tr>
<tr><td>14</td><td>In the last column of the truth table for $\neg(p \vee \neg q)$ the number of final outcomes of the truth value $'F'$ are
(1) 1 (2) 2 (3) 3 (4) 4</td><td>Ans: 3</td></tr>
<tr><td>15</td><td>Which one of the following is incorrect? For any two propositions p and q, we have
(1) $\neg(p \vee q) \equiv \neg p \wedge \neg q$ (2) $\neg(p \wedge q) \equiv \neg p \vee \neg q$
(3) $\neg(p \vee q) \equiv \neg p \vee \neg q$ (4) $\neg(\neg p) \equiv p$</td><td>Ans:
$\neg(p \vee q) \equiv \neg p \vee \neg q$</td></tr>
<tr><td>16</td><td>
<table>
<tr><th>p</th><th>q</th><th>$(p \wedge q) \rightarrow \neg p$</th></tr>
<tr><td>T</td><td>T</td><td>(a)</td></tr>
<tr><td>T</td><td>F</td><td>(b)</td></tr>
<tr><td>F</td><td>T</td><td>(c)</td></tr>
<tr><td>F</td><td>F</td><td>(d)</td></tr>
</table></td><td>Ans:
a b c d
F T T T</td></tr>
</table>

	Which one of the following is correct for the truth value of $(p \wedge q) \to \neg p$? (a) (b) (c) (d) (1) T T T T (2) F T T T (3) F F T T (4) T T T F	
17	The dual of $\neg(p \vee q) \vee [p \vee (p \wedge \neg r)]$ is (1) $\neg(p \wedge q) \wedge [p \vee (p \wedge \neg r)]$ (2) $(p \wedge q) \wedge [p \wedge (p \vee \neg r)]$ (3) $\neg(p \wedge q) \wedge [p \wedge (p \wedge r)]$ (4) $\neg(p \wedge q) \wedge [p \wedge (p \vee \neg r)]$	**Ans:** $\neg(p \wedge q) \wedge [p \wedge (p \vee \neg r)]$
18	The proposition $p \wedge (\neg p \vee q)$ is (1) a tautology (2) a contradiction (3) logically equivalent to $p \wedge q$ (4) logically equivalent to $p \vee q$	**Ans:** logically equivalent to $p \wedge q$
19	Determine the truth value of each of the following statements: (a) $4 + 2 = 5$ and $6 + 3 = 9$ (b) $3 + 2 = 5$ and $6 + 1 = 7$ (c) $4 + 5 = 9$ and $1 + 2 = 4$ (d) $3 + 2 = 5$ and $4 + 7 = 11$ (a) (b) (c) (d) (1) F T F T (2) T F T F (3) T T F F (4) F F T T	**Ans:** **a b c d** **F T F T**
20	Which one of the following is not true? (1) Negation of a negation of a statement is the statement itself. (2) If the last column of the truth table contains only T then it is a tautology. (3) If the last column of its truth table contains only F then it is a contradiction. (4) If p and q are any two statements then $p \leftrightarrow q$ is a tautology.	**Ans:** If p and q are any two statements then $p \leftrightarrow q$ is a tautology

CHAPTER : 12

DISCRETE MATHEMATICS

CONCEPT : TRUTH TABLE

A table showing the relationship between truth values of simple statements and truth values of compund statements is called truth table .

Table 1: Truth Table for 7p.

P	7P
T	F
F	T

P one statement
=>2 rows

Table 2 : Truth Table for p^q (P cap q)

p, q two statements ⟹ 4 rows

p	q	p ^ q
T_1	T_1	T
T_1	F_o	F
F_o	T_1	F
F_o	F_o	F

^ - small number

Table 3 : Truth Table for p v q (p cup q)

p	q	p v q
T_1	T_1	T
T_1	F_o	T
F_o	T_1	T
F_o	F_o	F

$\wedge$ $-big\ number$

Table 4 : Truth Table for $p \rightarrow q$ (p implies q)

p	q	$p \rightarrow q$
T	T	T
T	F	F
F	T	T
F	F	T

Note:
T ∪ FF
T, F⇒F
Otherwise T.

Table 5: Truth Table for $p \leftrightarrow q$

p	q	$p \leftrightarrow q$
T_+	T_+	T
T_+	F_-	F
F_-	T_+	F
F_-	F_-	T

Multiplication Rule
$+ \times + = +$
$+ \times - = -$
$- \times + = -$
$- \times - = +$

Table 6: Truth Table for $p \bar{v} q$, p cup not q (or) p EOR q

p	q	$p \bar{v} q$
T	T	F
T	F	T
F	T	T
F	F	F

If Single T comes then T comes otherwise F.

NOTE :

i. A Statement is said to be tautology if the last column of the truth table is T.
ii. A Statement is said to be contradiction it its truth value is always F..
iii. A Statement which is neither a tautology nor a contradiction is called contingency.

Concept :

Some properties of BINARY OPERATION

Let S be a non – empty set and * be the binary operation defined on it ...

1. **Closure property :**

   a, b ∈ s ⇒ a * b ∈ s   ∀ a, b ∈ s.

2. **Commutative property:**

   a * b = b * a   ∀ a, b ∈ s,

3. **Associative property:**

   (a * b) * c = a * (b * c)   ∀ a, b, c ∈ s,

4. **Identity property :**

   Let e be the identity element then...

   a * e = e * a = a,   ∀ a ∈ s.

5. **Inverse property :**

   Let inverse of a be a^{-1} Then.

$$a * a^{-1} = a^{-1} * a = e \quad \forall a \in s$$

TWO MARK QUESTIONS

1. **Example 12.8** **March 2023 , June 2023.**

   IF A = $\begin{bmatrix} 0 & 1 \\ 1 & 1 \end{bmatrix}$ **B =** $\begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$

   Find $\boldsymbol{A \vee B}$ **and ;** $\boldsymbol{A \wedge B}$

   ANS:

$$A \vee B = \begin{bmatrix} 0 & 1 \\ 1 & 1 \end{bmatrix} \vee \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 0 \vee 1 & 1 \vee 1 \\ 1 \vee 0 & 1 \vee 1 \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}$$

$$A \wedge B = \begin{bmatrix} 0 & 1 \\ 1 & 1 \end{bmatrix} \wedge \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 0 \wedge 1 & 1 \wedge 1 \\ 1 \wedge 0 & 1 \wedge 1 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 0 & 1 \end{bmatrix}$$

2. **Theorem: (Unique ness of Identity)** **March 2020.**

   In an algeabraic structure , prove that the identity elements (if exits) must be unique identity

   Proof:

   If possible , let e_1 and e_2 be two identity elements..

   If e_1 is identity and e_2 be any element .

   Then,

   $e_1 * e_2 = e_2$ ....(1)

   If $\mathbf{e_2}$ is identity and e_1 be any element.....

   Then,

   $e_1 * e_2 = e_1$ ....(2)

   From (i) and (ii) $e_1 = e_2$

   ⇒ Identity element is unique …

3. **Theorem : Uniqueness of inverse .**

   In an algebraic structure the inverse of an element (if exists) must be unique ….

   Proof :

   Suppose that a has two inverse say a_1 and a_2

   Let inverse of a be a_1 ⇒ $a * a_1 = e$ ....(1)

   Let inverse of a be $a_2 \Rightarrow a * a_2 = e$ ....(2)

   From 1 and 2 $a * a_1 = a * a_2$

   By left cancellation law , $a_1 = a_2$

   Inverse element is unique.

4. Example : 12.1 (1) **Sep 2020**

Examine the binary operation (Closure property) for the following .

$$a * b = a + 3\,ab - 5\,b^2 \quad \forall\ a, b \in z.$$

Ans :

let a,b ∈ z

$a * b = a + 3\,ab - 5b^2 \in z$

$a * b \in z$

⇒ * is a binary operation of z ..

⇒ * has closure property …

5. Examine the binary operation (closure property) for the following **Sep 2020**

$$a * b = \frac{a-b}{b-1}\ \forall\ a, b \in Q.$$

Ans:

* does not have closure property

Reason If b = 1 , a*b = $\frac{a-b}{1-1} = \cdots.$

= $\frac{a-b}{0}$ does not defined.

When b = 1, $a * b \notin Q$.

⇒ * Is not a binary operation.

6. Exercise 12.1 – (1)

Determine whether * is a binary operation on the sets given below ...

(i) a *b = $a|b|$ on R

(ii) a *b = min (a, b) on A = {1, 2, 3, 4, 5}

(iii) a *b = $a\sqrt{b}$ is binary on R.

Ans :

(i) Given a * b = $a|b|$ on R

a, b ϵ R ⇒ a*b ϵ R ⇒ * is binary operation

(ii) Given A = {1, 2, 3, 4, 5} a * b = min (a, b)

a, b ϵ A = a *b ϵ A

⇒ * is a binary operation..

iii) Given a * b = $a\sqrt{b}$ a,b ϵ R

a, b ∈ R ⇒ a * b = $a\sqrt{b}$ ∉ R

a, b ∈ R ⇒ a * b ∉ R

⇒ * is not a binary operation…

Note : -4∈ R but $\sqrt{-4} \notin R$

7. Exercise 12.1(2)

On z, define * by m ⊗ $n = m^n + n^m \quad \forall\ m, n \in z$. Is ⊗ binary on Z ?

Ans:

⊗ is not a binary operation on z.

m, n ∉z ⇒ $m \otimes n \notin z$

8. Exercise 12.1 (3)

Let * be defined on R by a * b = a + b + ab - 7 is * binary on R? If so find 3*$\left(\frac{-7}{15}\right)$?

Ans:

i) * is binary on R.

a, b ∈ R⇒ a* b ∈ R.

ii) $3 * \left(\frac{-7}{15}\right) = 3 + \left(\frac{-7}{15}\right) + 3\left(\frac{-7}{15}\right) - 7$

$= \frac{3}{1} - \frac{7}{15} - \frac{21}{15} - \frac{7}{1}$

$= \frac{45-7-21-105}{15} = \frac{45-133}{15}$

$= -\frac{88}{15}$

9. **Exercise 12.1 (4)**

Let A = $\{a + \sqrt{5}b,\ a, b \in z\}$ Check whether the usual multiplication is a binary operation on A.

Ans:

*is binary operation on A.

Let x, y ∈ A. * is usual multiplication.

Let $x = a + \sqrt{5}b,\ y = c + \sqrt{5}d$

$x\,.\,y = (a + \sqrt{5}b)\,(c + \sqrt{5}d)$

$= ac + ad\sqrt{5} + bc\sqrt{5} + 5bd$

$= (ac + 5bd) + \sqrt{5}\,(ad + bc) \in A$

$x, y \in A \Rightarrow x\,.\,y \in A$

⇒ * is binary operation on A..

3 MARKS

10. **Example :12.7** **March 2020.**

Establish the equivalence property

$$p \to q \equiv 7p \vee q$$

ANS:

p, q two statements ⇒ 4 Rows.

p	q	LHS p → q	7p	RHS $7p \vee q$
T	T	T	F	T
T	F	F	F	F
F	T	T	T	T
F	F	T	T	T

(1) … … …. (2)

(1) and (2) L.H.S = R.H.S.

$p \to q \equiv 7p \vee q$

11. **Prove that $((7p) \wedge p) \wedge q$ is contradictions** **Sep 2020**

Ans:

P , q two statements ⇒ 4 rows

p	q	7p	$(7p) \wedge p$	$((7p) \wedge p) \wedge q$
T	T	F	F	F
T	F	F	F	F
F	T	T	F	F
F	F	T	F	F

Since the last column contains only F,

` $((7p) \wedge p) \wedge q$ is contradiction…

12. Show that $p \rightarrow q$ and $q \rightarrow p$ are not equivalent. **March 2023**

p	q	$p \rightarrow q$	q	p	$q \rightarrow p$
T	T	T	T	T	T
T	F	F	F	T	T
F	T	T	T	F	F
F	F	T	F	F	T

(1) (2)

(1), (2) and $p \rightarrow q \not\equiv q \rightarrow p$

That is , $p \rightarrow q$ and $q \rightarrow p$ are not equivalent.

EXAMPLE : 12.16

13. Construct the truth Table for :

$$(p \,\bar{\vee}\, q) \wedge (p \,\bar{\vee}\, 7q)$$

Ans :

p, q two statements ⇒ 4 rows.

p	q	7q	$p \,\bar{\vee}\, q$	$p \,\bar{\vee}\, 7q$	$(p \,\bar{\vee}\, q) \wedge (p \,\bar{\vee}\, 7q)$
T	T	F	F	T	F
T	F	T	T	F	F
F	T	F	T	F	F
F	F	T	F	T	F

Note : since the last column contains F only, given statement is contradiction.

5 MARKS

14. Example : 12.18 **June 2023**

Establish the equivalence property connecting the bi – conditional with conditional.

$$p \leftrightarrow q \equiv (p \rightarrow q) \wedge (q \rightarrow p)$$

ANS:

p, q two statements ⇒ 4 rows

p	q	LHS $p \leftrightarrow q$	$p \rightarrow q$	$q \rightarrow p$	R.H.S $(p \rightarrow q) \wedge (q \rightarrow p)$
T	T	T	T	T	T
T	F	F	F	T	F
F	T	F	T	F	F
F	F	T	T	T	T

(1) (2)

From (1) and (2) LHS = RHS

$$p \leftrightarrow q \equiv (p \rightarrow q) \wedge (q \rightarrow p)$$

15. Exercise 12.2 – 7 (iii) **July 2022**

Verify $(p \rightarrow q) \leftrightarrow (7p \rightarrow q)$ is tautology or contradiction or contingency .

Ans :

p	q	$(p \rightarrow q)$	$7p$	$(7p \rightarrow q)$	$(p \rightarrow q) \leftrightarrow (7p \rightarrow q)$
T	T	T	F	T	T
T	F	F	F	T	F
F	T	T	T	T	T
F	F	T	T	F	F

From last column we say , given statement is contingency....

16. Example 12.19

Using the equivalance property show that

$p \leftrightarrow q \equiv (p \wedge q) \vee (7p \wedge 7q)$

p	q	LHS $p \leftrightarrow q$	$(p \wedge q)$	7p	7q	$(7p \wedge 7q)$	R.H.S
T	T	T	T	F	F	F	T
T	F	F	F	F	T	F	F
F	T	F	F	T	F	F	F
F	F	T	F	T	T	T	T
		(1)					(2)

From 1 and 2 L.H.S = R.H.S

$$p \leftrightarrow q \equiv (p \wedge q) \vee (7p \wedge 7q)$$

17. Exercise 12.2 (8) **Aug -21**

Prove that i) $7(p \wedge q) \equiv 7p \vee 7q$

ii) $7(p \rightarrow q) \equiv p \wedge 7q$

Ans: i)

p	q	$p \wedge q$	LHS $7(p \wedge q)$	7p	7q	R.H.S $7p \vee 7q$
T	T	T	F	F	F	F
T	F	F	T	F	T	T
F	T	F	T	T	F	T
F	F	F	T	T	T	T
			(1)			(2)

From (1) and (2) $7(p \wedge q) \equiv 7p \vee 7q$

ii)

p	q	$p \rightarrow q$	LHS $7(p \rightarrow q)$	7p	7q	R.H.S $p \wedge 7q$
T	T	T	F	F	F	F
T	F	F	T	F	T	T
F	T	T	F	T	F	F
F	F	T	F	T	T	F
			(1)		(2)	

From (1) and (2) $7(p \wedge q) \equiv 7p \vee 7q$.

18. Prove that $7(p \leftrightarrow q) \equiv p \leftrightarrow 7q$

Proof :

p, q two statements $\Rightarrow$ 4 rows

p	q	$p \leftrightarrow q$	LHS $7(p \leftrightarrow q)$	7q	R.H.S $p \leftrightarrow 7q$
T	T	T	F	F	F
T	F	F	T	T	T
F	T	F	T	F	T
F	F	T	F	T	F
			(1)		(2)

From (1) and (2) ..,

$$7(p \leftrightarrow q) \equiv p \leftrightarrow 7q$$

19. Using Truth Table prove that **March 2022**

$$\boldsymbol{p \rightarrow (q \rightarrow r) \equiv (p \wedge q) \rightarrow r}$$

Proof:

p, q, r Three statements $\Rightarrow$ 8 Rows.

p	q	r	$q \rightarrow r$	L.H.S. $p \rightarrow (q \rightarrow r)$	$p \wedge q$	R.H.S. $(p \wedge q) \rightarrow r$
T	T	T	T	T	T	T
T	T	F	F	F	T	F
T	F	T	T	T	F	T
T	F	F	T	T	F	T
F	T	T	T	T	F	T
F	T	F	F	T	F	T
F	F	T	T	T	F	T
F	F	F	T	T	F	T
				(1)		(2)

From 1 and 2 $p \rightarrow (q \rightarrow r) \equiv (p \wedge q) \rightarrow r$

20. Using Truth Table prove that March 2023

$$\boldsymbol{p \rightarrow (7q \vee r) \equiv 7p \vee (7q \vee r)}$$

Proof:

p,q,r $\Rightarrow$ 8 rows

P	q	r	7q	$7q \vee r$	L.H.S. $p \rightarrow (7q \vee r)$	$7p$	R.H.S. $7p \vee (7q \vee r)$
T	T	T	F	T	T	F	T
T	T	F	F	F	F	F	F
T	F	T	T	T	T	F	T
T	F	F	T	T	T	F	T
F	T	T	F	T	T	T	T
F	T	F	F	F	T	T	T
F	F	T	T	T	T	T	T
F	F	F	T	T	T	T	T
					(1)		(2)

From (1) and (2) : $p \rightarrow (7q \vee r) \equiv 7p \vee (7q \vee r)$

21. Do it yourself

Using Truth Table prove that

Associative laws

i) $p \vee (q \vee r) \equiv (p \vee q) \vee r$

ii) $p \wedge (q \wedge r) \equiv (p \wedge q) \wedge r$

22. **Distributive laws:**

(i) $p \vee (q \wedge r) \equiv (p \vee q) \wedge (p \vee r)$

(ii) $p \wedge (q \vee r) \equiv (p \wedge q) \vee (p \wedge r)$

23. Example 12.6 **July 2022**

Verify (i) Closure property (ii) Commutative Property (iii) associative property of the following operation on a given set..

a * b = a^b; a, b ∈ N

ANS :

Set: N. Natural numbers

a * b = a^b

i) Closure property:

Let a, b ∈ N⇒ a * b = a^b ∈ N

2, 3 ∈ N ⇒ 2 * 3 = 2^3 = 8 ∈ N

* have closure property.

ii) **Commutative property:**

a * b = a^b

also b * a = b^a

⇒$a * b \neq b * a$

⇒* have not commutative property..

iii) **Associative property**:

(a * b) * c ≠ a * (b * c) $\forall\, a, b, c \in N$

⇒ * have not Associative property.

Note :

iv) * have not Identity Property.

v) * have not Inverse property.

24. Exercise 12.1 (5)

Define on operation * on Q as follows:

$$\boldsymbol{a * b = \frac{a+b}{2} \quad \forall\, a, b \in Q}$$

i) Examine the closure, commutative and associative properties satisfied by * on Q.

ii) Examine the existence of identity and the existence of inverse for the operation * on Q.

Ans : Set : Q Relational numbers

⇒ * Is : a * b = $\frac{a+b}{2}$ $\forall a, b \in Q$

i) **Closure property**:

$a, b \in Q \Rightarrow$ a * b = $\frac{a+b}{2}$ ∈ Q

⇒ * have closure property

ii) **Commutative property**:

$a * b = \frac{a+b}{2}$

$= \frac{b+a}{2}$

$= b * a$

⇒ * have commutative property.

iii) **Associative property** :

$(a * b) * c \neq a * (b * c) \quad \forall\, a, b, c \in Q$

$\Rightarrow$ * have not Associative property.

iv) **Identity property**:

* does not satisfy uniqueness of Identity..

$\Rightarrow$ * have not Identity property.

v) **Inverse property** :

No Identity $\Rightarrow$ * have not Inverse property..

25. Example: 12.7

Verify (i) Closure property (ii) Commutative property (iii) associative property Existence of identity and (V) existence of inverse for the following operation on the given set.

m * n = m + n – mn; ∀ m, n ∈ z

ANS : set : z

* Is m * n = m + n – mn, ∀m, n ∈ z

1) Closure property --- TRUE
2) Commutative property – TRUE
3) Associative property ---TRUE
4) Identity property ---TRUE e = 0 ∈ Z
5) Inverse property – Does not Exist (Inverse element does not exist)

26. Exercise 12.1 (10- i, ii)

Let A be Q /{1} .Define * on A.

by x * y = x + y – xy. Is * binary on A? If so

i) Examine the commutative and associative property satisfied by * on A.

ii) Examine the existence of identity and existence of identity and existence of inverse properties for the operation * on A.

ANS: Set A = Q – {1}

* is x * y = x + y – xy

i) Closure property – TRUE
ii) Commutative property – TRUE
iii) Associative property ---TRUE
iv) Identity property ---TRUE e = 0 ∈ Z
v) Inverse property -- True…

(inverse of x) , $x^{-1} = \frac{-x}{1-x} \in A$

27. Example : 12.9

Verify (i) Closure property (ii) Commutative property (iii) Associative property (iv) Existence of identity and (v) existence of inverse for the operation $+_5$ on z_5 using table corresponding to addition modulo 5..

ANS:

SET $z_5 = \{0, 1, 2, 3, 4\}$ * is $+_5$

$+_5$	0	1	2	3	4
0	0	1	2	3	4
1	1	2	3	4	0
2	2	3	4	0	1
3	3	4	0	1	2
4	4	0	1	2	3

1. Closure property – True
2. Commutative property –True
3. Associative Property – True
4. Identity Property -True e = 0 $\in z_5$
5. Inverse property – True from table,

From Table, Inverse of 0 = 0 $\in z_5$
Inverse of 1 = 4 $\in z_5$
Inverse of 2 = 3 $\in z_5$
Inverse of 3= 2 $\in z_5$
Inverse of 4= 1 $\in z_5$

Note : All properties – TRUE

28. Example 12.10

Verify (i) closure property (ii) Commutative property (iii)Associative property iv) existence of identity and v) existence of inverse for the operation x_{11} on a subset A = { 1,3,4,5,9} of the set of remainders {0,1,2,3,4,5,6,7,8,9,10}?

ANS:

SET A = {1, 3, 4, 5, 9}

* = X_{11}

X_{11}	1	3	4	5	9
1	(1)	3	4	5	9
3	3	9	(1)	4	5
4	4	(1)	5	9	3
5	5	4	9	3	(1)
9	9	5	3	(1)	4

1. Closure property – True
2. Commutative property –True
3. Associative Property – True
4. Identity Property -True e = 1 $\in$ A
5. Inverse property – True ,

Inverse of 1 = 1 $\in$ A
Inverse of 3 = 4 $\in$ A
Inverse of 4 = 3 $\in$ A
Inverse of 5 = 9$\in$ A
Inverse of 9 = 5 $\in$ A

NOTE : All properties ------------**True**

29. Exercise 12.1 (9) **June 2023**

Let M = $\left\{\begin{pmatrix} x & x \\ x & x \end{pmatrix} : x \in R - \{0\}\right\}$ and let * be the matrix multiplication. Determine whether M is closed under *. If so (i) examine the commutative and associative for the operation * on M properties satisfied by * on M. (ii) Examine the existence of identity and existence of inverse properties.

Ans:

Let M = $\left\{\begin{pmatrix} x & x \\ x & x \end{pmatrix} : x \in R - \{0\}\right\}$

* is matrix Multiplication…

1. Closure property – True
2. Commutative property –True
3. Associative Property – True
4. Identify Property -True

$$E = \begin{pmatrix} 1/2 & 1/2 \\ 1/2 & 1/2 \end{pmatrix} \in M$$

5. Inverse property – True ,

Inverse element $= \begin{pmatrix} \frac{1}{4x} & \frac{1}{4x} \\ \frac{1}{4x} & \frac{1}{4x} \end{pmatrix} \in M$

	Questions	Closure Property	Commutative Property	Associate Property	Identity Property	Inverse Property
23	N a * b = a^b	√	x	x	x	x
24	Q $a * b = \frac{a+b}{2}$	√	√	x	x	x
25	Z m * n = m + n - mn	√	√	√	√ e = 0	x
26	Q – {1} x * y = x + y - xy	√	√	√	√ e = 0	$x^{-1} = \frac{-x}{1-x}$ √

Chapter 6. APPLICATION OF VECTOR ALGEBRA

5 MARKS

1. Prove by vector method $\cos(\alpha + \beta) = \cos\alpha \cos\beta - \sin\alpha \sin\beta$

$\vec{a} = \cos\alpha\, \vec{i} + \sin\alpha\, \vec{j}$

$\vec{b} = \cos\beta\, \vec{i} - \sin\beta\, \vec{j}$

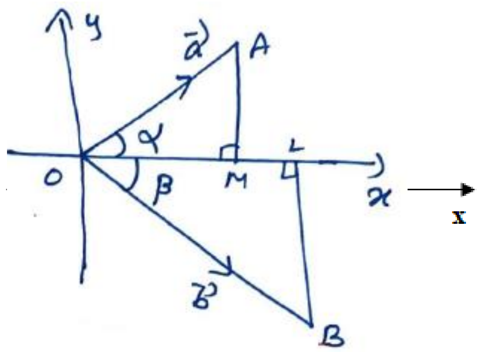
$\vec{b} \cdot \vec{a} = \cos(\alpha + \beta)$ ………….. (1)

$\vec{b} \cdot \vec{a} = \cos\alpha \cos\beta - \sin\alpha \sin\beta$ …….. (2)

From (1) and (2)

$\cos(\alpha + \beta) = \cos\alpha \cos\beta - \sin\alpha \sin\beta$

HENCE PROVED



2. $\cos(\alpha - \beta) = \cos\alpha \cos\beta + \sin\alpha \sin\beta$

$\vec{a} = \cos\alpha\, \vec{i} + \sin\alpha\, \vec{j}$

$\vec{b} = \cos\beta\, \vec{i} + \sin\beta\, \vec{j}$

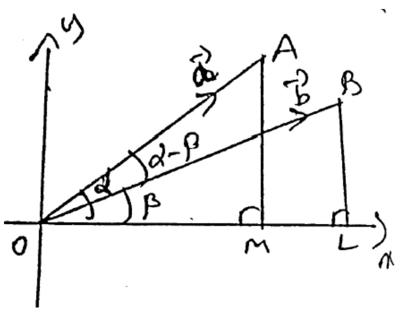
$\vec{b} \cdot \vec{a} = \cos(\alpha - \beta)$ ………….. (1)

$\vec{b} \cdot \vec{a} = \cos\alpha \cos\beta + \sin\alpha \sin\beta$ …….. (2)

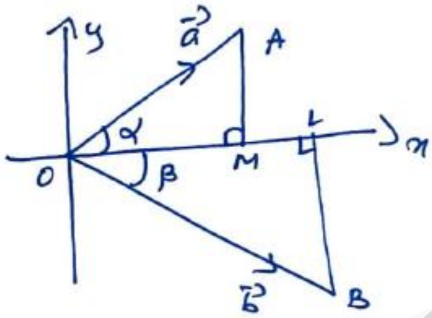
From (1) and (2)

$\cos(\alpha - \beta) = \cos\alpha \cos\beta + \sin\alpha \sin\beta$

HENCE PROVED



3. $\sin(\boldsymbol{\alpha} + \boldsymbol{\beta}) = \boldsymbol{sin\alpha\, cos\beta + cos\alpha\, sin\beta}$



$\vec{a} = \cos\alpha\,\vec{\imath} + sin\alpha\,\vec{\jmath}$

$\vec{b} = \cos\beta\,\vec{\imath} - sin\beta\,\vec{\jmath}$

$\vec{b} \times \vec{a} = \sin(\alpha + \beta)\,\vec{k}$ ………….. (1)

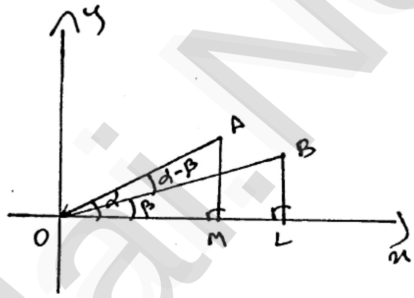
$\vec{b} \times \vec{a} = (sin\alpha\, cos\beta + cos\alpha\, sin\beta)\,\vec{k}$ …….. (2)

From (1) and (2)

$\sin(\alpha + \beta) = sin\alpha\, cos\beta + cos\alpha\, sin\beta$

HENCE PROVED

4. $\sin(\boldsymbol{\alpha} - \boldsymbol{\beta}) = \boldsymbol{sin\alpha\, cos\beta - cos\alpha\, sin\beta}$



$\vec{a} = \cos\alpha\,\vec{\imath} + sin\alpha\,\vec{\jmath}$

$\vec{b} = \cos\beta\,\vec{\imath} + sin\beta\,\vec{\jmath}$

$\vec{b} \times \vec{a} = \sin(\alpha - \beta)\,\vec{k}$ ………….. (1)

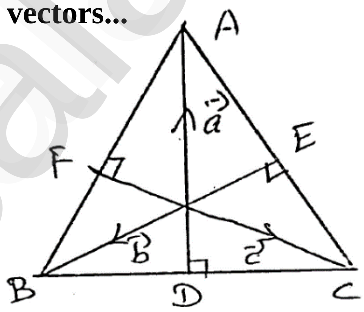
$\vec{b} \times \vec{a} = (sin\alpha\, cos\beta - cos\alpha\, sin\beta)\,\vec{k}$ ….. (2)

From (1) and (2)

$\sin(\alpha - \beta) = sin\alpha\, cos\beta - cos\alpha\, sin\beta$

Hence Proved.

5.` **Show that the attitudes of a triangle are concurrent by using vectors…**



$\overrightarrow{OA} = \vec{a},\ \overrightarrow{OB} = \vec{b},\ \overrightarrow{OC} = \vec{c}$

$\vec{a}.(\vec{b} - \vec{c}) = 0$ …………. **(1)**

$\vec{b}.\ (\vec{c} - \vec{a}) = 0$ …………. **(2)**

(1) + (2)

$\vec{c}.(\vec{a} - \vec{b}) = 0$

$\overrightarrow{OC} \perp \overrightarrow{AB}$ Hence

The attitudes of a triangle are concurrent hence proved.

6. $\boldsymbol{\vec{a} = \vec{\imath} - \vec{\jmath},\ \vec{b} = \vec{\imath} - \vec{\jmath} - 4\vec{k},\ \vec{c} = 3\vec{\jmath} - \vec{k}}$ ***and*** $\boldsymbol{\vec{d} = 2\vec{\imath} + 5\vec{\jmath} + \vec{k}}$ **verify that…**

(i) $\boldsymbol{(\vec{a} \times \vec{b}) \times (\vec{c} \times \vec{d}) = [\vec{a},\ \vec{b},\ \vec{d}]\vec{c} - [\vec{a},\ \vec{b},\ \vec{c}]\vec{d}}$

Solution: LHS $(\vec{a} \times \vec{b}) \times (\vec{c} \times \vec{d})$

$$\vec{a} \times \vec{b} = \begin{vmatrix} \vec{\imath} & \vec{\jmath} & \vec{k} \\ 1 & -1 & 0 \\ 1 & -1 & -4 \end{vmatrix} = 4\vec{\imath} + 4\vec{\jmath}$$

$$\vec{c} \times \vec{d} = \begin{vmatrix} \vec{\imath} & \vec{\jmath} & \vec{k} \\ 0 & 3 & -1 \\ 2 & 5 & 1 \end{vmatrix} = 8\vec{\imath} - 2\vec{\jmath} - 6\vec{k}$$

$$(\vec{a} \times \vec{b}) \times (\vec{c} \times \vec{d}) = \begin{vmatrix} \vec{\imath} & \vec{\jmath} & \vec{k} \\ 4 & 4 & 0 \\ 8 & -2 & -6 \end{vmatrix} = -24\vec{\imath} + 24\vec{\jmath} - 40\vec{k} \ \text{……. (1)}$$

RHS $[\vec{a},\ \vec{b},\ \vec{d}]\vec{c} - [\vec{a},\ \vec{b},\ \vec{c}]\vec{d}$

$$[\vec{a},\ \vec{b},\ \vec{d}] = \begin{vmatrix} 1 & -1 & 0 \\ 1 & -1 & -4 \\ 2 & 5 & 1 \end{vmatrix} = 28$$

$$[\vec{a},\ \vec{b},\ \vec{c}] = \begin{vmatrix} 1 & -1 & 0 \\ 1 & -1 & -4 \\ 0 & 3 & -1 \end{vmatrix} = 12$$

$[\vec{a},\ \vec{b},\ \vec{d}]\vec{c} - [\vec{a},\ \vec{b},\ \vec{c}]\vec{d} = -24\vec{\imath} + 24\vec{\jmath} - 40\vec{k}$ …… (2)

From (1) & (2)

$(\vec{a} \times \vec{b}) \times (\vec{c} \times \vec{d}) = [\vec{a}, \vec{b}, \vec{d}]\vec{c} - [\vec{a}, \vec{b}, \vec{c}]\vec{d}$

Hence proved

For practice:

$(\vec{a} \times \vec{b}) \times (\vec{c} \times \vec{d}) = [\vec{a}, \vec{c}, \vec{d}]\vec{b} - [\vec{b}, \vec{c}, \vec{d}]\vec{a}$

7. **If $\vec{a} = 2\vec{\imath} + 3\vec{\jmath} - \vec{k}$, $\vec{b} = 3\vec{\imath} + 5\vec{\jmath} + 2\vec{k}$, $\vec{c} = -\vec{\imath} - 2\vec{\jmath} + 3\vec{k}$; Then verify that**

(i) $(\vec{a} \times \vec{b}) \times \vec{c} = (\vec{a}.\vec{c})\vec{b} - (\vec{b}.\vec{c})\vec{a}$

Solution : L H S $(\vec{a} \times \vec{b}) \times \vec{c}$

$$\vec{a} \times \vec{b} = \begin{vmatrix} \vec{\imath} & \vec{\jmath} & \vec{k} \\ 2 & 3 & -1 \\ 3 & 5 & 2 \end{vmatrix} = 11\vec{\imath} - 7\vec{\jmath} + \vec{k}$$

$$(\vec{a} \times \vec{b}) \times \vec{c} = \begin{vmatrix} \vec{\imath} & \vec{\jmath} & \vec{k} \\ 11 & -7 & 1 \\ -1 & -2 & 3 \end{vmatrix} = -19\vec{\imath} - 34\vec{\jmath} - 29\vec{k} \quad \text{.......... (1)}$$

RHS $(\vec{a}.\vec{c})\vec{b} - (\vec{b}.\vec{c})\vec{a}$

$\vec{a}.\vec{c} = -11$

$\vec{b}.\vec{c} = -7$

/ $(\vec{a}.\vec{c})\vec{b} - (\vec{b}.\vec{c})\vec{a} = -19\vec{\imath} - 34\vec{\jmath} - 29\vec{k}$ .... (2)

From (1) & (2)

$(\vec{a} \times \vec{b}) \times \vec{c} = (\vec{a}.\vec{c})\vec{b} - (\vec{b}.\vec{c})\vec{a}$

HENCE PROVED

ii. $\vec{a} \times (\vec{b} \times \vec{c}) = (\vec{a}.\vec{c})\vec{b} - (\vec{a}.\vec{b})\vec{c}$

LHS $\vec{a} \times (\vec{b} \times \vec{c})$

$$\vec{b} \times \vec{c} = \begin{vmatrix} \vec{\imath} & \vec{\jmath} & \vec{k} \\ 3 & 5 & 2 \\ -1 & -2 & 3 \end{vmatrix} = 19\vec{\imath} - 11\vec{\jmath} - \vec{k}$$

$$\vec{a} \times (\vec{b} \times \vec{c}) = \begin{vmatrix} \vec{\imath} & \vec{\jmath} & \vec{k} \\ 2 & 3 & -1 \\ 19 & -11 & -1 \end{vmatrix} = -14\vec{\imath} - 17\vec{\jmath} - 79\vec{k}$$

$\vec{a}.\vec{c} = -11$

$\vec{a}.\vec{b} = 19$

RHS $(\vec{a}.\vec{c})\vec{b} - (\vec{a}.\vec{b})\vec{c} = -14\vec{\imath} - 17\vec{\jmath} - 79\vec{k}$

From (1) & (2) $(\vec{a}.\vec{c})\vec{b} - (\vec{a}.\vec{b})\vec{c} = -14\vec{\imath} - 17\vec{\jmath} - 79\vec{k}$

Hence proved.

One Point & Two vectors

Parametric vector equation : $\vec{r} = \vec{a} + t\vec{b} + s\vec{c}$

Non – Parametric vector equation : $(\vec{r} - \vec{a}).(\vec{b} \times \vec{c}) = 0$

Cartesian equation : $\begin{vmatrix} x - x_1 & y - y_1 & z - z_1 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix} = 0$

8. Find the non- parametricform form of vector equation and cartesian equation of the plane passing through the point (0, 1, -5) and parallel to the straight line

$$\vec{r} = (\vec{i} + 2\vec{j} - 4\vec{k}) + s(2\vec{i} + 3\vec{j} + 6\vec{k}) \text{ and}$$

$$\vec{r} = (\vec{i} - 3\vec{j} + 5\vec{k}) + t(\vec{i} + \vec{j} - \vec{k})$$

Solution : $\vec{a} = \vec{j} - 5\vec{k}$

$\vec{b} = 2\vec{i} + 3\vec{j} + 6\vec{k}$

$\vec{c} = \vec{i} + \vec{j} - \vec{k}$

(i) Parametric vector equation .

$$\vec{r} = \vec{a} + s\vec{b} + t\vec{c}$$

$$\vec{r} = (\vec{j} - 5\vec{k}) + s(2\vec{i} + 3\vec{j} + 6\vec{k}) + t(\vec{i} + \vec{j} - \vec{k})$$

(ii) Cartesian equation: $\begin{vmatrix} x - x_1 & y - y_1 & z - z_1 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix} = 0$

$$\begin{vmatrix} x - 0 & y - 1 & z + 5 \\ 2 & 3 & 6 \\ 1 & 1 & -1 \end{vmatrix} = 0$$

$(x_1, y_1, z_1) = (0, 1, -5)$
$(b_1, b_2, b_3) = (2, 3, 6)$
$(c_1, c_2, c_3) = (1, 1, -1)$

$(x - 0)(-3 - 6) - (y - 1)(-2 - 6) + (z + 5)(2 - 3) = 0$

$x(-9) + 8(y - 1) - 1(z + 5) = 0$

$-9x + 8y - z - 13 = 0$

∴ 9x – 8y + z + 13 = 0

(iii) Non – parametric vector equation..

$$(\vec{r} - \vec{a}).(\vec{b} \times \vec{c}) = 0$$

$$\vec{r}.(9\vec{i} - 8\vec{j} + \vec{k}) + 13 = 0$$

9. Find the non – parametric form of vector equation and Cartesian equation of the plane passing through the point (2,3,6) and parallel to the straight lines

$$\frac{x-1}{2} = \frac{y+1}{3} = \frac{z-3}{1} \quad \& \quad \frac{x+3}{2} = \frac{y-3}{-5} = \frac{z+1}{-3}.$$

Solution :

$\vec{a} = 2\vec{i} + 3\vec{j} + 6\vec{k}$

$\vec{b} = 2\vec{i} + 3\vec{j} + \vec{k}$

$\vec{c} = 2\vec{i} - 5\vec{j} - 3\vec{k}$

(i) Parametric vector equation : $\vec{r} = \vec{a} + s\vec{b} + t\vec{c}$

$$\vec{r} = (2\vec{i} + 3\vec{j} + 6\vec{k}) + s(2\vec{i} + 3\vec{j} + \vec{k}) + t(2\vec{i} - 5\vec{j} - 3\vec{k})$$

(ii) Cartesian Equation : $\begin{vmatrix} x - x_1 & y - y_1 & z - z_1 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix} = 0$

$$\begin{vmatrix} x - 2 & y - 3 & z - 6 \\ 2 & 3 & 1 \\ 2 & -5 & -3 \end{vmatrix} = 0$$

$(x_1, y_1, z_1) = (2, 3, 6)$
$(b_1, b_2, b_3) = (2, 3, 1)$
$(c_1, c_2, c_3) = (2, -5, -3)$

$(x - 2)(-9 + 5) - (y - 3)(-6 - 2) + (z - 6)(-10 - 6) = 0$

$(x - 2)(-4) - (y - 3)(-8) + (z - 6)(-16) = 0$

$(x - 2)(-1) + 2(y - 3) - 4(z - 6) = 0$

$-x + 2y - 4z + 20 = 0$

$x - 2y + 4z - 20 = 0$

(iii) Non – parametric vector equation : $(\vec{r} - \vec{a}).(\vec{b} \times \vec{c}) = 0$

$$\vec{r}.(\vec{i} - 2\vec{j} + 4\vec{k}) - 20 = 0`$$

10. Find the vector (parametric and non- parametric and Cartesian equation of the plane containing the line $\vec{r} = (\vec{i} - \vec{j} + 3\vec{k}) + t(2\vec{i} - \vec{j} + 4\vec{k})$ and perpendicular to the plane $\vec{r}.(\vec{i} + 2\vec{j} + \vec{k}) = 8$.

Solution : $\vec{a} = \vec{i} - \vec{j} + 3\vec{k}$ $\quad (x_1, y_1, z_1) = (1, -1, 3)$

$\vec{b} = 2\vec{i} - \vec{j} + 4\vec{k}$ $\quad (b_1, b_2, b_3) = (2, -1, 4)$

$\vec{c} = \vec{i} + 2\vec{j} + \vec{k}$ $\quad (c_1, c_2, c_3) = (1, 2, 1)$

(i) Parametric vector equation $\vec{r} = \vec{a} + s\vec{b} + t\vec{c}$

$$\vec{r} = (\vec{i} - \vec{j} + 3\vec{k}) + s(2\vec{i} - \vec{j} + 4\vec{k}) + t(\vec{i} + 2\vec{j} + \vec{k})$$

(iii) Cartesian equation : $\begin{vmatrix} x - x_1 & y - y_1 & z - z_1 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix} = 0$

$$\begin{vmatrix} x - 1 & y + 1 & z - 3 \\ 2 & -1 & 4 \\ 1 & 2 & 1 \end{vmatrix} = 0$$

$(x - 1)(-1 - 8) - (y + 1)(2 - 4) + (z - 3)(4 + 1) = 0$

$(x - 1)(-9) - (y + 1)(-2) + (z - 3)(5) = 0$

$-9x + 2y + 5z - 4 = 0$

ஃ $9x - 2y - 5z + 4 = 0$

(iii) Non parametric vector equation $(\vec{r} - \vec{a}).(\vec{b} \times \vec{c}) = 0$

$$\vec{r}.(9\vec{i} - 2\vec{j} - 5\vec{k}) + 4 = 0$$

11. Find the vector (parametric and non – parametric) and Cartesian equations of the plane passing through the point (1,-2,4) and perpendicular to the plane ; x + 2y – 3z = 11 and parallel to the line $\frac{x+7}{3} = \frac{y+3}{-1} = \frac{z}{1}$.

Solution : $\vec{a} = \vec{i} - 2\vec{j} + 4\vec{k}$ $\quad (x_1, y_1, z_1) = (1, -2, 4)$

$\vec{b} = \vec{i} + 2\vec{j} - 3\vec{k}$ $\quad (b_1, b_2, b_3) = (1, 2, -3)$

$\vec{c} = 3\vec{i} - \vec{j} + \vec{k}$ $\quad (c_1, c_2, c_3) = (3, -1, 1)$

(i) Parametric vector equation : $\vec{r} = \vec{a} + s\vec{b} + t\vec{c}$

$$\vec{r} = (\vec{i} - 2\vec{j} + 4\vec{k}) + s(\vec{i} + 2\vec{j} - 3\vec{k}) + t(3\vec{i} - \vec{j} + \vec{k})$$

(ii) Cartesian equation : $\begin{vmatrix} x - x_1 & y - y_1 & z - z_1 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix} = 0$

$$\begin{vmatrix} x - 1 & y + 2 & z - 4 \\ 1 & 2 & -3 \\ 3 & -1 & 1 \end{vmatrix} = 0$$

$(x - 1)(2 - 3) - (y + 2)(1 + 9) + (z - 4)(-1 - 6) = 0$

$(x - 1)(-1) - (y + 2)(10) + (z - 4)(-7) = 0$

$-x - 10y - 7z + 9 = 0$

ஃ $x + 10y + 7z - 9 = 0$

(iii) Non – Parametric vector equation : $(\vec{r} - \vec{a}).(\vec{b} \times \vec{c}) = 0$

$$\vec{r}.(\vec{i} + 10\vec{j} + 7\vec{k}) - 9 = 0$$

12. Find the non – parametric from of vector equation and Cartesian equation of the plane $\vec{r} = (6\vec{i} - \vec{j} + \vec{k}) + s(-\vec{i} + 2\vec{j} + \vec{k}) + t(-5\vec{i} - 4\vec{j} - 5\vec{k})$

Solution : $\vec{a} = 6\vec{i} - \vec{j} + \vec{k}$ $\quad (x_1, y_1, z_1) = (6, -1, 1)$

$\vec{b} = -\vec{i} + 2\vec{j} + \vec{k}$ $\quad (b_1, b_2, b_3) = (-1, 2, 1)$

$\vec{c} = -5\vec{i} - 4\vec{j} - 5\vec{k}$ $\quad (c_1, c_2, c_3) = (-5, -4, -5)$

(i) Cartesian equation $\begin{vmatrix} x - x_1 & y - y_1 & z - z_1 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix} = 0$

$$\begin{vmatrix} x-6 & y+1 & z-1 \\ -1 & 2 & 1 \\ -5 & -4 & -5 \end{vmatrix} = 0$$

$(x-6)(-10+4) - (y+1)(5+5) + (z-1)(4+10) = 0$

$(x-6)(-6) - (y+1)(10) + (z-1)(14) = 0$

$-3x - 5y + 7z + 6 = 0$

ஃ $3x + 5y - 7z - 6 = 0$

(ii) Non – parametric vector equation :

$$(\vec{r} - \vec{a}).(\vec{b} \times \vec{c}) = 0$$

$$\vec{r}.(3\vec{i} + 5\vec{j} - 7\vec{k}) - 6 = 0$$

TWO POINTS AND ONE VECTORS

1. Parametric vector equation : $\vec{r} = (1-s)\vec{a} + s\vec{b} + t\vec{c}$
2. Non – parametric vector : $(\vec{r} - \vec{a}).[(\vec{b} - \vec{a}) \times \vec{c}] = 0$
3. Cartesian equation: $\begin{vmatrix} x - x_1 & y - y_1 & z - z_1 \\ x_2 - x_1 & y_2 - y_1 & z_2 - z_1 \\ c_1 & c_2 & c_3 \end{vmatrix} = 0$

13. Find the parametric form of vector equation and Cartesian equation of the plane passing through the points (2,2,1) (9,3,6) and perpendicular to the plane 2x+6y+6z= 9.

Ans : $\vec{a} = 2\vec{i} + 2\vec{j} + \vec{k}$ $\quad (x_1, y_1, z_1) = (2, 2, 1)$

$\vec{b} = 9\vec{i} + 3\vec{j} + 6\vec{k}$ $\quad (x_2, y_2, z_2) = (9, 3, 6)$

$\vec{c} = 2\vec{i} + 6\vec{j} + 6\vec{k}$ $\quad (c_1, c_2, c_3) = (2, 6, 6)$

i. Parametric vector equation : $\vec{r} = (1-s)\vec{a} + s\vec{b} + t\vec{c}$

$$\vec{r} = (1-s)(2\vec{i} + 2\vec{j} + \vec{k}) + s(9\vec{i} + 3\vec{j} + 6\vec{k}) + t(2\vec{i} + 6\vec{j} + 6\vec{k})$$

ii. Cartesian equation : $\begin{vmatrix} x - x_1 & y - y_1 & z - z_1 \\ x_2 - x_1 & y_2 - y_1 & z_2 - z_1 \\ c_1 & c_2 & c_3 \end{vmatrix} = 0$

$$\begin{vmatrix} x-2 & y-2 & z-1 \\ 9-2 & 3-2 & 6-1 \\ 2 & 6 & 6 \end{vmatrix} = 0$$

$$\begin{vmatrix} x-2 & y-2 & z-1 \\ 7 & 1 & 5 \\ 2 & 6 & 6 \end{vmatrix} = 0$$

$(x-2)(6-30) - (y-2)(42-10) + (z-1)(42-2) = 0$

$(x-2)(-24) - (y-2)(32) + (z-1)(40) = 0$

$(x-2)(-3) - (y-2)(4) + (z-1)(5) = 0$

$-3x - 4y + 5z + 9 = 0$

ஃ $3x + 4y - 5z - 9 = 0$

iii. Non – parametric vector equation : $(\vec{r} - \vec{a}).[(\vec{b} - \vec{a}) \times \vec{c}] = 0$

$$\vec{r}.(3\vec{i} + 4\vec{j} - 5\vec{k}) - 9 = 0$$

14. Find the vector parametric , vector non- parametric and carterian form of the equation of the plane passing through the points (-1, 2, 0) (2, 2, -1) and parallel to the straight line ; $\frac{x-1}{1} = \frac{2y+1}{2} = \frac{z+1}{-1}$.

Ans : $\vec{a} = -\vec{i} + 2\vec{j}$ $\quad (x_1, y_1, z_1) = (-1, 2, 0)$

$\vec{b} = 2\vec{i} + 2\vec{j} - \vec{k}$ $\quad (x_2, y_2, z_2) = (2, 2, -1)$

$\vec{c} = \vec{i} + \vec{j} - \vec{k}$ $\quad (c_1, c_2, c_3) = (1, 1, -1)$

i. Parametric vector equation: $\vec{r} = (1-s)\vec{a} + s\vec{b} + t\vec{c}$

$$\vec{r} = (1-s)(-\vec{i} + 2\vec{j}) + s(2\vec{i} + 2\vec{j} - \vec{k}) + t(\vec{i} + \vec{j} - \vec{k})$$

ii. Cartesian equation: $\begin{vmatrix} x - x_1 & y - y_1 & z - z_1 \\ x_2 - x_1 & y_2 - y_1 & z_2 - z_1 \\ c_1 & c_2 & c_3 \end{vmatrix} = 0$

$$\begin{vmatrix} x+1 & y-2 & z-0 \\ 2+1 & 2-2 & -1-0 \\ 1 & 1 & -1 \end{vmatrix} = 0$$

$$\begin{vmatrix} x+1 & y-2 & z-0 \\ 3 & 0 & -1 \\ 1 & 1 & -1 \end{vmatrix} = 0$$

$$(x+1)(0+1) - (y-2)(-3+1) + z(3-0) = 0$$

$$(x+1) - (y-2)(-2) + 3z = 0$$

$$\therefore\ x + 2y + 3z - 3 = 0$$

iii. Non – parametric vector equation : $(\vec{r} - \vec{a}).[(\vec{b} - \vec{a}) \times \vec{c}] = 0$

$$\vec{r}.(\vec{i} + 2\vec{j} + 3\vec{k}) - 3 = 0$$

15. Find parametric from of vector equation and Cartesian equation of the plane passing through the points (2, 2, 1) (1, -2, 3)and parallel to the straight line passing through the points (2, 1, -3) and (-1, 5, -8).

Ans : $\vec{a} = 2\vec{i} + 2\vec{j} + \vec{k}$ $\quad (x_1, y_1, z_1) = (2,2,1)$

$\vec{b} = \vec{i} - 2\vec{j} + 3\vec{k}$ $\quad (x_2, y_2, z_2) = (1, -2, 3)$

$\vec{c} = (2, 1, -3) - (-1, 5, -8)$

$= (2+1, 1-5, -3+8)$

$= (3, -4, 5)$

$\therefore\ \vec{c} = 3\vec{i} - 4\vec{j} + 5\vec{k}$ $\quad (c_1, c_2, c_3) = (3, -4, 5)$

i. Parametric vector equation: $\vec{r} = (1-s)\vec{a} + s\vec{b} + t\vec{c}$

$$\vec{r} = (1-s)(2\vec{i} + 2\vec{j} + \vec{k}) + s(\vec{i} - 2\vec{j} + 3\vec{k}) + t(3\vec{i} - 4\vec{j} + 5\vec{k})$$

ii. Cartesian equation: $\begin{vmatrix} x - x_1 & y - y_1 & z - z_1 \\ x_2 - x_1 & y_2 - y_1 & z_2 - z_1 \\ c_1 & c_2 & c_3 \end{vmatrix} = 0$

$$\begin{vmatrix} x-2 & y-2 & z-1 \\ 1-2 & -2-2 & 3-1 \\ 3 & -4 & 5 \end{vmatrix} = 0$$

$$\begin{vmatrix} x-2 & y-2 & z-1 \\ -1 & -4 & 2 \\ 3 & -4 & 5 \end{vmatrix} = 0$$

$$(x-2)(-20+8)-(y-2)(-5-6)+(z-1)(4+12)=0$$
$$(x-2)(-12)-(y-2)(-11)+(z-1)(16)=0$$
$$-12x+11y+16z-14=0$$
$$\therefore\ 12x-11y-16z+14=0$$

iii. Non – parametric vector equation : $(\vec{r}-\vec{a}).[(\vec{b}-\vec{a})\times\vec{c}]=0$

$$\vec{r}.(12\vec{i}-11\vec{j}-16\vec{k})=-14$$

3 Points

1. Parametric vector equation: $\vec{r}=(1-s-t)\vec{a}+s\vec{b}+t\vec{c}$
2. Non – parametric vector equation : $[\vec{r}-\vec{a},\ \vec{b}-\vec{a},\ \vec{c}-\vec{a}]=0$
3. Cartesian equation: $\begin{vmatrix} x-x_1 & y-y_1 & z-z_1 \\ x_2-x_1 & y_2-y_1 & z_2-z_1 \\ x_3-x_1 & y_3-y_1 & z_3-z_1 \end{vmatrix}=0$

16. Find the vector (Parametric and non – parametric) and Cartesian from of the equations of the plane (3, 6, -2), (-1, -2, 6) and (6, 4, -2)

ANS: $\vec{a}=3\vec{i}+6\vec{j}-2\vec{k}$ $(x_1, y_1, z_1)=(3,6,-2)$

$\vec{b}=-\vec{i}-2\vec{j}+6\vec{k}$ $(x_2, y_2, z_2)=(-1,-2,6)$

$\vec{c}=6\vec{i}+4\vec{j}-2\vec{k}$ $(x_3, y_3, z_3)=(6,4,-2)$

1. Parametric vector equation: $\vec{r}=(1-s-t)\vec{a}+s\vec{b}+t\vec{c}$

$$\vec{r}=(1-s-t)3\vec{i}+6\vec{j}-2\vec{k}+s(-\vec{i}-2\vec{j}+6\vec{k})+t(6\vec{i}+4\vec{j}-2\vec{k})$$

2. Cartesian equation : $\begin{vmatrix} x-x_1 & y-y_1 & z-z_1 \\ x_2-x_1 & y_2-y_1 & z_2-z_1 \\ x_3-x_1 & y_3-y_1 & z_3-z_1 \end{vmatrix}=0$

$$\begin{vmatrix} x-3 & y-6 & z+2 \\ -1-3 & -2-6 & 6+2 \\ 6-3 & 4-6 & -2+2 \end{vmatrix}=0$$

$$\begin{vmatrix} x-3 & y-6 & z+2 \\ -4 & -8 & 8 \\ 3 & -2 & 0 \end{vmatrix}=0$$

$$(x-3)(0+16)-(y-6)(0-24)+(z+2)(8+24)=0$$
$$(x-3)(16)-(y-6)(-24)+(z+2)(32)=0$$
$$(x-3)(2)-(y-6)(-3)+(z+2)(4)=0$$
$$\therefore\ 2x+3y+4z-16=0$$

3. Non – parametric vector equation

$$[\vec{r}-\vec{a},\ \vec{b}-\vec{a},\ \vec{c}-\vec{a}]=0$$
$$\vec{r}.(2\vec{i}+3\vec{j}+4\vec{k})=16$$

17. Find the vector parametric and Cartesian equation of a straight line passing through the points (-5, 7, -4) and (13, -5, 2). Find the point where the straight line crosses the xy- plane..

Ans : A (-5, 7, -4) B (13, -5, 2)

AB (13 + 5, -5-7, 2+4) = AB (18, -12, 6)

= AB (3, -2, 1)

1. Parametric vector equation:

$$\vec{r}=(-5\vec{i}+7\vec{j}-4\vec{k})+t(3\vec{i}-2\vec{j}+\vec{k}) \text{ and}$$
$$\vec{r}=(13\vec{i}-5\vec{j}+2\vec{k})+s(3\vec{i}-2\vec{j}+\vec{k})$$

2. Cartesian equation :

$\frac{x+5}{3} = \frac{y-7}{-2} = \frac{z+4}{1}$ and $\frac{x-13}{3} = \frac{y+5}{-2} = \frac{z-2}{1}$

$$\frac{x+5}{3} = \frac{y-7}{-2} = \frac{z+4}{1} = t$$

(x, y, z) = (3t – 5, -2t + 7, t -4)

Crosses xy plane = 0

t – 4 = 0 $\Longrightarrow t = 4$

Crosses the xy plane (x, y, z) = (7,-1,0)

2 Marks questions

18. **Find the volume of the parallelopiped whose coterminus edges (adjacent sides) are given by the vectors $2\vec{\imath} - 3\vec{\jmath} + 4\vec{k}$, $\vec{\imath} + 2\vec{\jmath} - \vec{k}$ and $3\vec{\imath} - \vec{\jmath} + 2\vec{k}$.**

Ans : $[\vec{a}, \vec{b}, \vec{c}] = \begin{vmatrix} 2 & -3 & 4 \\ 1 & 2 & -1 \\ 3 & -1 & 2 \end{vmatrix} = -7$

Volume of parallelepiped $= |-7| =$ 7 cu.units

19. **The volume of the parallelepiped whose determine edges are $7\vec{\imath} + \lambda\vec{\jmath} - 3\vec{k}$, $\vec{\imath} + 2\vec{\jmath} - \vec{k}$, $-3\vec{\imath} + 7\vec{\jmath} + 5\vec{k}$ is 90 cubic units. Find the value of λ.**

Ans : $[\vec{a}, \vec{b}, \vec{c}] = \begin{vmatrix} 7 & \lambda & -3 \\ 1 & 2 & -1 \\ -3 & 7 & 5 \end{vmatrix} = 90$

$\lambda = -5$

20. **Show that the vectors $\vec{\imath} + 2\vec{\jmath} - 3\vec{k}$, $2\vec{\imath} - \vec{\jmath} + 2\vec{k}$ and $3\vec{\imath} + \vec{\jmath} - \vec{k}$ are coplanar.**

Ans : $[\vec{a}, \vec{b}, \vec{c}] = \begin{vmatrix} 1 & 2 & -3 \\ 2 & -1 & 2 \\ 3 & 1 & -1 \end{vmatrix} = 0$

$\therefore$ Given vectors are coplanar.

21. **If $2\vec{\imath} - \vec{\jmath} + 3\vec{k}$, $3\vec{\imath} + 2\vec{\jmath} + \vec{k}$, $\vec{\imath} + m\vec{\jmath} + 4\vec{k}$ are coplanar find the value of m.**

Ans : $\begin{vmatrix} 2 & -1 & 3 \\ 3 & 2 & 1 \\ 1 & m & 4 \end{vmatrix} = 0$

m = -3

22. **For any vector $\vec{a}$, prove that….**

$\vec{\imath} \times (\vec{a} \times \vec{\imath}) + \vec{\jmath} \times (\vec{a} \times \vec{\jmath}) + \vec{k} \times (\vec{a} \times \vec{k}) = 2\vec{a}$

Ans : $\vec{\imath} \times (\vec{a} \times \vec{\imath}) = (\vec{\imath}.\vec{\imath})\vec{a} - (\vec{\imath}.\vec{a})\vec{\imath}$

L.H.S.

$\vec{\imath} \times (\vec{a} \times \vec{\imath}) + \vec{\jmath} \times (\vec{a} \times \vec{\jmath}) + \vec{k} \times (\vec{a} \times \vec{k}) = 3\vec{a} - \vec{a}$

$= 2\vec{a}$ R.H.S

$\vec{\imath} \times (\vec{a} \times \vec{\imath}) + \vec{\jmath} \times (\vec{a} \times \vec{\jmath}) + \vec{k} \times (\vec{a} \times \vec{k}) = 2\vec{a}$

Hence proved..

23. **Prove that $[\vec{a} - \vec{b}, \vec{b} - \vec{c}, \vec{c} - \vec{a}] = 0$**

ANS: L.H.S

$[\vec{a} - \vec{b}, \quad \vec{b} - \vec{c}, \quad \vec{c} - \vec{a}] = \begin{vmatrix} 1 & -1 & 0 \\ 0 & 1 & -1 \\ -1 & 0 & 1 \end{vmatrix} [\vec{a}, \vec{b}, \vec{c}] = 0$

$[\vec{a} - \vec{b}, \vec{b} - \vec{c}, \vec{c} - \vec{a}] = 0$

Hence proved…

24. **For any vectors $\vec{a}, \vec{b}, \vec{c}$ then proved that $[\vec{a}+\vec{c},\ \vec{a}+\vec{b},\ \vec{a}+\vec{b}+\vec{c}] = [\vec{a}, \vec{b}, \vec{c}]$**

Ans : L.H.S

$$[\vec{a}+\vec{c},\quad \vec{a}+\vec{b},\quad \vec{a}+\vec{b}+\vec{c}] = \begin{vmatrix}1 & 0 & 1\\ 1 & 1 & 0\\ 1 & 1 & 1\end{vmatrix}[\vec{a}, \vec{b}, \vec{c}]$$

$$= [\vec{a}, \vec{b}, \vec{c}]$$

$[\vec{a}+\vec{c},\ \vec{a}+\vec{b},\ \vec{a}+\vec{b}+\vec{c}] = [\vec{a}, \vec{b}, \vec{c}]$ Hence proved.

25. **Prove that the distance from the origin to the plane $3x + 6y + 2z + 7 = 0$ is 1 units..**

Ans : origin $(x, y, z) = (0, 0, 0)$

distance $= \left|\frac{3(0)+6(0)+2(0)+7}{\sqrt{3^2+6^2+2^2}}\right| = \left|\frac{7}{7}\right| = 1$ units.

26. **Show that the lines $\frac{x-1}{4} = \frac{2-y}{6} = \frac{z-4}{12}$ and $\frac{x-3}{-2} = \frac{y-3}{3} = \frac{5-2}{6}$ are parallel.**

Ans : $4\vec{i} - 6\vec{j} + 12\vec{k} = -2(-2\vec{i} + 3\vec{j} - 6\vec{k})$

The given lines are parallel…

3 Marks

27. **Find the magnitude and the direction cosines of the torque about the point (2, 0, -1) of a force $2\vec{i} + \vec{j} - \vec{k}$ whose line of action passes through the origin.**

Ans: $\overrightarrow{OA} = 2\vec{i} - \vec{k}$

$\vec{r} = \overrightarrow{AO} = -2\vec{i} + \vec{k}$

$\vec{F} = 2\vec{i} + \vec{j} - \vec{k}$

Torque $\vec{t} = \vec{r} \times \vec{F} = \begin{vmatrix}\vec{i} & \vec{j} & \vec{k}\\ -2 & 0 & 1\\ 2 & 1 & -1\end{vmatrix} = -\vec{i} - 2\vec{k}$

Magnitude $|\vec{t}| = \sqrt{1^2 + 0 + 2^2} = \sqrt{5}$

Direction cosines; $\frac{-1}{\sqrt{5}},\ 0,\ \frac{-2}{\sqrt{5}}$

28. **Forces of magnitudes $5\sqrt{2}$ and $10\sqrt{2}$ units acting in the directions $3\vec{i} + 4\vec{j} + 5\vec{k}$ and $10\vec{i} + 6\vec{j} - 8\vec{k}$ respectively, act on a particle which is displaced from the point with position vector $4\vec{i} - 3\vec{j} - 2\vec{k}$ to the point with position vector $6\vec{i} + \vec{j} - 3\vec{k}$ find the work done by the solution…**

Ans : Force $\vec{F} = 5\sqrt{2}\left(\frac{3\vec{i}+4\vec{j}+5\vec{k}}{5\sqrt{2}}\right) + 10\sqrt{2}\left(\frac{10\vec{i}+6\vec{j}-8\vec{k}}{10\sqrt{2}}\right)$

$= 3\vec{i} + 4\vec{j} + 5\vec{k} + 10\vec{i} + 6\vec{j} - 8\vec{k}$

$\vec{F} = 13\vec{i} + 10\vec{j} - 3\vec{k}$

Displacement $\vec{d} = (6\vec{i} + \vec{j} - 3\vec{k}) - (4\vec{i} - 3\vec{j} - 2\vec{k})$

$= 2\vec{i} + 4\vec{j} - \vec{k}$

Work done by the force $= W = \vec{F} \cdot \vec{d}$

$(13\vec{i} + 10\vec{j} - 3\vec{k}) \cdot (2\vec{i} + 4\vec{j} - \vec{k}) = 69$ units.

29. Find the angle between the straight lines $\frac{x+3}{2} = \frac{y-1}{2} = -z$ with coordinate axes.

Ans : $=\frac{x+3}{2} = \frac{y-1}{2} = \frac{-z}{1}$

$\vec{b} = \frac{2\vec{i}+2\vec{j}-\vec{k}}{|2\vec{i}+2\vec{j}-\vec{k}|} = \frac{1}{3}(2\vec{i}+2\vec{j}-\vec{k})$

$\alpha = cos^{-1}\left(\frac{2}{3}\right), \ \beta = cos^{-1}(2/3), \ \gamma = cos^{-1}(-1/3)$

30. If $\vec{a}, \vec{b}, \vec{c}$ are coplanar. Then prove that $\vec{a}+\vec{b}, \vec{b}+\vec{c}, \vec{c}+\vec{a}$ are coplanar..

Ans : $[\vec{a}+\vec{b}, \vec{b}+\vec{c}, \vec{c}+\vec{a}] = \begin{vmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \\ 1 & 0 & 1 \end{vmatrix} [\vec{a}, \vec{b}, \vec{c}]$

$= 0$

$= \{\vec{a}+\vec{b}, \vec{b}+\vec{c}, \vec{c}+\vec{a}\}$ are coplanar..

31. Show that the points (2, 3, 4) (-1, 4, 5) and (8, 1, 2) are collinear

Ans : $\begin{vmatrix} 2 & 3 & 4 \\ -1 & 4 & 5 \\ 8 & 1 & 2 \end{vmatrix} = 0$

The given points are collinear..

32. Find the angle between the straight lines $\frac{x-4}{2} = \frac{y}{1} = \frac{z+1}{-2}$ and $\frac{x-1}{4} = \frac{y+1}{-4} = \frac{z-2}{2}$ and state whether they are parallel (or) perpendicular:

Ans: Angle between the two st.lines

$$\theta = cos^{-1}\left(\frac{a_1b_1 + a_2b_2 + a_3b_3}{\sqrt{a_1^2 + a_2^2 + a_3^2}\ \sqrt{b_1^2 + b_2^2 + b_3^2}}\right)$$

$\theta = cos^{-1}(0) = \pi/2$

The given lines are perpendicular

33. Find the vector equation of a plane which is at a distance of 7 units from the origin having 3, -4, 5 as direction ratios of a normal to it ….

Solution : $\vec{r}.\vec{n} = P$

$\vec{r}.\frac{(3\vec{i}-4\vec{j}+5\vec{k})}{\sqrt{9+16+25}} = 7$

$\vec{r}.\frac{(3\vec{i}-4\vec{j}+5\vec{k})}{5\sqrt{2}} = 7$

Cartesian equation is $3x - 4y + 5z = 35\sqrt{2}$

34. Find the distance from the point (2, 5, -3) to the plane

$\vec{r}.(6\vec{i} - 3\vec{j} + 2\vec{k}) = 5$

Ans: $\delta = \frac{|\vec{u}.\vec{n}-P|}{|\vec{n}|} = \frac{|(2\vec{i}+5\vec{j}-3\vec{k}).(6\vec{i}-3\vec{j}+2\vec{k})-5|}{|(6\vec{i}-3\vec{j}+2\vec{k})|}$

$\delta = 2$ units.

35. Find the distance between the two planes. x + 2y – 2z + 1 = 0 and 2x + 4y – 4z + 5 = 0

Ans : $\delta = \frac{|d_1 - d_2|}{\sqrt{a^2 + b^2 + c^2}}$

2x + 4y – 4z + 5 = 0

x + 2y – 2z + $\frac{5}{2}$ = 0

$\delta = \frac{|1 - 5/2|}{\sqrt{1^2 + 2^2 + 2^2}} = \frac{1}{2}$ units……

Chapter 2. COMPLEX NUMBERS

2 MARK QUESTIONS:

1. **Simplify :** $i^{-1924} + i^{2018}$

ANS : $i^0 + i^2$

$= 1 - 1$

$= 0$

$i^2 = -1$

$i^3 = -i$

$i^4 = 1$

2. **Simplify :**

$$\sum_{n=1}^{10} i^{n+50}$$

Note: Divide the last 2 digit of the power by 4 and write the remainder as power.

Ans:

$$\sum_{n=1}^{10} i^{n+50} = (i^{51} + i^{52} + i^{53} + i^{54}) + (i^{55} + i^{56} + i^{57} + i^{58}) + i^{59} + i^{60}$$

$= 0 + 0 + i^{59} + i^{60}$

$= i^3 + i^0$

$= -i + 1$

$= 1 - i$

Do yourself:

1. Simplify: $i.i^2.i^3 \ldots\ldots\ldots.i^{40}$ Ans: 1

2. $i^{1947} + i^{1950}$ Ans: $-i - 1$

3. **If $z_1 = 3 - 2i$ and; $z_2 = 6 + 4i$ find ; $\frac{z_1}{z_2}$ in the rectangular form …**

Ans: $\frac{z_1}{z_2} = \frac{3-2i}{6+4i}$

$= \frac{3-2i}{6+4i} \times \frac{6-4i}{6-4i}$

$= \frac{18-12i-12i+8i^2}{36+16}$

$= \frac{18-24i-8}{52}$

$= \frac{10-24i}{52}$

$\frac{z_1}{z_2} = \frac{10}{52} - \frac{24}{52}i = \frac{5}{26} - \frac{6}{13}i$

$z = x + iy$

$\overline{z} = x - iy$

$z\overline{z} = (x + iy)(x - iy)$

$z\overline{z} = x^2 + y^2$

4. **Find ; z^{-1} , if $z = (2 + 3i)(1 - i)$**

Solition :

$z = (2 + 3i)(1 - i)$

$= 2 - 2i + 3i - 3i^2$

$= 2 + i + 3$

$z = 5 + i$

$z^{-1} = \frac{1}{z} = \frac{1}{5+i} \times \frac{5-i}{5-i} = \frac{5-i}{25+1}$

$z^{-1} = \frac{5-i}{26}$

5. Prove the following properties; $Re(z) = \frac{z+\bar{z}}{2}$ (ii) $Im(z) = \frac{z-\bar{z}}{2i}$;:

Ans: $z = x + iy$

$Re(z) = x, \quad Im(z) = y$

$\bar{z} = x - iy$

$z + \bar{z} = 2x$

$x = \frac{z+\bar{z}}{2}$

$Re(z) = \frac{z+\bar{z}}{2}$

$z - \bar{z} = 2iy$

$y = \frac{z-\bar{z}}{2i}$

$Im(z) = \frac{z-\bar{z}}{2i}$

6. Show that $(2+i\sqrt{3})^{10} + (2-i\sqrt{3})^{10}$ is real.

Ans :

$z = (2+i\sqrt{3})^{10} + (2-i\sqrt{3})^{10}$

$\bar{Z} = (2-i\sqrt{3})^{10} + (2+i\sqrt{3})^{10}$

$\bar{Z} = Z$

$\therefore Z = (2+i\sqrt{3})^{10} + (2-i\sqrt{3})^{10}$ *is real ...*

7. Show that $(2+i\sqrt{3})^{10} - (2-i\sqrt{3})^{10}$ is purely imaginary

ANS :

$z = (2+i\sqrt{3})^{10} - (2-i\sqrt{3})^{10}$

$\bar{Z} = (2-i\sqrt{3})^{10} - (2+i\sqrt{3})^{10}$

$\bar{Z} = -Z$

$\therefore Z = (2+i\sqrt{3})^{10} - (2-i\sqrt{3})^{10}$ is purely imaginary

8. Find the square root of 6 -8i

Ans : $z = 6 - 8i$

$$\sqrt{a-ib} = \pm\left(\sqrt{\frac{|z|+a}{2}} - i\sqrt{\frac{|z|-a}{2}}\right)$$

$|z| = \sqrt{6^2 + 8^2} = \sqrt{36+64} = \sqrt{100}$

$|z| = 10, \quad a = 6, \quad b = -8$

$$\sqrt{6-8i} = \pm\left(\sqrt{\frac{10+6}{2}} - i\sqrt{\frac{10-6}{2}}\right)$$

$$= \pm\left(\sqrt{\frac{16}{2}} - i\sqrt{\frac{4}{2}}\right)$$

$$= \pm\left(\sqrt{8} - i\sqrt{2}\right)$$

$$= \pm\left(2\sqrt{2} - i\sqrt{2}\right)$$

9. Find the square root of $-5-12i$

Ans: $z = -5-12i$

$|z| = \sqrt{25+144} = \sqrt{169} = 13, \quad a = -5, \quad b = -12$

$\sqrt{-5-12i} = \pm\left(\sqrt{\frac{13-5}{2}} - i\sqrt{\frac{13+5}{2}}\right)$

$= \pm\left(\sqrt{\frac{8}{2}} - i\sqrt{\frac{18}{2}}\right)$

$= \pm(\sqrt{4} - i\sqrt{9})$

∴ $= \pm(2-3i)$

10. Find the square root $-6+8i$

Ans: $\sqrt{a+ib} = \pm\left(\sqrt{\frac{|z|+a}{2}} + i\sqrt{\frac{|z|-a}{2}}\right)$

$|z| = \sqrt{36+64} = \sqrt{100} = 10 \quad , \quad a = -6, \quad b = 8$

$\sqrt{-6+8i} = \pm\left(\sqrt{\frac{10-6}{2}} + i\sqrt{\frac{10+6}{2}}\right)$

$= \pm\left(\sqrt{\frac{4}{2}} + i\sqrt{\frac{16}{2}}\right)$

$= \pm(\sqrt{2} + i\sqrt{8})$

∴ $= \pm(\sqrt{2} + i\,2\sqrt{2})$

11. Show that the equation $|3z-6+12i| = 8$ represent a circle and final its centre and radius..

Ans : The equation of circle

$|z-a| = r$

Centre = a, radius; = r

$|3z-6+12i| = 8$

Divided by 3 on both side …

$|z-(2-4i)| = \frac{8}{3}$

∴ $|z.(2-4i)| = \frac{8}{3}$ represent a circle, centre = $2-4i$, radius = $\frac{8}{3}$

12. Find the modulus and principal arguments of $\sqrt{3}+i$

Ans: $z = \sqrt{3}+i$

Modulus $|z| = \sqrt{(\sqrt{3})^2 + 1^2}$

$= \sqrt{3+1}$

$= \sqrt{4}$

$|z| = 2$

Argument $\alpha = tan^{-1}\left(\frac{1}{\sqrt{3}}\right)$

$\alpha = \frac{\pi}{6}$

The principal argument $\theta = \alpha = \frac{\pi}{6}$

$z = \sqrt{3}+i$ is in first quadrant

Frmula

$z = x+iy$

modulus $|z| = \sqrt{x^2+y^2}$

argument $\alpha = tan^{-1}\left(\left|\frac{y}{x}\right|\right)$

To find principal

II	I
$\theta = \pi - \alpha$	$\theta = \alpha$
$\theta = -\pi + \alpha$	$\theta = -\alpha$
III	IV

13. Find the modules and principal arguments of $-\sqrt{3}-i$

Ans : $z = -\sqrt{3}-i$ is in III[rd] quadrant

$$|z| = \sqrt{\left(-\sqrt{3}\right)^2 + (-1)^2}$$

$$|z| = \sqrt{3+1} = \sqrt{4}$$

$$|z| = 2$$

Arguments $\alpha = tan^{-1}\left(\frac{1}{\sqrt{3}}\right) = \frac{\pi}{6}$

Principal argument $\theta = -\pi + \alpha$

$$= -\pi + \frac{\pi}{6} = \frac{-5\,\pi}{6}$$

3 Mark Questions

14. Show that $\left(\frac{19-7i}{9+i}\right)^{12} + \left(\frac{20-5i}{7-6i}\right)^{12}$ is real.

Ans :

$$\frac{19-7i}{9+i} = \frac{19-7i}{9+i} \times \frac{9-i}{9-i}$$

$$= \frac{(171-7)+i(-63-19)}{81+1}$$

$$= \frac{164-82i}{82}$$

$$\frac{19-7i}{9+i} = 2-i$$

Similarly : $\frac{20-5i}{7-6i} = 2+i$

$$z = \left(\frac{19-7i}{9+i}\right)^{12} + \left(\frac{20-5i}{7-6i}\right)^{12}$$

$$z = (2-i)^{12} + (2+i)^{12}$$

$$\overline{z} = (2+i)^{12} + (2-i)^{12}$$

$$\overline{z} = z$$

∴ z is real.......

15. If $|z| = 3$ *show that* $7 \le |z+6-8i| \le 13$ ..

ANS: $||z_1| - |z_2|| \le |z_1 + z_2| \le |z_1| + |z_2|$

$z_1 = z$ $z_2 = 6-8i$

$|z_1| = |z| = 3$ $|z_2| = \sqrt{36+64}$

$$|z_2| = \sqrt{100}$$

$$|z_2| = 10$$

$$|3-10| \le |z+6-8i| \le 3+10$$

$$|-7| \le |z+6-8i| \le 13$$

$$7 \le |z+6-8i| \le 13$$

16. IF $z = x + iy$ is a complex number $\left|\frac{z-4i}{z+4i}\right| = 1$ such that the locus of z is real axis

Proof: $\left|\frac{z-4i}{z+4i}\right| = 1$

$|z - 4i| = |z + 4i|$

$|x + iy - 4i| = |x + iy + 4i|$

$|x + i(y - 4)|^2 = |x + i(y + 4)|^2$

$x^2 + (y - 4)^2 = x^2 + (y + 4)^2$

$y^2 - 8y + 16 = y^2 + 8y + 16$

$0 = 16y$,

$y = 0$

The locus of z is real axis.

17. Find the Cartesian form of the complex number $\left(\cos\frac{\pi}{6} + i\sin\frac{\pi}{6}\right)\left(\cos\frac{\pi}{12} + i\sin\frac{\pi}{12}\right)$

Solution: $\left(\cos\frac{\pi}{6} + i\sin\frac{\pi}{6}\right)\left(\cos\frac{\pi}{12} + i\sin\frac{\pi}{12}\right)$

$= \cos\left(\frac{\pi}{6} + \frac{\pi}{12}\right) + i\sin\left(\frac{\pi}{6} + \frac{\pi}{12}\right)$

$= \cos\left(\frac{3\pi}{12}\right) + i\sin\left(\frac{3\pi}{12}\right)$

$= \cos\frac{\pi}{4} + i\sin\frac{\pi}{4}$

$= \frac{1}{\sqrt{2}} + i\frac{1}{\sqrt{2}}$

18. If $(x_1 + iy_1)(x_2 + iy_2)(x_3 + iy_3) \ldots\ldots\ldots (x_n + iy_n) = a + ib$ then show that

(i) $({x_1}^2 + {y_1}^2)({x_2}^2 + {y_2}^2)({x_3}^2 + {y_3}^2)\ldots({x_n}^2 + {y_n}^2) = a^2 + b^2$

(ii) $\sum_{r=1}^{n} \tan^{-1}\left(\frac{y_r}{x_r}\right) = \tan^{-1}\left(\frac{b}{a}\right) + 2k\pi \qquad k \in z$

Proof: $|(x_1 + iy_1)(x_2 + iy_2) \ldots\ldots\ldots (x_n + iy_n)| = |a + ib|$

$\sqrt{{x_1}^2 + {y_1}^2}\sqrt{{x_2}^2 + {y_2}^2} \ldots\ldots\ldots \sqrt{{x_n}^2 + {y_n}^2} = \sqrt{a^2 + b^2}$

$({x_1}^2 + {y_1}^2)({x_2}^2 + {y_2}^2) \ldots\ldots\ldots ({x_n}^2 + {y_n}^2) = (a^2 + b^2)$

$\arg(x_1 + y_1) + \arg(x_2 + y_2) + \cdots \ldots\ldots + \arg(x_n + iy_n) = \arg(a + ib)$

$\tan^{-1}\left(\frac{y_1}{x_1}\right) + \tan^{-1}\left(\frac{y_2}{x_2}\right) + \cdots \ldots\ldots\ldots + \tan^{-1}\left(\frac{y_n}{x_n}\right) = \tan^{-1}\left({}^{b}/_{a}\right)$

$\sum_{r=1}^{n} \tan^{-1}\left(\frac{y_r}{x_r}\right) = \tan^{-1}\left(\frac{b}{a}\right) + 2k\pi \qquad k \in z$

19. Simplify $\left(\frac{1+i}{1-i}\right)^3 - \left(\frac{1-i}{1+i}\right)^3$ into rectangular form

Solution: $\frac{1+i}{1-i} = \frac{1+i}{1-i} \times \frac{1+i}{1+i}$

$= \frac{1+2i+i^2}{1+1} = \frac{1+2i-1}{2}$

$= \frac{2i}{2}$

$\frac{1+i}{1-i} = i\,; \qquad \frac{1-i}{1+i} = -i$

$\left(\frac{1+i}{1-i}\right)^3 - \left(\frac{1-i}{1+i}\right)^3 = i^3 - (-i)^3$

$= i^3 + i^3$

$= -i - i$

$= -2i$

20. Represent the complex number $-1-i$ in polar form

Solution:

$$-1-i = r\,(cos\theta + i\,sin\theta)$$

$$r = \sqrt{(-1)^2 + (-1)^2} = \sqrt{1+1} = \sqrt{2}$$

$$\alpha = tan^{-1}\left(\frac{|y|}{|x|}\right) = tan^{-1}\left(\frac{1}{1}\right) = \frac{\pi}{4}$$

$-1-i$ is in 3rd quadrant.

$$\theta = -\pi + \alpha = -\pi + \frac{\pi}{4} = \frac{-3\pi}{4}$$

$$-1-i = \sqrt{2}\left(cos\left(\frac{-3\pi}{4}\right) + i\,sin\left(\frac{-3\pi}{4}\right)\right)$$

$$= \sqrt{2}\left(\cos\frac{3\pi}{4} - i\sin\frac{3\pi}{4}\right)$$

$$= \sqrt{2}\left(\cos\left(\frac{3\pi}{4} + 2km\right) - isin\left(\frac{3\pi}{4} + 2km\right)\right) \qquad k \in z$$

21. Express $1 + i\sqrt{3}$ in polar form

Solution:

$$1 + i\sqrt{3} = r(cos\theta + isin\theta)$$

$$r = \sqrt{1 + (3)^2} = \sqrt{1+3} = \sqrt{4} = 2$$

$$\alpha = tan^{-1}(\sqrt{3}) = \frac{\pi}{3}$$

$$1 + i\sqrt{3} = 2\left(cos\frac{\pi}{3} + isin\frac{\pi}{3}\right)$$

$$1 + i\sqrt{3} = 2\left(\cos\left(2k\pi + \frac{\pi}{3}\right) + i\,sin\left(2k\pi + \frac{\pi}{3}\right)\right)$$

22. Simplify : $\left(sin\frac{\pi}{6} + icos\frac{\pi}{6}\right)^{18}$

Solution: $\left(sin\frac{\pi}{6} + icos\frac{\pi}{6}\right)^{18}$

$$= \left(cos\left(\frac{\pi}{2} - \frac{\pi}{6}\right) + isin\left(\frac{\pi}{2} - \frac{\pi}{6}\right)\right)^{18}$$

$$= \left(cos\frac{\pi}{3} + isin\frac{\pi}{3}\right)^{18}$$

$$= cos\,18 \times \frac{\pi}{3} + isin\,18 \times \frac{\pi}{3}$$

$$= \cos 6\pi + i \sin 6\pi$$

$$= 1$$

23. If $z = cos\theta + i\,sin\theta$, show that $z^n + \frac{1}{z^n} = 2\cos n\theta$

Proof: $z = cos\theta + i\,sin\theta$

$$z^n = (cos\theta + i\,sin\theta)^n = \cos n\theta + isin\,n\theta$$

$$\frac{1}{z^n} = \cos n\theta - i\sin n\theta$$

$$z^n + \frac{1}{z^n} = 2\cos n\theta$$

24. If $z = cos\theta + i\,sin\theta$, show that $z^n - \frac{1}{z^n} = 2i\sin n\theta$

Proof:

$$z = cos\theta + i\,sin\theta$$

$$z^n = (cos\theta + i\,sin\theta)^n = \cos n\theta + isin\,n\theta$$

$$\frac{1}{z^n} = \cos n\theta - i\sin n\theta$$

$$z^n - \frac{1}{z^n} = 2\,\mathrm{i}\sin n\theta$$

25. **Simplify** $(\mathbf{1}+\boldsymbol{i})^{18}$

Solution: $1+i = r(cos\theta + isin\theta)$

$$r = \sqrt{1^2+1^2} = \sqrt{1+1} = \sqrt{2}$$

$$\theta = tan^{-1}\left(\frac{1}{1}\right) = tan^{-1}(1) = \frac{\pi}{4}$$

$$(1+i) = \sqrt{2}\left(cos\frac{\pi}{4} + i\,sin\frac{\pi}{4}\right)$$

$$(1+i)^{18} = (\sqrt{2})^{18}\left(cos\frac{\pi}{4} + i\,sin\frac{\pi}{4}\right)^{18}$$

$$= 2^9\left(cos\ \frac{\pi}{4}\times 18 + i\,sin\ \frac{\pi}{4}\times 18\right)$$

$$= 2^9\left(cos\ 9\frac{\pi}{2} + i\,sin\ 9\frac{\pi}{2}\right)$$

$$= 2^9 i$$

5 Mark Questions:

26. **Given the complex number $z = 3+2i$, represent the complex numbers z, iz, and $z+iz$ on one Avgand diagram. Show that these complex numbers form the vertices of an isosceles right triangle**

Solution:

$$z = 3+2i \implies A(3,2)$$

$$iz = i(3+2i)$$

$$iz = -2+3i \implies B(-2,3)$$

$$z+iz = 1+5i \implies C(1,5)$$

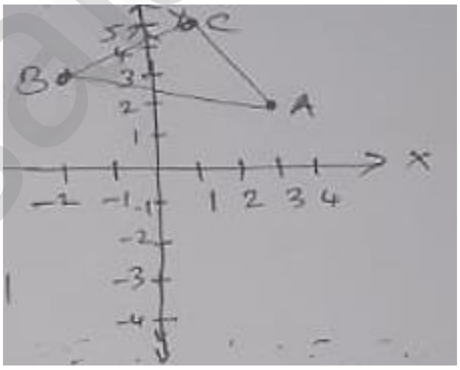
$$AB = |3+2i+2-3i|$$

$$AB = |5-i| = \sqrt{25+1} = \sqrt{26}$$

$$BC = |-3-2i| = \sqrt{9+4} = \sqrt{13}$$

$$CA = |1+5i-3-2i| = |-2+3i|$$

$$= \sqrt{4+9} = \sqrt{13}$$



$$AB^2 + BC^2 + CA^2$$

$$(\sqrt{26})^2 = (\sqrt{13})^2 + (\sqrt{13})^2$$

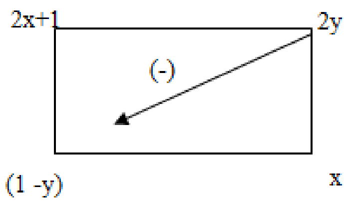
$$26 = 13+13$$

ΔABC is an isosceles right triangle.

27. **If $z = x+iy$ is a complex number such that $Im\left(\frac{2z+1}{iz+1}\right) = 0$, show that the locus of z is $2x^2+2y^2+x-2y = 0$**

Solution: $\frac{2z+1}{iz+1} = \frac{2(x+iy)+1}{i(x+iy)+1}$

$$= \frac{(2x+1)+2yi}{(1-y)+ix}$$



$$\text{Im} = \frac{2y(1-y)-x\,(2x+1)}{(1-y)^2+x^2} = 0$$

$$Im\left(\frac{2z+1}{iz+1}\right) = 0$$

$$2y(1-y) - x\,(2x+1) = 0$$

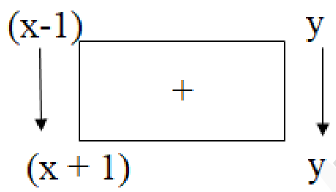
$$2y - 2y^2 - 2x^2 - x = 0$$

$$-2x^2 - 2y^2 - x + 2y = 0$$

(-) x $\quad 2x^2 + 2y^2 + x - 2y = 0$

28. **If $z = iy$ and $arg\left(\frac{z-1}{z+1}\right) = \frac{\pi}{2}$ show that $x^2 + y^2 = 1$**

Solution: $z = x + iy$



$arg\left(\frac{z-1}{z+1}\right) = \frac{\pi}{2}, \quad Re\left(\frac{z-1}{z+1}\right) = 0$

$\frac{z-1}{z+1} = \frac{(x+iy)-1}{(x+iy)+1}$

$= \frac{(x-1)+iy}{(x+1)+iy}$

$Re = \frac{(x-1)(x+1)+y^2}{(x+1)^2+y^2}$

$Re\left(\frac{z-1}{z+1}\right) = 0$, $(x-1)(x+1) + y^2 = 0$

ஃ $x^2 - 1 + y^2 = 0$

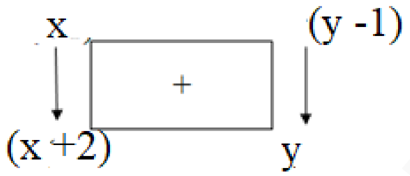
$x^2 + y^2 = 1$

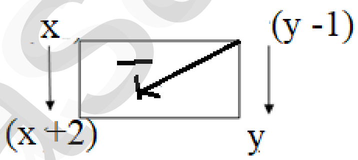
29. **If $z = x + iy$ and $arg\left(\frac{z-i}{z+2}\right) = \frac{\pi}{4}$, Then show that $x^2 + y^2 + 3x - 3y + 2 = 0$**

Solution: $\arg\left(\frac{z-i}{z+2}\right) = \frac{\pi}{4}$

$\Rightarrow Re\left(\frac{z-i}{z+2}\right) = Im\left(\frac{z-i}{z+2}\right)$ .... (1)

$\frac{z-i}{z+2} = \frac{x+iy-i}{x+iy+2} = \frac{x+i\,(y-1)}{(x+2)+iy}$





$Re = \frac{x(x+2)+y(y-1)}{(x+2)^2+y^2}$ $\quad Im = \frac{(x+2)\,(y-1)-xy}{(x+2)^2+y^2}$

(1) – Re = Im

$\Rightarrow x(x+2) + y(y-1) = (x+2)(y-1) - xy$

$x^2 + 2x + y^2 - y = xy - x + 2y - 2 - xy$

$x^2 + 2x + y^2 - y + x - 2y + 2 = 0$

$x^2 + y^2 + 3x - 3y + 2 = 0$

30. **Solve the equation $z^3 + 8i = 0$, where $z \in C$**

Solution: $z^3 + 8i = 0$

$z^3 = -8i$

$z = (-8i)^{1/3}$

$z = 8^{1/3}(-i)^{1/3}$

$z = (2^3)^{1/3}\left(cos\frac{\pi}{2} - isin\frac{\pi}{2}\right)^{1/3}$

$z = 2\left(cos\left(-\frac{\pi}{2}\right) + isin\left(-\frac{\pi}{2}\right)\right)^{1/3}$

$= 2\left[cos\left(2k\pi - \frac{\pi}{2}\right) + isin\left(2k\pi - \frac{\pi}{2}\right)\right]^{1/3}$

$= 2\left[cos(4k-1)\frac{\pi}{2} + isin\,(4k-1)\frac{\pi}{2}\right]^{1/3}$

$z = 2\left[cos(4k-1)\frac{\pi}{6} + isin(4k-1)\frac{\pi}{6}\right]$

$k = 0, 1, 2$

If k = 0 $\Rightarrow z = 2\left[cos\left(-\frac{\pi}{6}\right) + isin\left(-\frac{\pi}{6}\right)\right]$

$z = 2\left(\frac{\sqrt{3}}{2} - i\frac{1}{2}\right) = \sqrt{3} - i$

If k =1 $\Rightarrow$ $z = 2\left[\cos 3 \times \frac{\pi}{6} + isin\, 3 \times \frac{\pi}{6}\right]$

$z = 2i$

If k = 2 $\Rightarrow z = 2\left[cos\frac{7\pi}{6} + isin\frac{7\pi}{6}\right]$

$z = 2\left(cos\left(\pi + \frac{\pi}{6}\right) + isin\left(\pi + \frac{\pi}{6}\right)\right)$

$= 2\left(-cos\,{}^{\pi}/_{6} - isin\frac{\pi}{6}\right)$

$= 2\left(\frac{-\sqrt{3}}{2} - i\frac{1}{2}\right) = -\sqrt{3} - i$

31. Find all cube roots of $\sqrt{3} + i$

Solution: $z = \left(\sqrt{3} + i\right)^{1/3}$

$\sqrt{3} + i = r(cos\theta + isin\theta)$

$r = \sqrt{\left(\sqrt{3}\right)^2 + 1^2} = \sqrt{3+1} = \sqrt{4} = 2$

$\theta = tan^{-1}\left(\frac{1}{\sqrt{3}}\right) = \frac{\pi}{6}$

$\sqrt{3} + i = 2\left(cos\frac{\pi}{6} + isin\,\frac{\pi}{6}\right)$

$z = \left(\sqrt{3} + i\right)^{1/3} = 2^{1/3}\left(cos\frac{\pi}{6} + isin\,\frac{\pi}{6}\right)^{1/3}$

$= 2^{1/3}\left[cos\left(2k\pi + \frac{\pi}{6}\right) + isin\left(2k\pi + \frac{\pi}{6}\right)\right]^{1/3}$

$k = 0, 1, 2$

$= 2^{1/3}\left[\cos(12k + 1)\frac{\pi}{6} + isin(12k + 1)\frac{\pi}{6}\right]^{1/3}$

$= 2^{1/3}\left[\cos(12k + 1)\frac{\pi}{18} + isin(12k + 1)\frac{\pi}{18}\right]$

$k = 0, 1, 2$

$k = 0 \Rightarrow z = 2^{1/3}\left[cos\frac{\pi}{18} + isin\frac{\pi}{18}\right]$

$k = 1 \Rightarrow z = 2^{1/3}\left[cos\frac{13\pi}{18} + isin\frac{13\pi}{18}\right]$

$k = 2 \Rightarrow z = 2^{1/3}\left[cos\frac{25\pi}{18} + isin\frac{25\pi}{18}\right]$

$= 2^{1/3}\left[-cos\frac{7\pi}{18} - isin\frac{7\pi}{18}\right]$

32. Solve the Equation: $x^3 + 27 = 0$

Solution: $x^3 + 27 = 0$

$x^3 = -27$

$x^3 = 27(-1)$

$z = (27)^{1/3}(-1)^{1/3}$

$z = 3(-1)^{1/3}$

$z = 3(\,cos\pi + i\,sin\pi)^{1/3}$

$z = 3\,[cos(\pi + 2k\pi) + isin(\pi + 2k\pi)]^{1/3}$

$1 = cos0 + i\,sin0$

$-1 = cos\pi + i\,sin\pi$

$i = \cos\frac{\pi}{2} + isin\frac{\pi}{2}$

$-i = \cos\frac{\pi}{2} - isin\frac{\pi}{2}$

by De Moivre's theorem

$$= 3\left[cos\frac{\pi+2k\pi}{3} + isin\frac{\pi+2k\pi}{3}\right]$$

$k = 1, 2, 0$

Values: $k = 0$, $3cis\ \frac{\pi}{3}$

$k = 1$, $3cis\ \frac{3\pi}{3}$ (or) $3cis\pi$

$k = 2$, $3cis\ \frac{5\pi}{3}$

33. **If $2cos\alpha = x + \frac{1}{x}$ and $2cos\beta = y + \frac{1}{y}$, show that**

(i) $\frac{x}{y} + \frac{y}{x} = 2cos(\alpha - \beta)$

(ii) $xy - \frac{1}{xy} = 2i\sin(\alpha + \beta)$

(iii) $\frac{x^m}{y^n} - \frac{y^n}{x^m} = 2isin\ (m\alpha - n\beta)$

(iv) $x^m y^n = \frac{1}{x^m y^n} = 2\cos(m\alpha + n\beta)$

Proof:

$2cos\alpha = x + \frac{1}{x} \Rightarrow x = cos\alpha + isin\alpha$

$2cos\beta = y + \frac{1}{y} \Rightarrow y = cos\beta + isin\beta$

i. $\frac{x}{y} = \frac{cos\alpha + i\sin\alpha}{cos\beta + i\sin\beta} = cos(\alpha - \beta) + isin(\alpha - \beta)$

$\frac{y}{x} = cos(\alpha - \beta) - isin(\alpha - \beta)$

$\frac{x}{y} + \frac{y}{x} = 2\cos(\alpha - \beta)$

ii. $xy = (cos\alpha + isin\alpha)(cos\beta + isin\beta)$

$xy = cos(\alpha + \beta) + isin(\alpha + \beta)$

$\frac{1}{xy} = cos(\alpha + \beta) - isin(\alpha + \beta)$

$xy - \frac{1}{xy} = 2i\ sin\ (\alpha + \beta)$

iii. $x^m = (cos\alpha + isin\alpha)^m = cosm\alpha + isin\ m\alpha$

$y^n = (cos\beta + isin\beta)^n = cosn\beta + isin\ n\beta$

$\frac{x^m}{y^n} = \frac{cosm\alpha + isin\ m\alpha}{cosn\beta + isin\ n\beta}$

$\frac{x^m}{y^n} = cos(m\alpha - n\beta) + i\sin(m\alpha - n\beta)$

$\frac{y^n}{x^m} = cos(m\alpha - n\beta) - i\sin(m\alpha - n\beta)$

$\frac{x^m}{y^n} - \frac{y^n}{x^m} = 2isin\ (m\alpha - n\beta)$

iv. $x^m y^n = (cosm\alpha + isin\ m\alpha)\ (cosn\beta + isin\ n\beta)$

$x^m y^n = cos(m\alpha + n\beta) + isin(m\alpha + n\beta)$

$\frac{1}{x^m y^n} = cos\ (m\alpha + n\beta) - isin(m\alpha + n\beta)$

$x^m y^n + \frac{1}{x^m y^n} = 2cos(m\alpha + n\beta)$

Chapter – 5
Two Dimensional Analytical Geometry

1. Find the equation of the circle passing through the points (1, 1), (2, -1) and (3, 2) (5.10) March 20

$\boldsymbol{A(1,1)}$ $\boldsymbol{B(2,-1)}$ $\boldsymbol{C(3,2)}$

Slope of AB = $-2 = m_1$

Slope of AC = = $1/2 = m_2$

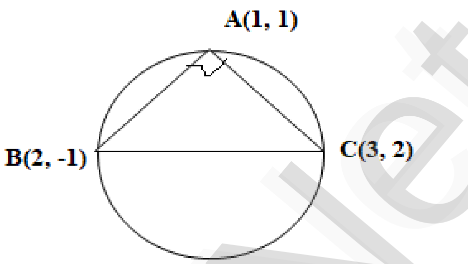
$m_1 m_2 = -1$

BC is diameter of circle $(2, -1)$ $(3, 2)$

$x_1 \; y_1 \quad x_2 \; y_2$

Equation of circle: $(x - x_1)(x - x_2) + (y - y_1)(y - y_2) = 0$

$x^2 + y^2 - 5x - y + 4 = 0$



2. Find the equation of circle through the points (1, 0) (-1, 0) and (0, 1) (5.1-6) June 23, Sep 23

$\boldsymbol{A(0,1)}$ $\boldsymbol{B(-1,0)}$ $\boldsymbol{C(1,0)}$

Slope of AB = $1 = m_1$

Slope of AC = $-1 = m_2$

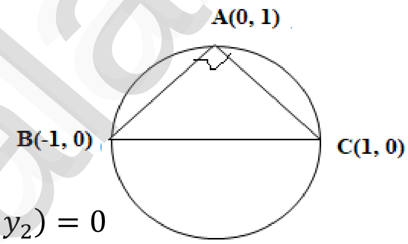
$m_1 m_2 = -1$

BC is diameter of circle $(-1, 0)$ $(1, 0)$

$x_1 \; y_1 \quad x_2 \; y_2$

Equation of circle: $(x - x_1)(x - x_2) + (y - y_1)(y - y_2) = 0$

$x^2 + y^2 = 1$



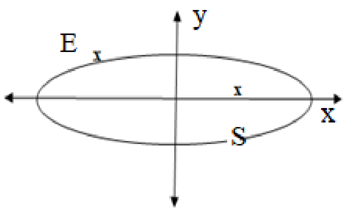
3. The maximum and minimum distance of the Earth from the sun respectively are 152×10^6 km and 94.5×10^6 km. The sun is at one focus of the elliptical orbit. Find the distance from the sun to the other focus (5.32) Mar 22 (5m), Mar – 23 (3m)

$a + c = 152 \times 10^6$ km .... (1)

$a - c = 94.5 \times 10^6\, km$ .... (2)

(1) – (2) $\Rightarrow$

$2c = 57.5 \times 10^6\, km$



Required distance $= 575 \times 10^5\, km$

4. Assume that water issuing from the end of horizontal pipe 7.5m above the ground describes a parabolic path. The vertex of the parabolic path is at the end of the pipe at a position 2.5m below the line of the pipe, the flow of water has curved outward 3m beyond the vertical line through the end of the pipe. How far beyond this vertical line will the water strike the ground? (5.5 – 8) Mar – 20 Mar – 24

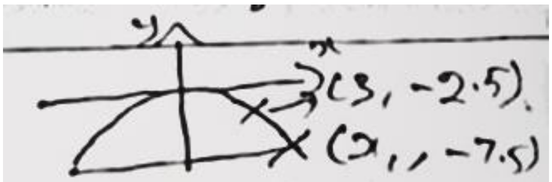
Equation of Parabola : $x^2 = 4ay$

$(3, -2.5) a = 9/10$

$x^2 = -4 \times \frac{9}{10} y$

$(x_1, -7.5)$

Required distance $= 3\sqrt{3}$ m



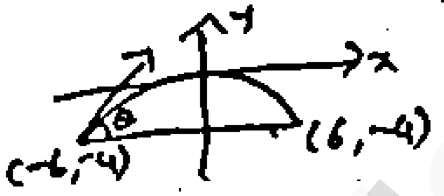
5. On lighting a rocket cracker, it gets projected in a parabolic path and reaches a maximum height of 4m when it is 6m away from the point of projection. Finally it reaches the ground 12m away from the starting point. Find the angle of projection. (5.5 – 9) Sep – 21.

Equation of parabola: $x^2 = -4ay$

$(-6,-4)$ $a = {}^9/_4$

$x^2 = -9y$

$\frac{dy}{dx} = \frac{-2x}{9}$



Angle = $\theta = tan^{-1}\left(\frac{4}{3}\right)$

6. Parabolic cable of a 60m portion of the road bed of a suspension bridge are positioned as shown below. Vertical cables are to be space every 6m along the portion of the road bed. Calculate the lengths of the first two of these vertical cables from the vertex. (5.5 - 5)

Equation of parabola:

$x^2 = 4ay$

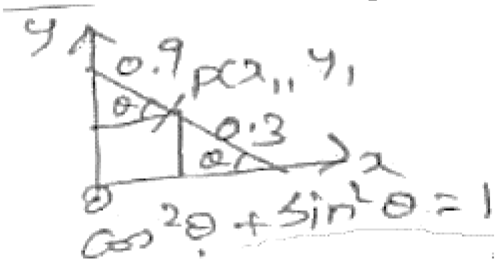
$(30, 13)$ $4a = \frac{900}{13}$

$x^2 = \frac{900}{13}\ y$

$(6,\ y_1)$ Req. Distance $= 3.52$ m

$(12,\ y_2)$ Req. Distance $= 5.08$ m

7. A rod of length 1.2 m moves with its ends always touching the co-ordinate axes. The locus of a point P on the rod, which is 0.3m from the end in contact with x – axis is an ellipse. Find the eccentricity (5.5 - 7) Sep – 20

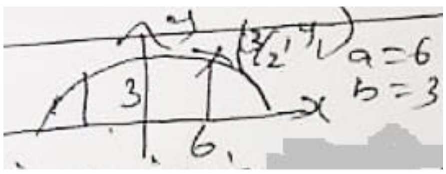


Equation of Ellipse : $\frac{x^2}{0.9^2} + \frac{y^2}{0.3^2} = 1$

$e = \sqrt{1 - \frac{b^2}{a^2}} = \frac{2\sqrt{2}}{3}$

8. A semi elliptical archway over a one-way road has a height of 3m and a width of 12m. The truck has a width of 3m and a height of 2.7m. Will the truck clear the opening of the arch way? (5.31)

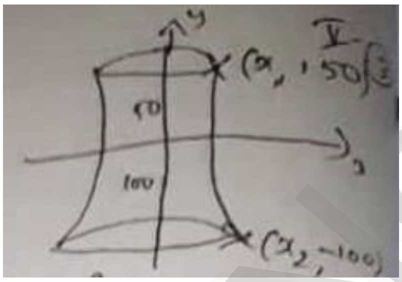
Equation of ellipse $\frac{x^2}{6^2} + \frac{y^2}{3^2} = 1$



$\left(\frac{3}{2},\ y_1\right)$ $y_1 = 2.9 > 2.7$

Truck will clear the archway.

9. **Cross section of a Nuclear cooling tower is in the shape of a hyperbola with equation $\frac{x^2}{30^2} - \frac{y^2}{44^2} = 1$ The tower is 150m tall and the distance from the top of the tower to the centre of the hyperbola is half the distance from the base of the tower to the centre of the hyperbola. Find the diameter of the top and base of the tower (5.5 – 6)**



$$\frac{x^2}{30^2} - \frac{y^2}{44^2} = 1$$

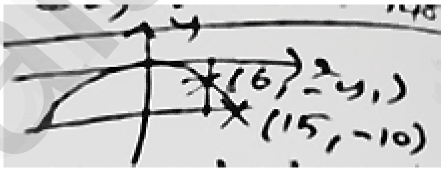
$(x_1 , 50)$ $\quad x_1 = 45.41$

Top diameter = 90.82 m

$(x_2 , -100)$ $\quad x_2 = 74.45$

Bottom diameter = 148.9 m

10. **A bridge has a parabolic arch that is 10m high at the centre and 30m wide at the bottom. Find the height of the arch 6m from the centre on either sides? (5.5 – 1) June - 24**



Equation of parabola : $x^2 = -4ay$

$(15, -10)$ $\quad 4a = \frac{225}{10}$

$$x^2 = -\frac{225}{10}y$$

$(6, -y_1)$ $\quad y_1 = \frac{8}{5} = 1.6$

Required height = 8.4 m

11. **If the equation of the ellipse is $\frac{(x-11)^2}{484} + \frac{y^2}{64} = 1$ (x and y are measured in c.m) where to the nearest cm, should the patient's kidney stone be placed so that the reflected sound hits the kidney stone) (5.39)**

$a^2 = 484$ $\quad b^2 = 64$ $\quad c^2 = a^2 - b^2 = 420$

$c = \sqrt{420} = 20.5 \ cm$

12. **Show that the line $x - y + 4 = 0$ is a tangent to the ellipse $x^2 + 3y^2 = 12$. Also find the co-ordinates of the point of contact. (5.4 -3)**

$x - y + 4 = 0 \implies m = 1, \ c = 4$

$\frac{x^2}{12} + \frac{y^2}{4} = 1 \implies a^2 = 12, \ b^2 = 4$

$a^2m^2 + b^2 = 16 = c^2$

It is a tangent point of contact $= \left(\frac{-a^2m}{c}, \frac{b^2}{c}\right) = (-3, 1)$

13. **Find the equations two tangents that can be drawn from (5, 2) to the ellipse $2x^2 + 7y^2 = 14$**
(5.4 – 1)

$a^2 = 7,\ b^2 = 2$

$(5,2)\quad y = mx \pm \sqrt{a^2m^2 + b^2}$

$9m^2 - 10m + 1 = 0 \qquad m = 1,\ {}^1/_9$

Equation of tangents: $x - y - 3 = 0$

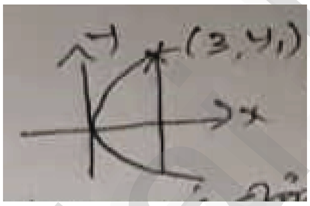
$x - 9y + 13 = 0$

14. **The parabolic communication antenna has a focus at 2m distance from the vertex of the antenna. Find the width of the antenna 3m from the vertex. (5.34) June 22**

$a = 2$, Equation of parabola $y^2 = 8x$

$(3, y_1) \quad y_1 = 2\sqrt{6}$

Width $= 4\sqrt{6}$ m

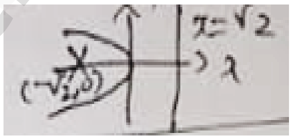


15. **Show that the equation of the parabola whose focus $(-\sqrt{2}, 0)$ and directrix $x = \sqrt{2}$ is $y^2 = -4\sqrt{2}x$ Mar. 22, Sep – 21**

Focus: $(-\sqrt{2}, 0)$; Directrix $x = \sqrt{2} \Rightarrow a = \sqrt{2}$

Equation of parabola : $y^2 = -4ax$

$y^2 = -4\sqrt{2}x$

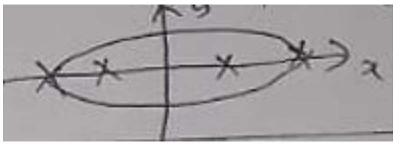


16. Find eccentricity, foci, vertices and centre of the ellipse $\frac{x^2}{25} + \frac{y^2}{9} = 1$. And draw approximate diagram (created) June – 22, Sep – 21

$\frac{x^2}{25} + \frac{y^2}{9} = 1 \qquad a^2 = 25,\ b^2 = 9$

$c^2 = a^2 - b^2 = 16$

$e = {}^c/_a = {}^4/_5$



Centre: $(0,0)$

Foci: $(\pm c, 0) = (\pm 4, 0)$

Vertices: $(\pm a, 0) = (\pm 5, 0)$

17. **Find the vertex, focus, equation of directrix and length of the latus rectum of $y^2 - 4y - 8x + 12 = 0$ (5.2 – 4(V) Mar – 24**

$y^2 - 4y - 8x + 12 = 0$

$y^2 - 4y = 8x - 12$

$(y-2)^2 = 8(x-1) \qquad 4a = 8\ ;\ a = 2$

Vertex: $(h,k) = (1,2)$

Focus: $(h + a,\ k + 0) = (3,2)$

Equation of Directrix: $x = h - a = -1$

Length of latus rectum $= 4a = 8$

18. **Find the vertex, focus, directrix and length of the latus rectum of the parabola $x^2 - 4x - 5y - 1 = 0$** **(5.19)**

$x^2 - 4x - 5y - 1 = 0$

$x^2 - 4x = 5y + 1$

$(x-2)^2 = 5(y+1)$ $\quad 4a = 5 \Rightarrow \quad a = {}^5/_4$

Vertex: $(2,-1) = (h,k)$

Focus: $(h+0,\ k+a) = \left(2, -1 + {}^5/_4\right) = \left(2,\ {}^1/_4\right)$

Equation of Directrix: $y = k - a = -1 - {}^5/_4 = -{}^9/_4$

Length of latus rectum $= 4a = 5$

19. **Identify the type of conic and find centre, foci, vertices and directrices of $18x^2 + 18x^2 + 12y^2 - 144x + 48y + 120 = 0$** **(5.2 -8(V)) Mar 23**

It is an ellipse.

$18x^2 - 144x + 12y^2 + 48y = -120$

$18(x^2 - 8x) + 12(y^2 + 4y) = -120$

$\frac{(x-4)^2}{12} + \frac{(y+2)^2}{18} = 1$; $\quad c^2 = a^2 - b^2$

$a^2 = 18,$ $\quad b^2 = 12$ $\quad c^2 = 6$

$a = 3\sqrt{2}$ $\quad c = \sqrt{6}$

$e = {}^c/_a = {}^1/_{\sqrt{3}}$

Centre: $(h,k) = (4,-2)$

Foci: $(h+0,\ k \pm c) = \left(4, -2 \pm \sqrt{6}\right)$

Vertices: $(h+0,\ k \pm a) = (4, -2 \pm 3\sqrt{2})$

Directrices: $: \ y = k \pm {}^a/_e$

$y = -2 \pm 3\sqrt{6}$

20. **Find the foci, vertices and length of major and minor axis of the conic $4x^2 + 36y^2 + 40x - 288y + 532 = 0$** **(5.22)**

$4x^2 + 36y^2 + 40x - 288y + 532 = 0$

$4(x^2 + 10x) + 36(y^2 - 8y) = -532$

$\frac{(x+5)^2}{36} + \frac{(y-4)^2}{4} = 1$; $\quad c^2 = a^2 - b^2$

$a^2 = 36,$ $\quad b^2 = 4$ $\quad c^2 = 32$

$a = 6$ $\quad b = 2$ $\quad c = 4\sqrt{2}$

Centre: $(h,k) = (-5,4)$

Foci: $(h \pm c,\ k+0) = \left(-5 \pm 4\sqrt{2}, 4\right)$

Vertices: $(h \pm a,\ 0) = (-5+6,\ 4)\ \&\ (-5-6, 4)$

$= (1,4),\ (-11,4)$

Length of major axis $= 2a = 12$

Length of minor axis $= 2b = 4$

21. **Find the centre, foci and eccentricity of the hyperbola $11x^2 - 25y^2 - 44x + 50y - 256 = 0$** **(5.26) *Sep – 20***

$11x^2 - 25y^2 - 44x + 50y - 256 = 0$

$11(x^2 - 4x) - 25(y^2 - 2y) = 256$

$\frac{(x-2)^2}{25} - \frac{(y-1)^2}{11} = 1$; $c^2 = a^2 + b^2$

$a^2 = 25,$ $b^2 = 11$ $c^2 = 36$

$a = 6$ $c = 6$

Centre: $(h, k) = (2, 1)$

Foci: $(h \pm c,\ k + 0) = (2 + 6, 1), (2 - 6, 1)$

$= (8, 1),\ (-4, 1)$

Vertices: $(h \pm a,\ k + 0) = (2 + 5,\ 4),\ (2 - 5, 1)$

$= (7, 1),\ (-3, 1)$

eccentricity = $e = {}^{c}/_{a} = {}^{6}/_{5}$

22. **Identify the type of conic and find centre, foci, vertices and directrices of $9x^2 - y^2 - 36x - 6y + 18 = 0$** **(5.2 – 8 (vi) Jun 24)**

It is a hyperbola.

$9x^2 - y^2 - 36x - 6y = -18$

$9(x^2 - 4x) - (y^2 + 6y) = -18$

$\frac{(x-2)^2}{1} - \frac{(y+3)^2}{9} = 1$

$a^2 = 1,$ $b^2 = 9$ $c^2 = a^2 + b^2 = 10$

$a = 1$ $c = \sqrt{10}$

$e = {}^{c}/_{a} = \sqrt{10}$

Centre: $(h, k) = (2, -3)$

foci: $(h \pm c,\ k + 0) = (2 \pm \sqrt{10}, -3)$

Vertices: $(h \pm a\ ,\ k = 0) = (2 + 1, -3)\ (2 - 1, -3)$

$= (3, -3),\ (1, -3)$

Direction: $x = h \pm {}^{a}/_{e}$

$x = 2 \pm {}^{1}/_{\sqrt{10}}$

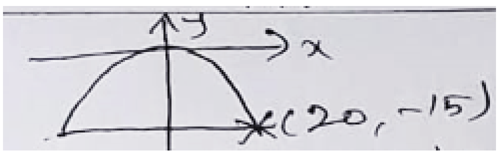
3 Marks:

23. **A concrete bridge is designed as a parabolic arch. The road over bridge is 40m long and the maximum height of the arch is 15m. Write the equation of the parabolic arch. (5.33) Mar – 20**

Equation of parabola: $x^2 = -4ay$

$(20, -15)$ $4a = \frac{400}{15} = \frac{80}{3}$

$x^2 = -\frac{80}{3}y$ $3x^2 = -80y$



24. **A circle of area 9π sq. units has two of its diameters along the lines $x + y = 5$ and $x - y = 1$. Find the equation of the circle (5.1 – 7) Sep -20**

$\pi r^2 = 9\pi \implies r = 3$

$x + y = 5$, $x - y = 1$

Centre: $(h, k) = (x, y) = (3, 2)$

Equation of Circle: $(x - h)^2 + (y - k)^2 = r^2$

$(x - 3)^2 + (y - 2)^2 = 3^2$

$x^2 + y^2 - 6x - 4y + 4 = 0$

25. **Show that the equation of parabola whose focus (4, 0) and directrix x = 4 is $y^2 = 16x$ (5.2-1(i)) March 21**

Focus $= (4, 0)$

Directrix: $x = -4$ $a = 4$

Equation of Parabola : $y^2 = 4ax$

$\boldsymbol{y^2 = 16x}$

26. **Find the equation of the parabola with vertex (-1, -2), axis parallel to y – axis and parsing through (3, 6) (5.18) March 23**

Vertex = (-1, -2) = (h, k)

Equation of Parabola $(x - h)^2 = 4a(y - k)$

$(x + 1)^2 = 4a(y + 2)$

$(3, 6)$ $4a = 2$

$(x + 1)^2 = 2(y + 2)$

$\boldsymbol{x^2 + 2x - 2y - 3 = 0}$

27. **Find the general equation of the circle whose diameter is the line segment joining the points (-4, -2) and (1, 1) (5.4) March 22**

Equation of Circle: $(x - x_1)(x - x_2) + (y - y_1)(y - y_2) = 0$

$(x + 4)(x - 1) + (y + 2)(y - 1) = 0$

$\boldsymbol{x^2 + y^2 + 3x + y - 6 = 0}$

28. **Find centre and radius of the circle $x^2 + y^2 + 6x - 4y + 4 = 0$ June 22**

Centre $= (-g, -f) = (-3, 2)$

Radius $= \sqrt{g^2 + f^2 - c}$

$= \sqrt{9 + 4 - 4} = 3\,m$

29. **If the equation $3x^2 + (3 - p)xy + qy^2 - 2px = 8pq$ represents a circle, find p and q. Also determine the centre and radius of the circle (5.1 – 12)**

$3x^2 + (3 - p)xy + qy^2 - 2px = 8pq$

co – eff of x^2 = co. eff. of $y^2 \implies q = 3$

co.eff of $xy = 0 \Rightarrow p = 3$

$3x^2 + 3y^2 - 6x - 72 = 0$

$x^2 + y^2 - 2x - 24 = 0$

centre: $(-g, -f) = (1, 0)$

radius $= \sqrt{g^2 + f^2 - c} = 5$

30. **Find the equation of ellipse whose foci $(\pm 3, 0)$ and eccentricity $= {}^{1}/_{2}$ (5.2-2(i) Jun 2024**

Foci $= (\pm 3, 0)$ $e = {}^{1}/_{2}$

Centre: $= (0, 0)$, $c = 3$

$c = ae \implies a = 6$

$c^2 = a^2 - b^2 \implies 36 - b^2 = 9$

$b^2 = 27$

Equation of ellipse : $\frac{x^2}{36} + \frac{y^2}{27} = 1$

31. **Prove that the point of intersection of the tangents at $'t_1'$ and $'t_2'$ on the parabola $y^2 = 4ax$ is $(at_1 t_2, a(t_1 + t_2))$** **(5.4 -7) June 23**

Equation of tangent : $yt = x + at^2$

$yt_1 = x + at_1^2$ .... (1)

$yt_2 = x + at_2^2$ .... (2)

(1) − (2) $y = a(t_1 + t_2)$

Sub in (1) , $x = at_1 t_2$

∴ Printing of intersection is $(at_1 t_2, a(t_1 + t_2))$

2 Marks:

32. **If $y = 4x + c$ is a tangent to the circle $x^2 + y^2 = 9$, find c (5.12) March 23**

$y = 4x + c = mx + c \Rightarrow m = 4$

$c = c$

$x^2 + y^2 = 9 = a^2$

$c^2 = a^2 (1 + m^2) = 9 (1 + 16)$

$c = \pm 3\sqrt{17}$

33. **Find the equation of the hyperbola with vertices $(0, \pm 4)$ and foci $(0, \pm 6)$** **(5.24)** ***Jun*. 23**

Vertices = $(0, \pm 4)$ $a = 4$ $a^2 = 16$

foci = $(0, \pm 6)$ $c = 6$ $c^2 = 36$

Centre (0, 0) $c^2 = a^2 + b^2$ $\Rightarrow b^2 = 20$

Equation of hyperbola : $\frac{y^2}{16} - \frac{x^2}{20} = 1$

34. **Find the general equation of a circle with centre $(-3, -4)$ and radius 3 units.** **(5.1) March 24**

Centre = $(h, k) = (-3, -4)$

radius = $r = 3$

Equation of circle : $(x - h)^2 + (y - k)^2 = r^2$

$(x + 3)^2 + (y + 4)^2 = 3^2$

$x^2 + y^2 + 6x + 8y + 16 = 0$

35. **Obtain the equation of the circle for which (3, 4) and (2, -7) are the ends of a diameter.** **(5.1-5) Jun 24**

Equation of circle: $(x - x_1)(x - x_2) + (y - y_1)(y - y_2) = 0$

$(x - 3)(x - 2) + (y - 4)(y + 7) = 0$

$\mathbf{x^2 + y^2 - 5x + 3y - 22 = 0}$

36. **Examine the position of the point (2, 3) with respect to the circle**

$x^2 + y^2 - 6x - 8y + 12 = 0$

sub (2, 3)

$= 4 + 9 - 12 - 24 + 12 = -11 < 0$

(2, 3) lies inside of the circle.

Chapter 11. Probability Distribution

2 Marks:

1. Suppose two coine are tossed once. If X denotes the number of tails. (i) Write down the sample space (ii) Find the inverse image of 1 (iii) the values of a random variable and number of elements in its inverse images

Ans:

$(i)\ S = (HH,\ HT,\ TH,\ TT\}$ $\qquad n(S) = 4$

$(ii)\ X^{-1}\{1\} = \{TH,\ HT\}$

(iii)

Values of the random variables	0	1	2	Total
Number of elements in inverse image	1	2	1	4

2. Suppose a pair of unbiased dice is rolled once. If X denotes the total score of two dice, write down (i) the sample space (ii) the values taken by the random variable X (iii) the inverse image of 10, and (iv) the number of elements in inverse image of X.

Ans:

(i) $n(S) = 36$

(ii) Then the random variable X takes on the values: 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12

(iii) The inverse images of 10 is $\{(4,6),(5,5),\ (6,4)\}$

(iv)

Values of the random variable	2	3	4	5	6	7	8	9	10	11	12	Total
Number of elements in inverse image	1	2	3	4	5	6	5	4	3	2	1	36

3. Suppose X is the number of tails occurred when three fair coins are tossed once simultaneously. Find the volume of the random variable X and number of points into inverse images.

Ans:

$S = \{HHH,\ HHT,\ HTH, THH, HTT, THT, TTH, TTT\}$

$n(S)\ =\ 8$

Values of the random variable	0	1	2	3	Total
Number of elements in inverse image	1	3	3	1	8

4. The Probability density function of X is given by $f(x) = \begin{cases} K\,xe^{-2x} & for\ x > 0 \\ 0 & for\ x\ \leq 0 \end{cases}$. Find the value of K?

Ans:

$$\int_{\infty}^{\infty} f(x)dx = 1$$

$$K\int_{0}^{\infty} xe^{-2x}dx = 1$$

$$K\ =\ 4$$

3 Marks

5. **An urn contains 5 mangoes and 4 apples. here fruits are taken at random. If the number of apples taken is a random variable, then find the values of the random variable and number of points in its inverse images.**

Ans:

Values of the random variable	0	1	2	3	Total
Number of elements in inverse image	$5C_3 = 10$	$5C_2 \times 4C_1 = 40$	$5C_1 \times 4C_2 = 30$	$4C_3 = 4$	84

6. **Two fair coins are tossed simultaneously (equivalent to a fair coin is tossed twice). Find the probability mass function for number of heads occurred.**

Ans:

$S = \{HH,\ HT,\ TH,\ TT\}$ $\quad\quad n(S) = 4$

Values of the random variable X	0	1	2	Total
Number of elements in inverse image	1	2	1	4

Probability mass function:

x	0	1	2
$f(x)$	$\frac{1}{4}$	$\frac{1}{2}$	$\frac{1}{4}$

7. **A pair of fair dice is tossed once. Find the probability of mass function to get the number of forms.**

Ans:

Values of Random variable X	0	1	2	Total
Number of elements in inverse images	25	10	1	36

The probability mass function is presented as

x	0	1	2
$f(x)$	$\frac{25}{36}$	$\frac{10}{36}$	$\frac{1}{36}$

8. **If X is the random variable with distribution function F(x) given by**

$$F(x) = \begin{cases} 0 & , \quad x < 0 \\ x & , \quad 0 \le x < 1 \\ 1 & , \quad 1 \le x \end{cases}$$

then find (i) the probability density function $f(x)$ (ii) $P(0.2 \le X \le 0.7)$

Ans:

(i) $$f(x) = F'(x) = \begin{cases} 0 & , \quad x < 0 \\ 1 & , \quad 0 \le x < 1 \\ 0 & , \quad 1 \le x \end{cases}$$

$$f(x) = \begin{cases} 1 & , \quad 0 \le x < 1 \\ 0 & , \quad Otherwise \end{cases}$$

(ii) $P(0.2 \le X \le 0.7) = F(0.7) - F(0.2)$

$= 0.7 - 0.2$

$= 0.5$

9. **If X is the random variable with distribution function $F(x)$ given by,**

$$F(x) = \begin{cases} 0 & , \quad -\infty < x < 0 \\ \frac{1}{2}(x^2 + x) & , \quad 0 \le x < 1 \\ 0 & , \quad 1 < x < \infty \end{cases}$$

then find (i) the probability density function $f(x)$

(ii) $P(0.3 \le X \le 0.6)$

Ans:

(i) $f(x) = F'(x) = \begin{cases} 0 & , \quad x < 0 \\ \frac{1}{2}(2x+1) & , \quad 0 \le x < 1 \\ 0 & , \quad 1 < x \end{cases}$

(ii) $P(0.3 \le X \le 0.6) = F(0.6) - F(0.3) = 0.285$

10. **Find the mean and variance of a random variable X, whose probability density function is**

$$f(x) = \begin{cases} \lambda e^{-\lambda x} & , \quad x \ge 0 \\ 0 & , \quad Otherwise \end{cases}$$

Ans:

Mean: $E(X) = \int_{-\infty}^{\infty} x f(x) dx$

$= \frac{1}{\lambda}$

Variance: $E(X^2) = \int_{-\infty}^{\infty} x^2 f(x) dx$

$= \frac{2}{\lambda^2}$

$Var(X) = E(X^2) - [E(X)]^2$

$= \frac{2}{\lambda^2} - \left(\frac{1}{\lambda}\right)^2$

$= \frac{1}{\lambda^2}$

Mean: $\frac{1}{\lambda}$, Variance = $\frac{1}{\lambda^2}$

5 Marks

11. **A six sided die is marked '1' on one face, '2' on two of its faces, and '3' on remaining three faces. The die is rolled twice. If X denotes the total score in two throws.**

(i) Find the probability mass function

(ii) Find the cumulative distribution function

(iii) Find $P(3 \le X < 6)$

(iv) Find $P(X \ge 4)$

Ans:

I \ II	1	2	2	3	3	3
1	2	3	3	4	4	4
2	3	4	4	5	5	5
2	3	4	4	5	5	5
3	4	5	5	6	6	6
3	4	5	5	6	6	6
3	4	5	5	6	6	6

(i) Probability mass function is:

x	2	3	4	5	6	Total
$f(x)$	$\frac{1}{36}$	$\frac{4}{36}$	$\frac{10}{36}$	$\frac{12}{36}$	$\frac{9}{36}$	1

(ii) Cumulative distribution function

x	2	3	4	5	6
$F(x)$	$\frac{1}{36}$	$\frac{5}{36}$	$\frac{15}{36}$	$\frac{27}{36}$	$\frac{36}{36}$

(iii) $P(3 \le X < 6) = \frac{4}{36} + \frac{10}{36} + \frac{12}{36} = \frac{26}{36}$

(iv) $P(X \ge 4) = \frac{10}{36} + \frac{12}{36} + \frac{9}{36} = \frac{31}{36}$

12. A random variable X has the following probability mass function

x	1	2	3	4	5	6
$f(x)$	**K**	**2K**	**6K**	**5K**	**6K**	**10K**

Find (i) $P(2 < X < 6)$ (ii) $P(2 \le X < 5)$

(iii) $P(X \le 4)$ (iv) $P(3 < X)$

Ans: $\Sigma f(x) = 1$

$K + 2K + 6K + 5K + 6K + 10K = 1$

$30K = 1 \Rightarrow K = \frac{1}{30}$

(i) $P(2 < X < 6) = 6K + 5K + 6K = 17K = \frac{17}{30}$

(ii) $P(2 \le X < 5) = 2K + 6K + 5K = 13K = \frac{13}{30}$

(iii) $P(X \le 4) = K + 2K + 6K + 5K = 14K = \frac{14}{30}$

(iv) $P(3 < X) = 5K + 6K + 10K = 21K = \frac{21}{30}$

13. A six sided die is marked '1' on one face, '3' on two of its faces, and '5' on remaining three faces. The die is thrown twice. If X denotes the total score in two throws, find

(i) the probability mass function **(ii) the cumulative distribution function**

(iii) $P(4 \le X < 10)$ (iv) $P(X \ge 6)$

Ans:

I \ II	1	3	3	5	5	5
1	2	4	4	6	6	6
3	4	6	6	8	8	8
3	4	6	6	8	8	8
5	6	8	8	10	10	10
5	6	8	8	10	10	10
5	6	8	8	10	10	10

(i) The probability mass function is:

x	2	4	6	8	10	Total
$f(x)$	$\frac{1}{36}$	$\frac{4}{36}$	$\frac{10}{36}$	$\frac{12}{36}$	$\frac{9}{36}$	1

(ii) Cumulative distribution function:

x	2	4	6	8	10
$F(x)$	$\frac{1}{36}$	$\frac{5}{36}$	$\frac{15}{36}$	$\frac{27}{36}$	$\frac{36}{36}$

(iii) $P(4 \leq X < 10) = \frac{4}{36} + \frac{10}{36} + \frac{12}{36} = \frac{26}{36} = \frac{13}{18}$

(iv) $P(X \geq 6) = \frac{10}{36} + \frac{12}{36} + \frac{9}{36} = \frac{31}{36}$

14. The cumulative distribution function of a discrete random variable is given by

$$F(x) = \begin{cases} 0 & , -\infty < x < -1 \\ 0.15 & , -1 < x < 0 \\ 0.35 & , 0 \leq x < 1 \\ 0.60 & , 1 \leq x < 2 \\ 0.85 & , 2 \leq x < 3 \\ 1 & , 3 \leq x < \infty \end{cases}$$

Find (i) the probability mass function (ii) $P(X < 1)$ and $P(X \geq 2)$

Ans: (i) Probability mass function is

x	-1	0	1	2	3
$f(x)$	0.15	0.20	0.25	0.25	0.15

(ii) $P(X < 1) = 0.14 + 0.20 = 0.35$

(ii) $P(X \geq 2) = 0.25 + 0.15 = 0.40$

15. A random variable X has the following probability mass function

x	1	2	3	4	5
$f(x)$	K^2	$2K^2$	$3K^2$	$2K$	$3K$

Find (i) the value of K (ii) $P(2 \leq X < 5)$ (iii) $P(3 < x)$

Ans: (i) $\Sigma f(x) = 1$

$6K^2 + 5K = 1 \implies 6K^2 + 5K - 1 = 0$

$K = -1$ $(Not\ Possibe)$ $K = \frac{1}{6}$

(ii) $P(2 \leq X < 5) = 2K^2 + 3K^2 + 2K = 5K^2 + 2K$

$= \frac{5}{36} + \frac{2}{6}$

$= \frac{5}{36} + \frac{12}{36} = \frac{17}{36}$

(iii) $P(3 < x) = 2K + 3K = 5K = \frac{5}{6}$

16. The cumulative distribution function of a discrete random variable is given by

$$F(x) = \begin{cases} 0 & , -\infty < x < 0 \\ \frac{1}{2} & , 0 \leq x < 1 \\ \frac{3}{5} & , 1 \leq x < 2 \\ \frac{4}{5} & , 2 \leq x < 3 \\ \frac{9}{10} & , 3 \leq x < 4 \\ 1 & , 4 \leq x < \infty \end{cases}$$

Find (i) the probability mass function (ii) $P(X < 3)$ **and (iii)** $P(X \geq 2)$

Ans:

(*i*)

x	0	1	2	3	4
$f(x)$	$\frac{1}{2}$	$\frac{1}{10}$	$\frac{1}{5}$	$\frac{1}{10}$	$\frac{1}{10}$

(*ii*) $P(X < 3) = \frac{1}{2} + \frac{1}{10} + \frac{1}{5} = \frac{8}{10} = \frac{4}{5}$

(*iii*) $P(X \geq 2) = \frac{1}{5} + \frac{1}{10} + \frac{1}{10} = \frac{4}{10} = \frac{2}{5}$

17. Suppose the amount of milk sold daily at a milk 600 is distribution with a minimum of 200 litres and a maximum of 600 litres with probability density function

$$f(x) = \begin{cases} K & , \quad 200 \leq x < 600 \\ 0 & , \quad Otherwise \end{cases}$$

Find (i) the value of K (ii) the distribution function

(iii) the probability that daily sales will fall between 300 litres on 500 litres.

Ans:

$$(i) \quad \int_{-\infty}^{\infty} f(x)dx = 1$$

$$K \int_{200}^{600} dx = 1$$

$$K = \frac{1}{400}$$

$$(ii) \quad F(x) = \begin{cases} 0 & , \quad x < 200 \\ \frac{x}{400} - \frac{1}{2} & , \quad 200 \leq x \leq 600 \\ 1 & , \quad x > 600 \end{cases}$$

$$(iii) \quad \int_{300}^{500} f(x)dx = \frac{200}{400} = \frac{1}{2}$$

18. For the random variable X with the given probability mass function as below, find the mean and variance

$$f(x) = \begin{cases} 2(x-1) & , \quad 1 < x < 2 \\ 0 & , \quad Otherwise \end{cases}$$

Ans:

Mean: $E(x) = \int_{-\infty}^{\infty} xf(x)dx$

$$= 2\int_{1}^{2} (x^2 - x)dx$$

$$= 2\left(\frac{5}{6}\right) = \frac{5}{3}$$

$$E(x^2) = \int_{-\infty}^{\infty} x^2 f(x)dx$$

$$= 2\int_{1}^{2} x^2(x-1)dx$$

$$= \frac{17}{6}$$

Variance: $Var(x) = E(x^2) - [E(x)]^2$

$$= \frac{17}{6} - \frac{25}{9}$$

$$= \frac{1}{18}$$

19. The mean and variance of a binomial variate x are respectively 2 and 1.5.
Find $(i)\ P(x=0)$ $(ii)\ P(x=1)$ $(iii)\ P(x \geq 1)$

Ans: Mean = $np = 2$ Variance = $npq = 1.5$

$\frac{npq}{np} = \frac{1.5}{2} = \frac{3}{4} = q \quad p = \frac{1}{4} \quad n = 8$

$P(X = x) = nc_x\, p^x q^{n-x}$

$= 8c_x \left(\frac{1}{4}\right)^x \left(\frac{3}{4}\right)^{8-x} \qquad x = 0, 1, 2, \ldots \ldots \ldots 8$

$(i)\ P(x=0) = 8c_0 \left(\frac{1}{4}\right)^0 \left(\frac{3}{4}\right)^{8-0} = \left(\frac{3}{4}\right)^8$

$(ii)\ P(x=1) = 2\left(\frac{3}{4}\right)^7$

$(iii)\ P(x \geq 1) = 1 - P(x < 1) = 1 - \left(\frac{3}{4}\right)^8$

20. If $X \sim B(n,p)$ such that $4P(X=4) = P(X=2)$ and n = 6. Find the distribution, mean standard deviation of X

Ans:

$n = 6 \quad p = \frac{1}{3} \quad q = \frac{2}{3}$

$p(x = x) = f(x) = nc_x\ p^x\ q^{n-x} \qquad x = 0, 1, 2, \ldots \ldots \ldots n$

$f(x) = 6c_x \left(\frac{1}{3}\right)^x \left(\frac{2}{3}\right)^{6-x} \qquad x = 0, 1, 2, \ldots \ldots \ldots 6$

Mean $= np = 6 \times \frac{1}{3} = 2$

Standard deviation $= \sqrt{npq} = \sqrt{6 \times \frac{1}{3} \times \frac{2}{3}}$

$= \frac{2}{\sqrt{3}}$

21. If the probability that a fluorescent light has a useful life of at least 600 hours in 0.9, find the probabilities that among 12 such lights

i. Exactly 10 will have a useful life of at least 600 hours;
ii. at least 11 will have useful life of at least 600 hours;
iii. at least 2 will not have a useful life of at least 600 hours.

Ans:

$p = 0.9 = \frac{9}{10} \qquad q = 1 - p = 1 - \frac{9}{10} = \frac{1}{10} \qquad n = 12$

$P(X = x) = nC_x p^x q^{n-x}, \quad x = 0, 1, 2, \ldots \ldots \ldots, n$

$(i)\ P(X = 10) = 12C_{10} \left(\frac{9}{10}\right)^{10} \left(\frac{1}{10}\right)^{12-10}$

$= 12C_{10}(0.9)^{10}(0.1)^{12-10}$

$(ii)\ P(X \geq 11) = P(X = 11) + P(X = 12)$

$= 12C_{11}\left(\frac{9}{10}\right)^{11}\left(\frac{1}{10}\right)^{12-11} - 12C_{12}\left(\frac{9}{10}\right)^{12}\left(\frac{1}{10}\right)^{12-12}$

$= 12\left(\frac{9}{10}\right)^{11}\left(\frac{1}{10}\right)^{1} + \left(\frac{9}{10}\right)^{12}$

$= \left(\frac{9}{10}\right)^{11}\left[12 \times \frac{1}{10} + \frac{9}{10}\right]$

$= (0.9)^{11}[1.2 + 0.9]$

$= 2.1(0.9)^{11}$

$(iii)\ P(X \geq 2) = 1 - P(X \geq 11)$

$= 1 - 2.1(0.9)^{11}$

22. **A multiple choice examination has ten questions, each question has four distance with exactly one correct answer. Suppose a student answers by guessing and if X denotes the number of correct answers, find (i) binomial distribution (ii) probability that the student will get seven correct answers (iii) the probability of getting at least one correct answer.**

Ans:

$n = 10; \quad p = \frac{1}{4} \Rightarrow q = 1 - p = 1 - \frac{1}{4} = \frac{3}{4}$

$(i) \;\; P(X = x) = nC_x p^x q^{n-x}, \quad x = 0, 1, 2, \dots\dots\dots, n$

$P(X = x) = 10C_x p^x q^{10-x}, \quad n = 0, 1, 2, \dots\dots\dots, 10$

$(ii) \;\; P(X = 7) = 10C_7 \left(\frac{1}{4}\right)^7 \left(\frac{3}{4}\right)^{10-7}$

$= 10C_3 \left(\frac{1}{4}\right)^7 \left(\frac{3}{4}\right)^3$

$= \frac{10.9.8}{1.2.3} . \left(\frac{1}{4}\right)^7 \left(\frac{3}{4}\right)^3$

$= 120 \left(\frac{3^3}{4^{10}}\right)$

$(iii) \;\; P(X \geq 1) = 1 - P(X < 1)$

$= 1 - P(X = 0)$

$= 1 - 10C_0 \left(\frac{1}{4}\right)^0 \left(\frac{3}{4}\right)^{10-0}$

$= 1 - \left(\frac{3}{4}\right)^{10}$

23. **On the average, 20% of the products manufactured by ABC Company are found to be defective. If we select 6 of these products at random and X denotes the number of defective products find the probability that (i) two products are defective (ii) at most one product is defective (iii) at least two products are defective.**

Ans:

$p = 20\% = \frac{20}{100} = \frac{1}{5} \qquad q = 1 - p = 1 - \frac{1}{5} = \frac{4}{5} \qquad n = 6$

$P(X = x) = nC_x p^x q^{n-x}, \quad x = 0, 1, 2, \dots\dots\dots, n$

$(i) \;\; P(X = 2) = 6C_2 \left(\frac{1}{5}\right)^2 \left(\frac{4}{5}\right)^{6-2}$

$= \frac{6.5}{1.2} . \left(\frac{1}{5}\right)^2 \left(\frac{4}{5}\right)^4$

$= 15 \left(\frac{4^4}{5^6}\right)$

$(ii) \; (X \leq 1) = P(X = 0) + P(X = 1)$

$= 6C_0 \left(\frac{1}{5}\right)^0 \left(\frac{4}{5}\right)^{6-0} + 6C_1 \left(\frac{1}{5}\right)^1 \left(\frac{4}{5}\right)^{6-1}$

$= \left(\frac{4}{5}\right)^6 + 6 \left(\frac{1}{5}\right)^1 \left(\frac{4}{5}\right)^5$

$= \frac{4^5}{5^6} (4 + 6)$

$= \frac{4^5}{5^6} \times 10$

$= 2 \left(\frac{4}{5}\right)^5$

$(iii) \; P\,(X \geq 2) = 1 - P(X \leq 1)$

$= 1 - 2 \left(\frac{4}{5}\right)^5$

Chapter – 4
Inverse trigonometric Function

2 & 3 Marks:

1. Find the value of $sin^{-1}(\sin {2\pi}/{3})$

Solution: $= sin^{-1}(\sin {2\pi}/{3})$

$= sin^{-1}(\sin(\pi - {\pi}/{3}))$

$= sin^{-1}(\sin {\pi}/{3})$

$= {\pi}/{3}\ \varepsilon\ [-{\pi}/{2},\ {\pi}/{2}]$

2. Find the value of $cos^{-1}\left(\cos\left(\frac{7\pi}{6}\right)\right)$

Solution: $= cos^{-1}(\cos(\pi + {\pi}/{6})) = cos^{-1}\left(-cos\frac{\pi}{6}\right) = cos^{-1}\left(cos\left(\pi - \frac{\pi}{6}\right)\right)$

$= cos^{-1}(\cos {5\pi}/{6})$

$= {5\pi}/{6}\ \ \varepsilon\ [0,\ \pi]$

3. $sin^{-1}(2 - 3x^2)$ in Domain

Solution: $-1\ \leq 2 - 3x^2\ \leq 1$

(-2) $\quad -3\ \leq\ -3x^2\ \leq\ -1$

$(\div 3)$ $\quad -1\ \leq\ -x^2\ \leq\ -{1}/{3}$

(-1) $\quad 1\ \geq\ x^2\ \geq\ {1}/{3}$

$1\ \geq |x| \geq\ {1}/{\sqrt{3}}$

$\frac{1}{\sqrt{3}}\ \leq\ |x|\ \leq 1$

$x\ \in\ \left[-1, -\frac{1}{\sqrt{3}}\right] \cup \left[\frac{1}{\sqrt{3}}\ ,\ 1\right]$

4. $cos^{-1}\left(cos({-\pi}/{6})\right) \neq {-\pi}/{6}$ True? Justify your answer:

Solution: NOT TRUE $\quad cos^{-1}(\cos x) = x$ only if $x \in\ [0,\ \pi]$

$cos^{-1}\left(cos({-\pi}/{6})\right) = cos^{-1}\left(cos({\pi}/{6})\right) = {\pi}/{6}$

ஃ $cos^{-1}\left(cos({-\pi}/{6})\right) \neq {-\pi}/{6}$

5. Find the value

$$sin^{-1}\left(\sin\frac{5\pi}{9}\cdot cos\frac{\pi}{9} + cos\frac{5\pi}{9}\cdot \sin\frac{\pi}{9}\right)$$

Solution: $\sin(A + B) = sinA\ cosB + cosA \sin B$

$= \quad sin^{-1}\left(\sin\left(\frac{5\pi}{9} + \frac{\pi}{9}\right)\right)$

$= \quad sin^{-1}\left(\sin\left(\frac{6\pi}{9}\right)\right)\ =\ sin^{-1}\left(\sin\left(\frac{2\pi}{3}\right)\right)$

$= \quad sin^{-1}\left(\sin\left(\pi - \frac{\pi}{3}\right)\right) \quad \Rightarrow sin(\pi - \theta) = \sin\theta$

$= \quad sin^{-1}\ \left(\sin\frac{\pi}{3}\right)$

$= \quad {\pi}/{3}\ \varepsilon[{-\pi}/{2}, {\pi}/{2}]$

6. **Find the value of $tan^{-1}\left(\tan\frac{5\pi}{4}\right)$**

Solution:

$$tan^{-1}\left(\tan\frac{5\pi}{4}\right) = tan^{-1}\left(tan\left(\pi + \frac{\pi}{4}\right)\right)$$
$$= tan^{-1}\left(\tan\frac{\pi}{4}\right)$$
$$= \frac{\pi}{4}\ \varepsilon\left[{}^{-\pi}/_{2}, {}^{\pi}/_{2}\right]$$

7. **$cot^{-1}({}^{1}/_{7}) = \theta$ Find the value of $\cos\theta$**

Solution:

$$\theta = cot^{-1}({}^{1}/_{7})$$
$$\cot\theta = \frac{1}{7} \Longrightarrow tan\theta = 7$$
$$sec\theta = \sqrt{1 + tan^2\theta} = \sqrt{1+49} = \sqrt{50}$$
$$sec\theta = 5\sqrt{2}$$
$$cos\theta = \frac{1}{5\sqrt{2}}$$

8. **Find the Principal value $cosec^{-1}(-\sqrt{2})$**

Solution: $cosec^{-1}(-\sqrt{2}) = sin^{-1}\left(-{}^{1}/_{\sqrt{2}}\right)$

$$= -sin^{-1}\left({}^{1}/_{\sqrt{2}}\right) \qquad sin^{-1}(-x) = -\,sin^{-1}x$$
$$= -{}^{\pi}/_{4} \qquad x\ \varepsilon\ [-1, 1]$$

5 Mark:

9. **Find Domain $f(x) = sin^{-1}\left(\frac{|x|-2}{3}\right) + cos^{-1}\left(\frac{1-|x|}{4}\right)$**

Solution:

$$-1 \le \frac{|x|-2}{3} \le 1 \qquad -1 \le \frac{1-|x|}{4} \le 1$$
$$-3 \le |x| - 2 \le 3 \qquad -4 \le 1 - |x| \le 4$$
$$-1 \le |x| \le 5 \qquad -5 \le -|x| \le 3$$
$$|x| \le 5 - (1) \qquad 5 \ge |x| \ge -3$$
$$-3 \le |x| \le 5\text{-}(2)$$

From (1) & (2) $-5 \le x \le 5$

10. **Find the value of**

$$cos^{-1}\left(\cos\frac{4\pi}{3}\right) + cos^{-1}\left(\cos\frac{5\pi}{4}\right)$$

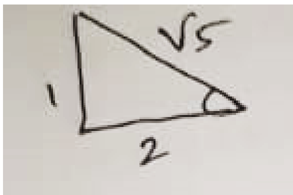
Ans:

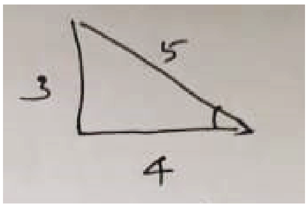
$$= cos^{-1}\left(cos\left(\pi + \frac{\pi}{3}\right)\right) + cos^{-1}\left(cos\left(\pi + \frac{\pi}{4}\right)\right)$$
$$[\cos(\pi+\theta) = \cos(\pi-\theta) = -cos\theta]$$
$$= cos^{-1}\left(cos\left(\pi - \frac{\pi}{3}\right)\right) + cos^{-1}\left(cos\left(\pi - \frac{\pi}{4}\right)\right)$$
$$= cos^{-1}\left(cos({}^{2\pi}/_{3})\right) + cos^{-1}\left(cos({}^{3\pi}/_{4})\right)$$
$$= \frac{2\pi}{3} + \frac{3\pi}{4} = \frac{17\pi}{12}$$

11. Find the value $\sin\left(\tan^{-1}\left({}^{1}/_{2}\right)\right) - \cos^{-1}\left({}^{4}/_{5}\right)$

Ans:

$\tan^{-1} {}^{1}/_{2} = A$ $\cos^{-1} {}^{4}/_{5} = B$





$\sin A = {}^{1}/_{\sqrt{5}}$ $\sin B = {}^{3}/_{5}$

$\cos A = {}^{2}/_{\sqrt{5}}$ $\cos B = {}^{4}/_{5}$

$$\sin\left(\tan^{-1}\left({}^{1}/_{2}\right) - \cos^{-1}\left({}^{4}/_{5}\right)\right) = \sin(A - B)$$

$$= \sin A \cos B - \cos A \sin B$$

$$= \frac{1}{\sqrt{5}} \cdot \frac{4}{5} - \frac{2}{\sqrt{5}} \cdot \frac{3}{5}$$

$$= \frac{-2}{5\sqrt{5}} \times \frac{\sqrt{5}}{\sqrt{5}} = \frac{-2\sqrt{5}}{25}$$

12. Find the value

$$\cot^{-1}(1) + \sin^{-1}\left(-{}^{\sqrt{3}}/_{2}\right) - \sec^{-1}(-\sqrt{2})$$

Ans:

$$= \tan^{-1}(1) - \sin^{-1}\left({}^{\sqrt{3}}/_{2}\right) - \cos^{-1}\left({}^{-1}/_{\sqrt{2}}\right)$$

$$= \frac{\pi}{4} - \frac{\pi}{3} - \left(\pi - {}^{\pi}/_{4}\right)$$

$$= \frac{\pi}{4} - \frac{\pi}{3} - \pi + \frac{\pi}{4}$$

$$= \frac{\pi}{4} - \frac{\pi}{3} - \frac{3\pi}{4}$$

$$= \frac{-5\pi}{6}$$

13. $\cos^{-1}x + \cos^{-1}y + \cos^{-1}z = \pi$ **and** $0 < x, y, z < 1$ **s.t.** $x^2 + y^2 + z^2 + 2xyz = 1$

Solution:

$$\cos^{-1}x + \cos^{-1}y + \cos^{-1}z = \pi$$

$$\cos^{-1}x + \cos^{-1}y = \pi - \cos^{-1}z$$

$$\cos^{-1}\left(xy - \sqrt{1-x^2} . \sqrt{1-y^2}\right) = \cos^{-1}(-z)$$

$$xy - \sqrt{1-x^2} . \sqrt{1-y^2} = -z$$

$$xy + z = \sqrt{1-x^2} . \sqrt{1-y^2}$$

$$(xy+z)^2 = (1-x^2)(1-y^2)$$

$$x^2y^2 + z^2 + 2xyz = 1 - y^2 - x^2 + x^2y^2$$

$$x^2 + y^2 + z^2 + 2xyz = 1$$

14. **If $a_1 a_2 a_3 ....a_n$ is an AP with common difference is d Prove that**

$$\tan\left[\tan^{-1}\left(\frac{d}{1+a_1a_2}\right)+\tan^{-1}\left(\frac{d}{1+a_2a_3}\right)+ ....+\tan^{-1}\left(\frac{d}{1+a_na_{n-1}}\right)\right] = \frac{a_n-a_1}{1+a_na_1}$$

Solution:

$$d = a_2 - a_1 = a_3 - a_2 = a_n - a_{n-1}$$

LHS = $\tan\left[\tan^{-1}\left(\frac{d}{1+a_1a_2}\right)+\tan^{-1}\left(\frac{d}{1+a_2a_3}\right)+ ......+\tan^{-1}\left(\frac{d}{1+a_na_{n-1}}\right)\right]$

$$= \tan(\tan^{-1}a_2 - \tan^{-1}a_1 + \tan^{-1}a_3 - \tan^{-1}a_2 + \cdots..+ \tan^{-1}a_n - \tan^{-1}a_n - 1)$$

$$= \tan(\tan^{-1}a_n - \tan^{-1}a_1)$$

$$= \tan\left(\tan^{-1}\left(\frac{a_n-a_1}{1+a_na_1}\right)\right)$$

$$= \frac{a_n-a_1}{1+a_na_1} = \text{RHS}$$

15. Solve: $\tan^{-1}\left(\frac{x-1}{x-2}\right)+\tan^{-1}\left(\frac{x+1}{x+2}\right) = \pi/4$

Solution: $\tan^{-1}\left(\frac{x-1}{x-2}\right)+\tan^{-1}\left(\frac{x+1}{x+2}\right) = \tan^{-1}\left(\frac{\frac{x-1}{x-2}+\frac{x+1}{x+2}}{1-\left(\frac{x-1}{x-2}\right)\left(\frac{x+1}{x+2}\right)}\right) = \pi/4$

$$= \frac{(x-1)(x+2)+(x+1)(x-2)}{(x-2)(x+2)-(x-1)(x+1)} = \tan \pi/4$$

$$\frac{2x^2-4}{x^2-4+x^2+1} = 1 \Longrightarrow \frac{2x^2-4}{-3} = 1$$

$$2x^2 - 4 = -3 \Longrightarrow x^2 = 1/2$$

$$x = \pm 1/\sqrt{2}$$

16. The number at solution at the earn $\tan^{-1}(x-1) + \tan^{-1}x + \tan^{-1}(x+1) = \tan^{-1}(3x)$

Solution: $\tan^{-1}(x-1) + \tan^{-1}(x+1) = \tan^{-1}3x - \tan^{-1}x$

$$\tan^{-1}\left(\frac{x-1+x+1}{1-(x-1)(x+1)}\right) = \tan^{-1}\left(\frac{3x-x}{1+3x^2}\right)$$

$$\frac{2x}{1-(x^2-1)} = \frac{2x}{1+3x^2}$$

$$x(1+3x^2) = x(1-x^2+1)$$

$$x + 3x^3 = 2x - x^3$$

$4x^3 - x = 0$ which is cubic earn

no. of solution = 3

Chapter – 3: THEORY OF EQUATIONS

Concepts:

1. General form of the Quadratic equation is $ax^2 + bx + c = 0$, where $a \neq 0$

Solution: $x = \frac{-b \pm \sqrt{b^2-4ac}}{2a}$, Let $\Delta = b^2 - 4ac$ is called discriminant

(i) $\Delta = 0$ if, and only if, the roots are equal.

(ii) $\Delta > 0$ if, and only if, the roots are real and distinct

(iii) $\Delta < 0$ if, and only if, the Quadratic equations has no real roots. (ie., the roots are imaginary)

2. Vieta's formula for Quadratic Equations:

Let α and β be the roots of the Quadratic equation $ax^2 + bx + c = 0$

The sum of the roots $\alpha + \beta = \frac{-b}{a}$

The product of the roots = $\alpha\ \beta = \frac{c}{a}$

Converse,

The Quadratic equation whose roots are α and β is $x^2 - (\alpha + \beta)x + \alpha\beta = 0$

3. Vieta's formula for polynomial equation of degree 3

Let α, β and γ be the roots of the cubic equation $ax^3 + bx^2 + cx + d = 0$, then

$i. \Sigma\alpha = \alpha + \beta + \gamma = \frac{-b}{a}$ $ii. \Sigma\alpha\beta = \alpha\beta + \beta\gamma + \gamma\alpha = \frac{c}{a}$

$iii. \Sigma\alpha\beta\gamma = \alpha\beta\gamma = \frac{-d}{a}$

Converse,

The cubic equation whose roots are α, β, γ is $x^3 - (\Sigma\alpha)x^2 + (\Sigma\alpha\beta)x - \alpha\beta\gamma = 0$

4. **DESCARTE'S RULE:**

Let $P(x)$ be the polynomial of degree 'n'.

i. Let 'm' denote the number of sign changes in coefficients of P(x)

ii. Let 'k' denote the number of sign changes in coefficients of P(-x)

iii. Then there are at least n-(m +k) imaginary roots for the polynomial P(x).

2 Mark Sums:

1. If α and β are the roots of the Quadratic equation $17x^2 + 43x - 73 = 0$, construct a Quadratic equation whose roots are $\alpha + 2$ and $\beta + 2$

Solution:

(i) $17x^2 + 43x - 73 = 0$

$a = 17,\ b = 43\ \ c = -73$

$\alpha + \beta = \frac{-b}{a} = \frac{-43}{17}$ $\alpha\beta = \frac{c}{a} = \frac{-73}{17}$

(ii) The given roots are $\alpha + 2$ and $\beta + 2$

The sum of the roots $= \alpha + \beta + 4$

$= \frac{-43}{17} + 4 = \frac{25}{17}$

(iii) The product of the roots $= \alpha\beta + 2(\alpha + \beta) + 4$

$= \frac{-73}{17} + 2\left(\frac{-43}{17}\right) + 4 = \frac{-91}{17}$

(iv) Hence a Quadratic equation is

$x^2 - (\alpha + \beta)x + \alpha\beta = 0$

$\Rightarrow x^2 - \frac{25}{17}x - \frac{91}{17} = 0$

(x) by 17, $17x^2 - 25x - 91 = 0$

2. **If α and β are the roots of the Quadratic equation $2x^2 - 17x + 13 = 0$, construct a Quadratic equation whose roots are α^2 and β^2**

Solution:

(i) $2x^2 - 17x + 13 = 0$

$a = 2, \ b = -17 \ \ c = 13$

$\alpha + \beta = \frac{-b}{a} = \frac{7}{2}$ $\qquad \alpha\beta = \frac{c}{a} = \frac{13}{2}$

(ii) The given roots are $\alpha^2 and \ \beta^2$, then

The sum of the roots $= \alpha^2 + \beta^2$

$= (\alpha + \beta)^2 - 2\alpha\beta$

$= \left(\frac{7}{2}\right)^2 - 2\left(\frac{13}{2}\right) = \frac{49}{4} - \frac{13}{1} = \frac{-3}{4}$

(iii) The product of the roots

$\alpha^2 \ \beta^2 = (\alpha\beta)^2 = \left(\frac{13}{2}\right)^2 = \frac{169}{4}$

(iv) Hence a Quadratic equation is

$x^2 - (\alpha^2 + \beta^2)x + \alpha^2\beta^2 = 0$

$x^2 - \left(\frac{-3}{4}\right)x + \frac{169}{4} = 0$

$x^2 + \frac{3}{4}x + \frac{169}{4} = 0$

$(x)4,$ $\qquad 4x^2 + 3x + 169 = 0$

3. **If $\alpha, \ \beta \ and \ \gamma$ are the roots of the equation $x^3 + px^2 + qx + r = 0$, find the value of $\Sigma\frac{1}{\beta\gamma}$ in terms of the coefficients.**

Solution: (i) $x^3 + px^2 + qx + r = 0$ $\qquad a = 1, \ b = p, \ c = q, \ d = r$

(ii) The sum of the roots $= \alpha + \beta + \gamma = \frac{-b}{a} = -p$

(iii) The product of the roots $= \alpha\beta + \beta\gamma + \gamma\alpha = \frac{c}{a} = q$

$\alpha \, \beta \, \gamma = \frac{-d}{a} = -r$

(iv) $\Sigma\frac{1}{\beta\gamma} = \frac{1}{\alpha\beta} + \frac{1}{\beta\gamma} + \frac{1}{\gamma\alpha} = \frac{\gamma + \alpha + \beta}{\alpha\,\beta\,\gamma} = \frac{-p}{-r} = \frac{p}{r}$

4. **Show that the equation $2x^2 - 6x + 7 = 0$ cannot be satisfied by any real values of x.**

Solution: (i) $(i) \ 2x^2 - 6x + 7 = 0$

$a = 2, \ b = -6 \ \ c = 7$

(ii) $\Delta = b^2 - 4ac = (6)^2 - 4(2)(7) = 36 - 56$

$\Delta = -20 < 0$

(iii) The roots are imaginary numbers. Hence proved.

5. **Show that, if p, q, r are rational the roots of the equation**

$x^2 - 2px + p^2 - q^2 + 2qr - r^2 = 0$ are rational.

Solution: (i) $x^2 - 2px + p^2 - q^2 + 2qr - r^2$

$a = 1 \ \ b = -2p, \ c = p^2 - q^2 + 2qr - r^2$

(ii) $\Delta = b^2 - 4ac$

$= (-2p)^2 - 4(1)(p^2 - q^2 + 2qr - r^2)$

$= 4(q^2 - 2qr + r^2)$

$= [2(q - r)]^2 > 0$ which is a perfect square.

Hence the roots are rational.

6. If α, β, γ and δ are the roots of the polynomial equation $2x^4 + 5x^3 - 7x^2 + 8 = 0$, find a Quadratic equation with integer coefficients whose roots are $\alpha + \beta + \gamma + \delta$ and $\alpha\beta\gamma\delta$.

Solution: (i) $2x^4 + 5x^3 - 7x^2 + 8 = 0$

$a = 2, \quad b = 5, \quad c = -7, \quad d = 8$

$\alpha + \beta + \gamma + \delta = \frac{-b}{2} = \frac{-5}{2}, \quad \alpha\beta\gamma\delta = \frac{d}{a} = \frac{8}{2} = 4$

(ii) The sum of the roots are

$(\alpha + \beta + \gamma + \delta) + (\alpha\beta\gamma\delta) = \frac{-5}{2} + 4 = \frac{3}{2}$

(iii) The product of the roots are

$(\alpha + \beta + \gamma + \delta)(\alpha\beta\gamma\delta) = \left(\frac{-5}{2}\right)4 = -10$

(iv) Hence a Quadratic equation is

$x^2 - (\alpha + \beta + \gamma + \delta)x + \alpha\beta\gamma\delta = 0$

$x^2 - \frac{3}{2}x - 10 = 0,$ (x) by 2

$2x^2 - 3x - 20 = 0$

7. Find a polynomial equation of minimum degree with rational coefficients, having $2 - \sqrt{3}$ *as a* root.

Solution: (i) The given roots is $2 - \sqrt{3}$

Then $2 + \sqrt{3}$ is also a root.

(ii) The sum of the roots = $\alpha + \beta$

$= (2 - \sqrt{3}) + (2 + \sqrt{3}) = 4$

The product of the roots $\alpha\beta$

$= (2 - \sqrt{3})(2 + \sqrt{3}) = 2^2 - (\sqrt{3})^2 = 4 - 3 = 1$

(iii) Hence a Quadratic equation is

$x^2 - (\alpha + \beta)x + \alpha\beta = 0$

$x^2 - 4x + 1 = 0$

8. If p and q are the roots of the equation $lx^2 + nx + n = 0$, show that $\sqrt{p/q} + \sqrt{q/p} + \sqrt{n/l} = 0$

Solution: (i) $lx^2 + nx + n = 0, \quad a = l, \quad b = n, \quad c = n$

(ii) The sum of the roots: $p + q = -\frac{b}{a} = \frac{-n}{l}$

The product of the roots: $pq = \frac{c}{a} = \frac{n}{l}$

(iii) $\sqrt{p/q} + \sqrt{q/p} = \frac{\sqrt{p}}{\sqrt{q}} + \frac{\sqrt{q}}{\sqrt{p}} = \frac{p+q}{\sqrt{pq}}$

$= \frac{-n/l}{\sqrt{n/l}} = \frac{-\sqrt{n/l}\cdot\sqrt{n/l}}{\sqrt{n/l}} = -\sqrt{n/l}$

$\sqrt{p/q} + \sqrt{q/p} + \sqrt{n/l} = 0.$ Hence proved.

9. Find a polynomial equation of minimum degree with rational coefficients having $2+\sqrt{3}i$ as a root

Solution: (i) Given roots is $2+\sqrt{3}i$. Then $2-\sqrt{3}i$ is also a root.

(ii) The sum of the roots $(\alpha+\beta)$

$$=(2+\sqrt{3}i)+(+2-\sqrt{3}i)=4$$

The product of the roots $(\alpha\beta)$

$$=(2+\sqrt{3}i)(+2-\sqrt{3}i)=2^2+(\sqrt{3})^2=4+3=7$$

(iii) Hence a Quadratic equation is

$x^2-(\alpha+\beta)x+\alpha\beta=0$

$x^2-4x+7=0$

10. If α and β are roots of $x^2+5x+6=0$ then prove that $\alpha^2+\beta^2=13$.

Solution: (i) $x^2+5x+6=0$

$a=1\ \ b=5\ \ c=6$

(ii) The sum of the roots are

$$\alpha+\beta=-{}^{b}/_{a}=-\frac{5}{1}=-5$$

The product of the roots are

$$\alpha\,\beta=\frac{6}{1}=6$$

(iii) $\alpha^2+\beta^2=(\alpha+\beta)^2-2\alpha\,\beta$

$=(-5)^2-2(6)=25-12$

$\alpha^2+\beta^2=13$. Hence proved

3 Mark Sums:

11. If the sides of a cubic box are increased by 1, 2, 3 units respectively to form a cuboid, then the volume is increased by 52 cubic units. Find the volume of the cuboid.

Solution: (i) The volume of the cuboid

= The volume of the cubic +52

$(x+1)(x+2)(x+3)=x^3+52$

$(x^2+3x+2)(x+3)=x^3+52$

$6x^2+11x-46=0\ \ \Rightarrow x=2$

(ii) The volume of the cuboid

$=(x+1)(x+2)(x+3)=(2+1)(2+2)(2+3)$

= 60 cubic units

12. If α,β and γ are the roots of the cubic equation $x^3+2x^2+3x+4=0$ form a cubic equation whose roots are $\frac{1}{\alpha},\frac{1}{\beta},\frac{1}{\gamma}$

Solution: $i)\ \Sigma\frac{1}{\alpha}=\frac{1}{\alpha}+\frac{1}{\beta}+\frac{1}{\gamma}$

$=\frac{\beta\gamma+\alpha\gamma+\alpha\beta}{\alpha\beta\gamma}=\frac{c}{-d}=\frac{3}{-4}=\frac{-3}{4}$ $[a=1,\ b=2,\ c=3,\ d=4]$

$ii)\ \Sigma\frac{1}{\alpha}\frac{1}{\beta}=\frac{1}{\alpha\beta}+\frac{1}{\beta\gamma}+\frac{1}{\gamma\alpha}$

$=\frac{\gamma+\alpha+\beta}{\alpha\beta\gamma}=\frac{-b}{-d}=\frac{-2}{-4}=\frac{1}{2}$

$iii)\ \Sigma\frac{1}{\alpha}\cdot\frac{1}{\beta}\cdot\frac{1}{\gamma}=\Sigma\frac{1}{\alpha\beta\gamma}=\frac{1}{-d}=\frac{1}{-4}=\frac{-1}{4}$

(iv) Hence a cubic equation is

$$x^3 - \left(\Sigma \frac{1}{\alpha}\right) x^2 + \left(\Sigma \frac{1}{\alpha} . \frac{1}{\beta}\right) x - \Sigma \frac{1}{\alpha} . \frac{1}{\beta} . \frac{1}{\gamma} = 0$$

$$x^3 - \left(\frac{-3}{4}\right) x^2 + \left(\frac{1}{2}\right) x - \left(\frac{-1}{4}\right) = 0, \text{ (x) by 4}$$

$$4x^3 + 3x^2 + 2x + 1 = 0$$

13. If α, β and γ are the roots of the polynomial equation $ax^3 + bx^2 + cx + d = 0$, find the value of $\Sigma \frac{\alpha}{\beta\gamma}$ in terms of the coefficients.

Solution: i) $ax^3 + bx^2 + cx + d = 0$

$\Sigma\alpha = \frac{-b}{a}$, $\Sigma\alpha\beta = \frac{c}{a}$, $\Sigma\alpha\beta\gamma = \frac{-d}{a}$

(ii) $\Sigma \frac{\alpha}{\beta\gamma} = \frac{\alpha}{\beta\gamma} + \frac{\beta}{\gamma\alpha} + \frac{\gamma}{\alpha\beta} = \frac{\alpha^2 + \beta^2 + \gamma^2}{\alpha\beta\gamma}$

$= \frac{(\Sigma\alpha)^2 - 2\Sigma\alpha\beta}{\alpha\beta\gamma} = \frac{b^2/a^2 - 2(c/a)}{(-d/a)}$

$\Sigma \frac{\alpha}{\beta\gamma} = \frac{2ac - b^2}{ad}$

14. Form a polynomial equation with integer coefficients with $\sqrt{\frac{\sqrt{2}}{\sqrt{3}}}$ as a root.

Solution: (i) Let $x = \sqrt{\frac{\sqrt{2}}{\sqrt{3}}}$

Squaring on both sides, $x^2 = \frac{\sqrt{2}}{\sqrt{3}}$

(ii) Again squaring on both sides, $x^4 = \frac{2}{3}$

$\Rightarrow 3x^4 = 2$

$3x^4 - 2 = 0$

15. Show that the polynomial $9x^9 + 2x^5 - x^4 - 7x^2 + 2 = 0$ has atleast six imaginary roots.

Solution: (i) $P(x) = 9x^9 + 2x^5 - x^4 - 7x^2 + 2$

Degree n = 9, Sign changes m = 2

(ii) $P(-x) = 9(-x)^9 + 2(-x)^5 - (-x)^4 - 7(-x)^2 + 2$

$= -9x^9 - 2x^5 - x^4 - 7x^2 + 2$ sign change

for $P(-x)$ is $k = 1$

(iii) By Descartes Rule,

$n - (m + k) = 9 - (2 + 1) = 9 - 3 = 6$

Hence P(x) has at least six imaginary roots.

16. If P is real, discuss the nature of the roots of the equation $4x^2 + 4px + p + 2 = 0$, in terms of p.

Solution: (i) $4x^2 + 4px + p + 2 = 0$

$a = 4, \quad b = 4p, \quad c = p + 2$

(ii) $\Delta = b^2 - 4ac = (4p)^2 - 4(4)(p + 2)$

$= 16(p^2 - p - 2) = 16(p + 1)(p - 2)$

(iii) $\Delta < 0$ if $-1 < p < 2$ then it has imaginary roots.

$\Delta = 0$ if $p = -1$ (or) p =2 then it has equal real roots.

$\Delta > 0$ if $-\infty < p < -1$ (or) $2 < p < \infty$ then it has distinct real roots.

17. Obtain the condition that the roots of $x^3 + px^2 + qx + r = 0$ are in A.P.

Solution: (i) $x^3 + px^2 + qx + r = 0$ ……………………. (1)

$a = 1, \ b = p, \ c = q, \ d = r$. Let the roots be in A.P.

Then, we can assume them in the form a-d, a, a +d.

(ii) The sum of the roots = $-b/_a = -p$

$(a-d) + a + (a+d) = -p$

$3a = -p \Rightarrow a = {}^{-p}/_3$

(iii) Sub the value of a in (1)

$(1) \Rightarrow \left(\frac{-p}{3}\right)^3 + p\left(\frac{-p}{3}\right)^2 + q\left(\frac{-p}{3}\right) + r = 0$

$\frac{-p^3}{27} + \frac{p^3}{9} - \frac{pq}{3} + r = 0$ (x) by 27

$2p^3 - 9pq + 27r = 0$

$9pq = 2p^3 + 27r$

18. Prove that the roots of the equation $x^4 - 3x^2 - 4 = 0$ are $\pm 2, \pm i$

Solution: (i) $x^4 - 3x^2 - 4 = 0$

(ii) Let $x^2 = y$, $(x^2)^2 - 3x^2 - 4 = 0$

$y^2 - 3y - 4 = 0$

$(y+1)(y-4) = 0$

$y = -1, \; y = 4$

(iii) If $y = -1$ then

$x^2 = -1$

$x = \pm i$

If $y = 4$ then

$x^2 = 4$

$x = \pm 2$

Hence proved.

5 Mark Sums:

19. Discuss the maximum possible number of positive and negative roots of the polynomial equation $9x^9 - 4x^8 + 4x^7 - 3x^6 + 2x^5 + x^3 + 7x^2 + 7x + 2 = 0$

Solution: (i) $P(x) = 9x^9 - 4x^8 + 4x^7 - 3x^6 + 2x^5 + x^3 + 7x^2 + 7x + 2$

There is 4 sign changes for $P(x)$, $m = 4$

(ii) $P(-x) = -9x^9 - 4x^8 - 4x^7 - 3x^6 - 2x^5 - x^3 + 7x^2 - 7x + 2$

There is 3 sign changes for P(-x)

(iii) $P(x)$ has maximum number of real roots is 4 and hence there are 3 negative roots.

20. If $2 + i$ and $3 - \sqrt{2}$ are roots of the equation $x^6 - 13x^5 + 62x^4 - 126x^3 + 65x^2 + 127x - 140 = 0$. Find all roots

Solution: (i) Given roots are $2 + i$ and $3 - \sqrt{2}$

Another roots are $2 - i$ and $3 + \sqrt{2}$

(ii) The Degree of the given equation is 6

It has 6 roots. Let α, β be 2 roots.

(iii) The sum of the roots $(\alpha + \beta) = \frac{-b}{a} = 13$

$(2+i) + (2-i) + (3+\sqrt{2}) + (3-\sqrt{2}) + (\alpha+\beta) = 13$

$\alpha + \beta = 3$

(iv) The product of the roots $(\alpha\beta) = \frac{Constant}{a} = -140$

$(2+i)(2-i)(3+\sqrt{2})(3-\sqrt{2})(\alpha\beta) = -140$

$(2^2 + 1^2)(3^2 - \sqrt{2}^2)\alpha\beta = -140$

$(5)(7)\alpha\beta = -140 \Rightarrow \alpha\beta = -4$

(v) The Quadratic equation whose roots are α and β is

$x^2 - (\alpha + \beta)x + \alpha\beta = 0$

$x^2 - 3x - 4 = 0$ $\quad a = 1, \ b = -3, \ c = -4$

$x = \frac{-b \pm \sqrt{b^2-4ac}}{2a} = \frac{3 \pm \sqrt{(-3)^2-4(1)(-4)}}{2(1)}$

$= \frac{3 \pm \sqrt{25}}{2} = \frac{3 \pm 5}{2} = \frac{3+5}{2}, \frac{3-5}{2}$

$= \frac{8}{2}, \frac{-2}{2} = 4, -1$

Result: Thus $2 + i$, $2 - i$, $3 + \sqrt{2}$, $3 - \sqrt{2}$, 4 and -1 are the roots of the given polynomial equation

21. Find all zeros of the polynomial $x^6 - 3x^5 - 5x^4 + 22x^3 - 39x^2 - 39x + 135$, if it is known that $1 + 2i$ and $\sqrt{3}$ are two of its zeros.

Solution: (i) Given roots are $1 + 2i$ and $\sqrt{3}$

Another roots are $1 - 2i$ and $-\sqrt{3}$

(ii) The Degree of the given equation is 6

It has 6 roots. Let α, β be 2 roots.

(iii) The sum of the roots $(\alpha + \beta) = \frac{-b}{a} = 3$ $\quad a = 1, \ b = -3$; Constant = 135

$(1 + 2i) + (1 - 2i) + \sqrt{3} - \sqrt{3} + (\alpha + \beta) = 3$

$2 + (\alpha + \beta) = 3 \Rightarrow \alpha + \beta = 1$

(iv) The product of the roots $(\alpha\beta) = \frac{Constant}{a} = 135$

$(1 + 2i)(1 - 2i)\left(\sqrt{3}\right)\left(-\sqrt{3}\right)(\alpha\beta) = 135$

$(5)\ (-3)\ \alpha\beta = 135 \Rightarrow \alpha\beta = -9$

(v) The Quadratic equation whose roots are α and β is

$x^2 - (\alpha + \beta)x + \alpha\beta = 0$

$x^2 - (1)x + (-9) = 0$ $\quad a = 1, b = -1, \ c = -9$

$x = \frac{-b \pm \sqrt{b^2-4ac}}{2a} = \frac{(-1) \pm \sqrt{(-1)^2-4(1)(-9)}}{2 \times 1}$

$= \frac{1 \pm \sqrt{37}}{2}$

Result: Thus $1 + 2i, 1 - 2i, \sqrt{3}, -\sqrt{3}, \frac{1+\sqrt{37}}{2}, \frac{1-\sqrt{37}}{2}$ are the zeros of the given polynomial equation.

22. Solve the equation: $6x^4 - 5x^3 - 38x^2 - 5x + 6 = 0$ if it is know that $\frac{1}{3}$ is a solution.

Solution: (i) $P(x) = 6x^4 - 5x^3 - 38x^2 - 5x + 6$ $\quad a = 6, \ b = -5$

(ii) The another solution of $\frac{1}{3}$ is 3. It is a Reciprocal equation.

Let x and $\frac{1}{x}$ are the two solutions.

(iii) The sum of the roots $= \frac{-b}{a} = \frac{5}{6}$

$3 + \frac{1}{3} + x + \frac{1}{x} = \frac{5}{6}$

$\frac{10}{3} + x + \frac{1}{x} = \frac{5}{6}$

$x + \frac{1}{x} = \frac{5}{6} - \frac{10}{3} = \frac{5-20}{6} = \frac{-15}{6} = \frac{-5}{2}$

$x + \frac{1}{x} = \frac{-5}{2} = -2 - \frac{1}{2}$

$x = -2, \ x = -\frac{1}{2}$

Result: Thus ${}^{1}/_{3}$, 3, -2, $-{}^{1}/_{2}$ are the solutions of the given equation.

23. Solve the equation: $2x^3 + 11x^2 - 9x - 18 = 0$

Solution: (i) $2x^3 + 11x^2 - 9x - 18 = 0$

(ii) The sum of the coefficients of the odd powers

$= 2 - 9 = -7$ ………………. (1)

The sum of the coefficients of the even powers

$= 11 - 18 = -7$ …………………(2)

$(1) = (2)$

(iii) $x = -1$ is a root of the equation

-1	2	11	-9	-18
	0	-2	-9	+18
	2	9	-18	0

$2x^2 + 9x - 18 = 0$ as the Quotient.

(iv) Factorize: $(x + 6)\left(x - {}^3\!/_2\right) = 0$

$x = -6, x = \frac{3}{2}$

Result: Thus $-6, -1, {}^3\!/_2$ are the roots (or) solutions of the given equation.

24. Solve the equation $x^4 - 9x^2 + 20 = 0$

Solution: (i) $x^4 - 9x^2 + 20 = 0, \Longrightarrow (x^2)^2 - 9x^2 + 20 = 0$

(ii) Let $x^2 = y$, then $y^2 - 9y + 20 = 0$

Factorize, $(y - 5)(y - 4) = 0$

$\Longrightarrow y = 5, \quad y = 4$

(iii) If $y = 5$ then $x^2 = 5 \Longrightarrow x = \pm\sqrt{5}$

$y = 4$ then $x^2 = 4 \Longrightarrow x = \pm 2$

Result: Thus $\sqrt{5}, -\sqrt{5}, 2, -2$ are the solutions of the given equation.

25. Solve the equation $x^3 - 3x^2 - 33x + 35 = 0$

Solution: (i) $x^3 - 3x^2 - 33x + 35 = 0$

(ii) The sum of the coefficients of the polynomial

$= 1 - 3 - 33 + 35 = 36 - 36 = 0$

$x = 1$ is a root of the given equation.

(iii)

1	1	-3	-33	35
	0	1	-2	-35
	1	-2	-35	0

$x^2 - 2x - 35 = 0$ as the Quotient.

(iv) Factorize: $(x - 7)(x + 5) = 0$

$\Longrightarrow x = 7, \quad x = -5$

Result: Thus $1, 7, -5$ are the solutions of the given equation.

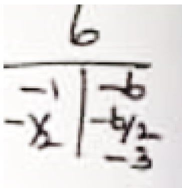
26. Solve the cubic equation: $2x^3 - 9x^2 + 10x = 3$, if 1 is a roots, find the other roots (Modified)

Solution: (i) $2x^3 - 9x^2 + 10x - 3 = 0$

(ii) The sum of the coefficients of the polynomial

$2 - 9 + 10 - 3 = 12 - 12 = 0$

$\Rightarrow x = 1$ is a root of the given equation.



(iii)

-1	2	-9	10	-3
	0	2	-7	3
	2	-7	3	0

$2x^2 - 7x + 3 = 0$ as the Quotient.

(iv) Factorize: $\left(x - \frac{1}{2}\right)(x - 3) = 0$

$\Rightarrow x = \frac{1}{2}, \quad x = 3$

Result: Thus, 1, $^1/_2$, 3 are the roots of the cubic equation.

27. Solve the equation: $(x - 2)(x - 7)(x - 3)(x + 2) + 19 = 0$

Solution: (i) we can rewriting the given equation as the coefficients of x^2 and x are equal.

$[(x-2)(x-3)]\,[(x-7)(x+2)] + 19 = 0$

$(x^2 - 5x + 6)(x^2 - 5x - 14) + 19 = 0$

(ii) Let $x^2 - 5x = y$

$(y + 6)(y - 14) + 19 = 0$

$y^2 - 8y - 65 = 0$

(iii) Factorize: $(y - 13)(y + 5) = 0$

$y = 13$ and $y = -5$

(iv) If $y = 13$ then

$x^2 - 5x = 13$

$x^2 - 5x - 13 = 0$

$a = 1, \; b = -5, \; c = -13$

$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$

$= \frac{5 \pm \sqrt{(-5)^2 - 4(1)(-13)}}{2 \times 1}$

$= \frac{5 \pm \sqrt{77}}{2}$

If $y = -5$ then

$x^2 - 5x = -5$

$x^2 - 5x + 5 = 0$

$a = 1, \; b = -5, \; c = 5$

$x = \frac{5 \pm \sqrt{25 - 20}}{2}$

$= \frac{5 \pm \sqrt{5}}{2}$

Result: Thus $\frac{5 \pm \sqrt{77}}{2}, \frac{5 \pm \sqrt{5}}{2}$ are solutions of the given equation.

28. Solve the equation: $(2x - 3)(6x - 1)(3x - 2)(x - 2) - 5 = 0$

Solutions: (i) We can rewriting the given equation as the coefficients of x^2 and x are equal.

$[(2x-3)(3x-2)]\,[(6x-1)(x-2)] - 5 = 0$

$(6x^2 - 13x + 6)(6x^2 - 13x + 2) - 5 = 0$

(ii) Let $6x^2 - 13x = y$

$(y + 6)(y + 2) - 5 = 0$

$y^2 + 8y + 7 = 0$

(iii) Factorize: $(y+1)(y+7)=0$

$y = -1,\ y = -7$

(iv) If $y = -1$ then

$6x^2 - 13x = -1$

$6x^2 - 13x + 1 = 0$

$a = 6,\ b = -13,\ c = 1$

$x = \frac{-b \pm \sqrt{b^2-4ac}}{2a}$

$= \frac{13 \pm \sqrt{(-13)^2-4(6)(1)}}{2 \times 6}$

$x = \frac{13 \pm \sqrt{145}}{12}$

If $y = -7$ then

$6x^2 - 13x = -7$

$6x^2 - 13x + 7 = 0$

Factorize

$(x-1)\left(x - {}^{7}/_{6}\right) = 0$

$x = 1,$

$x = {}^{7}/_{6}$

Result: Thus $1,\ {}^{7}/_{6},\ \frac{13+\sqrt{145}}{12},\ \frac{13-\sqrt{145}}{12}$ are the solutions of the given equation.

TRY THIS SUMS:

1. Solve: $(x-5)(x-7)(x+6)(x+4) = 504$
2. Solve: $(x-4)(x-7)(x-2)(x+1) = 16$
3. Solve: $(2x-1)(x+3)(x-2)(2x+3) + 20 = 0$

10. Ordinary Differential Equation

2 Marks:

1. **The order and Degree of the Differential equation $y' + (y'')^2 = x(x+y'')^2$**

Ans:

Order -2, Degree – 2

2. **The order and Degree of the Differential equation $\left(\frac{d^2y}{dx^2}\right)^2 + \left(\frac{dy}{dx}\right)^2 = x \sin\left(\frac{d^2y}{dx^2}\right)$**

Ans:

Order – 2, Degree is not Defined.

3. **Show that the solution of $\frac{dy}{dx} = \sqrt{\frac{1-y^2}{1-x^2}}$ is $\sin^{-1}y = \sin^{-1}x + c$ (OR)**

Solve: $\frac{dy}{dx} = \sqrt{\frac{1-y^2}{1-x^2}}$

Ans:

$\frac{dy}{dx} = \frac{\sqrt{1-y^2}}{\sqrt{1-x^2}}$

$\frac{dy}{\sqrt{1-y^2}} = \frac{dx}{\sqrt{1-x^2}}$

$\int \frac{dy}{\sqrt{1-y^2}} = \int \frac{dx}{\sqrt{1-x^2}}$

$\sin^{-1}y = \sin^{-1}x + c$

4. **$y = ae^x + be^{-x}$ is a solution of the differential equation $y'' - y = 0$**

Ans:

$y = ae^x + be^{-x}$

$y' = ae^x - be^{-x}$ ……………………… (1)

$y'' = ae^x - be^{-x}(-1)$

$y'' = ae^x + be^{-x}$

$y'' = y$

$y'' - y = 0$

5. **Form the differential equation of the curve $y = ax^2 + bx + c$ where a, b and c are arbitrary constant.**

Ans:

$$y = ax^2 + bx + c$$

$$y' = 2ax + b \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \quad (1)$$

$$x^2 - y'' - 2xy' + 2y = 0$$

6. **The Population P of a city increase at a rate proportional to the product of population and to the difference between 5,00,000 and the population physical statements in the form of differential equation.**

Ans:

$$\frac{dP}{dt} \alpha P(500000 - P)$$

$$\frac{dP}{dt} = KP(500000 - P)$$

where K is constant

7. **Find the differential equation of the family of parabola $y^2 = 4ax$ where a is an arbitrary constant.**

Ans:

$$y^2 = 4ax \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots \quad (1)$$

$$2y\frac{dy}{dx} = 4a \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \quad (2)$$

$$2y\frac{dy}{dx} = \frac{y^2}{x}$$

$$\frac{dy}{dx} = \frac{y^2}{2xy} = \frac{y}{2x}$$

$$\therefore \frac{dy}{dx} = \frac{y}{2x}$$

8. **Show that the solution of the differential equation $yx^3dx + e^{-x}dy = 0$ is $(x^3 - 3x^2 + 6x - 6)e^x + \log y = c$**

Ans:

$$x^3e^xdx + \frac{dy}{y} = 0$$

$$(x^3 - 3x^2 + 6x - 6)e^x + \log y = c$$

9. **Solve: $\frac{dy}{dx} + y = e^{-x}$**

Ans:

$$I.F = e^{\int pdx} = e^x$$

Solution $ye^x = x + c$

10. **Show that the differential equation for the function $y = e^{-x} + Mx + n$ where M and n are arbitrary constants $e^x\left(\frac{d^2y}{dx^2}\right) - 1 = 0$**

Ans:

$$y = e^{-x} + Mx + n$$

$$\frac{dy}{dx} = -e^{-x} + M$$

$$\frac{d^2y}{dx^2} = e^{-x} \Rightarrow \frac{d^2y}{dx^2} = {}^1/_{e^x}$$

$$e^x\left(\frac{d^2y}{dx^2}\right) - 1 = 0$$

11. Show that the differential equation corresponding to $y = A\,sinx$, where A is an arbitrary constant is $y = y'\tan x$

Ans:

$$y = A\,sinx$$

$$y' = A\,cosx$$

$$\frac{y'}{\cos x} = A$$

$$y = y'\tan x$$

3 Marks:

12. Show that each of the following expression is a solution of the corresponding given Differential equation $y = ae^{x} + be^{-x}$; $y'' - y = 0$

Ans:

$$y = ae^{x} + be^{-x}$$

$$y' = ae^{x} + be^{-x}\,(-1) \qquad \ldots\ldots\ldots\ldots\ldots\ldots\ldots (1)$$

$$y' = ae^{x} - be^{-x}$$

$$y'' = ae^{x} - be^{-x}\,(-1)$$

$$y'' = ae^{x} + be^{-x}$$

$$y'' = y$$

$$y'' - y = 0$$

13. Solve : $(e^{y} + 1)\cos x\, dx + e^{y}\, sinx\, dy = 0$

Ans:

$$(e^{y} + 1)\cos x\, dx = -e^{y}\, sinx\, dy$$

$$\frac{\cos x}{\sin x}dx = -\frac{e^{y}}{e^{y}+1}dy$$

$$\int\frac{\cos x}{\sin x}\,dx = -\int\frac{e^{y}}{e^{y}+1}dy$$

$$\log \text{sinx} = -\log(e^{y} + 1) + \log c$$

$$(e^{y} + 1)\sin x = c$$

14. Form the differential equations by eliminating arbitrary constants given in bracket

$y = e^{3x}(C\,cos2x + D\sin 2x)$, {C, D}

Ans:

$$y = e^{3x}(C\,cos2x + D\sin 2x)$$

$$y\,e^{-3x} = C\,cos2x + Dsin\,2x$$

Difference W. r to x

$$y\,e^{-3x}(-3) + e^{-3x}y' = -2\,C\,sin2x + 2D\,cos2x$$

$$e^{-3x}[-3y + y'] = -2C\,sin2x + 2D\,cos2x$$

Difference r to x

$$e^{-3x}[y'' - 3y'] + [y' - 3y]e^{-3x}(-3) = -4\,C\cos 2x - 4D\,sin2x$$

$$y'' - 6y' + 13y = 0$$

15. Assume that a spherical rain drop evaporates at a rate proportional to its surface area. Form a differential equation involving the rate of change of the radius of the rain drop.

Ans:

$V = {}^{4}/_{3}\, \pi r^3$ — Volume of sphere

$A = 4\pi r^2$ — surface Area

$\frac{dV}{dt} = -kA$

$\int \frac{d({}^{4}/_{3}\, \pi r^3)}{dt} = -k\,(4\,\pi r^2)$

${}^{4}/_{3}\, \pi\, 3r^2\, \frac{dv}{dt} = -k(4\,\pi r^2)$

$\frac{dV}{dt} = -k$

16. Find the differential equation corresponding to the family of curves represented by the equation $y = Ae^{8x} + Be^{-8x}$ where A and B are arbitrary constants

Ans:

$y = Ae^{8x} + Be^{-8x}$

$y' = 8Ae^{8x} - 8Be^{-8x}$

$y'' = 8A(8)e^{8x} - 8B\,(-8)e^{-8x}$

$y'' = 64Ae^{8x} + 64Be^{-8x}$

$y'' = 64(y)$

$y'' - 64y = 0$

5 Marks:

17. Show that the solution of the Differential equation $(1 + x^2)\frac{dy}{dx} = 1 + y^2$ is $tan^{-1}y = tan^{-1}x + c$ (or) $tan^{-1}x = tan^{-1}y + c$

Ans:

$(1 + x^2)\frac{dy}{dx} = 1 + y^2$

$\frac{dy}{1+y^2} = \frac{dx}{1+x^2}$

$\int \frac{dy}{1+y^2} = \int \frac{dx}{1+x^2}$

$tan^{-1}y = tan^{-1}x + c$

18. Solve: $\frac{dy}{dx} = e^{x+y} + x^3e^y$

Ans:

$\frac{dy}{dx} = e^y[e^x + x^3]$

$\frac{dy}{e^y} = [e^x + x^3]dx$

$\int e^{-y}dy = \int(e^x + x^3)dx$

$-e^{-y} = e^x + \frac{x^4}{4} + c$

$e^x + e^{-y} + \frac{x^4}{4} = c$

19. Solve the Differential equation: $\frac{dy}{dx}+\frac{y}{x}=sinx$

Ans:

$$\frac{dy}{dx}+\frac{y}{x}= sinx$$

$$\int Pdx = \int {}^{1}/_{x}\, dx=\log x$$

$$I.F = e^{\int Pdx} = e^{\log x}=x$$

Solution: $ye^{\int Pdx} = \int Qe^{\int Pdx}dx+c$

$$xy+x\cos x = \sin x+c$$

20. If F is the constant force Generated by the motor of an automobile of mass M its velocity V is given by $M\frac{dV}{dt}=F-KV$ where K is a constant, Express V interms of t given that v = 0 when t = 0

Ans:

$$M\frac{dV}{dt}=F-KV$$

$$\frac{dV}{F-KV} = \frac{dt}{M}$$

$$\int\frac{dV}{F-KV} = \int\frac{dt}{M}$$

$$\frac{\log(F-KV)}{-K}= {}^{1}/_{M}\ t+\log c$$

$$\log(F-KV) = -\frac{Kt}{M}+\log c$$

$$t=0,\quad V=0$$

$$c=\frac{-M}{K}\log F$$

$$F=(F-KV)e^{\frac{kt}{m}}$$

21. The rate of increase in the number of bacteria in a certain bacteria culture is proportional to the number present. Given that the number triples in 5 hours, find how man bacteria will be present after 10 hours?

Ans:

t	A
0	A_0
5	$3A_0$
10	?

$$A=Ce^{Kt}$$

$$t=0,\quad A= A_0$$

$$A_0=Ce^{K(0)}$$

$$C= A_0 \Rightarrow A = A_0e^{Kt}$$

$$t=5,\quad A=3A_0$$

$$3A_0 = A_0$$

$$e^{5k} = 3$$

$$t=10,\quad A= ? \Rightarrow A = A_0(e^{5K})$$

$$A=9A_0$$

22. Find the population of a city at any time t given that the rate of increase of population is proportional to the population at that instant and that in *a* period of 40 years the population increased from 3,00,000 to 4,00,000

Ans:

$A = Ce^{Kt}$

$t = 0, \quad A = 3{,}00{,}000$

t	A
0	3,00,000
40	4,00,000

$3{,}00{,}000 = Ce^{K(0)}$

$C = 3{,}00{,}000$

$A = 3{,}00{,}000\, e^{Kt}$

$t = 40, \quad A = 4{,}00{,}000$

$e^{40k} = \frac{4}{3}$

$(e^K)^{40} = \frac{4}{3} \Rightarrow e^K = \left(\frac{4}{3}\right)^{\frac{1}{40}}$

$k = {}^1\!/_{40} \log\left({}^4\!/_3\right)$

$\Rightarrow A = 3{,}00{,}000 \left(\frac{4}{3}\right)^{\frac{t}{40}}$

23. Suppose a person deposits Rs.10,000 in a bank account at the rate of 5% per annum compounded continuously. How much money will be in his bank account 18 months later?

Ans:

$A = Ce^{Kt}$

$K = 5\% = \frac{5}{100} = 0.05$

t	A
0	10000
1.5	?

$A = Ce^{0.05t}$

$t = 0, \quad A = 10000$

$10000 = 10000\, e^{0.05t}$

$t = 1.5, \quad A = ?$

$A = 10000\, e^{0.05(1.5)}$

$A = 10000\, e^{0.075}$

24. Assume that the rate at which radioactive nuclei decay is proportional to the number of such nucleic that are present in each sample. In a certain sample 10% of the original number of radioactive nuclei have undergone disintegration in a period of 100 years. What percentage of the original radioactive nuclei will remain after 1000 years?

Ans:

$A = Ce^{Kt}$

t	A
0	100
100	9/100
1000	?

$t = 0, \quad A = 100$

$100 = Ce^{K(0)}$

$C = 100$

$A = 100\, e^{Kt}$

$t = 100, \quad e^{100K} = \frac{9}{10}$

$t = 1000, \quad A = \frac{9^{10}}{10^8}\,\%$

25. **A radioactive isotope has an initial mass 200mg which two years later is 150mg. Find the expression for the isotope remaining at any time. What is its half – life? (half – life means the time taken for the radioactivity of a specified isotope to fall to half its original value)**

Ans:

t	A
0	200
2	150
?	100

$$A = Ce^{Kt}$$

$t = 0,\ \ A = 200$

$$200 = Ce^{K(0)} \Rightarrow C = 200$$

$$A = 200\, e^{Kt}$$

$t = 2,\ \ A = 150$

$$150 = 200e^{2K}$$

$$K = -{}^{1}/_{2} \log({}^{4}/_{3})$$

$A = 100,\ \ \ t = ?$

$$A = 200\, e^{-{}^{t}/_{2} \log({}^{4}/_{3})}$$

$$t = \frac{2\log({}^{1}/_{2})}{log\left(\frac{4}{3}\right)}$$

26. **Water at temperature 100^oC cools in 10 minutes to 80^oC in a room temperature of 25^oC. Find (i) The temperature of water after 20 minutes**

(ii) The time when the temperature is 40^oC

Ans:

t	T
0	100^0C
10	80^0C
20	?
?	40

$$T - S = Ce^{Kt}$$

$$T - 25 = Ce^{Kt}$$

$t = 0,\ \ T = 100^oC$

$$100 - 25 = C\, e^{K(0)}$$

$$C = 75$$

$$T - 25 = 75\, Ce^{Kt}$$

$t = 10,\ \ T = 80^oC$

$$55 = 75\, e^{-10K}$$

$$K = -{}^{1}/_{10}\ log\left(\frac{15}{11}\right)$$

$t = 20,\ \ T = ?$

$$T - 25 = 75\, e^{20K}$$

$$T - 25 = 75\, (e^{-10K})^2$$

$$T - 25 = 75\left(\frac{11}{15}\right)^2$$

$$T = 65.33^0C$$

$t = 40,\ \ 40 - 25 = 75e^{-Kt}$

$$e^{-Kt} = {}^{15}/_{75} = {}^{1}/_{5}$$

$$Kt = \log 5$$

$$t = \frac{1.6095}{0.03101} = 53.46 \ \text{minus}$$

27. **A pot of boiling water at 100^oC is removed from a stove at time t = 0 and left the cool in the kitchen after 5 minutes the water temperature has decreased to 80^oC and another 5 minutes later it has dropped to 65^oC. Determine the temperature of the kitchen?**

Ans:

$$\frac{dT}{dT} = K(T-S)$$
$$T = S + Ce^{Kt}$$

t	T
0	100^0C
5	80^0C
10	65^0C

$t = 0,\ T = 100^0$
$$T - S = Ce^{Kt}$$
$$100 - S = Ce^{K(0)}$$
$$100 - S = C$$
$$T - S = (100 - S)e^{Kt}$$

$t = 5,\ T = 80^0$
$$80 - S = (100 - S)\, e^{5K}$$
$$e^{5K} = \frac{80-S}{100-S}$$

Temperature of the Kitchen S = 20^0

28. **In Murder investigation, a corpse was found by detective at exactly 8 P.M. Being alert, the detective also measured the body temperature and found it to be $70^0\,F$. Two hours later the detective measured the body temperature again and found it to be $60^0\,F$. If the room temperature is $50^0\,F$ assuming that the body temperature of the person before death was 98.6F at what time did the murder occur?**

Ans:

$$\left[\frac{\log(2.43)}{log2} = \mathbf{1.28}\right]$$

$$\frac{dT}{dt} = k(T-S)$$
$$T - S = Ce^{Kt}$$
$$T - 50 == Ce^{Kt}$$

t	T	
0	70^0C	8 PM
2	60^0C	10 PM
?	98.6	

$t = 0,\quad T = 70$
$$70 - 50 == Ce^{K(0)}$$
$$C = 20$$
$$T - 50 = 20\,e^{Kt}$$

$t = 2,\quad T = 60$
$$60 - 50 = 20e^{2K}$$
$$e^{2K} = {}^1/_2$$
$$K = {}^1/_2\, log({}^1/_2)$$

$T = 98.6,\qquad t = ?$

$$98.6 - 50 = 20\, e^{log\left(\frac{1}{2}\right)t/2}$$
$$t/_2 \log({}^1/_2) = \log\left(\frac{48.6}{20}\right)$$
$$t = 2\frac{\log\left(\frac{48.6}{20}\right)}{log\left(\frac{1}{2}\right)}$$
$$t = 2.56 \quad (or) \quad -2.30\ hours$$

Time of death is 5.30 P.M. (8.00 – 2.30 PM)

29. A tank initially contains 50 liters of pure water starting at time t = 0 a brine containing with 2 grams of dissolved salt per liters flows into the tank at the rate of 3 liters per minute. The mixture is kept uniform by stirring and the well – stirred mixture simultaneously flows out of the tank at the same rate. Find the amount of salt present in the tank at any time t > 0

Ans:

$$\frac{dx}{dt} = IN - OUT$$

$$\frac{dx}{dt} = 6 - \frac{3}{50}x$$

$$x = 100 + Ce^{\frac{-3t}{100}}$$

$$t = 0, \quad C = -100$$

Amount of salt at time t

$$x = 100 - 100 \ e^{\frac{-3t}{50}}$$

Type: 1 Increase (or) Decrease in the Amount quantity t is the Amount of quantity A

$$\frac{dA}{dt} \propto A \Rightarrow \frac{dA}{dt} = KA \Rightarrow A = Ce^{Kt}$$

Increase if K > 0

Decrease if K < 0

Type: 2 Newton's law of cooling / warming

$\frac{dT}{dt} \alpha T - S$ S – Room Temperature

$$\Rightarrow \frac{dT}{dt} = K(T - S)$$

$$\Rightarrow T = S + Ce^{Kt}$$

Type: 3 **Mixture Problems**

Letting x to denote the amount of S present at time t and the derivative $\frac{dx}{dt}$ to denote the rate of change of x w.r. to. x

If IN Denotes the rate at which S enters the mixture and OUT denote the rate at which it leaves, then we have the equation $\frac{dx}{dt}$ = IN – OUT

S.No.	PROBLEMS	ANSWERS
1	Number of Bacteria	9 times the original number of bacteria $A - 9A_0$
2	The Population of a city	$A = 3,00,000 \left(\frac{4}{3}\right)^{t/40}$
3	Electromotice force for an Electric circuit	$I = \frac{E}{R} + Ce^{-\frac{Rt}{L}}$
4	Engine of a Motor boat moving at 10m/s is shut off	$V = 10e^{-2}$
5	At the rate of 5% per annum compounded continuously	$A = 10000 \ e^{0.075}$
6	The rate at which ratioactive nuclei Decay	$\frac{9^{10}}{10^8}$% of ratioactive nuclei will remain after 1000 years
7	The growth of a population	$t = 50 \left(\frac{\log 3}{\log 2}\right)$

8	A radioactive isotope	$K = \frac{1}{2} log\left(\frac{4}{3}\right)$
9	Room Temperature of $25^o C$	$t = 53.46\ minutes$
10	At 10.00 AM a woman took a cup of hot instant coo coffee	$T = 151.4\ F$
11	A pot of boiling water at $100^o C$	Room Temperature S = $20^o C$
12	In a Murder investigation corpse	The person was Murdered = 5.30 PM
13	A tank initially contains 50 litres of pure water	$A = 100 - 100\ e^{-3t/50}$
14	A tank contains 1000 litres of water in which 100 gram of salt	$A = 5000 - 4900\ e^{-0.01t}$

Chapter 8. Differentials and Partial derivative

2 and 3 Marks:

1. Use linear approximation to find an approximate value of $\sqrt{9.2}$ without using a calculator.

Solutions:

$f(x) = \sqrt{x}$ Given: x = 9.2, $x_0 = 9$

$f'(x) = \frac{1}{2\sqrt{x}}$

$L(x) = f(x_0) + f'(x_0)(x - x_0)$

$\sqrt{9.2} = 3 + \frac{0.2}{6} = 3.033$

2. Let $f(x) = \sqrt[3]{x}$. Find the linear approximation at x = 27. Use the linear approximation to approximate $\sqrt[3]{27.2}$

Solution:

$f(x) = \sqrt[3]{x}$ Given: $x = 27.2,\ x_0 = 27$

$f'(x) = \frac{1}{3(\sqrt[3]{x})^2}$

$L(x) = f(x_0) + f'(x_0)(x - x_0)$

$\sqrt[3]{27.2} = 3 + \frac{0.2}{27} = 3.0074$

3. Use the linear approximation to find approximate value of $(123)^{2/3}$

Ans:

$f(x) = x^{2/3}$ Given: $x = 123,\ x_0 = 125$

$f'(x) = \frac{2}{3\sqrt[3]{x}}$

$L(x) = f(x_0) + f'(x_0)(x - x_0)$

$(123)^{2/3} = 25 + \frac{2}{15}(-2) = 24.73$

4. Use the linear approximation to find approximate value of $\sqrt[4]{15}$

Solution:

$f(x) = \sqrt[4]{x}$ Given: $x = 15,\ x_0 = 16$

$f'(x) = \frac{1}{4(\sqrt[4]{x})^3}$

$L(x) = f(x_0) + f'(x_0)(x - x_0)$

$\sqrt[4]{15} = 2 + \frac{1}{32}(-1) = 1.9688$

5. Use the linear approximation to find approximate value of $\sqrt[3]{26}$

Solution:

$f(x) = \sqrt[3]{x}$ Given: $x = 26,\ x_0 = 27$

$f'(x) = \frac{1}{3\left(\sqrt[3]{x}\right)^2}$ $L(x) = f(x_0) + f'(x_0)(x - x_0)$

$\sqrt[3]{26} = 3 + \frac{1}{27}(-1) = 2.963$

6. Assuming $\log_{10} e = 0.4343$, find an approximate value of $\log_{10} 1003$

Solution:

$(x) = \log_{10} x$ Given: $x = 1003,\ x_0 = 1000$

$f'(x) = \frac{1}{x}\log_{10} e$ $L(x) = f(x_0) + f'(x_0)(x - x_0)$

$\log_{10} 1003 = 3 + \frac{3}{1000} \times 0.4343 = 3.0013$

7. The time T, taken for a complete oscillation of a simple pendulum with length l, is given by the equation $T = 2\pi\sqrt{\frac{l}{g}}$, where g is a constant. Find the approximate percentage error in the calculated value of T corresponding to an error of 2 percent in the value of l.

Solution:

$\log T = \log\left(\frac{2\pi}{\sqrt{g}}\right) + \frac{1}{2}\log l$

$\frac{dT}{T} \times 100 = \frac{1}{2}\left(\frac{dl}{l} \times 100\right)$

Percentage error in T = $\frac{1}{2} \times Percentage\ error\ in\ l$

$= \frac{1}{2} \times 2\% = 1\%$

8. Show that the percentage error in the n^{th} root of a number in approximately $\frac{1}{n}$ times the percentage error in the number.

Solution:

$y = x^{1/n}$

$\log y = \frac{1}{n}\log x$

$\frac{dy}{y} \times 100 = \frac{1}{n}\left(\frac{dx}{x} \times 100\right)$

Percentage error in y = $\frac{1}{n} \times$ Percentage error in x

9. Let $g(x) = x^2 + \sin x$. Calculate the differential dg.

Solution:

$g(x) = x^2 + \sin x$

$dg = g'(x)dx$

$dg = (2x + cosx)dx$

10. Find df for $f(x) = x^2 + 3x$ and evaluate it for (i) $x = 2, and\ dx = 0.1$ (ii) $x = 3\ and\ dx = 0.02$

Solution:

$f(x) = x^2 + 3x$

$df = f'(x)dx$

$= (2x + 3)dx$

(i) $x = 2, dx = 0.1$

$df = 7 \times 0.1 = 0.7$

(ii) $x = 3, dx = 0.02$

$df = 9 \times 0.02 = 0.18$

11. If the radius of a sphere, with radius 10cm, has to decrease by 0.1cm, approximately how much will its volume decrease?

Solution:

$v = \frac{4}{3}\pi r^3$

Given: $dr = -0.1, r = 10$

$dv = 4\pi r^2 dr = 4\pi(100)(-0.1) = -40\pi cm^2$

12. If $\cup(x, y, z) = \log(x^3 + y^3 + z^3)$, find $\frac{\partial U}{\partial x} + \frac{\partial U}{\partial y} + \frac{\partial U}{\partial z}$

Solution:

$$\frac{\partial U}{\partial x} + \frac{\partial U}{\partial y} + \frac{\partial U}{\partial z} = \frac{3x^2}{x^3+y^3+z^3} + \frac{3y^2}{x^3+y^3+z^3} + \frac{3z^2}{x^3+y^3+z^3}$$

$$= \frac{3(x^2+y^2+z^2)}{x^3+y^3+z^3}$$

5 Marks:

13. If $f(x, y) = tan^{-1}\left(\frac{x}{y}\right)$, find f_x, f_y and show that $f_{xy} = f_{yx}$

Solution:

$f_x = \frac{y}{x^2+y^2}$ $\qquad$ $f_y = \frac{-x}{x^2+y^2}$

$f_{xy} = \frac{x^2-y^2}{(x^2+y^2)^2}$ —(1) $\qquad$ $f_{yx} = \frac{x^2-y^2}{(x^2+y^2)^2}$ —(2)

$\therefore f_{xy} = f_{yx}$

14. If $u = \sin^{-1}\left[\frac{x+y}{\sqrt{x}+\sqrt{y}}\right]$, show that $x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} = \frac{1}{2}\tan u$

Solution:

$f(x, y) = \frac{x+y}{\sqrt{x}+\sqrt{y}} = \sin u$

f is homogeneous with degree $n = \frac{1}{2}$

By Euler's theorem,

$x\frac{\partial f}{\partial x} + y\frac{\partial f}{\partial y} = nf$

$x\frac{\partial(\sin u)}{\partial x} + y\frac{\partial(\sin u)}{\partial y} = \frac{1}{2}\sin u$

$xcos\, u\frac{\partial u}{\partial x} + ycos\, u\frac{\partial u}{\partial y} = \frac{1}{2}\sin u$

$x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} = \frac{1}{2}\tan u$

15. If $u(x, y) = \frac{x^2+y^2}{\sqrt{x+y}}$, prove that $x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} = \frac{3}{2}u$

Solution:

$u(x, y) = \frac{x^2+y^2}{\sqrt{x+y}}$

u is homogeneous with degree $n = \frac{3}{2}$

By Euler's theorem,

$$x\frac{\partial f}{\partial x} + y\frac{\partial f}{\partial y} = nf$$

$$x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} = \frac{3}{2}u$$

16. **If $v(x,y) = \log\left(\frac{x^2+y^2}{x+y}\right)$, prove that $x\frac{\partial v}{\partial x} + y\frac{\partial v}{\partial y} = 1$**

Solution:

$$f(x,y) = \frac{x^2+y^2}{x+y} = e^v$$

f is homogeneous with degree n = 1

By Euler's theorem,

$$x\frac{\partial f}{\partial x} + y\frac{\partial f}{\partial y} = nf$$

$$x\frac{\partial (e^v)}{\partial x} + y\frac{\partial (e^v)}{\partial y} = 1.e^v$$

$$xe^v\frac{\partial v}{\partial x} + ye^v\frac{\partial v}{\partial y} = e^v$$

$$x\frac{\partial v}{\partial x} + y\frac{\partial v}{\partial y} = 1$$

17. **Prove that $f(x,y) = x^3 - 2x^2y + 3xy^2 + y^3$ is homogeneous what is the degree? verify Euler's theorem for f**

Solution:

$$f(\lambda x, \lambda y) = \lambda^3 f(x,y)$$

f is homogeneous with degree n = 3

By Euler's theorem, $x\frac{\partial f}{\partial x} + y\frac{\partial f}{\partial y} = 3f$

$$\frac{\partial f}{\partial x} = 3x^2 - 4xy + 3y^2$$

$$\frac{\partial f}{\partial y} = -2x^2 + 6xy + 3y^2$$

$$x\frac{\partial f}{\partial x} + y\frac{\partial f}{\partial y} = 3f$$

Hence verified.

18. **If $w(x,y,z) = \log\left(\frac{5x^3y^4+7y^2xz^4-75y^3z^4}{x^2+y^2}\right)$, find $x\frac{\partial w}{\partial x} + y\frac{\partial w}{\partial y} + z\frac{\partial w}{\partial z}$**

Solution:

$$f = \frac{5x^3y^4+7y^2xz^4-75y^3z^4}{x^2+y^2} = e^w$$

f is homogenous with degree n = 5

By Euler's theorem,

$$x\frac{\partial f}{\partial x} + y\frac{\partial f}{\partial y} + z\frac{\partial f}{\partial z} = nf$$

$$x\frac{\partial (e^w)}{\partial x} + y\frac{\partial (e^w)}{\partial y} + z\frac{\partial (e^w)}{\partial z} = 5e^w$$

$$xe^w\frac{\partial w}{\partial x} + ye^w\frac{\partial w}{\partial y} + ze^w\frac{\partial w}{\partial z} = 5e^w$$

$$x\frac{\partial w}{\partial x} + y\frac{\partial w}{\partial y} + z\frac{\partial w}{\partial z} = 5$$

19. Prove that $g(x,y) = x\log\left(\frac{y}{x}\right)$ is homogeneous, what is the degree? Verify Euler's theorem for g

Solution:

$$g(\lambda x\,,\ \lambda y) = \lambda^1 g(x,y)$$

g is homogeneous with degree n = 1

By Euler's theorem,

$$x\frac{\partial g}{\partial x} + y\frac{\partial g}{\partial y} = g$$

$$\frac{\partial g}{\partial x} = \log\left(\frac{y}{x}\right) - 1$$

$$\frac{\partial g}{\partial y} = \frac{x}{y}$$

$$x\frac{\partial g}{\partial x} + y\frac{\partial g}{\partial y} = g$$

Hence verified.

3 Marks:

20. Assume that the cross section of the artery of human is circular. A drug is given to a patient to dilate his arteries. If the radius of an artery is increased from 2mm to 2.1mm, how much is cross – sectional area increased approximately?

Solution:

$r = 2, dr = 2.1 - 2 = 0.1$

$A = \pi r^2$

$dA = 2\pi r\ dr$

$= 2\pi(2)(0.1)$

$= 0.4\pi\ \ mm^2$

21. An egg of a particular bird is very nearly spherical. If the radius to the inside of the shell is 5mm and radius to the outside of the shell is 5.3 m.m, find the volume of the shell approximately

Solution:

$r = 5, dr = 5.3 - 5 = 0.3$

$V = \frac{4}{3}\pi r^3$

$dV = 4\pi r^2 dr = 4\pi(5)^2(0.3) = 30\pi\ \ mm^3$

22. A circular plate expands uniformly under the influence of heat. If it's radius increases from 10.5cm to 10.75cm, then find an approximate change in the area and the approximate percentage change in the area.

Solution:

$r = 10.5, dr = 10.75 - 10.5 = 0.25$

$A = \pi r^2$

$dA = 2\pi r\ dr$

$= 2\pi(10.5)(0.25) = 5.25\pi\ \ cm^2$

Approximate percentage change $= \frac{dA}{A} \times 100\%$

$= 4.76\%$

Chapter – 7
Applications of Differential calculus

5 Marks:

1. Example: 7.5

A particle is fired straight up from the ground to reach a height of s feet in t seconds, where s(t) = $128t - 16t^2$

(i) Compute the maximum height of the particle reached

(ii) What is the velocity when the particle hits the ground

Solution:

i. At maximum height $v(t) = 0$

$v(t) = \frac{ds}{dt} = 128 - 32t$

$v(t) = 0$

$128 - 32t = 0 \Longrightarrow$ t = 4

The height t = 4 is s(t) = 128(4) – 16(4)^2 = 256 ft.

ii. When the particle hits the ground then S = 0

S = 0

$128t - 16t^2 = 0$

t = 0, t = 8 seconds

The particle hits the ground at t = 8 seconds

The velocity when it hit the ground is $v(8) = 128 - 32(8)$

$= -128\ ft/s$

2. Exercise 7.1 -2

A camera is accidentally knocked off an edge of a cliff 400ft high. The camera falls a distance of S = $16t^2$ in t seconds

i. How long does the camera fall before it hits the ground?

ii. What is the average velocity high which the camera falls during the rest 2 seconds?

iii. What is the instantaneous velocity of the camera when it hits the ground?

Solution:

i. To hit the ground the camera has to travel 400ft

S = 400

$16t^2 = 400$

$t^2 = \frac{400}{16} = 25$

$t = 5\ second$

ii. The average velocity in last 2 seconds

$= \frac{s(5)-s(3)}{5-3}$

$= 128\ ft/sec$

iii. The instantaneous velocity when it hits the ground

$V = \frac{ds}{dt} = 32t$

when t = 5, $v = 32 \times 5$

$v = 160 ft/sec$

3. Exercise: 7.1 -3

A particle moves along a line according to the law $s(t) = 2t^3 - 9t^2 + 12t - 4$ where $t \geq 0$

i. At what times the particle changes direction?

ii. Find the total distance travelled by the particle in the first 4 seconds

iii. Find the particle acceleration each time the velocity is zero

Solution:

$v(t) = 6t^2 - 18t + 12$

$v(t) = 6(t-1)(t-2)$

$a(t) = 12t - 18$

i. $v(t) = 0$

$6(t-1)(t-2) = 0$

$t = 1, 2$

The particle change its direction at $t = 1$ sec and $t = 2$ sec

ii. The total distance travelled by the particle

$= |s(0) - s(1)| + |s(1) - s(2)| + |s(2) - s(4)|$

$= |-4 - 1| + |1 - 0| + |0 - 28|$

$= 5 + 1 + 28$

= 34 meters

iii. The acceleration when v = 0 is $a(1) = -6\ m/s^2$

$a(2) = 6\ m/s^2$

Do yourself:

Example 7.6

A particle moves along a horizontal line such that its position at any time $t > 0$ is given by $s(t) = t^3 - 6t^2 + 9t + 1$ where s is measured in metres and t is seconds?

i. At what time the particle is at rest?

ii. At what time the particle changes its direction?

iii. Find the total distance travelled by the particle in the first 2 seconds/

4. Example: 7.9

Salt is poured from a conveyer belt at a rate of 30 cubic metre per minute forming a conical pile with a circular base whose height and diameter of base are always equal. How fast is the height of the pile increasing when the pile is 10 metre high?

Solution:

$2r = h$

$r = h/2$

$V = \frac{1}{3}\pi r^2 h$

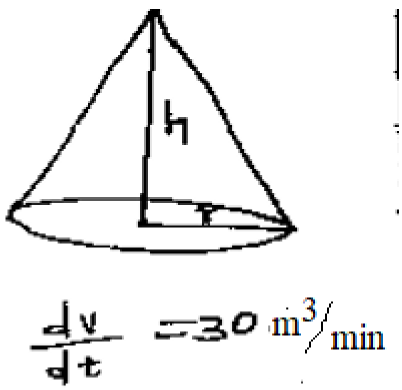
$V = \frac{1}{12}\pi h^3$

$\frac{dv}{dt} = \frac{1}{4}\pi h^2 \frac{dh}{dt}$

$\frac{dh}{dt} = 4\frac{dv}{dt} \cdot \frac{1}{\pi h^2}$

$\frac{dh}{dt} = 4 \times 30 \times \frac{1}{100\pi}$

$\frac{dh}{dt} = \frac{6}{5\pi}\ m/min$

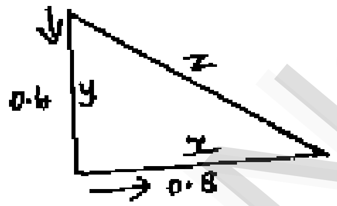


5. **Exercise: 7.1-10**

A police jeep approaching an orthogonal intersection from the northern direction is chasing a speeding car that has turned and moving straight east. When the jeep is 0.6km north of the intersection and the car is 0.8km to the east. The police determine with a radar that the distance between them and the car is increasing at 20 km/hr. It the jeep is moving at 60 km/hr at the instant of measurement, what is the speed fo the car?

Solution:

Given $\frac{dy}{dt} = -60\ km/hr,\quad \frac{dz}{dt} = 20\ km/hr$



$z^2 = x^2 + y^2$

$= (0.8)^2 + (0.6)^2$

$z^2 = 1$

$z = 1$

$x^2 + y^2 = z^2$

$2x\frac{dx}{dt} + 2y\frac{dy}{dt} = 2z\frac{dz}{dt}$

$(0.8)\frac{dx}{dt} + (0.6)(-60) = (1)(20)$

$\frac{dx}{dt} = 70\ km/hr$

6. **Exercise: 7.1-9.** **A ladder 17 metre long is learning against the wall. The base of the ladder is pulled away from the wall at a rate of 5m/s when the base of the ladder is 8 metres from the well**

i. How fast is the top fo the ladder moving down the well?

ii. At what rate the area of the triangle formed by the ladder, wall and the floor is changing?

Solution:

$(i)\quad x^2 + y^2 = 17^2$

$2x\frac{dx}{dt} + 2y\frac{dy}{dt} = 0$

$\frac{dy}{dt} = \frac{-x}{y}\frac{dx}{dt}$

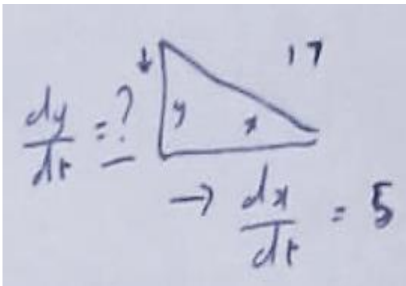
$\frac{dy}{dt} = \frac{-5x}{y}$

$\frac{dy}{dt} = \frac{-5(8)}{15} = -8/_3\ m/s$

$x^2 + y^2 = z^2$

$8^2 + y^2 = 17^2$

$y = 15$



$ii).\quad A = \frac{1}{2}xy$

$\frac{dA}{dt} = \frac{1}{2}(x\frac{dy}{dt} + y\frac{dx}{dt})$

$= \frac{1}{2}[8\,(\frac{-8}{3}) + 15(5)]$

$= \frac{1}{6}[-64 + 225]$

$= \frac{161}{6}$

$= 26.83\ m^2/sec$

Do yourself:

Example: 7.7 -If we blow air into a balloon of spherical shape at a rate of 1000 cm^3 per second, at what rate the radius of the balloon changes when the radius is 7cm. Also compute the rate at which the surface area changes.

Example: 7.8: The price of a product is related to the number of units available (supply) by the equation px + 3p – 16x = 234, where p is the price of the product per unit in rupees (Rs). and x is the number of units. Find the rate at which the price is changing with respect to time when 90 units are available and the supply is increasing at the rate of 15 units / week.

Example: 7.10: A road running north to south crosses a road going east to west at the point P. Car A is driving north along the first road, and car B is driving east along the second road. At a particular time car A is 10 kilometres to the north of P and traveling of 80km/hr while car B is 15 kilometres to the east of P and braseling at 100km/hr. How fast is the distance between the two cars changing?

Exercise: 7.1 – 7: A beacon makes one revolution every 10 seconds. It is located on a ship which is anchored 5km from a straight shore line. How fast is the beam moving along the shore line when it makes an angle of 45° with the shore?

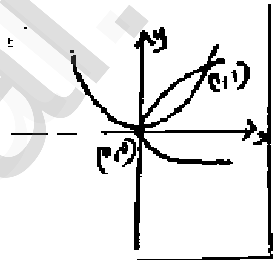
Exercise: 7.1-8: A conical water tank with vertex down of 12 metres height has a radius of 5 metres at the top. If water flows into the tank at a rate 10 cubic m/min, how fast is the depth of the water increases when the water is 8 metres deep?

7. Example: 7.15

Find the angle between the curves $y = x^2$ and $x = y^2$ of their point of intersection (0, 0) and (1, 1)

Solution:

The tangent at (0, 0) are x axis and y axis. Angle between x axis and y axis is 90°.



$$\theta = 90^0$$

$y = x^2$ $\qquad$ $y^2 = x$

$\frac{dy}{dx} = 2x$ $\qquad$ $\frac{dy}{dx} = \frac{1}{2y}$

$m_1 = \frac{dy}{dx\,(1,1)} = 2$ $\qquad$ $m_2 = \frac{dy}{dx\,(1,1)} = \frac{1}{2}$

$$tan\,\theta = \left|\frac{2 - \frac{1}{2}}{1 + 2(\frac{1}{2})}\right|$$

$$tan\,\theta = 3/4$$

$$\theta = tan^{-1}(3/4)$$

8. Example: 7.17

If the curves $ax^2 + by^2 = 1$ and $cx^2 + dy^2 = 1$ intersect each other orthogonally then show that $\frac{1}{a} - \frac{1}{b} = \frac{1}{c} - \frac{1}{d}$

Solution:

Let the two curves intersect at (x_1, y_1)

$(a - c)x_1^2 + (b - d)y_1^2 = 0$ ……………………. (1)

$ax^2 + by^2 = 1$ $\qquad$ $x^2 + dy^2 = 1$

$\frac{dy}{dx} = -\frac{ax}{by}$ $\qquad$ $\frac{dy}{dx} = -\frac{cx}{dy}$

$m_1 = \frac{dy}{dx}(x_1, y_1) = -\frac{ax_i}{by_i}$ $\qquad$ $m_2 = \frac{dy}{dx}(x_1, y_1) = -\frac{cx_1}{dy_1}$

Two curve cut orthogonally

$$m_1 m_2 = -1$$

$$\left(\frac{-ax_1}{by_1}\right) \times \left(\frac{-cx_1}{dy_1}\right) = -1$$

$acx_1^2 + bdy_1^2 = 0$ …………………………… (2)

$$\frac{(1)}{(2)} \Rightarrow \frac{a-c}{ac} = \frac{b-d}{bd}$$

$$\Rightarrow \frac{1}{c} - \frac{1}{a} = \frac{1}{d} - \frac{1}{b}$$

$$\Rightarrow \frac{1}{a} - \frac{1}{b} = \frac{1}{c} - \frac{1}{d}$$

9. Example 7.18:

Prove that ellipse $x^2 + 4y^2 = 8$ and hyperbola $x^2 - 2y^2 = 4$ intersect orthogonally

Solution: Let point of intersection be (a, b)

$a^2 + 4b^2 = 8$

$a^2 - 2b^2 = 4$

(-) (+) (-)

$\mathbf{6b^2 = 4}$

$\mathbf{b^2 = 4/6}$

$\mathbf{b^2 = 2/3}$

$\mathbf{a^2 = 16/3}$

$\mathbf{x^2 + 4y^2 = 8}$

$2x + 8y\frac{dy}{dx} = 0$

$m_1 = \frac{dy}{dx_{(a,b)}} = -a/4b$

$x^2 - 2y^2 = 4$

$2x - 4y\frac{dy}{dx} = 0$

$m_2 = \frac{dy}{dx_{(a,b)}} = a/2b$

$m_1\, m_2 = \left(-\frac{a}{4b}\right)\left(\frac{a}{2b}\right)$

$= -\frac{a^2}{8b^2}$

$= \frac{-\frac{16}{3}}{8\left(\frac{2}{3}\right)}$

$m_1\, m_2 = -1$

The curves cut orthogonally

10. Exercise: 7.2 – 9

Find the angle between the curves $xy = 2$ and $x^2 + 4y = 0$

Solution:

The point of intersection of two curve is (-2, -1)

$xy = 2$

$xy' + y = 0$

$y' = \frac{-y}{x}$

$m_1 = y'(-2, -1) = -1/2$

$x^2 + 4y = 0$

$2x + 4y' = 0$

$y' = -\frac{2x}{4}$

$m_2 = y'(-2, -1) = 1$

$tan\theta = \left|\frac{\frac{-1}{2} - 1}{1 + \left(\frac{-1}{2}\right)(1)}\right| = \left|\frac{-3/2}{1/2}\right|$

$\boldsymbol{\theta = tan^{-1}(3)}$

11. Exercise 7.2-10

Show that the two curves $x^2 - y^2 = r^2$ and $xy = c^2$ where c, r are constant, cut orthogonally.

Solution:

Let the point of intersection be (x_1, y_1)

$$x^2 - y^2 = r^2$$

$$2x - 2yy' = 0$$

$$y' = \frac{x}{y}$$

$$m_1 = y'_{(x_1,y_1)} = \frac{x_1}{y_1}$$

$$xy = c^2$$

$$xy' + y = 0$$

$$y' = -\frac{y}{x}$$

$$m_2 = y'_{(x_1,y_1)} = -\frac{y_1}{x_1}$$

$$m_1\, m_2 = \left(\frac{x_1}{y_1}\right)\left(-\frac{y_1}{x_1}\right)$$

$$m_1\, m_2 = -1$$

Two curves cut orthogonally.

12. Exercise: 7.4.1(V)

Write the Maclaurin series expansion of $tan^{-1}x : -1 \le x \le 1$

Solution:

$f(x) = tan^{-1}x$ $\quad$ $f(0) = 0$

$f'(x) = \frac{1}{1+x^2} = (1+x^2)^{-1} = 1 - x^2 + x^4 - x^6 + \cdots$ $\quad$ $f'(0) = 1$

$f''(x) = -2x + 4x^3 - 6x^5 + \ldots.$ $\quad$ $f''(0) = 0$

$f'''(x) = -2 + 12x^2 - 30x^4 + \cdots$ $\quad$ $f'''(0) = -2$

$f^4(x) = 24x - 120x^3 + \ldots$ $\quad$ $f^4(0) = 0$

$f^5(x) = 24 - 360x^2 + \cdots..$ $\quad$ $f^5(0) = 24$

$$tan^{-1}x = 0 - \frac{x}{1!}(1) + \frac{x^2}{2!}(0) + \frac{x^3}{3!}(-2) + \frac{x^4}{4!}(0) + \frac{x^5}{5!}(24) + \cdots ....$$

$$tan^{-1}x = x - \frac{x^3}{3} + \frac{x^5}{5} - \ldots\ldots\ldots\ldots$$

13. Exercise : 7.5 -10

Evaluate : $\lim\limits_{x \to \frac{\pi}{2}} (sinx)^{tanx}$

Solution:

Let $y = (sinx)^{tanx}$

$$log\, y = tan\, x\, log\, sin\, x$$

$$\log y = \frac{logsinx}{cotx}$$

$$\lim_{x \to \frac{\pi}{2}} \log y = \lim_{x \to \frac{\pi}{2}} \frac{logsinx}{cotx} \quad \left(\frac{0}{0}\ form\right)$$

$$= \lim_{x \to \frac{\pi}{2}} \frac{\frac{1}{sinx}.cosx}{-cosec^2 x}$$

$$= \lim_{x \to \frac{\pi}{2}} -cosxsinx$$

$$\lim_{x \to \frac{\pi}{2}} \log y = 0$$

$$log_e \lim_{x \to \frac{\pi}{2}} y = 0$$

$$\lim_{x \to \frac{\pi}{2}} y = e^0$$

$$\lim_{x \to \frac{\pi}{2}} (sinx)^{tanx} = 1$$

Do yourself:

Example: 7.43: Using the $L'h\hat{o}\,pital$ rule, prove that $\lim\limits_{x \to 0^+} (1+x)^{1/x} = e$

Example: 7.44: Evaluate : $\lim\limits_{x \to \infty} (1+2x)^{\frac{1}{2logx}}$

Example 7.45: Evaluate: $\lim\limits_{x \to 1} x^{\frac{1}{1-x}}$

Exercise: 7.5-8 Evaluate: $\lim\limits_{x \to 0^+} x^x$

Exercise: 7.5-9: Evaluate: $\lim\limits_{x \to \infty} (1+\frac{1}{x})^x$

Exercise: 7.5-11 Evaluate: $\lim\limits_{x \to 0^+} (cosx)^{\frac{1}{x^2}}$

Exercise: 7.5 – 12

If an initial amount A_o of money is invested at an interest rate r compounded n times a year, the value of the investment after t years in $A = A_o\left(1+\frac{r}{n}\right)^{nt}$. If the interest is compounded continuously, (that is as x → ∞), show that the amount after t years is $A = A_o\, e^{rt}$

14. Example: 7.60

Find the local extrema of the function $f(x) = 4x^6 - 6x^4$

Solution:

$f'(x) = 24x^5 - 24x^3$

$= 24x^3(x^2 - 1)$

$f'(x) = 24x^3(x+1)(x-1)$

$f'(x) = 0$

$x = -1, 0, 1$

$f''(x) = 120x^4 - 72x^2$

$f''(x) = 24x^2(5x^2 - 3)$

$f''(-1) = 48 > 0,$

$f''(1) = 48 > 0$,

$f''(0) = 0$

Interval	Sign of $f'(x)$	Monotonicity
$x \in (-\infty, -1)$	-	Strictly decreasing
$x \in (-1, 0)$	+	Strictly increasing
$x \in (0, 1)$	-	Strictly decreasing
$x \in (1, \infty)$	+	Strictly increasing

$f(x)$ has local maximum of $x = -1$ and $x = 1$

Local minimum value is -2

$f(x)$ has local maximum at $x = 0$

Local maximum value is 0

Do yourself:

Exercise: 7.7-3

Find the function $f(x) = 4x^3 + 3x^2 - 6x + 1$ find the intervals of monotonicity, local extrema intervals of conctrity and point of inflection.

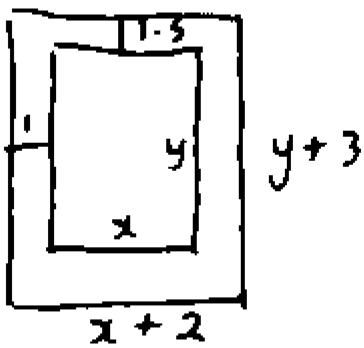
15. Exercise: 7.8 -5:

A rectangular page is to contain 24cm^2 of print. The margins of the top and bottom of the page are 1.5cm and the margins at other sides of the page is 1cm. What should be the dimensions of the page so that the area of the paper used is minimum?

Solution:

Let x and y be the dimensions of the printed portion

The poster dimensions are $(x+2)(y+3)$



$A = (x+2)(y+3)$

$A(x) = (x+2)\left(\frac{24}{x} + 3\right)$

$A(x) = 24 + \frac{48}{x} + 3x + 6$

$A'(x) = \frac{-48}{x^2} + 3$

$A''(x) = \frac{96}{x^3}$

$A'(x) = 0$

$x = \pm 4$

When $x = 4$ $A''(x) > 0$

When $x = 4$ A is minimum

When $x = 4$ $y = 6$

The dimension of the poster are 6cm, 9cm.

16. Exercise: 7.8-(8)

Prove that among all the rectangles of the given perimeter, the square has the maximum area.

Solution:

Let x and y be the dimensions of rectangle.

$$P = 2x + 2y$$

$$y = \frac{P-2x}{2}$$

$$A = xy$$

$$A(x) = x\left(\frac{P-2x}{2}\right)$$

$$A(x) = \frac{p}{2}x - x^2$$

$$A'(x) = \frac{p}{2} - 2x$$

$$A''(x) = -2$$

$$A'(x) = 0$$

$$x = {}^{p}/_{4}$$

When $x = {}^{p}/_{4}$ $A''(x) < 0$

When $x = {}^{p}/_{4}$ The area is maximum

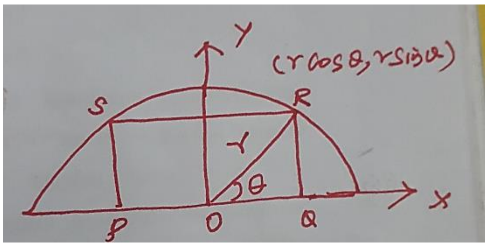
When $x = {}^{p}/_{4}$, $y = {}^{p}/_{4}$

Thus it is a square

17. Exercise 7.8-9.

Find the dimensions of the largest rectangle that can be inscribed in a semicircle of radius *r cm*

Solution:



$$PQ = 2r\cos\theta$$

$$QR = r\sin\theta$$

Area of the rectangle $A = 2rcos\theta . rsin\theta$

$$(\theta) = r^2 sin2\theta$$

$$A'(\theta) = 2r^2 cos2\theta$$

$$A''(\theta) = -4r^2 sin2\theta$$

$$A'(\theta) = 0 => cos2\theta = 0 => \theta = \frac{\pi}{4}$$

When $\theta = \frac{\pi}{4}$, $A''(\theta) < 0$

When $\theta = \frac{\pi}{4}$, The area is maximum

The dimensions are $2r\cos\frac{\pi}{4}$, $r\sin\frac{\pi}{4}$

The dimensions are $\sqrt{2}r$, $\frac{r}{\sqrt{2}} cm$

18. Example: 7.65

Prove that among all the rectangles of the given area square has the least perimeter.

Solution:

Area of the rectangle is $xy = k$

Perimeter of the rectangle $p(x) = 2(x + y)$

$p(x) = 2\left(x + {}^{k}/_{x}\right)$

$p'(x) = 2\left(1 - {}^{k}/_{x^2}\right)$

$p'(x) = 0$

$x = \pm\sqrt{k}$

$p''(x) = {}^{4k}/_{x^3}$

$p''(\sqrt{k}) > 0$

P(x) has its minimum value at $x = \sqrt{k}$

$x = \sqrt{k}$, y$= \sqrt{k}$

The minimum perimeter rectangle of a given area is a square

Do yourself:

Exercise: 7.8 – 12

A hollow cone with base radius a cm and height b cm is placed on a table show that the volume of the largest cylinder that can be hidden underneath is 4/9 times volume of the cone.

3 Marks:

19. Example 7.11

Find the equation of the tangent and normal to the curve $y = x^2 + 3x - 2$ at the point (1, 2).

Solution:

$\frac{dy}{dx} = 2x + 3$

$m = \frac{dy}{dx_{(1,2)}} = 5$

The equation of the tangent is

$(y - 2) = 5(x - 1)$

$5x - y - 3 = 0$

The equation of the normal is

$y - 2 = -\frac{1}{5}(x - 1)$

$x + 5y - 11 = 0$

Do yourself:

Example: 7.13

Find the equation of the tangent and normal at any point to the Lissejons curve given by $x = 2\, cos\, 3t$ and $y = 3\, sin\, 2t$, $t \in R$

20. Exercise 7.2-2:

Find the point on the curve $y = x^2 - 5x + 4$ at which the tangent is parallel to the line $3x + y = 7$

Solution:

The slope of the tangent is $m = -3$

$y = x^2 - 5x + 4$

$\frac{dy}{dx} = 2x - 5$

$2x - 5 = -3$

$x = 1$

$x = 1 \Rightarrow y = 0$

The point is (1, 0)

21. Exercise: 7.3 – 8

Does there exist a differentiable function $f(x)$ such that $f(0) = -1; f(2) = 4$ and $f'(x) \leq 2$ for all x. Justify your answer

Solution:

$f'(x) = \frac{f(2)-f(0)}{2-0}$

$f'(x) = \frac{4+1}{2} = 2.5$

Does not exist since it is given that $f'(x) \leq 2$

22. Example: 7.30

Expand $\log(1 + x)$ as a Maclaurin's series upto 4 non-zero terms for $-1 < x \leq 1$

Solution:

$f(x) = \log(1 + x)$ $\quad f(0) = 0$

$f'(x) = \frac{1}{1+x}$ $\quad f'(0) = 1$

$f''(x) = -\frac{1}{(1+x)^2}$ $\quad f''(0) = -1$

$f'''(x) = \frac{2}{(1+x)^3}$ $\quad f'''(0) = 2$

$f^4(x) = \frac{-6}{(1+x)^4}$ $\quad f^4(0) = -6$

Substituting the values and on simplification

$$\log(1 + x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \cdots \ldots . -1 < x \leq 1$$

23. Example: 7.37

If $\lim_{\theta\to0}\left(\frac{1-cosm\theta}{1-cosn\theta}\right)=1$ then prove that $m=\pm n$

Solution:

$$\lim_{\theta\to0}\left(\frac{1-cosm\theta}{1-cosn\theta}\right)=\lim_{\theta\to0}\frac{m\,sinm\theta}{n\,sinn\theta}$$

$$=\lim_{\theta\to0}\frac{m}{n}\times\frac{\frac{sinm\theta}{\theta}}{\frac{sinn\theta}{\theta}}$$

$$=\frac{m^2}{n^2}$$

$$\lim_{\theta\to0}\left(\frac{1-cosm\theta}{1-cosn\theta}\right)=1$$

$$\frac{m^2}{n^2}=1$$

$$m^2=n^2$$

$$m=\pm n$$

24. Example: 7.40

Evaluate: $\lim_{x\to0^+}x\,logx$

Solution:

$$\lim_{x\to0^+}x\,logx=\lim_{x\to0^+}\left(\frac{\log x}{1/x}\right)=\left(\frac{\infty}{\infty}form\right)$$

$$=\lim_{x\to0^+}\left(\frac{\frac{1}{x}}{-\frac{1}{x^2}}\right)$$

$$=\lim_{x\to0}(-x)$$

$$=0$$

25. Exercise: 7.6-1(i)

Find the absolute extrema of $f(x)=x^2-12x+10$; $[1,2]$

Solution:

$$f(x)=x^2-12x+10$$

$$f'(x)=2x-12$$

$$f'(x)=0$$

$$x=6\notin[1,2]$$

$$f(1)=-1$$

$$f(2)=-10$$

Absolute maximum is -1

Absolute minimum is -10

26. Example: 7.52

Prove that the function $f(x)=x-sinx$ is increasing on the real line. Also discuss for the existence of local extrema.

Solution:

$$f'(x)=1-cosx\ge0$$

$$f'(x)=0$$

$$x=2n\pi,\qquad n\epsilon z$$

The function is increasing an the real line no sign change in $f'(x)$ first derivative best there is no local extrema

2 Marks:

27. Exercise: 7.1 -4:

If the volume of a cube of side length x is $\boldsymbol{v = x^3}$. Find the rate of change of the volume with respect to x when x = 5 units.

Solution:

$\frac{dv}{dx} = 3x^2$

$= 3(25)$

$\frac{dv}{dx} = 75$

28. Example: 7.16

Find the angle of intersection of the curve y = sinx with the positive x axis.

Solution:

$y = sinx$ $\quad\quad$ $y = 0$

$\frac{dy}{dx} = cosx$ $\quad\quad$ $x = n\pi$

$m_1 = slope\ at\ x = n\pi\ are\ cosn\pi = (-1)^n$ $\quad\quad$ $m_2 = slope\ of\ x\ axis$

$\tan\theta = \left|\frac{(-1)^n - 0}{1+(-1)^n(0)}\right| = 1 \quad \forall\, n$ $\quad\quad$ $m_2 = 0$

$\theta = \frac{\pi}{4}$

29. Exercise: 7.3 –1 (i)

Explain why Rolle's theorem is not applicable to the function $\boldsymbol{f(x) = \left|\frac{1}{x}\right| : x \in [-1, 1]}$

Solution:

$f(x)$ is not continuous at x= 0 in [-1, 1]

Do yourself:

Exercise: 7.1 – 1(ii)

Explain why Rolle's theorem is not applicable to the function $f(x) = tanx,\ x \in [0, \pi]$

30. Exercise: 7.3 – 3(i)

Explain why Lagrange's mean value theorem is not applicable to the function

$\boldsymbol{f(x) = \frac{x+1}{x};\ x \in [-1, 2]}$

Solution:

$f(x)$ is not continuous at $x = 0 \in [-1,2]$

Do yourself:

Exercise: 7.3 -3(ii)

Explain why Lagrange's mean value theorem is not applicable to the function

$f(x) = |3x + 1|; x \in [-1,3]$

31. Example 7.21

Compute the value of c satisfied by Rolle's theorem for the function $f(x) = \log\left(\frac{x^2+6}{5x}\right)$ in the interval [2, 3]

Solution:

$f(x)$ is continuous in [2, 3]

$f(x)$ is differentiable in (2, 3)

$f(2) = f(3) = 0$

$f'(x) = \frac{x^2-6}{x(x^2+6)}$

$f'(C) = 0$

$\frac{C^2-6}{C(C^2+6)} = 0$

$C = \pm\sqrt{6}$

$C = -\sqrt{6} \notin (2,3)$

$C = \sqrt{6}$ satisfies the Rolle's theorem

32. Exercise: 7.4 -1(1)

Write the Maclaurin series expansion of the function e^x

Solution:

$f(x) = e^x \qquad f(0) = 1$

$f'(x) = e^x \qquad f'(0) = 1$

$f''(x) = e^x \qquad f''(0) = 1$

$f(x) = f(0) + \frac{x}{1!}f'(0) + \frac{x^2}{2!}f''(0) + \cdots \ldots\ldots$

$e^x = 1 + \frac{x}{1!} + \frac{x^2}{2!} + \cdots \ldots\ldots$

Do yourself:

Exercise: 7.4 – 1(ii)

Write the Maclaurin series expansion of the function sinx

Exercise : 7.4 1(iii)

Write the Maclaurin series expansion of the function cosx

33. Example: 7.33

Evaluate : $\lim_{x \to 1}\left(\frac{x^2-3x+2}{x^2-4x+3}\right)$

Solution:

$\lim_{x \to 1}\left(\frac{x^2-3x+2}{x^2-4x+3}\right) = \lim_{x \to 1}\left(\frac{2x-3}{2x-4}\right)$

$= {}^1/_2$

34. Example 7.34

Compute the limit : $\lim_{x \to a}\left(\frac{x^n - a^n}{x-a}\right)$

Solution:

$\lim_{x \to a}\left(\frac{x^n - a^n}{x-a}\right) = \lim_{x \to a}\left(\frac{nx^{n-1}}{1}\right)$

$= na^{n-1}$

35. **Exercise: 7.5 – 2**

Evaluate: $\lim_{x \to \infty} \frac{2x^2-3}{x^2-5x+3}$

Solution:

$$\lim_{x \to \infty} \frac{2x^2-3}{x^2-5x+3} = \lim_{x \to \infty} \frac{2-\frac{3}{x^2}}{1-\frac{5}{x}+\frac{3}{x^2}}$$

$$= 2$$

Do yourself:

Exercise 7.5 -1 Evaluate: $\lim_{x \to 0} \left(\frac{1-\cos x}{x^2}\right)$

Exercise: 7.5 -3 Evaluate: $\lim_{x \to \infty} \frac{x}{\log x}$

Exercise: 7.5 -4 Evaluate: $\lim_{x \to \frac{\pi}{2}^-} \frac{\sec x}{\tan x}$

Exercise: 7.5 – 6 Evaluate: $\lim_{x \to 0} \left(\frac{1}{\sin x} - \frac{1}{x}\right)$

Exercise: 7.35

Evaluate the limit: $\lim_{x \to 0} \frac{\sin mx}{x}$

Exercise: 7.39

Evaluate: $\lim_{x \to 0^+} \left(\frac{1}{x} - \frac{1}{e^x-1}\right)$

36. **Example: 7.46**

Prove that the function $f(x) = x^2 + 2$ is strictly increasing in the interval (2,7) and strictly decreasing in the interval (-2, 0)

Solution:

$f'(x) = 2x > 0 \;; \quad \forall x \in (2,7)$

$f'(x) = 2x < 0 \;; \quad \forall x \in (-2,0)$

and hence the proof is completed.

37. **Example: 7.47:**

Prove that the function $f(x) = x^2 - 2x - 3$ is strictly increasing in $(2, \infty)$

Solution:

$f(x) = x^2 - 2x - 3$

$f'(x) = 2x - 2 > 0 \;; \quad \forall x \in (2, \infty)$

$f(x)$ is strictly increasing in $(2, \infty)$

38. **Example: 7.50**

Find the intervals of monotonicity and hence find the local extrema for the function $f(x) = x^2 - 4x + 4$

Solution:

$f(x) = = x^2 - 4x + 4$

$f(x) = (x-2)^2$

$f'(x) = 2(x-2) = 0$

$x = 2$

Interval	Sign of $f'(x)$	Monotonicity
$x \in (-\infty, 2)$	$-$	Strictly decreasing
$x \in (2, \infty)$	$+$	Strictly increasing

Local minimum at $x = 2$

Local minimum value is $f(2) = 0$

Chapter – 1
Applications of Matrices and Determinants

2 Mark and 3 Mark Question Answers:

1. **If $A = \begin{bmatrix} 8 & -4 \\ -5 & 3 \end{bmatrix}$ verify that $A(adj\ A) = (adj\ A)A = |A|I_2$**

Solution: $Sep - 2020,\ Mar - 2021,\ Sep - 2021$

$adj\ A = \begin{bmatrix} 3 & 4 \\ 5 & 8 \end{bmatrix}$

$|A| = 24 - 20 = 4$

$A\ (adj\ A) = \begin{bmatrix} 4 & 0 \\ 0 & 4 \end{bmatrix}$ …………………….. (1)

$(adj\ A)A = \begin{bmatrix} 4 & 0 \\ 0 & 4 \end{bmatrix}$ …………………….. (2)

$|A|I_2 = \begin{bmatrix} 4 & 0 \\ 0 & 4 \end{bmatrix}$ …………………….. (3)

From (1), (2) & (3)

$A\ (adj\ A) = (adj\ A)A = |A|I_2$

2. **Prove that $\begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}$ is orthogonal.**

Solution: March 2023

$A = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}$

$A^T = \begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix}$

$AA^T = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = I_2$ …………………….. (1)

$A^TA = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = I_2$ ………………….. (2)

From (1) & (2)

$AA^T = A^TA = I_2$

∴ A is orthogonal.

3. **Verify the property $(A^T)^{-1} = (A^{-1})^T$ with $A = \begin{bmatrix} 2 & 9 \\ 1 & 7 \end{bmatrix}$**

Solution: March 2020

$A = \begin{bmatrix} 2 & 9 \\ 1 & 7 \end{bmatrix}$

$A^T = \begin{bmatrix} 2 & 1 \\ 9 & 7 \end{bmatrix}$

$|A| = |A^T| = 14 - 9 = 5$

$A^{-1} = \frac{1}{|A|} adj\ A$

$(A^T)^{-1} = \frac{1}{|A^T|} adj\ A^T$

$(A^T)^{-1} = \frac{1}{5}\begin{bmatrix} 7 & -1 \\ -9 & 2 \end{bmatrix}$ ……………………….. (1)

$A^{-1} = \frac{1}{5}\begin{bmatrix} 7 & -9 \\ -1 & 2 \end{bmatrix}$

$(A^{-1})^T = \frac{1}{5}\begin{bmatrix} 7 & -1 \\ -9 & 2 \end{bmatrix}$ …………………….. (2)

From (1) & (2)

$(A^T)^{-1} = (A^{-1})^T$

4. **If** $\boldsymbol{adj\ (A)} = \begin{bmatrix} \mathbf{0} & \mathbf{-2} & \mathbf{0} \\ \mathbf{6} & \mathbf{2} & \mathbf{-6} \\ \mathbf{-3} & \mathbf{0} & \mathbf{6} \end{bmatrix}$**, find** $\boldsymbol{A^{-1}}$

Solution: *June – 23, Sep – 23, June 24*

$$A^{-1} = \pm \frac{1}{\sqrt{|adj\ A|}}\ adj\ (A)$$

$$|adj\ A| = 36$$

$$\sqrt{|adj\ A|} = \sqrt{36} = 6$$

$$A^{-1} = \pm \frac{1}{6}\begin{bmatrix} 0 & -2 & 0 \\ 6 & 2 & -6 \\ -3 & 0 & 6 \end{bmatrix}$$

5. **If** $\boldsymbol{adj\ (A)} = \begin{bmatrix} \mathbf{-1} & \mathbf{2} & \mathbf{2} \\ \mathbf{1} & \mathbf{1} & \mathbf{2} \\ \mathbf{2} & \mathbf{2} & \mathbf{1} \end{bmatrix}$**, find** $\boldsymbol{A^{-1}}$

Solution:

$$A^{-1} = \pm \frac{1}{\sqrt{|adj\ A|}}\ adj\ (A)$$

$$|adj\ A| = 9$$

$$\sqrt{|adj\ A|} = \sqrt{9} = 3$$

$$A^{-1} = \pm \frac{1}{3}\begin{bmatrix} -1 & 2 & 2 \\ 1 & 1 & 2 \\ 2 & 2 & 1 \end{bmatrix}$$

6. **If** $\boldsymbol{adj\ (A)} = \begin{bmatrix} \mathbf{2} & \mathbf{-4} & \mathbf{2} \\ \mathbf{-3} & \mathbf{12} & \mathbf{-7} \\ \mathbf{-2} & \mathbf{0} & \mathbf{2} \end{bmatrix}$**, find A**

Solution:

$$A = \pm \frac{1}{\sqrt{|adj\ A|}}\ adj\ (adj\ A)$$

$$|adj\ A| = 16$$

$$\sqrt{|adj\ A|} = \sqrt{16} = 4$$

$$adj\ (adj\ A) = \begin{bmatrix} 24 & 8 & 4 \\ 20 & 8 & 8 \\ 24 & 8 & 12 \end{bmatrix}$$

$$A = \pm \frac{1}{4}\begin{bmatrix} 24 & 8 & 4 \\ 20 & 8 & 8 \\ 24 & 8 & 12 \end{bmatrix}$$

$$A = \pm \begin{bmatrix} 6 & 2 & 1 \\ 5 & 2 & 2 \\ 6 & 2 & 3 \end{bmatrix}$$

7. **If** $adj\ (A) = \begin{bmatrix} 7 & 7 & -7 \\ -1 & 11 & 7 \\ 11 & 5 & 7 \end{bmatrix}$**, find A**

Solution:

$$A = \pm \frac{1}{\sqrt{|adj\ A|}}\ adj\ (adj\ A)$$

$$|adj\ A| = 1764$$

$$\sqrt{|adj\ A|} = \sqrt{1764} = 42$$

$$adj\ (adj\ A) = \begin{bmatrix} 42 & -84 & +126 \\ 84 & 126 & -42 \\ -126 & 42 & 84 \end{bmatrix}$$

$$A = \pm \frac{1}{42} \begin{bmatrix} 42 & -84 & +126 \\ 84 & 126 & -42 \\ -126 & 42 & 84 \end{bmatrix}$$

$$A = \pm \begin{bmatrix} 1 & -2 & 3 \\ 2 & 3 & -1 \\ -3 & 1 & 2 \end{bmatrix}$$

8. **If** $F(\alpha) = \begin{bmatrix} cos\alpha & 0 & sin\alpha \\ 0 & 1 & 0 \\ -sin\alpha & 0 & cos\alpha \end{bmatrix}$ **show that** $[F(\alpha)]^{-1} = F(-\alpha)$

Solution: March – 2023

To Prove: $F(\alpha)\ F(-\alpha) = I_3$

$$F(\alpha)\ F(-\alpha) = \begin{bmatrix} cos\alpha & 0 & sin\alpha \\ 0 & 1 & 0 \\ -sin\alpha & 0 & cos\alpha \end{bmatrix} \begin{bmatrix} \cos(-\alpha) & 0 & \sin(-\alpha) \\ 0 & 1 & 0 \\ -\sin(-\alpha) & 0 & \cos(-\alpha) \end{bmatrix}$$

$$= \begin{bmatrix} cos\alpha & 0 & sin\alpha \\ 0 & 1 & 0 \\ -sin\alpha & 0 & cos\alpha \end{bmatrix} \begin{bmatrix} cos\alpha & 0 & -sin\alpha \\ 0 & 1 & 0 \\ sin\alpha & 0 & cos\alpha \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = I_3$$

ஃ $[F(\alpha)]^{-1} = F(-\alpha)$

9. **Solve the linear equation using matrix Inversion method**

i) $5x + 2y = 3$ ***Mar – 2022*** ***Sep – 2023***

$3x + 2y = 5$

Solution: $A = \begin{bmatrix} 5 & 2 \\ 3 & 2 \end{bmatrix}$

$$|A| = \begin{vmatrix} 5 & 2 \\ 3 & 2 \end{vmatrix} = 10 - 6 = 4$$

$$adj\ A = \begin{bmatrix} 2 & -2 \\ -3 & 5 \end{bmatrix}$$

$$A^{-1} = \frac{1}{|A|}\ adj\ A$$

$$A^{-1} = \frac{1}{4} \begin{bmatrix} 2 & -2 \\ -3 & 5 \end{bmatrix}$$

$$X = A^{-1}B$$

$$X = \frac{1}{4} \begin{bmatrix} 2 & -2 \\ -3 & 5 \end{bmatrix} \begin{bmatrix} 3 \\ 5 \end{bmatrix} = \frac{1}{4} \begin{bmatrix} 6 - 10 \\ -9 + 25 \end{bmatrix} = \frac{1}{4} \begin{bmatrix} -4 \\ 16 \end{bmatrix}$$

$$\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} -1 \\ 4 \end{pmatrix}$$

$x = -1 \ , \qquad y = 4$

For Practice: (Solve the linear equation using matrix Inversion method)

$ii. \quad 2x - y = 8$

$3x + 2y = -2$

$iii. \ 2x + 5y = -2 \qquad June\ 2023$

$x + 2y = -3$

10. **Verify $(AB)^{-1} = B^{-1}A^{-1}$ with $A = \begin{bmatrix} 0 & -3 \\ 1 & 4 \end{bmatrix}$, $B = \begin{bmatrix} -2 & -3 \\ 0 & -1 \end{bmatrix}$**

Solution: $Sep - 2020 \qquad July - 2022$

$$AB = \begin{bmatrix} 0 & -3 \\ 1 & 4 \end{bmatrix} \times \begin{bmatrix} -2 & -3 \\ 0 & -1 \end{bmatrix} = \begin{bmatrix} 0 & 3 \\ -2 & -7 \end{bmatrix}$$

$$(AB)^{-1} = \frac{1}{|AB|} adj\ (AB)$$

$$(AB)^{-1} = \frac{1}{6}\begin{bmatrix} -7 & -3 \\ 2 & 0 \end{bmatrix} \qquad \ldots\ldots\ldots\ldots\ldots\ldots\ldots (1)$$

$$A^{-1} = \frac{1}{|A|} adj\ A$$

$$A^{-1} = \frac{1}{3}\begin{bmatrix} 4 & 3 \\ -1 & 0 \end{bmatrix}$$

$$B^{-1} = \frac{1}{|B|} adj\ B$$

$$B^{-1} = \frac{1}{2}\begin{bmatrix} -1 & 3 \\ 0 & -2 \end{bmatrix}$$

$$B^{-1}A^{-1} = \frac{1}{2}\begin{bmatrix} -1 & 3 \\ 0 & -2 \end{bmatrix} \frac{1}{3}\begin{bmatrix} 4 & 3 \\ -1 & 0 \end{bmatrix}$$

$$B^{-1}A^{-1} = \frac{1}{6}\begin{bmatrix} -7 & -3 \\ 2 & 0 \end{bmatrix} \qquad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (2)$$

From (1) & (2)

$(AB)^{-1} = B^{-1}A^{-1}$ verified.

(ii) For Practice: If $A = \begin{bmatrix} 3 & 2 \\ 7 & 5 \end{bmatrix}$ and $B = \begin{bmatrix} -1 & -3 \\ 5 & 2 \end{bmatrix}$ verify that $(AB)^{-1} = B^{-1}A^{-1}$

11. **If $A = \frac{1}{9}\begin{bmatrix} -8 & 1 & 4 \\ 4 & 4 & 7 \\ 1 & -8 & 4 \end{bmatrix}$, Prove that $A^{-1} = A^T$**

Solution:

$$A^T = \frac{1}{9}\begin{bmatrix} -8 & 4 & 1 \\ 1 & 4 & -8 \\ 4 & 7 & 4 \end{bmatrix}$$

$$AA^T = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = I$$

$$A^{-1} = A^T$$

12. **Find the rank of the matrix $\begin{bmatrix} 1 & -2 & 3 \\ 2 & 4 & -6 \\ 5 & 1 & -1 \end{bmatrix}$** **July – 2022**

Solution:

$$\begin{vmatrix} 1 & -2 & 3 \\ 2 & 4 & -6 \\ 5 & 1 & -1 \end{vmatrix} = 4 \neq 0$$

Rank of the matrix = 3.

13. Find the rank of the matrix $\begin{bmatrix} 0 & 1 & 2 & 1 \\ 0 & 2 & 4 & 3 \\ 8 & 1 & 0 & 2 \end{bmatrix}$ **Mar – 2021**

Solution:

$$\begin{vmatrix} 1 & 2 & 1 \\ 2 & 4 & 3 \\ 1 & 0 & 2 \end{vmatrix} = 1(8-0) - 2(4-3) + 1(0-4) = 1(8) - 2(1) + 1(-4) = 8 - 2 - 4 = 8 - 6 = 2$$

$$\begin{vmatrix} 1 & 2 & 1 \\ 2 & 4 & 3 \\ 1 & 0 & 2 \end{vmatrix} = 2 \neq 0$$

Rank of the matrix = 3.

14. Find the rank of the matrix $\begin{bmatrix} 1 & -2 & -1 \\ 3 & -1 & 2 \\ 1 & -2 & 3 \\ 1 & -1 & 1 \end{bmatrix}$ **March 2022**

Solution:

$$\begin{vmatrix} 1 & 2 & -1 \\ 3 & -1 & 2 \\ 1 & -2 & 3 \end{vmatrix} = -8 \neq 0$$

Rank of the matrix = 3

15. Find the rank of the matrix $\begin{bmatrix} -1 & 3 \\ 4 & -7 \\ 3 & -4 \end{bmatrix}$

Solution:

$$\begin{vmatrix} -1 & 3 \\ 4 & -7 \end{vmatrix} = 7 - 12 = -5 \neq 0$$

Rank = 2

16. Find the rank of the matrix $\begin{bmatrix} 1 & -2 & -1 & 0 \\ 3 & -6 & -3 & 1 \end{bmatrix}$

Solution:

$$\begin{vmatrix} -1 & 0 \\ -3 & 1 \end{vmatrix} = -1 \neq 0$$

Rank = 2

17. Find the rank of the matrix by row reduction method. (row – echelon form) $\begin{bmatrix} 1 & 2 & 3 \\ 2 & 1 & 4 \\ 3 & 0 & 5 \end{bmatrix}$

Solution:

$$A = \begin{bmatrix} 1 & 2 & 3 \\ 2 & 1 & 4 \\ 3 & 0 & 5 \end{bmatrix}$$

$$A \sim \begin{bmatrix} 1 & 2 & 3 \\ 0 & -3 & -2 \\ 0 & -6 & -4 \end{bmatrix} \begin{matrix} R_2 \to R_2 - 2R_1 \\ R_3 \to R_3 - 3R_1 \end{matrix}$$

$$\sim \begin{bmatrix} 1 & 2 & 3 \\ 0 & -3 & -2 \\ 0 & 0 & 0 \end{bmatrix} R_3 \to R_3 - 2R_2$$

$\rho(A) = 2$

For Practice: Find the rank of the following matrices

(i) $\begin{bmatrix} 3 & -8 & 5 & 2 \\ 2 & -5 & 1 & 4 \\ -1 & 2 & 3 & -2 \end{bmatrix}$ (ii) $\begin{bmatrix} 1 & 1 & 1 & 3 \\ 2 & -1 & 3 & 4 \\ 5 & -1 & 7 & 11 \end{bmatrix}$

(iii) $\begin{bmatrix} 2 & -2 & 4 & -3 \\ -3 & 4 & -2 & -1 \\ 6 & 2 & -1 & 7 \end{bmatrix}$

18. Solve by Cramer's rule

$$\mathbf{5x - 2y + 16 = 0, \quad x + 3y - 7 = 0}$$

Solution:

$$5x - 2y = -16$$

$$x + 3y = 7$$

$$\Delta = \begin{vmatrix} 5 & -2 \\ 1 & 3 \end{vmatrix} = 15 + 2 = 17$$

$$\Delta_1 = \begin{vmatrix} -16 & -2 \\ 7 & 3 \end{vmatrix} = -48 + 14 = -34$$

$$\Delta_2 = \begin{vmatrix} 5 & -16 \\ 1 & 7 \end{vmatrix} = 35 + 16 = 51$$

By Cramer's Rule

$$x = \frac{\Delta_1}{\Delta} = \frac{-34}{17} = -2$$

$$y = \frac{\Delta_2}{\Delta} = \frac{51}{17} = 3$$

Solution: $(x, y) = (-2, 3)$

For Practice: Solve by Cramer's rule

$$\frac{3}{x} + 2y = 12$$

$$\frac{2}{x} + 3y = 13$$

5 Mark question and answers:

19. Test the consistency of the following system of linear equations.

$$\mathbf{x - y + z = -9, 2x - y + z = 4, \quad 3x - y + z = 6, \quad 4x - y + 2z = 7}$$

Solution: March – 2020

$$\text{Aug mat. } [A\,|B] = \left[\begin{array}{ccc|c} 1 & -1 & 1 & -9 \\ 2 & -1 & 1 & 4 \\ 3 & -1 & 1 & 6 \\ 4 & -1 & 2 & 7 \end{array}\right]$$

$$\sim \left[\begin{array}{ccc|c} 1 & -1 & 1 & -9 \\ 0 & 1 & -1 & 22 \\ 0 & 2 & -2 & 33 \\ 0 & 3 & -2 & 43 \end{array}\right] \begin{array}{l} R_2 \to R_2 - 2R_1 \\ R_3 \to R_3 - 3R_1 \\ R_4 \to R_4 - 4R_1 \end{array}$$

$$\sim \left[\begin{array}{ccc|c} 1 & -1 & 1 & -9 \\ 0 & 1 & -1 & 22 \\ 0 & 0 & 0 & -11 \\ 0 & 0 & 1 & -23 \end{array}\right] \begin{array}{l} R_3 \to R_3 - 2R_2 \\ R_4 \to R_4 - 3R_2 \end{array}$$

$$\sim \left[\begin{array}{ccc|c} 1 & -1 & 1 & -9 \\ 0 & 1 & -1 & 22 \\ 0 & 0 & 1 & -23 \\ 0 & 0 & 0 & -11 \end{array}\right] R_3 \leftrightarrow R_4$$

$$\rho(A) \neq \rho(A/B)$$

System of equations is **inconsistent** and has **no solution.**

20. Test for consistency and if possible, solve the systems of equations by rank method,

$$2x + 2y + z = 5, \quad x - y + z = 1, \quad 3x + y + 2z = 4$$

Solution: Sep – 2020

$$\text{Aug Matrix } [A\,|B] = \begin{bmatrix} 2 & 2 & 1 & 5 \\ 1 & -1 & 1 & 1 \\ 3 & 1 & 2 & 4 \end{bmatrix}$$

$$\sim \left[\begin{array}{ccc|c} 1 & -1 & 1 & 1 \\ 2 & 2 & 1 & 5 \\ 3 & 1 & 2 & 4 \end{array}\right] R_1 \leftrightarrow R_2$$

$$\sim \left[\begin{array}{ccc|c} 1 & -1 & 1 & 1 \\ 0 & 4 & -1 & 3 \\ 0 & 4 & -1 & 1 \end{array}\right] \begin{array}{l} R_2 \to R_2 - 2R_1 \\ R_3 \to R_3 - 3R_1 \end{array}$$

$$\sim \left[\begin{array}{ccc|c} 1 & -1 & 1 & 1 \\ 0 & 4 & -1 & 3 \\ 0 & 0 & 0 & -2 \end{array}\right] R_3 \to R_3 - R_2$$

$\rho(A) \neq \rho(A/B)$

Short Cut

2	2	1	5
1	-1	1	1 (+)
3	1	2	6
			+4

Given system is inconsistent and has no solution.

For Practice:

$$2x - y + z = 2, \quad 6x - 3y + 3z = 6, \quad 4x - 2y + 2z = 4$$

Test for consistency, solve the system of equations by rank method.

21. Solve by Cramer's rule:

$$x - y + 2z = 2, \quad 2x + y + 4z = 7, \quad 4x - y + z = 4$$

Solution: March – 2021

$$\Delta = \begin{vmatrix} 1 & -1 & 2 \\ 2 & 1 & 4 \\ 4 & -1 & 1 \end{vmatrix} = -21 \neq 0$$

$$\Delta_1 = \begin{vmatrix} 2 & -1 & 2 \\ 7 & 1 & 4 \\ 4 & -1 & 1 \end{vmatrix} = -21$$

$$\Delta_2 = \begin{vmatrix} 1 & 2 & 2 \\ 2 & 7 & 4 \\ 4 & 4 & 1 \end{vmatrix} = -21$$

$$\Delta_3 = \begin{vmatrix} 1 & -1 & 2 \\ 2 & 1 & 7 \\ 4 & -1 & 4 \end{vmatrix} = -21$$

$x = \frac{\Delta_1}{\Delta}$ $\quad y = \frac{\Delta_2}{\Delta}$ $\quad z = \frac{\Delta_3}{\Delta}$

$x = \frac{-21}{-21}$ $\quad y = \frac{-21}{-21}$ $\quad z = \frac{-21}{-21}$

$x = 1$ $\quad y = 1$ $\quad z = 1$

$(x, y, z) = (1, 1, 1)$

22. Cramer's rule is not applicable to solve the system of equations $3x + y + z = 2$, $x - 3y + 2z = 1$, $7x - y + 4z = 5$. Why?

Solution: March 2022

$$\Delta = \begin{vmatrix} 3 & 1 & 1 \\ 1 & -3 & 2 \\ 7 & -1 & 4 \end{vmatrix} = 0$$

∴ Cramer's rule cannot be applied.

23. Solve by Cramer's rule:

$$3x + 3y - z = 11$$
$$2x - y + 2z = 9$$
$$4x + 3y + 2z = 25$$

Solution:

$$\Delta = \begin{vmatrix} 3 & 3 & -1 \\ 2 & -1 & 2 \\ 4 & 3 & 2 \end{vmatrix} = -22 \neq 0$$

$$\Delta_1 = \begin{vmatrix} 11 & 3 & -1 \\ 9 & -1 & 2 \\ 25 & 3 & 2 \end{vmatrix} = -44$$

$$\Delta_2 = \begin{vmatrix} 3 & 11 & -1 \\ 2 & 9 & 2 \\ 4 & 25 & 2 \end{vmatrix} = -66$$

$$\Delta_3 = \begin{vmatrix} 3 & 3 & 11 \\ 2 & -1 & 9 \\ 4 & 3 & 25 \end{vmatrix} = -88$$

by Cramer's rule

$$x = \frac{\Delta_1}{\Delta} = \frac{-44}{-22} = 2$$

$$y = \frac{\Delta_2}{\Delta} = \frac{-66}{-22} = 3$$

$$z = \frac{\Delta_3}{\Delta} = \frac{-88}{-22} = 4$$

$\therefore$ *Solution* $(x, y, z) = (2, 3, 4)$

24. A boy is walking along the path $y = ax^2 + bx + c$, through the points (-6, 8), (-2, -12) and (3, 8). He wants to meet his friend at P(7, 60). Will be meet his friend? (Use Gaussian elimination method)

Solution: March 2023

$$y = ax^2 + bx + c$$

$(-6, 8) \implies 36a - 6b + c = 8$ .... (1)

$(-2, -12) \implies 4a - 2b + c = -12$ .... (2)

$(3, 8) \implies 9a + 3b + c = 8$ .... (3)

Aug matrix $[A\,|B] = \left[\begin{array}{ccc|c} 36 & -6 & 1 & 8 \\ 4 & -2 & 1 & -12 \\ 9 & 3 & 1 & 8 \end{array}\right]$

$$\sim \left[\begin{array}{ccc|c} 36 & -6 & 1 & 8 \\ 0 & -12 & 8 & -116 \\ 0 & 18 & 3 & 24 \end{array}\right] \begin{array}{l} R_2 \to 9R_2 - R_1 \\ R_3 \to 4R_3 - R_1 \end{array}$$

$$\sim \left[\begin{array}{ccc|c} 36 & -6 & 1 & 8 \\ 0 & -3 & 2 & -29 \\ 0 & 6 & 1 & 8 \end{array}\right] \begin{array}{l} R_2 \to R_2/4 \\ R_3 \to R_3/3 \end{array}$$

$$\sim \left[\begin{array}{ccc|c} 36 & -6 & 1 & 8 \\ 0 & -3 & 2 & -29 \\ 0 & 0 & 5 & -50 \end{array}\right] R_3 \to R_3 + 2R_2$$

$(a, b, c) = (1, 3, -10)$

$\therefore y = x^2 + 3x - 10$

P(7, 60) satisfies the path, he can meet his friend at P(7, 60).

25. **Solve by Cramer's rule, the system of equations**

$\boldsymbol{x_1 - x_2 = 3,\ 2x_1 + 3x_2 + 4x_3 = 17,\ x_2 + 2x_3 = 7}$

Solution: *Sep* 2021, *June* 2023, *Sep* 2023

$$\Delta = \begin{vmatrix} 1 & -1 & 0 \\ 2 & 3 & 4 \\ 0 & 1 & 2 \end{vmatrix} = 6 \neq 0$$

$$\Delta_1 = \begin{vmatrix} 3 & -1 & 0 \\ 17 & 3 & 4 \\ 7 & 1 & 2 \end{vmatrix} = 12$$

$$\Delta_2 = \begin{vmatrix} 1 & 3 & 0 \\ 2 & 17 & 4 \\ 0 & 7 & 2 \end{vmatrix} = -6$$

$$\Delta_3 = \begin{vmatrix} 1 & -1 & 3 \\ 2 & 3 & 17 \\ 0 & 1 & 17 \end{vmatrix} = 24$$

by Cramer's rule,

$x_1 = \frac{\Delta_1}{\Delta} = \frac{12}{6}$, $x_1 = 2$

$x_2 = \frac{\Delta_2}{\Delta} = \frac{-6}{6}$, $x_2 = -1$

$x_3 = \frac{\Delta_3}{\Delta} = \frac{24}{6}$, $x_3 = 4$

Solution: $(x_1, x_2, x_3) = (2, -1, 4)$

26. **Solve by Cramer's rule:** **March – 2024**

$\boldsymbol{\frac{3}{x} - \frac{4}{y} - \frac{2}{z} - 1 = 0,\ \frac{1}{x} + \frac{2}{y} + \frac{1}{z} - 2 = 0,\ \frac{2}{x} - \frac{5}{y} - \frac{4}{z} + 1 = 0}$

Solution:

Let $\frac{1}{x} = a,\ \frac{1}{y} = b$ and $\frac{1}{z} = c$

$$3a - 4b - 2c = 1$$

$$a + 2b + c = 2$$

$$2a - 5b - 4c = -1$$

$$\Delta = \begin{vmatrix} 3 & -4 & -2 \\ 1 & 2 & 1 \\ 2 & -5 & -4 \end{vmatrix} = -15 \neq 0$$

$$\Delta_1 = \begin{vmatrix} 1 & -4 & -2 \\ 2 & 2 & 1 \\ -1 & -5 & -4 \end{vmatrix} = -15$$

$$\Delta_2 = \begin{vmatrix} 3 & 1 & -2 \\ 1 & 2 & 1 \\ 2 & -1 & -4 \end{vmatrix} = -5$$

$$\Delta_3 = \begin{vmatrix} 3 & -4 & 1 \\ 1 & 2 & 2 \\ 2 & -5 & -1 \end{vmatrix} = -5$$

$a = \frac{\Delta_1}{\Delta} = \frac{-15}{-15}$, $a = \frac{1}{1} = 1$

$b = \frac{\Delta_2}{\Delta} = \frac{-5}{-15}$, $b = \frac{1}{3}$

$c = \frac{\Delta_3}{\Delta} = \frac{-5}{-15}$, $c = \frac{1}{3}$

Solution: $(x, y, z) = (1, 3, 3)$

27. Solve by matrix inversion method:

$2x + 3y - z = 9,\ x + y + z = 9,\ \ 3x - y - z = -1$ ***June 2024***

Solution: $\begin{bmatrix} 2 & 3 & -1 \\ 1 & 1 & 1 \\ 3 & -1 & -1 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 9 \\ 9 \\ -1 \end{bmatrix}$

$A\,X = B$

$|A| = \begin{vmatrix} 2 & 3 & -1 \\ 1 & 1 & 1 \\ 3 & -1 & -1 \end{vmatrix} = 16 \neq 0$

$adj\ A = \begin{bmatrix} 0 & 4 & 4 \\ 4 & 1 & -3 \\ -4 & 11 & -1 \end{bmatrix}$

$A^{-1} = \frac{1}{|A|}\, adj\ A$

$A^{-1} = \frac{1}{16} \begin{bmatrix} 0 & 4 & 4 \\ 4 & 1 & -3 \\ -4 & 11 & -1 \end{bmatrix}$

$X = A^{-1}B$

$= \frac{1}{16} \begin{bmatrix} 0 & 4 & 4 \\ 4 & 1 & -3 \\ -4 & 11 & -1 \end{bmatrix} \begin{bmatrix} 9 \\ 9 \\ -1 \end{bmatrix}$

$= \frac{1}{16} \begin{pmatrix} 32 \\ 48 \\ 64 \end{pmatrix}$

$\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 2 \\ 3 \\ 4 \end{pmatrix}$

Solution: $(x, y, z) = (2, 3, 4)$

28. Investigate the values of λ *and* μ the system of linear equations $2x + 3y + 5z = 9$, $7x + 3y - 5z = 8$, $2x + 3y + \lambda z = \mu$ have

(i) no solution (ii) a unique solution (iii) an infinite number of solutions

Solution:

Aug matrix $[A\,|B] = \left[\begin{array}{ccc|c} 2 & 3 & 5 & 9 \\ 7 & 3 & -5 & 8 \\ 2 & 3 & \lambda & \mu \end{array}\right]$

$\sim \left[\begin{array}{ccc|c} 2 & 3 & 5 & 9 \\ 0 & -15 & -45 & -47 \\ 0 & 0 & \lambda - 5 & \mu - 9 \end{array}\right] \begin{matrix} R_2 \to 2R_2 - 7R_1 \\ R_3 \to R_3 - R_1 \end{matrix}$

(i) no solution:

$\lambda = 5,\ \ \mu \neq 9$

$\rho(A) \neq \rho(A/B)$

(ii) Unique solution:

$\lambda \neq 5,\ \ \mu \neq 9$

$\rho(A) = \rho(A/B) = 3$

(iii) Infinite number of solution:

$\lambda = 5,\ \ \mu = 9$

$\rho(A) = \rho(A/B) = 2 < 3$

29. **Find the value of K for which the equation $kx - 2y + z = 1$, $x - 2ky + z = -2$, $x - 2y + kz = 1$ have**

(i) no solution (ii) unique solution (iii) infinitely many solution

Solution:

$$\text{Aug Matrix } [A\,|B] = \left[\begin{array}{ccc|c} k & -2 & 1 & 1 \\ 1 & -2k & 1 & -2 \\ 1 & -2 & k & 1 \end{array}\right]$$

$$\sim \left[\begin{array}{ccc|c} 1 & -2 & k & 1 \\ 1 & -2k & 1 & -2 \\ k & -2 & 1 & 1 \end{array}\right] R_1 \leftrightarrow R_3$$

$$\sim \left[\begin{array}{ccc|c} 1 & -2 & k & 1 \\ 0 & -2k+2 & 1-k & -3 \\ 0 & -2+2k & 1-k^2 & 1-k \end{array}\right] \begin{array}{l} R_2 \to R_2 - R_1 \\ R_3 \to R_3 - kR_1 \end{array}$$

$$\sim \left[\begin{array}{ccc|c} 1 & -2 & k & 1 \\ 0 & -2k+2 & 1-k & -3 \\ 0 & 0 & (k-1)(k+2) & -(k+2) \end{array}\right] R_3 \to R_3 + R_2$$

(i) no solution:

$k = 1$

$\Longrightarrow P(A) \neq P(A/B)$

(ii) Unique solution:

$k \neq 1,\; k \neq -2$

$\Longrightarrow \rho(A) = \rho(A/B) = 3$

(iii) infinitely many solution

$k = -2$

$\Longrightarrow \rho(A) = \rho(A/B) = 2 < 3$

30. **Determine the values of λ for which the following system of equations $x + y + 3z = 0$, $4x + 3y + \lambda z = 0$, $2x + y + 2z = 0$ has**

(i) a unique solution (ii) a non-trivial solution

Solution: $$\text{Aug Matrix } [A\,|B] = \left[\begin{array}{ccc|c} 1 & 1 & 3 & 0 \\ 4 & 3 & \lambda & 0 \\ 2 & 1 & 2 & 0 \end{array}\right]$$

$$\sim \left[\begin{array}{ccc|c} 1 & 1 & 3 & 0 \\ 2 & 1 & 2 & 0 \\ 4 & 3 & \lambda & 0 \end{array}\right] R_2 \leftrightarrow R_3$$

$$\sim \left[\begin{array}{ccc|c} 1 & 1 & 3 & 0 \\ 0 & -1 & -4 & 0 \\ 0 & -1 & \lambda-12 & 0 \end{array}\right] \begin{array}{l} R_2 \to R_2 - 2R_1 \\ R_3 \to R_3 - 4R_1 \end{array}$$

$$\sim \left[\begin{array}{ccc|c} 1 & 1 & 3 & 0 \\ 0 & -1 & -4 & 0 \\ 0 & 0 & \lambda-8 & 0 \end{array}\right] R_3 \to R_3 - R_2$$

(i) Unique solution (trivial solution)

$\lambda \neq 8$

$\rho(A) = \rho(A/B) = 3$

(ii) a non-trivial solution:

$\lambda = 8$

$\rho(A) = \rho(A/B) = 2 < 3$

CHAPTER - 9

Applications of Integral Calculus

Important points:

- The curve y = f(x), Area lies above the x - axis A = $\int_a^b ydx$
- The curve y = f(x) Area lies below the x - axis A = $\int_a^b -ydx$
- The curve x = g(y), Area lies right of y - axis A= $\int_c^d xdy$
- The curve x = g(y), Area lies Left of y – axis A = $\int_c^d -xdy$
- Common area of the region bounded by the curves $y_u = f(x)$, $y_L = g(x)$ about x-axis A= $\int_a^b (y_u - y_L)dx$
- Common area of the region bounded by the curves $x_u = f(y)$, $x_L = g(y)$ about y-axis A = $\int_c^d (x_u - x_L)dy$
- If $f(-x) = -f(x)$ then f is an odd function
- If $f(-x) = f(x)$ then f is an even function
- Gamma Integral : $\int_0^\infty x^n e^{-ax}dx = \frac{n!}{a^{n+1}}$
- $\int_0^a f(x)dx = \int_0^a f(a-x)dx$
- $\int_a^b f(x)dx = \int_a^b f(a+b-x)dx$
- $\int_0^1 x^m(1-x)^n dx = \frac{m! \times n!}{(m+n+1)!}$

2 and 3 marks question

- **Reduction formulae:**

$$I_n = \int_0^{\frac{\pi}{2}} \sin^n xdx = \int_0^{\frac{\pi}{2}} \cos^n xdx = \begin{cases} \frac{n-1}{n} \cdot \frac{n-3}{n-2} \ldots\ldots \frac{2}{3}.1 ; & n \text{ is odd} \\ \frac{n-1}{n} \cdot \frac{n-3}{n-2} \ldots\ldots \frac{1}{2}.\frac{\pi}{2} ; & n \text{ is even} \end{cases}$$

1. **Evaluate:** $\int_0^{\frac{\pi}{2}} \sin^{10} xdx$

Solution:

n = 10, $I_{10} = \frac{9}{10}.\frac{7}{8}.\frac{5}{6}.\frac{3}{4}.\frac{1}{2}.\frac{\pi}{2} = \frac{63\pi}{512}$

2. **Evaluate:** $\int_0^{\frac{\pi}{2}} \cos^7 x dx$

Solution:

$n = 7, \quad I_7 = \frac{6}{7}.\frac{4}{5}.\frac{2}{3}.1 = \frac{16}{35}$

3. **Evaluate::** $\int_0^{\frac{\pi}{2}} (\sin^2 x + \cos^4 x) dx$

Solution:

$$I = \int_0^{\frac{\pi}{2}} \sin^2 x dx + \int_0^{\frac{\pi}{2}} \cos^4 x dx$$

$= I_2 + I_4$

$= \frac{1}{2}.\frac{\pi}{2} + \frac{3}{4}.\frac{1}{2}.\frac{\pi}{2} = \frac{\pi}{4} + \frac{3\pi}{16} = \frac{7\pi}{16}$

4. **Evaluate** $\int_0^{\frac{\pi}{2}} (\sin^2 x \times \cos^4 x) dx$

Solution:

$$I = \int_0^{\frac{\pi}{2}} (1 - \cos^2 x)\cos^4 x dx$$

$$= \int_0^{\frac{\pi}{2}} \cos^4 x dx - \int_0^{\frac{\pi}{2}} \cos^6 x dx$$

$= I_4 - I_6$

$= \frac{3}{4}.\frac{1}{2}.\frac{\pi}{2} - \frac{5}{6} . \frac{3}{4}.\frac{1}{2}.\frac{\pi}{2} = \frac{3\pi}{16}\left(1 - \frac{5}{6}\right) = \frac{\pi}{32}$

5. **Evaluate::** $\int_0^{\frac{\pi}{2}} \begin{vmatrix} \cos^4 x & 7 \\ \sin^5 x & 3 \end{vmatrix} dx$

Solution:

$$I = \int_0^{\frac{\pi}{2}} (3\cos^4 x - 7\sin^5 x) dx$$

$$= 3\int_0^{\frac{\pi}{2}} \cos^4 x dx - 7\int_0^{\frac{\pi}{2}} \sin^5 x dx$$

$= 3I_4 - 7I_5$

$= 3\left(\frac{3}{4}.\frac{1}{2}.\frac{\pi}{2}\right) - 7\left(\frac{4}{5}.\frac{2}{3}.1\right) = \frac{9\pi}{16} - \frac{56}{15}$

6. Evaluate: $\int_0^{\frac{\pi}{4}} \sin^6(2x)dx$

x	0	$\pi/4$
t	0	$\pi/2$

Solution:

$Let\ \ t = 2x,\ \frac{dt}{dx} = 2, \frac{dt}{2} = dx$

$I = \int_0^{\frac{\pi}{2}} \sin^6 t.\frac{dt}{2} = \frac{1}{2}\int_0^{\frac{\pi}{2}} \sin^6 t.dt = \frac{1}{2} I_6$

$I = \frac{1}{2}\left(\frac{5}{6} \cdot \frac{3}{4} \cdot \frac{1}{2} \cdot \frac{\pi}{2}\right) = \frac{5\pi}{64}$

7. Evaluate: $\int_0^{\frac{\pi}{6}} \sin^5(3x)dx$

x	0	$\pi/6$
t	0	$\pi/2$

Solution:

$Let\ \ t = 3x,\ \frac{dt}{dx} = 3, \frac{dt}{3} = dx$

$I = \int_0^{\frac{\pi}{2}} \sin^5 t.\frac{dt}{3} = \frac{1}{3}\int_0^{\frac{\pi}{2}} \sin^5 t.dt = \frac{1}{3} I_5$

$I = \frac{1}{3}\left(\frac{4}{5} \cdot \frac{2}{3} \cdot 1\right) = \frac{8}{45}$

8. Evaluate: $\int_0^{2\pi} \sin^7 \frac{x}{4}dx$

x	0	2π
t	0	$\pi/2$

Solution:

$Let\ \ t = \frac{x}{4},\ \frac{dt}{dx} = \frac{1}{4}, 4dt = dx$

$I = \int_0^{\frac{\pi}{2}} \sin^7 t(4dt) = 4\int_0^{\frac{\pi}{2}} \sin^7 t.dt = 4\, I_7$

$I = 4\left[\frac{6}{7} \cdot \frac{4}{5} \cdot \frac{2}{3} \cdot 1\right] = \frac{64}{35}$

❖ Gamma Integration: $\int_0^{\infty} x^n e^{-ax}dx = \frac{n!}{a^{n+1}}$, a > 0

9. Evaluate: $\int_0^{\infty} x^5 e^{-3x}dx$

Solution: n = 5, a = 3

$I = \frac{5\,!}{3^{5+1}} = \frac{5\,!}{3^6}$

10. Evaluate $\int_0^{\infty} x^3 e^{-\alpha x^2} dx =$ **32, α > 0 Find α**

Solution:: $I = \int_0^{\infty} x^2 e^{-\alpha x^2}(xdx) = 32$

x	0	∞
t	0	∞

$t = x^2, \frac{dt}{dx} = 2x, \frac{dt}{2} = x\,dx$

$I = \int_0^{\infty} t.e^{-\alpha t}.\frac{dt}{2} = 32 \Rightarrow \frac{1}{2}\int_0^{\infty} t^1 e^{-\alpha t}\,dt = 32$

$\int_0^{\infty} t^1 e^{-\alpha t}\,dt = 64$ (By Gamma Integral)

$\frac{1!}{\alpha^{1+1}} = 64 \Rightarrow \alpha^2 = \frac{1}{64} \Rightarrow \alpha = \frac{1}{8} (\because \alpha > 0)$

❖ $\int_0^1 x^m (1-x)^n dx = \frac{m! \times n!}{(m+n+1)!}$ (Where m and n – are positive integers)

11. Evaluate: $\int_0^1 x^3 (1-x)^4 dx$

Solution: m= 3 , n = 4 m + n + 1 = 8

$I = \frac{3! \times 4!}{8!} = \frac{6 \times 4!}{(8.7.6.5)\times 4!} = \frac{1}{8.7.5} = \frac{1}{280}$

12. Evaluate: $\int_0^1 x^2 (1-x)^3 dx$

Solution: m = 2, n = 3, m + n + 1 = 6

$I = \frac{2! \times 3!}{6!} = \frac{2 \times 6}{720} = \frac{12}{720} = \frac{1}{60}$

❖ Properties of Integral:

i. $\int_0^a f(x)dx = \int_0^a f(a-x)dx$ ii. $\int_a^b f(x)dx = \int_a^b f(a+b-x)dx$

13. Evaluate: $\int_0^a \frac{f(x)}{f(x)+f(a-x)} dx$

Solution $I = \int_0^a \frac{f(x)}{f(x)+f(a-x)} dx$ …………… (1)

By property, $\int_0^a f(x)dx = \int_0^a f(a-x)dx$

$I = \int_0^a \frac{f(a-x)}{f(a-x)+f(x)} dx$ …………………… (2)

x↔ a – x

(1) + (2) ⇒ $2I = \int_0^a \frac{f(x)+f(a-x)}{f(x)+f(a-x)} dx$

$2I = \int_0^a dx = [x]_0^a = a$

$I = \frac{a}{2}$

14. Evaluate: $\int_{2}^{3} \frac{\sqrt{x}}{\sqrt{5-x}+\sqrt{x}}\,dx$

Solution $\quad I = \int_{2}^{3} \frac{\sqrt{x}}{\sqrt{5-x}+\sqrt{x}}\,dx$ .... (1)

By property $\int_{a}^{b} f(x)dx = \int_{a}^{b} f(a+b-x)dx$

$I = \int_{2}^{3} \frac{\sqrt{5-x}}{\sqrt{x}+\sqrt{5-x}}\,dx$ .... (2) $\quad x \leftrightarrow 5 - x$

$(1) + (2) \Rightarrow 2I = \int_{2}^{3} \frac{\sqrt{x}+\sqrt{5-x}}{\sqrt{x}+\sqrt{5-x}}\,dx$

$2I = \int_{2}^{3} dx = (x)_2^{\,3} = 3 - 2 = 1$

$I = \frac{1}{2}$

15. Evaluate: $\int_{\pi/8}^{3\pi/8} \frac{1}{1+\sqrt{\tan x}}\,dx$

Solution $\quad I = \int_{\pi/8}^{3\pi/8} \frac{1}{1+\frac{\sqrt{\sin x}}{\sqrt{\cos x}}}$

$I = \int_{\pi/8}^{3\pi/8} \frac{\sqrt{\cos x}}{\sqrt{\cos x}+\sqrt{\sin x}}dx$ .... (1)

By property, $\int_{a}^{b} f(x)dx = \int_{a}^{b} f(a+b-x)dx$

$I = \int_{\pi/8}^{3\pi/8} \frac{\sqrt{\sin x}}{\sqrt{\sin x}+\sqrt{\cos x}}dx$ .... (2)

sin x ↔ cos x

$(1) + (2) \Rightarrow 2I = \int_{\pi/8}^{3\pi/8} \frac{\sqrt{\cos x}+\sqrt{\sin x}}{\sqrt{\cos x}+\sqrt{\sin x}}dx$

$= \int_{\pi/8}^{3\pi/8} dx$

$= (x)_{\pi/8}^{3\pi/8}$

$2I = \frac{3\pi}{8} - \frac{\pi}{8} = \frac{2\pi}{8}$

$2I = \frac{2\pi}{8}, \qquad I = \frac{\pi}{8}$

❖ $$\int_{-a}^{a} f(x)dx = \begin{cases} 0 & ; f \text{ is an odd function} \\ 2\int_{0}^{a} f(x)dx & ; f \text{ is an even function} \end{cases}$$

16. Evaluate $\int_{-\frac{\pi}{4}}^{\frac{\pi}{4}} \sin^2 x dx$

Solution: $f(x) = sin^2 x = (\sin x)^2$

$f(-x) = [\sin(-x)]^2 = (-\sin x)^2 = sin^2 x = f(x)$

f(-x) = f(x) / fis an even function

$$I = 2\int_{0}^{\frac{\pi}{4}} \sin^2 x dx = 2\int_{0}^{\frac{\pi}{4}} \frac{(1-\cos 2x)}{2} dx$$

$$I = \left[x - \frac{\sin 2x}{2}\right]_0^{\pi/4} = \left(\frac{\pi}{4} - \frac{1}{2}\right) - 0 = \frac{\pi - 2}{4}$$

17. Evaluate: $\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} x \cos x dx$

Solution: $f(x) = x\, cosx$ $\quad f(-x) = (-x)\, cos\,(-x) = -xcos\, x$

$/ f(-x) = -f(x)$

$f(x)$ is an odd function

$$I = \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} x \cos x dx$$

18. Evaluate: $\int_{3}^{4} \frac{dx}{x^2 - 4}$

Solution: $I = \int_3^4 \frac{dx}{x^2 - 2^2}$ $\quad \because \int \frac{dx}{x^2 - a^2} = \frac{1}{2a}\log\left(\frac{x-a}{x+a}\right)$

$I = \left[\frac{1}{2(2)}\log\left(\frac{x-2}{x+2}\right)\right]_3^4$ $\quad$ a = 2

$= \frac{1}{4}\left[\log\frac{2}{6} - \log\frac{1}{5}\right]$

$= \frac{1}{4}\left[\log\left(\frac{\frac{1}{3}}{\frac{1}{5}}\right)\right]$

$= \frac{1}{4}\log\left(\frac{5}{3}\right)$

19. Evaluate: $\int_0^{\frac{\pi}{3}} \frac{\sec x \tan x}{1+sec^2 x}\, dx$

x	0	$\pi/3$
t	1	2

Solution: $t = \sec x$, $\frac{dt}{dx} = \sec x\, tanx,$ dt = secx tan dx

$I = \int_1^2 \frac{dt}{1+t^2}$ $\quad\because \int \frac{dx}{1+x^2} = tan^{-1}x$

$=[tan^{-1}(t)]_1^{\,2} = tan^{-1}(2) - tan^{-1}(1)$

$= tan^{-1}(2) - \frac{\pi}{4}$

20. Evaluate: $\int_0^9 \frac{1}{x+\sqrt{x}}\, dx$

Solution: $\sqrt{x} = t$, x = t^2, $\frac{dt}{dx} = 2t, dx = 2tdt$

x	0	9
t	0	3

$I = \int_0^3 \frac{1}{t^2+t}(2t\, dt) = \int_0^3 \frac{dt}{1+t} = 2\,[\log(1+t)]_0^{\,3}$

I = 2[log 4 – log 1] = 2 log 4 – 0 = log 4^2 = log 16

21. Evaluate: $\int_1^2 \frac{x}{(x+1)(x+2)}\, dx$

Solution: $I = \int_1^2 \frac{x}{(x+1)(x+2)}\, dx$

By partial Fractions, $\frac{x}{(x+1)(x+2)} = \frac{A}{x+1} + \frac{B}{x+2} = \frac{-1}{x+1} + \frac{2}{x+2}$

$I = \int_1^2 \left(\frac{-1}{x+1} + \frac{2}{x+2}\right) dx$

$=[-\log(x+1) + 2\log(x+2)]_1^2$

$=[\log(x+2)^2 - \log(x+1)]_1^2$

$=\left[\log \frac{(x+2)^2}{(x+1)}\right]_1^2$

$=\log\left(\frac{16}{3}\right) - \log\left(\frac{9}{2}\right)$

$=\log\left(\frac{16/3}{9/2}\right) \quad = log\left(\frac{32}{27}\right)$

22. Evaluate: $\int_{-4}^{4} |x+3|dx$

Solution: $|x+3| = \begin{cases} -(x+3) & ; \quad x < -3 \\ x+3 & ; \quad x \geq -3 \end{cases}$

$$I = \int_{-4}^{-3} -(x+3)dx + \int_{-3}^{4} (x+3)dx$$

$$= -\left[\frac{x^2}{2} + 3x\right]_{-4}^{-3} + \left[\frac{x^2}{2} + 3x\right]_{-3}^{4}$$

$$= -\left[\left(\frac{9}{2} - 9\right) - (8 - 12)\right] + \left[(8+12) - \left(\frac{9}{2} - 9\right)\right]$$

$$= -\left[-\frac{9}{2} + 4\right] + \left[20 + \frac{9}{2}\right]$$

$$= \frac{9}{2} - 4 + 20 + \frac{9}{2} = 16 + 9 = 25$$

- If n is odd and m is any positive integer (even or odd)
- $\int_0^{\frac{\pi}{2}} \sin^m x \cos^n x \, dx = \frac{n-1}{m+n} \cdot \frac{n-3}{m+n-2} \cdot \frac{n-5}{m+n-4} \cdots \cdot \frac{2}{m+3} \cdot \frac{1}{m+1}$
- $\int_0^{\frac{\pi}{2}} \sin^m x \cos^n x \, dx = \int_0^{\frac{\pi}{2}} \sin^n x \cos^m x \, dx$

$$I = \frac{m-1}{n+m} \cdot \frac{m-3}{n+m-2} \cdot \frac{m-5}{n+m-2} \cdots \cdot \frac{2}{n+3} \cdot \frac{1}{n+1}$$

- If n or m any one odd

23. Evaluate: $\int_0^{\frac{\pi}{2}} \sin^3 x \cos^5 x \, dx$

Solution: Here m = 3, n = 5 , m + n = 8

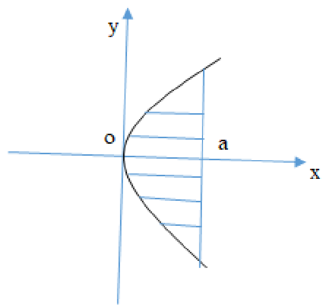
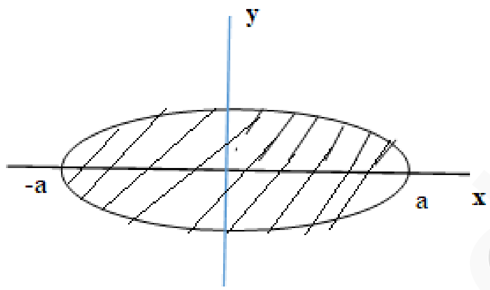
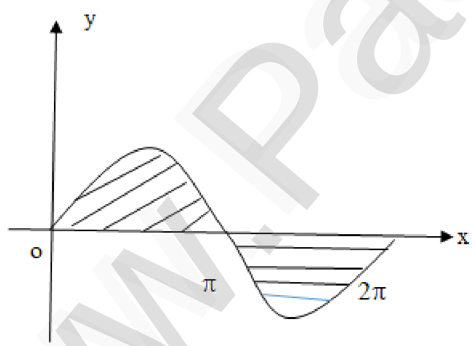
$$I = \frac{4}{8} \cdot \frac{2}{6} \cdot \frac{1}{4} = \frac{1}{24}$$

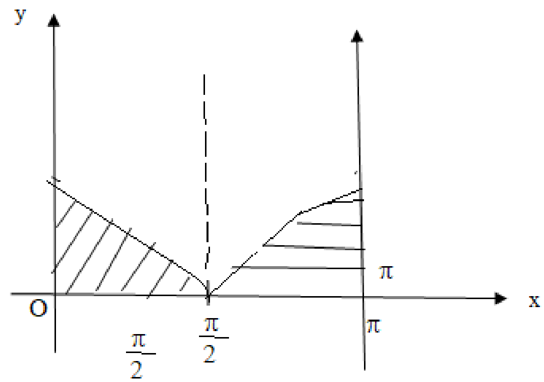
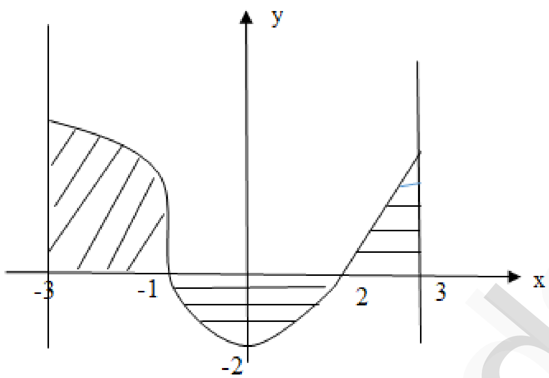
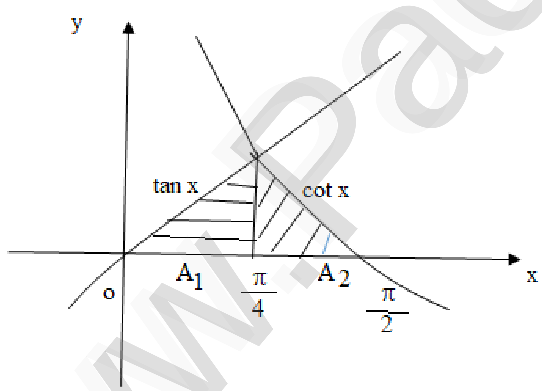
24. Evaluate: $\int_0^{\frac{\pi}{2}} \sin^5 x \cos^4 x \, dx = \int_0^{\frac{\pi}{2}} \sin^4 x \cos^5 x \, dx$

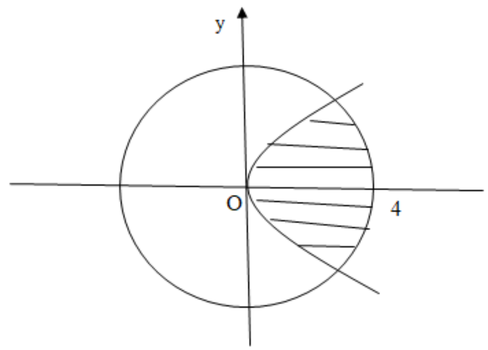
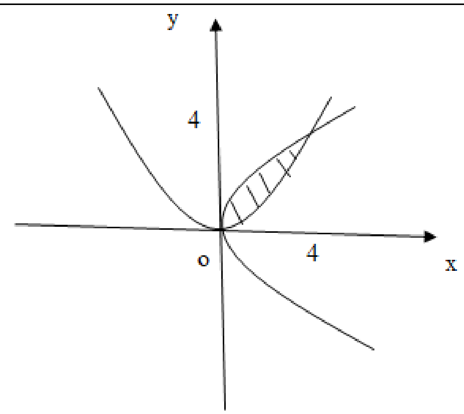
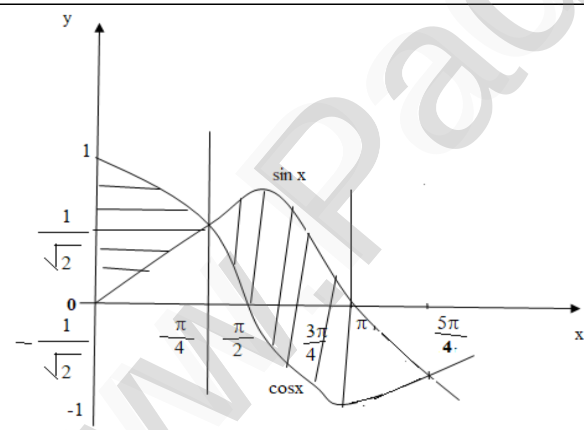
Solution: $m = 5, n = 4, m + n = 9$

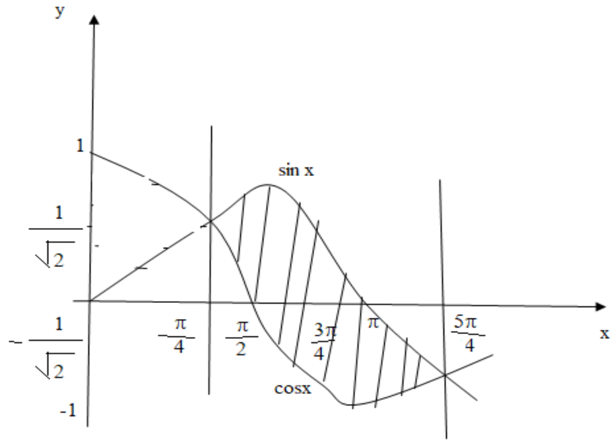
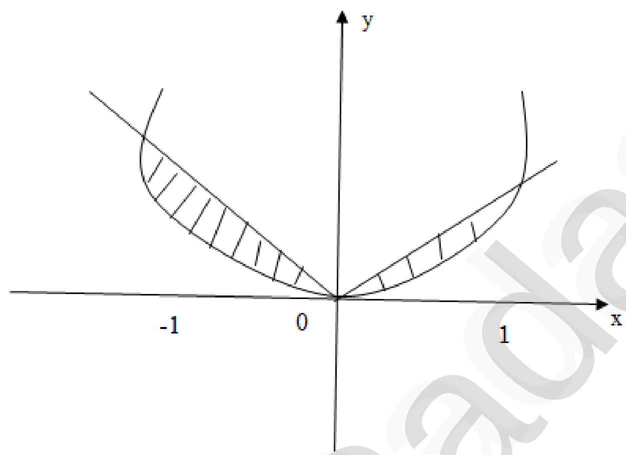
$$I = \frac{4}{9} \cdot \frac{2}{7} \cdot \frac{1}{5} = \frac{8}{315}$$

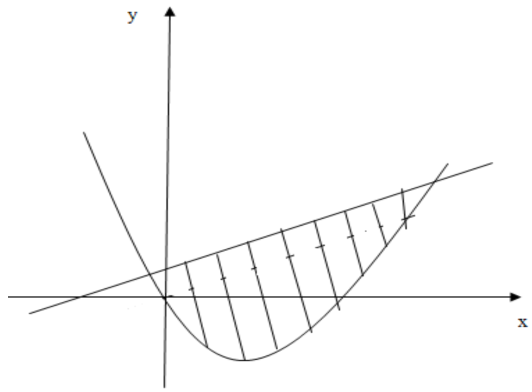
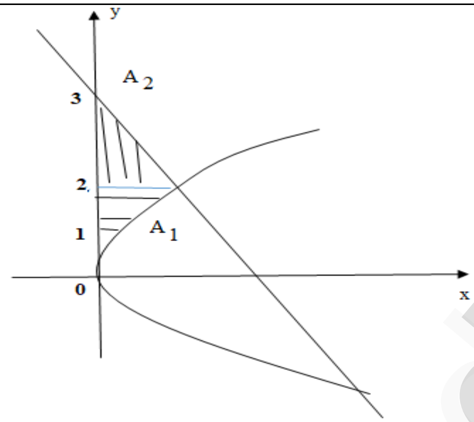
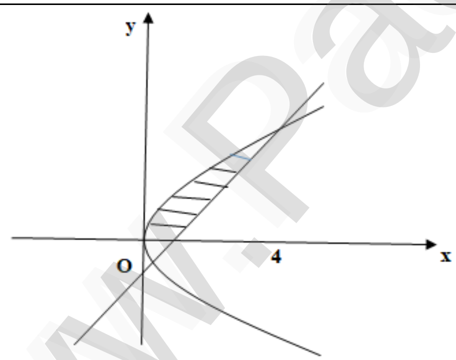
Five marks questions

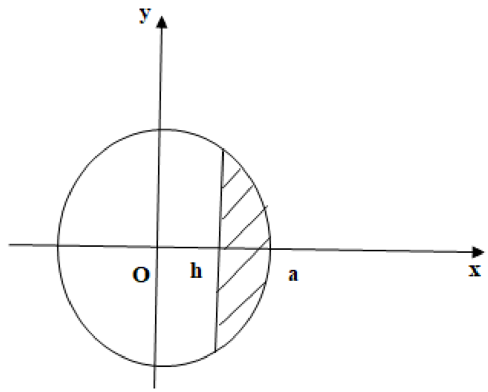
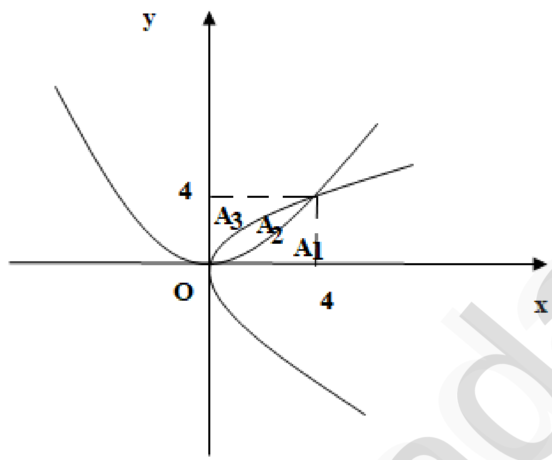
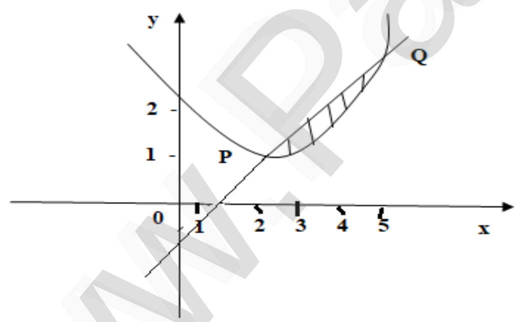
	Equation of the Lines /curves	Diagram	Formula	Area
1	Eqn.of parabola $y^2 = 4ax$ eqn.of Latus rectum; $x = a$		$A = \int_a^b y\,dx$	$\frac{8}{3}a^2$
2	Eqn. of the Ellipse: $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$		$A = \int_a^b y\,dx$	πab
3	Eqn. of the curve $y = \sin x$		$A = A_1 + A_2$ $A = \int_a^b y\,dx + \int_b^c -y\,dx$	4

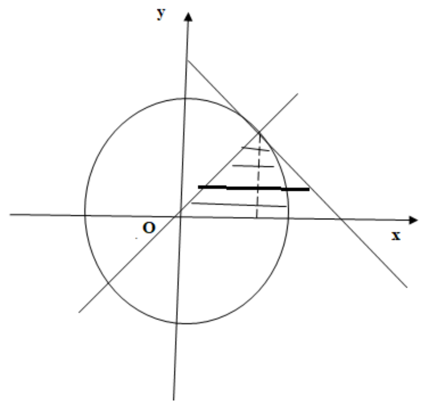
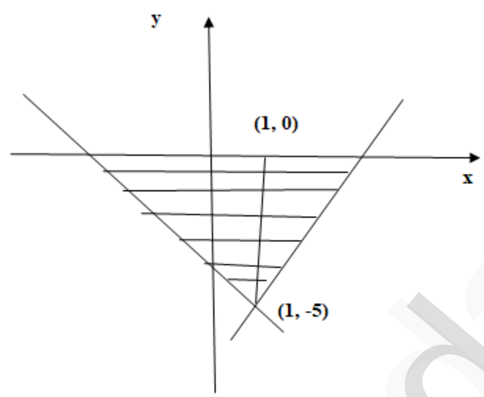
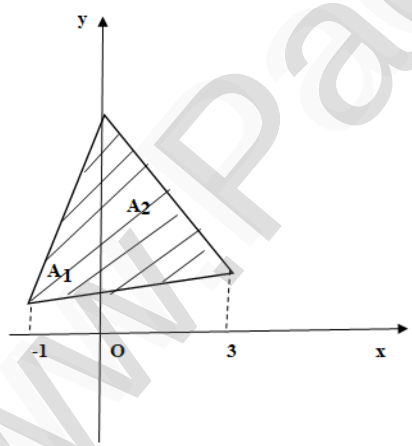
4	Eqn. of the curve $y = \|\cos x\|$		$A = A_1 + A_2$ $A = \int_a^b y\,dx$ $A = \int_0^{\frac{\pi}{2}} \cos x\,dx + \int_{\frac{\pi}{2}}^{\pi} -\cos x\;dx$	2
5	Eqn. of the curve $2+x-x^2+y=0$ $y = x^2 - x - 2$		$A = A_1 + A_2 + A_3$ $A = \int_{-3}^{-1} y\,dx + \int_{-1}^{2} -y\,dx + \int_{2}^{3} y\,dx$	15
6	Eqn. of the curves y = tan x y = cot x		$A = A_1 + A_2$ $A = \int_a^b y\,dx + \int_b^c y\,dx$ $A = \int_0^{\frac{\pi}{4}} \tan x\,dx + \int_{\frac{\pi}{4}}^{\frac{\pi}{2}} cotx\,dx$	*Log* 2

7	Eqn. of the curves $x^2 + y^2 = 16$ $y^2 = 16x$		$A = \int_c^d (x_u - x_L)dy$	$\frac{4}{3}(4\pi + \sqrt{3})$
8	Eqn. of the curves $y^2 = 4x$ $x^2 = 4y$		$A = \int_a^b (y_u - y_L)dx$	$\frac{16}{3}$
9	Eqn. of the curves $y = \sin x$ $y = \cos x$ $x = 0, and\ x = \pi$ About x-axis		$A = \int_a^b (y_u - y_L)dx$	$2\sqrt{2}$

10	Eqn. of the curves y = sinx x = cos x $x = \frac{\pi}{4}$ and $x = \frac{5\pi}{4}$ about x-axis		$A = \int_a^b (y_u - y_L)dx$	$2\sqrt{2}$
11	Eqn. of the curve $x^2 = y$ Eqn. of the lines $y = \|x\|$		$A = \int_a^b (y_u - y_L)dx$	$\frac{1}{3}$

12	Eqn. of the line $y = x^2 - 2x + 5$ Eqn. of the curve $y = x^2 - 2x$		$A = \int_a^b (y_u - y_L)dx$	36
13	Eqn. of the line x +y =3 Eqn. of the curve $y^2 = 4x$		$A = A_1 + A_2$ $A = \int_0^2 xdy + \int_2^3 xdy$	$\frac{7}{6}$
14	Eqn. of the line y = x-2 Eqn. of the curve $y^2 = x$		$A = \int_c^d (x_u - x_L)dy$	$\frac{9}{2}$

15	Eqn. of the circle $x^2 + y^2 = 4$ Eqn. of the line x = h		$A = 2\int_a^b y dx$	$a^2 cos^{-1}\left(\frac{h}{a}\right)$ $- h\sqrt{a^2 - h^2}$
16	Eqn. of the curves $y^2 = 4x$, $x^2 = 4y$ x = 0, x = 4 and y = 0, y = 4		$A = A_1 + A_2 + A_3$ $A = \int_a^b y\, dx + \int_a^b (y_u - y_L)dx + \int_a^b xdy$	$A_1 = \frac{16}{3}$ $A_2 = \frac{16}{3}$ $A_3 = \frac{16}{3}$
17	Eqn. of the curve $y = (x-2)^2 + 1$ Slope of PQ = 2		$A = \int_a^b (y_u - y_L)dx$	$\frac{4}{3}$

18	Tangent and normal to the circle $x^2 + y^2 = 4$ drawn at $(1, \sqrt{3})$		$A = \int_c^d (x_u - x_L)dy$	$2\sqrt{3}$
19	Eqn. of the lines $5x - 2y = 15$ $x + y = -4$		$A = \int_c^d (x_u - x_L)dy$	$\frac{35}{2}$
20	The triangle ABC formed by the vertices (-1, 1) (3, 2) and (0, 5)		$A = A_1 + A_2 - A_3$ $A = \int_a^b y\,dx + \int_b^c ydx - \int_a^c ydx$	$\frac{15}{2}$