If A is nonsingular, then prove that $|A^{-1}| = \frac{1}{|A|}$!

proof:-

Let A be non-singular. Then $|A| \neq 0$ and A^{-1} exists.

$$AA^{-1} = A^{-1}A = I_n$$

$$|AA^{-1}| = |A^{-1}A| = |I_n|$$

$$|A||A^{-1}| = 1 \Rightarrow |A^{-1}| = \frac{1}{|A|}.$$

If A is non singlular, then prove that $(A^T)^{-1} = (A^{-1})^T$

proof:-

Let A be non-singular. Then $|A| \neq 0$ and A^{-1} exists.

$$AA^{-1} = A^{-1}A = I_n$$

$$(AA^{-1})^T = (A^{-1}A)^T = (I_n)^T$$

$$(A^{-1})^T A^T = A^T (A^{-1})^T = I_n$$

$$\Rightarrow (A^T)^{-1} = (A^{-1})^T$$

If A is non singular, then prove that $(\lambda A)^{-1} = \frac{1}{\lambda} A^{-1}$ where λ is non-zero number.

proof:-

Let A be non-singular. Then $|A| \neq 0$ and A^{-1} exists.

$$AA^{-1} = A^{-1}A = I_n$$

$$(\lambda A)\left(\frac{1}{\lambda} A^{-1}\right) = \left(\frac{1}{\lambda} A^{-1}\right)(\lambda A) = I_n$$

$$\Rightarrow (\lambda A)^{-1} = \frac{1}{\lambda} A^{-1}$$

State and prove Left Cancellation Law:-

Let A, B and C be square matrices of order n. If A is non-singular and $AB = AC$, then $B = C$

proof:-

Let A be non-singular. Then $|A| \neq 0$ and A^{-1} exists

$$AA^{-1} = A^{-1}A = I_n$$

Taking $AB = AC$

$$A^{-1}(AB) = A^{-1}(AC)$$

$$(A^{-1}A)B = (A^{-1}A)C$$

$$IB = IC$$

$$\boxed{B = C}$$

State and Prove Right Cancellation Law:-

Let A, B and C be square matrices of order n. If A is non-singular and $BA = CA$, then $B = C$

proof:-

Let A be non-singular. Then $|A| \neq 0$ and A^{-1} exists.

$$AA^{-1} = A^{-1}A = I_n$$

Taking $BA = CA$

$$(BA)A^{-1} = (CA)A^{-1}$$

$$B(AA^{-1}) = C(AA^{-1})$$

$$BI = CI$$

$$\boxed{B = C}$$

State and prove Reversal Law for Inverses:-

If A and B are non singular matrices of the same order, then the product AB is also non singular and $(AB)^{-1} = B^{-1}A^{-1}$

Proof:-

Assume that A and B are non singular matrices of same order 'n'. Then $|A| \neq 0, |B| \neq 0$ both A^{-1} and B^{-1} exist.

$$|AB| = |A||B| \neq 0$$

So, AB is non singular and

$$(AB)(B^{-1}A^{-1}) = (A(BB^{-1}))A^{-1}$$

$$= (AI_n)A^{-1}$$

$$= AA^{-1} = I_n$$

$$(B^{-1}A^{-1})(AB) = (B^{-1}(A^{-1}A))B$$

$$= (B^{-1}I_n)B$$

$$= B^{-1}B = I_n$$

Hence $(AB)^{-1} = B^{-1}A^{-1}$

State and Prove Law of Double Inverse:-

If A is non singular, then A^{-1} is also nonsingular and $(A^{-1})^{-1} = A$.

Proof:-

Assume that A is non-singular, Then $|A| \neq 0$ and A^{-1} exists.

$$AA^{-1} = A^{-1}A = I_n$$

$$(AA^{-1})^{-1} = (I_n)^{-1}$$

$$(A^{-1})^{-1}A^{-1} = I_n$$

$$(A^{-1})^{-1}(A^{-1}A) = I_nA$$

$$(A^{-1})^{-1}I_n = I_nA$$

$$(A^{-1})^{-1} = A$$

If A and B are any two non-singular square matrices of order n, then

$$\operatorname{adj}(AB) = (\operatorname{adj}B)(\operatorname{adj}A)$$

Proof:-

$$A^{-1} = \frac{1}{|A|}(\operatorname{adj}A)$$

$$\operatorname{adj}A = |A|A^{-1}$$

Replacing A by AB.

$$\operatorname{adj}(AB) = |AB|(AB)^{-1}$$

$$= |A||B|(B^{-1}A^{-1})$$

$$= (|B|B^{-1})(|A|A^{-1})$$

$$\operatorname{adj}(AB) = (\operatorname{adj}B)(\operatorname{adj}A)$$

For every square matrix A of order n, $A(\operatorname{adj}A) = (\operatorname{adj}A)A = |A|I_n$

proof:-

we prove the theorem for n=3

$$A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$$

$$A(\operatorname{adj}A) = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix} \begin{bmatrix} A_{11} & A_{21} & A_{31} \\ A_{12} & A_{22} & A_{32} \\ A_{13} & A_{23} & A_{33} \end{bmatrix}$$

$$= \begin{bmatrix} |A| & 0 & 0 \\ 0 & |A| & 0 \\ 0 & 0 & |A| \end{bmatrix} = |A| \begin{bmatrix}1&0&0\\0&1&0\\0&0&1\end{bmatrix} = |A| I_3 \rightarrow (1)$$

$$(adj A) A = \begin{bmatrix} A_{11}&A_{21}&A_{31}\\A_{12}&A_{22}&A_{32}\\A_{13}&A_{23}&A_{33}\end{bmatrix}\begin{bmatrix}a_{11}&a_{12}&a_{13}\\a_{21}&a_{22}&a_{23}\\a_{31}&a_{32}&a_{33}\end{bmatrix}$$

$$= \begin{bmatrix} |A| & 0 & 0 \\ 0 & |A| & 0 \\ 0 & 0 & |A| \end{bmatrix} = |A| \begin{bmatrix}1&0&0\\0&1&0\\0&0&1\end{bmatrix} = |A| I_3 \rightarrow (2)$$

from (1), (2)

$A(adj A) = (adj A)A = |A| I_3$.

If a square matrix of order n. Then A^{-1} exists if and only if A is non-singular.

proof:-

Suppose that A^{-1} exists. Then

$AA^{-1} = A^{-1}A = I_n$

$|AA^{-1}| = |A||A^{-1}| = |A^{-1}||A| = |I_n| = 1$

So, $|A| \neq 0$. Hence A is non-singular.

Conversely,

A is non-singular, then $|A| \neq 0$

$A(adj A) = (adj A)A = |A| I_n$

$A\left(\frac{1}{|A|}(adj A)\right) = \left(\frac{1}{|A|}(adj A)\right)A = I_n$

Hence A^{-1} exists and $A^{-1} = \frac{1}{|A|} adj A$

If a square matrix has an inverse, then it is unique.

Proof:-

Let A be a square matrix of order n such that A^{-1} exists.

Let A has two inverses, say, B and C.

Let B is inverse of A.

$AB = BA = I_n \rightarrow (1)$

Let C is inverse of A.

$AC = CA = I_n \rightarrow (2)$

$B = BI_n$

$= B(AC)$

$= (BA)C$

$= I_nC$

$\boxed{B = C}$

Hence the uniqueness proved.

Prove that $|z| = |\bar z|$

If $z = x+iy$ then $\bar z = x - iy$.

$|z| = |x+iy| = \sqrt{x^2+y^2} \rightarrow (1)$

$|\bar z| = |x - iy| = \sqrt{x^2+(-y)^2}$

$= \sqrt{x^2+y^2} \rightarrow (2)$

from (1) & (2)

$|z| = |\bar z|$

Prove that $|z_1z_2| = |z_1||z_2|$

$|z_1 z_2|^2 = (z_1z_2)(\overline{z_1z_2})$

$= (z_1)(z_2)(\bar z_1)(\bar z_2)$

$= (z_1\bar z_1)(z_2\bar z_2)$

$|z_1z_2|^2 = |z_1|^2|z_2|^2$

$\Rightarrow |z_1z_2| = |z_1||z_2|$

Prove that $\left|\frac{z_1}{z_2}\right| = \frac{|z_1|}{|z_2|}, z_2\neq 0$

$z_1 = \frac{z_1}{z_2}\cdot z_2, \quad z_2 \neq 0$

$|z_1| = \left|\frac{z_1}{z_2}\cdot z_2\right|$

$|z_1| = \left|\frac{z_1}{z_2}\right||z_2|$ (Property)

$\frac{|z_1|}{|z_2|} = \left|\frac{z_1}{z_2}\right| \Rightarrow \left|\frac{z_1}{z_2}\right| = \frac{|z_1|}{|z_2|}$

Prove that $|z^n| = |z|^n$

$|z^n| = |z\cdot z \cdots z|$

$= |z||z|\cdots|z|$ (n times)

$|z^n| = |z|^n$

Prove that $Re(z) \le |z|$

If $z = x+iy$ and $y\neq0, x\neq0$

$Re(z) = x$

$|z| = \sqrt{x^2+y^2}$

$x < \sqrt{x^2+y^2} \Rightarrow Re(z) < |z| \rightarrow (1)$

If $z = x+iy$ and $y=0, x\neq 0$

$Re(z) = x$

$|z| = \sqrt{x^2} = x$

$x = x \Rightarrow Re(z) = |z| \rightarrow (2)$

from (1) & (2)

$Re(z) \le |z|$

Prove that $Im(z) \le |z|$

If $z = x+iy$ and $x\neq0, y\neq0$

$Im(z) = y$

$|z| = \sqrt{x^2+y^2}$

$y < \sqrt{x^2+y^2} \Rightarrow Im(z) < |z| \rightarrow (1)$

If $z = x+iy$ and $x=0, y\neq0$

$Im(z) = y$

$|z| = \sqrt{y^2} = y$

$y = y \Rightarrow Im(z) = |z| \rightarrow (2)$

from (1), (2)

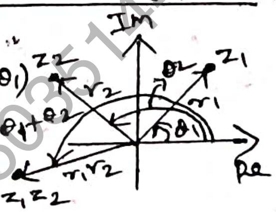
$Im(z) \le |z|$

If z_1 and z_2 are two complex numbers, (i) $\arg(z_1z_2) = \arg z_1 + \arg z_2$

proof:-

$z_1 = r_1(\cos\theta_1 + i\sin\theta_1)$

$z_2 = r_2(\cos\theta_2 + i\sin\theta_2)$



$\Rightarrow$

$z_1z_2 = r_1(\cos\theta_1+i\sin\theta_1)\, r_2(\cos\theta_2+i\sin\theta_2)$

$= r_1r_2[(\cos\theta_1\cos\theta_2 - \sin\theta_1\sin\theta_2) + i(\sin\theta_1\cos\theta_2+\cos\theta_1\sin\theta_2)]$

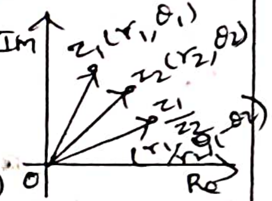
$z_1z_2 = r_1r_2(\cos(\theta_1+\theta_2)+i\sin(\theta_1+\theta_2))$

$\arg(z_1z_2) = \theta_1+\theta_2 = \arg z_1 + \arg z_2$

(ii) $\arg\left(\frac{z_1}{z_2}\right) = \arg z_1 - \arg z_2$

$z_1 = r_1(\cos\theta_1+i\sin\theta_1)$

$z_2 = r_2(\cos\theta_2 + i\sin\theta_2)$



$\frac{z_1}{z_2} = \frac{r_1(\cos\theta_1+i\sin\theta_1)}{r_2(\cos\theta_2+i\sin\theta_2)}$

$= \frac{r_1}{r_2}\frac{(\cos\theta_1+i\sin\theta_1)}{(\cos\theta_2+i\sin\theta_2)} \times \frac{(\cos\theta_2 - i\sin\theta_2)}{(\cos\theta_2-i\sin\theta_2)}$

$= \frac{r_1}{r_2}\frac{(\cos\theta_1\cos\theta_2+\sin\theta_1\sin\theta_2)+i(\sin\theta_1\cos\theta_2-\cos\theta_1\sin\theta_2)}{\cos^2\theta+\sin^2\theta}$

$\frac{z_1}{z_2} = \frac{r_1}{r_2}(\cos(\theta_1-\theta_2)+i\sin(\theta_1-\theta_2))$

$\arg\left(\frac{z_1}{z_2}\right) = \theta_1-\theta_2 = \arg z_1 - \arg z_2$

State and prove Jacobi's Identity

For any three vectors $\vec a, \vec b, \vec c$ we have $\vec a\times(\vec b\times\vec c) + \vec b\times(\vec c\times\vec a) + \vec c\times(\vec a\times\vec b) = \vec 0$

Proof:-

$\vec a\times(\vec b\times\vec c) = (\vec a\cdot\vec c)\vec b - (\vec a\cdot\vec b)\vec c$

$\vec b\times(\vec c\times\vec a) = (\vec b\cdot\vec a)\vec c - (\vec b\cdot\vec c)\vec a$

$\vec c\times(\vec a\times\vec b) = (\vec c\cdot\vec b)\vec a - (\vec c\cdot\vec a)\vec b$

Adding the above equations and using commutative property, we get

$\vec a \times(\vec b\times\vec c)+\vec b\times(\vec c\times\vec a)+\vec c\times(\vec a\times\vec b) = \vec 0$

State and prove Lagrange's identity.

For any four vectors $\vec a,\vec b,\vec c,\vec d$, we have $(\vec a\times\vec b)\cdot(\vec c\times\vec d) = \begin{vmatrix}\vec a\cdot\vec c & \vec a\cdot\vec d\\ \vec b\cdot\vec c & \vec b\cdot\vec d\end{vmatrix}$

Proof: $(\vec a\times\vec b)\cdot(\vec c\times\vec d) = \vec a\cdot(\vec b\times(\vec c\times\vec d))$

$= \vec a\cdot((\vec b\cdot\vec d)\vec c - (\vec b\cdot\vec c)\vec d)$

$= (\vec a\cdot\vec c)(\vec b\cdot\vec d) - (\vec a\cdot\vec d)(\vec b\cdot\vec c)$

$= \begin{vmatrix}\vec a\cdot\vec c & \vec a\cdot\vec d\\ \vec b\cdot\vec c & \vec b\cdot\vec d\end{vmatrix}$

State Intermediate Value theorem?

If f is continuous in the closed interval $[a,b]$ and $f(a) \le k \le f(b)$, then there exists atleast one $c \in [a,b]$ such that $f(c)=k$.

State Rolle's theorem?

If a function $f(x)$ is

i) continuous in the closed interval $[a,b]$

ii) differentiable in the open interval (a,b) and

iii) $f(a)=f(b)$

then there exist atleast one $c \in (a,b)$ such that $f'(c)=0$

State Lagrange's Mean value theorem?

If a function $f(x)$ is

i) continuous in the closed interval $[a,b]$

ii) differentiable in the open interval (a,b)

then there exists atleast one $c \in (a,b)$ such that

$$f'(c) = \frac{f(b)-f(a)}{b-a}$$

State Exterem value theorem?

If $f(x)$ is continuous on a closed interval $[a,b]$, then f has both an absolute maximum and an absolute minimum (extreme values) in the interval $[a,b]$.

State Fermat theorem?

If $f(x)$ has a relative extrema $x=c$ then c is a critical number.

Invariably there will be critical number of the function obtained as solution of the equation $f'(x)=0$ or values of x which $f'(x)$ does not exist.

Prove that $E(ax+b)=aE(X)+b$?

Let X be the discrete random variable.

$$E(ax+b) = \sum_{i=1}^{\infty} (ax_i+b) f(x_i)$$

$$= \sum_{i=1}^{\infty} a x_i f(x_i) + \sum_{i=1}^{\infty} b f(x_i)$$

$$= a \sum_{i=1}^{\infty} x_i f(x_i) + b \sum_{i=1}^{\infty} f(x_i)$$

$$= aE(X) + b(1) = aE(X)+b$$

Prove that $var(ax+b) = a^2 var(x)$?

$$var(ax+b) = E((ax+b) - E(ax+b))^2$$

$$= E(ax+\cancel{b} - aE(x) - \cancel{b})^2$$

$$= E(ax - aE(x))^2$$

$$= E(a^2 (x-E(x))^2)$$

$$= a^2 E(x-E(x))^2$$

$$= a^2 var(x)$$

Prove that $\int_a^b f(x)dx = \int_a^b f(a+b-x)dx$

Let $u = a+b-x \mid x = a+b-u$

$\frac{du}{dx} = -1 \Rightarrow dx = -du$

$x=a \Rightarrow u=b \mid x=b \Rightarrow u=a$

$$\int_a^b f(x)dx = \int_b^a f(a+b-u)(-du)$$

$$= \int_a^b f(a+b-u)du = \int_a^b f(a+b-x)dx$$

Prove that $\int_0^{2a} f(x)dx = \int_0^a [f(x)+f(2a-x)]dx$

$$\int_0^{2a} f(x)dx = \int_0^a f(x)dx + \int_a^{2a} f(x)dx.$$

Put $x=2a-u$ in $\int_a^{2a} f(x)dx$

$\frac{dx}{du} = -1 \Rightarrow dx = -du$

$x=a \Rightarrow u=a \mid x=2a \Rightarrow u=0$

$$\int_a^{2a} f(x)dx = \int_a^0 f(2a-u)(-du)$$

$$= \int_0^a f(2a-u)du = \int_0^a f(2a-x)dx$$

$$\therefore \int_0^{2a} f(x)dx = \int_0^a f(x)dx + \int_0^a f(2a-x)dx$$

$$= \int_0^a [f(x)+f(2a-x)]dx.$$

If $f(x)$ is an even function, then prove that $\int_{-a}^a f(x)dx = 2\int_0^a f(x)dx$

$$\int_{-a}^a f(x)dx = \int_{-a}^0 f(x)dx + \int_0^a f(x)dx$$

Put $x=-u$ in $\int_{-a}^0 f(x)dx$

$\frac{dx}{du} = -1 \Rightarrow dx = -du$

$x=-a \Rightarrow u=a \mid x=0 \Rightarrow u=0$

$$\int_{-a}^0 f(x)dx = \int_a^0 f(-u)(-du)$$

$$= \int_0^a f(-u)du = \int_0^a f(-x)dx$$

$$= \int_0^a f(x)dx \quad [\because f(-x)=f(x)]$$

$$\int_{-a}^a f(x)dx = \int_0^a f(x)dx + \int_0^a f(x)dx$$

$$= 2\int_0^a f(x)dx$$

If $f(x)$ is odd function, then prove that

$\int_{-a}^{a} f(x)dx = 0$

$\int_{-a}^{a} f(x)dx = \int_{-a}^{0} f(x)dx + \int_{0}^{a} f(x)dx$

put $x = -u$ in $\int_{-a}^{0} f(x)dx$

$\frac{dx}{du} = -1 \Rightarrow dx = -du$

$x = -a \Rightarrow u = a \mid x = 0 \Rightarrow u = 0$

$\int_{-a}^{0} f(x)dx = \int_{a}^{0} f(-u)(-du)$

$= \int_{0}^{a} f(-u)du = \int_{0}^{a} f(-x)dx$

$= -\int_{0}^{a} f(x)dx \quad [\because f(-x) = -f(x)]$

$\therefore \int_{-a}^{a} f(x)dx = -\int_{0}^{a} f(x)dx + \int_{0}^{a} f(x)dx$

$= 0.$

If $f(2a-x) = f(x)$, then prove that

$\int_{0}^{2a} f(x)dx = 2\int_{0}^{a} f(x)dx.$

W.K.T.

$\int_{0}^{2a} f(x)dx = \int_{0}^{a} [f(x) + f(2a-x)]dx$

If $f(2a-x) = f(x)$,

$= \int_{0}^{a} [f(x) + f(x)]dx$

$= 2\int_{0}^{a} f(x)dx.$

If $f(2a-x) = -f(x)$, then prove that

$\int_{0}^{2a} f(x)dx = 0.$

W.K.T.

$\int_{0}^{2a} f(x)dx = \int_{0}^{a} [f(x) + f(2a-x)]dx$

If $f(2a-x) = -f(x)$

$= \int_{0}^{a} [f(x) - f(x)]dx$

$= \int_{0}^{a} (0)dx = 0.$

Prove that $Var(X) = E(X^2) - (E(X))^2$

W.K.T. $E(X) = \mu$

$Var(X) = E(X-\mu)^2$

$= E(X^2 - 2X\mu + \mu^2)$

$= E(X^2) - 2\mu E(X) + \mu^2$

$= E(X^2) - 2\mu\mu + \mu^2$

$= E(X^2) - 2\mu^2 + \mu^2$

$= E(X^2) - \mu^2$

$= E(X^2) - (E(X))^2$

State and prove uniqueness of Identity?

In an algebraic structure the identity element (if exists) must be unique.

proof:-

Let $(S,*)$ be an algebraic structure.

Assume that e_1 and e_2 be any two identity elements $\in S$.

Treat 'e_1' as identity element

$e_1 * e_2 = e_2 * e_1 = e_2 \rightarrow (1)$

Treat 'e_2' as identity element.

$e_1 * e_2 = e_2 * e_1 = e_1 \rightarrow (2)$

From (1), (2)

$e_1 = e_2$

Hence the identity element is unique.

State and prove uniqueness of Inverse?

In an algebraic structure the inverse of an element (if exists) must be unique.

proof:-

Let $(S,*)$ be an algebraic structure and $a \in S$.

Assume that a has two inverses, say, a_1, a_2.

Treat 'a_1' as an inverse of 'a'.

$a * a_1 = a_1 * a = e \rightarrow (1)$

Treat 'a_2' as an inverse of 'a'.

$a * a_2 = a_2 * a = e \rightarrow (2)$

$a_1 = a_1 * e$

$= a_1 * (a * a_2)$

$= (a_1 * a) * a_2$

$= e * a_2$

$= a_2$

$a_1 = a_2$

Hence the inverse of 'a' is unique.

P. NESAMANI. M.Sc., B.Ed.,
PG ASST IN MATHS.
'NESA JEE MATHS'
9750351441.