Part – IV
(5 Mark Questions)

EXERCISE 1.1

3. If $F(\alpha)=\begin{bmatrix}\cos\alpha & 0 & \sin\alpha\\ 0 & 1 & 0\\ -\sin\alpha & 0 & \cos\alpha\end{bmatrix}$, show that $[F(\alpha)]^{-1}=F(-\alpha)$.

14. If $A=\begin{bmatrix}0 & 1 & 1\\ 1 & 0 & 1\\ 1 & 1 & 0\end{bmatrix}$, show that $A^{-1}=\frac{1}{2}\left(A^2-3I\right)$.

15. Decrypt the received encoded message $\begin{bmatrix}2 & -3\end{bmatrix}\begin{bmatrix}20 & 4\end{bmatrix}$ with the encryption matr $\begin{bmatrix}-1 & -1\\ 2 & 1\end{bmatrix}$ and the decryption matrix as its inverse, where the system of codes a described by the numbers 1-26 to the letters $A-Z$ respectively, and the number 0 to blank space.

EXERCISE 1.2

3. Find the inverse of each of the following by Gauss-Jordan method: (each 5 marks)

(i) $\begin{bmatrix}2 & -1\\ 5 & -2\end{bmatrix}$ (ii) $\begin{bmatrix}1 & -1 & 0\\ 1 & 0 & -1\\ 6 & -2 & -3\end{bmatrix}$ (iii) $\begin{bmatrix}1 & 2 & 3\\ 2 & 5 & 3\\ 1 & 0 & 8\end{bmatrix}$

EXERCISE 1.3

1. Solve the following system of linear equations by matrix inversion method:

(iii) $2x+3y-z=9,\ x+y+z=9,\ 3x-y-z=-1$

(iv) $x+y+z-2=0,\ 6x-4y+5z-31=0,\ 5x+2y+2z=13$

2. If $A=\begin{bmatrix}-5 & 1 & 3\\ 7 & 1 & -5\\ 1 & -1 & 1\end{bmatrix}$ and $B=\begin{bmatrix}1 & 1 & 2\\ 3 & 2 & 1\\ 2 & 1 & 3\end{bmatrix}$, find the products AB and BA and hence solve the system of equations $x+y+2z=1, 3x+2y+z=7, 2x+y+3z=2$.

3. A man is appointed in a job with a monthly salary of certain amount and a fixed amount of annual increment. If his salary was ₹.19,800 per month at the end of the first month after 3 years of service and ₹.23,400 per month at the end of the first month after 9 years of service, find his starting salary and his annual increment. (Use matrix inversion method to solve the problem.)

4. 4 men and 4 women can finish a piece of work jointly in 3 days while 2 men and 5 women can finish the same work jointly in 4 days. Find the time taken by one man alone and that of one woman alone to finish the same work by using matrix inversion method.

5. The price of three commodities A, B and C are ₹.x, y and z per units respectively. A person P purchases 4 units of B and sells two units of A and 5 units of C. Person Q purchases 2 units of C and sells 3 units of A and one unit of B. Person R purchases one unit of A and sells 3 unit of B and one unit of C. In the process, P, Q and R earn ₹.15,000, ₹.1,000 and ₹.4,000 respectively. Find the prices per unit of A, B and C. (Use matrix inversion method to solve the problem.)

EXERCISE 1.4

1. Solve the following systems of linear equations by Cramer's rule:

   (iii) $3x+3y-z=11,\ 2x-y+2z=9,\ 4x+3y+2z=25$

   (iv) $\frac{3}{x}-\frac{4}{y}-\frac{2}{z}-1=0, \frac{1}{x}+\frac{2}{y}+\frac{1}{z}-2=0, \frac{2}{x}-\frac{5}{y}-\frac{4}{z}+1=0$

3. A chemist has one solution which is 50% acid and another solution which is 25% acid. How much each should be mixed to make 10 litres of a 40% acid solution ? (Use Cramer's rule to solve the problem).

4. A fish tank can be filled in 10 minutes using both pumps A and B simultaneously. However, pump B can pump water in or out at the same rate. If pump B is inadvertently run in reverse, then the tank will be filled in 30 minutes. How long would it take each pump to fill the tank by itself ?

5. A family of 3 people went out for dinner in a restaurant. The cost of two dosai, three idlies and two vadais is ₹.150. The cost of the two dosai, two idlies and four vadais is ₹.200. The cost of five dosai, four idlies and two vadais is ₹.250. The family has ₹. 350 in hand and they ate 3 dosai and six idlies and six vadais. Will they be able to manage to pay the bill within the amount they had ?

EXERCISE 1.5

1. Solve the following systems of linear equations by Gaussian elimination method:

   (i) $2x-2y+3z=2,\ \ x+2y-z=3,\ \ 3x-y+2z=1$

   (ii) $2x+4y+6z=22,\ \ 3x+8y+5z=27,\ \ -x+y+2z=2$

2. If ax^2+bx+c is divided by $x+3, x-5$, and $x-1$, the remainders are $21, 61$ and 9 respectively. Find a, b and c. (Use Gaussian elimination method.)

3. An amount of ₹. 65,000 is invested in three bonds at the rates of 6%, 8% and 9% per annum respectively. The total annual income is ₹.4,800. The income from the third bond is ₹.600 more than that from the second bond. Determine the price of each bond. (Use Gaussian elimination method.)

4. A boy is walking along the path $y=ax^2+bx+c$ through the points $(-6,8), (-2,-12)$, and $(3,8)$. He wants to meet his friend at $P(7,60)$. Will he meet his friend? (Use Gaussian elimination method.)

EXERCISE 1.6

1. Test for consistency and if possible, solve the following systems of equations by ra
method.
   (i) $x-y+2z=2,\quad 2x+y+4z=7,\quad 4x-y+z=4$
   (ii) $3x+y+z=2,\quad x-3y+2z=1,\quad 7x-y+4z=5$
   (iv) $2x-y+z=2,\quad 6x-3y+3z=6,\quad 4x-2y+2z=4$
2. Find the value of k for which the equations $kx-2y+z=1,\quad x-2ky+z=-2,\quad x-2y+kz=$
have
   (i) no solution (ii) unique solution (iii) infinitely many solution
3. Investigate the values of λ and μ the system of linear equations $2x+3y+5z=9$,
$7x+3y-5z=8,\ 2x+3y+\lambda z=\mu$, have
   (i) no solution (ii) a unique solution (iii) an infinite number of solutions.

EXERCISE 1.7

1. Solve the following system of homogenous equations.
   (i) $3x+2y+7z=0,\quad 4x-3y-2z=0,\quad 5x+9y+23z=0$
   (ii) $2x+3y-z=0,\quad x-y-2z=0,\quad 3x+y+3z=0$
2. Determine the values of λ for which the following system of equations
$x+y+3z=0,\ 4x+3y+\lambda z=0,\ 2x+y+2z=0$ has
   (i) a unique solution (ii) a non-trivial solution.

Example 1.1

If $A=\begin{bmatrix}8 & -6 & 2\\ -6 & 7 & -4\\ 2 & -4 & 3\end{bmatrix}$, verify that $A(\text{adj } A)=(\text{adj } A)A = |A|I_3$.

Example 1.12

If $A=\frac{1}{7}\begin{bmatrix}6 & -3 & a\\ b & -2 & 6\\ 2 & c & 3\end{bmatrix}$ is orthogonal, find a, b and c, and hence A^{-1}.

Example 1.20

Find the inverse the non-singular matrix $A=\begin{bmatrix}0 & 5\\ -1 & 6\end{bmatrix}$, by Gauss-Jordan method.

Example 1.21

Find the inverse of $A=\begin{bmatrix}2 & 1 & 1\\ 3 & 2 & 1\\ 2 & 1 & 2\end{bmatrix}$ by Gauss-Jordan method.

Example 1.23

Solve the following system of equations, using matrix inversion method:
$2x_1+3x_2+3x_3=5,\quad x_1-2x_2+x_3=-4,\quad 3x_1-x_2-2x_3=3.$

Example 1.24

If $A=\begin{bmatrix}-4 & 4 & 4\\ -7 & 1 & 3\\ 5 & -3 & -1\end{bmatrix}$ and $B=\begin{bmatrix}1 & -1 & 1\\ 1 & -2 & -2\\ 2 & 1 & 3\end{bmatrix}$, find the products AB and BA and hence solve the system of equations $x-y+z=4, x-2y-2z=9, 2x+y+3z=1$.

Example 1.25

Solve, by Cramer's rule, the system of equations

$x_1-x_2=3, 2x_1+3x_2+4x_3=17, x_2+2x_3=7$.

Example 1.26

In a T20 match, a team needed just 6 runs to win with 1 ball left to go in the last over. The last ball was bowled and the batsman at the crease hit it high up. The ball traversed along a path in a vertical plane and the equation of the path is $y=ax^2+bx+c$ with respect to a $xy-$ coordinate system in the vertical plane and the ball traversed through the points $(10,8),(20,16),(40,22)$, can you conclude that the team won the match? Justify your answer. (All distances are measured in metres and the meeting point of the plane of the path with the farthest boundary line is $(70,0)$.)

Example 1.27

Solve the following system of linear equations, by Gaussian elimination method :

$4x+3y+6z=25,\ x+5y+7z=13,\ 2x+9y+z=1$.

Example 1.28

The upward speed $v(t)$ of a rocket is approximated by $v(t)=at^2+bt+c$, where $a,b,$ and c are constants. It has been found that the speed at times $t=3, t=6$, and $t=9$ seconds are respectively, 64, 133, and 208 miles per second respectively. Find the speed at time $t=15$ seconds.

Example 1.29

Test for consistency of the following system of linear equations and if possible solve: $x+2y-z=3,\ 3x-y+2z=1,\ x-2y+3z=3,\ x-y+z+1=0$.

Example 1.30

Test for consistency of the following system of linear equations and if possible solve:

$4x-2y+6z=8,\ x+y-3z=-1,\ 15x-3y+9z=21$.

Example 1.31

Test for consistency of the following system of linear equations and if possible solve: $x-y+z=-9,\ 2x-2y+2z=-18,\ 3x-3y+3z+27=0$.

Example 1.32

Test the consistency of the following system of linear equations

$x-y+z=-9,\ 2x-y+z=4,\ 3x-y+z=6, 4x-y+2z=7$.

Example 1.33

Find the condition on a,b and c so that the following system of linear equations has one parameter family of solutions: $x+y+z=a,\ x+2y+3z=b,\ 3x+5y+7z=c$.

Example 1.34

Investigate for what values of λ and μ the system of linear equations

$x+2y+z=7,\ x+y+\lambda z=\mu, x+3y-5z=5$

has (i) no solution (ii) a unique solution (iii) an infinite number of solutions.

Example 1.35

Solve the following system:

$x+2y+3z=0,\ 3x+4y+4z=0,\ 7x+10y+12z=0.$

Example 1.36

Solve the system: $x+3y-2z=0,\ 2x-y+4z=0,\ x-11y+14z=0.$

Example 1.37

Solve the system:

$x+y-2z=0, 2x-3y+z=0, 3x-7y+10z=0, 6x-9y+10z=0.$

Example 1.38

Determine the values of λ for which the following system of equations

$(3\lambda-8)x+3y+3z=0,\ 3x+(3\lambda-8)y+3z=0,\ 3x+3y+(3\lambda-8)z=0$

has a non-trivial solution.

Example 1.40

If the system of equations $px+by+cz=0,\ ax+qy+cz=0,\ ax+by+rz=0$ has a non-trivi solution and $p\neq a,\ q\neq b,\ r\neq c$, prove that $\frac{p}{p-a}+\frac{q}{q-b}+\frac{r}{r-c}=2.$

EXERCISE 2.5

7. If z_1, z_2, and z_3 are three complex numbers such that $|z_1|=1, |z_2|=2, |z_3|=3$ and $|z_1+z_2+z_3|=1$, show that $|9z_1z_2+4z_1z_3+z_2z_3|=6$.

EXERCISE 2.6

2. If $z=x+iy$ is a complex number such that $Im\left(\frac{2z+1}{iz+1}\right)=0$, show that the locus of z is $2x^2+2y^2+x-2y=0.$

EXERCISE 2.7

3. If $(x_1+iy_1)(x_2+iy_2)(x_3+iy_3)\cdots(x_n+y_n)=a+ib$, show that

   (i) $(x_1^2+y_1^2)(x_2^2+y_2^2)(x_3^2+y_3^2)\cdots(x_n^2+y_n^2)=a^2+b^2$

   (ii) $\sum_{r=1}^{n}\tan^{-1}\left(\frac{y_r}{x_r}\right)=\tan^{-1}\left(\frac{b}{a}\right)+k\pi, k\in\mathbb{Z}.$

5. If $\cos\alpha+\cos\beta+\cos\gamma=\sin\alpha+\sin\beta+\sin\gamma=0$, show that

   (i) $\cos3\alpha+\cos3\beta+\cos3\gamma=3\cos(\alpha+\beta+\gamma)$ and

   (ii) $\sin3\alpha+\sin3\beta+\sin3\gamma=3\sin(\alpha+\beta+\gamma).$

6. If $z=x+iy$ and $\arg\left(\frac{z-i}{z+2}\right)=\frac{\pi}{4}$, show that $x^2+y^2+3x-3y+2=0.$

EXERCISE 2.8

3. Find the value of $\left(\dfrac{1+\sin\frac{\pi}{10}+i\cos\frac{\pi}{10}}{1+\sin\frac{\pi}{10}-i\cos\frac{\pi}{10}}\right)^{10}$.

4. If $2\cos\alpha = x+\dfrac{1}{x}$ and $2\cos\beta = y+\dfrac{1}{y}$, show that

(i) $\dfrac{x}{y}+\dfrac{y}{x}=2\cos(\alpha-\beta)$

(ii) $xy-\dfrac{1}{xy}=2i\sin(\alpha+\beta)$ (any 2 carry 5 marks)

(iii) $\dfrac{x^m}{y^n}-\dfrac{y^n}{x^m}=2i\sin(m\alpha-n\beta)$

(iv) $x^m y^n+\dfrac{1}{x^m y^n}=2\cos(m\alpha+n\beta)$

5. Solve the equation $z^3+27=0$.

6. If $\omega\neq 1$ is a cube root of unity, show that the roots of the equation $(z-1)^3+8=0$ are $-1, 1-2\omega, 1-2\omega^2$.

Example 2.12

If z_1, z_2 and z_3 are complex numbers such that $|z_1|=|z_2|=|z_3|=|z_1+z_2+z_3|=1$, find the value of $\left|\dfrac{1}{z_1}+\dfrac{1}{z_2}+\dfrac{1}{z_3}\right|$.

Example 2.15

Let z_1, z_2 and z_3 be complex numbers such that $|z_1|=|z_2|=|z_3|=r>0$ and $z_1+z_2+z_3\neq 0$. Prove that $\left|\dfrac{z_1z_2+z_2z_3+z_3z_1}{z_1+z_2+z_3}\right|=r$.

Example 2.18

Given the complex number $z=3+2i$, represent the complex numbers z, iz, and $z+iz$ on one Argand diagram. Show that these complex numbers form the vertices of an isosceles right triangle.

Example 2.27

If $z=x+iy$ and $\arg\left(\dfrac{z-1}{z+1}\right)=\dfrac{\pi}{2}$, show that $x^2+y^2=1$.

Example 2.30

Simplify $\left(\dfrac{1+\cos 2\theta+i\sin 2\theta}{1+\cos 2\theta-i\sin 2\theta}\right)^{30}$.

Example 2.31

Simplify (ii) $(-\sqrt{3}+3i)^{31}$.

Example 2.32

Find the cube roots of unity.

Example 2.33

Find the fourth roots of unity.

Example 2.34

Solve the equation $z^3 + 8i = 0$, where $z \in \mathbb{C}$.

Example 2.35

Find all cube roots of $\sqrt{3} + i$.

Example 2.36

Suppose z_1, z_2, and z_3 are the vertices of an equilateral triangle inscribed in the circ $|z| = 2$. If $z_1 = 1 + i\sqrt{3}$, then find z_2 and z_3.

Chapter 3

Theory of Equations

EXERCISE 3.1

4. Solve the equation $3x^3 - 16x^2 + 23x - 6 = 0$ if the product of two roots is 1.
5. Find the sum of squares of roots of the equation $2x^4 - 8x^3 + 6x^2 - 3 = 0$.
6. Solve the equation $x^3 - 9x^2 + 14x + 24 = 0$ if it is given that two of its roots are in the rati 3:2.
10. If the equations $x^2 + px + q = 0$ and $x^2 + p'x + q' = 0$ have a common root, then show tha $\dfrac{pq' - p'q}{q - q'} = \dfrac{q - q'}{p' - p}$.

EXERCISE 3.2

1. If k is real, discuss the nature of the roots of the polynomial equation $2x^2 + kx + k = 0$, i terms of k.
5. Prove that a straight line and parabola cannot intersect at more than two points.

EXERCISE 3.3

1. Solve the cubic equation : $2x^3 - x^2 - 18x + 9 = 0$ if sum of two of its roots vanishes.
2. Solve the equation $9x^3 - 36x^2 + 44x - 16 = 0$ if the roots form an arithmetic progression.
3. Solve the equation $3x^3 - 26x^2 + 52x - 24 = 0$ if its roots form a geometric progression.
4. Determine k and solve the equation $2x^3 - 6x^2 + 3x + k = 0$ if one of its roots is twice the sum of the other two roots.
5. Find all zeros of the polynomial $x^6 - 3x^5 - 5x^4 + 22x^3 - 39x^2 - 39x + 135$, if it is known that $1 + 2i$ and $\sqrt{3}$ are two of its zeros.
6. Solve the cubic equations : (i) $2x^3 - 9x^2 + 10x = 3$, (ii) $8x^3 - 2x^2 - 7x + 3 = 0$. (each 5 marks)
7. Solve the equation : $x^4 - 14x^2 + 45 = 0$.

EXERCISE 3.4

1. Solve : (i) $(x-5)(x-7)(x+6)(x+4) = 504$ (ii) $(x-4)(x-7)(x-2)(x+1) = 16$. (each 5)
2. Solve : $(2x-1)(x+3)(x-2)(2x+3) + 20 = 0$.

EXERCISE 3.5

1. Solve the following equations:

   (ii) $12x^3 + 8x = 29x^2 - 4$

2. Examine for the rational roots of (each 5 marks)

(i) $2x^3 - x^2 - 1 = 0$ (ii) $x^8 - 3x + 1 = 0$

3. Solve : $8x^{\frac{3}{2n}} - 8x^{\frac{-3}{2n}} = 63$.

4. Solve : $2\sqrt{\frac{x}{a}} + 3\sqrt{\frac{a}{x}} = \frac{b}{a} + \frac{6a}{b}$.

5. Solve the equations (each 5 marks)

(i) $6x^4 - 35x^3 + 62x^2 - 35x + 6 = 0$ (ii) $x^4 + 3x^3 - 3x - 1 = 0$

6. Find all real numbers satisfying $4^x - 3(2^{x+2}) + 2^5 = 0$.

7. Solve the equation $6x^4 - 5x^3 - 38x^2 - 5x + 6 = 0$ if it is known that $\frac{1}{3}$ is a solution.

EXERCISE 3.6

1. Discuss the maximum possible number of positive and negative roots of the polynomial equation $9x^9 - 4x^8 + 4x^7 - 3x^6 + 2x^5 + x^3 + 7x^2 + 7x + 2 = 0$.
2. Discuss the maximum possible number of positive and negative zeros of the polynomials $x^2 - 5x + 6$ and $x^2 - 5x + 16$. Also draw rough sketch of the graphs.
4. Determine the number of positive and negative roots of the equation $x^9 - 5x^8 - 14x^7 = 0$.

Example 3.5

Find the condition that the roots of cubic equation $x^3 + ax^2 + bx + c = 0$ are in the ratio $p:q:r$.

Example 3.6

From the equation whose roots are the squares of the roots of the cubic equation $x^3 + ax^2 + bx + c = 0$.

Example 3.7

If p is real, discuss the nature of the roots of the equation $4x^2 + 4px + p + 2 = 0$, in terms of p.

Example 3.14

Prove that a line cannot intersect a circle at more than two points.

Example 3.15

If $2+i$ and $3-\sqrt{2}$ are roots of the equation $x^6 - 13x^5 + 62x^4 - 126x^3 + 65x^2 + 127x - 140 = 0$ find all roots.

Example 3.16

Solve the equation $x^4 - 9x^2 + 20 = 0$.

Example 3.17

Solve the equation $x^3 - 3x^2 - 33x + 35 = 0$.

Example 3.18

Solve the equation $2x^3 + 11x^2 - 9x - 18 = 0$.

Example 3.19

Obtain the condition that the roots of $x^3 + px^2 + qx + r = 0$ are in A.P.

Example 3.20

Find the condition that the roots of $ax^3 + bx^2 + cx + d = 0$ are in geometric progression. Assume $a, b, c, d \neq 0$.

Example 3.21

If the roots of $x^3+px^2+qx+r=0$ are in H.P., prove that $9pqr=27r^2+2q^3$. Assum
$p,q,r\neq 0$ (Remark : HP is not defined).

Example 3.22

It is known that the roots of the equation $x^3-6x^2-4x+24=0$ are in arithmetic progressio
Find its roots.

Example 3.23

Solve the equation $(x-2)(x-7)(x-3)(x+2)+19=0$.

Example 3.24

Solve the equation $(2x-3)(6x-1)(3x-2)(x-2)-5=0$.

Example 3.26

Find the roots of $2x^3+3x^2+2x+3=0$.

Example 3.27

Solve the equation $7x^3-43x^2=43x-7$.

Example 3.28

Solve the following equation : $x^4-10x^3+26x^2-10x+1=0$.

Example 3.31 (each 5 marks)

Discuss the nature of the roots of the following polynomials:

(i) $x^{2018}+1947x^{1950}+15x^8+26x^6+2019$

(ii) $x^5-19x^4+2x^3+5x^2+11$

EXERCISE 4.2

6. Find the domain of

(i) $f(x)=\sin^{-1}\left(\frac{|x|-2}{3}\right)+\cos^{-1}\left(\frac{1-|x|}{4}\right)$.

EXERCISE 4.5

5. Prove that $\tan^{-1}x+\tan^{-1}y+\tan^{-1}z=\tan^{-1}\left[\frac{x+y+z-xyz}{1-xy-yz-zx}\right]$.

6. If $\tan^{-1}x+\tan^{-1}y+\tan^{-1}z=\pi$, show that $x+y+z=xyz$.

7. Prove that $\tan^{-1}x+\tan^{-1}\frac{2x}{1-x^2}=\tan^{-1}\left(\frac{3x-x^3}{1-3x^2}\right), |x|<\frac{1}{\sqrt{3}}$.

9. (i) $\sin^{-1}\frac{5}{x}+\sin^{-1}\frac{12}{x}=\frac{\pi}{2}$ (ii) $2\tan^{-1}x=\cos^{-1}\frac{1-a^2}{1+a^2}-\cos^{-1}\frac{1-b^2}{1-b^2}, a>0, b>$

(iii) $2\tan^{-1}(\cos x)=\tan^{-1}(2\operatorname{cosec} x)$ (iv) $\cot^{-1}x-\cot^{-1}(x+2)=\frac{\pi}{12}, x>0$.

10. Find the number of solutions of the equation
$\tan^{-1}(x-1)+\tan^{-1}x+\tan^{-1}(x+1)=\tan^{-1}3x$.

Example 4.11

Prove that $\tan\left(\sin^{-1}x\right)=\dfrac{x}{\sqrt{1-x^2}}$, $-1<x<1$.

Example 4.18

Find the value of (iii) $\tan\left[\frac{1}{2}\sin^{-1}\left(\frac{2a}{1+a^2}\right)+\frac{1}{2}\cos^{-1}\left(\frac{1-a^2}{1+a^2}\right)\right]$.

Example 4.20

Evaluate $\sin\left[\sin^{-1}\left(\frac{3}{5}\right)+\sec^{-1}\left(\frac{5}{4}\right)\right]$.

Example 4.22

If $\cos^{-1}x+\cos^{-1}y+\cos^{-1}z=\pi$ and $0<x,y,z<1$, then show that $x^2+y^2+z^2+2xyz=1$.

Example 4.23

If $a_1, a_2, a_3, \dots a_n$ is an arithmetic progression with common difference d, then prove that $\tan\left[\tan^{-1}\left(\frac{d}{1+a_1a_2}\right)+\tan^{-1}\left(\frac{d}{1+a_2a_3}\right)+\dots+\tan^{-1}\left(\frac{d}{1+a_{n-1}a_n}\right)\right]=\frac{a_n-a_1}{1+a_1a_n}$.

Example 4.24

Solve $\tan^{-1}\frac{1-x}{1+x}=\frac{1}{2}\tan^{-1}x$ for $x>0$.

Example 4.28

Solve : $\tan^{-1}\left(\frac{x-1}{x-2}\right)+\tan^{-1}\left(\frac{x+1}{x+2}\right)=\frac{\pi}{4}$.

Example 4.29

Solve : $\cos\left(\sin^{-1}\left(\frac{x}{\sqrt{1+x^2}}\right)\right)=\sin\left\{\cot^{-1}\left(\frac{3}{4}\right)\right\}$.

Chapter 4

(Created from the Text Book)

(1) Draw $y=\sin x$ in $\left[-\frac{\pi}{2},\frac{\pi}{2}\right]$ and $y=\sin^{-1}x$ in $[-1,1]$

(2) Draw $y=\cos x$ in $[0,\pi]$ and $y=\cos^{-1}x$ in $[-1,1]$

(3) Draw $y=\tan x$ in $\left(-\frac{\pi}{2},\frac{\pi}{2}\right)$ and $y=\tan^{-1}x$ in $(-\infty,\infty)$

(4) Draw $y=\operatorname{cosec} x$ in $\left[-\frac{\pi}{2},\frac{\pi}{2}\right]\setminus\{0\}$ and $y=\operatorname{cosec}^{-1}x$ in $\mathbb{R}\setminus(-1,1)$

(5) Draw $y=\sec x$ in $[0,\pi]\setminus\left\{\frac{\pi}{2}\right\}$ and $y=\sec^{-1}$ in $\mathbb{R}\setminus(-1,1)$

(6) Draw $y=\cot x$ in $(0,\pi)$ and $y=\cot^{-1}$ in $(-\infty,\infty)$

EXERCISE 5.1

6. Find the equation of the circle passing through the points $(1,0),(-1,0)$, and $(0,1)$.
9. Find the equation of the tangent and normal to the circle $x^2+y^2-6x+6y-8=0$ at $(2,2)$.

EXERCISE 5.2

4. Find the vertex, focus, equation of directrix and length of the latus rectum of the following:

   (iv) $x^2-2x+8y+17=0$ (v) $y^2-4y-8x+12=0$

5. Identify the type of conic and find centre, foci, vertices and directrices of each of the following:

   (i) $\frac{x^2}{25}+\frac{y^2}{9}=1$ (ii) $\frac{x^2}{3}+\frac{y^2}{10}=1$ (iii) $\frac{x^2}{25}-\frac{y^2}{144}=1$ (iv) $\frac{y^2}{16}-\frac{x^2}{9}=1$

8. Identify the type of conic and find centre, foci, vertices, and directrices of each of the following : (any three informations carry 5 marks)

   (i) $\frac{(x-3)^2}{225}+\frac{(y-4)^2}{289}=1$ (ii) $\frac{(x+1)^2}{100}+\frac{(y-2)^2}{64}=1$

   (iii) $\frac{(x+3)^2}{225}-\frac{(y-4)^2}{64}=1$ (iv) $\frac{(y-2)^2}{25}-\frac{(x+1)^2}{16}=1$

   (v) $18x^2+12y^2-144x+48y+120=0$ (vi) $9x^2-y^2-36x-6y+18=0$

EXERCISE 5.4

1. Find the equations of the two tangents that can be drawn from $(5,2)$ to the ellipse $2x^2+7y^2=14$.
2. Find the equation of tangents to the hyperbola $\frac{x^2}{16}-\frac{y^2}{64}=1$ which are parallel to $10x-3y+9=0$.
3. Show that the line $x-y+4=0$ is a tangent to the ellipse $x^2+3y^2=12$. Also find the coordinates of the point of contact.
4. Find the equation of the tangent to the parabola $y^2=16x$ perpendicular to $2x+2y+3=0$.
5. Find the equation of tangent at $t=2$ to the parabola $y^2=8x$.
6. Find the equation of the tangent and normal to hyperbola $12x^2-9y^2=108$ at $\theta=\frac{\pi}{3}$. (Hint : Use parametric form)
7. Prove that the point of intersection of the tangents at 't_1' and 't_2' on the parabola $y^2=4ax$ is $\left[at_1t_2, a(t_1+t_2)\right]$.
8. If the normal at the point 't_1' on the parabola $y^2=4ax$ meets the parabola again at the point 't_2' then prove that $t_2=-\left(t_1+\frac{2}{t_1}\right)$.

EXERCISE 5.5

1. A bridge has a parabolic arch that is 10m height in the centre and 30m wide at the bottom. Find the height of the arch 6m from the centre, on either sides.
2. A tunnel through a mountain for a four lane highway is to have a elliptical opening. The total width of the highway (not the opening) is to be 16m, and the height at the edge of the road must be sufficient for a truck 4m high to clear if the highest point of the opening is to be 5m approximately. How wide must the opening be?

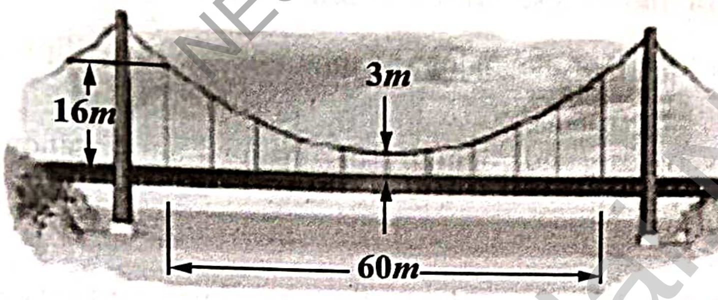
3. At a water fountain, water attains a maximum height of 4m at horizontal distance of 0.5m from its origin. If the path of water is a parabola, find the height of water at a horizontal distance of 0.75m from the point of origin.

4. An engineer designs a satellite dish with a parabolic cross section. The dish is 5m wide at the opening, and the focus is placed 1.2m from the vertex.

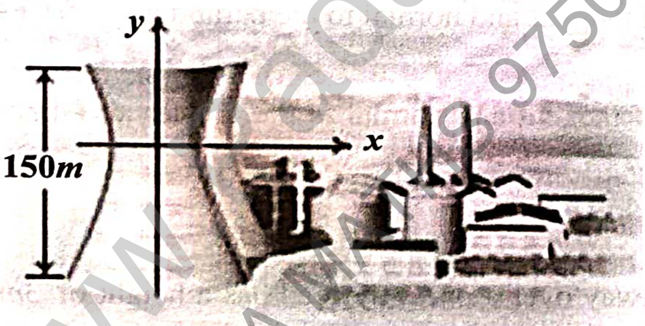
   (a) Position a coordinate system with the origin at the vertex and the x-axis on the parabola's axis of symmetry and find an equation of the parabola.

   (b) Find the depth of the satellite dish at the vertex.

5. Parabolic cable of 60m portion of the roadbed of a suspension bridge are positioned as shown below. Vertical cables are to be spaced every 6m along this portion of the roadbed. Calculate the lengths of first two of these vertical cables from the vertex.



6. Cross section of a Nuclear cooling tower is in the shape of a hyperbola with equation $\frac{x^2}{30^2}-\frac{y^2}{44^2}=1$. The tower is 150m tall and the distance from the top of the tower to the centre of the hyperbola is half the distance from the base of the tower to the centre of the hyperbola. Find the diameter of the top and base of the tower.



7. A rod of length 1.2m moves with its ends always touching the coordinate axes. The locus of a point P on the rod, which is 0.3m from the end in contact with x-axis is an ellipse. Find the eccentricity.

8. Assume that water issuing from the end of a horizontal pipe, 7.5m above the ground, describes a parabolic path. The vertex of the parabolic path is at the end of the pipe. At a position 2.5m below the line of the pipe, the flow of water has curved outward 3m beyond the vertical line through the end of the pipe. How far beyond this vertical line will the water strike the ground?

9. On lighting a rocket cracker it gets projected in a parabolic path and reaches a maximum height of 4m when it is 6m away from the point of projection. Finally it reaches the ground 12m away from the starting point. Find the angle of projection.

10. Points A and B are 10km apart and it is determined from the sound of an explosion heard at these points at different times that the location of the explosion is 6km closer to A than B. Show that the location of the explosion is restricted to a particular curve and find an equation of it.

Example 5.10

Find the equation of the circle passing through the points $(1,1)$, $(2,-1)$, and $(3,2)$.

Example 5.19

Find the vertex, focus, directrix, and length of the latus rectum of the parabola $x^2-4x-5y-1=0$.

Example 5.21

Find the equation of the ellipse whose eccentricity is $\frac{1}{2}$, one of the foci is $(2,3)$ and a directrix is $x=7$. Find the length of the major and minor axes of the ellipse.

Example 5.22

Find the foci, vertices and length of major and minor axis of the conic $4x^2+36y^2+40x-288y+532=0$.

Example 5.23

For the ellipse $4x^2+y^2+24x-2y+21=0$, find the centre, vertices, and the foci. Also prove that the length of latus rectum is 2.

Example 5.26

Find the centre, foci and eccentricity of the hyperbola $11x^2-25y^2-44x+50y-256=0$.

Example 5.29

Find the equations of tangent and normal to the parabola $x^2+6x+4y+5=0$ at $(1,-3)$.

Example 5.30

Find the equations of tangent and normal to the ellipse $x^2+4y^2=32$ in cartesian form and parametric form when $\theta=\frac{\pi}{4}$.

Example 5.31

A semielliptical archway over a one-way road has a height of $3m$ and a width of $12m$. The truck has a width of $3m$ and a height of $2.7m$. Will the truck clear the opening of the archway?

Example 5.40

Two coast guard stations are located 600 km apart at points $A(0,0)$ and $B(0,600)$. A distress signal from a ship at P is received at slightly different times by two stations. It is determined that the ship is 200 km farther from station A than it is from station B. Determine the equation of hyperbola that passes through the location of the ship.

Example 5.41

Certain telescopes contain both parabolic mirror and a hyperbolic mirror. In the telescope shown in figure the parabola and hyperbola share focus F_1 which is $14m$ above the vertex of the parabola. The hyperbola's second focus F_2 is $2m$ above the parabola's vertex. The vertex of the hyperbolic mirror is $1m$ below F_1. Position a coordinate system with the origin at the centre of the hyperbola and with the foci on the y-axis. Then find the equation of the hyperbola.

EXERCISE 6.1

8. If G is the centroid of a ΔABC, prove that (area of ΔGAB) =(area of ΔGBC) = (area of ΔGCA) $=\frac{1}{3}$ (area of ΔABC).
9. Using vector method, prove that $\cos(\alpha-\beta)=\cos\alpha\cos\beta+\sin\alpha\sin\beta$.
10. Prove by vector method that $\sin(\alpha+\beta)=\sin\alpha\cos\beta+\cos\alpha\sin\beta$.

EXERCISE 6.3

4. If $\vec{a}=2\hat{i}+3\hat{j}-\hat{k}, \vec{b}=3\hat{i}+5\hat{j}+2\hat{k}, \vec{c}=-\hat{i}-2\hat{j}+3\hat{k}$, verify that
(i) $(\vec{a}\times\vec{b})\times\vec{c}=(\vec{a}\cdot\vec{c})\vec{b}-(\vec{b}\cdot\vec{c})\vec{a}$ (ii) $\vec{a}\times(\vec{b}\times\vec{c})=(\vec{a}\cdot\vec{c})\vec{b}-(\vec{a}\cdot\vec{b})\vec{c}$

EXERCISE 6.5

2. Show that the lines $\vec{r}=(6\hat{i}+\hat{j}+2\hat{k})+s(\hat{i}+2\hat{j}-3\hat{k})$ and $\vec{r}=(3\hat{i}+2\hat{j}-2\hat{k})+t(2\hat{i}+4\hat{j}-5\hat{k})$ are skew lines and hence find the shortest distance between them.
4. Show that the lines $\frac{x-3}{3}=\frac{y-3}{-1}, z-1=0$ and $\frac{x-6}{2}=\frac{z-1}{3}, y-2=0$ intersect. Also find the point of intersection.
5. Show that the straight lines $x+1=2y=-12z$ and $x=y+2=6z-6$ are skew and hence find the shortest distance between them.
6. Find the parametric form of vector equation of the straight line passing through $(-1,2,1)$ and parallel to the straight line $\vec{r}=(2\hat{i}+3\hat{j}-\hat{k})+t(\hat{i}-2\hat{j}+\hat{k})$ and hence find the shortest distance between the lines.
7. Find the foot of the perpendicular drawn from the point $(5,4,2)$ to the line $\frac{x+1}{2}=\frac{y-3}{3}=\frac{z-1}{-1}$. Also, find the equation of the perpendicular.

EXERCISE 6.7

1. Find the non-parametric form of vector equation, and Cartesian equation of the plane passing through the point $(2,3,6)$ and parallel to the straight lines $\frac{x-1}{2}=\frac{y+1}{3}=\frac{z-3}{1}$ and $\frac{x+3}{2}=\frac{y-3}{-5}=\frac{z+1}{-3}$.
2. Find the parametric form of vector equation and Cartesian equation of the plane passing through the points $(2,2,1),(9,3,6)$ and perpendicular to the plane $2x+6y+6z=9$.
3. Find parametric form of vector equation and Cartesian equation of the plane passing through the points $(2,2,1),(1,-2,3)$ and parallel to the straight line passing through the points $(2,1,-3)$ and $(-1,5,-8)$.
4. Find the vector (parametric and non-parametric) and Cartesian equations of the plane passing through the point $(1,-2,4)$ and perpendicular to the plane $x+2y-3z=11$ and parallel to the line $\frac{x+7}{3}=\frac{y+3}{-1}=\frac{z}{1}$.
5. Find the vector (parametric and non-parametric) and Cartesian equations of the plane containing the line $\vec{r}=(\hat{i}-\hat{j}+3\hat{k})+t(2\hat{i}-\hat{j}+4\hat{k})$ and perpendicular to plane $\vec{r}\cdot(\hat{i}+2\hat{j}+\hat{k})=8$.

6. Find the vector (parametric and non-parametric)and Cartesian form of the equations of the plane passing through the three non-collinear points $(3,6,-2),(-1,-2,6)$, and $(6,4,-2)$.

7. Find the non-parametric form of vector equation, and Cartesian equations of the plane $\vec{r}=(6\hat{i}-\hat{j}+\hat{k})+s(-\hat{i}+2\hat{j}+\hat{k})+t(-5\hat{i}-4\hat{j}-5\hat{k})$.

EXERCISE 6.8

1. Show that the straight lines $\vec{r}=(5\hat{i}+7\hat{j}-3\hat{k})+s(4\hat{i}+4\hat{j}-5\hat{k})$ and $\vec{r}=(8\hat{i}+4\hat{j}+5\hat{k})+t(7\hat{i}+\hat{j}+3\hat{k})$ are coplanar. Find the vector equation of the plane in which they lie.

2. Show that the lines $\frac{x-2}{1}=\frac{y-3}{1}=\frac{z-4}{3}$ and $\frac{x-1}{-3}=\frac{y-4}{2}=\frac{z-5}{1}$ are coplanar. Also, find the cartesian equation of the plane containing these lines.

3. If the straight lines $\frac{x-1}{1}=\frac{y-2}{2}=\frac{z-3}{m^2}$ and $\frac{x-3}{1}=\frac{y-2}{m^2}=\frac{z-1}{2}$ are coplanar, find the distinct real values of m.

4. If the straight lines $\frac{x-1}{2}=\frac{y+1}{\lambda}=\frac{z}{2}$ and $\frac{x+1}{5}=\frac{y+1}{2}=\frac{z}{\lambda}$ are coplanar, find λ and Cartesian equation of the plane containing these two lines.

EXERCISE 6.9

2. Find the equation of the plane passing through the line of intersection of the planes $x+2y+3z=2$ and $x-y+z=3$, and at a distance $\frac{2}{\sqrt{3}}$ from the point $(3,1-1)$.

7. Find the point of intersection of the line $x-1=\frac{y}{2}=z+1$ with the plane $2x-y+2z=2$. Also, find the angle between the line and the plane.

8. Find the coordinates of the foot of the perpendicular and length of the perpendicular from the point $(4,3,2)$ to the plane $x+2y+3z=2$.

Example 6.1(Cosine formulae)

With usual notations, in any triangle ABC, prove the following by vector method.

(i) $a^2=b^2+c^2-2bc\cos A$ (ii) $b^2=c^2+a^2-2ca\cos B$

(iii) $c^2=a^2+b^2-2ab\cos C$

Example 6.2

With usual notations, in any triangle ABC, prove the following by vector method.

(i) $a=b\cos C+c\cos B$ (ii) $b=c\cos A+a\cos C$

(iii) $c=a\cos B+b\cos A$

Example 6.3

By vector method, prove that $\cos(\alpha+\beta)=\cos\alpha\cos\beta-\sin\alpha\sin\beta$.

Example 6.5

Prove by vector method that $\sin(\alpha-\beta)=\sin\alpha\cos\beta-\cos\alpha\sin\beta$.

Example 6.6

If D is the midpoint of the side BC of a triangle ABC, then show by vector method that $|\overrightarrow{AB}|^2+|\overrightarrow{AC}|^2=2(|\overrightarrow{AD}|^2+|\overrightarrow{BD}|^2)$.

Example 6.7

Show that the altitudes of a triangle are concurrent by using vectors.

Example 6.8

In triangle ABC, the points D, E, F are the midpoints of the sides BC, CA and AB, respectively. Using vector method, show that the area of ΔDEF is equal to $\frac{1}{4}$ (area of ΔABC).

Example 6.16

Show that the four points $(6,-7,0)$, $(16,-19,-4)$, $(0,3,-6)$, $(2,-5,10)$ lie on a same plane.

Example 6.21 (each)

For any four vectors $\vec{a},\vec{b},\vec{c},\vec{d}$, verify

(i) $(\vec{a}\times\vec{b})\times(\vec{c}\times\vec{d})=[\vec{a},\vec{b},\vec{d}]\vec{c}-[\vec{a},\vec{b},\vec{c}]\vec{d}$

(ii) $(\vec{a}\times\vec{b})\times(\vec{c}\times\vec{d})=[\vec{a},\vec{c},\vec{d}]\vec{b}-[\vec{b},\vec{c},\vec{d}]\vec{a}$.

Example 6.22

If $\vec{a}=-2\hat{i}+3\hat{j}-2\hat{k}$, $\vec{b}=3\hat{i}-\hat{j}+3\hat{k}$, $\vec{c}=2\hat{i}-5\hat{j}+\hat{k}$, find $(\vec{a}\times\vec{b})\times\vec{c}$ and $\vec{a}\times(\vec{b}\times\vec{c})$. State whether they are equal.

Example 6.23

If $\vec{a}=\hat{i}-\hat{j}$, $\vec{b}=\hat{i}-\hat{j}-4\hat{k}$, $\vec{c}=3\hat{j}-\hat{k}$ and $\vec{d}=2\hat{i}+5\hat{j}+\hat{k}$, verify that

(i) $(\vec{a}\times\vec{b})\times(\vec{c}\times\vec{d})=[\vec{a},\vec{b},\vec{d}]\vec{c}-[\vec{a},\vec{b},\vec{c}]\vec{d}$

(ii) $(\vec{a}\times\vec{b})\times(\vec{c}\times\vec{d})=[\vec{a},\vec{c},\vec{d}]\vec{b}-[\vec{b},\vec{c},\vec{d}]\vec{a}$

Example 6.24

A straight line passes through the point $(1,2,-3)$ and parallel to $4\hat{i}+5\hat{j}-7\hat{k}$. Find (i) vector equation in parametric form (ii) vector equation in non-parametric form (iii) Cartesian equations of the straight line.

Example 6.25

The vector equation in parametric form of a line is $\vec{r}=(3\hat{i}-2\hat{j}+6\hat{k})+t(2\hat{i}-\hat{j}+3\hat{k})$. Find (i) the direction cosines of the straight line (ii) vector equation in non-parametric form of the line (iii) Cartesian equations of the line

Example 6.27

Find the vector parametric and Cartesian equations of a straight passing through the points $(-5,7,-4)$ and $(13,-5,2)$. Find the point where the straight line crosses the xy,plane.

Example 6.37

Find the coordinates of the foot of the perpendicular drawn from the point $(-1,2,3)$ to the straight line $\vec{r}=(\hat{i}-4\hat{j}+3\hat{k})+t(2\hat{i}+3\hat{j}+\hat{k})$. Also, find the shortest distance from the point to the straight line.

Example 6.43

Find the non-parametric form of vector equation and Cartesian equation of the plane passing through the point $(0,1,-5)$ and parallel to the straight lines $\vec{r}=(\hat{i}+2\hat{j}-4\hat{k})+s(2\hat{i}+3\hat{j}+6\hat{k})$ and $\hat{r}=(\hat{i}-3\hat{j}+5\hat{k})+t(\hat{i}+\hat{j}-\hat{k})$.

Example 6.44

Find the vector parametric, vector non-parametric and Cartesian form of the equation of the plane passing through the points $(-1,2,0)$, $(2,2-1)$ and parallel to the straight line $\frac{x-1}{1}=\frac{2y+1}{2}=\frac{z+1}{-1}$.

Example 6.46

Show that the lines $\vec{r}=\left(-\hat{i}-3\hat{j}-5\hat{k}\right)+s\left(3\hat{i}+5\hat{j}+7\hat{k}\right)$ and $\vec{r}=\left(2\hat{i}+4\hat{j}+6\hat{k}\right)+t\left(\hat{i}+4\hat{j}+7\hat{k}\right)$ are coplanar. Also,find the vector equation in non-parametric form of the plane containing these lines.

Example 6.50

Find the distance of the point $(5,-5,-10)$ from the point of intersection of a straight line passing through the points $A(4,1,2)$ and $B(7,5,4)$ with the plane $x-y+z=5$.

Example 6.53

Find the equation of the plane passing through the intersection of the planes $\vec{r}\cdot\left(\hat{i}+\hat{j}+\hat{k}\right)+1=0$ and $\vec{r}\cdot\left(2\hat{i}-3\hat{j}+5\hat{k}\right)=2$ and the point $(-1,2,1)$.

Example 6.54

Find the equation of the plane passing through the intersection of the planes $2x+3y-z+7=0$ and $x+y-2z+5=0$ and perpendicular to the plane $x+y-3z-5=0$.

Example 6.55

Find the image of the point whose position vector is $\hat{i}+2\hat{j}+3\hat{k}$ in the plane $\vec{r}\cdot\left(\hat{i}+2\hat{j}+4\hat{k}\right)=38$.

EXERCISE 7.1

2. A camera is accidentally knocked off an edge of a cliff 400 ft. high. The camera falls a distance of $s=-16t^2$ in t seconds.
   (i) How long does the camera fall before it hits the ground?
   (ii) What is the average velocity with which the camera falls during the last 2 seconds?
   (iii) What is the instantaneous velocity of the camera when it hits the ground?
3. A particle moves along a line according to the law $s(t)=2t^3-9t^2+12t-4$, where $t\geq 0$.
   (i) At what times the particle changes direction?
   (ii) Find the total distance travelled by the particle in the first 4 seconds.
   (iii) Find the particle's acceleration each time the velocity is zero.
7. A beacon makes one revolution every 10 seconds. It is located on a ship which is anchored 5 km from a straight shore line. How fast is the beam moving along the shore line when it makes an angle of 45° with the shore?
8. A conical water tank with vertex down of 12 metres height has a radius of 5 metres at the top. If water flows into the tank at a rate 10 cubic m/min, how fast is the depth of the water increases when the water is 8 metres deep?

9. A ladder 17 metre long is leaning against the wall. The base of the ladder is pulled away from the wall at a rate of 5 m/s. When the base of the ladder is 8 metres from the wall.

(i) How fast is the top of the ladder moving down the wall?

(ii) At what rate, the area of the triangle formed by the ladder, wall, and the floor, is changing?

10. A police jeep, approaching an orthogonal intersection from the northern direction, is chasing a speeding car that has turned and moving straight east. When the jeep is 0.6 km north of the intersection and the car is 0.8 km to the east. The police determine with a radar that the distance between the jeep and the car is increasing at 20 km/hr. If the jeep is moving at 60 km/hr at the instant of measurement, what is the speed of the car?

EXERCISE 7.2

9. Find the angle between the curves $xy = 2$ and $x^2 + 4y = 0$.

10. Show that the two curves $x^2 - y^2 = r^2$ and $xy = c^2$ where c, r are constants, cut orthogonally.

EXERCISE 7.4

1. Write the Maclaurin series expansion of the following functions:

(v) $\tan^{-1}(x)$; $-1 \le x \le 1$ (vi) $\cos^2 x$

EXERCISE 7.5

Evaluate the following limits, if necessary use l'Hôpital Rule :

8. $\lim\limits_{x \to 0^+} x^x$

9. $\lim\limits_{x \to \infty} \left(1 + \frac{1}{x}\right)^x$

10. $\lim\limits_{x \to \frac{\pi}{2}} (\sin x)^{\tan x}$

11. $\lim\limits_{x \to 0^+} (\cos x)^{\frac{1}{x^2}}$

12. If an initial amount A_0 of money is invested at an interest rate r compounded n times a year, the value of the investment after t years is $A = A_0\left(1 + \frac{r}{n}\right)^{nt}$. If the interest is compounded continuously, (that is as $n \to \infty$), show that the amount after t years is $A = A_0 e^{rt}$.

EXERCISE 7.6

1. Find the absolute extrema of the following functions on the given closed interval.

(ii) $f(x) = 3x^4 - 4x^3$; $[-1, 2]$

2. Find the intervals of monotonicities and hence find the local extremum for the following functions:

(i) $f(x) = 2x^3 + 3x^2 - 12x$

(ii) $f(x) = \frac{x}{x-5}$

(iii) $f(x) = \frac{e^x}{1-e^x}$

(iv) $f(x) = \frac{x^3}{3} - \log x$

(v) $f(x) = \sin x \cos x + 5$, $x \in \left[0, \frac{\pi}{2}\right]$

EXERCISE 7.7

1. Find intervals of concavity and points of inflexion for the following functions:

   (i) $f(x)=x(x-4)^3$ (ii) $f(x)=\sin x+\cos x,\ 0<x<2\pi$

   (iii) $f(x)=\frac{1}{2}\left(e^{x}-e^{-x}\right)$

2. Find the local extrema for the following functions using second derivative test:

   (i) $f(x)=-3x^5+5x^3$ (ii) $f(x)=x\log x$

3. For the function $f(x)=4x^3+3x^2-6x+1$ find the intervals of monotonicity, local extre intervals of concavity and points of inflection.

EXERCISE 7.8

1. Find two positive numbers whose sum is 12 and their product is maximum.
3. Find the smallest possible value of x^2+y^2 given that $x+y=10$.
4. A garden is to be laid out in a rectangular area and protected by wire fence. What is largest possible area of the fenced garden with 40 metres of wire.
5. A rectangular page is to contain 24cm^2 of print. The margins at the top and bottom of page are 1.5 cm and the margins at other sides of the page is 1 cm. What should be dimensions of the page so that the area of the paper used is minimum?
6. A farmer plans to fence a rectangular pasture adjacent to a river. The pasture must con 1,80,000 sq.mtrs in order to provide enough grass for herds. No fencing is needed along river. What is the length of the minimum needed fencing material?
7. Find the dimensions of the rectangle with maximum area that can be inscribed in a circl radius 10 cm.
8. Prove that among all the rectangles of the given perimeter, the square has the maximum ar
9. Find the dimensions of the largest rectangle that can be inscribed in a semi circle of radiu cm.
10. A manufacturer wants to design an open box having a square base and a surface area of sq.cm. Determine the dimensions of the box for the maximum volume.
11. The volume of a cylinder is given by the formula $V=\pi r^2h$. Find the greatest value of V $r+h=6$.
12. A hollow cone with base radius a cm and height b cm is placed on a table. Show that volume of the largest cylinder that can be hidden underneath is $\frac{4}{9}$ times volume of the cor

EXERCISE 7.9

2. Sketch the graphs of the following functions :

   (I) $y=-\frac{1}{3}(x^3-3x+2)$.

   (II) $y=x\sqrt{4-x}$.

   (III) $y=\frac{x^2+1}{x^2-4}$.

   (IV) $y=\frac{1}{1+e^{-x}}$

Note : The above 4 sub-divisions have more than 10 stages apart from the diagram. Hence an points is enough for 5 marks.

Example 7.1

For the function $f(x)=x^2, x\in[0,2]$, compute the average rate of changes in the subintervals $[0,0.5],[0.5,1],[1,1.5],[1.5,2]$ and the instantaneous rate of changes at the points $x=0.5,1,1.5,2$.

Example 7.6

A particle moves along a horizontal line such that its position at any time $t\geq 0$ is given by $s(t)=t^3-6t^2+9t+1$, where s is measured in metres and t in seconds.

(1) At what time the particle is at rest.

(2) At what time the particle changes direction.

(3) Find the total distance travelled by the particle in the first 2 seconds.

Example 7.7

If we blow air into a balloon of spherical shape at a rate of $1000cm^3$ per second. At what rate the radius of the baloon changes when the radius is 7cm? Also compute the rate at which the surface area changes.

Example 7.8

The price of a product is related to the number of units' available (supply) by the equation $Px+3P-16x=234$, where P is the price of the product per unit in Rupees and x is the number of units. Find the rate at which the price is changing with respect to time when 90 units are available and the supply is increasing at a rate of 15 units/week.

Example 7.9

Salt is poured from a conveyer belt at a rate of 30 cubic metre per minute forming a conical pile with a circular base whose height and diameter of base are always equal. How fast is the height of the pile increasing when the pile is 10 metre high?

Example 7.10 (Two variable related rate problem)

A road running north to south crosses a road going east to west at the point P. Car A is driving north along the first road, and car B is driving east along the second road. At a particular time car A is 10 kilometres to the north of P and traveling at 80 km/hr, while car B is 15 kilometres to the east of P and traveling at 100 km/hr. How fast is the distance between the two cars changing?

Example 7.14

Find the acute angle between $y=x^2$ and $y=(x-3)^2$.

Example 7.15

Find the acute angle between the curves $y=x^2$ and $x=y^2$ at their point of intersections $(0,0)$ and $(1,1)$.

Example 7.17

If the curves $ax^2+by^2=1$ and $cx^2+dy^2=1$ intersect each other orthogonally if,

$$\frac{1}{a}-\frac{1}{b}=\frac{1}{c}-\frac{1}{d}.$$

Example 7.18

Prove that the ellipse $x^2+4y^2=8$ and the hyperbola $x^2-2y^2=4$ intersect orthogonally.

Example 7.30

Expand $\log(1+x)$ as a Maclaurin's series upto 4 non-zero terms for $-1<x\leq 1$.

Example 7.31

Expand $\tan x$ in ascending powers of x upto 5^{th} power for $-\frac{\pi}{2} < x < \frac{\pi}{2}$.

Example 7.32

Write the Taylor series expansion of $\frac{1}{x}$ about $x = 2$ by finding the first three non-zero ter

Example 7.43

Using the l'Hôpital rule prove that, $\lim_{x \to 0^+}(1+x)^{\frac{1}{x}} = e$.

Example 7.44

Evaluate : $\lim_{x \to \infty}(1+2x)^{\frac{1}{2\log x}}$.

Example 7.45

Evaluate : $\lim_{x \to 1^+} x^{\frac{1}{1-x}}$.

Example 7.51

Find the intervals of monotonicity and hence find the local extrema for the funct $f(x) = x^{\frac{2}{3}}$.

Example 7.53

Discuss the monotonicity and local extrema of the function

$f(x) = \log(1+x) - \frac{x}{1+x}, x > -1$ and hence find the domain where, $\log(1+x) > \frac{x}{1+x}$.

Example 7.54

Find the intervals of monotonicity and local extrema of the function $f(x) = x\log x + 3x$.

Example 7.55

Find the intervals of monotonicity and local extrema of the function $f(x) = \frac{1}{1+x^2}$.

Example 7.56

Find the intervals of monotonicity and local extrema of the function $f(x) = \frac{x}{1+x^2}$.

Example 7.57

Determine the intervals of concavity of the curve $f(x) = (x-1)^3 \cdot (x-5), x \in \mathbb{R}$ and, points inflection if any.

Example 7.58

Determine the intervals of concavity of the curve $y = 3 + \sin x$.

Example 7.59

Find the local extremum of the function $f(x) = x^4 + 32x$.

Example 7.60

Find the local extrema of the function $f(x) = 4x^6 - 6x^4$.

Example 7.61

Find the local maximum and minimum of the function x^2y^2 on the line $x + y = 10$.

Example 7.62

A 12 square unit piece of thin material is to be made an open box by cutting small squares from the four corners and folding the sides up. Find the length of the side of the square to be removed when the volume is maximum?

Example 7.63

Find the points on the unit circle $x^2 + y^2 = 1$ nearest and farthest from $(1,1)$.

Example 7.64

A steel plant is capable of producing x tonnes per day of low-grade steel and y tonnes per day of a high-grade steel, where $y = \frac{40-5x}{10-x}$. If the fixed market price of low-grade steel is half that of high-grade steel, then what should be optimal productions in low-grade steel and high-grade steel in order to have maximum receipts (gains).

Example 7.65

Prove that among all the rectangles of the given area square has the least perimeter.

Note :

The following 4 examples have more than 10 stages apart from diagram. Hence any 5 points is enough for 5 marks.

Example 7.69

Sketch the curve $y = f(x) = x^2 - x - 6$.

Example 7.70

Sketch the curve $y = f(x) = x^3 - 6x - 9$.

Example 7.71

Sketch the curve $y = \frac{x^2 - 3x}{(x-1)}$.

Example 7.72

Sketch the graph of the function $y = \frac{3x}{x^2 - 1}$.

EXERCISE 8.1

1. Let $f(x) = \sqrt[3]{x}$. Find the linear approximation at $x = 27$. Use the linear approximation to approximate $\sqrt[3]{27.2}$.

EXERCISE 8.2

5. The trunk of a tree has diameter 30cm. During the following year, the circumference grew 6cm.

   (i) Approximately, how much did the tree's diameter grow?

   (ii) What is the percentage increase in area of the tree's cross-section?

EXERCISE 8.4

2. For each of the following functions find the f_x, f_y, and show that $f_{xy} = f_{yx}$. (each 5 marks)

   (i) $f(x, y) = \frac{3x}{y + \sin x}$ (ii) $f(x, y) = \tan^{-1}\left(\frac{x}{y}\right)$ (iii) $f(x, y) = \cos(x^2 - 3xy)$

3. If $U(x,y,z)=\frac{x^2+y^2}{xy}+3z^2y$, find $\frac{\partial U}{\partial x}, \frac{\partial U}{\partial y}$, and $\frac{\partial U}{\partial z}$.

5. For each of the following functions find the g_{xy}, g_{xx}, g_{yy} and g_{yx}.

(i) $g(x,y)=xe^y+3x^2y$ (ii) $g(x,y)=\log(5x+3y)$

(iii) $g(x,y)=x^2+3xy-7y+\cos(5x)$ (each 5 marks)

6. Let $w(x,y,z)=\frac{1}{\sqrt{x^2+y^2+z^2}},(x,y,z)\neq(0,0,0)$. Show that $\frac{\partial^2 w}{\partial x^2}+\frac{\partial^2 w}{\partial y^2}+\frac{\partial^2 w}{\partial z^2}=0$.

7. If $V(x,y)=e^x(x\cos y-y\sin y)$, then prove that $\frac{\partial^2 V}{\partial x^2}+\frac{\partial^2 V}{\partial y^2}=0$.

9. If $v(x,y,z)=x^3+y^3+z^3+3xyz$, show that $\frac{\partial^2 v}{\partial y\partial z}=\frac{\partial^2 v}{\partial z\partial y}$.

EXERCISE 8.6

1. If $u(x,y)=x^2y+3xy^4, x=e^t$ and $y=\sin t$, find $\frac{du}{dt}$ and evaluate it at $t=0$.

2. If $v(x,y)=x\sin(xy^2), x=\log t$ and $y=e^t$, find $\frac{dv}{dt}$.

3. If $w(x,y,z)=x^2+y^2+z^2, x=e^t, y=e^t\sin t$ and $z=e^t\cos t$, find $\frac{dw}{dt}$.

4. Let $U(x,y,z)=xyz, x=e^{-t}, y=e^{-t}\cos t, z=\sin t, t\in\mathbb{R}$. Find $\frac{dU}{dt}$.

5. If $w(x,y)=6x^3-3xy+2y^2, x=e^s; y=\cos s, s\in\mathbb{R}$, find $\frac{dw}{ds}$, and evaluate at $s=0$,

6. If $z(x,y)=x\tan^{-1}(xy), x=t^2, y=se^t, s,t\in\mathbb{R}$. Find $\frac{\partial z}{\partial s}$ and $\frac{\partial z}{\partial t}$ at $s=t=1$.

7. Let $u(x,y)=e^x\sin y$, where $x=st^2, y=s^2t, s,t\in\mathbb{R}$. Find $\frac{\partial u}{\partial s},\frac{\partial u}{\partial t}$ and evaluate them at (1,

8. Let $z(x,y)=x^3-3x^2y^3$, where $x=se^t, y=se^{-t}, s,t\in\mathbb{R}$. Find $\frac{\partial z}{\partial s}$ and $\frac{\partial z}{\partial t}$.

9. $W(x,y,z)=xy+yz+zx, x=u-v, y=uv, z=u+v, u,v\in\mathbb{R}$. Find $\frac{\partial W}{\partial u},\frac{\partial W}{\partial v}$, and evalu

them at $\left(\frac{1}{2},1\right)$.

EXERCISE 8.7

2. Prove that $f(x,y)=x^3-2x^2y+3xy^2+y^3$ is homogeneous; what is the degree? Verify Euler Theorem for f.

3. Prove that $g(x,y)=x\log\left(\frac{y}{x}\right)$ is homogeneous; what is the degree? Verify Euler's Theore for g.

5. If $v(x,y)=\log\left(\frac{x^2+y^2}{x+y}\right)$, prove that $x\frac{\partial v}{\partial x}+y\frac{\partial v}{\partial y}=1$.

6. If $w(x,y,z)=\log\left(\frac{5x^3y^4+7y^2xz^4-75y^3z^4}{x^2+y^2}\right)$, find $x\frac{\partial w}{\partial x}+y\frac{\partial w}{\partial y}+z\frac{\partial w}{\partial z}$.

Example 8.4

A right circular cylinder has radius $r=10$ cm. and height $h=20$ cm. Suppose that the radius of the cylinder is increased from 10 cm to 10.1 cm and the height does not change. Estimate the change in the volume of the cylinder. Also, calculate the relative error and percentage error.

Example 8.11

Let $f(x,y)=0$ if $xy\neq 0$ and $f(x,y)=1$ if $xy=0$.

(i) Calculate : $\frac{\partial f}{\partial x}(0,0), \frac{\partial f}{\partial y}(0,0)$.

(ii) Show that f is not continuous at $(0,0)$.

Example 8.12

Let $F(x,y)=x^3y+y^2x+7$ for all $(x,y)\in\mathbb{R}^2$. Calculate $\frac{\partial F}{\partial x}(-1,3)$ and $\frac{\partial F}{\partial y}(-2,1)$.

Example 8.13

Let $f(x,y)=\sin(xy^2)+e^{x^3+5y}$ for all $(x,y)\in\mathbb{R}^2$. Calculate $\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}$ and $\frac{\partial^2 f}{\partial y\partial x}$, and $\frac{\partial^2 f}{\partial x\partial y}$,

Example 8.14

Let $w(x,y)=xy+\frac{e^y}{y^2+1}$ for all $(x,y)\in\mathbb{R}^2$. Calculate $\frac{\partial^2 w}{\partial y\partial x}$ and $\frac{\partial^2 w}{\partial x\partial y}$.

Example 8.15

Let $u(x,y)=e^{-2y}\cos(2x)$ for all $(x,y)\in\mathbb{R}^2$. Prove that u is a harmonic function in $\mathbb{R}^2$.

Example 8.18

Verify $\frac{dF}{dt}=\frac{\partial F}{\partial x}\frac{dx}{dt}+\frac{\partial F}{\partial y}\frac{dy}{dt}$ for the function $F(x,y)=x^2-2y^2+2xy$ where $x(t)=\cos t, y(t)=\sin t, t\in[0,2\pi]$.

Example 8.19

Let $g(x,y)=x^2-yx+\sin(x+y), x(t)=e^{3t}, y(t)=t^2, t\in\mathbb{R}$. Find $\frac{dg}{dt}$.

Example 8.20

Let $g(x,y)=2y+x^2, x=2r-s, y=r^2+2s, r,s\in\mathbb{R}$. Find $\frac{\partial g}{\partial r}, \frac{\partial g}{\partial r}$.

Example 8.22

If $u=\sin^{-1}\left(\frac{x+y}{\sqrt{x}+\sqrt{y}}\right)$, Show that $x\frac{\partial u}{\partial x}+y\frac{\partial u}{\partial y}=\frac{1}{2}\tan u$.

EXERCISE 9.1

1. Find an approximate value of $\int_1^{1.5} x dx$ by applying the left-end rule with the part $\{1.1, 1.2, 1.3, 1.4, 1.5\}$.

2. Find an approximate value of $\int_1^{1.5} x^2 dx$ by applying the right-end rule with the part $\{1.1, 1.2, 1.3, 1.4, 1.5\}$.

3. Find an approximate value of $\int_1^{1.5} (2-x)dx$ by applying the mid-point rule with the part $\{1.1, 1.2, 1.3, 1.4, 1.5\}$.

EXERCISE 9.2

1. Evaluate the following integrals as the limit of sums:

(i) $\int_0^1 (5x+4)dx$ (ii) $\int_1^2 (4x^2-1)dx$

EXERCISE 9.3

2. Evaluate the following integrals using properties of integration:

(ii) $\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} (x^5 + x\cos x + \tan^3 x + 1)\, dx$

(v) $\int_0^{2\pi} \sin^4 x \cos^3 x\, dx$ (vi) $\int_0^1 |5x-3|\, dx$

(vii) $\int_0^{\sin^2 x} \sin^{-1}\sqrt{t}\, dt + \int_0^{\cos^2 x} \cos^{-1}\sqrt{t}\, dt$ (viii) $\int_0^1 \frac{\log(1+x)}{1+x^2} dx$

(ix) $\int_0^{\pi} \frac{x\sin x}{1+\sin x} dx$

(xi) $\int_0^{\pi} x\left[\sin^2(\sin x) + \cos^2(\cos x)\right] dx$

EXERCISE 9.7

1. Evaluate the following

(ii) $\int_0^{\frac{\pi}{2}} \frac{e^{-\tan x}}{\cos^6 x} dx$

2. If $\int_0^{\infty} e^{-ax^2} x^3 dx = 32, a > 0$, find α.

EXERCISE 9.8

3. Find the area of the region bounded by the curve $2+x-x^2+y=0$, x-axis, $x=-3$ and x

4. Find the area of the region bounded by the line $y=2x+5$ and the parabola $y=x^2-2x$.

5. Find the area of the region bounded between the curves $y=\sin x$ and $y=\cos x$ and the lines $x=0$ and $x=\pi$.

6. Find the area of the region bounded by $y=\tan x$, $y=\cot x$ and the lines $x=0$, $x=\frac{\pi}{2}$, $y=0$.

7. Find the area of the region bounded by the parabola $y^2=x$ and the line $y=x-2$.

8. Father of a family wishes to divide his square field bounded by $x=0$, $x=4$, $y=4$ and $y=0$ along the curve $y^2=4x$ and $x^2=4y$ into three equal parts for his wife, daughter and son. Is it possible to divide? If so, find the area to be divided among them.

9. The curve $y=(x-2)^2+1$ has a minimum point at P. A point Q on the curve is such that the slope of PQ is 2. Find the area bounded by the curve and the chord PQ.

10. Find the area of the region common to the circle $x^2+y^2=16$ and the parabola $y^2=6x$.

EXERCISE 9.9

2. Find the volume of the solid generated by revolving the region enclosed by $y=e^{-2x}, y=0, x=0$ and $x=1$ about x-axis, using integration.

3. Find the volume of the solid generated by revolving the region enclosed by $x^2=1+y$ and $y=3$ about y-axis, using integration.

4. The region enclosed between the graphs of $y=x$ and $y=x^2$ is denoted by R, find the volume generated by R when R is rotated through 360° about x-axis, using integration.

5. Find the volume of the container which is in the shape of a right circular concial frustum as shown in the given figure by using integration.

6. A watermelon is ellipsoid in model. Its major and minor axes are 20 cm and 10 cm respectively. Find its volume by revolving the area about its major axis.

Example 9.1

Estimate the value of $\int_0^{0.5} x^2 dx$ using the Riemann sum corresponding to 5 subintervals of equal width and applying (i) left-end rule (ii) right-end rule (iii) the mid-point rule.

Example 9.2

Evaluate $\int_0^1 x dx$, as a limit of a sum.

Example 9.3

Evaluate $\int_0^1 x^3 dx$, as a limit of a sum.

Example 9.4

Evaluate $\int_1^4 (2x^2+3) dx$, as a limit of a sum.

Example 9.10

Evaluate: $\int_1^2 \frac{x}{(x+1)(x+2)} dx$.

Example 9.11

Evaluate : $\int_0^{\frac{\pi}{2}} \frac{\cos\theta}{(1+\sin\theta)(2+\sin\theta)} d\theta$.

Example 9.12

Evaluate : $\int_0^{\frac{1}{\sqrt{2}}} \frac{\sin^{-1} x}{(1-x^2)^{\frac{3}{2}}} dx$.

Example 9.13

Evaluate : $\int_0^{\frac{\pi}{2}} \left(\sqrt{\tan x} + \sqrt{\cot x}\right) dx$.

Example 9.14

Evaluate : $\int_0^{1.5} \left[x^2\right] dx$, where $[x]$ is the greatest integer function.

Example 9.15

Evaluate : $\int_{-4}^{4} |x+3| \, dx$.

Example 9.16

Show that $\int_0^{\frac{\pi}{2}} \frac{dx}{4+5\sin x} = \frac{1}{3}\log_e 2$.

Example 9.17

Prove that $\int_0^{\frac{\pi}{4}} \frac{\sin 2x \, dx}{\sin^4 x + \cos^4 x} = \frac{\pi}{4}$.

Example 9.18

Prove that $\int_0^{\frac{\pi}{4}} \frac{dx}{a^2 \sin^2 x + b^2 \cos^2 x} = \frac{1}{ab} \tan^{-1}\left(\frac{a}{b}\right)$, where $a, \; b > 0$.

Example 9.19

Evaluate $\int_0^{\frac{\pi}{4}} \frac{1}{\sin x + \cos x} dx$.

Example 9.21

Evaluate $\int_0^{\pi} \frac{x}{1+\sin x} dx$.

Example 9.27

Prove that $\int_0^{\frac{\pi}{4}} \log(1+\tan x) dx = \frac{\pi}{8} \log 2$.

Example 9.28

Show that $\int_0^1 \left(\tan^{-1} x + \tan^{-1}(1-x)\right) dx = \frac{\pi}{2} - \log_e 2$.

Example 9.30

Evaluate $\int_{-\pi}^{\pi} \frac{\cos^2 x}{1+a^x} dx$.

Example 9.36

Evaluate $\int_{0}^{\frac{\pi}{2}} \frac{dx}{4\sin^2 x + 5\cos^2 x}$.

Example 9.40

Evaluate $\int_0^{2a} x^2\sqrt{2ax - x^2}\,dx$.

Example 9.43

Prove that $\int_0^{\infty} e^{-x}x^n dx = n!$, where n is a positive integer.

Example 9.45

Show that $\Gamma(n) = 2\int_0^{\infty} e^{-x^2}x^{2n-1}dx.$

Example 9.46

Evaluate $\int_0^{\infty}\frac{x^n}{n^x}dx$, where n is a positive integer ≥ 2.

Example 9.49

Find the area of the region bounded by the ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$.

Example 9.50

Find the area of the region bounded between the parabola $y^2 = 4ax$ and its latus rectum.

Example 9.51

Find the area of the region bounded by the y-axis and the parabola $x = 5 - 4y - y^2$.

Example 9.52

Find the area of the region bounded by $x-$ axis, the sine curve $y = \sin x$, the lines $x = 0$ and $x = 2\pi$.

Example 9.53

Find the area of the region bounded by $x-$ axis, the curve $y = |\cos x|$, the lines $x = 0$ and $x = \pi$.

Example 9.54

Find the area of the regioun bounded between the parabolas $y^2 = 4x$ and $x^2 = 4y$.

Example 9.55

Find the area of the region bounded between the parabola $x^2 = y$ and the curve $y = |x|$.

Example 9.56

Find the area of the region bounded by $y = \cos x, y = \sin x$, the lines $x = \frac{\pi}{4}$ and $x = \frac{5\pi}{4}$.

Example 9.57

The region enclosed by the circle $x^2 + y^2 = a^2$ is divided into two segments by the line $x = h$. Find the area of the smaller segment.

Example 9.58

Find the area of the region in the first quadrant bounded by the parabola $y^2 = 4x$, the lin $x + y = 3$ and y-axis.

Example 9.59

Find, by integration, the area of the region bounded by the lines $5x - 2y = 15$, $x + y + 4 =$ and the x-axis.

Example 9.60

Using integration find the area of the region bounded by triangle ABC, whose vertices A, and C are $(-1,1)$, $(3,2)$ and $(0,5)$ respectively.

Example 9.61

Using integration, find the area of the region which is bounded by x-axis, the tangent an normal to the circle $x^2 + y^2 = 4$ drawn at $(1, \sqrt{3})$.

Example 9.63

Find the volume of a right-circular cone of base radius r and height h.

Example 9.64

Find the volume of the spherical cap of height h cut of from a sphere of radius r.

Example 9.65

Find the volume of the solid formed by revolving the region bounded by the parabola $y = x^2$ x-axis, ordinates $x = 0$ and $x = 1$ about the x-axis.

Example 9.66

Find the volume of the solid formed by revolving the region bounded by the ellips $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1, a > b$ about the major axis.

Example 9.68

Find, by integration, the volume of the solid generated by revolving about y-axis the regio bounded between the curve $y = \frac{3}{4}\sqrt{x^2 - 16}$, $x \geq 4$, the y-axis, and the lines $y = 1$ and $y = 6$.

Example 9.69

Find, by integration, the volume of the solid generated by revolving about y-axis the region bounded by the curves $y = \log x$, $y = 0$, $x = 0$ and $y = 2$.

EXERCISE 10.5

1. If F is the constant force generated by the motor of an automobile of mass M, its velocity v is given by $M\frac{dv}{dt} = F - kv$, where k is a constant. Express v in terms of t given that $v = 0$ when $t = 0$.

2. The velocity v, of a parachute falling vertically satisfies the equation $v\frac{dv}{dx} = g\left(1 - \frac{v^2}{k^2}\right)$, where g and k are constants. If v and x are both initially zero, find v in terms of x.

3. Find the equation of the curve whose slope is $\frac{y-1}{x^2+x}$ and which passes through the point $(1,0)$.

4. Solve the following differential equations:

(ii) $ydx+(1+x^2)\tan^{-1}x\,dy=0$

(v) $(e^y+1)\cos x\,dx+e^y\sin x\,dy=0$

(vi) $(ydx-xdy)\cot\left(\frac{x}{y}\right)=ny^2dx$

(vii) $\frac{dy}{dx}-x\sqrt{25-x^2}=0$

(viii) $x\cos y\,dy=e^x(x\log x+1)dx$

(ix) $\tan y\frac{dy}{dx}=\cos(x+y)+\cos(x-y)$

(x) $\frac{dy}{dx}=\tan^2(x+y)$

EXERCISE 10.6

Solve the following differential equations:

1. $\left[x+y\cos\left(\frac{y}{x}\right)\right]dx=x\cos\left(\frac{y}{x}\right)dy$
2. $(x^3+y^3)dy-x^2ydx=0$
3. $ye^{\frac{x}{y}}dx=\left(xe^{\frac{x}{y}}+y\right)dy$
4. $2xydx+(x^2+2y^2)dy=0$
5. $(y^2-2xy)dx=(x^2-2xy)dy$
6. $x\frac{dy}{dx}=y-x\cos^2\left(\frac{y}{x}\right)$
7. $\left(1+3e^{\frac{y}{x}}\right)dy+3e^{\frac{y}{x}}\left(1-\frac{y}{x}\right)dx=0$, given that y = 0 when $x=1$.
8. $(x^2+y^2)dy=xy\,dx$. It is given that $y(1)=1$ and $y(x_0)=e$. Find the value of x_0.

EXERCISE 10.7

Solve the following Linear differential equations:

2. $(1-x^2)\frac{dy}{dx}-xy=1$
3. $\frac{dy}{dx}+\frac{y}{x}=\sin x$
4. $(x^2+1)\frac{dy}{dx}+2xy=\sqrt{x^2+4}$
5. $(2x-10y^3)dy+ydx=0$
6. $x\sin x\frac{dy}{dx}+(x\cos x+\sin x)y=\sin x$
7. $\left(y-e^{\sin^{-1}x}\right)\frac{dx}{dy}+\sqrt{1-x^2}=0$
8. $\frac{dy}{dx}+\frac{y}{(1-x)\sqrt{x}}=1-\sqrt{x}$
9. $(1+x+xy^2)\frac{dy}{dx}+(y+y^3)=0$
10. $\frac{dy}{dx}+\frac{y}{x\log x}=\frac{\sin 2x}{\log x}$
11. $(x+a)\frac{dy}{dx}-2y=(x+a)^4$
12. $\frac{dy}{dx}=\frac{\sin^2 x}{1+x^3}-\frac{3x^2}{1+x^3}y$
13. $x\frac{dy}{dx}+y=x\log x$
14. $x\frac{dy}{dx}+2y-x^2\log x=0$
15. $\frac{dy}{dx}+\frac{3y}{x}=\frac{1}{x^2}$, given that $y=2$ when $x=1$

EXERCISE 10.8

1. The rate of increase in the number of bacteria in a certain bacteria culture is proportional the number present. Given that the number triples in 5 hours, find how many bacteria w be present after 10 hours?
2. Find the population of a city at any time t, given that the rate of increase of population proportional to the population at that instant and that in a period of 40 years the populati increased from 3,00,000 to 4,00,000.
3. The equation of electromotive force for an electric circuit containing resistance ar self-inductance is $E = Ri + L\frac{di}{dt}$, where E is the electromotive force given to the circuit, the resistance and L, the coefficient of induction. Find the current i at time t when E = 0.
4. The engine of a motor boat moving at 10 m/s is shut off. Given that the retardation at ar subsequent time (after shutting off the engine) equal to the velocity at that time. Find th velocity after 2 seconds of switching off the engine.
5. Suppose a person deposits 10,000 Indian rupees in a bank account at the rate of 5% p annum compounded continuously. How much money will be in his bank account 18 month later?
6. Assume that the rate at which radioactive nuclei decay is proportional to the number o such nuclei that are present in a given sample. In a certain sample 10% of the origin number of radioactive nuclei have undergone disintegration in a period of 100 years.Wha percentage radioactive nuclei will remain after 1000 years? (Take the initial amount as A_0
7. Water at temperature $100^\circ C$ cools in 10 minutes to $80^\circ C$ in a room temperature of $25^\circ C$ Find
   (i) The temperature of water after 20 minutes
   (ii) The time when the temperature is $40^\circ C$ $\left[\log_e \frac{11}{15} = -0.3101;\ \log_e 5 = 1.6094\right]$
8. At 10.00 A.M. a woman took a cup of hot instant coffee from her microwave oven an placed it on a nearby Kitchen counter to cool. At this instant the temperature of the coffe was $180^\circ F$, and 10 minutes later it was $160^\circ F$. Assume the constant temperature of th kitchen was $70^\circ F$. What was the temperature of the coffee at 10.15A.M.?
9. A pot of boiling water at $100^\circ C$ is removed from a stove at time $t = 0$ and left to cool in th kitchen. After 5minutes, the water temperature has decreased to $80^\circ C$, and another minutes later it has dropped to $65^\circ C$. Determine the temperature of the kitchen.
10. A tank initially contains 50 litres of pure water. Starting at time $t = 0$ a brine containing with 2 grams of dissolved salt per litre flows into the tank at the rate of 3 litres per minute The mixture is kept uniform by stirring and the well-stirred mixture simultaneously flows out of the tank at the same rate. Find the amount of salt present in the tank at any time $t > 0$.

Example 10.12

Find the particular solution of $(1+x^3)dy - x^2 y dx = 0$ satisfying the condition $y(1) = 2$.

Example 10.13

Solve $y' = \sin^2(x - y + 1)$.

Classification of Questions – 5 marks

Example 10.14

Solve : $\frac{dy}{dx}=\sqrt{4x+2y-1}$.

Example 10.15:

Solve $\frac{dy}{dx}=\frac{x-y+5}{2(x-y)+7}$.

Example 10.17

Solve $(x^2-3y^2)dx+2xydy=0$.

Example 10.18

Solve $\left(y+\sqrt{x^2+y^2}\right)dx-xdy=0,\quad y(1)=0$.

Example 10.19

Solve $(2x+3y)dx+(y-x)dy=0$.

Example 10.21

Solve $\left(1+2e^{x/y}\right)dx+2e^{x/y}\left(1-\frac{x}{y}\right)dy=0$.

Example 10.23

Solve $\left[y(1-x\tan x)+x^2\cos x\right]dx-xdy=0$.

Example 10.24

Solve : $\frac{dy}{dx}+2y\cot x=3x^2\operatorname{cosec}^2 x$.

Example 10.25

Solve $(1+x^3)\frac{dy}{dx}+6x^2y=1+x^2$.

Example 10.26

Solve $ye^y dx=(y^3+2xe^y)dy$.

Example 10.27

The growth of a population is proportional to the number present. If the population of a colony doubles in 50 years, in how many years will the population become triple?

Example 10.28

A radioactive isotope has an initial mass $200mg$, which two years later is $50mg$. Find the expression for the amount of the isotope remaining at any time. What is its half-life? (half-life means the time taken for the radioactivity of a specified isotope to fall to half its original value).

Example 10.29

In a murder investigation, a corpse was found by a detective at exactly 8 P.M. Being alert, the detective also measured the body temperature and found it to be 70°F. Two hours later, the detective measured the body temperature again and found it to be 60°F. If the room temperature is 50°F, and assuming that the body temperature of the person before death was 98.6°F, at what time did the murder occur?

$[\log(2.43)=0.88789;\quad \log(0.5)=-0.69315]$

Example 10.30

A tank contains 1000 litres of water in which 100 grams of salt is dissolved. Brine (*Brine is high-concentration solution of salt (usually sodium chloride) in water*) runs in a rate of 10 litres per minute, and each litre contains 5grams of dissolved salt. The mixture of the tank is kept uniform by stirring. Brine runs out at 10 litres per minute. Find the amount of salt at any time t.

EXERCISE 11.2

2. A six sided die is marked '1' on one face, '3' on two of its faces, and '5' on remaining three faces. The die is thrown twice. If X denotes the total score in two throws, find
   (i) the probability mass function (ii) the cumulative distribution function
   (iii) $P(4 \le X \le 10)$ (ii) $P(X \ge 6)$
3. Find the probability mass function and cumulative distribution function of number of girl child in families with 4 children, assuming equal probabilities for boys and girls.
4. Suppose a discrete random variable can only take the values $0,1,$ and 2.
   The probability mass function is defined by

$$f(x)=\begin{cases}\dfrac{x^2+1}{k}, \text{ for } x=0,1,2\\ 0 \quad \text{otherwise}\end{cases}$$

   Find (i) the value of k (ii) cumulative distribution function (iii) $P(X \ge 1)$.
5. The cumulative distribution function of a discrete random variable is given by

$$F(x)=\begin{cases}0 & -\infty < x < -1\\ 0.15 & -1 \le x < 0\\ 0.35 & 0 \le x < 1\\ 0.60 & 1 \le x < 2\\ 0.85 & 2 \le x < 3\\ 1 & 3 \le x < \infty\end{cases}$$

   Find (i) the probability mass function (ii) $P(X<1)$ and (iii) $P(X \ge 2)$.
6. A random variable X has the following probability mass function:

X	1	2	3	4	5
$f(x)$	k^2	$2k^2$	$3k^2$	$2k$	$3k$

   Find (i) the value of k (ii) $P(2 \le X < 5)$ (iii) $P(3 < X)$
7. The cumulative distribution function of a discrete random variable is given by

$$F(x)=\begin{cases}0 & \text{for } -\infty < x < 0\\ \frac{1}{2} & \text{for } 0 \le x < 1\\ \frac{3}{5} & \text{for } 1 \le x < 2\\ \frac{4}{5} & \text{for } 2 \le x < 3\\ \frac{9}{10} & \text{for } 3 \le x < 4\\ 1 & \text{for } 4 \le x < \infty\end{cases}$$

   Find (i) the probability mass function (ii) $P(X<3)$ and (iii) $PX \ge 2)$.

EXERCISE 11.3

3. Suppose the amount of milk sold daily at a milk booth is distributed with a minimum of 200 litres and a maximum of 600 litres with probability density function

$$f(x)=\begin{cases}k & 200\le x\le 600\\ 0 & \text{otherwise}\end{cases}$$

Find (i) the value of k (ii) the distribution function
(iii) the probability that daily sales will fall between 300 litres and 500 litres?

4. The probability density function of X is given by $f(x)=\begin{cases}ke^{-\frac{x}{3}} & \text{for } x>0\\ 0 & \text{for } x\le 0\end{cases}$

Find (i) the value of k (ii) the distribution function (iii) $P(X<3)$
(iv) $P(5\le X)$ (v) $P(X\le 4)$

5. If X is the random variable with probability density function,

$$f(x)=\begin{cases}x+1, & -1\le x<0\\ -x+1, & 0\le x<1\\ 0 & \text{otherwise}\end{cases}$$

then find (i) the distribution function $F(x)$ (ii) $P(-0.5\le X\le 0.5)$

6. If X is the random variable with distribution function,

$$F(x)=\begin{cases}0, & x<0\\ \frac{1}{2}(x^2+x) & 0\le x<1\\ 1, & x\ge 1\end{cases}$$

then find (i) the probability density function $f(x)$ (ii) $P(0.3\le X\le 0.6)$.

EXERCISE 11.4

2. Two balls are drawn in succession without replacement from an urn containing four red balls and three black balls. Let X be the possible outcomes of drawing red balls. Find the probability mass function and mean for X.

4. Four fair coins are tossed once. Find the probability mass function, mean and variance for number of heads occurred.

7. The probability density function of the random variable X is given by

$$f(x)=\begin{cases}16xe^{-4x} & \text{for } x>0\\ 0 & \text{for } x\le 0\end{cases}$$

find the mean and variance of X.

EXERCISE 11.5

6. If the probability that a fluorescent light has a useful life of at least 600 hours is 0.9, find the probabilities that among 12 such lights

(i) exactly 10 will have a useful life of at least 600 hours;

(ii) at least 11 will have a useful life of at least 600 hours.

(iii) atleast 2 will not have a useful life of at least 600 hours.

7. The mean and standard deviation of a binomial variate X are respectively 6 and 2. Find (i) the probability mass function (ii) $P(X=3)$ (iii) $P(X \geq 2)$.

Example 11.7

If the probability mass function $f(x)$ of a random variable X is

x	1	2	3	4
$f(x)$	$\frac{1}{12}$	$\frac{5}{12}$	$\frac{5}{12}$	$\frac{1}{12}$

find (i) its cumulative distribution function, hence find (ii) $P(X \leq 3)$ and, (iii) $P(X \geq 2)$

Example 11.8

A six sided die is marked '1' on one face, '2' on two of its faces, and '3' on remaining three faces. The die is rolled twice. If X denotes the total score in two throws.

(i) Find the probability mass function

(ii) Find the cumulative distribution function

(iii)Find $P(3 \leq X < 6)$ (iv) Find $P(X \geq 4)$

Example 11.9

Find the probability mass function $f(x)$ of the discrete random variable X whose cumulative distribution function

$$F(x) = \begin{cases} 0 & -\infty < x < -2 \\ 0.25 & -2 \leq x < -1 \\ 0.60 & -1 \leq x < 0 \\ 0.90 & 0 \leq x < 1 \\ 1 & 1 \leq x < \infty \end{cases}$$

Also find (i) $P(X < 0)$ and (ii) $P(X \geq -1)$

Example 11.10

A random variable X has the following probability mass function.

x	1	2	3	4	5	6
$f(x)$	k	$2k$	$6k$	$5k$	$6k$	$10k$

Find (i) $P(2 < X < 6)$ (ii) $P(2 \leq X < 5)$ (iii) $P(X \leq 4)$ (iv) $P(3 < X)$.

Example 11.12

If X is the random variable with probability density function,

$$f(x) = \begin{cases} x-1, & 1 \leq x < 2 \\ -x+3, & 2 \leq x < 3 \\ 0 & \text{otherwise} \end{cases}$$

find (i) the distribution function $F(x)$

(ii) $P(1.5 \leq X \leq 2.5)$

Example 11.14

The probability density function of X is given by $f(x) = \begin{cases} k & 1 \leq x \leq 5 \\ 0 & \text{otherwise} \end{cases}$

find (i) distribution function (ii) $P(X < 3)$ (iii) $P(2 < X < 4)$ (iv) $P(3 \leq X)$

Example 11.17

Two balls are chosen randomly from an urn containing 8 white and 4 black balls. Suppose that we win Rs 20 for each black ball selected and we lose Rs10 for each white ball selected. Find the expected winning amount and variance.

Example 11.18

Find the mean and variance of a random variable X, whose probability density function is $f(x)=\begin{cases}\lambda e^{-\lambda x} & \text{for } x \geq 0\\ 0 & \text{otherwise}\end{cases}$

Example 11.20

A multiple choice examination has ten questions, each question has four distractors with exactly one correct answer. Suppose a student answers by guessing and if X denotes the number of correct answers, find (i) binomial distribution (ii) probability that the student will get seven correct answers (iii) at least one correct answer.

Example 11.22

On the average, 20% of the products manufactured by ABC Company are found to be defective. If we select 6 of these products at random and X denote the number of defective products find (i) two products are defective (ii) at most one product is defective (iii) at least two products are defective.

EXERCISE 12.1

5. (i) Define an operation $*$ on $\mathbb{Q}$ as follows: $a*b=\left(\frac{a+b}{2}\right); a,b\in\mathbb{Q}$. Examine the closure, commutative, associative properties satisfied by $*$ on $\mathbb{Q}$.

   (ii) Define an operation $*$ on $\mathbb{Q}$ as follows: $a*b=\left(\frac{a+b}{2}\right); a,b\in\mathbb{Q}$. Examine the existence of identity and the existence of inverse for the operation $*$ on $\mathbb{Q}$.

9. Let $M=\left\{\begin{pmatrix} x & x\\ x & x\end{pmatrix}: x\in R-\{0\}\right\}$ and let $*$ be the matrix multiplication. Determine whether M is closed under $*$. If so, examine the closure, commutative, associative, existence of identity and inverse properties.

10. Let A be $Q-\{1\}$. Define $*$ on A by $x*y=x+y-xy$. Is $*$ a binary on A. If so, examine the closure, commutative, associative, the existence of identity and existence of inverse properties.

EXERCISE 12.2

7. Verify whether the following compound propositions are tautologies or contradictions or contingency

   (i) $(p\wedge q)\wedge\neg(p\vee q)$ (ii) $((p\vee q)\wedge\neg p)\rightarrow q$

   (iii) $(p\rightarrow q)\leftrightarrow(\neg p\rightarrow q)$ (iv) $((p\rightarrow q)\wedge(q\rightarrow r))\rightarrow(p\rightarrow r)$

8. Show that (i) $\neg(p\wedge q)\equiv\neg p\vee\neg q$ (ii) $\neg(p\rightarrow q)\equiv p\wedge\neg q$.

10. Show that $p\rightarrow q$ and $q\rightarrow p$ are not equivalent.

11. Show that $\neg(p\leftrightarrow q)\equiv p\leftrightarrow\neg q$.

12. Check whether the statement $p \to (q \to p)$ is a tautology or a contradiction without using the truth table.

13. Using truth table check whether the statements $\neg(p \vee q) \vee (\neg p \wedge q)$ and $\neg p$ are logically equivalent.

14. Prove $p \to (q \to r) \equiv (p \wedge q) \to r$ without using truth table.

15. Prove that $p \to (\neg q \vee r) \equiv \neg p \vee (\neg q \vee r)$ using truth table.

Example 12.2

Verify the (i) closure property, (ii) commutative property, (iii) associative property (iv) existence of identity and (v) existence of inverse for the arithmetic operation + on $\mathbb{Z}$.

Example 12.3

Verify the (i) closure property, (ii) commutative property, (iii) associative property (iv) existence of identity and (v) existence of inverse for the arithmetic operation − on $\mathbb{Z}$.

Example 12.4

Verify the (i) closure property, (ii) commutative property, (iii) associative property, (iv) existence of identity and (v) existence of inverse for the arithmetic operation + on $\mathbb{Z}_e$ = the set of all even integers.

Example 12.5

Verify the (i) closure property, (ii) commutative property, (iii) associative property (iv) existence of identity and (v) existence of inverse for the arithmetic operation + on $\mathbb{Z}_o$ = the set of all odd integers.

Example 12.6

Verify (i) closure property, (ii) commutative property, and (iii) associative property of the following operation on the given set.

$(a*b) = a^b; \forall a,b \in \mathbb{N}$ (exponentiation property).

Example 12.7

Verify (i) closure property, (ii) commutative property, (iii) associative property (iv) existence of identity, and (v) existence of inverse for following operation on the given set.

$m*n = m+n-mn; m,n \in \mathbb{Z}$

Example 12.9

Verify (i) closure property, (ii) commutative property, (iii) associative property, (iv) existence of identity, and (v) existence of inverse for the operation $+_5$ on $\mathbb{Z}_5$ using multiplication table corresponding to addition modulo 5.

Example 12.10

Verify (i) closure property, (ii) commutative property, (iii) associative property, (iv) existence of identity, and (v) existence of inverse for the operation $\times_{11}$ on a subset $A = \{1,3,4,5,9\}$ of the set of remainders $\{0,1,2,3,4,5,6,7,8,9,10\}$.

Example 12.16

Construct the truth table for $(p \,\bar{\vee}\, q) \wedge (p \,\bar{\vee}\, \neg q)$.

Example 12.18

Establish the equivalence property connecting the bi-conditional with conditional:

$p \leftrightarrow q \equiv (p \to q) \wedge (q \to p)$

Example 12.19

Using the equivalence property, show that $p \leftrightarrow q \equiv (p \wedge q) \vee (\neg p \wedge \neg q)$.