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Volume - I

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CHAPTER 1

APPLICATION OF MATRICES AND DETERMINANTS

MUST KNOW DEFINITIONS

- If $|A| \neq 0$, then A is a non-singular matrix and if $|A| = 0$, then A is a singular matrix.
- The adjoint matrix of A is defined as the transpose of the matrix of co-factors of A (adj A).
- If $AB = BA = I_n$, then the matrix B is called an inverse of A.
- If a square matrix has an inverse, then it is unique.
- A^{-1} exists if and only if A is non-singular.
- Singular matrix has no inverse.
- If A is non – singular and AB = AC, then B = C (left cancellation law).
- If A is non – singular and BA = CA then B = C (Right cancellation law).
- If A and B are any two non-singular square matrices of order n, then adj (AB) = (adj B) (adj A)
- A square matrix A is called orthogonal if $AA^T = A^TA = I$
- Two matrices A and B of same order are said to the **equivalent** if one can be obtained from the other by the applications of elementary transformations (A~B).
- A non – zero matrix is in a **row - echelon** form if all zero rows occur as bottom rows of the matrix and if the first non – zero element in any lower row occurs to the right of the first non – zero entry in the higher row.
- The **rank** of a matrix A is defined as the order of a highest order non – vanishing minor of the matrix A [$\rho(A)$].
- The **rank** of a non – zero matrix is equal to the number of non – zero rows in a row – echelon form of the matrix.
- An **elementary matrix** is a matrix which is obtained from an identity matrix by applying only one elementary transformation. Every non-singular matrix can be transformed to an identity matrix by a sequence of elementary row operations.
- A system of linear equations having atleast one solution is said to be **consistent**.
- A system of linear equations having no solutions is said to be **inconsistent**.

EXERCISE 1.3

1. Solve the following system of linear equations by matrix inversion method.

(i) $2x + 5y = -2, x + 2y = -3$

(ii) $2x - y = 8, 3x + 2y = -2$ **[PTA -3]**

(iii) $2x + 3y - z = 9, x + y + z = 9, 3x - y - z = -1$

(iv) $x + y + z - 2 = 0, 6x - 4y + 5z - 31 = 0, 5x + 2y + 2z = 13.$

Sol. **(i)** $2x + 5y = -2, x + 2y = -3$

The matrix form of the system is

$$= \begin{pmatrix} 2 & 5 \\ 1 & 2 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} -2 \\ -3 \end{pmatrix}$$

$\Rightarrow$ AX = B where

$$A = \begin{pmatrix} 2 & 5 \\ 1 & 2 \end{pmatrix}, B = \begin{pmatrix} -2 \\ -3 \end{pmatrix},$$

$$X = \begin{pmatrix} x \\ y \end{pmatrix}$$

$\Rightarrow$ $X = A^{-1} B$

$$|A| = \begin{vmatrix} 2 & 5 \\ 1 & 2 \end{vmatrix} = 4 - 5 = -1 \neq 0.$$

$$\therefore A^{-1} = \frac{1}{|A|} \text{adj } A = \frac{1}{-1}\begin{bmatrix} 2 & -5 \\ -1 & 2 \end{bmatrix}$$

$$= \begin{bmatrix} -2 & 5 \\ 1 & -2 \end{bmatrix}$$

$$\therefore X = A^{-1} B = \begin{bmatrix} -2 & 5 \\ 1 & -2 \end{bmatrix}\begin{bmatrix} -2 \\ -3 \end{bmatrix}$$

$$= \begin{bmatrix} 4-15 \\ -2+6 \end{bmatrix} = \begin{bmatrix} -11 \\ 4 \end{bmatrix}$$

$\therefore$ Solution set is $\{-11, 4\}$.

(ii) $2x - y = 8, 3x + 2y = -2$

The matrix form of the system is

$$\begin{bmatrix} 2 & -1 \\ 3 & 2 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 8 \\ -2 \end{bmatrix}$$

$\Rightarrow$ AX = B where $A = \begin{bmatrix} 2 & -1 \\ 3 & 2 \end{bmatrix}$,

$$B = \begin{bmatrix} 8 \\ -2 \end{bmatrix}$$

$\Rightarrow$ $X = A^{-1} B.$

$$\text{Now, } |A| = \begin{bmatrix} 2 & -1 \\ 3 & 2 \end{bmatrix} = 4 + 3 = 7$$

$$\therefore A^{-1} = \frac{1}{|A|} \text{adj } A$$

$$= \frac{1}{7}\begin{bmatrix} 2 & 1 \\ -3 & 2 \end{bmatrix}$$

$$\therefore X = A^{-1} B = \frac{1}{7}\begin{bmatrix} 2 & 1 \\ -3 & 2 \end{bmatrix}\begin{bmatrix} 8 \\ -2 \end{bmatrix}$$

$$= \frac{1}{7}\begin{bmatrix} 16-2 \\ -24-4 \end{bmatrix}$$

$$= \frac{1}{7}\begin{bmatrix} 14 \\ -28 \end{bmatrix} = \begin{bmatrix} \frac{14}{7} \\ \frac{-28}{7} \end{bmatrix} = \begin{bmatrix} 2 \\ -4 \end{bmatrix}$$

$$\therefore x = 2, y = -4$$

Hence, the solution set is $\{2, -4\}$

(iii) $2x + 3y - z = 9, x + y + z = 9, 3x - y - z = -1.$

The matrix form of the system is

$$\begin{bmatrix} 2 & 3 & -1 \\ 1 & 1 & 1 \\ 3 & -1 & -1 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 9 \\ 9 \\ -1 \end{bmatrix}$$

$\Rightarrow$ AX = B where $A = \begin{bmatrix} 2 & 3 & -1 \\ 1 & 1 & 1 \\ 3 & -1 & -1 \end{bmatrix}$,

$$X = \begin{bmatrix} x \\ y \\ z \end{bmatrix} \text{ and } B = \begin{bmatrix} 9 \\ 9 \\ -1 \end{bmatrix}$$

$\Rightarrow$ $X = A^{-1} B$

$$|A| = \begin{vmatrix} 2 & 3 & -1 \\ 1 & 1 & 1 \\ 3 & -1 & -1 \end{vmatrix} = 2\begin{vmatrix} 1 & 1 \\ -1 & -1 \end{vmatrix} - 3\begin{vmatrix} 1 & 1 \\ 3 & -1 \end{vmatrix} - 1\begin{vmatrix} 1 & 1 \\ 3 & -1 \end{vmatrix}$$

[Expanded along R_1]

$= 2(-1 + 1) - 3(-1 - 3) - 1(-1 - 3)$
$= 0 - 3(-4) - 1(-4) = 12 + 4 = 16.$

$$\text{adj } A = \begin{bmatrix} +\begin{vmatrix} 1 & 1 \\ -1 & -1 \end{vmatrix} & -\begin{vmatrix} 1 & 1 \\ 3 & -1 \end{vmatrix} & +\begin{vmatrix} 1 & 1 \\ 3 & -1 \end{vmatrix} \\ -\begin{vmatrix} 3 & -1 \\ -1 & -1 \end{vmatrix} & +\begin{vmatrix} 2 & -1 \\ 3 & -1 \end{vmatrix} & -\begin{vmatrix} 2 & 3 \\ 3 & -1 \end{vmatrix} \\ +\begin{vmatrix} 3 & -1 \\ 1 & 1 \end{vmatrix} & -\begin{vmatrix} 2 & -1 \\ 1 & 1 \end{vmatrix} & +\begin{vmatrix} 2 & 3 \\ 1 & 1 \end{vmatrix} \end{bmatrix}^T$$

PTA QUESTION & ANSWERS

1 MARK

1. If A and B are orthogonal, then $(AB)^T$ (AB) is [PTA - 1]

(1) A (2) B (3) I (4) A^T

[Ans: (3) I]

Hint :
$$AA^T = A^TA = I$$
$$BB^T = B^TB = I$$
$$(AB)^T(AB) = B^TA^T(AB) = B^T(A^TA)B = B^T(IB) = I$$

2. The adjoint of 3×3 matrix P is $\begin{vmatrix} -1 & 2 & 2 \\ 1 & 1 & 2 \\ 2 & 2 & 1 \end{vmatrix}$, then the possible value(s) of the determinant P is (are) [PTA - 4]

(1) 3 (2) –3 (3) ± 3 (4) $\pm\sqrt{3}$

[Ans: (3) I]

Hint : $\begin{vmatrix} -1 & 2 & 2 \\ 1 & 1 & 2 \\ 2 & 2 & 1 \end{vmatrix} = -1[1-4] - 2[1-4] + 2[2-2]$

$= -1(-3) - 2(-3) + 0 = 3 + 6 = 9$

$|P| = \pm\sqrt{9} = \pm 3$

3. If A is a 3 × 3 matrix such that |3*adj* A| = 3 then |A| is equal to [PTA - 5]

(1) $\frac{1}{3}$ (2) $-\frac{1}{3}$ (3) $\pm\frac{1}{3}$ (4) ± 3

[Ans: (3) $\pm\frac{1}{3}$]

Hint : $|3\, adj\, A| = 27\, |adj\, A|$

$3 = 27|A|^2$

$|A|^2 = \frac{1}{9}$

$|A| = \pm\frac{1}{3}$

4. Let A be a non-singular matrix then which one of the following is false

(1) $(adjA)^{-1} = \frac{A}{|A|}$
(2) I is an orthogonal matrix
(3) $adj(adjA) = |A|^n A$
(4) If A is symmetric then *adj* A is symmetric

[Ans: (3) I]

Hint : $adj\,(adjA) = |A|^{n-2} A$

2 MARKS

1. Prove that $\begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}$ is orthogonal. [PTA - 1]

Sol. Let A $= \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}$

Then, $A^T = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}^T = \begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix}$

$= \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}\begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix}$

$= \begin{bmatrix} \cos^2\theta + \sin^2\theta & \cos\theta\sin\theta - \sin\theta\cos\theta \\ \sin\theta\cos\theta - \cos\theta\sin\theta & \sin^2\theta + \cos^2\theta \end{bmatrix}$

$= \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = I_2.$

Similarly, we get $A^TA = I_2$.

Hence $AA^T = A^TA = I_2 \Rightarrow$ A is orthogonal.

2. If A = $\begin{bmatrix} 2 & 3 \\ 1 & 2 \end{bmatrix}$, B = $\begin{bmatrix} 4 & 0 \\ 2 & 5 \end{bmatrix}$, find *adj* (AB). [PTA - 3]

Sol. $AB = \begin{bmatrix} 2 & 3 \\ 1 & 2 \end{bmatrix}\begin{bmatrix} 4 & 0 \\ 2 & 5 \end{bmatrix} = \begin{bmatrix} 8+6 & 0+15 \\ 4+4 & 0+10 \end{bmatrix} = \begin{bmatrix} 14 & 15 \\ 8 & 10 \end{bmatrix}$

$adj\,(AB) = \begin{bmatrix} 10 & -15 \\ -8 & 14 \end{bmatrix}$

3 MARKS

1. If A = $\begin{bmatrix} 3 & -2 \\ \lambda & -2 \end{bmatrix}$, find the value of λ so that $A^2 = \lambda A - 2I$. [PTA - 2]

Sol. $A^2 = \begin{bmatrix} 3 & -2 \\ \lambda & -2 \end{bmatrix}\begin{bmatrix} 3 & -2 \\ \lambda & -2 \end{bmatrix}$

$= \begin{bmatrix} 9-2\lambda & -6+4 \\ 3\lambda-2\lambda & -2\lambda+4 \end{bmatrix}$

$= \begin{bmatrix} 9-2\lambda & -2 \\ \lambda & -2\lambda+4 \end{bmatrix}$

$\lambda A - 2I = \lambda\begin{bmatrix} 3 & -2 \\ \lambda & -2 \end{bmatrix} - 2\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$

$= \begin{bmatrix} 3\lambda & -2\lambda \\ \lambda^2 & -2\lambda \end{bmatrix} - \begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix}$

10. If the rank of the matrix $\begin{bmatrix} \lambda & -1 & 0 \\ 0 & \lambda & -1 \\ -1 & 0 & \lambda \end{bmatrix}$ is 2, then find λ.

Sol. Given rank of $\begin{bmatrix} \lambda & -1 & 0 \\ 0 & \lambda & -1 \\ -1 & 0 & \lambda \end{bmatrix}$ is 2

⇒ The value of the third order determinant is zero

$$\Rightarrow \begin{vmatrix} \lambda & -1 & 0 \\ 0 & \lambda & -1 \\ -1 & 0 & \lambda \end{vmatrix} = 0$$

$$\Rightarrow \lambda\begin{vmatrix} \lambda & -1 \\ 0 & \lambda \end{vmatrix} + 1\begin{vmatrix} 0 & -1 \\ -1 & \lambda \end{vmatrix} + 0 = 0$$

$$\Rightarrow \lambda(\lambda^2 - 0) + 1(0 - 1) = 0$$

$$\Rightarrow \lambda^3 - 1 = 0$$

$$\Rightarrow \lambda^3 = 1$$

$$\Rightarrow \lambda = 1$$

$$\therefore \lambda = 1$$

5 MARKS

1. Using determinants, find the quadratic defined by $f(x) = ax^2 + bx + c$, if $f(1) = 0$, $f(2) = -2$ and $f(3) = -6$.

Sol. Given $f(x) = ax^2 + bx + c$

$f(1) = 0$

$\Rightarrow a(1)^2 + b(1) + c = 0$

$\Rightarrow a + b + c = 0$ ...(1)

$f(2) = -2$

$\Rightarrow a(2^2) + b(2) + c = -2$

$\Rightarrow 4a + 2b + c = -2$ ...(2)

$f(3) = -6$

$\Rightarrow a(3^2) + b(3) + c = -6$

$\Rightarrow 9a + 3b + c = -6$ ...(3)

$$\Delta = \begin{vmatrix} 1 & 1 & 1 \\ 4 & 2 & 1 \\ 9 & 3 & 1 \end{vmatrix} = 1\begin{vmatrix} 2 & 1 \\ 3 & 1 \end{vmatrix} - 1\begin{vmatrix} 4 & 1 \\ 9 & 1 \end{vmatrix} + 1\begin{vmatrix} 4 & 2 \\ 9 & 3 \end{vmatrix}$$

$= 1(2 - 3) - 1(4 - 9) + 1(12 - 18)$

$= -1 + 5 - 6 = -2 \neq 0$

$$\Delta_1 = \begin{vmatrix} 0 & 1 & 1 \\ -2 & 2 & 1 \\ -6 & 3 & 1 \end{vmatrix} = 0 - 1\begin{vmatrix} -2 & 1 \\ -6 & 1 \end{vmatrix} + 1\begin{vmatrix} -2 & 2 \\ -6 & 3 \end{vmatrix}$$

$= -1(-2 + 6) + 1(-6 + 12)$

$= -1(4) + 1(6) = 2$

$$\Delta_2 = \begin{vmatrix} 1 & 0 & 1 \\ 4 & -2 & 1 \\ 9 & -6 & 1 \end{vmatrix} = 1\begin{vmatrix} -2 & 1 \\ -6 & 1 \end{vmatrix} + 0 + 1\begin{vmatrix} 4 & -2 \\ 9 & -6 \end{vmatrix}$$

$= -1(-2 + 6) + 1(-24 + 18) = 4 - 6 = -2$

$$\Delta_3 = \begin{vmatrix} 1 & 1 & 0 \\ 4 & 2 & -2 \\ 9 & 3 & -6 \end{vmatrix} = 1\begin{vmatrix} 2 & -2 \\ 3 & -6 \end{vmatrix} - 1\begin{vmatrix} 4 & -2 \\ 9 & -6 \end{vmatrix}$$

$= 1(-12 + 6) - 1(-24 + 18) = -6 + 6 = 0$

∴ By Cramer's rule,

$$a = \frac{\Delta_1}{\Delta} = \frac{2}{-2} = -1$$

$$b = \frac{\Delta_2}{\Delta} = \frac{-2}{-2} = 1$$

$$c = \frac{\Delta_3}{\Delta} = \frac{0}{-2} = 0$$

$$\therefore f(x) = (-1)x^2 + 1x + 0$$

$$\Rightarrow f(x) = x^2 + x$$

2. Solve:

$$\frac{2}{x} + \frac{3}{y} + \frac{10}{z} = 4,\ \frac{4}{x} - \frac{6}{y} + \frac{5}{z} = 1,\ \frac{6}{x} + \frac{9}{y} - \frac{20}{z} = 2$$

Sol. Put $\frac{1}{x} = a$, $\frac{1}{y} = b$, $\frac{1}{z} = c$

$\therefore 2a + 3b + 10c = 4$ ...(1)

$4a - 6b + 5c = 1$ ...(2)

$6a + 9b - 20c = 2$ ...(3)

$$\Delta = \begin{vmatrix} 2 & 3 & 10 \\ 4 & -6 & 5 \\ 6 & 9 & -20 \end{vmatrix}$$

$$= 2\begin{vmatrix} -6 & 5 \\ 9 & -20 \end{vmatrix} - 3\begin{vmatrix} 4 & 5 \\ 6 & -20 \end{vmatrix} + 10\begin{vmatrix} 4 & -6 \\ 6 & 9 \end{vmatrix}$$

$= 2(120 - 45) - 3(-80 - 30) + 10(36 + 36)$

$= 150 + 330 + 720 = 1200$

$$\Delta_1 = \begin{vmatrix} 4 & 3 & 10 \\ 1 & -6 & 5 \\ 2 & 9 & -20 \end{vmatrix}$$

$$= 4\begin{vmatrix} -6 & 5 \\ 9 & -20 \end{vmatrix} - 3\begin{vmatrix} 1 & 5 \\ 2 & -20 \end{vmatrix} + 10\begin{vmatrix} 1 & -6 \\ 2 & 9 \end{vmatrix}$$

CHAPTER 2

COMPLEX NUMBERS

MUST KNOW DEFINITIONS

- If a Complex number is of the form $x + iy$ where x is a real part and y is the imaginary part of the complex number.
- $z_1 = z_2$ iff Re (z_1) = Re (z_2) and Im (z_1) = Im (z_2)

Properties of complex numbers:

Under Additions

- Let z_1, z_2 and z_3 are complex numbers.
  - (i) Closure property ($z_1 + z_2$ is a complex number)
  - (ii) Commutative property $(z_1 + z_2 = z_2 + z_1)$
  - (iii) Associative property $(z_1 + z_2) + z_3 = z_1 + (z_2 + z_3)$
  - (iv) Additive identity $(z + 0 = 0 + z = z)$
  - (v) Additive inverse $(z + -z) = (-z + z) = 0$
- **Under Multiplication:**
  - (i) Closure property ($z_1 z_2$ is also a complex number)
  - (ii) Commutative property $(z_1 z_2 = z_2 z_1)$
  - (iii) Associative property $(z_1 z_2) z_3 = z_1 (z_2 z_3)$
  - (iv) Multiplicative identity $(z_1 . 1 = 1. z_1 = z_1)$
  - (v) Multiplicative inverse $z.w = w.z = 1 \Rightarrow w - z^{-1}$
- **Distributive property (Multiplication distributes over addition)**

  $z_1 (z_2 + z_3) = z_1 z_2 + z_1 z_3$

  Also, $(z_1 + z_2) z_3 = z_1 z_3 + z_2 z_3$
- Conjugate of $x + iy$ is $x - iy$
- If $z = x + iy$ then $|z| = \sqrt{x^2 + y^2}$
- $|z - z_0| = r$ is the equation of circle where z_0 is a fixed complex number and r is the distance form z_0 to z.
- Polar form of $z = x + iy$ is $z = r(\cos\theta + i\sin\theta)$

  where $r = \sqrt{x^2 + y^2}$ and $\tan\theta = \frac{y}{x}$.

De Moivre's theorems

- Given any complex number $\cos\theta + i\sin\theta$ and any integer n, then $(\cos\theta + i\sin\theta)^n = \cos n\theta + i\sin n\theta$.
- $z^{1/n} = r^{1/n}\left[\cos\left(\frac{\theta+2k\pi}{n}\right) + i\sin\left(\frac{\theta+2k\pi}{n}\right)\right]$, $k = 0, 1, 2, \ldots\ldots n-1$.

IMPORTANT FORMULA TO REMEMBER

- $i^0 = 1, i^1 = i, i^4 = 1, i^{-4} = 1, (i)^{-1} = -i, (i)^{-2} = -1, (i)^{-3} = i$
- $\sqrt{ab} = \sqrt{a}.\sqrt{b}$ is valid only if atleast one of a, b is non-negative.
- When $z_1 = x_1 + iy_1$ and $z_2 = x_2 + iy_2$ then

$$z_1 + z_2 = (x_1 + x_2) + i(y_1 + y_2)$$
$$z_1 - z_2 = (x_1 - x_2) + i(y_1 - y_2)$$
$$z_1 z_2 = (x_1 x_2 - y_1 y_2) + i(x_1 y_2 + x_2 y_1)$$

Properties of complex conjugates

(1) $\overline{z_1 + z_2} = \bar{z}_1 + \bar{z}_2$

(2) $\overline{z_1 - z_2} = \bar{z}_1 - \bar{z}_2$

(3) $\overline{z_1 z_2} = \bar{z}_1\,\bar{z}_2$

(4) $\overline{\left(\frac{z_1}{z_2}\right)} = \frac{\bar{z}_1}{\bar{z}_2}$, $z_2 \neq 0$

(5) $\text{Re}(z) = \frac{z+\bar{z}}{2}$

(6) $\text{Im}(z) = \frac{z-\bar{z}}{2i}$

(7) $\overline{\left(z^n\right)} = \left(\bar{z}\right)^n$, where n is an integer.

(8) z is real iff $z = \bar{z}$

(9) z is purely imaginary iff $z = -\bar{z}$

(10) $\bar{\bar{z}} = z$

Properties of modulus of a complex number

(1) $|z| = |\bar{z}|$

(2) $|z_1 + z_2| \leq |z_1| + |z_2|$ (Triangle inequality)

(3) $|z_1 z_2| = |z_1|\,|z_2|$

(4) $|z_1 - z_2| \geq ||z_1| - |z_2||$

(5) $\left|\frac{z_1}{z_2}\right| = \frac{|z_1|}{|z_2|}$, $z_2 \neq 0$

(6) $|z^n| = |z|^n$, where n is an integer

(7) $\text{Re}(z) \leq |z|$

(8) $\text{Im}(z) \leq |z|$

- $|z_1 - z_2| = \sqrt{(x_1 - x_2)^2 + (y_1 - y_2)^2}$
- $|z_1 - z_2| \leq |z_1| + |z_2|$
- $||z_1| - |z_2|| \leq |z_1 + z_2| \leq |z_1| + |z_2|$
- $||z_1| - |z_2|| \leq |z_1 - z_2| \leq |z_1| + |z_2|$
- $|z_1 z_2| = |z_1|\,|z_2|$

(ii) Since z is purely imaginary

$$z = -\bar{z}$$

$$\therefore 2^n\left[\cos\frac{n\pi}{6}+i\sin\frac{n\pi}{6}\right]=-2^n\left[\cos\frac{n\pi}{6}-i\sin\frac{n\pi}{6}\right]$$

[From (1) & (2)]

$$\Rightarrow \cos\frac{n\pi}{6}+i\sin\frac{n\pi}{6} = -\cos\frac{n\pi}{6}+i\sin\frac{n\pi}{6}$$

$$\Rightarrow 2\cos\frac{n\pi}{6} = 0$$

$$\Rightarrow \cos n\frac{\pi}{6} = 0 = \cos\frac{\pi}{2} \qquad [\because \cos\frac{\pi}{2} = 0]$$

$$\Rightarrow \frac{n\pi}{6} = \frac{\pi}{2} \Rightarrow n = \frac{6}{2}$$

$$\Rightarrow n = 3.$$

7. Show that

(i) $(2+i\sqrt{3})^{10}-(2-i\sqrt{3})^{10}$ **is purely imaginary.** **[PTA-3]**

(ii) $\left(\frac{19-7i}{9+i}\right)^{12}+\left(\frac{20-5i}{7-6i}\right)^{12}$ **is real.**

Sol. (i) Let $z = (2+i\sqrt{3})^{10} - (2-i\sqrt{3})^{10}$

Now $\bar{z} = \overline{(2+i\sqrt{3})^{10}-(2-i\sqrt{3})^{10}}$

$$\bar{z} = \overline{(2+i\sqrt{3})^{10}} - \overline{(2-i\sqrt{3})^{10}}$$

$[\because \overline{z_1-z_2} = \bar{z_1} - \bar{z_2}]$

$$= (2-i\sqrt{3})^{10}-(2+i\sqrt{3})^{10}$$

$$= -\left[(2+i\sqrt{3})^{10}-(2-i\sqrt{3})^{10}\right] = -z$$

$\therefore \bar{z} = -\bar{z} \Rightarrow z$ is purely imaginary

Hence $(2+i\sqrt{3})^{10}-(2-i\sqrt{3})^{10}$ is purely imaginary

(ii) Consider $\frac{19-7i}{9+i} = \frac{19-7i}{9+i}\times\frac{9-i}{9-i}$

$$= \frac{171-19i-63i+7i^2}{(9)^2-i^2}$$

$$= \frac{171-82i-7}{81+1} = \frac{164-82i}{82}$$

$$= \frac{82(2-i)}{82} = 2-i$$

Also $\frac{20-5i}{7-6i} = \frac{20-5i}{7-6i}\times\frac{7+6i}{7+6i}$

$$= \frac{140+120i-35i-30i^2}{7^2-(6i)^2}$$

$$= \frac{140+85i+30}{49+36} = \frac{170+85i}{85}$$

$$= \frac{85(2+i)}{85} = 2+i$$

$$\therefore \left(\frac{19-7i}{9+i}\right)^{12}+\left(\frac{20-5i}{7-6i}\right)^{12} = (2-i)^{12}+(2+i)^{12}$$

Let $z = (2-i)^{12} + (2+i)^{12}$

$$\therefore \bar{z} = \overline{(2-i)^{12}+(2+i)^{12}}$$

$$= \overline{(2-i)^{12}}+\overline{(2+i)^{12}}$$

$[\because \overline{z_1+z_2} = \bar{z_1}+\bar{z_2}]$

$$= (2+i)^{12}+(2-i)^{12} = z$$

$\therefore \bar{z} = z \Rightarrow z$ is purely real

$\therefore \left(\frac{19-7i}{9+i}\right)^{12}+\left(\frac{20-5i}{7-6i}\right)^{12}$ is real

EXERCISE 2.5

1. Find the modulus of the following complex numbers

(i) $\frac{2i}{3+4i}$ **(ii)** $\frac{2-i}{1+i}+\frac{1-2i}{1-i}$

(iii) $(1-i)^{10}$ **(iv)** $2i(3-4i)(4-3i)$

Sol. (i) $\frac{2i}{3+4i}$

Let $z = \frac{2i}{3+4i}$

$$|z| = \left|\frac{2i}{3+4i}\right| = \frac{|2i|}{|3+4i|} = \frac{\sqrt{2^2}}{\sqrt{3^2+4^2}} = \frac{2}{\sqrt{9+16}}$$

$$= \frac{2}{\sqrt{25}} = \frac{2}{5}$$

GOVT. EXAM QUESTION & ANSWERS

1 MARK

I. Choose the Correct or the most suitable answer from the given four alternatives :

1. The least value of n satisfying $\left[\frac{\sqrt{3}}{2}+\frac{i}{2}\right]^n = 1$ is [Govt. MQP-2019]

(1) 30 (2) 24 (3) 12 (4) 18

[Ans: (3) 12]

Hint :

$$r = \sqrt{\frac{3}{4}+\frac{1}{4}} = 1$$

$$\alpha = \tan^{-1}\left(\frac{\frac{1}{2}}{\frac{\sqrt{3}}{2}}\right) = \tan^{-1}\left(\frac{1}{\sqrt{3}}\right) = 30^\circ$$

$$\left(\frac{\sqrt{3}}{2}\right)+\frac{i}{2} = (\cos 30^\circ + i\sin 30^\circ)^n$$
$$= \cos n30^\circ + i\sin n\,30^\circ$$
$$= \cos 360^\circ + i\sin 360^\circ = 1$$
$$n = 12$$

2. Which one of the points i, $-2+i$, 2 and 3 is farthest from the origin? [Qy-2019]

(1) 3 (2) $-2+i$ (3) i (4) 2

[Ans: (1) 3]

Hint :

$$|i| = 1$$
$$|-2+i| = \sqrt{4+1} = \sqrt{5}$$
$$|2| = 2$$
$$|3| = 3$$

3. If z is a complex number such that Re (z) = Im(z), then [Qy - 2019]

(1) Re(z^2) = 0 (2) Im(z^2) = 0

(3) Re(z^2) = Im (z^2) (4) Re(z^2) = – Im(z^2)

[Ans: (1) Re(z^2) = 0]

Hint :

$$z = a + ia$$
$$\Rightarrow \quad z^2 = a^2 - a^2 + 2ia^2 = 0 + 2ia^2$$

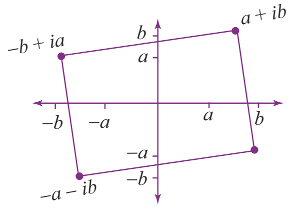
4. If z is any complex number, then the points z, iz, $-z$, $-iz$ [Qy - 2019]

(1) form a square (2) form a trapezium

(3) are collinear

(4) lie on a circle $|z| = \sqrt{2}$ with centre (0,0) and radius $\sqrt{2}$

[Ans: (1) form a square]



Hint :

$$z = a + ib$$
$$\Rightarrow \quad iz = ia - b = -b + ia$$
$$-z = -a - ib$$
$$-iz = b - ia$$

5. If $x^2 + y^2 = 1$, then the value of $\frac{1+x+iy}{1+x-iy}$ is : [Hy - 2019]

(1) $x - iy$ (2) $2x$ (3) $-2iy$ (4) $x + iy$

[Ans: (4) $x + iy$]

Hint :

$$x + iy = e^{i\theta}\ ; \ x - iy = e^{-i\theta}$$

$$\frac{1+x+iy}{1+x-iy} = \frac{1+e^{i\theta}}{1+e^{-i\theta}} = \frac{e^{i\theta}}{e^{-\frac{i\theta}{2}}}$$

$$= e^{\frac{i\theta}{2}} = e^{-i\theta} = x + iy$$

2 MARKS

1. If z_1, z_2 and z_3 are complex numbers such that $|z_1| = |z_2| = |z_3| = |z_1 + z_2 + z_3| = 1$, find the value of $\left|\frac{1}{z_1}+\frac{1}{z_2}+\frac{1}{z_3}\right|$. [Govt. MQP-2019]

Sol.

Since, $|z_1| = |z_2| = |z_3| = 1$,

$$|z_1|^2 = 1 \Rightarrow z_1\,\overline{z_1} = 1,$$
$$|z_2|^2 = 1$$
$$z_2\,\overline{z_2} = 1, \text{ and } |z_3|^2 = 1$$
$$\Rightarrow \quad z_3\,\overline{z_3} = 1,$$

Therefore, and hence $\overline{z_1} = \frac{1}{z_1}$, $\overline{z_2} = \frac{1}{z_2}$, and $\overline{z_3} = \frac{1}{z_3}$

$$\left|\frac{1}{z_1}+\frac{1}{z_2}+\frac{1}{z_3}\right| = |\overline{z_1}+\overline{z_2}+\overline{z_3}|$$
$$= |\overline{z_1+z_2+z_3}|$$
$$= |z_1+z_2+z_3| = 1$$

2. Simplify : $\left(\sin\frac{\pi}{6}+i\cos\frac{\pi}{6}\right)^{18}$. [Qy - 2019]

Sol.

$$\left(\sin\frac{\pi}{6}-i\cos\frac{\pi}{6}\right)^{18} = \left[i\left(\cos\frac{\pi}{6}-i\sin\frac{\pi}{6}\right)\right]^{18}$$
$$= -(\cos 3\pi - i\sin 3\pi) = 1$$

CHAPTER 3

THEORY OF EQUATIONS

MUST KNOW DEFINITIONS

- For the quadratic equation $ax^2 + bx + c = 0$,
  (i) $D = b^2 - 4ac > 0$ iff the roots are real and distinct
  (ii) $D = b^2 - 4ac < 0$ iff the equation has no real roots

Fundamental theorem of algebra :

- Every polynomial equation of degree n has at least one root in C.

Complex conjugate root theorem :

- If a complex number z_0 is a root of a polynomial equation with real co-efficients, then complex conjugate $\overline{z_0}$ is also a root.
- If $p + \sqrt{q}$ is a root of a quadratic equation then $p - \sqrt{q}$ is also a root of the same equation where p. q are rational and $\sqrt{q}$ is irrational.
- If $\sqrt{p} + \sqrt{q}$ is a root of a polynomial equation then $\sqrt{p} - \sqrt{q}$, $-\sqrt{p} + \sqrt{q}$, and $-\sqrt{p} - \sqrt{q}$ are also roots of the same equation.
- If the sum of the co-efficients in $p(x) = 0$ is $p(1)$. Then 1 is a root of $p(x)$.
- If the sum of the co-efficients of odd powers = sum of the co-efficients of even powers, then –1 is a root of $p(x)$.

Rational root theorem :

- Let $a_n x^n + ...+ a_1x + a_0$ with $a_n \neq 0$, $a_0 \neq 0$ be a polynomial with integer co-efficients. If $\frac{p}{q}$ with $(p, q) = 1$, is a root of the polynomial, then p is a factor of a_0 and q is a factor of a_n

Reciprocal polynomial :

- A polynomial $p(x)$ of degree n is said to be a reciprocal polynomial if one of the conditions is true
  (i) $p(x) = x^n \; p\left(\frac{1}{x}\right)$ (ii) $p(x) = -x^n. \; p\left(\frac{1}{x}\right)$
- A change of sign in the co-efficients is said to occur at the j^{th} power of x in $p(x)$ if the co-efficient of x^{j+1} and the co-efficient. of x^j (or) co-efficient. of x^{j-1}, the co-efficient of x^j are of different signs.

Hint : $p(x) = x^3 + 2x + 3$

$p(x)$ has no sign change

$p(-x) = (-x)^3 + 2(-x) + 3 = -x^3 - 2x + 3$

$p(-x)$ has only one sign change ⇒ at most one negative root.

∴ $p(x)$has no positive root and at most one negative root since its degree is 3, it has one negative and two imaginary roots.

10. The number of positive zeros of the polynomial $\sum_{j=0}^{n} {}^nC_r(-1)^r x^r$ is

(1) 0 (2) n (3) $< n$ (4) r

[Ans : (2) n]

Hint : $\sum_{j=0}^{n} {}^nC_r(-1)^r x^r = {}^nC_0(-1)^0 x^0 + {}^nC_1(-1)^1 x^1 + {}^nC_2(-1)^2 x^2 + ... + {}^nC_n(-1)^n x^n$

$= 1 - n\,x^1 + {}^nC_2\, x^2 - {}^nC_3\, x^3 + ... + x^n$

Since its degree is n and it has n change of sign, the number of positive roots are n.

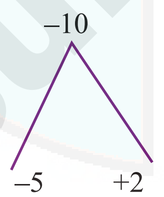
PTA QUESTION & ANSWERS

3 MARKS

1. Solve the equation $x^3 - 5x^2 - 4x + 20 = 0$. **[PTA - 1]**

Sol.

2	1	–5	–4	20
	0	2	–6	–20
	1	–3	–10	0



$(x-5)\quad(x+2)$

$(x - 2)\,(x^2 - 3x - 10)$

$(x -)\,(x - 5)\,(x + 2)$

2. Solve the equation $2x^3 + 11x^2 - 9x - 18 = 0$ **[PTA - 5]**

Sol. We observe that the sum of the coefficients of the odd powers and that of the even powers are equal. Hence −1 is a root of the equation.

To find other roots, we divide

$2x^3 + 11x^2 - 9x - 18$ by $x + 1$

and get $2x^2 + 9x - 18$ as the quotient.

Solving this we get $\frac{3}{2}$ and −6 as roots.

Thus $-6, -1, \frac{3}{2}$, are the roots or solutions of the given equation.

5 MARKS

1. Solve the equation $6x^4 - 35x^3 + 62x^2 - 35x + 6 = 0$. **[PTA - 1]**

Sol. $6x^2 - 35x^3 + 62x^2 - 35x + 6 = 0$

Dividing this factor by x^2 and rearranging the terms we get

$$6\left(x^2 + \frac{1}{x^2}\right) - 35\left(x + \frac{1}{x}\right) + 62$$

Setting $u = \left(x + \frac{1}{x}\right)$ it becomes a quadratic polynomial as

$6(y^2 - 2) - 35u + 62$ which reduces to $6y^2 - 35u + 50$. Solving we obtain $u = \frac{10}{3}, \frac{5}{2}$. Taking $u = \frac{10}{3}$ gives $x = 3, \frac{1}{3}$ and taking $u = \frac{5}{2}$ gives $x = 2, \frac{1}{2}$. So the required solutions are $2, \frac{1}{2}, 3, \frac{1}{3}$.

2. Investigate for what values of λ and μ the system of linear equations $x + 2y + z = 7, x + y + \lambda z = \mu$ has (i) no solution (ii) a unique solution (iii) an infinite number of solutions. **[PTA - 2]**

Sol. Here the number of unknowns is 3.

The matrix form of the system is AX = B, where

$$A = \begin{bmatrix} 1 & 2 & 1 \\ 1 & 1 & \lambda \\ 1 & 3 & -5 \end{bmatrix}, X = \begin{bmatrix} x \\ y \\ z \end{bmatrix}, B = \begin{bmatrix} 7 \\ \mu \\ 5 \end{bmatrix}$$

Applying elementary row operations on the augmented matrix [A|B], we get

$$[A|B] = \left[\begin{array}{ccc|c} 1 & 2 & 1 & 7 \\ 1 & 1 & \lambda & \mu \\ 1 & 3 & -5 & 5 \end{array}\right] \xrightarrow{R_2 \leftrightarrow R_3} \left[\begin{array}{ccc|c} 1 & 2 & 1 & 7 \\ 1 & 3 & -5 & 5 \\ 1 & 1 & \lambda & \mu \end{array}\right]$$

$$\xrightarrow[R_3 \to R_3 - R_1,]{R_2 \to R_2 - R_1,} \left[\begin{array}{ccc|c} 1 & 2 & 1 & 7 \\ 0 & 1 & -6 & -2 \\ 0 & -1 & \lambda - 1 & \mu - 7 \end{array}\right]$$

$$\xrightarrow{R_3 \to R_3 + R_2} \left[\begin{array}{ccc|c} 1 & 2 & 1 & 7 \\ 0 & 1 & -6 & -2 \\ 0 & 0 & \lambda - 7 & \mu - 9 \end{array}\right]$$

CHAPTER 4

INVERSE TRIGONOMETRIC FUNCTIONS

MUST KNOW DEFINITIONS

- f is **periodic** if there exists $p > 0$ such that for all x in the domain of f, $x + p$ is in the domain of f and $f(x + p) = f(x)$.
- The smallest of all such number is called the **period** of the function f.
- A real values function f is an **even** function if for all x in the domain of f, $-x$ is also in the domain of f and $f(-x) = f(x)$.
- A real values function f is an **odd** function if for all x in the domain of f, $-x$ is also in the domain of f and $f(-x) = -f(x)$.
- **Amplitude** of a function is the height from the x-axis to its maximum or minimum.
- The period is the distance required for the function to complete one full cycle.
- The inverse sine function $\sin^{-1}: [-1, 1] \to \left[\frac{-\pi}{2}, \frac{\pi}{2}\right]$ is defined by $\sin^{-1}(x) = y$ if and only if $\sin y = x$ and $y \in \left[\frac{-\pi}{2}, \frac{\pi}{2}\right]$
- The inverse tangent function $\tan^{-1} (-\infty, \infty) \to \left[\frac{-\pi}{2}, \frac{\pi}{2}\right]$ is defined by $\tan^{-1}(x) = y$ if and only if $\tan y = x$ and $y \in \left[\frac{-\pi}{2}, \frac{\pi}{2}\right]$
- The inverse cosecant function $\text{cosec}^{-1}: (-\infty, -1] \cup [1, \infty) \left[\frac{-\pi}{2}, 0\right) \cup \left[0, \frac{\pi}{2}\right)$ is defined by $\text{cosec}^{-1}(x) = y$ if and only if $\text{cosec}\, y = x$ and $y \in \left[\frac{-\pi}{2}, 0\right) \cup \left[0, \frac{\pi}{2}\right)$
- The inverse secant function $\sec^{-1}: R\backslash(-1, 1) \to [0, \pi]/\left\{\frac{\pi}{2}\right\}$ is defined by $\sec^{-1}(x) = y$ whenever $\sec y = x$ and $y \in [0, \pi]/\left\{\frac{\pi}{2}\right\}$
- The inverse cotangent function $\cot^{-1}: (-\infty, \infty) \to [0, \pi]$ is defined by $\cot^{-1}(x) = y$ if and only if $\cot y = x$ and $y \in [0, \pi]$.

IMPORTANT FORMULA TO REMEMBER

Properties of Inverse Trigonometric Functions

✦ **Property-I**

(i) $\sin^{-1}(\sin\theta) = \theta$, if $\theta \in \left[\frac{-\pi}{2}, \frac{\pi}{2}\right]$

(ii) $\cos^{-1}(\cos\theta) = \theta$, if $\theta \in [0, \pi]$

(iii) $\tan^{-1}(\tan\theta) = \theta$, if $\theta \in \left(\frac{-\pi}{2}, \frac{\pi}{2}\right)$

(iv) $\text{cosec}^{-1}(\text{cosec}\,\theta) = \theta$, if $\theta \in \left[\frac{-\pi}{2}, \frac{\pi}{2}\right] \setminus \{0\}$

(v) $\sec^{-1}(\sec\theta) = \theta$, if $\theta \in [0, \pi] \setminus \left\{\frac{\pi}{2}\right\}$

(vi) $\cot^{-1}(\cot\theta) = \theta$, if $\theta \in [0, \pi]$

✦ **Property-II**

(i) $\sin(\sin^{-1}x) = x$, if $x \in [-1, 1]$

(ii) $\cos(\cos^{-1}x) = x$, if $x \in [-1, 1]$

(iii) $\tan(\tan^{-1}x) = x$, if $x \in \mathbb{R}$

(iv) $\text{cosec}(\text{cosec}^{-1}x) = x$, if $x \in \mathbb{R}\setminus(-1, 1)$

(v) $\sec(\sec^{-1}x) = x$, if $x \in \mathbb{R}\setminus(-1, 1)$

(vi) $\cot(\cot^{-1}x) = x$, if $x \in \mathbb{R}$

✦ **Property-III (Reciprocal inverse identities)**

(i) $\sin^{-1}\left(\frac{1}{x}\right) = \text{cosec}\,x$, if $x \in \mathbb{R} \setminus(-1, 1)$

(ii) $\cos^{-1}\left(\frac{1}{x}\right) = \sec x$, if $x \in \mathbb{R} \setminus(-1, 1)$

(iii) $\tan^{-1}\left(\frac{1}{x}\right) = \begin{cases} \cot^{-1}x & \text{if } x > 0 \\ -\pi+\cot^{-1}x & \text{if } x < 0 \end{cases}$

Property-IV (Reflection identities)

(i) $\sin^{-1}(-x) = -\sin^{-1}x$, if $x \in [-1, 1]$

(ii) $\tan^{-1}(-x) = -\tan^{-1}x$, if $x \in \mathbb{R}$

(iii) $\text{cosec}^{-1}(-x) = -\text{cosec}^{-1}x$, if $|x| \geq 1$ or $x \in \mathbb{R}\setminus(-1, 1)$

(iv) $\cos^{-1}(-x) = \pi - \cos^{-1}x$, if $x \in [-1, 1]$

(v) $\sec^{-1}(-x) = \pi - \sec^{-1}x$, if $x \geq 1$ or $x \in \mathbb{R}\setminus(-1, 1)$

(vi) $\cot^{-1}(-x) = \pi - \cot^{-1}x$, if $x \in \mathbb{R}$

Also from the definition of $\cos^{-1}x$,

$-1 \le x \le 1$ ... (2)

$\therefore$ From (1) & (2),

Domain of $g(x)$ $= [-1, 1] \cup [-1, 1]$

$= [-1, 1]$

Hence the domain of $g(x)$ is $[-1, 1]$.

7. For what value of x, the inequality

$\frac{\pi}{2} < \cos^{-1}(3x-1) < \pi$ holds?

Sol. Given $\frac{\pi}{2} < \cos^{-1}(3x-1) < \pi$

$\cos\frac{\pi}{2} < 3x - 1 < \cos\pi$

$\Rightarrow 0 < 3x - 1 < -1$

$0 + 1 < 3x < -1 + 1$

$1 < 3x < 0$

$\frac{1}{3} < x < 0$

This inequality is true, only when $0 < x < \frac{1}{3}$.

8. Find the value of

(i) $\cos\left(\cos^{-1}\left(\frac{4}{5}\right) + \sin^{-1}\left(\frac{4}{5}\right)\right)$

(ii) $\cos^{-1}\left(\cos\left(\frac{4\pi}{3}\right)\right) + \cos^{-1}\left(\cos\left(\frac{5\pi}{4}\right)\right)$

Sol. **(i)** $\cos\left(\cos^{-1}\left(\frac{4}{5}\right) + \sin^{-1}\left(\frac{4}{5}\right)\right)$

Let $\cos^{-1}\left(\frac{4}{5}\right) = \theta \Rightarrow \frac{4}{5} = \cos\theta$ ...(1)

Also $\sin^{-1}\left(\frac{4}{5}\right) = \sin^{-1}(\cos\theta)$ [using (1)]

$= \sin^{-1}\left(\sin\left(\frac{\pi}{2} - \theta\right)\right)$ $\left[\because \cos\theta = \sin\left(\frac{\pi}{2} - \theta\right)\right]$

$= \frac{\pi}{2} - \theta$ ...(2)

$\therefore \cos\left(\cos^{-1}\left(\frac{4}{5}\right) + \sin^{-1}\left(\frac{4}{5}\right)\right)$

$= \cos\left(\theta + \frac{\pi}{2} - \theta\right)$ [using (1) and (2)]

$= \cos\frac{\pi}{2} = 0$

Sol. **(ii)** $\cos^{-1}\left(\cos\left(\frac{4\pi}{3}\right)\right) + \cos^{-1}\left(\cos\left(\frac{5\pi}{4}\right)\right)$

$\cos^{-1}\left(\cos\left(\frac{4\pi}{3}\right)\right) = \cos^{-1}\left(\cos\left(2\pi - \frac{2\pi}{3}\right)\right)$

[$\because$ the period of cosine in 2π]

$= \cos^{-1}\left(\cos\frac{2\pi}{3}\right)$ [$\because \cos(2\pi - \theta) = \cos\theta$]

$= \frac{2\pi}{3}$ $\left[\because \frac{2\pi}{3} \in [0, \pi]\right]$ ... (1)

$\cos^{-1}\left(\cos\left(\frac{5\pi}{4}\right)\right) = \cos^{-1}\left(\cos\left(2\pi - \frac{3\pi}{4}\right)\right)$

[$\because$ Period of cosine is 2π]

$= \cos^{-1}\left(\cos\left(3\frac{\pi}{4}\right)\right)$ [$\because \cos(2\pi - \theta) = \cos\theta$]

$= 3\frac{\pi}{4}$ $\left[\because 3\frac{\pi}{4} \in [0, \pi]\right]$ ... (2)

$\therefore \cos^{-1}\left(\cos\left(\frac{4\pi}{3}\right)\right) + \cos^{-1}\left(\cos\left(\frac{5\pi}{4}\right)\right)$

$= \frac{2\pi}{3} + 3\frac{\pi}{4}$

$= \frac{8\pi + 9\pi}{12} = \frac{7\pi}{12}$

$\cos^{-1}\left(\cos\left(\frac{4\pi}{3}\right)\right) + \cos^{-1}\left(\cos\left(\frac{5\pi}{4}\right)\right) = \frac{7\pi}{12}$

EXERCISE 4.3

1. Find the domain of the following functions :

(i) $\tan^{-1}\sqrt{9 - x^2}$

(ii) $\frac{1}{2}\tan^{-1}(1 - x^2) - \frac{\pi}{4}$

(i) Let $f(x) = \tan^{-1}\sqrt{9 - x^2}$

$\sqrt{9 - x^2} \in \mathbb{R}$ but $\sqrt{9 - x^2} \ge 0$

$\therefore 9 - x^2 \ge 0$

$\Rightarrow x^2 - 9 \le 0$

$\Rightarrow (x + 3)(x - 3) \le 0$

$\Rightarrow$ Domain is $[-3, 3]$

18. If $\sin^{-1} x + \cot^{-1}\left(\frac{1}{2}\right) = \frac{\pi}{2}$, then x is equal to

(1) $\frac{1}{2}$ (2) $\frac{1}{\sqrt{5}}$ (3) $\frac{2}{\sqrt{5}}$ (4) $\frac{\sqrt{3}}{2}$

[Ans: (2) $\frac{1}{\sqrt{5}}$]

19. If $\sin^{-1}\frac{x}{5} + \text{cosec}^{-1}\frac{5}{4} = \frac{\pi}{2}$, then the value of x is [PTA - 5]

(1) 4 (2) 5 (3) 2 (4) 3

[Ans: (4) 3]

20. $\sin(\tan^{-1} x)$, $|x| < 1$ is equal to

(1) $\frac{x}{\sqrt{1-x^2}}$ (2) $\frac{1}{\sqrt{1-x^2}}$

(3) $\frac{1}{\sqrt{1+x^2}}$ (4) $\frac{x}{\sqrt{1+x^2}}$

[Ans: (4) $\frac{x}{\sqrt{1+x^2}}$]

PTA QUESTION & ANSWERS

1 MARK

1. The range of $\sec^{-1} x$ is [PTA - 2; Qy - 2019]

(1) $[-\pi, \pi]\backslash\left\{\frac{\pi}{2}\right\}$ (2) $[0, \pi]\backslash\left\{\frac{\pi}{2}\right\}$

(3) $(0, \pi)\backslash\left\{\frac{\pi}{2}\right\}$ (4) $(-\pi, \pi)\backslash\left\{\frac{\pi}{2}\right\}$

[Ans:(2) $[0, \pi]\backslash\left\{\frac{\pi}{2}\right\}$]

Hint : For the given function $y = \sec^{-1}x$ if we draw a graph we find that it's discontinuous at $x = -1$ and 1

Therefore domain of the function will be $= (-\infty, -1]\ [1, \infty)$ and the range of the function will be $\left[0, \frac{\pi}{2}\right) \cup \left(\frac{\pi}{2}, \pi\right]$.

2 MARKS

1. Write the principal value of $\tan^{-1}\left(\sin\left(-\frac{\pi}{2}\right)\right)$

Sol. $\tan^{-1}\left[\sin\left(-\frac{\pi}{2}\right)\right] = \tan^{-1}(-1)$ [PTA -4]

$= -\tan^{-1}(1) = -\frac{\pi}{4}$

2. Find the value of $\cos^{-1}\left(\frac{1}{2}\right) - 2\sin^{-1}\left(-\frac{1}{2}\right)$

Sol. $\cos^{-1}\left(\frac{1}{2}\right) - 2\sin^{-1}\left(-\frac{1}{2}\right)$ [PTA -6]

$= \frac{\pi}{3} + 2\frac{\pi}{6} = \frac{2\pi}{3}$

5 MARKS

1. If $\cos^{-1} x + \cos^{-1} y + \cos^{-1} z = \pi$ and $0 < x, y, z < 1$, then show that $x^2 + y^2 + z^2 + 2xyz = 1$. [PTA -1]

Sol. Let $\cos^{-1} x = \alpha$ and $\cos{-1}\ y = \beta$. Then, $x = \cos\alpha$ and $y = \beta$.

$\cos^{-1} x + \cos^{-1} y + \cos^{-1} z = \pi$

gives $\alpha + \beta = \pi - \cos^{-1} z$. ... (1)

Now, $\cos(\alpha + \beta) = \cos\alpha\cos\beta - \sin\alpha\sin\beta$

$= xy - \sqrt{1-x^2}\sqrt{1-y^2}$.

From (1), we get $\cos(\pi - \cos^{-1} z)$

$= xy - \sqrt{1-x^2}\sqrt{1-y^2}$.

$-\cos(\cos^{-1} z) = xy - \sqrt{1-x^2}\sqrt{1-y^2}$

So, $-z = xy - \sqrt{1-x^2}\sqrt{1-y^2}$, which gives $-xy - z = -\sqrt{1-x^2}\sqrt{1-y^2}$.

Squaring on both sides and simplifying, we get $-x^2 + y^2 + z^2 + 2xyz = 1$.

2. Evaluate : $\sin\left(\sin^{-1}\left(\frac{3}{5}\right) + \sec^{-1}\left(\frac{5}{4}\right)\right)$ [PTA -2]

Sol. Let $\sec^{-1}\frac{5}{4} = \theta$ Then $\sec\theta = \frac{5}{4}$ and hence, $\cos\theta = \frac{4}{5}$.

Also, $\sin\theta = \sqrt{1-\cos^2\theta} = \sqrt{1-\left(\frac{4}{5}\right)^2} = \frac{3}{5}$,

which gives $\theta = \sin^{-1}\left(\frac{3}{5}\right)$

Thus, $\sec^{-1}\left(\frac{5}{4}\right) = \sin^{-1}\left(\frac{3}{5}\right)$ and $\sin^{-1}\frac{3}{5} +$

$\sec^{-1}\left(\frac{5}{4}\right) = 2\sin^{-1}\left(\frac{3}{5}\right)$

We know that $\sin^{-1}\left(2x\sqrt{1-x^2}\right) = 2\sin^{-1} x$,

CHAPTER 5

TWO DIMENSIONAL ANALYTICAL GEOMETRY - II

MUST KNOW DEFINITIONS

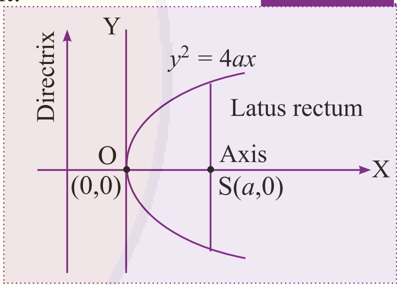
- A circle is the locus of a point in a plane which moves such that its distance from a fixed point is always constant.
- Tangent of a circle is a line which touches the circle at only one point.
- Normal is a line perpendicular to the tangent.
- From any point outside the circle two tangents can be drawn.
- **Parabola**

  S(a, 0) → Focus   O(0, 0) → Vertex

  Y = 0 → Axis

  Length of latus rectum = 4a

Parabola



- **Ellipse**

  $$\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$$

  S(ae, 0), S′($-ae$, 0) → Foci

  A(a, 0), A′($-a$, 0) → Vertices

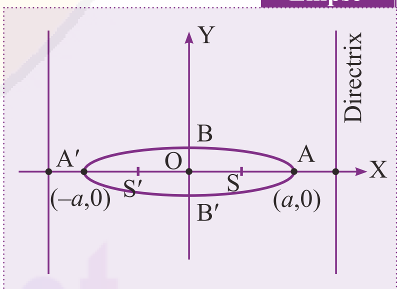
  O (0, 0)   Centre

  AA′ → Major axis → $2a$; BB′ → Minor axis → $2b$

  $b^2 = a^2(1-e^2)$

  Directrices $x=\frac{a}{e},\ x=\frac{-a}{e}$

Ellipse



- Sum of the focal distances of any point on the ellipse is equal to length of the major axis.
- **Hyperbola**

  $$\frac{x^2}{a^2}-\frac{y^2}{b^2}=1$$

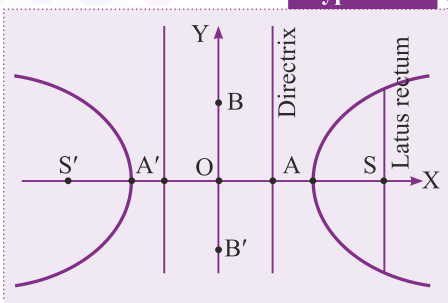
  AA′ = $2a$ → Transverse axis; BB′ = $2b$ → Conjuate axis

  O(0, 0) → centre

  A(a, 0), A′($-a$, 0) → vertices; S(ae, 0), S′($-ae$, 0) → Focus

  Directrices $x=\pm\frac{a}{e}$; Latus rectum = $\frac{2b^2}{a}$; $b^2=a^2(e^2-1)$

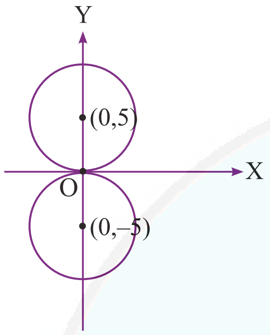
Hyperbola



EXERCISE 5.1

1. Obtain the equation of the circles with radius 5 cm and touching *x*-axis at the origin in general form.

Sol. Given $r = 5$ cm



Since the circle touches the x axis, its centre is $(0, \pm 5)$

Equation of the circle is $(x-h)^2 + (y-k)^2 = r^2$

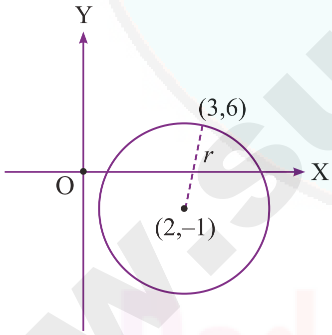
$\Rightarrow \quad (x-0)^2 + (y \pm 5)^2 = 5^2$

$\Rightarrow \quad x^2 + y^2 + \cancel{25} \pm 10y = \cancel{25}$

$\Rightarrow \quad x^2 + y^2 + 10y = 0$

2. Find the equation of the circle with centre (2, –1) and passing through the point (3, 6) in standard form.

Sol.



Given centre is (2, –1) and passing through the point (3, 6)

$\therefore r$ = distance between (2, –1) and (3, 6)

$= \sqrt{(2-3)^2 + (-1-6)^2}$

$= \sqrt{(-1)^2 + (-7)^2}$

$= \sqrt{1+49} = \sqrt{50}$

$\therefore$ Equation of the circle is

$(x-h)^2 + (y-k)^2 = r^2$

$(x-2)^2 + (y+1)^2 = (\sqrt{50})^2$

$\Rightarrow (x-2)^2 + (y+1)^2 = 50$

3. Find the equation of circles that touch both the axes and pass through (– 4, – 2) in general form.

Sol. Since the circle touch both the axis. Its equation will be $(x-a)^2 + (y-a)^2 = a^2$ ...(1)

It passes through (–4, –2)

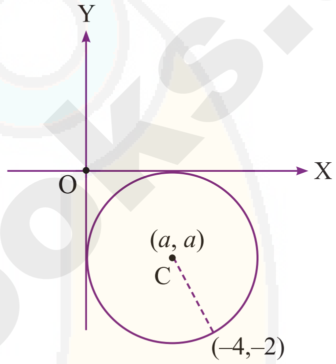
$\therefore (-4-a)^2 + (-2-a)^2 = a^2$

$16 + \cancel{a^2} + 8a + 4 + a^2 + 4a = \cancel{a^2}$

$\Rightarrow \quad a^2 + 12a + 20 = 0$

$\Rightarrow \quad (a+10)(a+2) = 0$

$a = -10$ or -2



Case (i)

When $a = -10$, (1) becomes

$(x+10)^2 + (y+10)^2 = 10^2$

$\Rightarrow \quad x^2 + 100 + 20x + y^2 + \cancel{100} + 20y = \cancel{100}$

$\Rightarrow \quad x^2 + y^2 + 20x + 20y + 100 = 0$

Case (ii)

When $a = -2$, (1) becomes

$(x+2)^2 + (y+2)^2 = 2^2$

$\Rightarrow \quad x^2 + 4x + 4 + y^2 + 4y + \cancel{4} = \cancel{4}$

$\Rightarrow \quad x^2 + y^2 + 4x + 4y + 4 = 0$

Hence, equation of the circles are

$x^2 + 4^2 + 4x + 4y + 4 = 0$

or $\quad x^2 + y^2 + 20x + 20y + 100 = 0$

4. Find the equation of the circle with centre (2, 3) and passing through the intersection of the lines $3x - 2y - 1 = 0$ and $4x + y - 27 = 0$.

Sol. Given centre is (2, 3)

Let us solve $\quad 3x - 2y = 1 \quad$ ...(1)

and $\quad 4x + y = 27 \quad$ ...(2)

(1) → $\quad 3x - 2y = 1$

(2) × 2 → $\quad 8x + 2y = 54$

$11x = 55 \Rightarrow x = 5$

$\therefore 3(5) - 2y = 1$

24. The values of m for which the line $y = mx + 2\sqrt{5}$ touches the hyperbola $16x^2 - 9y^2 = 144$ are the roots of $x^2 - (a + b)x - 4 = 0$, then the value of $(a + b)$ is

(1) 2 (2) 4 (3) 0 (4) –2

[Ans. (3) 0]

Hint :

$$y = mx + 2\sqrt{5}$$

$$\Rightarrow m = m,\ c = 2\sqrt{5}$$

$$\Rightarrow 16x^2 - 9y^2 = 144$$

$$\Rightarrow \frac{16x^2}{144} - \frac{9y^2}{144} = 1 \Rightarrow \frac{x^2}{9} - \frac{y^2}{16} = 1$$

$$a^2 = 9,\ b^2 = 16$$

Condition is $c^2 = a^2m^2 - b^2$

$$\Rightarrow (2\sqrt{5})^2 = 9m^2 - 16$$

$$\Rightarrow 20 = 9m^2 - 16$$

$$\Rightarrow 9m^2 = 36 \Rightarrow m^2 = 4$$

$$\Rightarrow m = 2, -2$$

$$\therefore a = 2,\ b = -2$$

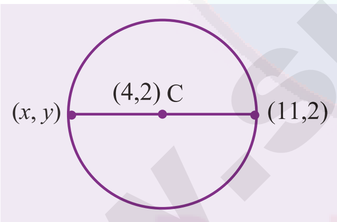
[$\because$ a and b are the roots of $x^2 - (a + b)x - 4 = 0$]

$\therefore a + b = 2 - 2 = 0$

25. If the coordinates at one end of a diameter of the circle $x^2 + y^2 - 8x - 4y + c = 0$ are (11, 2) the coordinates of the other end are

(1) (–5, 2) (2) (–3, 2)
(3) (5, –2) (4) (–2, 5) **[Ans.(–3, 2)]**

Hint : Equation of circle is $x^2 + y^2 - 8x - 4y + c = 0$



$2g = -8 \Rightarrow g = -4$

$2f = -4 \Rightarrow f = -2$

$\therefore$ Centre is $(-g, -f) = (4, 2)$

Let the other end be (x, y)

Centre is the mid-point of (x, y) and (11, 2)

$$\therefore (4, 2) = \left(\frac{x+11}{2}, \frac{y+2}{2}\right)$$

$$\Rightarrow \frac{x+11}{2} = 4 \Rightarrow x + 11 = 8 \Rightarrow x = 8 - 11 = -3$$

$$\frac{y+2}{2} = 2 \Rightarrow y + 2 = 4 \Rightarrow y = 4 - 2 = 2$$

$\therefore$ The other end is (–3, 2)

PTA QUESTION & ANSWERS

1 MARK

1. The equation of the directrix of the parabola $y^2 = x + 4$ is **[PTA - 1]**

(1) $x = \frac{15}{4}$ (2) $x = \frac{15}{4}$

(3) $x = -\frac{17}{4}$ (4) $x = -\frac{17}{4}$

[Ans. (3) (–5, 0)**]**

Hint :

$$y^2 = 1\,[x - (-4)]$$

$$4a = 1 \Rightarrow a = \frac{1}{4}$$

Equation of directrix : $x = -4 - \frac{1}{4}$

$$x = -\frac{17}{4}$$

2. The vertex of the parabola $x^2 = 8y - 1$ is

(1) $\left(-\frac{1}{8}, 0\right)$ (2) $\left(\frac{1}{8}, 0\right)$ **[PTA - 3]**

(3) $\left(-6, \frac{9}{2}\right)$ (4) $\left(\frac{9}{2}, -6\right)$

[Ans. ★ $\left(0, \frac{1}{8}\right)$**]**

Hint : $x^2 = 8y - 1$

$$x^2 = 8\left(y - \frac{1}{8}\right)$$

$$(x-h)^2 = 4a(y - k)$$

$$\text{Vertex} = (h, k) = \left(0, \frac{1}{8}\right)$$

3. If P (x, y) be any point on $4x^2 + 9y^2 = 36$, then the sum of the distances of P from the points $(\sqrt{5}, 0)$ and $(-\sqrt{5}, 0)$ **[PTA - 4]**

(1) 4 (2) 8 (3) 6 (4) 18

[Ans. (3) 6**]**

Hint : $4x^2 + 9y^2 = 36$

$$\frac{x^2}{9} + \frac{y^2}{4} = 1$$

$$\Rightarrow a^2 = 9$$

$$\Rightarrow a = 3$$

By focal property, $PF_1 + PF_2 = 2a = 2(3) = 6$

The Correct match is

	(i)	(ii)	(iii)	(iv)
(1)	d	c	b	a
(2)	d	b	c	a
(3)	c	d	b	a
(4)	d	c	a	b

[Ans : (1) i – d ii – c iii – b iv – a]

5. Conics and its point of contact

List - A (Conic)		List - B (Point of contact	
i.	circle	a)	$\left(\frac{a}{m^2}, \frac{2a}{m}\right)$
ii.	parabola	b)	$\left(-\frac{a^2m}{e}, \frac{-b^2}{c}\right)$
iii.	ellipse	c)	$\left(-\frac{a^2m}{c}, \frac{b^2}{c}\right)$
iv.	hyperbola	d)	$\left(\frac{\mp a^2m}{\sqrt{1+m^2}}, \frac{\pm a}{\sqrt{1+m^2}}\right)$

The Correct match is

	(i)	(ii)	(iii)	(iv)
(1)	a	b	c	d
(2)	b	c	d	a
(3)	d	a	c	b
(4)	d	a	b	c

[Ans : (3) i – d ii – a iii – c iv – b]

2 MARKS

1. Find the equation of tangent to the circle $x^2 + y^2 + 2x - 3y - 8 = 0$ at (2, 3).

Sol. Given circle is $x^2 + y^2 + 2x - 3y - 8 = 0$

Equation of tangent is $xx_1 + yy_1 + 1(x + x_1) - \frac{3}{2}(y + y_1) - 8 = 0$

At (2, 3), the tangent is

$$x(2) + y(3) + x + 2 - \frac{3}{2}(y + 3) - 8 = 0$$

$$\Rightarrow \quad 3x + 3y + 2 - \frac{3y}{2} - \frac{9}{2} - 8 = 0$$

Multiply by 2 we get,

$$\Rightarrow \quad 6x + 6y + 4 - 3y - 9 - 16 = 0$$

$$\Rightarrow \quad 6x + 3y - 21 = 0$$

2. Find the length of the tangent from (2, –3) to the circle $x^2 + y^2 - 8x - 9y + 12 = 0$.

Sol. Given circle is $x^2 + y^2 - 8x - 9y + 12 = 0$

Length of the tangent =

$$\sqrt{2^2 + (-3)^2 - 8(2) - 9(-3) + 12}$$

$$= \sqrt{4 + 9 - 16 + 27 + 12} = \sqrt{36} = 6 \text{ units.}$$

3. Find the equation of the parabola with vertex at the origin, passing through (2, –3) and symmetric about x-axis.

Sol. Since the parabola is symmetric about x-axis, it is either open upward or downward.

Let the equation be $x^2 = 4ay$ ...(1)

Since (2, –3) lies on the parabola,

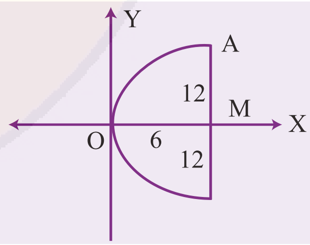
$$2^2 = 4a(-3) \Rightarrow a = \frac{-1}{3}$$

Substituting $a = \frac{-1}{3}$ in (1) we get,

$x^2 = 4\left(\frac{-1}{3}\right)y \Rightarrow 3x^2 = -4y$. Which is the required equation of the parabola.

4. If a parabolic reflector is 24 cm in diameter and 6 cm deep, find its locus.

Sol. Let AOB be the vertical section of the reflector and m is the mid-point of AB. Let the equation of the parabola be $y^2 = 4ax$ A(6, 12) lies on (1)



$\therefore\ 12^2 = 4a(6) \Rightarrow a = 6$

$\therefore$ Focus is $(a, 0) = (b, 0)$

Hence focus coincides with m, the mid-point of AB.

5. If the line $y = 3x + 1$, touches the parabola $y^2 = 4ax$, find the length of the latus rectum?

Sol. Given equation of tangent is $y = 3x + 1$

The condition for any line $y = mx + c$ to be a tangent to $y^2 = 4ax$ is $c = \frac{a}{m}$

$1 = \frac{a}{m} \Rightarrow a = m \Rightarrow a = 3$

$\therefore$ Length of the latus rectum is $4a = 4(3) = 12$ units.

CHAPTER 6

APPLICATIONS OF VECTOR ALGEBRA

MUST KNOW DEFINITIONS

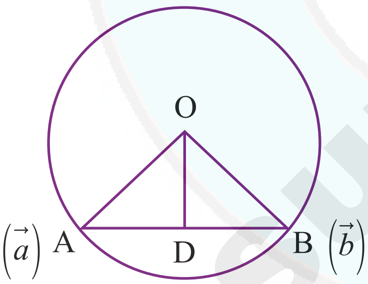
- If $\vec{a} = a_1\hat{i} + a_2\hat{j} + a_3\hat{k}$ and $\vec{b} = b_1\hat{i} + b_2\hat{j} + b_3\hat{k}$ then $\vec{a}\cdot\vec{b} = a_1b_1 + a_2b_2 + a_3b_3$ – (a scalar)
  and $\vec{a}\times\vec{b} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{vmatrix}$ (a vector)
- $\vec{a}\,\vec{b} = 0$ iff $\vec{a}\perp\vec{b}$
- $\vec{a}\times\vec{b} = \vec{0}$ iff $\vec{a}\,||\,\vec{b}$
- For any 3 vector $\vec{a}$, $\vec{b}$ and $\vec{c}$, $(\vec{a}\times\vec{b})\cdot\vec{c} = \vec{a}\cdot(\vec{b}\times\vec{c})$
- $(\vec{a}\times\vec{b})\cdot\vec{c} = 0 \Leftrightarrow \vec{a}, \vec{b}, \vec{c}$ are co-planar.
- $\vec{a}$, $\vec{b}$, $\vec{c}$ are co-planar iff there exist scalars $r, s, t \in$ R such atleast one of them is non-zero and $r\vec{a} + r\vec{b} + r\vec{c} = \vec{0}$
- Vector triple product is not associative
- Two lines are said to be **coplanar** if they lie in the same plane.
- Two lines in space are called **skew lines** if they are not parallel and do not intersect.
- A straight line which is perpendicular to a plane is called a **normal** to the plane.
- Angle between $\vec{a}$ and $\vec{b}$ is $\theta = \cos^{-1}\left(\frac{\vec{a}\cdot\vec{b}}{|\vec{a}|\,|\vec{b}|}\right)$
- $\vec{a}\,||\,\vec{b}$ if $\theta = 0$ or π
- $\vec{a}\perp\vec{b}$ if $\theta = \frac{\pi}{2}$ or $\frac{3\pi}{2}$
- $w = \vec{F}\cdot\vec{d}$ where $\hat{d}$ – displacement vector
  $\vec{F}$ – Force
  w – work done.

- The perpendicular distance from a point $\vec{u}$ to the plane $\vec{r}\cdot\vec{n}=p$ is $\delta=\dfrac{|\vec{u}\cdot\vec{n}-p|}{|\vec{n}|}$.
- Distance between two parallel planes $ax+by+cz+d_1=0$ and $ax+by+cz+d_2=0$ is $\dfrac{|d_1-d_2|}{\sqrt{a^2+b^2+c^2}}$.
- Vector equation of a plane which passes through the line of intersection by the planes $\vec{r}\cdot\vec{n}_1=d_1$ and $\vec{r}\cdot\vec{n}_2=d_2$ is $(\vec{r}\cdot\vec{n}_1-d_1)+d(\vec{r}\cdot\vec{n}_2-d_2)=0\cdot$
$(a_1x+b_1y+c_1z-d_1)+\lambda(a_2x+b_2y+c_2z-d_2)=0$ [cartesian form]
- Image of a point in a plane is
$\vec{v}=\vec{u}+\dfrac{2[p-(\vec{u}\cdot\vec{n})]}{|\vec{n}|^2}\ \vec{n}$
- Meeting point of a line $\vec{r}=\vec{a}+t\vec{b}$ and a plane $\vec{r}\cdot\vec{n}=p$ is $\vec{a}+\left(\dfrac{p-(\vec{a}\cdot\vec{n})}{\vec{b}\cdot\vec{n}}\right)\vec{b}$.

EXERCISE 6.1

1. Prove by vector method that if a line is drawn from the centre of a circle to the midpoint of a chord, then the line is perpendicular to the chord.

Sol.



Let the position vectors of the parts A and B on the circle lie $\vec{a}$ and $\vec{b}$ respectively.
Since O is the centre of the circle

$$|\overrightarrow{OA}|=|\overrightarrow{OB}|\Rightarrow|\vec{a}|=|\vec{b}| \quad ...(1)$$

Also D is the mid-point of AB,

$$\Rightarrow \overrightarrow{OD}=\frac{\vec{a}+\vec{b}}{2}$$

(mid-point formula)

$$=\left(\frac{\vec{a}+\vec{b}}{2}\right)\cdot(\overrightarrow{OB}-\overrightarrow{OA})$$

$$=\left(\frac{\vec{a}+\vec{b}}{2}\right)\cdot(\vec{b}-\vec{a})$$

$$=\frac{1}{2}\left[|\vec{b}|^2-|\vec{a}|^2\right]$$

$$[\because (\vec{a}+\vec{b})\cdot(\vec{b}-\vec{a})=|\vec{b}|^2-|\vec{a}|^2]$$

$$=\frac{1}{2}\left[|b|^2-|b|^2\right] \text{ (using (1))}$$

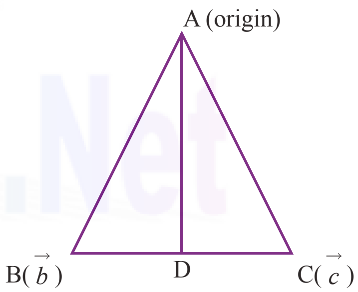
$$=\frac{1}{2}(0)=0$$

$$\Rightarrow \overrightarrow{OD}\cdot\overrightarrow{AB}=0\Rightarrow\overrightarrow{OD}\perp\overrightarrow{AB}$$

Hence, if a line is drawn from the centre of a circle to the mid-point of a chord, then that line is perpendicular to the chord.

2. Prove by vector method that the median to the base of an isosceles triangle is perpendicular to the base.

Sol.



Let ABC be an isosceles triangle with AB = AC and let D be the mid-point of BC.
Taking A as the origin, let the position vectors of B and C be $\vec{b}$ and $\vec{c}$ respectively.

The p.v. of D $=\dfrac{\vec{b}+\vec{c}}{2}$, $\overrightarrow{AB}=\vec{b}$, $\overrightarrow{AC}=\vec{c}$

The Cartesian equation of the plane containing the given lines is

$$\begin{vmatrix} x-x_2 & y-y_2 & z-z_2 \\ b_1 & b_2 & b_3 \\ d_1 & d_2 & d_3 \end{vmatrix} = 0$$

$$\Rightarrow \begin{vmatrix} x+1 & y+1 & z \\ 2 & 2 & 2 \\ 5 & 2 & 2 \end{vmatrix} = 0 \qquad [\because \lambda = 2]$$

$\Rightarrow (x+1)(4-4) - (y+1)(4-10) + z(4-10) = 0$

$\Rightarrow (x+1)(0) - (y+1)(-6) + z(-6) = 0$

$\Rightarrow 6(y+1) - 6z = 0$

$\Rightarrow y + 1 - z = 0 \qquad [\div 6]$

$\Rightarrow y - z + 1 = 0$ which is the required equation of the plane containing the given lines.

Exercise 6.9

1. Find the equation of the plane passing through the line of intersection of the planes $\vec{r}\cdot(2\hat{i}-7\hat{j}+4\hat{k}) = 3$ and $3x - 5y + 4z + 11 = 0$, and the point $(-2, 1, 3)$. **[PTA -3]**

Sol. Given equation of planes are

$\vec{r}\cdot(2\hat{i}-7\hat{j}+4\hat{k}) = 3$ and $3x - 5y + 4z + 11 = 0$.

The vector equation of a plane passing through the line of intersection of the planes $\vec{r}\cdot\vec{n}_1 = d_1$ and $\vec{r}\cdot\vec{n}_2 = d_2$ is given by

$(\vec{r}\cdot\vec{n}_1 - d_1) + \lambda(\vec{r}\cdot\vec{n}_2 - d_2) = 0 \qquad \ldots(1)$

put $\vec{r} = x\hat{i} + y\hat{j} + z\hat{k}$

$\vec{n}_1 = 2\hat{i} - 7\hat{j} + 4\hat{k}$, $\vec{n}_2 = 3\hat{i} - 5\hat{j} + 4\hat{k}$,

$d_1 = +3$, $d_2 = -11$ in (1) we get

$$\left[(x\hat{i}+y\hat{j}+z\hat{k})\cdot(2\hat{i}-7\hat{j}+4\hat{k}) - 3\right] + \lambda\left[(x\hat{i}+y\hat{j}+z\hat{k})\cdot(3\hat{i}-5\hat{j}+4\hat{k}) + 11\right] = 0$$

$\Rightarrow (2x - 7y + 4z - 3) + \lambda(3x - 5y + 4z + 11) = 0 \qquad \ldots(2)$

Since the plane passes through the point $(-2, 1, 3)$ we get,

$[2(-2) - 7(1) + 4(3) - 3] + \lambda[-6 - 5 + 12 + 11] = 0$

$\Rightarrow (-4 - 7 + 12 - 3) + \lambda(12) = 0$

$\Rightarrow -2 + 12\lambda = 0 \Rightarrow 12\lambda = 2 \Rightarrow \lambda = \frac{1}{6}$

Substituting $\lambda = \frac{1}{6}$ in (1) we get,

$(2x - 7y + 4z - 3) + \frac{1}{6}(3x - 5y + 4z + 11) = 0$

Multiplying by 6, we get,

$\Rightarrow 12x - 42y + 24z - 18 + 3x - 5y + 4z + 11 = 0$

$\Rightarrow 15x - 47y + 28z - 7 = 0$ which is the required equation of the plane.

2. Find the equation of the plane passing through the line of intersection of the planes $x + 2y + 3z = 2$ and $x - y + z = 3$, and at a distance $\frac{2}{\sqrt{3}}$ from the point $(3,1,-1)$.

Sol. Given equation of planes are $x + 2y + 3z = 2$ and $x - y + z = 3$

The Cartesian equation of a plane which passes through the line of intersection of the planes is

$(a_1x + b_1y + c_1z - d_1) + \lambda(a_2x + b_2y + c_2z - d_2) = 0$

$\therefore$ The required equation of the plane is

$(x + 2y + 3z - 2) + \lambda(x - y + z - 3) = 0 \qquad \ldots(1)$

$x(\lambda + 1) + y(2 - \lambda) + z(3 + \lambda) - 2 - 3\lambda = 0$

The distance from $(3, 1, -1)$ to this plane is $\frac{2}{\sqrt{3}}$

$$\therefore \frac{3(\lambda+1)+1(2-\lambda)-1(3+\lambda)-2-3\lambda}{\sqrt{(\lambda+1)^2+(2-\lambda)^2+(3+\lambda)^2}} = \frac{2}{\sqrt{3}}$$

$$\frac{3\lambda + \cancel{3} + \cancel{2} - 2\lambda - \cancel{3} - \lambda - \cancel{2} - 3\lambda}{\sqrt{\lambda^2+1+2\lambda+4+\lambda^2-4\lambda+9+\lambda^2+6\lambda}} = \frac{2}{\sqrt{3}}$$

$$\Rightarrow \frac{-3\lambda}{\sqrt{3\lambda^2+4\lambda+14}} = \frac{2}{\sqrt{3}}$$

Squaring on both sides

$$\Rightarrow \frac{9\lambda^2}{3\lambda^2+4\lambda+14} = \frac{4}{3}$$

$\Rightarrow 27\lambda^2 = 12\lambda^2 + 16\lambda + 56$

$\Rightarrow 15\lambda^2 - 16\lambda - 56 = 0$

To find λ, solve the equation. But this equation has irrational roots.

3. Find the angle between the line $\vec{r} = (2\hat{i} - \hat{j} + \hat{k}) + t(\hat{i} + 2\hat{j} - 2\hat{k})$ and the plane $\vec{r}\cdot(6\hat{i} + 3\hat{j} + 2\hat{k}) = 8$

Sol. Equation of given plane is

$\vec{r}\cdot(6\hat{i} + 3\hat{j} + 2\hat{k}) = 8 \qquad \therefore \vec{n}_1 = 6\hat{i} + 3\hat{j} + 2\hat{k}$

and the line is $\vec{r} = (2\hat{i} - \hat{j} + \hat{k}) + t(\hat{i} + 2\hat{j} - 2\hat{k})$

$\therefore \vec{b} = \hat{i} + 2\hat{j} - 2\hat{k}$

Angle between a line and a plane is

points A, B, C, D lie on a plane, we have to prove that the three vectors $\overrightarrow{AB}, \overrightarrow{AC}, \overrightarrow{AD}$ are coplanar.

Now $\overrightarrow{AB}$

$$= \overrightarrow{OB} - \overrightarrow{OA} = (16\hat{i} - 19\hat{j} - 4\hat{k}) - (6\hat{i} - 7\hat{j})$$

$$= 10\hat{i} - 12\hat{j} - 4\hat{k}$$

$\overrightarrow{AC} = \overrightarrow{OC} - \overrightarrow{OA} = 6\hat{i} + 10\hat{j} - 6\hat{k}$ and

$\overrightarrow{AD} = \overrightarrow{OD} - \overrightarrow{OA} = 4\hat{i} + 2\hat{j} + 10\hat{k}$

We have $[\overrightarrow{AB}, \overrightarrow{AC}, \overrightarrow{AD}] = \begin{vmatrix} 10 & -12 & -4 \\ -6 & 10 & -6 \\ -4 & 2 & 10 \end{vmatrix} = 0$

Therefore, the three vectors $\overrightarrow{AB}, \overrightarrow{AC}, \overrightarrow{AD}$ are coplanar and hence the four points A,B,C and D lie on a plane.

GOVT. EXAM QUESTION & ANSWERS

1 MARK

I. Choose the Correct or the most suitable answer from the given four alternatives :

1. The foot of the perpendicular from A (1, 0, 0) to the line $\frac{x-1}{2} = \frac{y+1}{-3} = \frac{z+10}{8}$ is [Qy - 2019]

(1) (3, –4, –2) (2) (5, –8, –4)
(3) (–3, 4, 2) (4) (2, –3, 4)

[Ans : (1) (3, –4, –2)]

Hint: The given line l is

$$\frac{x-1}{2} = \frac{y+1}{-3} = \frac{z+10}{8}$$

Any point Q on the line l is given by

$$\frac{x-1}{2} = \frac{y+1}{-3} = \frac{z+10}{8} = \lambda$$

$$x = 2\lambda + 1,$$

$$y = -3\lambda - 1, z = 8\lambda - 10$$

Direction ratios of AQ are

$2\lambda + 1 - 1$, $-3\lambda - 1 - 0$, $8\lambda - 10 - 0$ or 2λ, $-3\lambda - 1$, $8\lambda - 10$

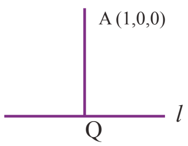
Since AQ $\perp l$

$2(2\lambda) + (-3)(-3\lambda - 1) + 8(8\lambda - 10) = 0$

$\Rightarrow 4\lambda + 9\lambda + 3 + 64\lambda - 80 = 0$

$77\lambda - 77 = 0$

$\lambda = 1$



Hence the coordinates of Q are

$x = 2(1) + 1 = 3$

$y = -3(1) - 1 = -4$

$z = 8(1) - 10 = -2$ (3, –4, –2)

2. The value of $[\hat{i}+\hat{j}, \hat{j}+\hat{k}, \hat{k}+\hat{i}]$ is equal to : [Hy - 2019]

(1) 0 (2) 1 (3) 2 (4) 4

[Ans : (3) 2]

Hint: $\begin{vmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \\ 1 & 0 & 1 \end{vmatrix} = 1[1-0] - 1[0-1]$

$1 + 1 = 2$

2 MARKS

1. If $2\vec{i} - \vec{j} + 3\vec{k}$, $3\vec{i} + 2\vec{j} + \vec{k}$, $\vec{i} + m\vec{j} + 4\vec{k}$ are coplanar, find the value of m. [Qy & Hy - 2019]

Sol. Since the given three vectors are coplanar, we have $\begin{vmatrix} 2 & -1 & 3 \\ 3 & 2 & 1 \\ 1 & m & 4 \end{vmatrix} = 0 \Rightarrow m = -3$

2. Determine whether the pair of straight lines $\vec{r} = (2\hat{i} + 6\hat{j} + 3\hat{k}) + t(2\hat{i} + 3\hat{j} + 4\hat{k})$, $\vec{r} = (2\hat{j} - 3\hat{k}) + s(\hat{i} + 2\hat{j} + 3\hat{k})$ [Hy - 2019]

Sol. Comparing the given two equations with

$\vec{r} = \vec{a} + s\vec{b}$ and $\vec{r} = \vec{c} + s\vec{d}$

we have $\vec{a} = 2\vec{i} + 6\vec{j} + 3\vec{k}$,

$\vec{b} = 2\vec{i} + 3\vec{j} + 4\vec{k}$,

$\vec{c} = 2\vec{j} - 3\vec{k}$,

$\vec{d} = \vec{i} + 2\vec{j} + 3\vec{k}$

Clearly, $\vec{b}$ is not a scalar multiple of $\vec{d}$. So, the two vectors are not parallel and hence the two lines are not parallel.

The shortest distance between the two straight lines is given by

$$\delta = \frac{(\vec{c} - \vec{a}) \cdot (\vec{b} - \vec{d})}{|\vec{b} \times \vec{d}|}$$

Now, $\vec{b} \times \vec{d} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 2 & 3 & 4 \\ 1 & 2 & 3 \end{vmatrix} \hat{i} - 2\vec{j} + \vec{k}$

Volume - II

MATHEMATICS

CONTENTS

VOLUME - II

CHAPTER 7

APPLICATIONS OF DIFFERENTIAL CALCULUS

MUST KNOW DEFINITIONS

- The **tangent line** to a plane curve at a given point is the straight line that just touches the curve at that point.
- The **normal** at a point on the curve is the straight line which is perpendicular to the tangent at the that point.

Intermediate value theorem :

- If f is continuous on $[a, b]$ and c is any number between $f(a)$ and $f(b)$, then there is atleast one number x in $[a, b]$ such that $f(x) = c$.

Rolle's theorem :

- Let $f(x)$ be continuous on $[a,b]$ and differentiable in (a, b). If $f(a) = f(b)$ then there is atleast one point $c \in (a, b)$ where $f'(c) = 0$.

Langrange's mean value theorem :

- Let $f(x)$ be continuous in $[a, b]$ and differentiable in (a, b). Then there exists atleast one point $c \in (a, b)$ such that $f'(c) = \dfrac{f(b) - f(a)}{b - a}$
- If $f(x)$ is continuous in $[a, b]$ and differentiable in (a, b) and if $f'(x) > 0$ $\forall x \in (a, b)$ then for $x_1, x_2 \in [a, b]$ such that $x_1 < x_2$ we have $f(x_1) < f(x_2)$
- Let $\lim\limits_{x \to \alpha} g(x)$ exists and let it be L and let $f(x)$ be continuous at $x = \text{L}$.

  Then $\lim\limits_{x \to \alpha} f(g(x)) = f\left(\lim\limits_{x \to \alpha} g(x)\right)$
- A stationary point $(x_0, f(x_0))$ of a differentiable function $f(x)$ is where $f'(x_0) = 0$
- A critical point $(x_0, f(x_0))$ of a function $f(x)$ is where $f'(x_0) = 0$ or does not exist.
- If $f(x)$ is continuous on a closed interval $[a, b]$, then f has both an absolute maximum and an absolute minimum on $[a, b]$.

Fermat's theorem :

- If $f(x)$ has a relative extrema at $x = c$ then c is critical number.

✦ **Text for Concavity :**

(i) If $f''(x) > 0$ on an open interval I, then $f(x)$ is concave up on I.

(ii) If $f''(x) < 0$ on an open interval I, then $f(x)$ is concave down on I.

✦ **Text for points of inflection :**

(i) If $f''(c)$ exists and $f''(c)$ changes sign when passing through $x = c$, then the point $(c, f(c))$ is a point of inflection of the graph of f.

(ii) If $f''(c)$ exists at the point of inflection, then $f''(c) = 0$.

✦ **Second derivative test :**

Suppose that c is a critical point at which $f'(c) = 0$, that $f'(x)$ exists in a neighbourhood of c and that $f'(c)$ exists.

Then f has a relative maximum value at c if $f''(c) < 0$ and a relative minimum value at c if $f''(c) > 0$. If $f''(c) = 0$, the test is not informative.

✦ Symmetric with respect to y - axis if $f(x, y) = f(-x, y)\ \forall\ x, y$

✦ Symmetric with respect to x - axis if $f(x, y) = f(x, -y)\ \forall\ x, y$

✦ Symmetric with respect to origin if $f(x, y) = f(-x, -y)\ \forall\ x, y$

✦ **Horizontal Asymptote :**

$y = L$ is said to be horizontal asymptote if $\lim_{x\to\infty} f(x) = L$ or $\lim_{x\to-\infty} f(x) = L$

✦ **Vertical Asymptote :**

$x = a$ is said to be vertical asymptote if $\lim_{x\to a^-} f(x) = \pm\infty$ or $\lim_{x\to a^+} f(x) = \pm\infty$

✦ **Slant Asymptote :**

A slant (oblique) asymptote occurs when the polynomial in the numerator is a higher degree than the polynomial in the denominator.

EXERCISE 7.1

1. A point moves along a straight line in such a way that after t seconds its distance from the origin is $s = 2t^2 + 3t$ meters.

(i) Find the average velocity of the points between $t = 3$ and $t = 6$ seconds.

(ii) Find the instantaneous velocities at $t = 3$ and $t = 6$ seconds.

Sol. (i)

$$\text{Given } s = 2t^2 + 3t$$

$$s(3) = 2 \times 3^2 + 3(3) = 2 \times 9 + 9 = 27 \text{ m} \quad \dots (1)$$

$$s(6) = 2 \times 6^2 + 3(6) = 72 + 18 = 90 \text{ m} \quad \dots (2)$$

$$\text{Average velocity} = \frac{s(6) - s(3)}{6 - 3} = \frac{90 - 27}{3} = \frac{63}{3} = 21 \text{ m/s}$$

(ii) Instantaneous Velocity $V(t) = \frac{ds}{dt} = 4t + 3$

Instantaneous Velocity at $t = 3$

$= V(3) = 15$ m/sec [From (1)]

Instantaneous Velocity at $t = 6$

$= V(6) = 27$ m/sec [From (2)]

Interval	$\left(0, \frac{\pi}{4}\right)$	$\left(\frac{\pi}{4}, \frac{3\pi}{4}\right)$	$\left(\frac{3\pi}{4}, \frac{5\pi}{4}\right)$	$\left(\frac{5\pi}{4}, \frac{7\pi}{4}\right)$	$\left(\frac{7\pi}{4}, 2\pi\right)$
Sign of $f'(x)$	Say $x = \frac{\pi}{6}$ $\cos 2 \times \frac{\pi}{6}$ $= \cos\frac{\pi}{3} = \frac{1}{2}$ + ve	Say $x = \frac{\pi}{2}$ $\cos 2 \times \frac{\pi}{2}$ $= \cos \pi = -1$ – ve	Say $x = \pi$ $\cos 2\pi = 1$ + ve	Say $x = \frac{3\pi}{2}$ $\cos 2 \times \frac{3\pi}{2}$ $= \cos 3\pi = -1$ = – ve	Say $x = 320°$ $\cos 2 \times 320°$ $= \cos 640$ $= \cos(360 + 280)$ $= \cos 280°$ $= \cos(270 + 10)$ $= \sin 10°$ = + ve
monotonicity	Strictly increasing	Strictly decreasing	Strictly increasing	Strictly decreasing	Strictly increasing

$\therefore f(x)$ is strictly increasing in $\left(0, \frac{\pi}{4}\right)$ $\left(\frac{3\pi}{4}, 5\frac{\pi}{4}\right)\left(\frac{7\pi}{4}, 2\pi\right)$ and strictly decreasing in $\left(\frac{\pi}{4}, \frac{3\pi}{4}\right)\left(\frac{5\pi}{4}, \frac{7\pi}{4}\right)$.

Since $f'(x)$ changes its position from positive to negative at $x = \frac{\pi}{4}, \frac{5\pi}{4}$, there is a local maximum at $x = \frac{\pi}{4}, \frac{5\pi}{4}$.

$$f\left(\frac{\pi}{4}\right) = \sin\frac{\pi}{4} \cos\frac{\pi}{4} + 5$$
$$= \frac{1}{\sqrt{2}} \cdot \frac{1}{\sqrt{2}} + 5 = \frac{1}{2} + 5 = \frac{11}{2}$$
$$f\left(\frac{5\pi}{4}\right) = \sin\frac{5\pi}{4} \cos\frac{\pi}{4} + 5$$
$$= \left(\frac{-1}{\sqrt{2}}\right)\left(\frac{-1}{\sqrt{2}}\right) + 5$$
$$= \frac{1}{2} + 5 = \frac{11}{2}$$

Also $f'(x)$ changes its position from negative to positive at $x = \frac{3\pi}{4}, \frac{7\pi}{4}$, there is a local minimum at $x = \frac{3\pi}{4}, \frac{7\pi}{4}$.

$$\therefore f\left(\frac{3\pi}{4}\right) = \cos\frac{3\pi}{4} \sin\frac{3\pi}{4} + 5$$
$$= \left(\frac{-1}{\sqrt{2}}\right)\left(\frac{+1}{\sqrt{2}}\right) + 5 = 5 - \frac{1}{2} = \frac{9}{2}$$

$$f\left(\frac{7\pi}{4}\right) = \cos\frac{7\pi}{4} \sin\frac{7\pi}{4} + 5$$
$$= \cos\left(2\pi - \frac{\pi}{4}\right)\sin\left(2\pi - \frac{\pi}{4}\right) + 5$$
$$= \left(\frac{1}{\sqrt{2}}\right)\left(\frac{-1}{\sqrt{2}}\right) + 5$$
$$= \frac{-1}{2} + 5 = \frac{9}{2}$$

EXERCISE 7.7

1. Find intervals of concavity and points of inflexion for the following functions:

(i) $f(x) = x(x-4)^3$

(ii) $f(x) = \sin x + \cos x, 0 < x < 2\pi$

(iii) $f(x) = \frac{1}{2}(e^x - e^{-x})$ **[PTA -2]**

Sol. (i) $f(x) = x(x-4)^3$

$$\text{Given } f(x) = x(x-4)^3$$
$$f'(x) = x \cdot 3(x-4)^2 + (x-4)^3(1)$$
$$= 3x(x-4)^2 + (x-4)^3$$
$$= (x-4)^2(3x + x - 4)$$
$$= (x-4)^2(4x-4)$$
$$= 4(x-1)(x-4)^2$$
$$f''(x) = 4[(x-1)2(x-4) + (x-4)^2(1)]$$
$$= 4[2(x-4)(x-1) + (x-4)^2]$$
$$= 4(x-4)[2x - 2 + x - 4)]$$
$$= 4(x-4)(3x-6)$$
$$= 12(x-4)(x-2)$$

PTA QUESTION & ANSWERS

1 MARK

1. If the rate of increase of the radius of a circle is 5 cm/sec, then the rate of increase of its area when the radius is 20 cm, will be [PTA - 2]

(1) 10π (2) 20π (3) 200π (4) 400π

Hint : $\frac{dr}{dt} = 5$ cm / sec **[Ans. (3) 200π]**

$\Rightarrow$ $A = \pi r^2$

$$\frac{dA}{dt} = \pi 2r\frac{dr}{dt}$$

$$= 2\pi(20)5 = 200\pi$$

2. The slope of the curve $y^3 - xy^2 = 4$ at the point where $y = 2$ is [PTA - 5]

(1) -2 (2) $-\frac{1}{2}$ (3) $\frac{1}{4}$ (4) $\frac{1}{2}$

Hint : $y^3 - xy^2 = 4$ **[Ans. (4) $\frac{1}{2}$]**

$$3y^2\frac{dy}{dx} = \left[x2y\frac{dy}{dx} + y^2\right] = 0$$

$$3y^2\frac{dy}{dx} - 2xy\frac{dy}{dx} - y^2 = 0$$

$$\frac{dy}{dx} = \frac{y^2}{3y^2 - 2xy}$$

At $y = 2$
$8 - x4 = 4$
$4x = 4$
$x = 1$

$$\left(\frac{dy}{dx}\right)(1,2) = \frac{2^2}{3\times 2^2 - 2\times 1\times 2} = \frac{4}{12-4} = \frac{4}{8} = \frac{1}{2}$$

2 MARKS

1. A particle is fired straight up from the ground to reach a height of s feet in t seconds, where $s(t)=128t - 16t^2$. Compute the maximum height of the particle reached. [PTA - 2]

Sol. At the maximum height, the velocity $v(t)$ of the particle is zero.

Now, we find the velocity of the particle at time t.

$$v(t)\frac{ds}{dt} = 128 - 32t$$

$$v(t) = 0 \Rightarrow 128 - 32t = 0t = 4.$$

After 4 seconds, the particle reaches the maximum height.

The height at $t = 4$ is $s(4)$

$$= 128(4) - 16(4)^2 = 256 \text{ ft.}$$

2. Explain why Largrange mean value theorem is not applicable to the function $f(x) = \left|\frac{1}{x}\right|$, $x \in [-1, 1]$ [PTA - 6]

Sol. $f(x) = \left|\frac{1}{x}\right|$, $x \in [-1, 1]$

Lagrange;s mean value theorem is not applicable since $f(x)$ is not continuous at $x = 0$

3 MARKS

1. Evaluate $\lim_{x\to 0}\frac{xe^x - \sin x}{x}$. [PTA - 1]

Sol. $\lim_{x\to 0} e^x - \lim_{x\to 0}\frac{\sin x}{x} = \lim_{x\to 0} e^x - \lim_{x\to 0}\frac{\sin x}{x}$

$= 1 - 1 = 0$

2. Examine the concavity for the function $f(x) = x^4 - 4x^3$. [PTA - 2]

Sol.

$$f(x) = x^4 - 4x^3$$
$$f'(x) = 4x^3 - 12x^2$$
$$f''(x) = 12x^2 - 24x$$
$$f''(x) = 0 \Rightarrow 12x^2 - 24x$$
$$12x(x-2) = 0$$
$$x = \text{ or } 2$$

Intervals of concavity

Interval	$(-\infty, 0)$	$(0, 2)$	$(2, \infty)$
Sign of $f''(x)$	+	–	+
Concavity	Concave up	Concave down	Concave up

3. For the function $f(x)$ 1 $x^4 - 2x^2$, find all the values of c in $(-2, 2)$ such that $f'(c) = 0$ [PTA - 3]

Sol. $f(x) = x^4 - 2x^2$

$f'(x) = 4x^3 - 4x$

$f'(x) = 0 \Rightarrow 4x^3 - 4x = 0$

$4x(x^2 - 1) = 0$

$x = 0$ (or) $x = \pm 1$

4. Find the equation of the tangent to the curve $x^2y - x = y^3 - 8$ at $x = 0$ [PTA -6]

Sol. $x = 0$; $x^2y - x = y^3 - 8$

$$y^3 = 8$$
$$y = 2$$

$$2xy + x^2\frac{dy}{dx} - 1 = 3y^2\frac{dy}{dx}$$

CHAPTER 8

DIFFERENTIALS AND PARTIALS DERIVATIVES

MUST KNOW DEFINITIONS

- **Linear approximation :**

  Let $f : (a, b) \to \mathbb{R}$ be a differentiable function and $x_0 \in (a,b)$

  $L(x) = f(x_0) + f'(x_0)(x - x_0)\ \forall\ x \in (a,b)$

- $\dfrac{df}{dx} = \lim\limits_{\Delta x \to 0} \dfrac{f(x+\Delta x) - f(x)}{\Delta x} = f'(x_0)$

- $df = f'(x)\,\Delta x$

- **Continuity :** Let $A = \{(x, y) / a < x < b, c < y < d\} \subset R^2$, F: $\mathbb{A} \to \mathbb{R}$

  F is continuous at (u, v) if

  a) F is defined at (u, v)

  b) $\lim\limits_{(x,y)\to(u,v)} F(x,y) = L$ exists

  c) $L = F(u, v)$

- **Clairaut's Theorem :** Let $A = \{(x, y) / a < x < b, c < y < d\} \subset R^2$, F: $\mathbb{A} \to \mathbb{R}$

  If f_{xy} and f_{yx} exist in A are continuous in A, then $f_{xy} = f_{gx}$ in A

- **Laplace's Equation :** Let $A = \{(x, y) / a < x < b, c < y < d\} \subset R^2$, F: $\mathbb{A} \to \mathbb{R}$. A function U : $\mathbb{A} \to \mathbb{R}^2$ is said to be harmonic in A if it satisfies $\dfrac{\partial^2 u}{\partial x^2} + \dfrac{\partial^2 y}{\partial y^2} = 0\ \forall\ (x, y) \in \mathbb{A}$

- F is a homogeneous function on A, if there exists *a* constant P such that $F(\lambda x, \lambda y) = \lambda^p F(x, y)\ \forall\ \lambda \in \mathbb{R}$ such that $(\lambda x, \lambda y) \in \mathbb{A}$. This constant *p* is called degree of F.

IMPORTANT FORMULA TO REMEMBER

- Absolute error = Actual Value – Approximate value
- Relative error = $\dfrac{\text{Actual value} - \text{Approximate value}}{\text{Acutal Value}}$
- Percentage error = Relative error × 100
- $df = f'(x)\,\Delta x$
- Linear approximation of F at $(x_0, y_0, z_0) \in \mathbb{A}$ is

$$F(x, y, z) = F(x_0, y_0, z_0) + \left.\frac{\partial F}{\partial x}\right|_{(x_0,y_0,z_0)}(x - x_0) + \left.\frac{\partial F}{\partial y}\right|_{(x_0,y_0,z_0)}(y - y_0) + \left.\frac{\partial F}{\partial z}\right|_{(x_0,y_0,z_0)}(z - z_0)$$

- Differential of F is

$$dF = \frac{\partial F}{\partial x}(x, y, z)\,dx + \frac{\partial F}{\partial y}(x, y, z)\,dy + \frac{\partial F}{\partial z}(x, y, z)\,dz$$

Where $dx = \Delta x$, $dy = \Delta y$ and $dz = \Delta z$.

- If $w(x, y)$ is a function of two variables x, y then $\dfrac{\partial w}{dt} = \dfrac{\partial w}{\partial x}.\dfrac{dx}{dt} + \dfrac{\partial w}{\partial y}.\dfrac{dy}{dx}$
- If $w(x, y)$ is a function of two variables (x, y) then $\dfrac{\partial w}{\partial s} = \dfrac{\partial w}{\partial x}.\dfrac{\partial x}{\partial s} + \dfrac{\partial w}{\partial y}.\dfrac{\partial y}{\partial s}$
- **Euler's theorem :** If F is having continuous partial derivatives and homogeneous on A with degree p, then

$$x.\frac{\partial F}{\partial x}(x, y, z) + y\frac{\partial F}{\partial y}(x, y, z) + z\frac{\partial F}{\partial z}(x, y, z) = p\,F(x, y, z)\ \forall\ (x, y, z) \in \mathbb{B}.$$

EXERCISE 8.1

1. Let $f(x) = \sqrt[3]{x}$. Find the linear approximation at $x = 27$. Use the linear approximation to approximate $\sqrt[3]{27.2}$.

Sol. Given $f(x) = \sqrt[3]{x}$

Let $x_0 = 27$ and $\Delta x = 0.2$

We know $L(x) = f(x_0) + f'(x_0)(x - x_0)\ \forall\ x \in (a,b)$

$\therefore \sqrt[3]{27.2} = f(27) + f'(27)(0.2)$ ...(1)

Now $f(27) = \sqrt[3]{27} = 3$

$$f'(x) = \frac{1}{3}x^{\frac{1}{3}-1} = \frac{1}{3}x^{-\frac{2}{3}} = \frac{1}{3x^{\frac{2}{3}}}$$

$$\therefore f'(27) = \frac{1}{3(27)^{\frac{2}{3}}} = \frac{1}{3(3^3)^{\frac{2}{3}}}\ \frac{1}{3(3^2)} = \frac{1}{27}$$

$\therefore$ (1) becomes,

$$\sqrt[3]{27.2} = 3 + \frac{1}{27}(0.2)$$
$$= 3 + .0074 = 3.0074$$
$$\therefore \sqrt[3]{27.2} = 3.0074$$

ADDITIONAL QUESTION & ANSWERS

1 MARK

I. Choose the Correct or the most suitable answer from the given four alternatives :

1. If $y = x^4 - 10$ and if x changes from 2 to 1.99, the approximate change in y is

(1) –32 (2) –0.32
(3) – 10 (4) 10 **[Ans: (2) –0.32]**

Hint : $dy = 4x^3\, dx$

$= 4(2)^3 (-0.01) = -0.32$

2. If the radius of the sphere is measured as 9 cm with an error of 0.03 cm, the approximate error in calculating its volume is

(1) 9.72 cm^3 (2) 0.972 cm^3
(3) 0.972π cm^3 (4) 9.72π cm^3

[Ans: (4) 9.72π cm^3]

Hint : Let r be the radius of the sphere

Given $r = 9$ cm

Error in measurement of radius = Δr

$\Delta r = 0.03$ cm

$v = \frac{4}{3}\pi r^3$

We need to find error in calculating the volume i.e. Δv.

$$\Delta v = \frac{dv}{dr}\Delta r = \frac{d\left(\frac{4}{3}\pi r^3\right)}{dr}\Delta r$$

$$v = \frac{4}{3}\pi (3r^2)(0.03)$$

$$= 4\pi (9)^2 (0.03) = 9.72\pi \text{ cm}^3$$

3. If $\log_e 4 = 1.3868$, then $\log_e 4.01$ =

(1) 1.3968 (2) 1.3898
(3) 1.3893 (4) none

[Ans: (3) 1.3893]

Hint : $y(x) = f(x) + f(x_0)(x - x_0)$

$$\log_e 4.01 = 1.3863 + \frac{1}{4}(0.01)$$

$$= 1.3893$$

4. If $u = \log\sqrt{x^2+y^2}$, then $\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2}$ is

(1) $\sqrt{x^2+y^2}$ (2) 0
(3) u (4) $2u$ **[Ans: (2) 0]**

Hint :

$$\frac{\partial u}{\partial x} = \frac{1}{\sqrt{x^2+y^2}}\frac{1}{2\sqrt{x^2+y^2}}\times 2x$$

$$= \frac{x}{x^2+y^2}$$

$$\frac{\partial^2 u}{\partial x^2} = \frac{(x^2+y^2) - x(2x)}{(x^2+y^2)^2}$$

$$= \frac{y^2-x^2}{(x^2+y^2)^2}$$

$$\frac{\partial u}{\partial y} = \frac{1}{\sqrt{x^2+y^2}}\frac{1}{2\sqrt{x^2+y^2}}\times 2y$$

$$= \frac{y}{x^2+y^2}$$

$$\frac{\partial^2 u}{\partial y^2} = \frac{(x^2+y^2) - y(2y^2)}{(x^2+y^2)^2}$$

$$= \frac{x^2-y^2}{(x^2+y^2)^2}$$

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0$$

5. If $u = x^y + y^x$ then $u_x + u_y$ at $x = y = 1$ is

(1) 0 (2) 2 (3) 1 (4) ∞

[Ans: (2) 2]

Hint : $u_x + u_y = yx^{y-1} + y^x \log y - x^y \log x + xy^{x-1}$

At $x = y = 1$

$u_x + u_y = 1 + 0 + 0 + 1 = 2$

6. If $u = (x-y)^4 + (y-z)^4 + (z-x)^4$ then $\sum\frac{\partial u}{\partial x}$ =

(1) 4 (2) 1 (3) 0 (4) – 4

[Ans: (3) 0]

Hint :

$$\frac{\partial u}{\partial x} = 4(x-y)^3 + 4(z-x)^3(-1)$$

$$= 4(x-y)^3 - 4(z-x)^3$$

$$\frac{\partial u}{\partial y} = 4(x-y)^3(-1) + 4(y-z)^3$$

$$= -4(x-y)^3 + 4(y-z)^3$$

$$\frac{\partial u}{\partial z} = 4(x-y)^3(-1) + 4(y-z)^3$$

$$= -4(y-z)^3 + 4(z-x)^3$$

$$\sum\frac{dy}{dx} = 0$$

CHAPTER 9

APPLICATIONS OF INTEGRATION

MUST KNOW DEFINITIONS

- First fundamental Theorem of Integral Calculus $F(x) = \int_a^x f(u)du$, $a < x < b$ then $\frac{d}{dx}(F(x)) = f(x)$
- Second fundamental Theorem of Integral Calculus $\int_a^b f(x)dx = F(b) - F(a)$
- $\int_0^\infty e^{-x}x^{n-1}dx$ is the gamma integral $\Gamma(n)$

IMPORTANT FORMULA TO REMEMBER

- **Right end rule for Riemann Integral:**

$$\int_a^b f(x)dx = \lim_{\substack{n\to\infty \\ \max(x_i - x_{i-1})\to 0}} \sum_{i=1}^{n} f(x_i)(x_i - x_{i-1})$$

- **Left end rule for Riemann Integral:**

$$\int_a^b f(x)dx = \lim_{\substack{n\to\infty \\ \max(x_i - x_{i-1})\to 0}} \sum_{i=1}^{n} f(x_{i-1})(x_i - x_{i-1})$$

- **Mid point rule for Riemann Integral:**

$$\int_a^b f(x)dx = \lim_{\substack{n\to\infty \\ \max(x_i - x_{i-1})\to 0}} \sum_{i=1}^{n} f\left(\frac{x_{i-1}+x_i}{2}\right)(x_i - x_{i-1})$$

$$= \lim_{n\to\infty}\left[\frac{1}{n}\sum_{r=1}^{n}\frac{5r}{n}+\frac{1}{n}\sum_{r=1}^{n}4\right]$$

$$= \lim_{n\to\infty}\left[\frac{1}{n}\cdot\frac{5}{n}\cdot(1+2+3+...+n)+\frac{1}{n}\cdot 4n\right]$$

$$= \lim_{n\to\infty}\left[\frac{5}{n^2}\frac{n(n+1)}{2}+4\right]$$

$$\left[\because \Sigma r = \frac{n(n+1)}{2};\ \Sigma r^2 = \frac{n(n+1)(2n+1)}{6}\right]$$

$$= \lim_{n\to\infty}\frac{5}{n^2}n^2\frac{(1+\frac{1}{n})}{2}+4$$

$$= \frac{5}{2}(1+0)+4 = \frac{5}{2}+4$$

$[\because$ when $n\to\infty$, $1/n\to 0]$

$$= \frac{5+8}{2} = \frac{13}{2}$$

$$\therefore \int_0^1 (5x+4)dx = \frac{13}{2}$$

(ii) $\int_1^2 \left(4x^2-1\right)dx$

Here $a = 1$, $b = 2$, $f(x) = 4x^2 - 1$

$$\therefore f(a+(b-a)\tfrac{r}{n}) = f(1+1(\tfrac{r}{n}))$$

$$= f(1+\tfrac{r}{n})$$

$$= 4(1+\tfrac{r}{n})^2 - 1 = 4\left(1+\tfrac{r^2}{n^2}+\tfrac{2r}{n}\right) - 1$$

$$= 4+\frac{4r^2}{n^2}+\frac{8r}{n}-1 = 3+\frac{4r^2}{n^2}+\frac{8r}{n}$$

$$\int_a^b f(x)dx = \lim_{n\to\infty}\frac{b-a}{n}\sum_{r=1}^{n}f\left(a+(b-a)\frac{r}{n}\right)$$

$$\therefore \int_1^2 (4x^2-1)\,dx = \lim_{n\to\infty}\frac{1}{n}\sum_{r=1}^{n}\left(3+\frac{4r^2}{n^2}+\frac{8r}{n}\right)$$

$$= \lim_{n\to\infty}\left[\frac{1}{n}\sum_{r=1}^{n}3+\frac{1}{n}\sum_{r=1}^{n}\frac{4r^2}{n^2}+\frac{1}{n}\sum_{r=1}^{n}\frac{8r}{n}\right]$$

$$= \left[\lim_{n\to\infty}\frac{1}{n}.3n+\frac{1}{n}\frac{4}{n^2}(1^2+2^2+...+n^2)+\frac{1}{n}\cdot\frac{8}{n}(1+2+3....+n)\right]$$

$$= \lim_{n\to\infty}\left[3+\frac{4}{n^3}\frac{n(n+1)(2n+1)}{6}+\frac{8}{n^2}\frac{n(n+1)}{2}\right]$$

$$\left[\because \Sigma r = \frac{n(n+1)}{2}\quad \Sigma r^2 = \frac{n(n+1)(2n+1)}{6}\right]$$

$$= \lim_{n\to\infty}\left[3+\frac{4}{n^3}\frac{n^3(1+\frac{1}{n})(2+\frac{1}{n})}{6}+\frac{8}{n^2}\frac{n^2(1+\frac{1}{n})}{2}\right]$$

$$= [3+\tfrac{2}{3}(1+0)(2+0)+4(1+0)]$$

$[\because$ when $n\to\infty$, $1/n\to 0]$

$$= 3+\frac{4}{3}+4 = \frac{9+4+12}{3} = \frac{25}{3}$$

$$\therefore \int_1^2 (4x^2-1)\,dx = \frac{25}{3}$$

EXERCISE 9.3

1. Evaluate the following definite integrals :

(i) $\int_3^4 \frac{dx}{x^2-4}$ **(ii)** $\int_{-1}^{1}\frac{dx}{x^2+2x+5}$

(iii) $\int_0^1 \sqrt{\frac{1-x}{1+x}}\,dx$ **(iv)** $\int_0^{\frac{\pi}{2}} e^x\left(\frac{1+\sin x}{1+\cos x}\right)dx$

(v) $\int_0^{\frac{\pi}{2}}\sqrt{\cos\theta}\,\sin^3\theta\,d\theta$ **(vi)** $\int_0^1 \frac{1-x^2}{\left(1+x^2\right)^2}dx$

Sol. **(i)** $\int_3^4 \frac{dx}{x^2-4}$

$$\int_3^4\frac{dx}{x^2-4} = \int_3^4\frac{dx}{x^2-2^2}$$

$$\left[\because \int\frac{dx}{x^2-a^2} = \frac{1}{2a}\log\left|\frac{x-a}{x+a}\right|\right]$$

$$= \left[\frac{1}{2(2)}\log\left|\frac{x-2}{x+2}\right|\right]_3^4$$

$$= \frac{1}{4}\left[\log\left(\frac{2}{6}\right)-\log\left(\frac{1}{5}\right)\right]$$

$$= \frac{1}{4}\log\left[\left(\frac{2}{6}\right)\div\frac{1}{5}\right] = \frac{1}{4}\log\left(\frac{2}{6}\times\frac{5}{1}\right)$$

$$= \frac{1}{4}\log\left(\frac{5}{3}\right)$$

$$\therefore \int_3^4\frac{dx}{x^2-4} = \frac{1}{4}\log\left(\frac{5}{3}\right)$$

$$\Rightarrow \int_0^\infty e^{-\alpha x^2}\cdot x^2\cdot x\,dx = 32$$

x	0	∞
t	0	∞

put $x^2 = t$

$\Rightarrow \quad 2x\,dx = dt$

$\Rightarrow \quad x\,dx = \frac{dt}{2}$

$$\Rightarrow \frac{1}{2}\int_0^\infty e^{-\alpha t}\cdot t\,dt = 32$$

$$\Rightarrow \frac{1}{2}\int_0^\infty e^{-\alpha t}t^1\,dt = 32$$

$$\Rightarrow \frac{1!}{\alpha^2} = 32\times 2\left[\because \int_0^\infty e^{-ax}x^n\,dx = \frac{n!}{a^{n+1}}\right]$$

$$\Rightarrow \alpha^2 = \frac{1}{32\times 2} \Rightarrow \alpha = \frac{1}{\sqrt{64}}$$

$$\Rightarrow \alpha = \frac{1}{8}$$

EXERCISE 9.8

1. Find the area of the region bounded by $3x - 2y + 6 = 0$, $x = -3$, $x = 1$ and x-axis.

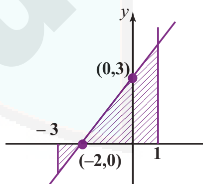
Sol. Given equation of line is $3x - 2y + 6 = 0$

$$2y = 3x + 6 \Rightarrow y = \frac{3x+6}{2}$$

x	0	-2
y	3	0

$$\therefore \text{Area} = \int_{-3}^{-2} -y\,dx + \int_{-2}^{1} y\,dx$$

[$\because$ the Area is below the x - axis]



$$= \frac{-1}{2}\int_{-3}^{-2}(3x+6)\,dx + \frac{1}{2}\int_{-2}^{1}(3x+6)\,dx$$

$$= \frac{-1}{2}\left[\frac{3x^2}{2}+6x\right]_{-3}^{-2} + \frac{1}{2}\left[\frac{3x^2}{2}+6x\right]_{-2}^{1}$$

$$= -\frac{1}{2}\left[\left(\frac{12}{2}-12\right)-\left(\frac{27}{2}-18\right)\right] + \frac{1}{2}\left[\left(\frac{3}{2}+6\right)-\left(\frac{12}{2}-12\right)\right]$$

$$= -\frac{1}{2}\left[(-6)-\left(\frac{27-36}{2}\right)\right] + \frac{1}{2}\left[\left(\frac{3+12}{2}\right)-(-6)\right]$$

$$= -\frac{1}{2}\left[-6+\frac{9}{2}\right] + \frac{1}{2}\left[\frac{15}{2}+6\right]$$

$$= -\frac{1}{2}\left[\frac{-3}{2}\right] + \frac{1}{2}\left[\frac{27}{2}\right] = \frac{3}{4}+\frac{27}{4} = \frac{30}{4} = \frac{15}{4}$$

$\therefore$ A = 7.5 sq.units

2. Find the area of the region bounded by $2x - y + 1 = 0$, $y = -1$, $y = 3$ and y-axis. [PTA - 5]

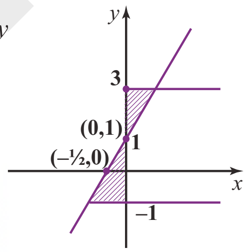
Sol. Given equation of line is $2x - y + 1 = 0$

$$\Rightarrow 2x = y - 1 \Rightarrow x = \frac{y-1}{2}$$

x	0	$-½$
y	1	0

[$\because$ the area is left of the x-axis]

$$\therefore \text{Area} = \int_{-1}^{1} -x\,dy + \int_{1}^{3} x\,dy$$



$$= -\frac{1}{2}\int_{-1}^{1}(y-1)\,dy + \frac{1}{2}\int_{1}^{3}(y-1)\,dy$$

$$= -\frac{1}{2}\left[\frac{y^2}{2}-y\right]_{-1}^{1} + \frac{1}{2}\left[\frac{y^2}{2}-y\right]_{1}^{3}$$

$$= \frac{1}{2}\left[\left(y-\frac{y^2}{2}\right)_{-1}^{1} + \left(\frac{y^2}{2}-y\right)_{1}^{3}\right]$$

$$= \frac{1}{2}\left[\left(1-\frac{1}{2}\right)-\left(1-\frac{1}{2}\right)+\left(\frac{9}{2}-3\right)-\left(\frac{1}{2}-1\right)\right]$$

$$= \frac{1}{2}\left[\frac{1}{2}+\frac{3}{2}+\frac{3}{2}+\frac{1}{2}\right] = \frac{1}{2}\left[\frac{8}{2}\right] = \frac{1}{2}(4)$$

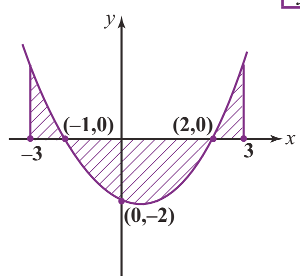
= 2 sq.units

3. Find the area of the region bounded by the curve $2 + x - x^2 + y = 0$, x-axis, $x = -3$ and $x = 3$.

Sol. Equation of the given curve is $2 + x - x^2 + y = 0$

$$\Rightarrow y = x^2 - x - 2$$

x	0	2	-1
y	-2	0	0



The Correct match is

	(i)	(ii)	(iii)	(iv)
(1)	b	c	a	d
(2)	a	b	c	d
(3)	d	a	b	c
(4)	b	c	d	a

[**Ans :** (4) i – b ii – c iii – d 4 – a]

2 MARKS

1. Prove that $\int_0^{\frac{\pi}{2}} \log(\tan x)\,dx = 0.$

Sol. Let $I = \int_0^{\pi/2} \log(\tan x)\,dx$ ...(1)

Applying the property $\int_0^a f(x)\,dx = \int_0^a f(a-x)\,dx$, we get

$$I = \int_0^{\pi/2} \log\left(\tan\left(\frac{\pi}{2}-x\right)\right)dx$$

$$= \int_0^{\pi/2} \log(\cot x)\,dx \quad ...(2)$$

$$(1)+(2) \to 2I = \int_0^{\pi/2} \log(\tan x) + \log(\cot x)\,dx$$

$$= \int_0^{\pi/2} \log \tan x \, . \cot x\,dx$$

$$= \int_0^{\pi/2} \log 1\,dx = 0$$

$\Rightarrow$ I = 0 Hence proved

2. Evaluate $\int_0^1 \frac{e^x}{1+e^{2x}}dx$.

Sol. Let $I = \int_0^1 \frac{e^x}{1+e^{2x}}dx$

Put $e^x = t \Rightarrow e^x\,dx = dt$

x	0	1
t	1	e

$$\therefore I = \int_1^e \frac{dt}{1+t^2} = \left[\tan^{-1}(t)\right]_1^e$$

$$= \tan^{-1}(e) - \tan^{-1}(1) = \tan^{-1}(e) - \frac{\pi}{4}$$

3. Evaluate $\int_1^2 \frac{3x}{9x^2-1}dx$

Sol. Let $I = \int_1^2 \frac{3x}{9x^2-1}dx \Rightarrow |A^3| = |I|$

Put $t = 9x^2 - 1 \Rightarrow dt = 18x\,dx \Rightarrow \frac{dt}{6} = 3x\,dx$

x	1	2
t	9	35

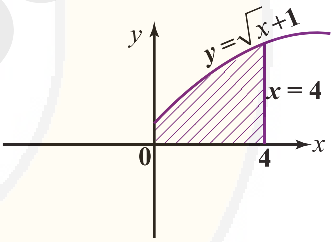
$$\therefore I = \int_8^{35} \frac{dt}{6t}$$

$$= \frac{1}{6}[\log t]_8^{35} = \frac{1}{6}[\log 35 - \log 8]$$

$$= \frac{1}{6}\left[\log\left(\frac{35}{8}\right)\right]$$

4. Find the area of the region enclosed by the curve $y = \sqrt{x} + 1$**, the axis of** x **and the lines** $x = 0, x = 4$.

Sol. Given curve is $y = \sqrt{x} + 1$



Required area

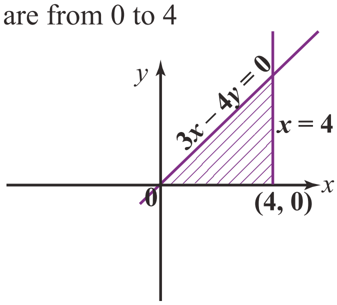
$$\int_0^4 y\,dx = \int_0^4 \left(\sqrt{x}+1\right)dx = \left[\frac{2}{3}x^{\frac{3}{2}} + x\right]_0^4$$

$$= \frac{2}{3}(4)^{\frac{3}{2}} + 4 = \frac{2}{3}(4)\sqrt{4} + 4 = \frac{16}{3} + 4$$

$$= \frac{16+12}{3} = \frac{28}{3} \text{ sq.units}$$

5. Find the volume of the solid obtained by revolving the area of the triangle whose sides are $x = 4$, $y = 0$ **and** $3x - 4y = 0$ **about x - axis**

Sol. Limits are from 0 to 4



CHAPTER 10

ORDINARY DIFFERENTIAL EQUATIONS

MUST KNOW DEFINITIONS

- The order of a differential equation is the highest **order** derivative present in the differential equation.
- The integral power of the highest order derivative appears is called the **degree** of the differential equation.
- If a differential equation contains only ordinary derivatives of one or more functions with respect to a single independent variable, it is an **ordinary differential equation** (ODE).
- An equation involving only partial derivatives of one or more functions of two or more independent variables is a **partial differential equation** (PDE)
- If $g(x) = 0$ in $a_n(x)y^{(n)} + a_{n-1}(x)\, y^{n-1} + \ldots + a_1(x)y + a_0 y = g(x)$ then it is said to be **homogeneous.**
- The solution which contains as many arbitrary constants as the order of the differential equation is called the general solution.
- Integrating factor I.F = $e^{\int pdx}$ in $\frac{dy}{dx} + \mathrm{P}y = \mathrm{Q}$.
- Solution of $\frac{dy}{dx} + \mathrm{P}y = \mathrm{Q}$ is $ye^{\int pdx} = \int \mathrm{Q}e^{\int pdx} dx + \mathrm{C}$

$$\Rightarrow \quad y'' e^{-3x} + 3y' e^{-3x} = 0$$
$$\Rightarrow \quad e^{-3x}(y'' + 3y) = 0$$
$$y'' + 3y' = 0 \quad [\because e^{-3x} \neq 0]$$

$\therefore y = ae^{-3x} + b$ is a solution of $y'' + 3y = 0$

7. Show that the differential equation representing the family of curves $y^2 = 2a\left(x + a^{\frac{2}{3}}\right)$ where a is a positive parameter, is $\left(y^2 - 2xy\frac{dy}{dx}\right)^3 = 8\left(y\frac{dy}{dx}\right)^5$

Sol. Consider $y^2 = 2a\left(x + a^{\frac{2}{3}}\right)$

$$\Rightarrow \quad y^2 = 2ax + 2a^{\frac{2}{3}}$$

Differentiating with respect to 'x' we get,

$$2yy' = 2a(1) \Rightarrow a = yy' \quad ...(2)$$

Substituting (2) in (1) we get,

$$y^2 = 2x(yy') + 2(yy')^{\frac{5}{3}}$$
$$\Rightarrow \quad y^2 - 2xyy' = (yy')^{\frac{2}{3}}$$

Taking power 3 both sides we get,

$$(y^2 - 2xyy')^3 = 2^3 (yy')^{\frac{5}{3}\times 3}$$
$$\Rightarrow \quad (y^2 - 2xyy')^3 = 8(yy')^5$$
$$y^2 = 2a(x + a^{\frac{2}{3}})$$

is a solution of

$$\left(y^2 - 2xy\frac{dy}{dx}\right)^3 = 8\left(y\frac{dy}{dx}\right)^5$$

8. Show that $y = a \cos bx$ is a solution of the differential equation $\frac{d^2y}{dx^2} + b^2 y = 0$.

Sol. Consider $y = a \cos bx \quad ...(1)$

Differentiating with respect to 'x' we get,

$$y' = -a \sin bx(b) = -ab \sin(bx)$$

Differentiating again with respect to 'x' we get,

$$y'' = -ab \cos(bx)(b)$$
$$= -ab^2 \cos(bx)$$
$$\Rightarrow \quad y'' = -b^2[a \cos bx]$$
$$\Rightarrow \quad y'' = -b^2 y \text{ [using (1)]}$$
$$\Rightarrow \quad y'' + b^2 y = 0$$

$\therefore y = a \cos bx$ is a solution of $\frac{d^2y}{dx^2} + b^2 y = 0$.

EXERCISE 10.5

1. If F is the constant force generated by the motor of an automobile of mass M, its velocity V is given by $M\frac{dV}{dt} = F - kV$, where k is a constant. Express V in terms of t given that V = 0 when $t = 0$.

Sol. Given equation is $m\frac{dv}{dt} = F - kv$ on separating the variables, we get

$$\frac{dv}{F - kv} = \frac{dt}{m}$$
$$\int \frac{dv}{F - kv} = \int \frac{dt}{m}$$
$$\Rightarrow \quad \frac{\log(F - kv)}{-k} = \frac{t}{m} + \log c$$
$$\Rightarrow \quad \log(F - kv) = \frac{-kt}{m} + \log c.$$
$$\Rightarrow \quad \log(F - kv) - \log c = \frac{-kt}{m}$$
$$\Rightarrow \quad \log\left(\frac{F - kv}{c}\right) = \frac{-kt}{m} \Rightarrow \frac{F - kv}{e} = e^{\frac{-kt}{m}}$$
$$\Rightarrow \quad \frac{F - kv}{c} = \frac{1}{e^{\frac{kt}{m}}}$$
$$\Rightarrow \quad c = (F - kv)e^{\frac{kt}{m}} \quad ...(1)$$

When $v = 0, t = 0 \Rightarrow c = (F - 0)e^0 \Rightarrow c = F$

$\therefore$ (1) becomes, $F = (F - kv)e^{\frac{kt}{m}}$

2. The velocity v, of a parachute falling vertically satisfies the equation $v\frac{dv}{dx} = g\left(1 - \frac{v^2}{k^2}\right)$, where g and k are constants. If v and x are both initially zero, find v in terms of x.

Sol. Given $v\frac{dv}{dx} = g\left(1 - \frac{v^2}{k^2}\right) = g\left(\frac{k^2 - v^2}{k^2}\right)$

On separating the variables we get,

$$\frac{vdv}{k^2 - v^2} = \frac{g}{k^2} \cdot dx$$

Multiplying by –2 both sides we get,

$$\frac{-2vdv}{k^2 - v^2} = \frac{-2g}{k^2} dx$$

2. Find the population of a city at any time t, given that the rate of increase of population is proportional to the population at that instant and that in a period of 40 years the population increased from 3,00,000 to 4,00,000.

Sol. Let P be denote the population of a city.

Given that $\frac{dP}{dt} \propto P$

$$\Rightarrow kP = \frac{dP}{dt}$$
$$\Rightarrow \frac{dP}{P} = k\,dt$$
$$\Rightarrow \int \frac{dP}{P} = k\int dt$$
$$\Rightarrow \log P = kt + \log c$$
$$\Rightarrow \log\left(\frac{P}{c}\right) = kt$$
$$\Rightarrow \frac{P}{c} = e^{kt}$$
$$\Rightarrow P = c.e^{kt} \quad \ldots (1)$$

Given when t = 0, P = 3,00,000

$\therefore (1) \rightarrow 3,00,000 = ce^0 \Rightarrow c = 3,00,000$

$$\therefore P = 3,00,000\, e^{kt} \quad \ldots (2)$$

Again when t = 40, P = 4,00,000

$\therefore (2) \Rightarrow 4,00,000 = 3,00,000\, e^{40k}$

$$\Rightarrow \frac{4}{3} = e^{40k}$$
$$\Rightarrow \log\left(\frac{4}{3}\right) = 40k$$
$$\Rightarrow k = \frac{1}{40}\log\left(\frac{4}{3}\right)$$
$$\Rightarrow k = \log\left(\frac{4}{3}\right)^{\frac{1}{40}} \quad \ldots (3)$$

$\therefore$ (2) becomes, $P = 3,00,000\, e^{\log\left(\frac{4}{3}\right)^{\frac{1}{40}t}}$

$$\Rightarrow P = 3,00,000\left(\frac{4}{3}\right)^{\frac{t}{40}}$$

3. The equation of electromotive force for an electric circuit containing resistance and self inductance is E = Ri L$\frac{di}{dt}$, where E is the electromotive force is given to the circuit, R the resistance and L, the coefficient of induction. Find the current i at time t when E = 0. [PTA -2]

Sol. Given $E = Ri + L\frac{di}{dt}$

$$\frac{E}{L} = \frac{Ri}{L} + \frac{di}{dt}$$
$$\Rightarrow \frac{Ri}{L} + \frac{di}{dt} = \frac{E}{L}$$

This is a linear differential equation

Here $P = \frac{R}{L}$ and $Q = \frac{E}{L}$

$$\therefore \int p\,dt = \int \frac{R}{L}\,dt = \frac{R}{L}t$$
$$\therefore \text{I.F} = e^{\int p\,dt} = e^{\frac{Rt}{L}}$$
$$\therefore \text{Solution is } ie^{\int p\,dt} = \int Qe^{\int p\,dt}\,dt + C$$
$$\Rightarrow i\,e^{\frac{Rt}{L}} = \int \frac{E}{L}\cdot e^{\frac{Rt}{L}}\,dt + C$$
$$\therefore i\,e^{\frac{Rt}{L}} = \frac{E}{L}\frac{e^{\frac{Rt}{L}}}{\frac{R}{L}} + C$$
$$= \frac{E}{R}e^{\frac{Rt}{L}} + C$$
$$i = \frac{E}{R} + ce^{-\frac{Rt}{L}}$$

When E = 0, $i = 0 + ce^{-\frac{Rt}{L}}$

$$\Rightarrow i = ce^{-\frac{Rt}{L}}$$

4. The engine of a motor boat moving at 10 m/s is shut off. Given that the retardation at any subsequent time (after shutting off the engine) equal to the velocity at that time. Find the velocity after 2 seconds of switching off the engine.

Sol. Let V be the velocity and the retardation (negative acceleration) be $-\frac{dv}{dt}$

Given $\frac{dv}{dt} = -V$

Separating the variables,

$$\frac{dv}{v} = -dt \Rightarrow \int\frac{dv}{v} = -\int dt$$
$$\Rightarrow \log v = -t + \log C$$
$$\Rightarrow \log v - \log C = -t$$
$$\Rightarrow \log\left(\frac{v}{C}\right) = -t \Rightarrow \frac{v}{C} = e^{-t}$$
$$\Rightarrow v = Ce^{-t} \quad \ldots(1)$$

Given when t = 0, v = 10 m/sec

By separating the variables,

$$\frac{v}{v+1}\,dv = \frac{dx}{x}$$

Integrating, we obtain

$v + \log|v| = \log|x| + \log|C|$ or

$$v = \log\left|\frac{Cx}{v}\right|$$

Replacing v by $\frac{y}{x}$, we get

$$\frac{y}{x} = \log\left|\frac{Cx^2}{y}\right|$$

$$\left|\frac{Cx^2}{y}\right| = e^{\frac{y}{x}}$$

$$kx^2 = ye^{\frac{y}{x}}$$

2. **Solve : $\frac{dy}{dx} = \frac{x-y}{2(x-y)+7}$** **[PTA - 5]**

Sol. Given that $\frac{dy}{dx} = \frac{x-y+5}{2(x-y)+7}$

Put $z = x - y$

$$\frac{dz}{dx} = 1 - \frac{dy}{dx}$$

$$\frac{dy}{dx} = 1 - \frac{dz}{dx}$$

Thus, the given equation reduces to

$$1 - \frac{dz}{dx} = \frac{z+5}{2z+7}$$

$$\frac{dz}{dx} = 1 - \frac{z+5}{2z+7}$$

$$\frac{dz}{dx} = \frac{z+2}{2z+7}$$

Separating the variables, we get

$$\frac{2z+7}{z+2}\,dz = dx$$

$$\frac{2(z+2)+3}{(z+2)}\,dz = dx$$

$$\left(2 + \frac{3}{z+2}\right)dz = dx$$

Integrating both sides, we get

$2z + 3\log|z+2| = x + C$

That is, $2(x-y) + 3\log|x-y+2| = x + C$

3. **An equation relating to the stability of an aircraft is given by $\frac{dv}{dt} = g\cos\alpha - kv$, where g, α, k are constants and v is the velocity. Obtain an expression in terms of v if $v = 0$ when $t = 0$.** **[PTA - 6]**

Sol. $\frac{dv}{dt} + kv = g\cos x$

$$P = k \qquad Q = g\cos\alpha$$

$$\int P\,dt = \int k\,dt = kt$$

$$\text{I.F} = e^{\int P\,dt} = e^{kt}$$

Hence solution,

$$ve^{\int Pdt} = \int Qe^{\int Pdt}\,dt + c$$

$$ve^{kt} = \int g\cos\alpha\, e^{kt}\,dt + c$$

$$ve^{kt} = g\cos\alpha\,\frac{e^{kt}}{k} + c$$

$t = 0$; $v = 0$ $\quad 0 = \frac{g\cos\alpha}{k} + c$

$$c = -\frac{g\cos\alpha}{k}$$

$$ve^{kt} = g\cos\alpha\,\frac{e^{kt}}{k} - \frac{g\cos\alpha}{k}$$

GOVT. EXAM QUESTION & ANSWERS

1 MARK

I. Choose the Correct or the most suitable answer from the given four alternatives :

1. **If m, n are the order and degree of the differential equation $\left[\frac{d^4y}{dx^4} + \frac{d^2y}{dx^2}\right]^{\frac{1}{2}} = a\frac{d^2y}{dx^2}$ respectively, then the value of $4m - n$ is** **[Govt. MQP-2019]**

(1) 15 (2) 12 (3) 14 (4) 13

[Ans. (1) 15]

Hint : $\left[\frac{d^4y}{dx^4} + \frac{d^2y}{dx^2}\right]^{\frac{1}{2}} = a\frac{d^2y}{dx^2}$

Squaring both sides

$$\frac{d^4y}{dx^4} + \frac{d^2y}{dx^2} = a^2\left(\frac{d^2y}{dx^2}\right)^2$$

$$m = 4,\ n = 1$$

$$4m - n = 4 \times 4 - 1 = 16 - 1 = 15$$

CHAPTER 11

PROBABILITY DISTRIBUTIONS

MUST KNOW DEFINITIONS

- A random variable. X is defined on a sample space S into the real numbers R is called **discrete random variable** if the range of X is countable.
- If X is a discrete random variable then the function

  $f(x_k) = P(x = x_k)$ for $k = 1, 2, 3 \ldots n$ is called the **probability mass function** of X. (p. m. f.)
- The **Cumulative distribution** function $F(x)$ of a discrete random variable X such that $x_1 < x_2 < \ldots$ with p. m. f. $f(x_i)$ is
- $$F(x) = P(X \le x) = \sum_{x_i \le x} f(x_i),\ x \in \mathbb{R}.$$
- Let S be a sample space and let a random variable $X = S \to \mathbb{R}$ that takes on any value in a set I or R. Then X is called a **continuous random variable** if $P(X = x) = 0$ for every x in I.
- Let x be a random variable associated with a Bernoulli trial by defining it as X (success) = 1 and X (failure) = 0 such that
- $$f(x) = \begin{cases} p & \text{if } x = 1 \\ q = 1 - p & \text{if } x = 0 \end{cases} \text{ where } 0 < p < 1.$$
- X is called a **Bernoulli random variable** and $f(x)$ is called the **Bernoulli distribution.**
- The binomial random variable X, equals the number of successes with probability p for a success and $q = 1 - p$ for a failure in n independent traits, has a binomial distribution
- The p.m. f. of X is $f(x) = nC_x\, p^x q^{n-x}$, $x = 0, 1, 2, \ldots n$.

One point distribution

✦ A random variable X has a one-point distribution if $\exists\, x_0$ such that p. m. f. $f(x)$ is $f(x) = p(X = x_0) = 1$

Two point distribution

✦ A random variable X has a two point distribution if $\exists$ two value x_1 and x_2 such that

✦ $f(x) = \begin{cases} p & \text{for} \quad x = x_1 \\ 1-p & \text{for} \quad x = x_2 \end{cases}, 0 < p < 1$

✦ $F(x) = \begin{cases} 0 & \text{if} \quad x < x_1 \\ p & \text{if} \quad x_1 \le x < x_2 \\ 1 & \text{if} \quad x \ge x_2 \end{cases}$

For Bernoulli's distribution

✦ $\mu = p$ and $\sigma^2 = pq$

For Binomial distribution

✦ $\mu = np$ and $\sigma^2 = np\,(1-p)$.

EXERCISE 11.1

1. Suppose X is the number of tails occurred when three fair coins are tossed once simultaneously. Find the values of the random variable X and number of points in its inverse images.

Sol. When 3 fair coins are tossed, sample space

s = {HHH, HHT, THH, HTH, HTT, THT, TTH, TTT}

Let X denote the number of tails occured.

X (no tail) = {HHH} = 1

X (1 tail) = {HHT, THT, HTH} = 3

X (2 tails) = {HTT, THT, TTH} = 3

X (3 tails) = {TTT} = 1

$\therefore$ X takes the values 0, 1, 2, 3.

Values of random variable X	0	1	2	3	Total
Number of elements in inverse images	1	3	3	1	8

2. In a pack of 52 playing cards, two cards are drawn at random simultaneously. If the number of black cards drawn is a random variable, find the values of the random variable and number of points in its inverse images.

Sol. Sample space s = {26 black cards, 26 red cards}.

Let X denote the number of black cards drawn.

X (no black card) = 26 × 25 = 650 [Taking 2 cards from red]

X (1 black card) = 26 × 26 = 676 [1 black card & 1 red card]

X (2 black cards) = 26 × 25 = 650 [Taking 2 cards from black]

$\therefore$ X takes the values 0, 1, 2.

Values of random variable X	0	1	2	Total
Number of elements in inverse images	650	676	650	1976

3. An urn contains 5 mangoes and 4 apples. Three fruits are taken at random. If the number of apples taken is a random variable, then find the values of the random variable and number of points in its inverse images.

Sol. Sample space S = {5 mangoes, 4 apples}

Let X denote the number of apples taken

$$X\text{ (no apple)} = 5C_3 = \frac{5 \times \overset{2}{\cancel{4}} \times \cancel{3}}{3 \times \cancel{2} \times 1} = 10$$

$$X\text{ (1 apple)} = 4C_1 \times 5C_2 = 4 \times \frac{5 \times \overset{2}{\cancel{4}}}{\cancel{2} \times 1} = 40$$

Hint : Given $p = \frac{1}{20}$, $q = \frac{19}{20}$ and $n = 3$

$$P(X=2) = 3C_2\left(\frac{1}{20}\right)^2\left(\frac{19}{20}\right)^1 = \frac{3\times19}{20} = \frac{57}{20^3}$$

PTA QUESTION & ANSWERS

1 MARKS

1. **If the mean of a binomial distribution is 5 and its variance is 4, then the value of n and p are** [PTA - 3]

(1) $\left(\frac{1}{5}, 25\right)$ (2) $\left(25, \frac{1}{5}\right)$

(3) $\left(25, \frac{4}{5}\right)$ (4) $\left(\frac{4}{5}, 25\right)$

[Ans. (3) $\left(25, \frac{4}{5}\right)$]

Hint : Mean = $np = 5$

Variance = $npq = 4$

$$\frac{npq}{np} = q = \frac{4}{5} \Rightarrow p = \frac{1}{5}$$

$$np = 5 \Rightarrow n = 25 \quad \left(25, \frac{4}{5}\right)$$

2 MARKS

2. **Find the mean of the distribution** [PTA - 3]

$$f(x) = \begin{cases} 3e^{-3x} & 0 < x < \infty \\ 0 & elsewhere \end{cases}$$

Sol. $\mu = E(X) = \int_{-0}^{\infty} xf(x)dx = \int_{-0}^{\infty} x(0)dx + \int_{0}^{\infty} 3e^{-3x}x\,dx$

$$= 0 + 3\int_0^{\infty} xe^{-3x}dx = 3\left[x\frac{e^{-3x}}{-3} - \frac{e^{-3x}}{9}\right]_0^{\infty}$$

$$= 3\left[0 - 0 - \left(0 - \frac{1}{9}\right)\right] = \frac{1}{3}$$

3 MARKS

1. **The probability distribution of a random variable is given below** [PTA - 2]

x	0	1	2	3	4	5	6	7
P (X = x)	0	k	$2k$	$2k$	$3k$	k^2	$2k^2$	$7k^2 + k$

Then find P(0 < X < 4).

Sol. $k + 2k + 2k + 3k + k^2 + 2k^2 + 7k^2 + k = 1$

$$10k^2 + 9k - 1 = 0$$
$$10k^2 + 10k - k - 1 = 0$$
$$10k\ (k+1) - (k+1) = 0$$
$$(k+1) - (10k-1) = 0$$
$$k = -1, \frac{1}{10}$$
$$P(0 < X < 4) = k + 2k + 2k$$
$$= 5k = 5\left(\frac{1}{10}\right) = 0.5$$

5 MARKS

1. **A multiple choice examination has ten questions, each question has four distractors with exactly one correct answer. Suppose a student answers by guessing and if X denotes the number of correct answers, find (i) binomial distribution (ii) probability that the student will get seven correct answers (iii) the probability of getting at least one correct answer.** [PTA - 1]

Sol. (i) Since X denotes the number of success, X can take the values 0,1,2,....,10

The probability for success is $p = \frac{1}{4}$ and for failure $q = 1 - p\ \frac{3}{4}$, and n =10.

Therefore X follows the binomial distribution

$X \sim B\left(10, \frac{1}{4}\right)$

This gives, $f(x) = \binom{10}{x}\left(\frac{1}{4}\right)^x\left(\frac{3}{4}\right)^{10-x}$,

$x = 0, 1, 2, ..., 10$

(ii) Probability for seven correct answers is

$$P(X=7) = f(7) = \binom{10}{x}\left(\frac{1}{4}\right)^x\left(\frac{3}{4}\right)^{10-x}$$
$$= 120\left(\frac{3^3}{4^{10}}\right)$$

Probability that the student will get seven correct answers is $120\left(\frac{3^3}{4^{10}}\right)$.

(ii) Probability for at least one correct answer is

$$P(X \geq 7) = 1 - P(X < 1) = 1 - P(X=0)$$
$$= 1 - \binom{10}{0}\left(\frac{1}{4}\right)^0\left(\frac{3}{4}\right)^{10} = 1 - \left(\frac{3}{4}\right)^{10}$$

Probability that the student will get for at least one correct answer is $1 - \left(\frac{3}{4}\right)^{10}$.

CHAPTER
12

DISCRETE MATHEMATICS

MUST KNOW DEFINITIONS

- A **binary operation** * on S is defined as follows: $\forall\, a, b \in$ S, $a * b$ is unique and $a * b \in$ S
- A binary operation * defined by *: S × S → S; $(a, b) = a * \in$ S must always lie in the given set and not in the complement of it. Then S is closed with respect to *.
- A binary operation * defined on a non empty set S is said to satisfy the **commutative property** if $a * b = b * a \;\forall\, a, b \in$ S
- If $a * (b * c) = (a * b) * c \;\forall\, a, b \in$ S, then S is said to satisfy the **associative property**.
- An element $e \in$ S is said to satisfy the **identity element** of s if $\forall\, a \in$ S, $a * e, = e * a = a$.
- If for every $a \in$ S, there exists b in S such that $a * b = b * a = e$ then $b \in$ S is said to be the inverse element of a.
- In an algebraic structure, the identity element and the inverse of an element must be unique.
- A Boolean matrix is a real matrix whose entries are either 0 or 1.
- Joint of A and B, $A \vee B = [a_{ij}] \vee [b_{ij}] = [a_{ij} \vee b_{ij}] = [c_{ij}]$ where $c_{ij} = \begin{cases} 1 & \text{if either } a_{ij} = 1 \text{ or } b_{ij} = 1 \\ 0 & \text{if both } a_{ij} = 0, \; b_{ij} = 0 \end{cases}$
- Meet of A and B, $A \vee B = [a_{ij} \wedge b_{ij}] = [c_{ij}]$ where $c_{ij} = \begin{cases} 1 & \text{if both } a_{ij} = 1, b_{ij} = 1 \\ 0 & \text{if either } a_{ij} = 0, b_{ij} = 0 \end{cases}$
- **Addition moduls** n

  Let $a, b \in \mathbb{Z}n$. Then $a +_n b$ = the remainder of $a + b$ on division by n.
- Multiplication moduls n

  Let $a, b \in \mathbb{Z}n$. Then $a \times_n b$ = the remainder of $a \times b$ on division by n.
- A statement is said to be a tautology ($\mathbb{T}$) if its truth value is always T irrespective of the truth values of its compound statements.
- A statement is a **contradiction** ($\mathbb{F}$) if its truth value is always F irrespective of the truth value of its compound statements.
- A dual is obtained by replacing $\mathbb{T}$ by $\mathbb{F}$ and $\mathbb{F}$ by $\mathbb{T}$

EXERCISE 12.3

Choose the Correct or the most suitable answer from the given four alternatives :

1. **A binary operation on a set S is a function from**
 (1) S → S (2) (S×S) → S
 (3) S → (S×S) (4) (S×S) → (S×S)
 [Ans. (2) (S×S) → S]

2. **Subtraction is not a binary operation in** [PTA-6]
 (1) ℝ (2) ℤ (3) ℕ (4) ℚ
 [Ans. (3) ℕ]

3. **Which one of the following is a binary operation on ℕ?**
 (1) Subtraction (2) Multiplication
 (3) Division (4) All the above
 [Ans. (2) Multiplication]

4. **In the set ℝ of real numbers '∗' is defined as follows. Which one of the following is not a binary operation on ℝ?** [PTA - 4]
 (1) $a*b = \min(a.b)$ (2) $a*b = \max(a,b)$
 (3) $a*b = a$ (4) $a*b = a^b$
 [Ans. (4) $a*b = a^b$]

Hint : Let $a = 0, b = 0, \Rightarrow o^\circ \notin R$. (4) $a * b = ab$

5. **The operation ∗ defined by $a*b = \frac{ab}{7}$ is not a binary operation on**
 (1) $\mathbb{Q}^+$ (2) ℤ (3) ℝ (4) ℂ
 [Ans. (3) ℤ]

Hint : Let $a = 3, b = 2$
$$\Rightarrow a * b = 3 * 2 = \frac{3(2)}{7} = \frac{6}{7} \notin z.$$

6. **In the set ℚ define $a \odot b = a + b + ab$. For what value of y, $3\odot(y\odot5) = 7$?**
 (1) $y = \frac{2}{3}$ (2) $y = \frac{-2}{3}$
 (3) $y = \frac{-3}{2}$ (4) $y = 4$ **[Ans. (2) $y = \frac{-2}{3}$]**

Hint : $\otimes b = a + b + ab$
$$\therefore 3 \otimes (y \otimes 5) = 7 \Rightarrow 3 \otimes (y + 5 + 5y) = 7$$
$$\Rightarrow 3 + y + 5 + 5y + 3(y + 5 + 5y) = 7$$
$$\Rightarrow 8 + 6y + 3y + 15 + 15y = 7$$
$$\Rightarrow 23 + 24y = 7 \Rightarrow 24y = 7 - 23 = -16$$
$$\Rightarrow y = \frac{-16}{24} = \frac{-2}{3}$$

7. **If $a * b = \sqrt{a^2 + b^2}$ on the real numbers then ∗ is** [PTA - 4]
 (1) commutative but not associative
 (2) associative but not commutative
 (3) both commutative and associative
 (4) neither commutative nor associative
 [Ans. (3) both commutative and associative]

Hint : $a * b = \sqrt{a^2 + b^2} = \sqrt{b^2 + a^2} = b * a$
Also $a * (b * c) = a * \sqrt{b^2 + c^2} = \sqrt{a^2 + b^2 + c^2}$
$(a * b) * c = \sqrt{a^2 + b^2} = \sqrt{a^2 + b^2 + c^2}$

8. **Which one of the following statements has the truth value T?** [PTA - 2]
 (1) sin x is an even function.
 (2) Every square matrix is non-singular
 (3) The product of complex number and its conjugate is purely imaginary
 (4) $\sqrt{5}$ is an irrational number
 [Ans. (4) $\sqrt{5}$ is an irrational number]

9. **Which one of the following statements has truth value F?**
 (1) Chennai is in India or $\sqrt{2}$ is an integer
 (2) Chennai is in India or $\sqrt{2}$ is an irrational number
 (3) Chennai is in China or $\sqrt{2}$ is an integer
 (4) Chennai is in China or $\sqrt{2}$ is an irrational number
 [Ans. (3) Chennai is in China or $\sqrt{2}$ is an integer]

10. **If a compound statement involves 3 simple statements, then the number of rows in the truth table is**
 (1) 9 (2) 8 (3) 6 (4) 5
 [Ans. (2) 8]

Hint : No. of rows = $2^3 = 8$

11. **Which one is the inverse of the statement $(p\vee q)\to(p\wedge q)$?** [PTA - 2;PTA -3]
 (1) $(p\wedge q) \to (p\vee q)$ (2) $\neg(p\vee q) \to (p\wedge q)$
 (3) $(\neg p\vee\neg q) \to (\neg p\wedge\neg q)$
 (4) $(\neg p\wedge\neg q) \to (\neg p\vee\neg q)$
 [Ans. (4) $(\neg p\wedge\neg q) \to (\neg p\vee\neg q)$]

Hint : $(p \vee q) \to (p \wedge q)$
$\sim[(p \vee q) \to (p \wedge q)] \to (\sim p \wedge \sim q) \to (\sim p \vee \sim q)$

12th STD

GOVT. MODEL QUESTION PAPER

Reg. No.

Mathematics

Time Allowed : 15 Minutes + 3.00 Hours] Maximum Marks : 90

Introductions :

(a) Check the question paper for fairness of printing. If there is any lack of fairness, inform the Half Supervisor immediately.

(b) Use **Blue** or **Black** ink to write and underline and pencil to draw diagrams

PART - I

Note : (i) Answer all questions are compulsory. **[20 × 1 = 20]**

(ii) Choose the most suitable answer from the given **four** correct alternatives and write the option code and the corresponding answer.

1. If $A = \begin{bmatrix} 3 & 5 \\ 1 & 2 \end{bmatrix}$, B = adj A and C = 3A, then $\frac{|\text{adj } B|}{|C|} =$

(a) $\frac{1}{3}$ (b) $\frac{1}{9}$ (c) $\frac{1}{4}$ (d) 1

2. If the inverse of the matrix $\begin{bmatrix} 1 & 2 \\ 3 & -5 \end{bmatrix}^n$ is $\frac{1}{11}\begin{bmatrix} a & b \\ c & d \end{bmatrix}$, then the ascending order of a, b, c, d is

(a) a, b, c, d (b) d, b, c, a

(c) c, a, b, d (d) b, a, c, d

3. The least value of n satisfying $\left[\frac{\sqrt{3}}{2}+\frac{i}{2}\right]^n = 1$ is

(a) 30 (b) 24 (c) 12 (d) 18

4. The principal argument of $\frac{3}{-1+i}$ is

(a) $\frac{-5\pi}{6}$ (b) $\frac{-2\pi}{3}$ (c) $\frac{-3\pi}{4}$ (d) $\frac{-\pi}{2}$

5. The polynomial equation $x^3 + 2x + 3 = 0$ has

(a) one negative and two real roots

(b) one positive and two imaginary roots

(c) three real roots

(d) no solution

6. The domain of the function defined by $f(x) = \sin^{-1}\left(\sqrt{x-1}\right)$ is

(a) [1, 2] (b) [–1, 1]

(c) [0, 1] (d) [–1,0]

7. If $x + y = k$ is a normal to the parabola $y^2 = 12x$, then the value of k is

(a) 3 (b) –1 (c) 1 (d) 9

8. The circle passing through(1, –2) and touching the x-axis of x at (3, 0), again passing through the point is

(a) (–5, 2) (b) (2, –5)

(c) (5, –2) (d) (–2, 5)

9. The volume of the parallelepiped with its edges represented by the vectors $\hat{i}+\hat{j}, \hat{i}+2\hat{j}, \hat{i}+\hat{j}+\pi\hat{k}$ is

(a) $\frac{\pi}{2}$ (b) $\frac{\pi}{3}$ (c) π (d) $\frac{\pi}{4}$

10. If the line $\frac{x-2}{3} = \frac{y-1}{-5} = \frac{z+2}{2}$ lies in the plane $x + 3y - \alpha z + \beta = 0$ then (α, β) is

(a) (– 5,5) (b) (– 6,7)

(c) (5, – 5) (d) (6, –7)

11. The function $\sin^4 x + \cos^4 x$ is increasing in the interval

(a) $\left[\frac{5\pi}{8}, \frac{3\pi}{4}\right]$ (b) $\left[\frac{\pi}{2}, \frac{5\pi}{8}\right]$

(c) $\left[\frac{\pi}{4}, \frac{\pi}{2}\right]$ (d) $\left[0, \frac{\pi}{4}\right]$

12. The curve $y = ax^4 + bx^2$ with $ab > 0$

(a) has no horizontal tangent

(b) is concave up

(c) is concave down

(d) has no points of inflection

13. If $u\ (x-y)^2$, then $\frac{\partial u}{\partial x} + \frac{\partial u}{\partial y}$ is

(a) 1 (b) –1 (c) 0 (d) 2

14. The value of $\int_0^{\pi} \frac{dx}{1+5^{\cos x}}$ is

(a) $\frac{\pi}{2}$ (b) π (c) $\frac{3\pi}{2}$ (d) 2π

12th STD

COMMON HALF YEARLY EXAMINATION - 2019

Reg. No.

Mathematics

TIME ALLOWED : 3.00 Hours MAXIMUM MARKS : 90

Instructions :

1. Check the question paper for fairness of printing. If there is any lack of fairness, inform the Hall Supervisor immediately.
2. Use **Blue** or **Black** ink to write and underline and pencil to draw diagrams

PART - I

Note : (i) Answer all the questions. **[20 × 1 = 20]**

(ii) Choose the most appropriate answer from the given four alternatives and write the option code and the corresponding answer.

1. **If A is a non-singular matrix such that $A^{-1} = \begin{bmatrix} 5 & 3 \\ -2 & -1 \end{bmatrix}$ then $(A^T)^{-1} =$**

(a) $\begin{bmatrix} -5 & 3 \\ 2 & 1 \end{bmatrix}$ (b) $\begin{bmatrix} 5 & 3 \\ -2 & -1 \end{bmatrix}$

(c) $\begin{bmatrix} -1 & -3 \\ 2 & 5 \end{bmatrix}$ (d) $\begin{bmatrix} 5 & -2 \\ 3 & -1 \end{bmatrix}$

2. **If adj $A = \begin{bmatrix} 2 & 3 \\ 4 & -1 \end{bmatrix}$ and adj $B = \begin{bmatrix} 1 & -2 \\ -3 & 1 \end{bmatrix}$ then adj (AB) is :**

(a) $\begin{bmatrix} -7 & -1 \\ 7 & -9 \end{bmatrix}$ (b) $\begin{bmatrix} -6 & 5 \\ -2 & -10 \end{bmatrix}$

(c) $\begin{bmatrix} -7 & 7 \\ -1 & -9 \end{bmatrix}$ (d) $\begin{bmatrix} -6 & -2 \\ 5 & -10 \end{bmatrix}$

3. **If $|z - 2 + i| \le 2$, then the greatest value of $|z|$**

(a) $\sqrt{3} - 2$ (b) $\sqrt{3} + 2$

(c) $\sqrt{5} - 2$ (d) $\sqrt{5} + 2$

4. **If $x^2 + y^2 = 1$, then the value of $\dfrac{1+x+iy}{1+x-iy}$.**

(a) $x - iy$ (b) $2x$ (c) $-2iy$ (d) $x + iy$

5. **The product of all four values of $\left(\cos\dfrac{\pi}{3} + i\sin\dfrac{\pi}{3}\right)^{\frac{3}{4}}$ is**

(a) -2 (b) -1 (c) 1 (d) 2

6. **The Polynomial $x^3 + 2x + 3$ has :**

(a) One negative and two imaginary zeros
(b) One positive and two imaginary zeros
(c) Three real zeros
(d) No zeros

7. **The value of $\sin^{-1}(\cos x)$, $0 \le x \le \pi$ is**

(a) $\pi - x$ (b) $x - \dfrac{\pi}{2}$ (c) $\dfrac{\pi}{2} - x$ (d) $x - \pi$

8. **An ellipse has OB, as semi minor axis, F and F′ its foci and the angle FBF′ is a right angle. Then the eccentricity of the ellipse is :**

(a) $\dfrac{1}{\sqrt{2}}$ (b) $\dfrac{1}{2}$ (c) $\dfrac{1}{4}$ (d) $\dfrac{1}{\sqrt{3}}$

9. **The vertex of the parabola $x^2 = 8y - 1$ is :**

(a) $\left(-\dfrac{1}{8}, 0\right)$ (b) $\left(\dfrac{1}{8}, 0\right)$

(c) $\left(0, \dfrac{1}{8}\right)$ (d) $\left(0, -\dfrac{1}{8}\right)$

10. **The value of $[\hat{i}+\hat{j}, \hat{j}+\hat{k}, \hat{k}+\hat{i}]$ is equal to :**

(a) 0 (b) 1 (c) 2 (d) 4

11. **The angle between the lines $\dfrac{x-2}{3} = \dfrac{y+1}{-2}$, $z = 2$ and $\dfrac{x-1}{1} = \dfrac{2y+3}{3} = \dfrac{z+5}{2}$ is :**

(a) $\dfrac{\pi}{6}$ (b) $\dfrac{\pi}{4}$ (c) $\dfrac{\pi}{3}$ (d) $\dfrac{\pi}{2}$

12. **A stone is thrown up vertically. The height it reaches at time t seconds is given by $x = 80t - 16t^2$. The stone reaches the maximum height when $t =$**

(a) 2 (b) 2.5 (c) 3 (d) 3.5

13. **The value of limit $\lim\limits_{x\to 0}\left(\cot x - \dfrac{1}{x}\right)$ is :**

(a) 0 (b) 1 (c) 2 (d) ∞

14. **If $f(x, y) = e^{xy}$, then $\dfrac{\partial^2 f}{\partial x \partial y}$ is :**

(a) xye^{xy} (b) $(1 + xy)e^{xy}$

(c) $(1 + y)e^{xy}$ (d) $(1 + x)e^{xy}$

Let x_1 be the distance between the bottom of the vertical line on the ground from the pipe end and the point on which this water touches the ground. But the height of the pipe from the ground is 7.5 m.

$\therefore$ $(x_1, -7.5)$ lies on ...(2)

$\therefore$ (2) becomes, $x_1^2 = \dfrac{-9}{2.5}(-7.5)$

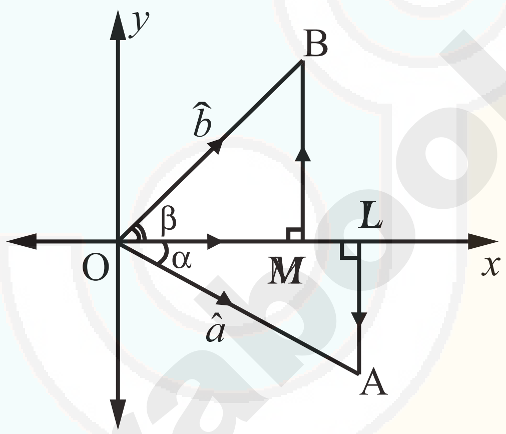
$\Rightarrow \quad x_1^2 = 9(3)$

$\Rightarrow x_1 = \sqrt{9 \times 3} = 3\sqrt{3}$ m

The water strikes the ground $3\sqrt{3}$ m beyond the vertical line.

(b) Solution :

Let $\hat{a} = \overrightarrow{OA}$ and $\hat{b} = \overrightarrow{OB}$ be the unit vectors and which make angles α and β, respectively, with positive x - axis, where A and B are as in the figure. Draw AL and BM perpendicular to the x-axis. Then $|\overrightarrow{OL}| = |\overrightarrow{OA}| \cos\alpha = \cos\alpha$, $|\overrightarrow{LA}| = |\overrightarrow{OA}| \sin\alpha = \sin\alpha$,



So, $\overrightarrow{OL} = |\overrightarrow{OL}|\hat{i} = \cos\alpha\,\hat{i}$, $\overrightarrow{LA} = \sin\alpha\,(-\hat{j})$

Therefore, $\hat{a} = \overrightarrow{OA} = \overrightarrow{OL} + \overrightarrow{LA}$

$= \cos\alpha\,\hat{i} - \sin\alpha\,\hat{j}$ ...(1)

Similarly, $\hat{b} = \cos\beta\,\hat{i} + \sin\beta\,\hat{j}$ ...(2)

The angle between $\hat{a}$ and $\hat{b}$ is $\alpha + \beta$ and so,

$\hat{a} \cdot \hat{b} = |\hat{a}||\hat{b}|\cos(\alpha+\beta) = \cos(\alpha+\beta)$ ...(3)

On the other hand, from (1) and (2)

$\hat{a} \cdot \hat{b} = (\cos a\,\hat{i} - \sin\alpha\,\hat{j}).(\cos\beta\,\hat{i} + \sin\beta\,\hat{j}) = \cos\alpha\cos\beta - \sin\alpha\sin\beta$ ...(4)

From (3) and (4), we get $\cos(\alpha+\beta) = \cos\alpha\cos\beta - \sin\alpha\sin\beta$