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# Chapter – IX – Differential Calculus – Limits and Continuity : Prasanth 8

## Exe (9.1)

In problems 1 – 6, complete the table using calculator and use the result to estimate the limit.

(1) $\lim\limits_{x \to 2} \dfrac{x-2}{x^2-x-2}$

| $x$ | 1.9 | 1.99 | 1.999 | 2.001 | 2.01 | 2.1 |
|---|---|---|---|---|---|---|
| $f(x)$ | | | | | | |

Soln

$$\underset{x\to 2}{Lt} \frac{x-2}{x^2-x-2} = \underset{x\to 2}{Lt} \frac{x-2}{(x-2)(x+1)}$$

$$= \underset{x\to 2}{Lt} \frac{1}{x+1}$$

| $x$ | 1.9 | 1.99 | 1.999 | 2.001 | 2.01 | 2.1 |
|---|---|---|---|---|---|---|
| $f(x)$ | $\frac{1}{2.9}$ $= 0.34$ | $\frac{1}{2.99}$ $= 0.33$ | $\frac{1}{2.999}$ $= 0.33$ | $\frac{1}{3.001}$ $= 0.33$ | $\frac{1}{3.01}$ $= 0.33$ | $\frac{1}{3.1}$ $= 0.32$ |

$\therefore \lim\limits_{x \to 2} \dfrac{x-2}{x^2-x-2} = 0.33$

(2) $\lim\limits_{x \to 2} \dfrac{x-2}{x^2-4}$

| $x$ | 1.9 | 1.99 | 1.999 | 2.001 | 2.01 | 2.1 |
|---|---|---|---|---|---|---|
| $f(x)$ | | | | | | |

Soln

$$\underset{x\to 2}{Lt} \frac{x-2}{x^2-4} = \underset{x\to 2}{Lt} \frac{x-2}{(x-2)(x+2)}$$

$$= \underset{x\to 2}{Lt} \frac{1}{x+2}$$

| $x$ | 1.9 | 1.99 | 1.999 | 2.001 | 2.01 | 2.1 |
|---|---|---|---|---|---|---|
| $f(x)$ | $\frac{1}{3.9}$ $= 0.256$ | $\frac{1}{3.99}$ $= 0.251$ | $\frac{1}{3.999}$ $= 0.250$ | $\frac{1}{4.001}$ $= 0.249$ | $\frac{1}{4.01}$ $= 0.249$ | $\frac{1}{4.1}$ $= 0.244$ |

$$\therefore \underset{x\to 2}{Lt} \frac{x-2}{x^2-4} = 0.25$$

(3) $\underset{x\to 0}{Lim} \frac{\sqrt{x+3}-\sqrt{3}}{x}$

| $x$ | $-0.1$ | $-0.01$ | $-0.001$ | $0.001$ | $0.01$ | $0.1$ |
|---|---|---|---|---|---|---|
| $f(x)$ | | | | | | |

Soln:

$\underset{x\to 0}{Lt} \frac{\sqrt{x+3}-\sqrt{3}}{x}$

| $x$ | $-0.1$ | $-0.01$ | $-0.001$ | $0.001$ | $0.01$ | $0.1$ |
|---|---|---|---|---|---|---|
| $f(x)$ | $\frac{\sqrt{2.9}-\sqrt{3}}{-0.1}$ $=0.29$ | $\frac{\sqrt{2.99}-\sqrt{3}}{-0.01}$ $=0.288$ | $\frac{\sqrt{2.999}-\sqrt{3}}{-0.001}$ $=0.288$ | $\frac{\sqrt{3.001}-\sqrt{3}}{+0.001}$ $\simeq 0.289$ | $\frac{\sqrt{3.01}-\sqrt{3}}{0.01}$ $=0.288$ | $\frac{\sqrt{3.1}-\sqrt{3}}{0.1}$ $=0.286$ |

$$\therefore \underset{x\to 0}{Lt} \frac{\sqrt{x+3}-\sqrt{3}}{x} = 0.288 \text{ (or) } 0.29$$

(4) $\underset{x\to -3}{Lim} \frac{\sqrt{1-x}-2}{x+3}$

| $x$ | $-3.1$ | $-3.01$ | $-3.001$ | $-2.999$ | $-2.99$ | $-2.9$ |
|---|---|---|---|---|---|---|
| $f(x)$ | | | | | | |

Soln:

$\underset{x\to -3}{Lt} \frac{\sqrt{1-x}-2}{x+3}$

| $x$ | $-3.1$ | $-3.01$ | $-3.001$ | $-2.999$ | $-2.99$ | $-2.9$ |
|---|---|---|---|---|---|---|
| $f(x)$ | $\frac{\sqrt{4.1}-2}{-0.1}$ $=-0.248$ | $\frac{\sqrt{4.01}-2}{-0.01}$ $=-0.249$ | $\frac{\sqrt{4.001}-2}{-0.001}$ $=-0.249$ | $\frac{\sqrt{3.999}-2}{0.001}$ $=-0.25$ | $\frac{\sqrt{3.99}-2}{0.01}$ $=-0.25$ | $\frac{\sqrt{3.9}-2}{0.1}$ $=-0.251$ |

$$\therefore \underset{x \to -3}{Lt} \frac{\sqrt{1-x} - 12}{x+3} = -0.25$$

(5) $\underset{x \to 0}{Lim} \frac{\sin x}{x}$

| $x$ | $-0.1$ | $-0.01$ | $-0.001$ | $0.001$ | $0.01$ | $0.1$ |
|---|---|---|---|---|---|---|
| $f(x)$ | | | | | | |

Soln

$\underset{x \to 0}{Lt} \frac{\sin x}{x}$

| $x$ | $-0.1$ | $-0.01$ | $-0.001$ | $0.001$ | $0.01$ | $0.1$ |
|---|---|---|---|---|---|---|
| $f(x)$ | $\frac{\sin(-0.1)}{-0.1} = 0.998$ | $\frac{\sin(-0.01)}{-0.01} = 0.999$ | $\frac{\sin(-0.001)}{-0.001} = 0.9999$ | $\frac{\sin(0.001)}{0.001} = 0.9999$ | $\frac{\sin(0.01)}{0.01} = 0.999$ | $\frac{\sin(0.1)}{0.1} = 0.998$ |

$$\therefore \underset{x \to 0}{Lt} \frac{\sin x}{x} = 1$$

(6) $\underset{x \to 0}{Lim} \frac{\cos x - 1}{x}$

| $x$ | $-0.1$ | $-0.01$ | $-0.001$ | $0.001$ | $0.01$ | $0.1$ |
|---|---|---|---|---|---|---|
| $f(x)$ | | | | | | |

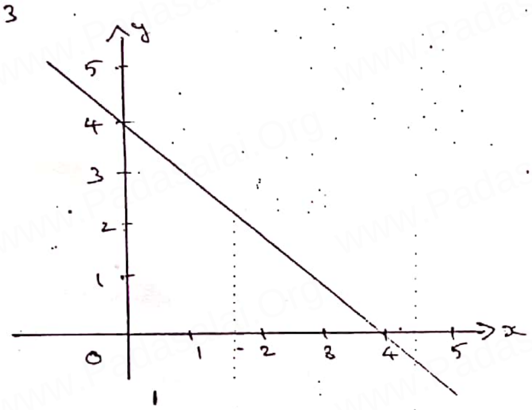
Soln $\underset{x \to 0}{Lt} \frac{\cos x - 1}{x}$

| $x$ | $-0.1$ | $-0.01$ | $-0.001$ | $0.001$ | $0.01$ | $0.1$ |
|---|---|---|---|---|---|---|
| $f(x)$ | $\frac{\cos(-0.1)-1}{-0.1} = 0.00001$ | $\frac{\cos(-0.01)-1}{-0.01} = 0.000000152$ | $\frac{\cos(-0.001)-1}{-0.001} = 0.000$ | $\frac{\cos(0.001)-1}{0.001} = 0.000$ | $\frac{\cos(0.01)-1}{0.01} = -0.000000152$ | $\frac{\cos(0.1)-1}{0.1} = -0.00001$ |

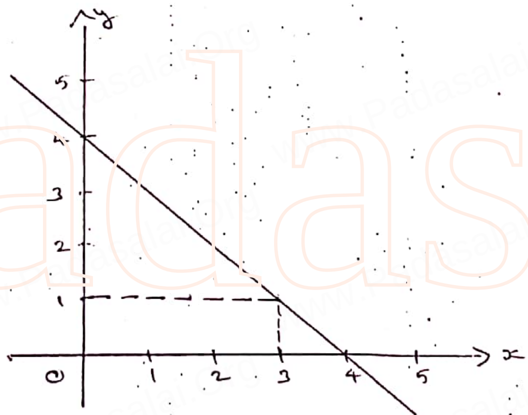
$$\therefore \lim_{x\to 0} \frac{\cos x - 1}{x} = 0$$

In problems 7-15, use the graph to find the limit (if it exists). If the limit does not exist, explain why?

(7) $\lim_{x\to 3} (4-x)$



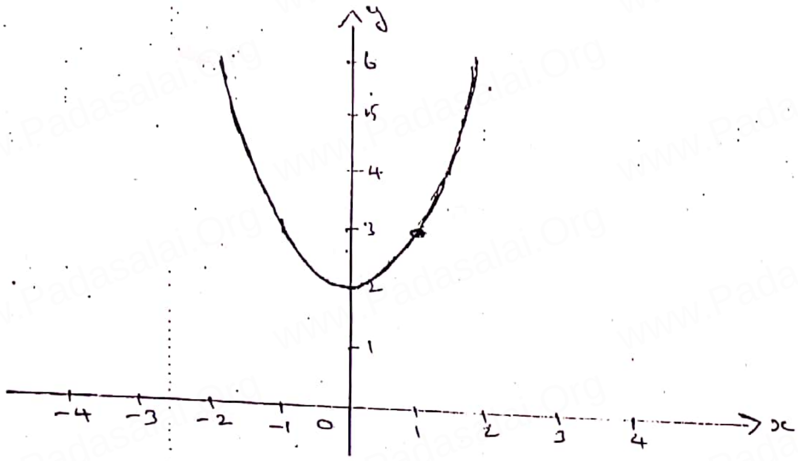

Soln



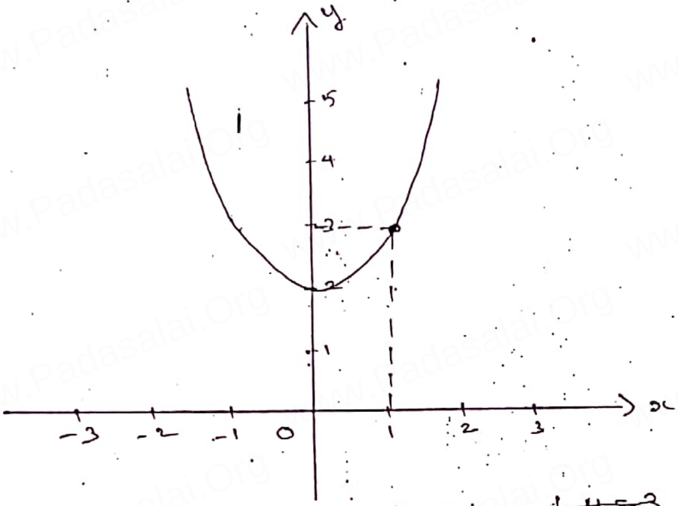

From the diagram, ~~at x=3, the value of y at~~ $\lim_{x\to 3^-} (4-x) = 1$ and $\lim_{x\to 3^+} (4-x) = 1$

$$\therefore \lim_{x\to 3} (4-x) = 1$$

(8) $\lim_{x\to 1} (x^2+2)$



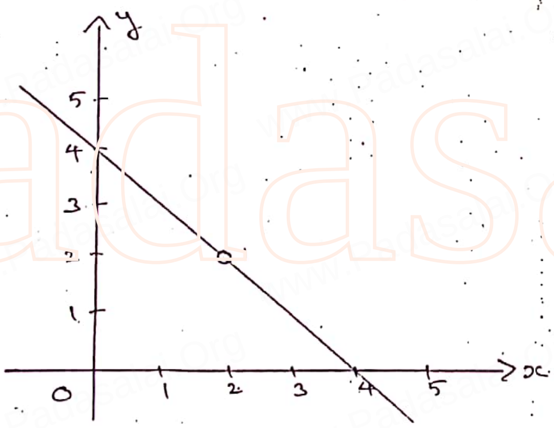
Sol^n



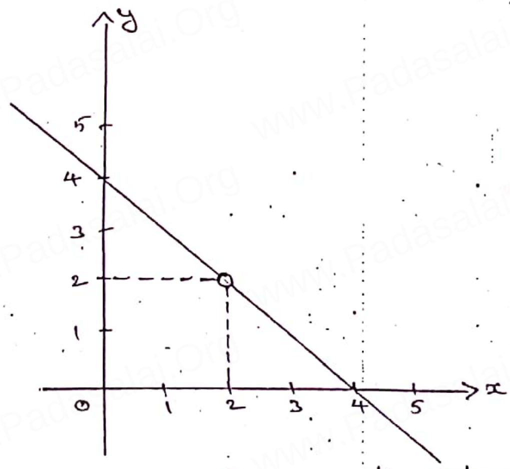

From the diagram, ~~at x = 1, the value of y = 3~~ $\longrightarrow \underset{x \to 1^-}{Lt} (x^2+2) = \underset{x \to 1^+}{Lt} (x^2+2) = 3$

$\therefore \underset{x \to 1}{Lt} (x^2+2) = 3$

(9) $\underset{x \to 2}{Lim} f(x)$ : where $f(x) = \begin{cases} 4-x & ; x \neq 2 \\ 0 & ; x = 2 \end{cases}$



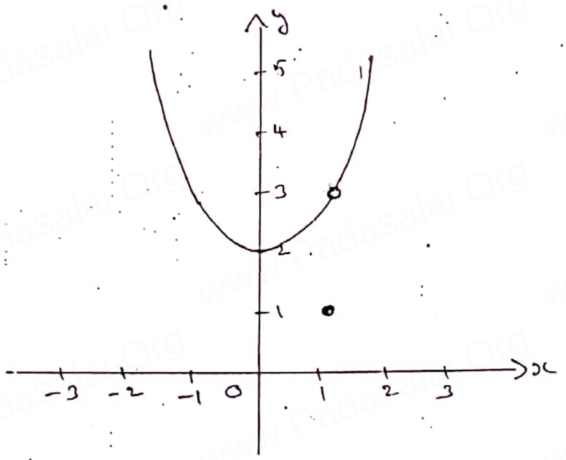

Sol^n



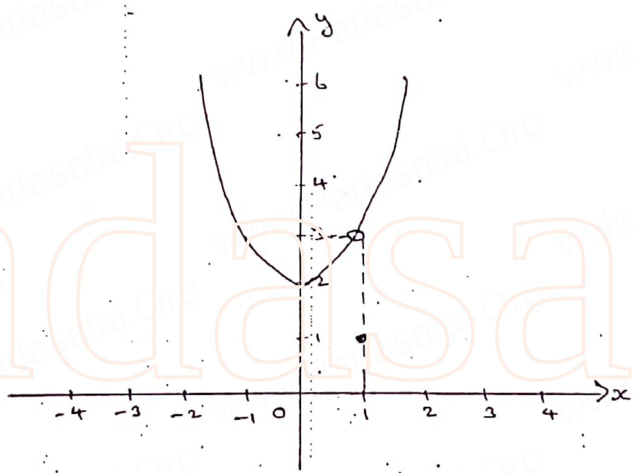

From the diagram, ~~at x = 2, the value of y = 2~~ $\longrightarrow \underset{x \to 2^-}{Lt} f(x) = 2 = \underset{x \to 2^+}{Lt} f(x)$

$\therefore \underset{x \to 2}{Lt} f(x) = 2$

(10) $\underset{x\to1}{Lt} f(x)$ ; where $f(x) = \begin{cases} x^2+2 & : x\neq1 \\ 1 & : x=1\end{cases}$



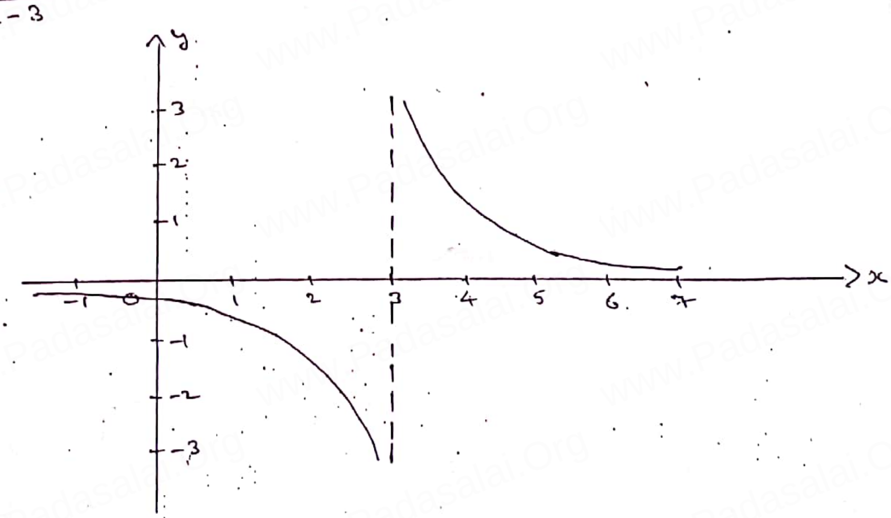

Soln:



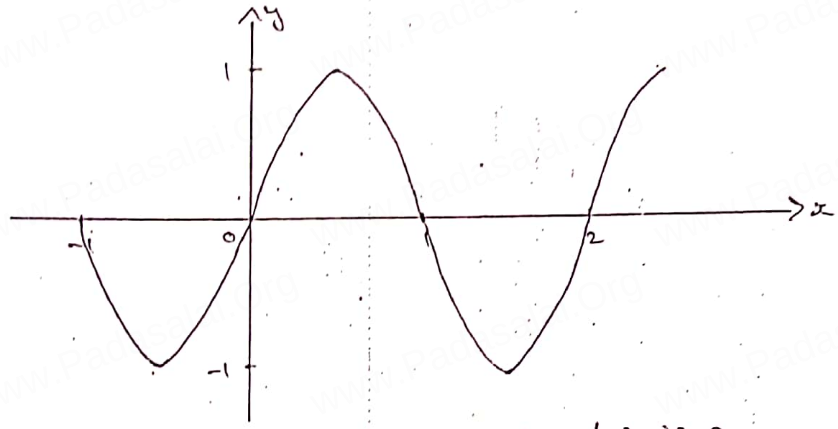

From the diagram, ~~at x = 1, the value of y = 3~~ → $\underset{x\to1^-}{Lt} f(x) = \underset{x\to1^+}{Lt} f(x) = 3$

$\therefore \underset{x\to1}{Lt} f(x) = 3$

(11) $\underset{x\to3}{Lt} \frac{1}{x-3}$



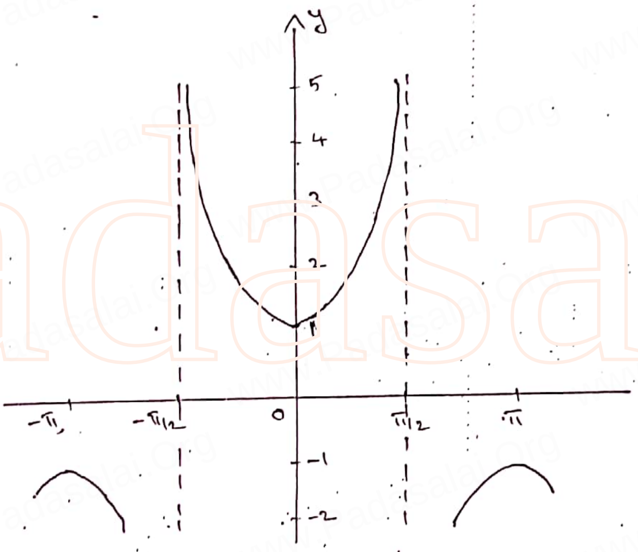
Soln



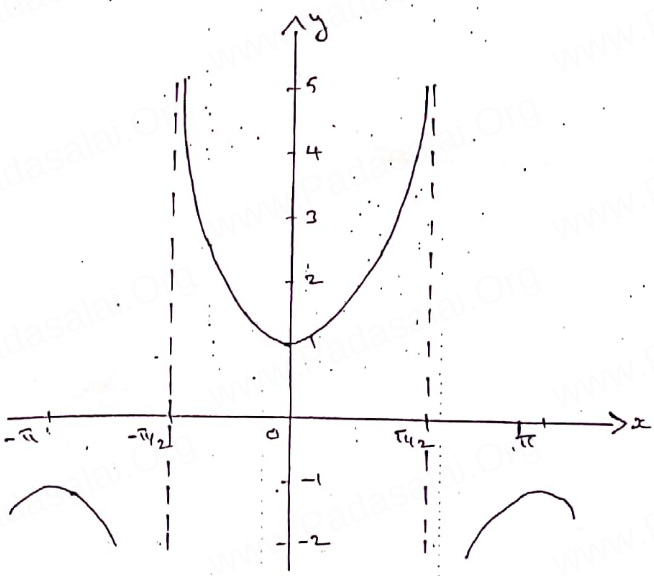

From the diagram, ~~at x = 1, the value of y is 0~~ $\Rightarrow \underset{x\to 1^-}{Lt} \sin \pi x = \underset{x \to 1^+}{Lt} \sin \pi x = 0$

$\therefore \underset{x \to 1}{Lt} \sin \pi x = 0.$

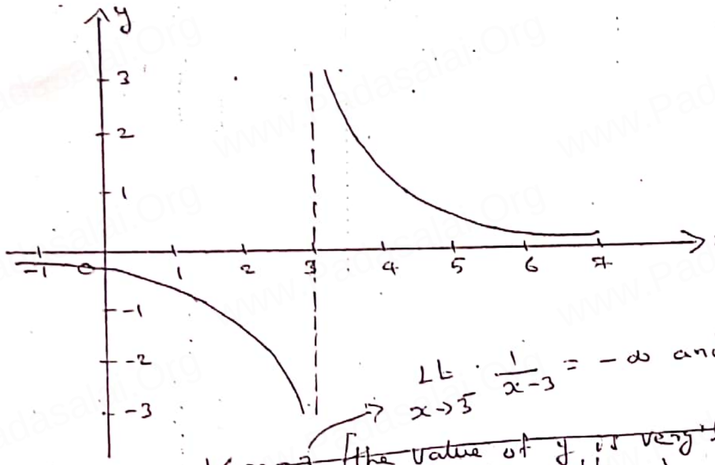
(14) $\underset{x \to 0}{Lim} \sec x.$




Soln



Soln.

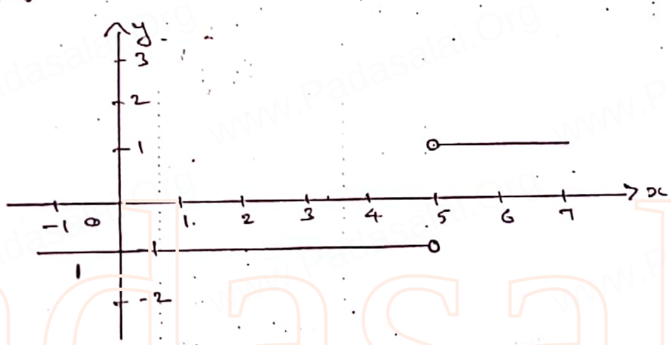


$\underset{x\to3^-}{Lt} \frac{1}{x-3} = -\infty$ and $\underset{x\to3^+}{Lt} \frac{1}{x-3} = \infty$

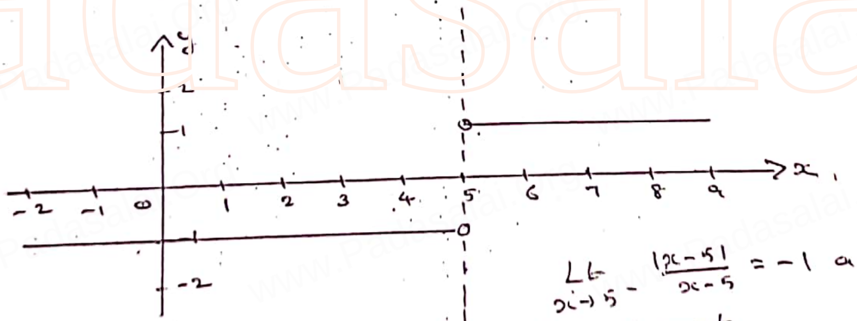
From the diagram, ~~at x=3,~~ [~~the value of y is very large (∞)~~] or $\underset{x\to3^-}{Lt} \frac{1}{x-3} \neq \underset{x\to3^+}{Lt} \frac{1}{x-3}$ ~~the curve does not exist~~

$\therefore \underset{x\to3}{Lt} \frac{1}{x-3}$ does not exist

(12) $\underset{x\to5}{Lim} \frac{|x-5|}{x-5}$



Soln.

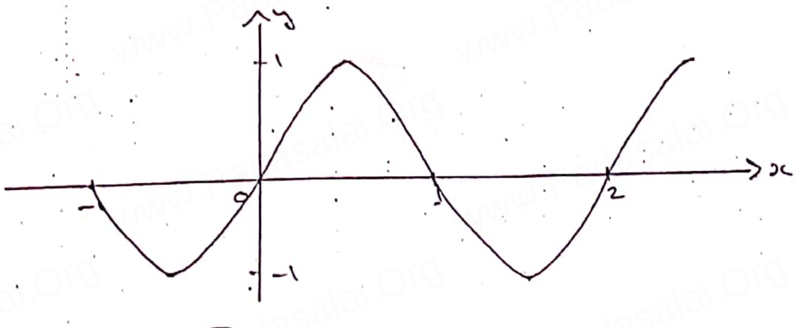


$\underset{x\to5^-}{Lt} \frac{|x-5|}{x-5} = -1$ and $\underset{x\to5^+}{Lt} \frac{|x-5|}{x-5} = 1$

From the diagram, ~~at x=5, the curve does not exists~~ $\underset{x\to5^-}{Lt} \frac{|x-5|}{x-5} \neq \underset{x\to5^+}{Lt} \frac{|x-5|}{x-5}$

$\therefore \underset{x\to5}{Lt} \frac{|x-5|}{x-5}$ does not exist

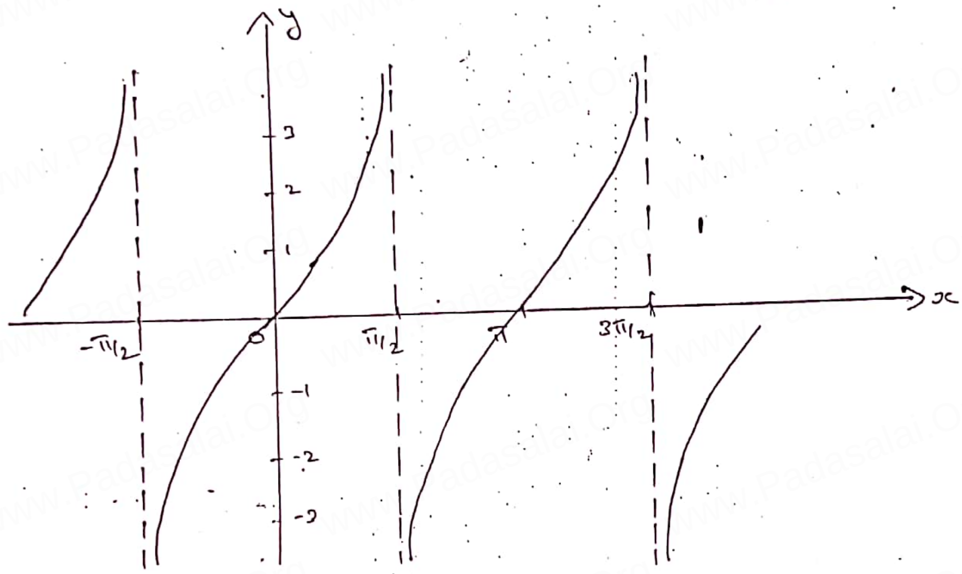
(13) $\underset{x\to1}{Lt} \sin\pi x$



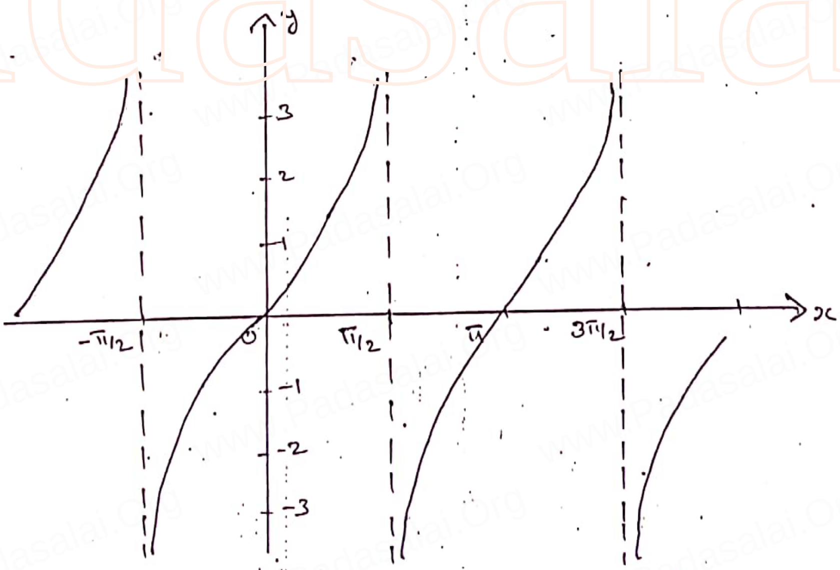
From the diagram, ~~at $x=0$, the value of $y=1$~~ $\Rightarrow \underset{x\to 0^-}{Lt} \sec x = \underset{x\to 0^+}{Lt} \sec x = 1$

$\therefore \underset{x\to 0}{Lt} \sec x = 1$

(15) $\underset{x\to \pi/2}{Lt} \tan x$




Soln




From the diagram, ~~at $x=\pi/2$, the curve does not exist~~ $\underset{x\to \pi/2^-}{Lt} \tan x = \infty$ and $\underset{x\to \pi/2^+}{Lt} \tan x = -\infty$

$\therefore \underset{x\to \pi/2}{Lt} \tan x$ does not exist $\quad \underset{x\to \pi/2^-}{Lt} \tan x \neq \underset{x\to \pi/2^+}{Lt} \tan x$

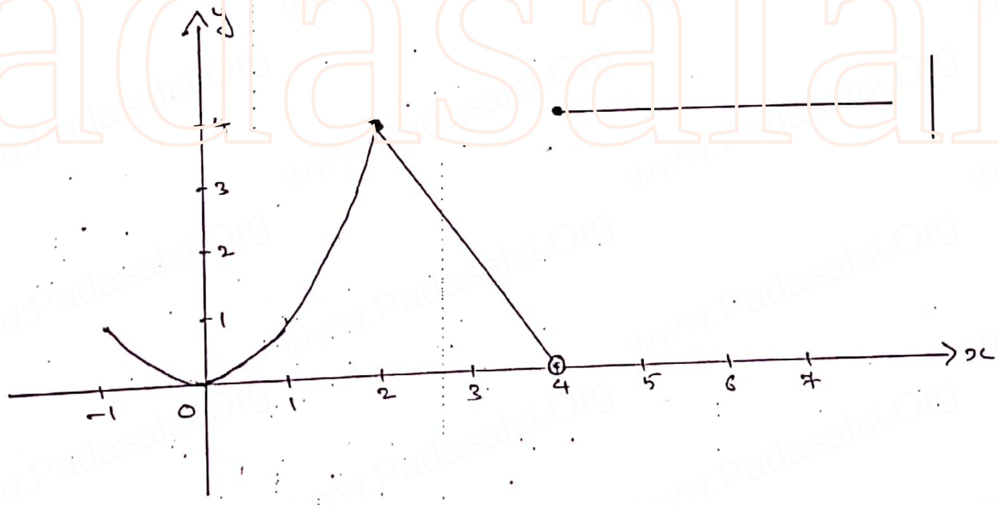
For 16 and 17 Sketch the graph of $f$, then identify the values of $x_0$ for which $\underset{x \to x_0}{Lt} f(x)$ exists.

(16) $f(x) = \begin{cases} x^2 & ; x \le 2 \\ 8-2x & ; 2 < x < 4 \\ 4 & ; x \ge 4 \end{cases}$

Soln.

$f(x) = \begin{cases} x^2 & ; x \le 2 \\ 8-2x & ; 2 < x < 4 \\ 4 & ; x \ge 4 \end{cases}$

| $x$ | -1 | 0 | 1 | 2 | 3 | 4 | 5 | 6 | 7 |
|---|---|---|---|---|---|---|---|---|---|
| $f(x)$ | $x^2$ | $x^2$ | $x^2$ | $x^2$ | $8-2x$ | 4 | 4 | 4 | 4 |
| Value of $f(x)$ | 1 | 0 | 1 | 4 | 2 | 4 | 4 | 4 | 4 |




From the diagram,

$\underset{x \to 4^-}{Lt} f(x) = 0$ and $\underset{x \to 4^+}{Lt} f(x) = 4$

$\therefore \underset{x \to 4^-}{Lt} f(x) \ne \underset{x \to 4^+}{Lt} f(x)$

$\therefore \underset{x \to 4}{Lt} f(x)$ does not exist

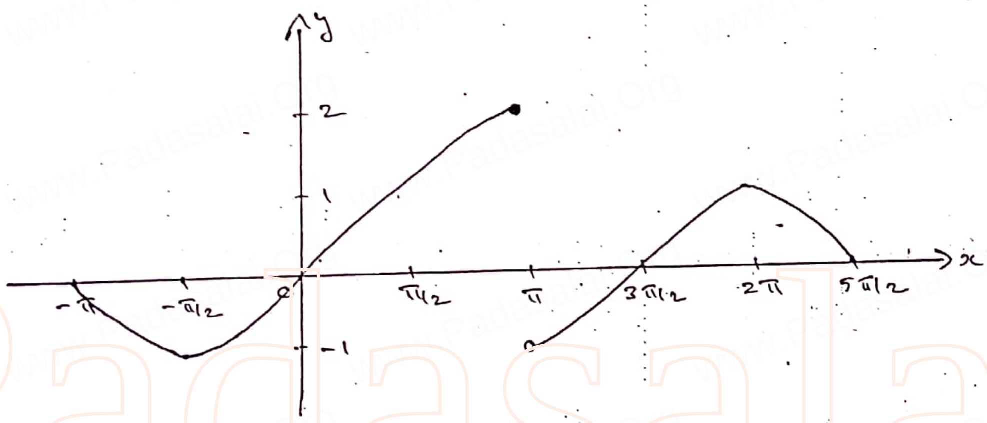
$\therefore$ The limit exist $\forall\, x \in R - \{4\}$

17) $f(x) = \begin{cases} \sin x & ; x < 0 \\ 1-\cos x & ; 0 \le x \le \pi \\ \cos x & ; x > \pi \end{cases}$

Soln/

$f(x) = \begin{cases} \sin x & ; x < 0 \\ 1-\cos x & ; 0 \le x \le \pi \\ \cos x & ; x > \pi \end{cases}$

| $x$ | $-\pi$ | $-\frac{\pi}{2}$ | 0 | $\frac{\pi}{2}$ | $\pi$ | $\frac{3\pi}{2}$ | $2\pi$ | $\frac{5\pi}{2}$ |
|---|---|---|---|---|---|---|---|---|
| $f(x)$ | $\sin x$ | $\sin x$ | $1-\cos x$ | $1-\cos x$ | $1-\cos x$ | $\cos x$ | $\cos x$ | $\cos x$ |
| The value of $f(x)$ | 0 | $-1$ | 0 | 1 | 2 | 0 | 1 | 0 |



From the diagram,

$\lim\limits_{x \to \pi^-} f(x) = 2$ and $\lim\limits_{x \to \pi^+} f(x) = -1$

$\therefore \lim\limits_{x \to \pi^-} f(x) \ne \lim\limits_{x \to \pi^+} f(x)$

$\therefore \lim\limits_{x \to \pi} f(x)$ does not exist

$\therefore$ The limit exist $\forall\, x \in R - \{\pi\}$

18) Sketch the graph of a function $f$ that satisfies the given values

(i)

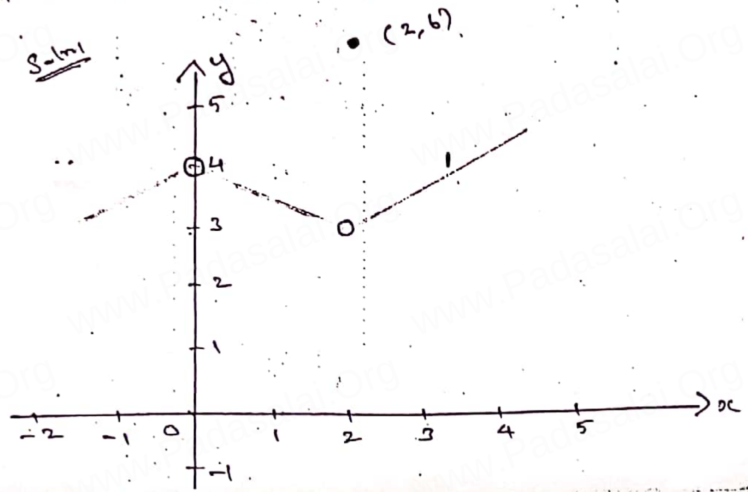
$f(0)$ is undefined

$\lim\limits_{x \to 0} f(x) = 4$

$f(2) = 6$

$\lim\limits_{x \to 2} f(x) = 3$

Soln/



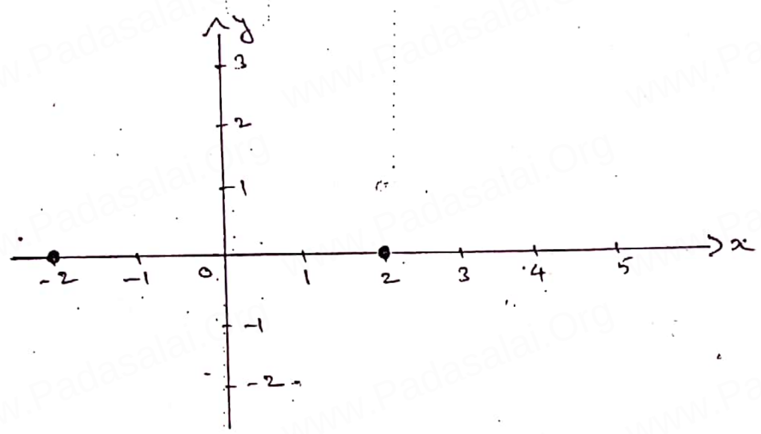
(ii) $f(-2)=0$

$f(2)=0$

$\lim_{x\to -2} f(x)=0$

$\lim_{x\to 2} f(x)$ does not exist

Soln




19) Write a brief description of the meaning of the notation

$\lim_{x\to 8} f(x)=25$

Soln

Given, $\lim_{x\to 8} f(x)=25$

$\Rightarrow \lim_{x\to 8^-} f(x) = \lim_{x\to 8^+} f(x) = 25$

20) If $f(2)=4$, Can you conclude anything about the limit of $f(x)$ as $x$ approaches 2?

Soln

Given $f(2)=4$

$\Rightarrow$ The value of $f(x)$ at $x=2$ is 4.

$\Rightarrow$ We cannot conclude ~~ab~~ anything about $\lim_{x\to 2} f(x)$

21) If the limit of $f(x)$ as $x$ approaches 2 is 4, can you conclude anything about $f(2)$? Explain reasoning

Soln

Given $\underset{x \to 2}{Lt} f(x) = 4$

∵ The limit of the function need not be equal to the value of the function,

∴ $f(2)$ need not be 4.

∴ we cannot conclude about $f(2)$

22) Evaluate : $\lim\limits_{x \to 3} \frac{x^2-9}{x-3}$ if it exists by finding $f(3^-)$ and $f(3^+)$.

Soln

$$\frac{x^2-9}{x-3} = \frac{(x-3)(x+3)}{x-3}$$

$$= x+3$$

$$f(3^-) = \underset{x \to 3^-}{Lt} \frac{x^2-9}{x-3}$$

$$= \underset{x \to 3^-}{Lt} (x+3)$$

$$= 3+3 = 6$$

$$f(3^+) = \underset{x \to 3^+}{Lt} \frac{x^2-9}{x-3}$$

$$= \underset{x \to 3^+}{Lt} (x+3)$$

$$= 3+3 = 6$$

∴ $f(3^-) = f(3^+)$

∴ $\underset{x \to 3}{Lt} \frac{x^2-9}{x-3} = 6$

23) verify the existance of $\lim\limits_{x \to 1} f(x)$, where $f(x) = \begin{cases} \frac{|x-1|}{x-1} & \text{for } x \neq 1 \\ 0 & \text{for } x = 1 \end{cases}$

Soln

Given, $f(x) = \begin{cases} \frac{|x-1|}{x-1} & \text{for } x \neq 1 \\ 0 & \text{for } x = 1 \end{cases}$

if $x > 1$
$|x-1| = +(x-1)$
if $x < 1$
$|x-1| = -(x-1)$.

$$\lim_{x\to 1^-} f(x) = \lim_{x\to 1^-} \frac{|x-1|}{x-1}$$

$$= \lim_{x\to 1^-} \frac{-(x-1)}{x-1}$$

$$= -1$$

$$\lim_{x\to 1^+} f(x) = \lim_{x\to 1^+} \frac{|x-1|}{x-1}$$

$$= \lim_{x\to 1^+} \frac{(x-1)}{x-1}$$

$$= 1$$

$$\therefore \lim_{x\to 1^-} f(x) \neq \lim_{x\to 1^+} f(x)$$

$\therefore \lim_{x\to 1} f(x)$ does not exist

Exe(9.2)

Evaluate the following limits:

1) $\lim_{x\to 2} \frac{x^4-16}{x-2} = \lim_{x\to 2} \frac{x^4-2^4}{x-2}$ $\left[\because \lim_{x\to a} \frac{x^n-a^n}{x-a} = na^{n-1}\right]$

$= 4(2)^{4-1}$

$= 4(8)$

$= 32$

2) $\lim_{x\to 1} \frac{x^m-1}{x^n-1} = \lim_{x\to 1} \frac{\frac{x^m-1}{x-1}}{\frac{x^n-1}{x-1}}$ $[m,n \in Z]$

$= \frac{\lim_{x\to 1} \frac{x^m-1}{x-1}}{\lim_{x\to 1} \frac{x^n-1}{x-1}}$

$= \frac{m(1)^{m-1}}{n(1)^{n-1}} = \frac{m}{n}$

3) $\lim_{\sqrt{x}\to 3} \frac{x^2-81}{\sqrt{x}-3} = \lim_{\sqrt{x}\to 3} \frac{(\sqrt{x})^4-3^4}{\sqrt{x}-3}$

$= 4(3)^{4-1}$

$= 4(27)$

$= 108$

4) $\lim_{h\to 0} \frac{\sqrt{x+h}-\sqrt{x}}{h} = \lim_{h\to 0} \left(\frac{\sqrt{x+h}-\sqrt{x}}{h} \times \frac{\sqrt{x+h}+\sqrt{x}}{\sqrt{x+h}+\sqrt{x}}\right)$ $[x>0]$

$= \lim_{h\to 0} \frac{(x+h)-x}{h(\sqrt{x+h}+\sqrt{x})} = \lim_{h\to 0} \frac{h}{h[\sqrt{x+h}+\sqrt{x}]}$

$= \lim_{h\to 0} \frac{1}{\sqrt{x+h}+\sqrt{x}}$

$= \lim_{h\to 0} \frac{1}{\sqrt{x+h}+\sqrt{x}} = \frac{1}{\sqrt{x}+\sqrt{x}}$

$= \frac{1}{2\sqrt{x}}$

5) $\lim_{x\to 5} \frac{\sqrt{x+4}-3}{x-5} = \lim_{x\to 5} \left(\frac{\sqrt{x+4}-3}{x-5} \times \frac{\sqrt{x+4}+3}{\sqrt{x+4}+3}\right)$

$= \lim_{x\to 5} \frac{(x+4)-9}{(x-5)[\sqrt{x+4}+3]}$

$$= \lim_{x \to 5} \frac{x-5}{(x-5)[\sqrt{x+4}+3]}$$

$$= \lim_{x \to 5} \frac{1}{\sqrt{x+4}+3}$$

$$= \frac{1}{\sqrt{5+4}+3}$$

$$= \frac{1}{6}$$

6) $\lim_{x \to 2} \frac{\frac{1}{x}-\frac{1}{2}}{x-2} = \lim_{x \to 2} \frac{\frac{2-x}{2x}}{x-2}$

$$= \lim_{x \to 2} \left[-\frac{(x-2)}{2x} \cdot \frac{1}{(x-2)}\right]$$

$$= \lim_{x \to 2} \frac{-1}{2x}$$

$$= -\frac{1}{4}$$

7) $\lim_{x \to 1} \frac{\sqrt{x}-x^2}{1-\sqrt{x}} = \lim_{x \to 1} \frac{x^2-\sqrt{x}}{\sqrt{x}-1}$

$$= \lim_{x \to 1} \frac{\sqrt{x}[x\sqrt{x}-1]}{\sqrt{x}-1}$$

$$= \left(\lim_{x \to 1} \sqrt{x}\right)\left(\lim_{x \to 1} \frac{(\sqrt{x})^3-1}{\sqrt{x}-1}\right)$$

$$= \left(\lim_{x \to 1} \sqrt{x}\right)\left(\lim_{\sqrt{x} \to 1} \frac{(\sqrt{x})^3-1}{\sqrt{x}-1}\right)$$

$$= \sqrt{1}\left[3(1)^{3-1}\right]$$

$$= 3$$

8) $\lim_{x \to 0} \frac{\sqrt{x^2+1}-1}{\sqrt{x^2+16}-4} = \lim_{x \to 0} \left[\frac{\sqrt{x^2+1}-1}{\sqrt{x^2+16}-4} \cdot \frac{\sqrt{x^2+16}+4}{\sqrt{x^2+16}+4} \times \frac{\sqrt{x^2+1}+1}{\sqrt{x^2+1}+1}\right]$

$$= \lim_{x \to 0} \left(\frac{(x^2+1)-1}{(x^2+16)-16}\right)\left(\frac{\sqrt{x^2+16}+4}{\sqrt{x^2+1}+1}\right)$$

$$= \lim_{x \to 0} \frac{\sqrt{x^2+16}+4}{\sqrt{x^2+1}+1}$$

$$= \frac{\sqrt{16}+4}{\sqrt{1}+1}$$

$$= \frac{8}{2}$$

$$= 4$$

9) $\underset{x\to 0}{Lt} \dfrac{\sqrt{1+x}-1}{x} = \underset{x\to 0}{Lt}\left[\dfrac{\sqrt{1+x}-1}{x} \times \dfrac{\sqrt{1+x}+1}{\sqrt{1+x}+1}\right]$

$= \underset{x\to 0}{Lt} \dfrac{(1+x)-1}{x\{\sqrt{1+x}+1\}}$

$= \underset{x\to 0}{Lt} \dfrac{1}{\sqrt{1+x}+1}$

$= \dfrac{1}{\sqrt{1}+1}$

$= 1/2$

10) $\underset{x\to 1}{Lt} \dfrac{\sqrt[3]{7+x^3}-\sqrt{3+x^2}}{x-1} = \underset{x\to 1}{Lt}\left[\dfrac{(7+x^3)^{1/3}}{x-1} - \dfrac{(3+x^2)^{1/2}}{x-1}\right]$

$= \underset{x\to 1}{Lt}\left[\dfrac{(7+x^3)^{1/3}-2}{x-1} - \dfrac{(3+x^2)^{1/2}-2}{x-1}\right]$

$= \underset{x\to 1}{Lt}\left[\dfrac{(7+x^3)^{1/3}-8^{1/3}}{7+x^3-8} \times \dfrac{x^3-1}{x-1} - \dfrac{(3+x^2)^{1/2}-4^{1/2}}{3+x^2-4} \times \dfrac{x^2-1}{x-1}\right]$

$= \underset{x\to 1}{Lt} \dfrac{(7+x^3)^{1/3}-8^{1/3}}{(7+x^3)-8} \times \dfrac{(x-1)(x^2+x+1)}{x-1} - \underset{x\to 1}{Lt} \dfrac{(3+x^2)^{1/2}-4^{1/2}}{(3+x^2)-4} \times \dfrac{(x-1)(x+1)}{x-1}$

$= \underset{7+x^3\to 8}{Lt}\left[\dfrac{(7+x^3)^{1/3}-8^{1/3}}{(7+x^3)-8}\right] \underset{x\to 1}{Lt}(x^2+x+1) - \underset{3+x^2\to 4}{Lt}\left[\dfrac{(3+x^2)^{1/2}-4^{1/2}}{(3+x^2)-4}\right] \underset{x\to 1}{Lt}(x+1)$

$= \dfrac{1}{3} 8^{1/3-1}(1+1+1) - \dfrac{1}{2} 4^{1/2-1}(1+1)$

$= 8^{-2/3} - 4^{-1/2}$

$= \dfrac{1}{4} - \dfrac{1}{2}$

$= -1/4$

$x \to 1 \Rightarrow x^3 \to 1$, $7+x^3 \to 8$

$x \to 1 \Rightarrow x^2 \to 1$, $3+x^2 \to 4$

$\underset{x\to 1}{Lt}(x^2+x+1) = 3$

$\underset{x\to 1}{Lt}(x+1) = 2$

$\underset{x\to a}{Lt} \dfrac{x^n-a^n}{x-a} = na^{n-1}$

11) $\underset{x\to 2}{Lt} \dfrac{2-\sqrt{x+2}}{\sqrt[3]{2}-\sqrt[3]{4-x}} = \underset{x\to 2}{Lt} \dfrac{(x+2)^{1/2}-4^{1/2}}{(4-x)^{1/3}-2^{1/3}}$

$= \underset{x\to 2}{Lt} \dfrac{(x+2)^{1/2}-4^{1/2}}{x+2-4} \times \dfrac{-(4-x-2)}{(4-x)^{1/3}-2^{1/3}}$

$= \underset{x\to 2}{Lt} \dfrac{(x+2)^{1/2}-4^{1/2}}{(x+2)-4} \times \underset{x\to 2}{Lt} \dfrac{-[(4-x)-2]}{(4-x)^{1/3}-2^{1/3}}$

$= \underset{x\to 2}{Lt} \dfrac{(x+2)^{1/2}-4^{1/2}}{(x+2)-4} \times \dfrac{-1}{\underset{x\to 2}{Lt} \dfrac{(4-x)^{1/3}-2^{1/3}}{(4-x)-2}}$

$= \underset{x+2\to 4}{Lt} \dfrac{(x+2)^{1/2}-4^{1/2}}{(x+2)-4} \times \dfrac{-1}{\underset{4-x\to 2}{Lt} \dfrac{(4-x)^{1/3}-2^{1/3}}{(4-x)-2}}$

$= \dfrac{1}{2} 4^{1/2-1} \times \dfrac{-1}{\frac{1}{3} 2^{1/3-1}} = \dfrac{1}{4} \times (-3 \times 2^{2/3})$

$= \dfrac{-3(\sqrt[3]{4})}{4}$

12) $\lim_{x\to 0} \frac{\sqrt{1+x^2}-1}{x} = \lim_{x\to 0}\left[\frac{\sqrt{1+x^2}-1}{x} \times \frac{\sqrt{1+x^2}+1}{\sqrt{1+x^2}+1}\right]$

$= \lim_{x\to 0} \frac{(1+x^2)-1}{x[\sqrt{1+x^2}+1]}$

$= \lim_{x\to 0} \frac{x}{\sqrt{1+x^2}+1}$

$= \frac{0}{\sqrt{1}+1} = 0$

13) $\lim_{x\to 0} \frac{\sqrt{1-x}-1}{x^2} = \lim_{x\to 0}\left[\frac{\sqrt{1-x}-1}{x^2} \times \frac{\sqrt{1-x}+1}{\sqrt{1-x}+1}\right]$

$= \lim_{x\to 0} \frac{(1-x)-1}{x^2[\sqrt{1-x}+1]}$

$= \lim_{x\to 0} \frac{-x}{x^2[\sqrt{1-x}+1]}$

$= \lim_{x\to 0} \frac{-1}{x[\sqrt{1-x}+1]}$

$= \frac{-1}{0}$

$= -\infty$ [$\because$ Limit does not exist]

14) $\lim_{x\to 5} \frac{\sqrt{x-1}-2}{x-5} = \lim_{x\to 5}\left[\frac{\sqrt{x-1}-2}{x-5} \times \frac{\sqrt{x-1}+2}{\sqrt{x-1}+2}\right]$

$= \lim_{x\to 5} \frac{(x-1)-4}{(x-5)[\sqrt{x-1}+2]}$

$= \lim_{x\to 5} \frac{1}{\sqrt{x-1}+2}$

$= \frac{1}{\sqrt{5-1}+2}$

$= 1/4$

15) $\lim_{x\to a} \frac{\sqrt{x-b}-\sqrt{a-b}}{x^2-a^2} = \lim_{x\to a}\left[\frac{\sqrt{x-b}-\sqrt{a-b}}{x^2-a^2} \times \frac{\sqrt{x-b}+\sqrt{a-b}}{\sqrt{x-b}+\sqrt{a-b}}\right]$ [$\because a > b$]

$= \lim_{x\to a} \frac{(x-b)-(a-b)}{(x-a)(x+a)[\sqrt{x-b}+\sqrt{a-b}]}$

$= \lim_{x\to a} \frac{x-a}{(x-a)(x+a)[\sqrt{x-b}+\sqrt{a-b}]}$

$= \lim_{x\to a} \frac{1}{(x+a)[\sqrt{x-b}+\sqrt{a-b}]}$

$= \frac{1}{(a+a)[\sqrt{a-b}+\sqrt{a-b}]}$

$$= \frac{1}{2a[2\sqrt{a-b}]}$$

$$= \frac{1}{4a\sqrt{a-b}}$$

Exer (9.3)

1) (a) Find the left and right limits of $f(x) = \frac{x^2-4}{(x^2+4x+4)(x+3)}$ at $x = -2$

Soln

$$f(x) = \frac{x^2-4}{(x^2+4x+4)(x+3)}$$

$$= \frac{(x-2)(x+2)}{(x+2)^2(x+3)}$$

$$= \frac{x-2}{(x+2)(x+3)}$$

$$\underset{x\to -2^-}{Lt} f(x) = \underset{x\to -2^-}{Lt} \frac{x-2}{(x+2)(x+3)}$$

$$= \frac{-4}{0(-ve)}$$

$$= +\infty$$

$$\underset{x\to -2^+}{Lt} f(x) = \underset{x\to -2^+}{Lt} \frac{x-2}{(x+2)(x+3)}$$

$$= \frac{-4}{0(+ve)}$$

$$= -\infty$$

$\therefore f(-2^-) \to \infty$ as $x \to -2^-$

$f(-2^+) \to -\infty$ as $x \to -2^+$

(b) $f(x) = \tan x$ at $x = \pi/2$

Soln

$f(x) = \tan x$

$$\underset{x\to \pi/2^-}{Lt} f(x) = \underset{x\to \pi/2^-}{Lt} \tan x$$

$$= \infty$$

$$\underset{x \to \pi/2^+}{Lt} f(x) = \underset{x \to \pi/2^+}{Lt} \tan x$$

$$= -\infty$$

[∵ In $2^{nd}$ quad, $\tan x = -ve$]

$\therefore f(\pi/2^-) \to \infty$ as $x \to \pi/2^-$

$f(\pi/2^+) \to -\infty$ as $x \to \pi/2^+$

(2) Evaluate the following limits [2 – 7]

$$\underset{x \to 3}{Lt} \frac{x^2 - 9}{x^2(x^2 - 6x + 9)}$$

Soln:

$$\text{Let } f(x) = \frac{x^2 - 9}{x^2(x^2 - 6x + 9)}$$

$$= \frac{(x-3)(x+3)}{x^2(x-3)^2}$$

$$= \frac{x+3}{x^2(x-3)}$$

$$= \frac{x(1 + 3/x)}{x^3(1 - 3/x)}$$

$$\underset{x \to 3^-}{Lt} f(x) = \underset{x \to 3}{Lt} \frac{x+3}{x^2(x-3)}$$

$$= \frac{6}{0(-ve)}$$

$$= -\infty$$

$$= \infty$$

$$\underset{x \to 3}{Lt} \frac{x+3}{x^2(x-3)}$$

$\therefore$ limit does not exist.

$$\underset{x \to 3^+}{Lt} f(x) = \underset{x \to 3^+}{Lt} \frac{x+3}{x^2(x-3)}$$

$$= \frac{6}{0(+ve)}$$

$$= \infty$$

$$\therefore \underset{x \to 3^-}{Lt} f(x) \neq \underset{x \to 3^+}{Lt} f(x)$$

$\therefore \underset{x \to 3}{Lt} f(x)$ does not exist

3) $$\underset{x \to \infty}{Lt} \left[\frac{3}{x-2} - \frac{2x+11}{x^2+x-6}\right]$$

Soln:

$$\text{Let } f(x) = \frac{3}{x-2} - \frac{2x+11}{x^2+x-6}$$

$$= \frac{3}{x-2} - \frac{2x+11}{(x+3)(x-2)}$$

$$= \frac{3(x+3)-(2x+11)}{(x+3)(x-2)}$$

$$= \frac{x-2}{(x+3)(x-2)}$$

$$= \frac{1}{x+3}$$

$$\underset{x\to\infty^-}{Lt} f(x) = \underset{x\to\infty^-}{Lt} \frac{1}{x+3}$$

$$= \frac{1}{\infty} = 0$$

$$\underset{x\to\infty^+}{Lt} f(x) = \underset{x\to\infty^+}{Lt} \frac{1}{x+3}$$

$$= \frac{1}{\infty} = 0$$

$$\therefore \underset{x\to\infty^-}{Lt} f(x) = \underset{x\to\infty^+}{Lt} f(x)$$

$$\therefore \underset{x\to\infty}{Lt} f(x) = 0$$

[OR]

$$\underset{x\to\infty}{Lt} f(x) = \underset{x\to\infty}{Lt} \frac{1}{x+3}$$

$$= \frac{1}{\infty}$$

$$= 0$$

4) $\underset{x\to\infty}{Lt} \dfrac{x^3+x}{x^4-3x^2+1}$

Soln

$$\text{Let } f(x) = \frac{x^3+x}{x^4-3x^2+1}$$

$$= \frac{x^3\left(1+\frac{1}{x^2}\right)}{x^4\left(1-\frac{3}{x^2}+\frac{1}{x^4}\right)}$$

$$= \frac{\left(1+\frac{1}{x^2}\right)}{x\left(1-\frac{3}{x^2}+\frac{1}{x^4}\right)}$$

$$\underset{x\to\infty}{Lt} f(x) = \underset{x\to\infty}{Lt} \frac{1+1/x^2}{x\left(1-\frac{3}{x^2}+\frac{1}{x^4}\right)}$$

$$= \frac{1}{\infty(1)}$$

$$= 0$$

5) $\underset{x\to\infty}{Lt} \dfrac{x^4-5x}{x^2-3x+1}$

Soln

$$\text{Let } f(x) = \frac{x^4-5x}{x^2-3x+1}$$

$$= \frac{x^4\left(1-\frac{5}{x^3}\right)}{x^2\left(1-\frac{3}{x}+\frac{1}{x^2}\right)}$$

$$= \frac{x^2(1-\frac{5}{x^3})}{1-\frac{3}{x}+\frac{1}{x^2}}$$

$$\underset{x\to\infty}{Lt} f(x) = \underset{x\to\infty}{Lt} \frac{x^2(1-5/x^3)}{1-\frac{3}{x}+\frac{1}{x^2}}$$

$$= \frac{\infty(1)}{1}$$

$$= \infty$$

6) $\underset{x\to\infty}{Lt} \frac{1+x-3x^3}{1+x^2+3x^3}$

Soln:

Let $f(x) = \frac{1+x-3x^3}{1+x^2+3x^3}$

$$= \frac{x^3\left[\frac{1}{x^3}+\frac{1}{x^2}-3\right]}{x^3\left[\frac{1}{x^3}+\frac{1}{x}+3\right]}$$

$$= \frac{\frac{1}{x^3}+\frac{1}{x^2}-3}{\frac{1}{x^3}+\frac{1}{x}+3}$$

$$\underset{x\to\infty}{Lt} f(x) = \underset{x\to\infty}{Lt} \frac{\frac{1}{x^3}+\frac{1}{x^2}-3}{\frac{1}{x^3}+\frac{1}{x}+3}$$

$$= \frac{-3}{3}$$

$$= -1$$

7) $\underset{x\to\infty}{Lt} \left(\frac{x^3}{2x^2-1} - \frac{x^2}{2x+1}\right)$

Soln:

Let $f(x) = \frac{x^3}{2x^2-1} - \frac{x^2}{2x+1}$

$$= \frac{x^3(2x+1) - x^2(2x^2-1)}{(2x^2-1)(2x+1)}$$

$$= \frac{2x^4+x^3-2x^4+x^2}{(2x^2-1)(2x+1)}$$

$$= \frac{x^3+x^2}{(2x^2-1)(2x+1)}$$

$$= \frac{x^3(1+\frac{1}{x})}{x^3(2-\frac{1}{x^2})(2+\frac{1}{x})}$$

$$= \frac{1+1/x}{(2-1/x^2)(2+1/x)}$$

$$\underset{x\to\infty}{Lt} f(x) = \underset{x\to\infty}{Lt} \frac{1+1/x}{(2-1/x^2)(2+1/x)}$$

$$= \frac{1}{2\times 2}$$

$$= 1/4$$

8) Show that

(i) $\underset{n\to\infty}{Lt} \dfrac{1+2+3+\dots+n}{3n^2+7n+2} = \dfrac{1}{6}$

Soln:

Let $f(n) = \dfrac{1+2+3+\dots+n}{3n^2+7n+2}$  $\left[\because 1+2+\dots+n = \dfrac{n(n+1)}{2}\right.$

$$= \frac{n(n+1)}{2[3n^2+7n+2]}$$

$$= \frac{n^2(1+\frac{1}{n})}{2n^2(3+\frac{7}{n}+\frac{2}{n^2})}$$

$$= \frac{(1+1/n)}{2(3+\frac{7}{n}+\frac{2}{n^2})}$$

$$\underset{n\to\infty}{Lt} f(n) = \underset{n\to\infty}{Lt} \frac{1+1/n}{2(3+\frac{7}{n}+\frac{2}{n^2})}$$

$$= \frac{1}{2(3)}$$

$$= 1/6$$

$$\therefore \underset{n\to\infty}{Lt} \frac{1+2+3+\dots+n}{3n^2+7n+2} = \frac{1}{6}$$

Hence proved

(ii) $\underset{n\to\infty}{Lt} \dfrac{1^2+2^2+\dots+(3n)^2}{(1+2+\dots+5n)(2n+3)} = \dfrac{9}{25}$

Soln:

Let $f(n) = \dfrac{1^2+2^2+\dots+(3n)^2}{(1+2+\dots+5n)(2n+3)}$  $\left[\begin{array}{l}\because 1^2+2^2+\dots+n^2 = \dfrac{n(n+1)(2n+1)}{6} \\ 1+2+\dots+n = \dfrac{n(n+1)}{2}\end{array}\right.$

$$= \frac{3n(3n+1)(6n+1)}{6\left[\frac{5n(5n+1)}{2}\right](2n+3)}$$

$$= \frac{n(3n+1)(6n+1)}{5n(5n+1)(2n+3)}$$

$$= \frac{n^3 (3+1/n)(6+1/n)}{5n^3 (5+1/n)(2+3/n)}$$

$$= \frac{(3+1/n)(6+1/n)}{5(5+1/n)(2+3/n)}$$

$$\underset{n\to\infty}{Lt} f(n) = \underset{n\to\infty}{Lt} \frac{(3+1/n)(6+1/n)}{5(5+1/n)(2+3/n)}$$

$$= \frac{(3)(6)}{5(5)(2)}$$

$$= \frac{9}{25}$$

$$\therefore \underset{n\to\infty}{Lt} \frac{1^2+2^2+\dots+(3n)^2}{(1+2+\dots+5n)(2n+3)} = \frac{9}{25}$$

Hence proved

(iii) $\underset{n\to\infty}{Lt}\left[\frac{1}{1.2}+\frac{1}{2.3}+\frac{1}{3.4}+\dots+\frac{1}{n(n+1)}\right] = 1$

Soln:

Let $f(n) = \frac{1}{1.2}+\frac{1}{2.3}+\frac{1}{3.4}+\dots+\frac{1}{n(n+1)}$

$$= \Sigma \frac{1}{n(n+1)}$$

$$= \Sigma \frac{n+1-n}{n(n+1)}$$

$$= \Sigma\left[\frac{1}{n}-\frac{1}{n+1}\right]$$

$$= \Sigma\frac{1}{n} - \Sigma\frac{1}{n+1}$$

$$= \left[\frac{1}{1}+\frac{1}{2}+\frac{1}{3}+\dots+\frac{1}{n}\right] - \left[\frac{1}{2}+\frac{1}{3}+\dots+\frac{1}{n+1}\right]$$

$$= 1 - \frac{1}{n+1}$$

$$= \frac{n+1-1}{n+1}$$

$$= \frac{n}{n+1}$$

$$= \frac{n}{n(1+1/n)}$$

$$= \frac{1}{1+1/n}$$

$$\underset{n\to\infty}{Lt} f(n) = \underset{n\to\infty}{Lt} \frac{1}{1+1/n}$$

$$= \frac{1}{1+0}$$

$$= 1$$

$$\therefore \lim_{n\to\infty}\left[\frac{1}{1.2}+\frac{1}{2.3}+\frac{1}{3.4}+\dots+\frac{1}{n(n+1)}\right]=1$$

Hence proved

9) An important problem in fishery Science is to estimate the number of fish presently Spawing in streams and use this information to predict the number of nature fish or "recruits" that will return to the rivers during the reproductive period; If $S$ is the number of Spawners and $R$ the number of recruits, "Beverton–Holt Spawner recruit function" is $R(S)=\frac{S}{\alpha S+\beta}$ where $\alpha$ and $\beta$ are positive constants. show that this function predicts approximately constant recruitment when the number of spawners is sufficiently large.

Soln:

Given, $R(S)=\frac{S}{\alpha S+\beta}$

$$\lim_{S\to\infty} R(S) = \lim_{S\to\infty}\frac{S}{\alpha S+\beta}$$

$$= \lim_{S\to\infty}\frac{S}{S(\alpha+\beta/S)}$$

$$= \lim_{S\to\infty}\frac{1}{\alpha+\beta/S}$$

$$= \frac{1}{\alpha+0}$$

$$= \frac{1}{\alpha}$$

10) A tank contains 5000 liters of pure water. Brine (very Salty water) that contains 30 grams of salt per litre of water is pumped into the tank at a rate of 25 litres per minute. The concentration of salt water after $t$ minutes (in grams per litre) is $c(t)=\frac{30t}{200+t}$. What happens to the concentration as $t\to\infty$?

10) Soln: Given, $c(t)=\frac{30t}{200+t}$

$$\lim_{t\to\infty} c(t) = \lim_{t\to\infty}\frac{30t}{t\left[\frac{200}{t}+1\right]}$$

$$= \lim_{t\to\infty}\frac{30}{\frac{200}{t}+1}$$

$$= \lim_{t\to\infty}\frac{30}{1+\frac{200}{t}}$$

$$= \frac{30}{1+0}$$

$$= 30$$

Exe (9.4)

Evaluate the following Limits

(1) $\underset{x \to \infty}{Lt} \left(1+\frac{1}{x}\right)^{7x}$

Soln

$$\underset{x \to \infty}{Lt} \left(1+\frac{1}{x}\right)^{7x} = \underset{x \to \infty}{Lt} \left[\left(1+\frac{1}{x}\right)^{x}\right]^{7}$$

$$= \left[\underset{x \to \infty}{Lt} \left(1+\frac{1}{x}\right)^{x}\right]^{7} \qquad \left[\because \underset{x \to \infty}{Lt} \left(1+\frac{1}{x}\right)^{x} = e\right]$$

$$= [e]^{7}$$

$$= e^{7}$$

(2) $\underset{x \to 0}{Lt} (1+x)^{1/3x}$

Soln

$$\underset{x \to 0}{Lt} (1+x)^{1/3x} = \underset{x \to 0}{Lt} \left[(1+x)^{1/x}\right]^{1/3}$$

$$= \left[\underset{x \to 0}{Lt} (1+x)^{1/x}\right]^{1/3} \qquad \left[\because \underset{x \to 0}{Lt} (1+x)^{1/x} = e\right]$$

$$= [e]^{1/3}$$

$$= e^{1/3}$$

(3) $\underset{x \to \infty}{Lt} \left(1+\frac{k}{x}\right)^{m/x}$

Soln

$$\underset{x \to \infty}{Lt} \left(1+\frac{k}{x}\right)^{m/x} = \underset{x \to \infty}{Lt} \left[\left(1+\frac{k}{x}\right)^{1/x}\right]^{m} \qquad \left[\begin{array}{l} \text{Put } t = 1/x \\ \text{As } x \to \infty \\ \quad t \to 0 \end{array}\right.$$

$$= \underset{t \to 0}{Lt} (1+kt)^{mt}$$

$$= (1+0)^{0}$$

$$= 1$$

(4) $\underset{x \to \infty}{Lt} \left(\frac{2x^2+3}{2x^2+5}\right)^{8x^2+3}$

Soln

$$\underset{x \to \infty}{Lt} \left(\frac{2x^2+3}{2x^2+5}\right)^{8x^2+3} = \underset{x \to \infty}{Lt} \left(\frac{4(2x^2+3)}{4(2x^2+5)}\right)^{8x^2+3}$$

$$= \underset{x \to \infty}{Lt} \left(\frac{8x^2+12}{8x^2+20}\right)^{8x^2+3}$$

$$= \underset{x\to\infty}{Lt} \left(\frac{8x^2+3+9}{8x^2+3+17}\right)^{8x^2+3}$$

[ Put $t = 8x^2+3$
As $x \to \infty$
$t \to \infty$ ]

$$= \underset{t\to\infty}{Lt} \left(\frac{t+9}{t+17}\right)^{t}$$

$$= \underset{t\to\infty}{Lt} \frac{t^t(1+9/t)^t}{t^t(1+17/t)^t}$$

$$= \frac{\underset{t\to\infty}{Lt}(1+9/t)^t}{\underset{t\to\infty}{Lt}(1+17/t)^t}$$

$\left[\because \underset{x\to\infty}{Lt}\left(1+\frac{\lambda}{x}\right)^x = e^{\lambda}\right]$

$$= \frac{e^9}{e^{17}}$$

$$= \frac{1}{e^8} \quad (\text{or}) \quad e^{-8}$$

5) $\underset{x\to\infty}{Lt} \left(1+\frac{3}{x}\right)^{x+2}$

Soln

$$\underset{x\to\infty}{Lt} \left(1+\frac{3}{x}\right)^{x+2} = \underset{x\to\infty}{Lt}\left(1+\frac{3}{x}\right)^x \cdot \underset{x\to\infty}{Lt}\left(1+\frac{3}{x}\right)^2$$

$$= e^3 \cdot (1+0)^2$$

$$= e^3$$

6) $\underset{x\to 0}{Lt} \frac{\sin^3(x/2)}{x^3}$

Soln

$$\underset{x\to 0}{Lt} \frac{\sin^3(x/2)}{x^3} = \underset{x\to 0}{Lt} \frac{\sin^3(x/2)}{\frac{x^3}{8}\cdot 8}$$

$$= \frac{1}{8} \underset{x\to 0}{Lt} \frac{\sin^3(x/2)}{(x/2)^3}$$

$$= \frac{1}{8}\left(\underset{\frac{x}{2}\to 0}{Lt} \frac{\sin(x/2)}{x/2}\right)^3$$

[ As $x \to 0$
$\frac{x}{2} \to 0$ ]

$\left[\because \underset{x\to 0}{Lt} \frac{\sin x}{x} = 1\right]$

$$= \frac{1}{8}(1^3)$$

$$= \frac{1}{8}$$

7) $\underset{x\to 0}{Lt} \frac{\sin\alpha x}{\sin\beta x}$

Soln

$$\underset{x\to 0}{Lt} \frac{\sin\alpha x}{\sin\beta x} = \underset{x\to 0}{Lt}\left[\frac{\sin\alpha x}{\alpha x}\cdot\frac{\beta x}{\sin\beta x}\cdot\frac{\alpha x}{\beta x}\right]$$

$$= \frac{\alpha}{\beta} \underset{x\to 0}{Lt} \left[ \frac{\sin \alpha x / \alpha x}{\sin \beta x / \beta x} \right]$$

$$= \frac{\alpha}{\beta} \cdot \frac{\underset{\alpha x\to 0}{Lt} \frac{\sin \alpha x}{\alpha x}}{\underset{\beta x\to 0}{Lt} \frac{\sin \beta x}{\beta x}}$$

[As $x \to 0$ $\Rightarrow \alpha x \to 0, \beta x \to 0$]

$$\left[\because \underset{x\to 0}{Lt} \frac{\sin x}{x} = 1\right]$$

$$= \frac{\alpha}{\beta} \cdot \frac{1}{1}$$

$$= \alpha / \beta$$

8) $\underset{x\to 0}{Lt} \frac{\tan 2x}{\sin 5x}$

Soln

$$\underset{x\to 0}{Lt} \frac{\tan 2x}{\sin 5x} = \underset{x\to 0}{Lt} \frac{\sin 2x}{\cos 2x} \cdot \frac{1}{\sin 5x}$$

$$= \underset{x\to 0}{Lt} \left[ \frac{1}{\cos 2x} \cdot \frac{\sin 2x}{2x} \cdot \frac{5x}{\sin 5x} \cdot \frac{2x}{5x} \right]$$

$$= \frac{2}{5} \frac{\underset{x\to 0}{Lt} \frac{1}{\cos 2x} \cdot \underset{2x\to 0}{Lt} \frac{\sin 2x}{2x}}{\underset{5x\to 0}{Lt} \frac{\sin 5x}{5x}}$$

$$\left[\because \underset{x\to 0}{Lt} \frac{\sin x}{x} = 1\right]$$

$$\underset{x\to 0}{Lt} \frac{1}{\cos 2x} = \frac{1}{\cos 0} = 1/1 = 1$$

$$= \frac{2}{5} \frac{(1)(1)}{1}$$

$$= 2/5$$

9) $\underset{\alpha\to 0}{Lt} \frac{\sin(\alpha^n)}{(\sin \alpha)^m}$

Soln

(i) If $m = n$

$$\underset{\alpha\to 0}{Lt} \frac{\sin(\alpha^n)}{(\sin \alpha)^m} = \underset{\alpha\to 0}{Lt} \frac{\sin(\alpha^m)}{(\sin \alpha)^m}$$

$$= \underset{\alpha\to 0}{Lt} \frac{\sin(\alpha^m)}{\alpha^m} \cdot \frac{\alpha^m}{(\sin \alpha)^m}$$

$$= \underset{\alpha\to 0}{Lt} \frac{\sin(\alpha^m)}{\alpha^m} \cdot \underset{\alpha\to 0}{Lt} \frac{\alpha^m}{(\sin \alpha)^m}$$

$$= \frac{\underset{\alpha^m\to 0}{Lt} \frac{\sin \alpha^m}{\alpha^m}}{\left(\underset{\alpha\to 0}{Lt} \frac{\sin \alpha}{\alpha}\right)^m} = \frac{1}{1^m} = 1$$

$$\left[\because \underset{x\to 0}{Lt} \frac{\sin x}{x} = 1\right]$$

(ii) If $m > n$

$$\underset{\alpha\to 0}{Lt} \frac{\sin(\alpha^n)}{(\sin\alpha)^m} = \underset{\alpha\to 0}{Lt} \frac{\sin(\alpha^n)}{\alpha^n} \cdot \frac{\alpha^m}{(\sin\alpha)^m} \cdot \frac{\alpha^n}{\alpha^m}$$

$$= \frac{\underset{\alpha\to 0}{Lt} \frac{\alpha^n}{\alpha^m} \cdot \underset{\alpha^n\to 0}{Lt} \frac{\sin(\alpha^n)}{\alpha^n}}{\left(\underset{\alpha\to 0}{Lt} \frac{\sin\alpha}{\alpha}\right)^m}$$

$$= \frac{\underset{\alpha\to 0}{Lt} \frac{\alpha^n}{\alpha^m} (1)}{1^m}$$

$$= \underset{\alpha\to 0}{Lt} \alpha^{n-m}$$

$$= \infty \qquad [\because m > n]$$

(iii) If $m < n$

$$\underset{\alpha\to 0}{Lt} \frac{\sin(\alpha^n)}{(\sin\alpha)^m} = \underset{\alpha\to 0}{Lt} \frac{\sin(\alpha^n)}{\alpha^n} \cdot \frac{\alpha^m}{(\sin\alpha)^m} \cdot \frac{\alpha^n}{\alpha^m}$$

$$= \frac{\underset{\alpha\to 0}{Lt} \frac{\alpha^n}{\alpha^m} \cdot \underset{\alpha^n\to 0}{Lt} \frac{\sin(\alpha^n)}{\alpha^n}}{\left(\underset{\alpha\to 0}{Lt} \frac{\sin\alpha}{\alpha}\right)^m}$$

$$= \frac{\underset{\alpha\to 0}{Lt} \frac{\alpha^n}{\alpha^m} (1)}{1^m}$$

$$= \underset{\alpha\to 0}{Lt} \alpha^{n-m}$$

$$= 0 \qquad [\because m < n]$$

10) $\underset{x\to 0}{Lt} \frac{\sin(a+x) - \sin(a-x)}{x}$

Soln

$$\underset{x\to 0}{Lt} \frac{\sin(a+x) - \sin(a-x)}{x} = \underset{x\to 0}{Lt} \frac{2\cos\left[\frac{a+x+a-x}{2}\right] \sin\left[\frac{a+x-a+x}{2}\right]}{x}$$

$$= \underset{x\to 0}{Lt} \frac{2\cos a \sin x}{x}$$

$$= 2\cos a \underset{x\to 0}{Lt} \frac{\sin x}{x} \qquad \left[\because \underset{x\to 0}{Lt} \frac{\sin x}{x} = 1\right]$$

$$= 2\cos a (1)$$

$$= 2\cos a$$

11) $\lim_{x\to 0} \dfrac{\sqrt{x^2+a^2}-a}{\sqrt{x^2+b^2}-b}$

Soln:

$$\lim_{x\to 0} \frac{\sqrt{x^2+a^2}-a}{\sqrt{x^2+b^2}-b} = \lim_{x\to 0} \frac{\sqrt{x^2+a^2}-a}{x^2} \cdot \frac{x^2}{\sqrt{x^2+b^2}-b}$$

$$= \lim_{x\to 0} \frac{\sqrt{x^2+a^2}-\sqrt{a^2}}{x^2+a^2-a^2} \cdot \frac{x^2+b^2-b^2}{\sqrt{x^2+b^2}-\sqrt{b^2}}$$

$$= \lim_{x\to 0} \frac{(x^2+a^2)^{1/2}-(a^2)^{1/2}}{(x^2+a^2)-a^2} \cdot \lim_{x\to 0} \frac{(x^2+b^2)-b^2}{(x^2+b^2)^{1/2}-(b^2)^{1/2}}$$

$$= \frac{\lim_{x^2+a^2\to a^2} \frac{(x^2+a^2)^{1/2}-(a^2)^{1/2}}{(x^2+a^2)-a^2}}{\lim_{x^2+b^2\to b^2} \frac{(x^2+b^2)^{1/2}-(b^2)^{1/2}}{(x^2+b^2)-b^2}}$$

$\left[ x\to 0 \Rightarrow x^2\to 0 \Rightarrow x^2+a^2\to a^2 \text{ and } x^2+b^2\Rightarrow b^2 \right]$

$$= \frac{\frac{1}{2}(a^2)^{-1/2}}{\frac{1}{2}(b^2)^{-1/2}}$$

$$= \frac{a^{-1}}{b^{-1}}$$

$$= b/a$$

12) $\lim_{x\to 0} \dfrac{2\arcsin x}{3x}$

Soln:

$$\lim_{x\to 0} \frac{2\arcsin x}{3x} = \frac{2}{3} \lim_{x\to 0} \frac{\arcsin(x)}{x}$$

$$= \frac{2}{3} \lim_{x\to 0} \frac{\sin^{-1}x}{x} \qquad \left[\because \lim_{x\to 0} \frac{\sin^{-1}x}{x} = 1\right]$$

$$= \frac{2}{3}(1)$$

$$= 2/3$$

13) $\lim_{x\to 0} \dfrac{1-\cos x}{x^2}$

Soln:

$$\lim_{x\to 0} \frac{1-\cos x}{x^2} = \lim_{x\to 0} \frac{2\sin^2(x/2)}{x^2}$$

$$= \lim_{x\to 0} \frac{2\sin^2(x/2)}{x^2/4} \cdot 1/4$$

$$= \frac{1}{2} \lim_{x\to 0} \frac{\sin^2(x/2)}{(x/2)^2}$$

$$= \frac{1}{2}\left[\lim_{x/2\to 0} \frac{\sin(x/2)}{x/2}\right]^2 \qquad \left[\because x\to 0 \Rightarrow x/2\to 0\right]$$

$$= \frac{1}{2}(1)^2 = 1/2$$

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14) $\lim_{x\to 0} \frac{\tan 2x}{x}$

Soln

$$\lim_{x\to 0} \frac{\tan 2x}{x} = \lim_{x\to 0} \frac{\sin 2x}{x} \cdot \frac{1}{\cos 2x}$$ $\left[\because \tan x = \frac{\sin x}{\cos x}\right]$

$$= \lim_{x\to 0} \frac{\sin 2x}{2x} \cdot \frac{2}{\cos 2x}$$

$$= 2 \lim_{x\to 0} \frac{1}{\cos 2x} \cdot \lim_{2x\to 0} \frac{\sin 2x}{2x}$$

$$= 2\left(\frac{1}{1}\right)(1)$$

$$= 2$$

15) $\lim_{x\to 0} \frac{2^x - 3^x}{x}$

Soln

$$\lim_{x\to 0} \frac{2^x - 3^x}{x} = \lim_{x\to 0} \frac{2^x - 1 + 1 - 3^x}{x}$$

$$= \lim_{x\to 0} \left[\frac{2^x - 1}{x} - \frac{3^x - 1}{x}\right]$$ $\left[\because \lim_{x\to 0} \frac{a^x - 1}{x} = \log a\right]$

$$= \lim_{x\to 0} \frac{2^x - 1}{x} - \lim_{x\to 0} \frac{3^x - 1}{x}$$

$$= \log 2 - \log 3$$

$$= \log (2/3)$$

16) $\lim_{x\to 0} \frac{3^x - 1}{\sqrt{x+1} - 1}$

Soln

$$\lim_{x\to 0} \frac{3^x - 1}{\sqrt{x+1} - 1} = \lim_{x\to 0} \frac{3^x - 1}{\sqrt{x+1} - 1} \times \frac{\sqrt{x+1} + 1}{\sqrt{x+1} + 1}$$

$$= \lim_{x\to 0} \frac{3^x - 1}{(x+1) - 1} \times \left(\sqrt{x+1} + 1\right)$$

$$= \lim_{x\to 0} \frac{3^x - 1}{x} \cdot \lim_{x\to 0} \left(\sqrt{x+1} + 1\right)$$ $\left[\because \lim_{x\to 0} \frac{a^x - 1}{x} = \log a\right]$

$$= \log 3 \cdot (1+1)$$

$$= 2 \log 3$$

$$= \log 3^2$$

$$= \log 9$$

17) $\underset{x\to 0}{Lt} \dfrac{1-\cos^2 x}{x\sin 2x}$

Soln

$\underset{x\to 0}{Lt} \dfrac{1-\cos^2 x}{x\sin 2x} = \underset{x\to 0}{Lt} \dfrac{\sin^2 x}{x \cdot 2\sin x\cos x}$

$\left[\because 1-\cos^2 x = \sin^2 x \quad \sin 2x = 2\sin x\cos x\right]$

$= \underset{x\to 0}{Lt} \dfrac{\sin x}{2x\cos x}$

$= \dfrac{1}{2} \underset{x\to 0}{Lt} \dfrac{\sin x}{x} \cdot \underset{x\to 0}{Lt} \dfrac{1}{\cos x}$

$= \dfrac{1}{2}(1)\left(\dfrac{1}{1}\right)$

$= 1/2$

18) $\underset{x\to\infty}{Lt} x\left[3^{1/x}+1-\cos(1/x)-e^{1/x}\right]$

Soln

$\underset{x\to\infty}{Lt} x\left[3^{1/x}+1-\cos(1/x)-e^{1/x}\right]$

$\left[\text{Put } y = 1/x \quad \text{As } x\to\infty \quad y\to 0\right]$

$= \underset{y\to 0}{Lt} \dfrac{1}{y}\left[3^y+1-\cos y-e^y\right]$

$= \underset{y\to 0}{Lt} \dfrac{1}{y}\left[3^y-1+1-\cos y-e^y+1\right]$

$= \underset{y\to 0}{Lt} \left[\dfrac{3^y-1}{y}+\dfrac{1-\cos y}{y}-\dfrac{(e^y-1)}{y}\right]$

$= \underset{y\to 0}{Lt} \dfrac{3^y-1}{y} + \underset{y\to 0}{Lt} \dfrac{2\sin^2(y/2)}{y} - \underset{y\to 0}{Lt} \dfrac{e^y-1}{y}$

$= \log 3 + \underset{y\to 0}{Lt} \dfrac{\sin(y/2)}{y/2} \cdot \sin(y/2) - 1$

$= \log 3 - 1 + \underset{y/2\to 0}{Lt} \dfrac{\sin(y/2)}{y/2} \cdot \underset{y\to 0}{Lt} \sin(y/2)$

$= \log 3 - 1 + (1)(0)$

$= \log 3 - 1$

19) $\underset{x\to\infty}{Lt} x\left[\log(x+a)-\log(x)\right]$

Soln

$\underset{x\to\infty}{Lt} x\left[\log(x+a)-\log(x)\right] = \underset{x\to\infty}{Lt} x\log\left(\dfrac{x+a}{x}\right)$

$= \underset{x\to\infty}{Lt} x\log\left(1+\dfrac{a}{x}\right)$

$$= \underset{x\to\infty}{Lt} \frac{\log(1+\frac{a}{x})}{1/x}$$

$$= \underset{x\to\infty}{Lt} \frac{a\log(1+\frac{a}{x})}{a/x}$$

[Put $y = a/x$

As $x \to \infty$

$y \to 0$ ]

$$= a \underset{y\to 0}{Lt} \frac{\log(1+y)}{y}$$

$$\left[\because \underset{x\to 0}{Lt} \frac{\log(1+x)}{x} = 1\right]$$

$$= a(1)$$

$$= a$$

20) $\underset{x\to\pi}{Lt} \frac{\sin 3x}{\sin 2x}$

Soln:

$$\underset{x\to\pi}{Lt} \frac{\sin 3x}{\sin 2x} = \underset{x\to\pi}{Lt} \frac{\sin(3\pi-3x)}{-\sin(2\pi-2x)}$$

[$\because \sin(3\pi-3x) = \sin 3x$

$\sin(2\pi-2x) = -\sin 2x$ ]

$$= -\underset{x\to\pi}{Lt} \frac{\sin(3\pi-3x)}{\pi-x} \cdot \frac{\pi-x}{\sin(2\pi-2x)}$$

$$= \frac{-3}{2} \underset{x\to\pi}{Lt} \frac{\sin(3\pi-3x)}{3(\pi-x)} \cdot \frac{2(\pi-x)}{\sin(2\pi-2x)}$$

[As $x \to \pi$

$x - \pi \to 0$

$\Rightarrow \pi - x \to 0$

$\Rightarrow 3\pi - 3x \to 0$

and $2\pi - 2x \to 0$ ]

$$= -\frac{3}{2} \frac{\underset{3\pi-3x\to 0}{Lt} \frac{\sin(3\pi-3x)}{3\pi-3x}}{\underset{2\pi-2x\to 0}{Lt} \frac{\sin(2\pi-2x)}{2\pi-2x}}$$

$$= -\frac{3}{2}\left(\frac{1}{1}\right)$$

$$= -3/2$$

21) $\underset{x\to\pi/2}{Lt} (1+\sin x)^{2\operatorname{cosec} x}$

Soln:

$$\underset{x\to\pi/2}{Lt} (1+\sin x)^{2\operatorname{cosec} x} = (1+\sin\pi/2)^{2\operatorname{cosec}\pi/2}$$

$$= (1+1)^2$$

$$= 4$$

22) $\underset{x\to 0}{Lt} \frac{\sqrt{2}-\sqrt{1+\cos x}}{\sin^2 x}$

Soln:

$$\underset{x\to 0}{Lt} \frac{\sqrt{2}-\sqrt{1+\cos x}}{\sin^2 x} = \underset{x\to 0}{Lt} \frac{\sqrt{2}+\sqrt{2\cos^2 x/2}}{\sin^2 x}$$

$$\left[\because 1+\cos 2\theta = 2\cos^2\theta\right]$$

$$= \lim_{x\to 0} \frac{\sqrt{2} - \sqrt{2}\cos(x/2)}{\sin^2 x}$$

$$= \sqrt{2} \lim_{x\to 0} \frac{1-\cos(x/2)}{\sin^2 x}$$

$$= \sqrt{2} \lim_{x\to 0} \frac{2\sin^2(x/4)}{\sin^2 x} \qquad \left[\because 1-\cos 2\theta = 2\sin^2\theta\right]$$

$$= 2\sqrt{2} \lim_{x\to 0} \frac{\sin^2(x/4)}{\sin^2 x}$$

$$= 2\sqrt{2} \lim_{x\to 0} \frac{\sin^2(x/4)}{(x/4)^2} \cdot \frac{(x/4)^2}{\sin^2 x}$$

$$= 2\sqrt{2} \lim_{x\to 0} \frac{\sin^2(x/4)}{(x/4)^2} \cdot \frac{1}{16} \cdot \frac{x^2}{\sin^2 x}$$

$$= \frac{2\sqrt{2}}{16} \frac{\left(\lim_{\frac{x}{4}\to 0} \frac{\sin(x/4)}{x/4}\right)^2}{\left(\lim_{x\to 0} \frac{\sin x}{x}\right)^2}$$

$$= \frac{\sqrt{2}}{8}\left(\frac{1}{1}\right)$$

$$= \frac{1}{4\sqrt{2}}$$

23) $\lim_{x\to 0} \frac{\sqrt{1+\sin x} - \sqrt{1-\sin x}}{\tan x}$

Soln:

$$\lim_{x\to 0} \frac{\sqrt{1+\sin x} - \sqrt{1-\sin x}}{\tan x} = \lim_{x\to 0} \frac{\sqrt{1+\sin x} - \sqrt{1-\sin x}}{\tan x} \times \frac{\sqrt{1+\sin x} + \sqrt{1-\sin x}}{\sqrt{1+\sin x} + \sqrt{1-\sin x}}$$

$$= \lim_{x\to 0} \frac{(1+\sin x) - (1-\sin x)}{\tan x\left[\sqrt{1+\sin x} + \sqrt{1-\sin x}\right]}$$

$$= \lim_{x\to 0} \frac{2\sin x}{\frac{\sin x}{\cos x}\left[\sqrt{1+\sin x} + \sqrt{1-\sin x}\right]}$$

$$= \lim_{x\to 0} \frac{2\cos x}{\sqrt{1+\sin x} + \sqrt{1-\sin x}}$$

$$= \frac{2(1)}{\sqrt{1} + \sqrt{1}}$$

$$= \frac{2}{2}$$

$$= 1$$

24) $\underset{x\to\infty}{Lt}\left(\dfrac{x^2-2x+1}{x^2-4x+2}\right)^x$

Soln:

$$\underset{x\to\infty}{Lt}\left(\frac{x^2-2x+1}{x^2-4x+2}\right)^x = \underset{x\to\infty}{Lt}\left[1+\frac{2x-1}{x^2-4x+2}\right]^x$$

$$x^2-4x+2 \,\overline{\big)\, x^2-2x+1 \,}\,1$$
$$\underline{x^2-4x+2}$$
$$2x-1$$

$$= \underset{x\to\infty}{Lt}\left[1+\frac{1}{\frac{x^2-4x+2}{2x-1}}\right]^x$$

$$= \underset{x\to\infty}{Lt}\left[1+\frac{1}{\frac{x^2-4x+2}{2x-1}}\right]^{x\left(\frac{x^2-4x+2}{2x-1}\right)\left(\frac{2x-1}{x^2-4x+2}\right)}$$

$$= \underset{x\to\infty}{Lt}\left[1+\frac{1}{\frac{x^2-4x+2}{2x-1}}\right]^{\frac{x(2x-1)}{x^2-4x+2}}$$

$$= \underset{x\to\infty}{Lt}\left[\left[1+\frac{1}{\frac{x^2-4x+2}{2x-1}}\right]^{\left(\frac{x^2-4x+2}{2x-1}\right)}\right]^{\underset{x\to\infty}{Lt}\frac{x(2x-1)}{x^2-4x+2}}$$

As $x\to\infty$

$$\frac{x^2-4x+2}{2x-1} = \frac{x(1-4/x+2/x^2)}{2-1/x} \to \infty$$

$$= \left[\underset{\frac{x^2-4x+2}{2x-1}\to\infty}{Lt}\left(1+\frac{1}{\frac{x^2-4x+2}{2x-1}}\right)^{\frac{x^2-4x+2}{2x-1}}\right]^{\underset{x\to\infty}{Lt}\frac{2x^2-x}{x^2-4x+2}}$$

$$\left[\underset{x\to\infty}{Lt}\left(1+\frac{1}{x}\right)^x = e\right]$$

$$= e^{\underset{x\to\infty}{Lt}\frac{2x^2-x}{x^2-4x+2}}$$

$$= e^{\underset{x\to\infty}{Lt}\frac{x^2(2-1/x)}{x^2(1-4/x+2/x^2)}}$$

$$= e^{\underset{x\to\infty}{Lt}\frac{2-1/x}{1-4/x+2/x^2}}$$

$$= e^{\frac{2-0}{1-0+0}}$$

$$= e^2$$

25) $\underset{x\to 0}{Lt} \frac{e^x - e^{-x}}{\sin x}$

Soln:

$$\underset{x\to 0}{Lt} \frac{e^x - e^{-x}}{\sin x} = \underset{x\to 0}{Lt} \frac{e^x - 1 - e^{-x} + 1}{\sin x}$$

$$= \underset{x\to 0}{Lt} \frac{\frac{e^x - 1 - (e^{-x} - 1)}{x}}{\frac{\sin x}{x}}$$

$$= \frac{\underset{x\to 0}{Lt} \frac{e^x - 1}{x} + \underset{-x\to 0}{Lt} \frac{e^{-x} - 1}{-x}}{\underset{x\to 0}{Lt} \frac{\sin x}{x}} \qquad \left[ \begin{array}{l} x \to 0 \\ \Rightarrow -x \to 0 \end{array} \right.$$

$$= \frac{1+1}{1}$$

$$= 2$$

26) $\underset{x\to 0}{Lt} \frac{e^{ax} - e^{bx}}{x}$

Soln:

$$\underset{x\to 0}{Lt} \frac{e^{ax} - e^{bx}}{x} = \underset{x\to 0}{Lt} \frac{e^{ax} - 1 - e^{bx} + 1}{x}$$

$$= \underset{x\to 0}{Lt} \frac{e^{ax} - 1}{x} - \underset{x\to 0}{Lt} \frac{e^{bx} - 1}{x}$$

$$= a \underset{x\to 0}{Lt} \frac{e^{ax} - 1}{ax} - b \underset{x\to 0}{Lt} \frac{e^{bx} - 1}{bx}$$

$$= a \left( \underset{ax\to 0}{Lt} \frac{e^{ax} - 1}{ax} \right) - b \left( \underset{bx\to 0}{Lt} \frac{e^{bx} - 1}{bx} \right)$$

$$= a(1) - b(1)$$

$$= a - b$$

27) $\underset{x\to 0}{Lt} \frac{\sin x (1 - \cos x)}{x^3}$

Soln:

$$\underset{x\to 0}{Lt} \frac{\sin x (1 - \cos x)}{x^3} = \underset{x\to 0}{Lt} \frac{\sin x}{x} \cdot \frac{2\sin^2(x/2)}{x^2}$$

$$= \underset{x\to 0}{Lt} \frac{\sin x}{x} \cdot \frac{2\sin^2(x/2)}{(x/2)^2 \cdot 4}$$

$$= \frac{1}{2} \underset{x\to 0}{Lt} \frac{\sin x}{x} \cdot \left( \underset{\frac{x}{2}\to 0}{Lt} \frac{\sin(x/2)}{x/2} \right)^2$$

$= \frac{1}{2}(1)(1)$

$= \frac{1}{2}$

28) $\underset{x\to 0}{Lt} \frac{\tan x - \sin x}{x^3}$

Soln:

$\underset{x\to 0}{Lt} \frac{\tan x - \sin x}{x^3} = \underset{x\to 0}{Lt} \frac{\frac{\sin x}{\cos x} - \sin x}{x^3}$

$= \underset{x\to 0}{Lt} \frac{\sin x\left(\frac{1-\cos x}{\cos x}\right)}{x^3}$

$= \underset{x\to 0}{Lt} \frac{\sin x}{x} \cdot \frac{(2\sin^2(x/2))}{x^2} \cdot \frac{1}{\cos x}$

$= \underset{x\to 0}{Lt} \frac{\sin x}{x} \cdot \frac{2\sin^2(x/2)}{(x/2)^2 \cdot 4} \cdot \frac{1}{\cos x}$

$= \frac{1}{2} \underset{x\to 0}{Lt} \frac{\sin x}{x} \cdot \left(\underset{\frac{x}{2}\to 0}{Lt} \frac{\sin(x/2)}{x/2}\right)^2 \underset{x\to 0}{Lt} \frac{1}{\cos x}$

$= \frac{1}{2}(1)(1)\left(\frac{1}{1}\right)$

$= \frac{1}{2}$

Exer (9.5)

1) Prove that $f(x) = 2x^2+3x-5$ is continuous at all points in R.

Soln:

$f(x) = 2x^2+3x-5$

$\because$ Algebraic function is continuous on R

$\Rightarrow f(x)$ is continuous at all points on R [$\because f(x)$ is algebraic fun]

2) Examine the continuity of the following:

(i) $x + \sin x$

Soln:

Let $f(x) = x + \sin x$

$\because$ $x$ is continuous on R

$\sin x$ is continuous on R

$x + \sin x$ is continuous on R.

$\Rightarrow x + \sin x$

$\therefore f(x)$ is continuous on R

(ii) $x^2 \cos x$

Soln

Let $f(x) = x^2 \cos x$

$\because x^2$ and $\cos x$ are continuous on R

$\Rightarrow x^2 \cos x$ is also continuous on R

$\therefore f(x)$ is continuous at all points on R

(iii) $e^x \tan x$

Soln

Let $f(x) = e^x \tan x$

$\because e^x$ is continuous on R

But $\tan x$ is not continuous at $x = (2n+1)\frac{\pi}{2}$ ; $n \in Z$

$\therefore f(x) = e^x \tan x$ is continuous for all $x \in R - \{(2n+1)\pi/2 \mid n \in Z\}$

(iv) $e^{2x} + x^2$

Soln

Let $f(x) = e^{2x} + x^2$

$\because e^{2x}$ and $x^2$ are continuous on R

$\therefore f(x) = e^{2x} + x^2$ is continuous at all points on R

(v) $x \log x$

Soln

Let $f(x) = x \log x$

$x$ is continuous on R.

But $\log x$ is continuous on $(0, \infty)$

$\therefore f(x) = x \log x$ is continuous for all $x \in (0, \infty)$

(vi) $\frac{\sin x}{x^2}$

Soln

Let $f(x) = \frac{\sin x}{x^2}$

$\because$ $\sin x$ and $x^2$ are continuous for all $x \in R$

But $f(x)$ does not exist at $x=0$

$\therefore f(x) = \frac{\sin x}{x^2}$ is continuous for all $x \in R-\{0\}$.

(vii) $\frac{x^2-16}{x+4}$

Soln

Let $f(x) = \frac{x^2-16}{x+4}$

$\because$ $x^2-16$ and $x+4$ are continuous for all $x \in R$

But $f(x)$ does not exist at $x=-4$

$\therefore f(x) = \frac{x^2-16}{x+4}$ is continuous for all $x \in R-\{-4\}$

(viii) $|x+2|+|x-1|$

Soln

Let $f(x) = |x+2|+|x-1|$

$\because$ modulus function is continuous for all $x \in R$

$\Rightarrow$ $|x+2|$ and $|x-1|$ are continuous for all $x \in R$

$\therefore f(x) = |x+2|+|x-1|$ is continuous for all $x \in R$

(ix) $\frac{|x-2|}{|x+1|}$

Soln

Let $f(x) = \frac{|x-2|}{|x+1|}$

$\because$ modulus function is continuous for all $x \in R$.

$\Rightarrow$ $|x-2|$ and $|x+1|$ are continuous for all $x \in R$

But $f(x) = \frac{|x-2|}{|x+1|}$ does not exist at $x=-1$

$\therefore f(x) = \frac{|x-2|}{|x+1|}$ is continuous for all $x \in R-\{-1\}$

(x) $\cot x + \tan x$

Soln

Let $f(x) = \cot x + \tan x$.

$\because$ $\cot x$ is not continuous at $x = n\pi \,;\, n \in Z$ $\rightarrow (1)$

$\tan x$ is not continuous at $x = (2n+1)\frac{\pi}{2} \,;\, n \in Z$ $\rightarrow (2)$

From (1) and (2)

$$\left\{\frac{2n\pi}{2} \,/\, n \in Z\right\} \cup \left\{(2n+1)\frac{\pi}{2} \,/\, n \in Z\right\}$$

$$= \left\{\frac{n\pi}{2} \,|\, n \in Z\right\}$$

$\therefore f(x) = \cot x + \tan x$ is not continuous at $x = \frac{n\pi}{2} \,;\, n \in Z$

$\Rightarrow f(x) = \cot x + \tan x$ is continuous for all $x \in R - \left\{\frac{n\pi}{2} \,|\, n \in Z\right\}$

3) Find the points of discontinuity of the function $f$, where,

(i) $f(x) = \begin{cases} 4x+5 & \text{if } x \le 3 \\ 4x-5 & \text{if } x > 3 \end{cases}$

Soln

$f(x) = \begin{cases} 4x+5 & \text{if } x \le 3 \\ 4x-5 & \text{if } x > 3 \end{cases}$

$$\lim_{x \to 3^-} f(x) = \lim_{x \to 3^-} (4x+5)$$

$$= 17$$

$$\lim_{x \to 3^+} f(x) = \lim_{x \to 3^+} (4x-5)$$

$$= 7$$

$$\lim_{x \to 3^-} f(x) \ne \lim_{x \to 3^+} f(x)$$

$\therefore f(x)$ is not continuous at $x = 3$

But $f(x)$ is continuous for all $x \in R - \{3\}$

(ii) $f(x) = \begin{cases} x+2 & \text{if } x \geq 2 \\ x^2 & \text{if } x < 2 \end{cases}$

Soln

$$f(x) = \begin{cases} x+2 & \text{if } x \geq 2 \\ x^2 & \text{if } x < 2 \end{cases}$$

$$\underset{x \to 2^-}{Lt} f(x) = \underset{x \to 2^-}{Lt} x^2$$

$$= 4$$

$$\underset{x \to 2^+}{Lt} f(x) = \underset{x \to 2^+}{Lt} (x+2)$$

$$= 4$$

$$f(2) = 4$$

$$\therefore \underset{x \to 2^-}{Lt} f(x) = \underset{x \to 2^+}{Lt} f(x) = f(2) = 4$$

$\therefore f(x)$ is continuous at $x = 2$

$[\because x+2$ and $x^2$ are continuous for all $x \in R$

$\Rightarrow f(x)$ is continuous for all $x \in R$

(iii) $f(x) = \begin{cases} x^3 - 3 & \text{if } x \leq 2 \\ x^2 + 1 & \text{if } x > 2 \end{cases}$

Soln

$$f(x) = \begin{cases} x^3 - 3 & \text{if } x \leq 2 \\ x^2 + 1 & \text{if } x > 2 \end{cases}$$

$$\underset{x \to 2^-}{Lt} f(x) = \underset{x \to 2^-}{Lt} (x^3 - 3)$$

$$= 8 - 3$$

$$= 5$$

$$\underset{x \to 2^+}{Lt} f(x) = \underset{x \to 2^+}{Lt} (x^2 + 1)$$

$$= 4 + 1$$

$$= 5$$

$$f(2) = 8 - 3$$

$$= 5$$

$$\therefore \underset{x \to 2^-}{Lt} f(x) = \underset{x \to 2^+}{Lt} f(x) = f(2) = 5$$

$\therefore f(x)$ is continuous at $x = 2$

$\Rightarrow f(x)$ is continuous for all $x \in R$

(iv) $f(x) = \begin{cases} \sin x & ; 0 \le x \le \pi/4 \\ \cos x & ; \frac{\pi}{4} < x < \pi/2 \end{cases}$

Soln

$$f(x) = \begin{cases} \sin x & ; 0 \le x \le \pi/4 \\ \cos x & ; \pi/4 < x < \pi/2 \end{cases}$$

$$\lim_{x \to \pi/4^-} f(x) = \lim_{x \to \pi/4} \sin x$$

$$= 1/\sqrt{2}$$

$$\lim_{x \to \pi/4^+} f(x) = \lim_{x \to \pi/4^+} \cos x$$

$$= 1/\sqrt{2}$$

$$f(\pi/4) = 1/\sqrt{2}$$

$$\therefore \lim_{x \to \pi/4^-} f(x) = \lim_{x \to \pi/4^+} f(x) = f(\pi/4) = 1/\sqrt{2}$$

$\therefore f(x)$ is continuous at $x = \pi/4$

$$\lim_{x \to 0^+} f(x) = \lim_{x \to 0^+} \sin x$$

$$= 0$$

$$f(0) = 0$$

$$\lim_{x \to 0^+} f(x) = f(0) = 0$$

$\therefore f(x)$ is continuous for all $x \in [0, \pi/2)$

4) At the given point $x_0$ discover whether the given function is continuous or discontinuous citing the reasons for your answer.

(i) $x_0 = 1$

$f(x) = \begin{cases} \frac{x^2-1}{x-1} & \text{if } x \ne 1 \\ 2 & \text{if } x = 1 \end{cases}$

Soln

$$f(x) = \begin{cases} \frac{x^2-1}{x-1} & \text{if } x \ne 1 \\ 2 & \text{if } x = 1 \end{cases}$$

$$\lim_{x\to 1^-} f(x) = \lim_{x\to 1^-} \frac{x^2-1}{x-1}$$ $\left[\because \frac{x^2-1}{x-1} = \frac{(x-1)(x+1)}{x-1} = (x+1)\right]$

$$= \lim_{x\to 1^-} (x+1)$$

$$= 2$$

$$\lim_{x\to 1^+} f(x) = \lim_{x\to 1^+} \frac{x^2-1}{x-1}$$

$$= \lim_{x\to 1^+} (x+1)$$

$$= 2$$

$$f(1) = 2$$

$$\therefore \lim_{x\to 1^-} f(x) = \lim_{x\to 1^+} f(x) = f(1) = 2$$

$\therefore f(x)$ is continuous at $x = x_0 = 1$

(ii) $x_0 = 3$

$$f(x) = \begin{cases} \frac{x^2-9}{x-3} & \text{if } x \neq 3 \\ 5 & \text{if } x = 3 \end{cases}$$

Soln

$$f(x) = \begin{cases} \frac{x^2-9}{x-3} & \text{if } x \neq 3 \\ 5 & \text{if } x = 3 \end{cases}$$

$$\lim_{x\to 3^-} f(x) = \lim_{x\to 3^-} \frac{x^2-9}{x-3}$$ $\left[\because \frac{x^2-9}{x-3} = \frac{(x-3)(x+3)}{x-3} = x+3\right]$

$$= \lim_{x\to 3^-} (x+3)$$

$$= 6$$

$$\lim_{x\to 3^+} f(x) = \lim_{x\to 3^+} \frac{x^2-9}{x-3}$$

$$= \lim_{x\to 3^+} (x+3)$$

$$= 6$$

$$f(3) = 5$$

$$\lim_{x\to 3^-} f(x) = \lim_{x\to 3^+} f(x) \neq f(3)$$

$\therefore f(x)$ is not continuous at $x = x_0 = 3$

5) Show that the function $\begin{cases} \dfrac{x^3-1}{x-1} & \text{if } x\neq 1 \\ 3 & \text{if } x=1 \end{cases}$ is continuous on $(-\infty, \infty)$

Soln

Let $f(x) = \begin{cases} \dfrac{x^3-1}{x-1} & \text{if } x\neq 1 \\ 3 & \text{if } x=1 \end{cases}$

$$\lim_{x\to 1^-} f(x) = \lim_{x\to 1^-} \frac{x^3-1}{x-1}$$

$$= \lim_{x\to 1^-} (x^2+x+1)$$

$$= 1+1+1$$

$$= 3$$

$\left[\because \dfrac{x^3-1}{x-1} = \dfrac{(x-1)(x^2+x+1)}{x-1} = x^2+x+1\right.$

$$\lim_{x\to 1^+} f(x) = \lim_{x\to 1^+} \frac{x^3-1}{x-1}$$

$$= \lim_{x\to 1^+} (x^2+x+1)$$

$$= 1+1+1$$

$$= 3$$

$$f(1) = 3$$

$$\therefore \lim_{x\to 1^-} f(x) = \lim_{x\to 1^+} f(x) = f(1) = 3$$

$\therefore f(x)$ is continuous at $x=1$

$\therefore f(x)$ is continuous for all $x \in R$

i.e., $f(x)$ is continuous on $(-\infty, \infty)$

6) For what value of $\alpha$ is this function $f(x) = \begin{cases} \dfrac{x^4-1}{x-1} & \text{if } x\neq 1 \\ \alpha & \text{if } x=1 \end{cases}$ continuous at $x=1$?

Soln

$f(x) = \begin{cases} \dfrac{x^4-1}{x-1} & \text{if } x\neq 1 \\ \alpha & \text{if } x=1 \end{cases}$

$$\lim_{x\to 1^-} f(x) = \lim_{x\to 1^-} \frac{x^4-1}{x-1}$$

$$= \lim_{x\to 1^-} (x+1)(x^2+1)$$

$\left[\because \dfrac{x^4-1}{x-1} = \dfrac{(x^2-1)(x^2+1)}{x-1} = \dfrac{\cancel{(x-1)}(x+1)(x^2+1)}{\cancel{x-1}}\right.$

$= 2 \times 2$

$= 4$

$\underset{x \to 1^+}{Lt} f(x) = \underset{x \to 1^+}{Lt} \frac{x^4 - 1}{x - 1}$

$= \underset{x \to 1^+}{Lt} (x+1)(x^2+1)$

$= 2 \times 2$

$= 4$

$\underset{x \to 1^-}{Lt} f(x) = \underset{x \to 1^+}{Lt} f(x) = 4 = f(1)$ [$\therefore$ f(x) is continuous at $x=1$]

$\Rightarrow$ At $x=1$, $f(x) = 4$

$\Rightarrow \boxed{\alpha = 4}$

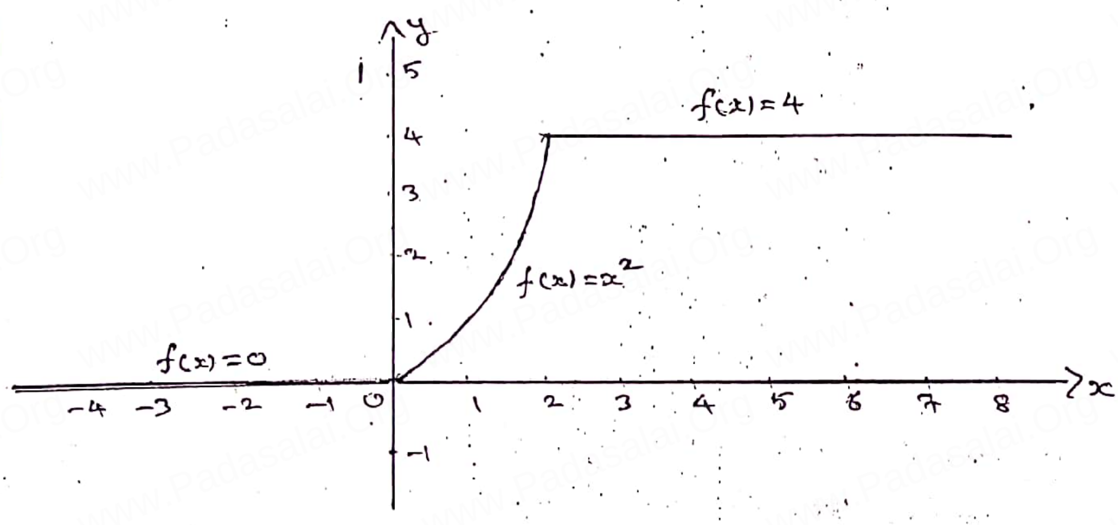
---

7) Let $f(x) = \begin{cases} 0 & \text{if } x < 0 \\ x^2 & \text{if } 0 \le x < 2 \\ 4 & \text{if } x \ge 2 \end{cases}$ Graph the function. Show that f(x) Continuous on $(-\infty, \infty)$

Soln

$f(x) = \begin{cases} 0 & \text{if } x < 0 \\ x^2 & \text{if } 0 \le x < 2 \\ 4 & \text{if } x \ge 2 \end{cases}$

Graph of the function:




$\underset{x \to 0^-}{Lt} f(x) = \underset{x \to 0^-}{Lt} 0$

$= 0$

$\underset{x \to 0^+}{Lt} f(x) = \underset{x \to 0^+}{Lt} x^2$

$= 0$

$f(0) = 0$

$$\lim_{x\to 0^-} f(x) = \lim_{x\to 0^+} f(x) = f(0) = 0$$

$\therefore f(x)$ is continuous at $x=0$

$$\lim_{x\to 2^-} f(x) = \lim_{x\to 2^-} x^2$$

$$= 4$$

$$\lim_{x\to 2^+} f(x) = \lim_{x\to 2^+} 4$$

$$= 4$$

$$f(2) = 4$$

$$\lim_{x\to 2^-} f(x) = \lim_{x\to 2^+} f(x) = f(2) = 4$$

$\therefore f(x)$ is continuous at $x=2$

$f(x)$ is continuous for all $x \in (-\infty, \infty)$

---

8) If $f$ and $g$ are continuous functions with $f(3)=5$ and $\lim_{x\to 3}[2f(x)-g(x)] = 4$, find $g(3)$.

Soln:

Given, $f(x)$ and $g(x)$ are continuous function

$$\lim_{x\to 3}[2f(x)-g(x)] = 4$$

$$\Rightarrow 2\lim_{x\to 3} f(x) - \lim_{x\to 3} g(x) = 4$$

$$\Rightarrow 2f(3) - g(3) = 4$$

$$\Rightarrow 2(5) - g(3) = 4$$

$$\Rightarrow g(3) = 10-4$$

$$\Rightarrow g(3) = 6$$

---

9) Find the points at which $f$ is discontinuous. At which of these points $f$ is continuous from the right, from the left, or neither? Sketch the graph of $f$.

(i) $$f(x) = \begin{cases} 2x+1 & \text{if } x \le -1 \\ 3x & \text{if } -1 < x < 1 \\ 2x-1 & \text{if } x \ge 1 \end{cases}$$

Soln:

$$f(x) = \begin{cases} 2x+1 & \text{if } x \le -1 \\ 3x & \text{if } -1 < x < 1 \\ 2x-1 & \text{if } x \ge 1 \end{cases}$$

$$\underset{x \to -1^-}{Lt} f(x) = \underset{x \to -1^-}{Lt} (2x+1)$$

$$= -1$$

$$\underset{x \to -1^+}{Lt} f(x) = \underset{x \to -1^+}{Lt} 3x$$

$$= -3$$

$$\underset{x \to -1^-}{Lt} f(x) \ne \underset{x \to -1^+}{Lt} f(x)$$

$$f(-1) = -1$$

$\therefore f(x)$ is not continuous at $x = -1$ but $f(x)$ is continuous from the left at $x = -1$

$$\underset{x \to 1^-}{Lt} f(x) = \underset{x \to 1^-}{Lt} 3x$$

$$= 3$$

$$\underset{x \to 1^+}{Lt} f(x) = \underset{x \to 1^+}{Lt} (2x-1)$$

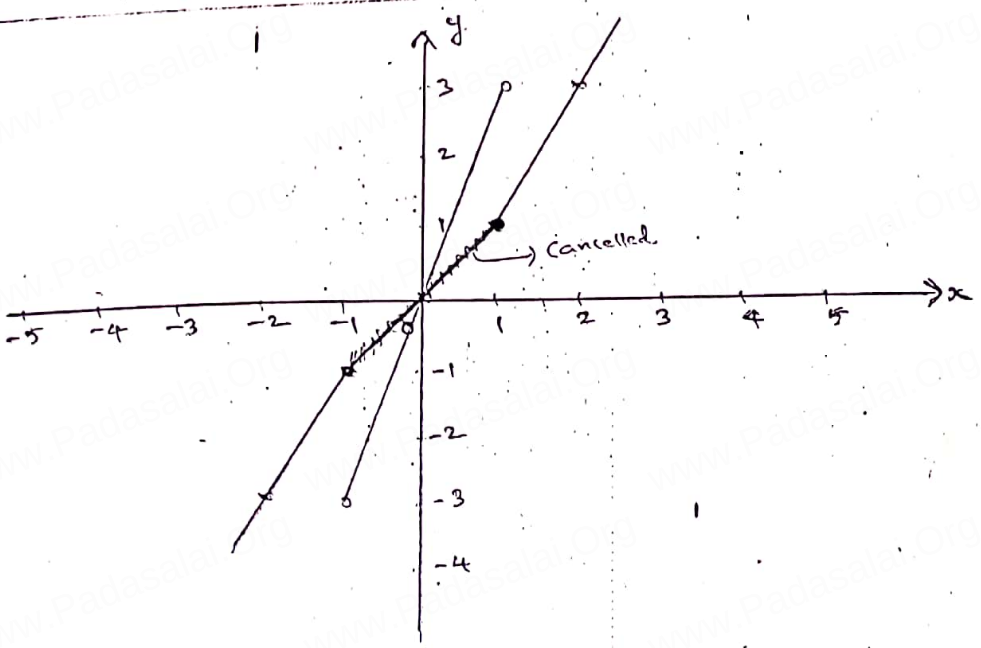
$$= 1$$

$$\underset{x \to 1^-}{Lt} f(x) \ne \underset{x \to 1^+}{Lt} f(x)$$

$$f(1) = 1$$

$\therefore f(x)$ is not continuous at $x = 1$ but $f(x)$ is continuous from the right at $x = 1$.

| $x$ | $-2$ | $-1$ | $0$ | $1$ | $2$ |
|---|---|---|---|---|---|
| $f(x)$ | $-3$ | $-1$ | $0$ | $1$ | $3$ |



(ii) $f(x) = \begin{cases} (x-1)^3 & \text{if } x < 0 \\ (x+1)^3 & \text{if } x \geq 0 \end{cases}$

Soln

$$f(x) = \begin{cases} (x-1)^3 & \text{if } x < 0 \\ (x+1)^3 & \text{if } x \geq 0 \end{cases}$$

$$\underset{x\to 0^-}{Lt} f(x) = \underset{x\to 0^-}{Lt} (x-1)^3$$

$$= -1$$

$$\underset{x\to 0^+}{Lt} f(x) = \underset{x\to 0^+}{Lt} (x+1)^3$$

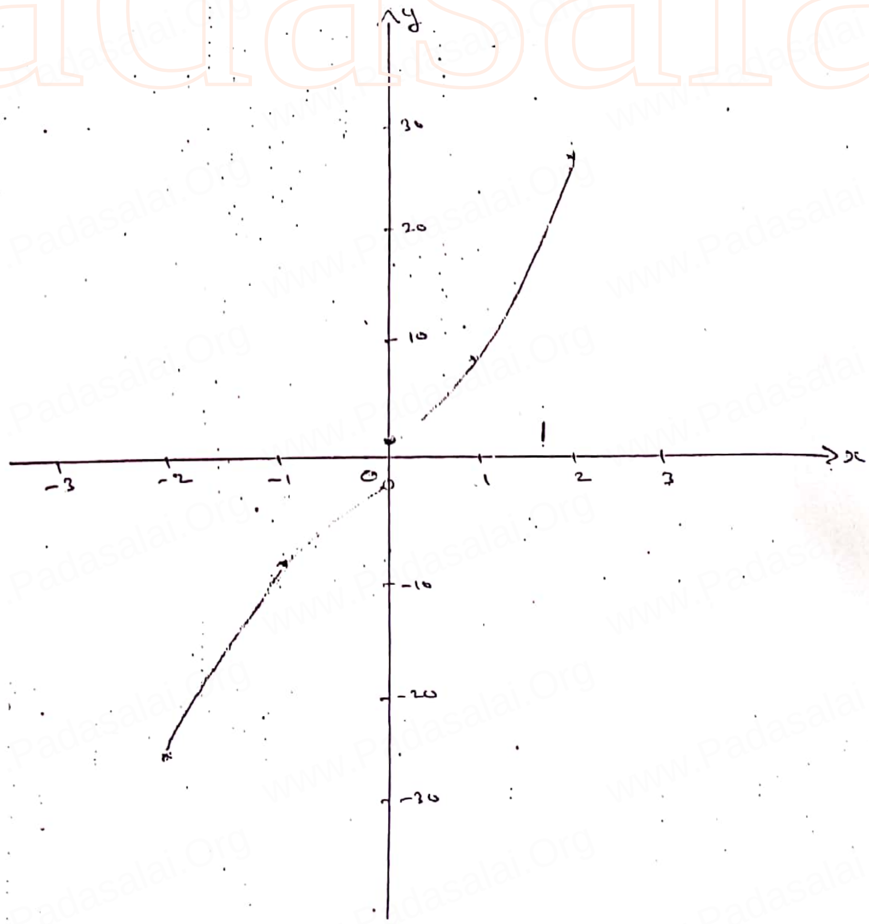
$$= 1$$

$$\underset{x\to 0^-}{Lt} f(x) \neq \underset{x\to 0^+}{Lt} f(x)$$

$$f(0) = 1$$

$\therefore f(x)$ is not continuous at $x=0$ but $f(x)$ is continuous from the right at $x=0$

| $x$ | $-2$ | $-1$ | $0$ | $1$ | $2$ |
|---|---|---|---|---|---|
| $f(x)$ | $-27$ | $-8$ | $1$ | $8$ | $27$ |



10) A function $f$ is defined as follows.

$$f(x) = \begin{cases} 0 & \text{if } x < 0 \\ x & \text{if } 0 \le x < 1 \\ -x^2+4x-2 & \text{if } 1 \le x < 3 \\ 4-x & \text{if } x \ge 3 \end{cases}$$

Is the function continuous?

Soln)

$$f(x) = \begin{cases} 0 & \text{if } x < 0 \\ x & \text{if } 0 \le x < 1 \\ -x^2+4x-2 & \text{if } 1 \le x < 3 \\ 4-x & \text{if } x \ge 3 \end{cases}$$

$$\lim_{x\to 0^-} f(x) = \lim_{x\to 0^-} 0 = 0$$

$$\lim_{x\to 0^+} f(x) = \lim_{x\to 0^+} x = 0$$

$$f(0) = 0$$

$$\therefore \lim_{x\to 0^-} f(x) = \lim_{x\to 0^+} f(x) = f(0) = 0$$

$\therefore$ $f(x)$ is continuous at $x=0$

$$\lim_{x\to 1^-} f(x) = \lim_{x\to 1^-} x = 1$$

$$\lim_{x\to 1^+} f(x) = \lim_{x\to 1^+} (-x^2+4x-2)$$

$$= -1+4-2 = 1$$

$$f(1) = -1+4-2$$

$$= 1$$

$$\therefore \lim_{x\to 1^-} f(x) = \lim_{x\to 1^+} f(x) = f(1) = 1$$

$\therefore$ $f(x)$ is continuous at $x=1$

$$\lim_{x\to 3^-} f(x) = \lim_{x\to 3^-} (-x^2+4x-2)$$

$$= -9+12-2$$

$$= 1$$

$$\lim_{x\to 3^+} f(x) = \lim_{x\to 3^+} (4-x)$$

$$= 4-3$$

$$= 1$$

$$f(3) = 1$$

$$\therefore \lim_{x\to 3^-} f(x) = \lim_{x\to 3^+} f(x) = f(3) = 1$$

$\therefore$ $f(x)$ is continuous at $x=3$.

$\Rightarrow$ $f(x)$ is continuous for all $x \in R$

11) Which of the following functions $f$ has a removable discontinuity at $x = x_0$? If the discontinuity is removable, find a function $g$ that agrees with $f$ for $x \neq x_0$ and is continuous on $R$.

(i) $f(x) = \frac{x^2 - 2x - 8}{x+2}$ ; $x_0 = -2$

(ii) $f(x) = \frac{x^3 + 64}{x+4}$ ; $x_0 = -4$

(iii) $f(x) = \frac{3 - \sqrt{x}}{9 - x}$ ; $x_0 = 9$.

Soln:

(i) $f(x) = \frac{x^2 - 2x - 8}{x+2}$

$f(x)$ does not exist at $x = -2$

$\therefore f(x)$ has removable discontinuity at $x = -2$

$$\lim_{x \to -2} f(x) = \lim_{x \to -2} \frac{x^2 - 2x - 8}{x+2}$$

$$= \lim_{x \to -2} \frac{(x+2)(x-4)}{x+2}$$

$$= \lim_{x \to -2} (x-4)$$

$$= -6$$

Now,

$$g(x) = \begin{cases} \frac{x^2 - 2x - 8}{x+2} & \text{if } x \neq -2 \\ -6 & \text{if } x = -2 \end{cases}$$

(ii) $f(x) = \frac{x^3 + 64}{x+4}$

$f(x)$ does not exist at $x = -4$

$\therefore f(x)$ has removable discontinuity at $x = -4$.

$$\lim_{x \to -4} f(x) = \lim_{x \to -4} \frac{x^3 + 64}{x+4}$$

$$= \lim_{x \to -4} \frac{(x+4)(x^2 - 4x + 16)}{x+4}$$

$$= \lim_{x \to -4} (x^2 - 4x + 16)$$

$$= 16 + 16 + 16 = 48$$

Now,

$$g(x) = \begin{cases} \dfrac{x^3+64}{x+4} & \text{if } x \neq -4 \\ 48 & \text{if } x = -4 \end{cases}$$

(iii)

$$f(x) = \frac{3-\sqrt{x}}{9-x}$$

f(x) does not exist at x=9

∴ f(x) has removable discontinuity at x=9

$$\lim_{x \to 9} f(x) = \lim_{x \to 9} \frac{3-\sqrt{x}}{9-x}$$

$$= \lim_{x \to 9} \frac{3-\sqrt{x}}{(3+\sqrt{x})(3-\sqrt{x})}$$

$$= \lim_{x \to 9} \frac{1}{3+\sqrt{x}}$$

$$= \frac{1}{3+3}$$

$$= 1/6$$

Now,

$$g(x) = \begin{cases} \dfrac{3-\sqrt{x}}{9-x} & \text{if } x \neq 9 \\ 1/6 & \text{if } x = 9 \end{cases}$$

12) Find the constant b that makes g continuous on $(-\infty, \infty)$

$$g(x) = \begin{cases} x^2 - b^2 & \text{if } x < 4 \\ bx + 20 & \text{if } x \geq 4 \end{cases}$$

Soln:

$$g(x) = \begin{cases} x^2 - b^2 & \text{if } x < 4 \\ bx + 20 & \text{if } x \geq 4 \end{cases}$$

$$\lim_{x \to 4^-} g(x) = \lim_{x \to 4^-} (x^2 - b^2)$$

$$= 16 - b^2$$

$$\lim_{x \to 4^+} g(x) = \lim_{x \to 4^+} (bx + 20)$$

$$= 4b + 20$$

$$g(4) = 4b + 20$$

g(x) is continuous on $(-\infty, \infty)$

$$\Rightarrow \lim_{x \to 4^-} g(x) = \lim_{x \to 4^+} g(x) = g(4)$$

$$\Rightarrow 16 - b^2 = 4b + 20$$

$\Rightarrow b^2+4b+4=0$

$\Rightarrow (b+2)^2=0$

$\Rightarrow \boxed{b=-2}$

13) Consider the function $f(x)=x\sin\frac{\pi}{x}$. what value must we give $f(0)$ in order to make the function continuous every where?

Soln:

$f(x)=x\sin\frac{\pi}{x}$

W.K.T. $-1\le \sin\frac{\pi}{x}\le 1$

$x^{ly}\ x \Rightarrow \quad -x\le x\sin\frac{\pi}{x}\le x$

$\lim_{x\to 0}(-x)\le \lim_{x\to 0} x\sin\frac{\pi}{x}\le \lim_{x\to 0} x$

$0\le \lim_{x\to 0} x\sin\frac{\pi}{x}\le 0$

By Sandwich theorem

$\lim_{x\to 0} x\sin\frac{\pi}{x}=0$

To make $f(x)$ continuous every where

$\Rightarrow \boxed{f(0)=0}$

14) The function $f(x)=\frac{x^2-1}{x^3-1}$ is not defined at $x=1$. what value must we give $f(1)$ inorder to make $f(x)$ continuous at $x=1$?

Soln:

$f(x)=\frac{x^2-1}{x^3-1}$

$=\frac{(x-1)(x+1)}{(x-1)(x^2+x+1)}$

$=\frac{x+1}{x^2+x+1}$

$\lim_{x\to 1^-} f(x)=\lim_{x\to 1^-}\frac{x+1}{x^2+x+1}$

$=\frac{2}{3}$

$\lim_{x\to 1^+} f(x)=\lim_{x\to 1^+}\frac{x+1}{x^2+x+1}$

$=2/3$

[OR]

$\lim_{x\to 1} f(x)=\lim_{x\to 1}\frac{x+1}{x^2+x+1}$

$=2/3$

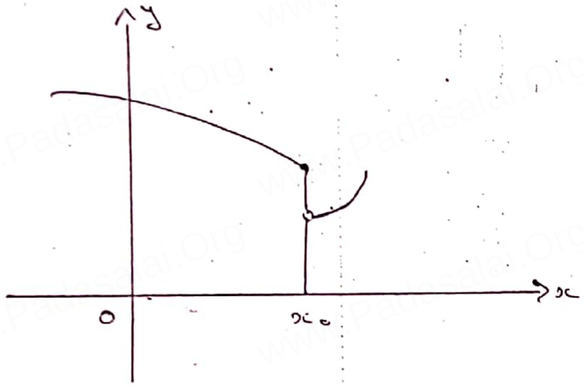
To make f(x) is continuous at $x=1$

$$\lim_{x\to1} f(x) = f(1) = 2/3$$

$$\Rightarrow \boxed{f(1) = 2/3}$$

(5) State how continuity is destroyed at $x=x_0$ for each of the following graphs.

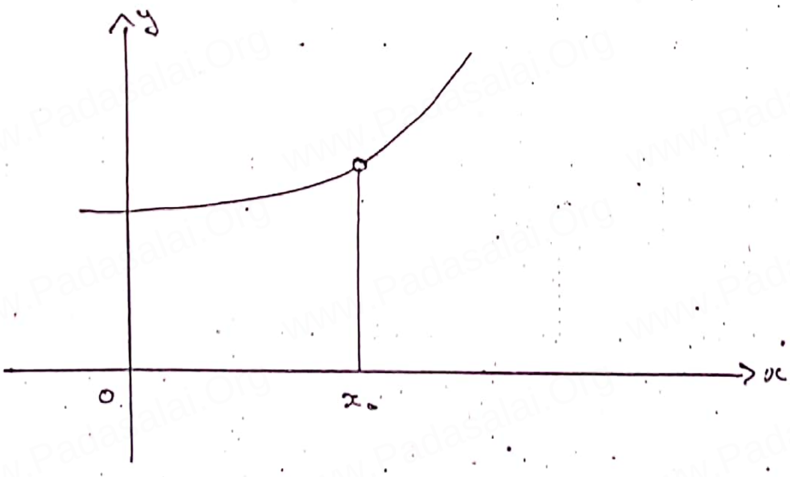
(a)




$\because$ The left limit and the right limit does not coincide at $x=x_0$

$\Rightarrow$ The continuity destroyed at $x=x_0$

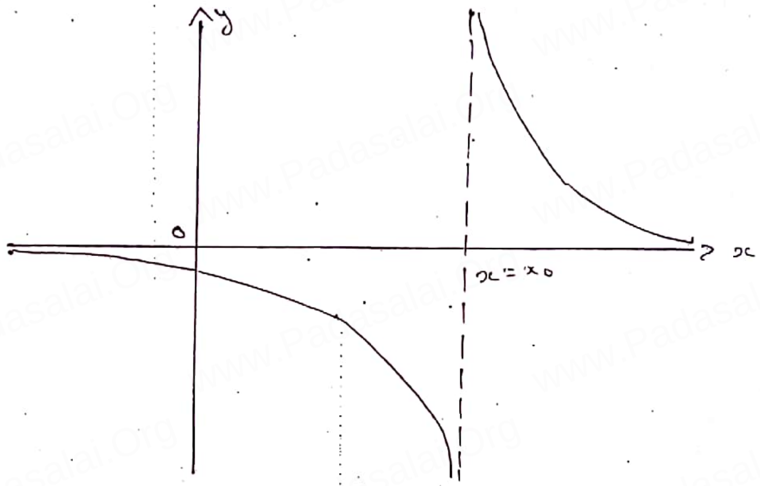
(b)




$\because$ The function is not defined at $x=x_0$

$\Rightarrow$ The continuity destroyed at $x=x_0$

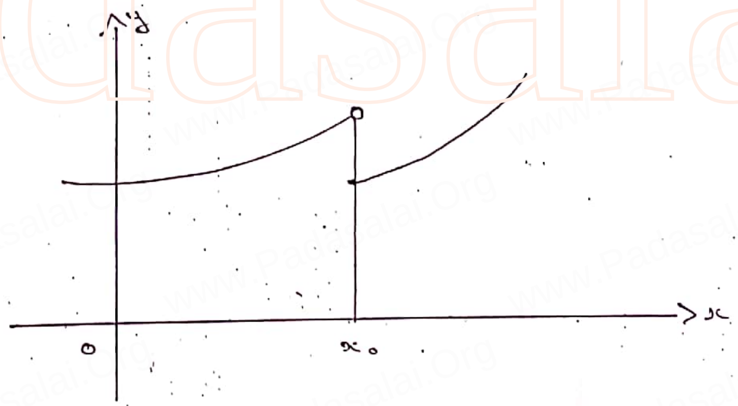
(c)




∵ The limit does not exists at $x = x_0$

$\Rightarrow$ The continuity destroyed at $x = x_0$

(d)




∵ The left limit and the right limit does not coincide at $x = x_0$

$\Rightarrow$ The continuity destroyed at $x = x_0$

$f'(1^+) = f'(1^-)$

$-1 = 2a$

$$\boxed{a = -\frac{1}{2}}$$

Now, $f(x)$ is continuous at $x=1$

$f(1^-) = f(1^+)$

$a - b = 1$

$b = a - 1$

$= -\frac{1}{2} - 1$

$$\boxed{b = -\frac{3}{2}}$$

$$\boxed{\therefore a = -\frac{1}{2},\ b = -\frac{3}{2}}$$

~~satisfies~~ Ans (3)

$$f(x) = \begin{cases} \frac{1}{|x|}, & x \le -1 \\ ax^2 - b, & -1 < x < 1 \\ \frac{1}{|x|}, & x \ge 1 \end{cases}$$

$$f(x) = \begin{cases} -\frac{1}{x}; & x \le -1 \\ ax^2 - b; & -1 < x < 1 \\ \frac{1}{x}; & x \ge 1 \end{cases}$$

$$f'(x) = \begin{cases} +\frac{1}{x^2}, & x \le -1 \\ 2ax, & -1 \le x < 1 \\ -\frac{1}{x^2}, & x \ge 1 \end{cases}$$

25) The number of points in R in which the function $f(x) = |x-1| + |x-3| + \sin x$ is not differentiable.

Solution:

$$f(x) = |x-1| + |x-3| + \sin x$$

$$= \begin{cases} -(x-1) - (x-3) + \sin x; & x < 1 \\ x-1 - (x-3) + \sin x; & 1 \le x < 3 \\ x-1 + (x-3) + \sin x; & x \ge 3 \end{cases}$$

$$f(x) = \begin{cases} -2x + 4 + \sin x; & x < 1 \\ 2 + \sin x; & 1 \le x < 3 \\ 2x - 4 + \sin x; & x \ge 3 \end{cases}$$

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$$f'(x) = \begin{cases} -2 + \cos x & ; x < 1 \\ \cos x & ; 1 \le x < 3 \\ 2 + \cos x & ; x \ge 3 \end{cases}$$

$\boxed{f'(1^-) = -2 + \cos 1}$

$\boxed{f'(1^+) = \cos 1}$

$\therefore f'(1^-) \neq f'(1^+)$

$\therefore f(x)$ is not differentiable at $x = 1$

$f'(3^-) = \cos 3$

$f'(3^+) = 2 + \cos 3$

$\therefore f'(3^-) \neq f'(3^+)$

$\therefore f(x)$ is not differentiable at $x = 3$.

Ans: (2)

$\therefore f(x)$ is not differentiable at two points in R