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Adjoint of a square matrix

The adjoint matrix of A is defined as the transpose of the matrix of cofactors of A . It is denoted by $\text{adj } A$ $\boxed{\text{adj } A = [A_{ij}]^T}$

For every square matrix of order n ,
 $A(\text{adj } A) = (\text{adj } A)A = |A|I_n$.

Example 1.1

If $A = \begin{bmatrix} 8 & -6 & 2 \\ -6 & 7 & -4 \\ 2 & -4 & 3 \end{bmatrix}$ verify that $A(\text{adj } A) = (\text{adj } A)A = |A|I_3$

Solution

$$\begin{aligned} |A| &= \begin{vmatrix} 8 & -6 & 2 \\ -6 & 7 & -4 \\ 2 & -4 & 3 \end{vmatrix} = 8(21-16) + 6(-18+8) + 2(24-14) \\ &= 8(5) + 6(-10) + 2(10) \\ &= 40 - 60 + 20 \\ &= 0 \end{aligned}$$

$$\text{adj } A = [A_{ij}]^T$$

$$= \begin{bmatrix} 21-16 & -(-18+8) & 24-14 \\ -(-18+8) & 24-4 & -(-32+12) \\ 24-14 & -(-32+12) & 56-36 \end{bmatrix}^T$$

$$= \begin{bmatrix} 5 & 10 & 10 \\ 10 & 20 & 20 \\ 10 & 20 & 20 \end{bmatrix}^T$$

$$= \begin{bmatrix} 5 & 10 & 10 \\ 10 & 20 & 20 \\ 10 & 20 & 20 \end{bmatrix}$$

(2)

$$\begin{aligned}
 A(\text{adj } A) &= \begin{bmatrix} 8 & -6 & 2 \\ -6 & 7 & -4 \\ 2 & -4 & 3 \end{bmatrix} \begin{bmatrix} 5 & 10 & 10 \\ 10 & 20 & 20 \\ 10 & 20 & 20 \end{bmatrix} \\
 &= \begin{bmatrix} 40-60+20 & 80-120+40 & 80-120+40 \\ -30+70-40 & -60+140-80 & -60+140-80 \\ 10-40+30 & 20-80+60 & 20-80+60 \end{bmatrix} \\
 &= \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \\
 &= 0 \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \\
 &= |A| I_3 \text{ ————— (1)}
 \end{aligned}$$

$$\begin{aligned}
 (\text{adj } A)A &= \begin{bmatrix} 5 & 10 & 10 \\ 10 & 20 & 20 \\ 10 & 20 & 20 \end{bmatrix} \begin{bmatrix} 8 & -6 & 2 \\ -6 & 7 & -4 \\ 2 & -4 & 3 \end{bmatrix} \\
 &= \begin{bmatrix} 40-60+20 & -30+70-40 & 10-40+30 \\ 80-120+40 & -60+140-80 & 20-80+60 \\ 80-120+40 & -60+140-80 & 20-80+60 \end{bmatrix} \\
 &= \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \\
 &= 0 \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \\
 &= |A| I_3 \text{ ————— (2)}
 \end{aligned}$$

From (1) and (2)

$$A(\text{adj } A) = (\text{adj } A)A = |A| I_3$$

02/06/20

(3)

Definition of Inverse of a Square matrix

Inverse of A

$$A^{-1} = \frac{1}{|A|} \text{adj } A$$

$|A| \neq 0$, i.e., A is a nonsingular matrix

Note: A singular matrix has no inverse

Example 1.2

If $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ is non singular, find A^{-1}

Given A is non singular

$\therefore A^{-1}$ exists

$$A^{-1} = \frac{1}{|A|} \text{adj } A$$

$$|A| = \begin{vmatrix} a & b \\ c & d \end{vmatrix} \\ = ad - bc$$

$$\text{adj } A = [A_{ij}]^T \\ = \begin{bmatrix} d & -c \\ -b & a \end{bmatrix}^T \\ = \begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$$

$$A^{-1} = \frac{1}{ad-bc} \begin{bmatrix} d & -b \\ -c & a \end{bmatrix} //$$

(4)

Example 1.3

Find the inverse of the matrix $\begin{bmatrix} 2 & -1 & 3 \\ -5 & 3 & 1 \\ -3 & 2 & 3 \end{bmatrix}$ Soln

$$\text{Let } A = \begin{bmatrix} 2 & -1 & 3 \\ -5 & 3 & 1 \\ -3 & 2 & 3 \end{bmatrix}$$

$$A^{-1} = \frac{1}{|A|} \text{adj } A$$

$$\begin{aligned} |A| &= \begin{vmatrix} 2 & -1 & 3 \\ -5 & 3 & 1 \\ -3 & 2 & 3 \end{vmatrix} = +2(9-2) + 1(-15+3) + 3(-10+9) \\ &= 2(7) + 1(-12) + 3(-1) \\ &= 14 - 12 - 3 \\ &= -1 \neq 0 \end{aligned}$$

A is non singular $\therefore A^{-1}$ is exist

$$\begin{aligned} \text{adj } A &= [A_{ij}]^T \\ &= \begin{bmatrix} 9-2 & -(-15+3) & -10+9 \\ -(-3-6) & 6+9 & -(4-3) \\ -1-9 & -(2+15) & 6-5 \end{bmatrix}^T \\ &= \begin{bmatrix} 7 & 12 & -1 \\ 9 & 15 & -1 \\ -10 & -17 & 1 \end{bmatrix}^T \\ &= \begin{bmatrix} 7 & 9 & -10 \\ 12 & 15 & -17 \\ -1 & -1 & 1 \end{bmatrix} \end{aligned}$$

$$A^{-1} = \frac{1}{-1} \begin{bmatrix} 7 & 9 & -10 \\ 12 & 15 & -17 \\ -1 & -1 & 1 \end{bmatrix} = \begin{bmatrix} -7 & -9 & 10 \\ -12 & -15 & 17 \\ 1 & 1 & -1 \end{bmatrix} //$$

(5)

Ex 1.1

3) If $F(\alpha) = \begin{bmatrix} \cos \alpha & 0 & \sin \alpha \\ 0 & 1 & 0 \\ -\sin \alpha & 0 & \cos \alpha \end{bmatrix}$ Show that $[F(\alpha)]^{-1} = F(-\alpha)$

Solution

$$[F(\alpha)]^{-1} = \frac{1}{|F(\alpha)|} \text{adj } F(\alpha)$$

$$|F(\alpha)| = \begin{vmatrix} \cos \alpha & 0 & \sin \alpha \\ 0 & 1 & 0 \\ -\sin \alpha & 0 & \cos \alpha \end{vmatrix}$$

$$= \cos \alpha (\cos \alpha - 0) + \sin \alpha (0 + \sin \alpha)$$

$$= \cos^2 \alpha + \sin^2 \alpha$$

$$= 1 \neq 0$$

$\therefore F(\alpha)$ is a non singular matrix

$\therefore$ Inverse of $F(\alpha)$ exists

$$\text{adj } F(\alpha) = \begin{bmatrix} \cos \alpha - 0 & -(0) & 0 + \sin \alpha \\ -(0) & \cos^2 \alpha + \sin^2 \alpha & -(0) \\ 0 - \sin \alpha & -(0) & \cos \alpha - 0 \end{bmatrix}^T$$

$$= \begin{bmatrix} \cos \alpha & 0 & \sin \alpha \\ 0 & 1 & 0 \\ -\sin \alpha & 0 & \cos \alpha \end{bmatrix}^T$$

$$= \begin{bmatrix} \cos \alpha & 0 & -\sin \alpha \\ 0 & 1 & 0 \\ \sin \alpha & 0 & \cos \alpha \end{bmatrix}$$

$$[F(\alpha)]^{-1} = \frac{1}{1} \begin{bmatrix} \cos \alpha & 0 & -\sin \alpha \\ 0 & 1 & 0 \\ \sin \alpha & 0 & \cos \alpha \end{bmatrix}$$

$$= \begin{bmatrix} \cos \alpha & 0 & -\sin \alpha \\ 0 & 1 & 0 \\ \sin \alpha & 0 & \cos \alpha \end{bmatrix} \quad \text{--- (1)}$$

(6)

$$F(-\alpha) = \begin{bmatrix} \cos(-\alpha) & 0 & \sin(-\alpha) \\ 0 & 1 & 0 \\ -\sin(-\alpha) & 0 & \cos(-\alpha) \end{bmatrix}$$

$$= \begin{bmatrix} \cos \alpha & 0 & -\sin \alpha \\ 0 & 1 & 0 \\ \sin \alpha & 0 & \cos \alpha \end{bmatrix} \quad \text{--- (2)}$$

$$\begin{aligned} \cos(-\theta) &= \cos \theta \\ \sin(-\theta) &= -\sin \theta \end{aligned}$$

From (1) and (2)

$$[F(\alpha)]^{-1} = F(-\alpha)$$

Properties of Inverse of a matrix.

$$* |A^{-1}| = \frac{1}{|A|}$$

$$* (A^{-1})^T = (A^T)^{-1}$$

$$* (AB)^{-1} = B^{-1}A^{-1}$$

$$* (A^{-1})^{-1} = A$$

$$* (\text{adj } A)^{-1} = \frac{1}{|A|} \cdot A$$

$$* |\text{adj } A| = |A|^{n-1}$$

$$* \text{adj}(\text{adj } A) = |A|^{n-2} A$$

(9)

If A is a nonsingular matrix of order 3,

then $|adj A| = |A|^{3-1}$

$$|adj A| = |A|^2$$

$$|A| = \pm \sqrt{|adj A|}$$

$$A^{-1} = \frac{1}{|A|} adj A$$

$$A^{-1} = \pm \frac{1}{\sqrt{|adj A|}} adj A$$

By property

$$adj(adj A) = |A|^{n-2} \cdot A$$

Here $n = 3$

$$adj(adj A) = |A|^{3-2} \cdot A$$

$$adj(adj A) = |A| A$$

$$A = \frac{1}{|A|} adj(adj A)$$

$$A = \pm \frac{1}{\sqrt{|adj A|}} adj(adj A)$$

(8)

Example 1.4

If A is a nonsingular matrix of odd order,
Prove that $|\text{adj} A|$ is positive

Soln

Let A be a nonsingular matrix of
order $n = 2m + 1$, where $m = 0, 1, 2, 3, \dots$

We know that, $|\text{adj} A| = |A|^{n-1}$

$$|\text{adj} A| = |A|^{(2m+1)-1}$$

$$= |A|^{2m}$$

$|A|^{2m}$ is always positive

$\therefore |\text{adj} A|$ is positive.

Example 1.5

Find a matrix A if $\text{adj} A = \begin{bmatrix} 7 & 7 & -7 \\ -1 & 11 & 7 \\ 11 & 5 & 7 \end{bmatrix}$

Soln

A is a matrix of order 3

$$\therefore A = \pm \frac{1}{\sqrt{|\text{adj} A|}} \text{adj}(\text{adj} A)$$

$$|\text{adj} A| = \begin{vmatrix} 7 & 7 & -7 \\ -1 & 11 & 7 \\ 11 & 5 & 7 \end{vmatrix} = 7(77 - 35) - 7(-7 - 77) - 7(-5 - 121)$$

$$= 7(42) - 7(-84) - 7(-126)$$

$$= 294 + 588 + 882$$

$$= 1764 > 0$$

3/6/2020

9

$$\sqrt{|\text{adj } A|} = \sqrt{1764}$$

$$= 42$$

$$\begin{array}{r} 42 \\ 4 \overline{) 1764} \\ \underline{16} \\ 164 \\ \underline{164} \\ 0 \end{array}$$

$$\text{adj}(\text{adj } A) = \begin{bmatrix} +(77-35) & -(-7-77) & +(-5-121) \\ -(49+35) & +(49+77) & -(35-77) \\ +(49+77) & -(49-7) & +(77+7) \end{bmatrix}^T$$

$$= \begin{bmatrix} 42 & 84 & -126 \\ -84 & 126 & 42 \\ 126 & -42 & 84 \end{bmatrix}^T$$

$$= \begin{bmatrix} 42 & -84 & 126 \\ 84 & 126 & -42 \\ -126 & 42 & 84 \end{bmatrix}$$

$$A = \pm \frac{1}{\sqrt{|\text{adj } A|}} \text{adj}(\text{adj } A)$$

$$= \pm \frac{1}{42} \begin{bmatrix} 42 & -84 & 126 \\ 84 & 126 & -42 \\ -126 & 42 & 84 \end{bmatrix}$$

$$= \pm \begin{bmatrix} 1 & -2 & 3 \\ 2 & 3 & -1 \\ -3 & 1 & 2 \end{bmatrix}$$

Example 1.6

If $\text{adj} A = \begin{bmatrix} -1 & 2 & 2 \\ 1 & 1 & 2 \\ 2 & 2 & 1 \end{bmatrix}$ find A^{-1}

Soln

$$A^{-1} = \pm \frac{1}{\sqrt{|\text{adj} A|}} \text{adj} A$$

$$|\text{adj} A| = \begin{vmatrix} -1 & 2 & 2 \\ 1 & 1 & 2 \\ 2 & 2 & 1 \end{vmatrix}$$

$$= -1(1-4) - 2(1-4) + 2(2-2)$$

$$= -1(-3) - 2(-3) + 2(0)$$

$$= 3 + 6$$

$$= 9$$

$$\sqrt{|\text{adj} A|} = \sqrt{9}$$

$$= 3$$

$$A^{-1} = \pm \frac{1}{\sqrt{|\text{adj} A|}} \text{adj} A$$

$$= \pm \frac{1}{3} \begin{bmatrix} -1 & 2 & 2 \\ 1 & 1 & 2 \\ 2 & 2 & 1 \end{bmatrix}$$

Exercise 1.1 (8, 9, 10) $\rightarrow$ H.W.

Result : $\text{adj}(A^T) = (\text{adj} A)^T$

Example 1.7

If A is symmetric, prove that $\text{adj} A$ is also symmetric ✓

Solution

A is symmetric

$$\therefore A^T = A$$

By result : $\text{adj}(A^T) = (\text{adj} A)^T$

$$\text{adj}(A) = (\text{adj} A)^T$$

$\therefore \text{adj} A$ is symmetric.

Example 1.8

Verify the property $(A^T)^{-1} = (A^{-1})^T$, $A = \begin{bmatrix} 2 & 9 \\ 1 & 7 \end{bmatrix}$

Solution

$$A^T = \begin{bmatrix} 2 & 1 \\ 9 & 7 \end{bmatrix}$$

$$(A^T)^{-1} = \frac{1}{|A^T|} \text{adj}(A^T)$$

$$\begin{aligned} |A^T| &= \begin{vmatrix} 2 & 1 \\ 9 & 7 \end{vmatrix} \\ &= 14 - 9 \\ &= 5 \end{aligned}$$

$$\text{adj}(A^T) = \begin{bmatrix} 7 & -9 \\ -1 & 2 \end{bmatrix}^T$$

$$= \begin{bmatrix} 7 & -1 \\ -9 & 2 \end{bmatrix}$$

$$(A^T)^{-1} = \frac{1}{5} \begin{bmatrix} 7 & -1 \\ -9 & 2 \end{bmatrix} \quad \text{--- (1)}$$

$$A^{-1} = \frac{1}{|A|} \text{adj} A$$

$$|A| = \begin{vmatrix} 2 & 9 \\ 1 & 7 \end{vmatrix}$$

$$= 14 - 9$$

$$= 5$$

$$\text{adj} A = \begin{bmatrix} 7 & -1 \\ -9 & 2 \end{bmatrix}^T$$

$$= \begin{bmatrix} 7 & -9 \\ -1 & 2 \end{bmatrix}$$

$$A^{-1} = \frac{1}{5} \begin{bmatrix} 7 & -9 \\ -1 & 2 \end{bmatrix}$$

$$(A^{-1})^T = \frac{1}{5} \begin{bmatrix} 7 & -1 \\ -9 & 2 \end{bmatrix} \quad \text{--- (2)}$$

From (1) and (2)

$$(A^T)^{-1} = (A^{-1})^T$$

Example 1.9

(13)

Verify $(AB)^{-1} = B^{-1}A^{-1}$ with $A = \begin{bmatrix} 0 & -3 \\ 1 & 4 \end{bmatrix}$; $B = \begin{bmatrix} -2 & -3 \\ 0 & -1 \end{bmatrix}$ Solution:

$$AB = \begin{bmatrix} 0 & -3 \\ 1 & 4 \end{bmatrix} \begin{bmatrix} -2 & -3 \\ 0 & -1 \end{bmatrix}$$

$$= \begin{bmatrix} 0+0 & 0+3 \\ -2+0 & -3-4 \end{bmatrix}$$

$$= \begin{bmatrix} 0 & 3 \\ -2 & -7 \end{bmatrix}$$

$$(AB)^{-1} = \frac{1}{|AB|} \text{adj}(AB)$$

$$|AB| = \begin{vmatrix} 0 & 3 \\ -2 & -7 \end{vmatrix}$$

$$= 0+6$$

$$= 6$$

$$\text{adj}(AB) = \begin{bmatrix} -7 & -(-2) \\ -3 & 0 \end{bmatrix}^T$$

$$= \begin{bmatrix} -7 & 2 \\ -3 & 0 \end{bmatrix}^T$$

$$= \begin{bmatrix} -7 & -3 \\ 2 & 0 \end{bmatrix}$$

$$(AB)^{-1} = \frac{1}{6} \begin{bmatrix} -7 & -3 \\ 2 & 0 \end{bmatrix} \text{--- (1)}$$

$$B^{-1} = \frac{1}{|B|} \text{adj } B$$

$$|B| = \begin{vmatrix} -2 & -3 \\ 0 & -1 \end{vmatrix}$$

$$= 2-0 = 2$$

$$\text{adj } B = \begin{bmatrix} -1 & 0 \\ -(-3) & -2 \end{bmatrix}^T$$

$$= \begin{bmatrix} -1 & 0 \\ 3 & -2 \end{bmatrix}^T$$

$$= \begin{bmatrix} -1 & 3 \\ 0 & -2 \end{bmatrix}$$

$$B^{-1} = \frac{1}{2} \begin{bmatrix} -1 & 3 \\ 0 & -2 \end{bmatrix}$$

$$A^{-1} = \frac{1}{|A|} \text{adj } A$$

$$|A| = \begin{vmatrix} 0 & -3 \\ 1 & 4 \end{vmatrix} = 0+3 = 3$$

$$\text{adj } A = \begin{bmatrix} 4 & -1 \\ 3 & 0 \end{bmatrix}^T$$

$$= \begin{bmatrix} 4 & 3 \\ -1 & 0 \end{bmatrix}$$

$$A^{-1} = \frac{1}{3} \begin{bmatrix} 4 & 3 \\ -1 & 0 \end{bmatrix}$$

$$B^{-1}A^{-1} = \frac{1}{2} \begin{bmatrix} -1 & 3 \\ 0 & -2 \end{bmatrix} \cdot \frac{1}{3} \begin{bmatrix} 4 & 3 \\ -1 & 0 \end{bmatrix}$$

$$= \frac{1}{6} \begin{bmatrix} -4-3 & -3+0 \\ 0+2 & 0+0 \end{bmatrix}$$

$$= \frac{1}{6} \begin{bmatrix} -7 & -3 \\ 2 & 0 \end{bmatrix} \text{--- (2)}$$

From (1) and (2) $(AB)^{-1} = B^{-1}A^{-1}$

Ex 1.1 (7) H.W

Example 1.10

If $A = \begin{bmatrix} 4 & 3 \\ 2 & 5 \end{bmatrix}$ find x and y such that

$$A^2 + xA + yI_2 = O_2. \text{ find } A^{-1}$$

Solution

$$A^2 = A \times A$$

$$= \begin{bmatrix} 4 & 3 \\ 2 & 5 \end{bmatrix} \begin{bmatrix} 4 & 3 \\ 2 & 5 \end{bmatrix}$$

$$= \begin{bmatrix} 16+6 & 12+15 \\ 8+10 & 6+25 \end{bmatrix}$$

$$= \begin{bmatrix} 22 & 27 \\ 18 & 31 \end{bmatrix}$$

$$A^2 + xA + yI_2 = O_2 \Rightarrow \begin{bmatrix} 22 & 27 \\ 18 & 31 \end{bmatrix} + x \begin{bmatrix} 4 & 3 \\ 2 & 5 \end{bmatrix} + y \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$$

$$\begin{bmatrix} 22 & 27 \\ 18 & 31 \end{bmatrix} + \begin{bmatrix} 4x & 3x \\ 2x & 5x \end{bmatrix} + \begin{bmatrix} y & 0 \\ 0 & y \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$$

$$\begin{bmatrix} 22+4x+y & 27+3x+0 \\ 18+2x+0 & 31+5x+y \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$$

$$27+3x=0$$

$$3x = -27$$

$$\boxed{x = -9}$$

$$22+4x+y=0$$

$$22+4(-9)+y=0$$

(15)

$$2x - 36 + y = 0$$

$$-14 + y = 0$$

$$\boxed{y = 14}$$

$$\therefore A^2 - 9A + 14I_2 = O_2$$

Post Multiplying this eqn by A^{-1}

$$A^2 A^{-1} - 9A A^{-1} + 14I_2 A^{-1} = O_2$$

$$A(A A^{-1}) - 9(A A^{-1}) + 14I_2 A^{-1} = O_2$$

$$A I_2 - 9I_2 + 14A^{-1} = O_2$$

$$A - 9I_2 + 14A^{-1} = O_2$$

$$14A^{-1} = 9I_2 - A$$

$$= 9 \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} - \begin{bmatrix} 4 & 3 \\ 2 & 5 \end{bmatrix}$$

$$= \begin{bmatrix} 9 & 0 \\ 0 & 9 \end{bmatrix} - \begin{bmatrix} 4 & 3 \\ 2 & 5 \end{bmatrix}$$

$$= \begin{bmatrix} 5 & -3 \\ -2 & 4 \end{bmatrix}$$

$$A^{-1} = \frac{1}{14} \begin{bmatrix} 5 & -3 \\ -2 & 4 \end{bmatrix}$$

Ex 1.1

4) If $A = \begin{bmatrix} 5 & 3 \\ -1 & -2 \end{bmatrix}$ show that $A^2 - 3A - 7I_2 = O_2$
Hence find A^{-1}

Solution

$$A^2 = A \times A$$

$$= \begin{bmatrix} 5 & 3 \\ -1 & -2 \end{bmatrix} \begin{bmatrix} 5 & 3 \\ -1 & -2 \end{bmatrix}$$

$$= \begin{bmatrix} 25-3 & 15-6 \\ -5+2 & -3+4 \end{bmatrix}$$

$$= \begin{bmatrix} 22 & 9 \\ -3 & 1 \end{bmatrix}$$

$$A^2 - 3A - 7I_2 = \begin{bmatrix} 22 & 9 \\ -3 & 1 \end{bmatrix} - 3 \begin{bmatrix} 5 & 3 \\ -1 & -2 \end{bmatrix} - 7 \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 22 & 9 \\ -3 & 1 \end{bmatrix} - \begin{bmatrix} 15 & 9 \\ -3 & -6 \end{bmatrix} - \begin{bmatrix} 7 & 0 \\ 0 & 7 \end{bmatrix}$$

$$= \begin{bmatrix} 22-15-7 & 9-9-0 \\ -3+3-0 & 1+6-7 \end{bmatrix}$$

$$= \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$$

$$= O_2.$$

$$A^2 - 3A - 7I_2 = O_2$$

Post multiplying by A^{-1}

$$A^2 A^{-1} - 3A A^{-1} - 7I_2 A^{-1} = O_2$$

$$A(AA^T) - 3(AA^T) - 7I_2 A^T = O_2$$

$$AI_2 - 3I_2 - 7A^T = O_2$$

$$A - 3I_2 - 7A^T = O_2$$

$$7A^T = A - 3I_2$$

$$7A^T = \begin{bmatrix} 5 & 3 \\ -1 & -2 \end{bmatrix} - \begin{bmatrix} 3 & 0 \\ 0 & 3 \end{bmatrix}$$

$$= \begin{bmatrix} 5-3 & 3-0 \\ -1-0 & -2-3 \end{bmatrix}$$

$$= \begin{bmatrix} 2 & 3 \\ -1 & -5 \end{bmatrix}$$

$$A^T = \frac{1}{7} \begin{bmatrix} 2 & 3 \\ -1 & -5 \end{bmatrix}$$

Ex 1.1

5) If $A = \frac{1}{9} \begin{bmatrix} -8 & 1 & 4 \\ 4 & 4 & 7 \\ 1 & -8 & 4 \end{bmatrix}$ prove that $A^{-1} = A^T$

Solution

$$A = \frac{1}{9} \begin{bmatrix} -8 & 1 & 4 \\ 4 & 4 & 7 \\ 1 & -8 & 4 \end{bmatrix}$$

$$= \begin{bmatrix} -8/9 & 1/9 & 4/9 \\ 4/9 & 4/9 & 7/9 \\ 1/9 & -8/9 & 4/9 \end{bmatrix}$$

$$A^{-1} = \frac{1}{|A|} \text{adj } A$$

$$|A| = \begin{vmatrix} -8/9 & 1/9 & 4/9 \\ 4/9 & 4/9 & 7/9 \\ 1/9 & -8/9 & 4/9 \end{vmatrix}$$

$$= \left(\frac{1}{9}\right)^3 \begin{vmatrix} -8 & 1 & 4 \\ 4 & 4 & 7 \\ 1 & -8 & 4 \end{vmatrix}$$

$$= \left(\frac{1}{9}\right)^3 \left[-8(16 + 56) - 1(16 - 7) + 4(-32 - 4) \right]$$

$$= \left(\frac{1}{9}\right)^3 \left[-8(72) - 1(9) + 4(-36) \right]$$

$$= \left(\frac{1}{9}\right)^3 \left[-576 - 9 - 144 \right]$$

$$= \frac{1}{729} \times -729$$

$$= -1 \neq 0$$

$$\text{adj } A = [A_{ij}]^T$$

$$= \begin{bmatrix} +\left(\frac{16}{81} + \frac{56}{81}\right) & -\left(\frac{16}{81} - \frac{7}{81}\right) & +\left(\frac{-32}{81} - \frac{4}{81}\right) \\ -\left(\frac{4}{81} + \frac{32}{81}\right) & +\left(\frac{32}{81} - \frac{4}{81}\right) & -\left(\frac{64}{81} - \frac{1}{81}\right) \\ +\left(\frac{7}{81} - \frac{16}{81}\right) & -\left(\frac{-56}{81} - \frac{16}{81}\right) & +\left(\frac{-32}{81} - \frac{4}{81}\right) \end{bmatrix}^T$$

$$= \begin{bmatrix} \frac{72}{81} & -\frac{9}{81} & -\frac{36}{81} \\ -\frac{36}{81} & -\frac{36}{81} & -\frac{63}{81} \\ -\frac{9}{81} & \frac{72}{81} & -\frac{36}{81} \end{bmatrix}^T$$

$$= \begin{bmatrix} 8/9 & -1/9 & -4/9 \\ -4/9 & -4/9 & -7/9 \\ -1/9 & 8/9 & -4/9 \end{bmatrix}^T$$

$$= \begin{bmatrix} 8/9 & -4/9 & -1/9 \\ -1/9 & -4/9 & 8/9 \\ -4/9 & -7/9 & -4/9 \end{bmatrix}$$

$$= \frac{1}{9} \begin{bmatrix} 8 & -4 & -1 \\ -1 & -4 & 8 \\ -4 & -7 & -4 \end{bmatrix}$$

$$A^{-1} = \frac{1}{-1} \cdot \frac{1}{9} \begin{bmatrix} 8 & -4 & -1 \\ -1 & -4 & 8 \\ -4 & -7 & -4 \end{bmatrix}$$

$$= \frac{1}{9} \begin{bmatrix} -8 & 4 & 1 \\ 1 & 4 & -8 \\ 4 & 7 & 4 \end{bmatrix} \quad \text{--- (1)}$$

$$A^T = \frac{1}{9} \begin{bmatrix} -8 & 4 & 1 \\ 1 & 4 & -8 \\ 4 & 7 & 4 \end{bmatrix} \quad \text{--- (2)}$$

From (1) and (2)

$$A^{-1} = A^T$$

Ex 1.1

(11) $A = \begin{bmatrix} 1 & \tan x \\ -\tan x & 1 \end{bmatrix}$ Show that $A^T A^{-1} = \begin{bmatrix} \cos 2x & -\sin 2x \\ \sin 2x & \cos 2x \end{bmatrix}$

Solution

$$A = \begin{bmatrix} 1 & \tan x \\ -\tan x & 1 \end{bmatrix}$$

$$A^{-1} = \frac{1}{|A|} \text{adj} A$$

$$|A| = \begin{vmatrix} 1 & \tan x \\ -\tan x & 1 \end{vmatrix}$$

$$= 1 + \tan^2 x$$

$$= \sec^2 x$$

$$\sec^2 x - \tan^2 x = 1$$

$$\sec^2 x = 1 + \tan^2 x$$

$$\text{adj} A = [A_{ij}]^T$$

$$= \begin{bmatrix} 1 & -(-\tan x) \\ -(\tan x) & 1 \end{bmatrix}^T$$

$$= \begin{bmatrix} 1 & \tan x \\ -\tan x & 1 \end{bmatrix}^T$$

$$= \begin{bmatrix} 1 & -\tan x \\ \tan x & 1 \end{bmatrix}$$

$$A^{-1} = \frac{1}{\sec^2 x} \begin{bmatrix} 1 & -\tan x \\ \tan x & 1 \end{bmatrix}$$

$$A^{-1} = \begin{bmatrix} 1 & -\tan x \\ \tan x & 1 \end{bmatrix}$$

(21)

$$A^T A^{-1} = \begin{bmatrix} 1 & -\tan x \\ \tan x & 1 \end{bmatrix} \cdot \frac{1}{\sec^2 x} \begin{bmatrix} 1 & -\tan x \\ \tan x & 1 \end{bmatrix}$$

$$= \frac{1}{\sec^2 x} \begin{bmatrix} 1 - \tan^2 x & -\tan x - \tan x \\ \tan x + \tan x & -\tan^2 x + 1 \end{bmatrix}$$

$$= \frac{1}{\sec^2 x} \begin{bmatrix} 1 - \tan^2 x & -2 \tan x \\ 2 \tan x & 1 - \tan^2 x \end{bmatrix}$$

$$= \begin{bmatrix} \frac{1}{\sec^2 x} - \frac{\tan^2 x}{\sec^2 x} & -\frac{2 \tan x}{\sec^2 x} \\ \frac{2 \tan x}{\sec^2 x} & \frac{1}{\sec^2 x} - \frac{\tan^2 x}{\sec^2 x} \end{bmatrix}$$

$$= \begin{bmatrix} \cos^2 x - \sin^2 x & -2 \sin x \cos x \\ 2 \sin x \cos x & \cos^2 x - \sin^2 x \end{bmatrix}$$

$$= \begin{bmatrix} \cos 2x & -\sin 2x \\ \sin 2x & \cos 2x \end{bmatrix}$$

$$\frac{1}{\sec^2 x} = \cos^2 x$$

$$\frac{\tan^2 x}{\sec^2 x} = \frac{\sin^2 x}{\cos^2 x} \times \cos^2 x$$

$$= \sin^2 x$$

$$\frac{\tan x}{\sec^2 x} = \frac{\sin x}{\cos^2 x} \times \cos^2 x$$

$$= \sin x \cos x$$

Ex 1.1

(2) Find the matrix A, for which.

$$A \begin{bmatrix} 5 & 3 \\ -1 & -2 \end{bmatrix} = \begin{bmatrix} 14 & 7 \\ 7 & 7 \end{bmatrix}$$

Solution:

$$A \begin{bmatrix} 5 & 3 \\ -1 & -2 \end{bmatrix} = \begin{bmatrix} 14 & 7 \\ 7 & 7 \end{bmatrix}$$

$$\text{Let } B = \begin{bmatrix} 5 & 3 \\ -1 & -2 \end{bmatrix} ; C = \begin{bmatrix} 14 & 7 \\ 7 & 7 \end{bmatrix}$$

$$AB = C$$

Post multiplying by B^{-1} on both sides

$$AB B^{-1} = C B^{-1}$$

$$A I = C B^{-1}$$

$$A = C B^{-1} \text{ — (i)}$$

$$\left[\begin{array}{l} \because B B^{-1} = I \\ A I = A \end{array} \right]$$

$$B^{-1} = \frac{1}{|B|} \text{adj } B$$

$$|B| = \begin{vmatrix} 5 & 3 \\ -1 & -2 \end{vmatrix}$$

$$= -10 + 3$$

$$= -7$$

$$\text{adj } B = [B_{ij}]^T$$

$$= \begin{bmatrix} +(-2) & -(-1) \\ -(3) & +(5) \end{bmatrix}^T$$

$$= \begin{bmatrix} -2 & 1 \\ -3 & 5 \end{bmatrix}^T$$

$$= \begin{bmatrix} -2 & -3 \\ 1 & 5 \end{bmatrix}$$

$$B^{-1} = \frac{1}{-7} \begin{bmatrix} -2 & -3 \\ 1 & 5 \end{bmatrix}$$

$$\Rightarrow A = \begin{bmatrix} 14 & 7 \\ 7 & 7 \end{bmatrix} \frac{1}{-7} \begin{bmatrix} -2 & -3 \\ 1 & 5 \end{bmatrix}$$

$$= \begin{bmatrix} 2 & 1 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} 2 & 3 \\ -1 & -5 \end{bmatrix}$$

$$= \begin{bmatrix} 4-1 & 6-5 \\ 2-1 & 3-5 \end{bmatrix}$$

$$= \begin{bmatrix} 3 & 1 \\ 1 & -2 \end{bmatrix}$$

$$A = \underline{\underline{\begin{bmatrix} 3 & 1 \\ 1 & -2 \end{bmatrix}}}$$

13. Given $A = \begin{bmatrix} 1 & -1 \\ 2 & 0 \end{bmatrix}$ $B = \begin{bmatrix} 3 & -2 \\ 1 & 1 \end{bmatrix}$; $C = \begin{bmatrix} 1 & 2 \\ 2 & 2 \end{bmatrix}$

find a matrix X such that $A \times B = C$

Solution

Given $A \times B = C$

Pre-multiplying by A^{-1} and post multiplying by B^{-1} on both sides

$$\begin{aligned} A^{-1}(A \times B)B^{-1} &= A^{-1}CB^{-1} \\ (A^{-1}A) \times (BB^{-1}) &= A^{-1}CB^{-1} \\ I \times I &= A^{-1}CB^{-1} \\ X &= A^{-1}CB^{-1} \quad \text{--- (1)} \end{aligned}$$

$$A^{-1} = \frac{1}{|A|} \text{adj } A$$

$$\begin{aligned} |A| &= \begin{vmatrix} 1 & -1 \\ 2 & 0 \end{vmatrix} \\ &= 0 + 2 \\ &= 2 \end{aligned}$$

$$\begin{aligned} \text{adj } A &= [A_{ij}]^T \\ &= \begin{bmatrix} +0 & -(2) \\ -(-1) & +1 \end{bmatrix}^T \\ &= \begin{bmatrix} 0 & -2 \\ 1 & 1 \end{bmatrix}^T \\ &= \begin{bmatrix} 0 & 1 \\ -2 & 1 \end{bmatrix} \end{aligned}$$

$$A^{-1} = \frac{1}{2} \begin{bmatrix} 0 & 1 \\ -2 & 1 \end{bmatrix}$$

$$B^{-1} = \frac{1}{|B|} \text{adj } B$$

$$\begin{aligned} |B| &= \begin{vmatrix} 3 & -2 \\ 1 & 1 \end{vmatrix} \\ &= 3 + 2 = 5 \end{aligned}$$

$$\begin{aligned} \text{adj } B &= [B_{ij}]^T \\ &= \begin{bmatrix} +1 & -(1) \\ -(-2) & +3 \end{bmatrix}^T \\ &= \begin{bmatrix} 1 & 2 \\ -1 & 3 \end{bmatrix} \end{aligned}$$

$$B^{-1} = \frac{1}{5} \begin{bmatrix} 1 & 2 \\ -1 & 3 \end{bmatrix}$$

$$\text{①} \Rightarrow X = A^{-1}CB^{-1}$$

$$\begin{aligned}
 X &= \frac{1}{2} \begin{bmatrix} 0 & 1 \\ -2 & 1 \end{bmatrix} \begin{bmatrix} 1 & 1 \\ 2 & 2 \end{bmatrix} \frac{1}{5} \begin{bmatrix} 1 & 2 \\ -1 & 3 \end{bmatrix} \\
 &= \frac{1}{10} \begin{bmatrix} 0+2 & 0+2 \\ -2+2 & -2+2 \end{bmatrix} \begin{bmatrix} 1 & 2 \\ -1 & 3 \end{bmatrix} \\
 &= \frac{1}{10} \begin{bmatrix} 2 & 2 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} 1 & 2 \\ -1 & 3 \end{bmatrix} \\
 &= \frac{1}{10} \begin{bmatrix} 2-2 & 4+6 \\ 0-0 & 0+0 \end{bmatrix} \\
 &= \frac{1}{10} \begin{bmatrix} 0 & 10 \\ 0 & 0 \end{bmatrix} \\
 &= \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}
 \end{aligned}$$

(14) If $A = \begin{bmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{bmatrix}$ s.t. $A^{-1} = \frac{1}{2} (A^2 - 3I)$

$$A^{-1} = \frac{1}{|A|} \text{adj} A$$

$$|A| = \begin{vmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{vmatrix}$$

$$= 0(0-1) - 1(0-1) + 1(1-0)$$

$$= 0 + 1 + 1$$

$$= 2$$

$$\text{adj} A = [A_{ij}]^T$$

$$= \begin{bmatrix} +(0-1) & -(0-1) & +(1-0) \\ -(0-1) & +(0-1) & -(0-1) \\ +(1-0) & -(0-1) & +(0-1) \end{bmatrix}^T$$

$$= \begin{bmatrix} -1 & 1 & 1 \\ 1 & -1 & 1 \\ 1 & 1 & -1 \end{bmatrix}^T$$

$$= \begin{bmatrix} -1 & 1 & 1 \\ 1 & -1 & 1 \\ 1 & 1 & -1 \end{bmatrix}$$

$$A^{-1} = \frac{1}{2} \begin{bmatrix} -1 & 1 & 1 \\ 1 & -1 & 1 \\ 1 & 1 & -1 \end{bmatrix} \text{ ——— ①}$$

$$A^2 = A \times A$$

$$= \begin{bmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{bmatrix} \begin{bmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{bmatrix}$$

$$= \begin{bmatrix} 0+1+1 & 0+0+1 & 0+1+0 \\ 0+0+1 & 1+0+1 & 1+0+0 \\ 0+1+0 & 1+0+0 & 1+1+0 \end{bmatrix}$$

$$= \begin{bmatrix} 2 & 1 & 1 \\ 1 & 2 & 1 \\ 1 & 1 & 2 \end{bmatrix}$$

$$A^2 - 3I = \begin{bmatrix} 2 & 1 & 1 \\ 1 & 2 & 1 \\ 1 & 1 & 2 \end{bmatrix} - \begin{bmatrix} 3 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 3 \end{bmatrix}$$

$$= \begin{bmatrix} -1 & 1 & 1 \\ 1 & -1 & 1 \\ 1 & 1 & -1 \end{bmatrix}$$

$$\frac{1}{2}(A^2 - 3I) = \frac{1}{2} \begin{bmatrix} -1 & 1 & 1 \\ 1 & -1 & 1 \\ 1 & 1 & -1 \end{bmatrix} \quad \text{--- (2)}$$

From (1) and (2)

$$A^{-1} = \frac{1}{2}(A^2 - 3I)$$

Defn : A square matrix A is called orthogonal if $AA^T = A^TA = I$

Note : A is orthogonal iff A is non singular and $A^{-1} = A^T$

Example 1.11

Prove that $\begin{bmatrix} \cos \theta & -\sin \theta \\ \sin \theta & \cos \theta \end{bmatrix}$ is orthogonal.

Solution

$$A = \begin{bmatrix} \cos \theta & -\sin \theta \\ \sin \theta & \cos \theta \end{bmatrix}; \quad A^T = \begin{bmatrix} \cos \theta & \sin \theta \\ -\sin \theta & \cos \theta \end{bmatrix}$$

$$\begin{aligned} AA^T &= \begin{bmatrix} \cos \theta & -\sin \theta \\ \sin \theta & \cos \theta \end{bmatrix} \begin{bmatrix} \cos \theta & \sin \theta \\ -\sin \theta & \cos \theta \end{bmatrix} \\ &= \begin{bmatrix} \cos^2 \theta + \sin^2 \theta & \cos \theta \sin \theta - \sin \theta \cos \theta \\ \sin \theta \cos \theta - \sin \theta \cos \theta & \sin^2 \theta + \cos^2 \theta \end{bmatrix} \\ &= \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = I_2 \end{aligned}$$

Similarly we can prove $A^TA = I_2$

$$\text{Hence } AA^T = A^TA = I_2$$

$\therefore A$ is orthogonal.

Example 1.12

If $A = \frac{1}{7} \begin{bmatrix} 6 & -3 & a \\ b & -2 & 6 \\ 2 & c & 3 \end{bmatrix}$ is orthogonal. find a, b, c and hence A^{-1} .

Solution

If A is orthogonal, then $AA^T = A^T A = I_3$

So we have,

$$AA^T = I_3 \Rightarrow \frac{1}{7} \begin{bmatrix} 6 & -3 & a \\ b & -2 & 6 \\ 2 & c & 3 \end{bmatrix} \cdot \frac{1}{7} \begin{bmatrix} 6 & b & 2 \\ -3 & -2 & c \\ a & 6 & 3 \end{bmatrix} = I_3$$

$$\frac{1}{49} \begin{bmatrix} 36+9+a^2 & 6b+b+6a & 12-3c+3a \\ 6b+b+6a & b^2+4+36 & 2b-2c+18 \\ 12-3c+3a & 2b-2c+18 & 4+c^2+9 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$\begin{bmatrix} 45+a^2 & 6a+6b+6 & 3a-3c+12 \\ 6a+6b+6 & b^2+40 & 2b-2c+18 \\ 3a-3c+12 & 2b-2c+18 & c^2+13 \end{bmatrix} = \begin{bmatrix} 49 & 0 & 0 \\ 0 & 49 & 0 \\ 0 & 0 & 49 \end{bmatrix}$$

$$45+a^2 = 49$$

$$6a+6b+6 = 0 \quad \text{--- (1)}$$

$$3a-3c+12 = 0 \quad \text{--- (2)}$$

$$b^2+40 = 49$$

$$2b-2c+18 = 0 \quad \text{--- (3)}$$

$$c^2+13 = 49$$

Solving ①, ② and ③

$$\textcircled{1} \Rightarrow 6a + 6b + 6 = 0$$

$$\textcircled{2} \times 2 \Rightarrow \begin{array}{r} 6a - 6c + 24 = 0 \\ \hline \end{array}$$

$$6b + 6c - 18 = 0$$

$$3 \times \textcircled{3} \Rightarrow \begin{array}{r} 6b - 6c + 54 = 0 \\ \hline \end{array}$$

$$12b + 36 = 0$$

$$12b = -36$$

$$\boxed{b = -3}$$

$$\textcircled{1} \Rightarrow 6a - 18 + 6 = 0$$

$$6a - 12 = 0$$

$$6a = 12$$

$$\boxed{a = 2}$$

$$\textcircled{2} \Rightarrow \begin{array}{r} 6 - 3c + 12 = 0 \\ -3c = -18 \end{array}$$

$$\boxed{c = 6}$$

$$\therefore a = 2; b = -3; c = 6$$

$$\therefore A = \frac{1}{7} \begin{bmatrix} 6 & -3 & 2 \\ -3 & -2 & 6 \\ 2 & 6 & 3 \end{bmatrix}$$

Find: A^{-1}

Given A is orthogonal.

$$\therefore A^{-1} = A^T$$

$$= \frac{1}{7} \begin{bmatrix} 6 & -3 & 2 \\ -3 & -2 & 6 \\ 2 & 6 & 3 \end{bmatrix}$$

Application of matrices to Cryptography

Ex 1.1

(15)

Decrypt the received encoded message

$[2 \ -3][20 \ 4]$ with the encryption matrix $\begin{bmatrix} -1 & -1 \\ 2 & 1 \end{bmatrix}$

and the decryption matrix as its inverse, where the system of codes are described by the numbers 1-26 to the letters A-Z respectively, and the number 0 to a blank space.

$$\text{Let } A = \begin{bmatrix} -1 & -1 \\ 2 & 1 \end{bmatrix}$$

$$A^{-1} = \frac{1}{|A|} \text{adj } A$$

$$|A| = \begin{vmatrix} -1 & -1 \\ 2 & 1 \end{vmatrix}$$

$$= -1 + 2$$

$$= 1$$

$$\text{adj } A = \begin{bmatrix} 1 & 1 \\ -2 & -1 \end{bmatrix}$$

$$A^{-1} = \frac{1}{1} \begin{bmatrix} 1 & 1 \\ -2 & -1 \end{bmatrix}$$

(30)

$$A^{-1} = \begin{bmatrix} 1 & 1 \\ -2 & -1 \end{bmatrix}$$

Coded row matrix

Decoding matrix

Decoded row matrix

$$\begin{bmatrix} 2 & -3 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 1 \\ -2 & -1 \end{bmatrix}$$

$$= \begin{bmatrix} 2+6 & 2+3 \\ 20-8 & 20-4 \end{bmatrix}$$

$$= \begin{bmatrix} 8 & 5 \end{bmatrix}$$

$$\begin{bmatrix} 20 & 4 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 1 \\ -2 & -1 \end{bmatrix}$$

$$= \begin{bmatrix} 20-8 & 20-4 \\ 12 & 16 \end{bmatrix}$$

$$= \begin{bmatrix} 12 & 16 \end{bmatrix}$$

So, the sequence of decoded row matrices is $\begin{bmatrix} 8 & 5 \end{bmatrix}$, $\begin{bmatrix} 12 & 16 \end{bmatrix}$

Thus the receiver reads the message as "HELP"

(31)

Elementary Transformations of a matrix

Elementary row and column operations

Elementary row (column) operations on a matrix are as follows:

- (i) The interchanging of any two rows (columns) of the matrix
- (ii) Replacing a row (column) of the matrix by a non zero scalar multiple of the row (column) by a non zero scalar
- (iii) Replacing a row (column) of the matrix by a sum of the row (column) with a non zero scalar multiple of another row (column) of the matrix

• We use the following notations for the elementary row transformations

- (i) Interchanging of i^{th} and j^{th} row ($R_i \leftrightarrow R_j$)
- (ii) The multiplication of each element of i^{th} row by a non zero constant λ is denoted by $R_i \rightarrow \lambda R_i$
- (iii) Addition to i^{th} row, a non zero constant λ multiple of j^{th} row is denoted by $R_i \rightarrow R_i + \lambda R_j$

Similar notations are used for elementary column transformations.

(32)

Row - Echelon form (x)

Defn 1.5

A non-zero matrix E is said to be in a row-echelon form if

- (i) All zero rows of E occur below every non zero row of E
- (ii) The first non zero element in any row i of E occurs in the j th column of E , then all other entries in the j th column of E below the first non zero element of row i are zeros.
- (iii) The first non zero entry in the i th row of E lies to the left of the first non zero entry in $(i+1)$ th row of E .

Examples for row echelon form

$$(i) \begin{bmatrix} 0 & 1 & 1 \\ 0 & 0 & 3 \\ 0 & 0 & 0 \end{bmatrix} \quad (ii) \begin{bmatrix} 1 & 0 & -1 & 2 \\ 0 & 0 & 2 & 8 \\ 0 & 0 & 0 & 6 \end{bmatrix}$$

Examples for the matrices not in row echelon form

$$(i) \begin{bmatrix} 1 & -2 & 0 \\ 0 & 0 & 5 \\ 0 & 1 & 0 \end{bmatrix} \quad (ii) \begin{bmatrix} 0 & 3 & -2 \\ 5 & 0 & 0 \\ 3 & 2 & 0 \end{bmatrix}$$

Example 1.13

Reduce the matrix $\begin{bmatrix} 3 & -1 & 2 \\ -6 & 2 & 4 \\ -3 & 1 & 2 \end{bmatrix}$ to a row echelon form

Solution

$$A = \begin{bmatrix} 3 & -1 & 2 \\ -6 & 2 & 4 \\ -3 & 1 & 2 \end{bmatrix}$$

$$\begin{array}{l} R_2 \rightarrow R_2 + 2R_1 \\ R_3 \rightarrow R_3 + R_1 \end{array} \rightarrow \begin{bmatrix} 3 & -1 & 2 \\ 0 & 0 & 8 \\ 0 & 0 & 4 \end{bmatrix}$$

$$R_3 \rightarrow 2R_3 - R_2 \rightarrow \begin{bmatrix} 3 & -1 & 2 \\ 0 & 0 & 8 \\ 0 & 0 & 0 \end{bmatrix}$$

This is the row echelon form of the given matrix

Note: A row echelon form of a matrix is not necessarily unique

Example 1.14

Reduce the matrix $\begin{bmatrix} 0 & 3 & 16 \\ -1 & 0 & 25 \\ 4 & 2 & 00 \end{bmatrix}$ to a row echelon form

Solution

$$A = \begin{bmatrix} 0 & 3 & 16 \\ -1 & 0 & 25 \\ 4 & 2 & 00 \end{bmatrix}$$

(34)

$$R_1 \leftrightarrow R_2 \rightarrow \begin{bmatrix} -1 & 0 & 2 & 5 \\ 0 & 3 & 1 & 6 \\ 4 & 2 & 0 & 0 \end{bmatrix}$$

$$R_3 \rightarrow R_3 + 4R_1 \rightarrow \begin{bmatrix} -1 & 0 & 2 & 5 \\ 0 & 3 & 1 & 6 \\ 0 & 2 & 8 & 20 \end{bmatrix}$$

$$R_3 \rightarrow 3R_3 - 2R_2 \rightarrow \begin{bmatrix} -1 & 0 & 2 & 5 \\ 0 & 3 & 1 & 6 \\ 0 & 0 & 22 & 48 \end{bmatrix}$$

This is the row echelon form of the given matrix.

Rank of a Matrix

Defn 1.6

The rank of a matrix A is defined as the order of a highest order non vanishing minor of the matrix A . It is denoted by the symbol $\rho(A)$. The rank of a zero matrix is defined to be 0.

Note: (i) If a matrix contains at least one non zero element, then $\rho(A) \geq 1$

(ii) The rank of the identity matrix I_n is n

(iii) If the rank of a matrix A is r , then there exists at least one minor of A of order r which does not vanish and every minor of A

- of order $r+1$ and higher order (if any) vanishes
- (iv) If A is an $m \times n$ matrix, then $\rho(A) \leq \min\{m, n\}$
- (v) A square matrix A of order n has inverse if and only if $\rho(A) = n$.

Example 1.15

Find the rank of the following matrices

(i) $\begin{bmatrix} 3 & 2 & 5 \\ 1 & 1 & 2 \\ 3 & 3 & 6 \end{bmatrix}$ (ii) $\begin{bmatrix} 4 & 3 & 1 & -2 \\ -3 & -1 & -2 & 4 \\ 6 & 7 & -1 & 2 \end{bmatrix}$

Solution

(i) Let $A = \begin{bmatrix} 3 & 2 & 5 \\ 1 & 1 & 2 \\ 3 & 3 & 6 \end{bmatrix}$

A is a matrix of order 3×3

$$\therefore \rho(A) \leq \min\{3, 3\} = 3$$

The highest order minor of A is 3,

It is $\begin{vmatrix} 3 & 2 & 5 \\ 1 & 1 & 2 \\ 3 & 3 & 6 \end{vmatrix} = 3(6-6) - 2(6-6) + 5(3-3) = 0$

The third order minor vanishes

$$\therefore \rho(A) \neq 3$$

$$\rho(A) < 3$$

Next Consider the second Order minor of A .

$$\begin{vmatrix} 3 & 2 \\ 1 & 1 \end{vmatrix} = 3 - 2 = 1 \neq 0$$

The 2nd order minor of A do not vanish

$$\therefore P(A) = 2$$

(ii) Let $A = \begin{bmatrix} 4 & 3 & 1 & -2 \\ -3 & -1 & -2 & 4 \\ 6 & 7 & -1 & 2 \end{bmatrix}$

A is a matrix of order 3×4

$$\therefore P(A) \leq \min \{3, 4\} = 3$$

The highest order minor of A is 3

$$\begin{vmatrix} 4 & 3 & 1 \\ -3 & -1 & -2 \\ 6 & 7 & -1 \end{vmatrix} = 4(1+14) - 3(3+12) + 1(-21+6) \\ = 4(15) - 3(15) + 1(-15) \\ = 60 - 45 - 15 \\ = 0$$

$$\begin{vmatrix} 4 & 1 & -2 \\ -3 & -2 & 4 \\ 6 & -1 & 2 \end{vmatrix} = 4(-4+4) - 1(-6-24) - 2(3+12) \\ = 4(0) - 1(-30) - 2(15) \\ = 30 - 30 \\ = 0$$

$$\begin{vmatrix} 4 & 3 & -2 \\ -3 & -1 & 4 \\ 6 & 7 & 2 \end{vmatrix} = 4(-2-28) - 3(-6-24) - 2(-21+6) \\ = 4(-30) - 3(-30) - 2(-15) \\ = -120 + 90 + 30 \\ = 0$$

$$\begin{vmatrix} 3 & 1 & -2 \\ -1 & -2 & 4 \\ 7 & -1 & 2 \end{vmatrix} = 3(-4+4) - 1(-2-28) - 2(1+14) \\ = 3(0) - 1(-30) - 2(15) \\ = 30 - 30 \\ = 0$$

(37)

All the 3rd order minors of A are Vanishes

$$\therefore \rho(A) \neq 3$$

$$\rho(A) < 3$$

Next Consider the second order minor of A

$$\begin{vmatrix} 4 & 3 \\ -3 & -1 \end{vmatrix} = -4 + 9 \\ = 5 \neq 0$$

The 2nd order minor of A doesnot Vanishes

$$\therefore \rho(A) = 2.$$

Example 1.16

Find the rank of the following matrices which are in row echelon form

$$(i) \begin{bmatrix} 2 & 0 & -7 \\ 0 & 3 & 1 \\ 0 & 0 & 1 \end{bmatrix}$$

$$(ii) \begin{bmatrix} -2 & 2 & -1 \\ 0 & 5 & 1 \\ 0 & 0 & 0 \end{bmatrix}$$

$$(iii) \begin{bmatrix} 6 & 0 & -9 \\ 0 & 2 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

Solution:-

$$(i) \text{ Let } A = \begin{bmatrix} 2 & 0 & -7 \\ 0 & 3 & 1 \\ 0 & 0 & 1 \end{bmatrix}$$

A is a matrix of order 3×3

$$\therefore \rho(A) \leq 3$$

The highest order minor of A is 3

$$\begin{vmatrix} 2 & 0 & -7 \\ 0 & 3 & 1 \\ 0 & 0 & 1 \end{vmatrix} = 2(3-0) - 7(0) \\ = 6 \neq 0$$

$$\therefore \rho(A) = 3$$

Note: There are 3 non zero rows

(ii) Let $A = \begin{bmatrix} -2 & 2 & -1 \\ 0 & 5 & 1 \\ 0 & 0 & 0 \end{bmatrix}$

A is a matrix of order 3×3

$$\therefore \rho(A) \leq 3$$

The highest order Minor of A is 3

$$\begin{vmatrix} -2 & 2 & -1 \\ 0 & 5 & 1 \\ 0 & 0 & 0 \end{vmatrix} = 0$$

The third order minor of A vanishes

$$\therefore \rho(A) \neq 3$$

$$\rho(A) < 3$$

Next Consider the second order minor of A

$$\begin{vmatrix} -2 & 2 \\ 0 & 5 \end{vmatrix} = -10 \neq 0$$

The 2nd order minor of A does not Vanishes

$$\therefore \rho(A) = 2$$

Note : There are 2 non zero rows

(iii) Let $A = \begin{bmatrix} 6 & 0 & -9 \\ 0 & 2 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$

(39)

A is a matrix of order 4×3

$$\rho(A) \leq \min \{4, 3\}$$

$$\rho(A) \leq 3$$

The highest order minor of A is 3

$$\begin{vmatrix} 6 & 0 & -9 \\ 0 & 9 & 0 \\ 0 & 0 & 0 \end{vmatrix} = 0$$

and all the third order minors of A are zero

$$\therefore \rho(A) \neq 3$$

$$\rho(A) < 3$$

Next consider the 2nd order minor of A

$$\begin{vmatrix} 6 & 0 \\ 0 & 2 \end{vmatrix} = 12 - 0 \\ = 12 \neq 0$$

The 2nd order minor of A does not vanish

$$\therefore \rho(A) = 2$$

Note: There are 2 non zero rows.

The rank of a matrix in row echelon form is the number of non zero rows in it

Example 1.17

Find the rank of the matrix $\begin{bmatrix} 1 & 2 & 3 \\ 2 & 1 & 4 \\ 3 & 0 & 5 \end{bmatrix}$ by

reducing it to a row-echelon form

Soln.:

$$\text{Let } A = \begin{bmatrix} 1 & 2 & 3 \\ 2 & 1 & 4 \\ 3 & 0 & 5 \end{bmatrix}$$

$$\begin{array}{l} R_2 \rightarrow R_2 - 2R_1 \\ R_3 \rightarrow R_3 - 3R_1 \end{array} \rightarrow \begin{bmatrix} 1 & 2 & 3 \\ 0 & -3 & -2 \\ 0 & -6 & -4 \end{bmatrix}$$

$$R_3 \rightarrow R_3 - 2R_2 \rightarrow \begin{bmatrix} 1 & 2 & 3 \\ 0 & -3 & -2 \\ 0 & 0 & 0 \end{bmatrix}$$

The last equivalent matrix is in row echelon form

It has two non zero rows

$$\therefore \rho(A) = 2$$

Example 1.18

Find the rank of the matrix $\begin{bmatrix} 2 & -2 & 4 & 3 \\ -3 & 4 & -2 & -1 \\ 6 & 2 & -1 & 7 \end{bmatrix}$ by reducing it to an echelon form

Solution :

(41)

$$\text{Let } A = \begin{bmatrix} 2 & -2 & 4 & 3 \\ -3 & 4 & -2 & -1 \\ 6 & 2 & -1 & 7 \end{bmatrix}$$

$$\begin{array}{l} R_2 \rightarrow 2R_2 + R_3 \\ R_3 \rightarrow R_3 - 3R_1 \end{array} \rightarrow \begin{bmatrix} 2 & -2 & 4 & 3 \\ 0 & 10 & -5 & 5 \\ 0 & 8 & -13 & -2 \end{bmatrix}$$

$$R_2 \rightarrow R_2/5 \rightarrow \begin{bmatrix} 2 & -2 & 4 & 3 \\ 0 & 2 & -1 & 1 \\ 0 & 8 & -13 & -2 \end{bmatrix}$$

$$R_3 \rightarrow R_3 - 4R_2 \rightarrow \begin{bmatrix} 2 & -2 & 4 & 3 \\ 0 & 2 & -1 & 1 \\ 0 & 0 & -9 & -6 \end{bmatrix}$$

The last equivalent matrix is in row echelon form.

It has 3 non zero rows

$$\therefore \rho(A) = 3.$$

Every non-singular matrix can be transformed to an identity matrix, by a sequence of elementary row operations

Example 1.19

Show that the matrix $\begin{bmatrix} 3 & 1 & 4 \\ 2 & 0 & -1 \\ 5 & 2 & 1 \end{bmatrix}$ is non singular and reduce it to the identity matrix by elementary row transformations.

Solution: Let $A = \begin{bmatrix} 3 & 1 & 4 \\ 2 & 0 & -1 \\ 5 & 2 & 1 \end{bmatrix}$

$$|A| = \begin{vmatrix} 3 & 1 & 4 \\ 2 & 0 & -1 \\ 5 & 2 & 1 \end{vmatrix} = 3(0+2) - 1(2+5) + 4(4-0) \\ = 6 - 7 + 16 \\ = 15 \neq 0$$

So, A is non singular matrix

$$A = \begin{bmatrix} 3 & 1 & 4 \\ 2 & 0 & -1 \\ 5 & 2 & 1 \end{bmatrix}$$

$$R_1 \rightarrow R_1 - R_2 \rightarrow \begin{bmatrix} 1 & 1 & 5 \\ 2 & 0 & -1 \\ 5 & 2 & 1 \end{bmatrix}$$

$$\begin{matrix} R_2 \rightarrow R_2 - 2R_1 \\ R_3 \rightarrow R_3 - 5R_1 \end{matrix} \rightarrow \begin{bmatrix} 1 & 1 & 5 \\ 0 & -2 & -11 \\ 0 & -3 & -24 \end{bmatrix}$$

$$\begin{matrix} R_2 \rightarrow R_2 \times \frac{1}{-2} \\ R_3 \rightarrow R_3 \times \frac{1}{-3} \end{matrix} \rightarrow \begin{bmatrix} 1 & 1 & 5 \\ 0 & 1 & 11 \\ 0 & 1 & 8 \end{bmatrix}$$

$$\underline{R_2 \rightarrow +R_2 - R_3} \rightarrow \begin{bmatrix} 1 & 1 & 5 \\ 0 & 1 & 3 \\ 0 & 1 & 8 \end{bmatrix}$$

$$\underline{R_3 \rightarrow R_3 - R_2} \rightarrow \begin{bmatrix} 1 & 1 & 5 \\ 0 & 1 & 3 \\ 0 & 0 & 5 \end{bmatrix}$$

$$\underline{R_3 \rightarrow R_3 / 5} \rightarrow \begin{bmatrix} 1 & 1 & 5 \\ 0 & 1 & 3 \\ 0 & 0 & 1 \end{bmatrix}$$

$$\underline{R_2 \rightarrow R_2 - 3R_3} \rightarrow \begin{bmatrix} 1 & 1 & 5 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$\underline{R_1 \rightarrow R_1 - R_2} \rightarrow \begin{bmatrix} 1 & 0 & 5 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$\underline{R_1 \rightarrow R_1 - 5R_3} \rightarrow \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Gauss - Jordan method (Finding Inverse)

Example 1.20

Find the inverse of the non singular matrix $A = \begin{bmatrix} 0 & 5 \\ -1 & 6 \end{bmatrix}$, by Gauss - Jordan method

Solution

Applying Gauss-Jordan method, we get

$$[A | I_2] = \left[\begin{array}{cc|cc} 0 & 5 & 1 & 0 \\ -1 & 6 & 0 & 1 \end{array} \right]$$

$$\xrightarrow{R_1 \rightarrow R_1 - R_2} \left[\begin{array}{cc|cc} 1 & -1 & 1 & -1 \\ -1 & 6 & 0 & 1 \end{array} \right]$$

$$\xrightarrow{R_2 \rightarrow R_2 + R_1} \left[\begin{array}{cc|cc} 1 & -1 & 1 & -1 \\ 0 & 5 & 1 & 0 \end{array} \right]$$

$$\xrightarrow{R_2 \rightarrow R_2 / 5} \left[\begin{array}{cc|cc} 1 & -1 & 1 & -1 \\ 0 & 1 & 1/5 & 0 \end{array} \right]$$

$$\xrightarrow{R_1 \rightarrow R_1 + R_2} \left[\begin{array}{cc|cc} 1 & 0 & 6/5 & -1 \\ 0 & 1 & 1/5 & 0 \end{array} \right]$$

so we get $A^{-1} = \begin{bmatrix} 6/5 & -1 \\ 1/5 & 0 \end{bmatrix}$

$$= \frac{1}{5} \begin{bmatrix} 6 & -5 \\ 1 & 0 \end{bmatrix}$$

Example 1.21

Find the inverse of $A = \begin{bmatrix} 2 & 1 & 1 \\ 3 & 2 & 1 \\ 2 & 1 & 2 \end{bmatrix}$ by

Gauss-Jordan method.

Applying Gauss-Jordan method, we get

$$[A|I_3] = \left[\begin{array}{ccc|ccc} 2 & 1 & 1 & 1 & 0 & 0 \\ 3 & 2 & 1 & 0 & 1 & 0 \\ 2 & 1 & 2 & 0 & 0 & 1 \end{array} \right]$$

$$R_1 \rightarrow -R_1 + R_2 \rightarrow \left[\begin{array}{ccc|ccc} 1 & 1 & 0 & -1 & 1 & 0 \\ 3 & 2 & 1 & 0 & 1 & 0 \\ 2 & 1 & 2 & 0 & 0 & 1 \end{array} \right]$$

$$\begin{array}{l} R_2 \rightarrow R_2 - 3R_1 \\ R_3 \rightarrow R_3 - 2R_1 \end{array} \rightarrow \left[\begin{array}{ccc|ccc} 1 & 1 & 0 & -1 & 1 & 0 \\ 0 & -1 & 1 & 3 & -2 & 0 \\ 0 & -1 & 2 & 2 & -2 & 1 \end{array} \right]$$

$$R_2 \rightarrow R_2 \times -1 \rightarrow \left[\begin{array}{ccc|ccc} 1 & 1 & 0 & -1 & 1 & 0 \\ 0 & 1 & -1 & -3 & 2 & 0 \\ 0 & -1 & 2 & 2 & -2 & 1 \end{array} \right]$$

$$R_3 \rightarrow R_3 + R_2 \rightarrow \left[\begin{array}{ccc|ccc} 1 & 1 & 0 & -1 & 1 & 0 \\ 0 & 1 & -1 & -3 & 2 & 0 \\ 0 & 0 & 1 & -1 & 0 & 1 \end{array} \right]$$

$$R_2 \rightarrow R_2 + R_3 \rightarrow \left[\begin{array}{ccc|ccc} 1 & 1 & 0 & -1 & 1 & 0 \\ 0 & 1 & 0 & -4 & 2 & 1 \\ 0 & 0 & 1 & -1 & 0 & 1 \end{array} \right]$$

$$R_1 \rightarrow R_1 - R_2 \rightarrow \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 3 & -1 & -1 \\ 0 & 1 & 0 & -4 & 2 & 1 \\ 0 & 0 & 1 & -1 & 0 & 1 \end{array} \right]$$

$$\text{So } A^{-1} = \begin{bmatrix} 3 & -1 & -1 \\ -4 & 2 & 1 \\ -1 & 0 & 1 \end{bmatrix}$$

Exercise 1.2

① (iv) Find the rank of the matrix $\begin{bmatrix} 1 & -2 & 3 \\ 2 & 4 & -6 \\ 5 & 1 & -1 \end{bmatrix}$
by minor method.

$$\text{Let } A = \begin{bmatrix} 1 & -2 & 3 \\ 2 & 4 & -6 \\ 5 & 1 & -1 \end{bmatrix}$$

A is a matrix of order 3×3

$$\therefore \rho(A) \leq 3$$

The highest order minor of A is 3

$$\begin{vmatrix} 1 & -2 & 3 \\ 2 & 4 & -6 \\ 5 & 1 & -1 \end{vmatrix} = 1(-4+6) + 2(-2+30) + 3(2-20) \\ = 1(2) + 2(28) + 3(-18) \\ = 2 + 56 - 54 \\ = 4 \neq 0$$

3rd order minor of A does not vanish

$$\therefore \rho(A) = 3$$

H.W

① (i) $\begin{bmatrix} 2 & -4 \\ -1 & 2 \end{bmatrix}$ (ii) $\begin{bmatrix} 1 & 3 \\ 4 & -7 \\ 3 & -4 \end{bmatrix}$ (iii) $\begin{bmatrix} 1 & -2 & -1 & 0 \\ 3 & 6 & -3 & 1 \end{bmatrix}$

(v) $\begin{bmatrix} 0 & 1 & 2 & 1 \\ 0 & 2 & 4 & 3 \\ 8 & 1 & 0 & 2 \end{bmatrix} \rightarrow \text{Find rank by minor method.}$

(47)

② (iii) Find the rank of the matrix

$$\begin{bmatrix} 3 & -8 & 5 & 2 \\ 2 & -5 & 1 & 4 \\ -1 & 2 & 3 & -2 \end{bmatrix} \text{ by row reduction (or) row echelon method.}$$

Solution

$$\text{Let } A = \begin{bmatrix} 3 & -8 & 5 & 2 \\ 2 & -5 & 1 & 4 \\ -1 & 2 & 3 & -2 \end{bmatrix}$$

$$\begin{array}{l} R_2 \rightarrow R_2 + 2R_3 \\ R_3 \rightarrow 3R_3 + R_1 \end{array} \rightarrow \begin{bmatrix} 3 & -8 & 5 & 2 \\ 0 & -1 & 7 & 0 \\ 0 & -2 & 14 & -4 \end{bmatrix}$$

$$R_3 \rightarrow R_3 - 2R_2 \rightarrow \begin{bmatrix} 3 & -8 & 5 & 2 \\ 0 & -1 & 7 & 0 \\ 0 & 0 & 0 & -4 \end{bmatrix}$$

The last equivalent matrix is in row echelon form

It has 3 non zero rows

$$\therefore \rho(A) = 3$$

H.W

$$\textcircled{2} \text{ (i) } \begin{bmatrix} 1 & 1 & 1 & 3 \\ 2 & -1 & 3 & 4 \\ 5 & -1 & 7 & 11 \end{bmatrix}, \text{ (ii) } \begin{bmatrix} 1 & 2 & -1 \\ 3 & -1 & 2 \\ 1 & -2 & 3 \\ 1 & -1 & 1 \end{bmatrix}$$

Find the rank using row echelon method.

③ (iii) Find the inverse of the matrix

$$\begin{bmatrix} 1 & 2 & 3 \\ 2 & 5 & 3 \\ 1 & 0 & 8 \end{bmatrix} \text{ by Gauss-Jordan method.}$$

Solution:

$$\text{Let } A = \begin{bmatrix} 1 & 2 & 3 \\ 2 & 5 & 3 \\ 1 & 0 & 8 \end{bmatrix}$$

Applying Gauss-Jordan method, we get,

$$[A | I_3] = \left[\begin{array}{ccc|ccc} 1 & 2 & 3 & 1 & 0 & 0 \\ 2 & 5 & 3 & 0 & 1 & 0 \\ 1 & 0 & 8 & 0 & 0 & 1 \end{array} \right]$$

$$\begin{array}{l} R_2 \rightarrow R_2 - 2R_1 \\ R_3 \rightarrow R_3 - R_1 \end{array} \rightarrow \left[\begin{array}{ccc|ccc} 1 & 2 & 3 & 1 & 0 & 0 \\ 0 & 1 & -3 & -2 & 1 & 0 \\ 0 & -2 & 5 & -1 & 0 & 1 \end{array} \right]$$

$$R_3 \rightarrow R_3 + 2R_2 \rightarrow \left[\begin{array}{ccc|ccc} 1 & 2 & 3 & 1 & 0 & 0 \\ 0 & 1 & -3 & -2 & 1 & 0 \\ 0 & 0 & -1 & -5 & 2 & 1 \end{array} \right]$$

$$R_3 \rightarrow R_3 \times (-1) \rightarrow \left[\begin{array}{ccc|ccc} 1 & 2 & 3 & 1 & 0 & 0 \\ 0 & 1 & -3 & -2 & 1 & 0 \\ 0 & 0 & 1 & 5 & -2 & -1 \end{array} \right]$$

$$R_2 \rightarrow R_2 + 3R_3 \rightarrow \left[\begin{array}{ccc|ccc} 1 & 2 & 3 & 1 & 0 & 0 \\ 0 & 1 & 0 & 13 & -5 & -3 \\ 0 & 0 & 1 & 5 & -2 & -1 \end{array} \right]$$

$$R_1 \rightarrow R_1 - 2R_2 \rightarrow \left[\begin{array}{ccc|ccc} 1 & 0 & 3 & -25 & 10 & 6 \\ 0 & 1 & 0 & 13 & -5 & -3 \\ 0 & 0 & 1 & 5 & -2 & -1 \end{array} \right]$$

$$R_1 \rightarrow R_1 - 3R_3 \rightarrow \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & -40 & 16 & 9 \\ 0 & 1 & 0 & 13 & -5 & -3 \\ 0 & 0 & 1 & 5 & -2 & -1 \end{array} \right]$$

$$\text{So } A^{-1} = \begin{bmatrix} -40 & 16 & 9 \\ 13 & -5 & -3 \\ 5 & -2 & -1 \end{bmatrix}$$

H.W

③

Find the inverse of the following by Gauss-Jordan method

(i) $\begin{bmatrix} 2 & -1 \\ 5 & -2 \end{bmatrix}$

(ii) $\begin{bmatrix} 1 & -1 & 0 \\ 1 & 0 & -1 \\ 6 & -2 & -3 \end{bmatrix}$

Applications of matrices: Solving system of Linear Equations

(i) Matrix Inversion method

This method can be applied only when the coefficient matrix is a square matrix and non-singular

Consider the matrix equation

$$AX = B$$

Pre multiplying A^{-1} by both sides

$$A^{-1}(AX) = A^{-1}B$$

$$(A^{-1}A)x = A^{-1}B$$

$$IX = A^{-1}B$$

$$X = A^{-1}B$$

Example 1.22

Solve the following system of linear equations using matrix inversion method:

$$5x + 2y = 3, \quad 3x + 2y = 5$$

Solution

The matrix form of the system is

$$\begin{bmatrix} 5 & 2 \\ 3 & 2 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 3 \\ 5 \end{bmatrix}$$

This is of the form $AX = B$

$$\text{Here } A = \begin{bmatrix} 5 & 2 \\ 3 & 2 \end{bmatrix}; \quad X = \begin{bmatrix} x \\ y \end{bmatrix}; \quad B = \begin{bmatrix} 3 \\ 5 \end{bmatrix}$$

$$AX = B \Rightarrow X = A^{-1}B$$

$$A^{-1} = \frac{1}{|A|} \text{adj } A$$

$$|A| = \begin{vmatrix} 5 & 2 \\ 3 & 2 \end{vmatrix} = 10 - 6 = 4 \neq 0$$

$$\text{adj } A = \begin{bmatrix} 2 & -2 \\ -3 & 5 \end{bmatrix}$$

$$A^{-1} = \frac{1}{4} \begin{bmatrix} 2 & -2 \\ -3 & 5 \end{bmatrix}$$

$$X = A^{-1}B$$

$$\begin{bmatrix} x \\ y \end{bmatrix} = \frac{1}{4} \begin{bmatrix} 2 & -2 \\ -3 & 5 \end{bmatrix} \begin{bmatrix} 3 \\ 5 \end{bmatrix}$$

(51)

$$= \frac{1}{4} \begin{bmatrix} 6-10 \\ -9+25 \end{bmatrix}$$

$$= \frac{1}{4} \begin{bmatrix} -4 \\ 16 \end{bmatrix}$$

$$= \begin{bmatrix} -1 \\ 4 \end{bmatrix}$$

$$\underline{x = -1; \quad y = 4}$$

Example 1.23

Solve the following system of equations,
using matrix inversion method.

$$2x_1 + 3x_2 + 3x_3 = 5; \quad x_1 - 2x_2 + x_3 = -4$$

$$3x_1 - x_2 - 2x_3 = 3$$

Solution

The matrix form of the system is

$$\begin{bmatrix} 2 & 3 & 3 \\ 1 & -2 & 1 \\ 3 & -1 & -2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 5 \\ -4 \\ 3 \end{bmatrix}$$

This is of the form $AX = B$

$$\text{Here } A = \begin{bmatrix} 2 & 3 & 3 \\ 1 & -2 & 1 \\ 3 & -1 & -2 \end{bmatrix}; \quad X = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}; \quad B = \begin{bmatrix} 5 \\ -4 \\ 3 \end{bmatrix}$$

$$AX = B \Rightarrow X = A^{-1}B$$

$$|A| = \begin{vmatrix} 2 & 3 & 3 \\ 1 & -2 & 1 \\ 3 & -1 & -2 \end{vmatrix}$$

$$= 2(4+1) - 3(-2-3) + 3(-1+6)$$

$$= 2(5) - 3(-5) + 3(5)$$

$$= 10 + 15 + 15$$

$$= 40 \neq 0$$

$$A^{-1} = \frac{1}{|A|} \text{adj } A$$

$$\text{adj } A = [A_{ij}]^T$$

$$= \begin{bmatrix} +(4+1) & -(-2-3) & +(-1+6) \\ -(-6+3) & +(-4-9) & -(-2-9) \\ +(3+6) & -(2-3) & +(-4-3) \end{bmatrix}^T$$

$$= \begin{bmatrix} 5 & 5 & 5 \\ 3 & -13 & 11 \\ 9 & 1 & -7 \end{bmatrix}^T$$

$$= \begin{bmatrix} 5 & 3 & 9 \\ 5 & -13 & 1 \\ 5 & 11 & -7 \end{bmatrix}$$

$$A^{-1} = \frac{1}{40} \begin{bmatrix} 5 & 3 & 9 \\ 5 & -13 & 1 \\ 5 & 11 & -7 \end{bmatrix}$$

$$X = A^{-1} B$$

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \frac{1}{40} \begin{bmatrix} 5 & 3 & 9 \\ 5 & -13 & 1 \\ 5 & 11 & -7 \end{bmatrix} \begin{bmatrix} 5 \\ -4 \\ 3 \end{bmatrix}$$

$$= \frac{1}{40} \begin{bmatrix} 25 - 12 + 27 \\ 25 + 52 + 3 \\ 25 - 44 - 21 \end{bmatrix}$$

$$= \frac{1}{40} \begin{bmatrix} 40 \\ 80 \\ -40 \end{bmatrix}$$

$$= \begin{bmatrix} 1 \\ 2 \\ -1 \end{bmatrix}$$

$$x_1 = 1 ; x_2 = 2 ; x_3 = -1$$

H.W	Solve by matrix inversion method	
Σ Ex 1.3	(i) $2x + 5y = -2$ $x + 2y = -3$	(ii) $2x - y = 8$ $3x + 2y = -2$
	(iii) $2x + 3y - z = 9$ $x + y + z = 9$ $3x - y - z = -1$	(iv) $x + y + z - 2 = 0$ $6x - 4y + 5z - 31 = 0$ $5x + 2y + 2z = 13$

Ex 1.24

$$\text{If } A = \begin{bmatrix} -4 & 4 & 4 \\ -7 & 1 & 3 \\ 5 & -3 & -1 \end{bmatrix} ; B = \begin{bmatrix} 1 & -1 & 1 \\ 1 & -2 & -2 \\ 2 & 1 & 3 \end{bmatrix}$$

find the products AB and BA and hence solve the system of equations $x - y + z = 4 ; x - 2y - 2z = 9$

$$2x + y + 3z = 1$$

$$AB = \begin{bmatrix} -4 & 4 & 4 \\ -7 & 1 & 3 \\ 5 & -3 & -1 \end{bmatrix} \begin{bmatrix} 1 & -1 & 1 \\ 1 & -2 & -2 \\ 2 & 1 & 3 \end{bmatrix}$$

$$= \begin{bmatrix} -4+4+8 & 4-8+4 & -4-8+12 \\ -7+1+6 & 7-2+3 & -7-2+9 \\ 5-3-2 & -5+6-1 & 5+6-3 \end{bmatrix}$$

$$= \begin{bmatrix} 8 & 0 & 0 \\ 0 & 8 & 0 \\ 0 & 0 & 8 \end{bmatrix}$$

$$= 8 \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$= 8I_3 \quad \text{--- (1)}$$

$$BA = \begin{bmatrix} 1 & -1 & 1 \\ 1 & -2 & -2 \\ 2 & 1 & 3 \end{bmatrix} \begin{bmatrix} -4 & 4 & 4 \\ -7 & 1 & 3 \\ 5 & -3 & -1 \end{bmatrix}$$

$$= \begin{bmatrix} -4+7+5 & 4-1-3 & 4-3-1 \\ -4+14-10 & 4-2+6 & 4-6+2 \\ -8-7+15 & 8+1-9 & 8+3-3 \end{bmatrix}$$

$$= \begin{bmatrix} 8 & 0 & 0 \\ 0 & 8 & 0 \\ 0 & 0 & 8 \end{bmatrix}$$

$$= 8 \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$= 8I_3 \quad \text{--- (2)}$$

From ① & ②

$$AB = BA = 8I_3$$

$$\left(\frac{1}{8}A\right)B = B\left(\frac{1}{8}A\right) = I_3$$

$$\therefore \boxed{B^{-1} = \frac{1}{8}A}$$

$$[\because BB^{-1} = B^{-1}B = I_3]$$

Writing the given system of equations in Matrix ~~form~~ form.

$$\begin{bmatrix} 1 & -1 & 1 \\ 1 & -2 & -2 \\ 2 & 1 & 3 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 4 \\ 9 \\ 1 \end{bmatrix}$$

This is of the form $Bx = C$

$$x = B^{-1}C$$

$$x = \frac{1}{8}AC$$

$$= \frac{1}{8} \begin{bmatrix} -4 & 4 & 4 \\ -7 & 1 & 3 \\ 5 & -3 & -1 \end{bmatrix} \begin{bmatrix} 4 \\ 9 \\ 1 \end{bmatrix}$$

$$= \frac{1}{8} \begin{bmatrix} -16 + 36 + 4 \\ -28 + 9 + 3 \\ 20 - 27 - 1 \end{bmatrix}$$

$$= \frac{1}{8} \begin{bmatrix} 24 \\ -16 \\ -8 \end{bmatrix}$$

$$= \begin{bmatrix} 3 \\ -2 \\ -1 \end{bmatrix}$$

$$x = 3 ; y = -2 ; z = -1$$

H.W

Ex 1.3(2)

If $A = \begin{bmatrix} -5 & 1 & 3 \\ 7 & 1 & -5 \\ 1 & -1 & 1 \end{bmatrix}$ and $B = \begin{bmatrix} 1 & 1 & 2 \\ 3 & 2 & 1 \\ 2 & 1 & 3 \end{bmatrix}$ Find the

Products AB and BA and hence solve the system of equations $x + y + 2z = 1$; $3x + 2y + z = 7$; $2x + y + 3z = 2$

Ex 1.3

- ③ A man is appointed in a job with monthly salary of certain amount and a fixed amount of annual increment. If his salary was ₹19,800 per month at the end of the first month after 3 years of service and ₹23,400 per month at the end of the first month after 9 years of service, find his starting salary and his annual increment (use matrix inversion method)

Solution

Let x be the starting salary and y be the fixed increment.

By the given data

$$x + 3y = 19800$$

$$x + 9y = 23400$$

Writing the given system of equations in matrix form

$$\begin{bmatrix} 1 & 3 \\ 1 & 9 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 19800 \\ 23400 \end{bmatrix}$$

This is of the form $AX = B$

$$X = A^{-1}B$$

$$A^{-1} = \frac{1}{|A|} \text{adj } A$$

$$|A| = \begin{vmatrix} 1 & 3 \\ 1 & 9 \end{vmatrix}$$

$$= 9 - 3$$

$$= 6$$

$$\text{adj } A = \begin{bmatrix} 9 & -3 \\ -1 & 1 \end{bmatrix}$$

$$A^{-1} = \frac{1}{6} \begin{bmatrix} 9 & -3 \\ -1 & 1 \end{bmatrix}$$

$$x = A^{-1} B$$

$$\begin{bmatrix} x \\ y \end{bmatrix} = \frac{1}{6} \begin{bmatrix} 9 & -3 \\ -1 & 1 \end{bmatrix} \begin{bmatrix} 19800 \\ 23400 \end{bmatrix}$$

$$= \begin{bmatrix} 9 & -3 \\ -1 & 1 \end{bmatrix} \begin{bmatrix} 3300 \\ 3900 \end{bmatrix}$$

$$= \begin{bmatrix} 29700 - 11700 \\ -3300 + 3900 \end{bmatrix}$$

$$= \begin{bmatrix} 18000 \\ 600 \end{bmatrix}$$

$$x = 18000 \quad ; \quad y = 600$$

∴ His Starting Salary is ₹ 18000

Annual increment is ₹ 600

$$\begin{array}{r} 33 \ 2 \\ \underline{297} \end{array}$$

$$\begin{array}{r} 39 \ 2 \\ \underline{33} \\ 600 \end{array}$$

Ex 13

- ④ Four men and 4 women can finish a piece of work jointly in 3 days while 2 men and 5 women can finish the same work jointly in 4 days. Find the time taken by one man alone and that of one woman alone to finish the same work by using matrix inversion method.

Solution:

Let x be the one man's one day work

Let y be the one woman's one day work

By the given data

$$4x + 4y = \frac{1}{3}$$

$$2x + 5y = \frac{1}{4}$$

Writing the given system of equations in Matrix form

$$\begin{bmatrix} 4 & 4 \\ 2 & 5 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} \frac{1}{3} \\ \frac{1}{4} \end{bmatrix}$$

This is of the form $Ax = B$

$$x = A^{-1}B$$

$$A^{-1} = \frac{1}{|A|} \text{adj } A$$

$$|A| = \begin{vmatrix} 4 & 4 \\ 2 & 5 \end{vmatrix}$$

$$= 20 - 8$$

$$= 12 \neq 0$$

$$\text{adj } A = \begin{bmatrix} 5 & -4 \\ -2 & 4 \end{bmatrix}$$

$$A^{-1} = \frac{1}{12} \begin{bmatrix} 5 & -4 \\ -2 & 4 \end{bmatrix}$$

$$X = A^{-1} B$$

$$\Rightarrow \begin{bmatrix} x \\ y \end{bmatrix} = \frac{1}{12} \begin{bmatrix} 5 & -4 \\ -2 & 4 \end{bmatrix} \begin{bmatrix} \frac{1}{3} \\ \frac{1}{6} \end{bmatrix}$$

$$= \frac{1}{12} \begin{bmatrix} 5/3 - 4/6 \\ -2/3 + 4/6 \end{bmatrix}$$

$$= \frac{1}{12} \begin{bmatrix} 5/3 - 2/3 \\ -2/3 + 2/3 \end{bmatrix}$$

$$= \frac{1}{12} \begin{bmatrix} 2/3 \\ 0 \end{bmatrix}$$

$$= \begin{bmatrix} 2/36 \\ 0 \end{bmatrix}$$

$$= \begin{bmatrix} 1/18 \\ 0 \end{bmatrix}$$

$$x = 1/18 ; y = 0$$

∴ One man's one day work = $1/18$

One woman's one day work = 0

∴ A man and a woman alone can do the work in 18 days and 36 days respectively.

Ex 1.3(5) The prices of three commodities A, B and C are ₹ x, y and z per units respectively. A person P purchases 4 units of B and sells 2 units of A and 5 units of C. Person Q purchases 2 units of C and sells 3 units of A and one unit of B. Person R purchases one unit of A and sells 3 units of B and one unit of C. In the process P, Q and R earn ₹ 15000, ₹ 1000 and

₹ 4,000 respectively. Find the prices per unit of A, B and C (using matrix inversion method)

Solution

By the given data,

$$2x - 4y + 5z = 15000 \quad (P)$$

$$3x + y - 2z = 1000 \quad (Q)$$

$$-x + 3y + z = 4000 \quad (R)$$

Writing the given system of equations in matrix form

$$\begin{bmatrix} 2 & -4 & 5 \\ 3 & 1 & -2 \\ -1 & 3 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 15000 \\ 1000 \\ 4000 \end{bmatrix}$$

This is of the form $Ax = B$

$$x = A^{-1}B$$

$$A^{-1} = \frac{1}{|A|} \text{adj} A$$

$$|A| = \begin{vmatrix} 2 & -4 & 5 \\ 3 & 1 & -2 \\ -1 & 3 & 1 \end{vmatrix}$$

$$= 2(1+6) + 4(3-2) + 5(9+1)$$

$$= 2(7) + 4(1) + 5(10)$$

$$= 14 + 4 + 50$$

$$= 68 \neq 0$$

$$\text{adj} A = [A_{ji}]^T$$

$$= \begin{bmatrix} +(1+6) & -(3-2) & +(9+1) \\ -(-4-15) & +(2+5) & -(6-4) \\ +(8-5) & -(-4-15) & 2+12 \end{bmatrix}^T$$

$$= \begin{bmatrix} 7 & -1 & 10 \\ 19 & 7 & -2 \\ 3 & 19 & 14 \end{bmatrix}^T$$

$$= \begin{bmatrix} 7 & 19 & 3 \\ -1 & 7 & 19 \\ 10 & -2 & 14 \end{bmatrix}$$

$$A^{-1} = \frac{1}{68} \begin{bmatrix} 7 & 19 & 3 \\ -1 & 7 & 19 \\ 10 & -2 & 14 \end{bmatrix}$$

$$\begin{array}{r} 105000 \\ 19000 \\ \hline 124000 \end{array}$$

$$X = A^{-1}B$$

$$\Rightarrow \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \frac{1}{68} \begin{bmatrix} 7 & 19 & 3 \\ -1 & 7 & 19 \\ 10 & -2 & 14 \end{bmatrix} \begin{bmatrix} 15000 \\ 1000 \\ 4000 \end{bmatrix}$$

$$= \frac{1}{68} \begin{bmatrix} 105000 + 19000 + 12000 \\ -15000 + 7000 + 76000 \\ 150000 - 2000 + 56000 \end{bmatrix}$$

$$= \frac{1}{68} \begin{bmatrix} 136000 \\ 68000 \\ 204000 \end{bmatrix}$$

$$= \frac{1000}{68} \begin{bmatrix} 136 \\ 68 \\ 204 \end{bmatrix}$$

$$= 1000 \begin{bmatrix} 2 \\ 1 \\ 3 \end{bmatrix}$$

$$= \begin{bmatrix} 2000 \\ 1000 \\ 3000 \end{bmatrix}$$

$$x = 2000; y = 1000; z = 3000$$

The prices per unit of A, B and C respectively ₹2000, ₹1000, ₹3000

(ii) Cramer's rule

Example 1.25

Solve by Cramer's rule the system of equations $x_1 - x_2 = 3$, $2x_1 + 3x_2 + 4x_3 = 17$
 $x_2 + 2x_3 = 7$

Solution

$$\Delta = \begin{vmatrix} 1 & -1 & 0 \\ 2 & 3 & 4 \\ 0 & 1 & 2 \end{vmatrix}$$

$$= 1(6-4) + 1(4) \\ = 2+4 \\ = 6$$

$$\Delta_1 = \begin{vmatrix} 3 & -1 & 0 \\ 17 & 3 & 4 \\ 7 & 1 & 2 \end{vmatrix}$$

$$= 3(6-4) + 1(34-28) \\ = 3(2) + 1(6) \\ = 12$$

$$\Delta_2 = \begin{vmatrix} 1 & 3 & 0 \\ 2 & 17 & 4 \\ 0 & 7 & 2 \end{vmatrix}$$

$$= 1(34-28) - 3(14-0) \\ = 1(6) - 12 \\ = -6$$

(63)

$$\Delta_3 = \begin{vmatrix} 1 & -1 & 3 \\ 2 & 3 & 17 \\ 0 & 1 & 7 \end{vmatrix}$$

$$= 1(21 - 17) + 1(14) + 3(2)$$

$$= 1(4) + 14 + 6$$

$$= 24$$

By Cramer's rule,

$$\begin{aligned} \text{we get } x_1 &= \frac{\Delta_1}{\Delta} \\ &= \frac{12}{6} \\ &= 2 \end{aligned}$$

$$\begin{aligned} x_2 &= \frac{\Delta_2}{\Delta} \\ &= \frac{-6}{6} = -1 \end{aligned}$$

$$\begin{aligned} x_3 &= \frac{\Delta_3}{\Delta} \\ &= \frac{24}{6} = 4 \end{aligned}$$

$$(x_1, x_2, x_3) = (2, -1, 4)$$

Example 1.26

In a T20 match, CSK needed just 6 runs to win with 1 ball left to go in the last over. The last ball was bowled and the batsman at the crease hit it high up. The ball traversed along a path in a vertical plane and the equation of the path is $y = ax^2 + bx + c$ with respect to a xy -coordinate system in the vertical plane and the ball traversed through the points $(10, 8)$, $(20, 16)$, $(40, 22)$. Can you conclude that CSK won the match?

Justify your answer (All distances are measured in metres and the meeting point of the plane of the path with the farthest boundary line is (70, 0)) (64)

Solution:

The path $y = ax^2 + bx + c$ passes through the points (10, 8), (20, 16) and (40, 22), so we get the system of equations,

$$8 = a(10)^2 + b(10) + c \Rightarrow 100a + 10b + c = 8$$

$$16 = a(20)^2 + b(20) + c \Rightarrow 400a + 20b + c = 16$$

$$22 = a(40)^2 + b(40) + c \Rightarrow 1600a + 40b + c = 22$$

$$\Delta = \begin{vmatrix} 100 & 10 & 1 \\ 400 & 20 & 1 \\ 1600 & 40 & 1 \end{vmatrix}$$

$$= 100 \times 10 \begin{vmatrix} 1 & 1 & 1 \\ 4 & 2 & 1 \\ 16 & 4 & 1 \end{vmatrix}$$

$$= 1000 [1(2-4) - 1(4-16) + 1(16-32)]$$

$$= 1000 [-2 + 12 - 16]$$

$$= 1000 \times -6$$

$$= -6000$$

$$\Delta_1 = \begin{vmatrix} 8 & 10 & 1 \\ 16 & 20 & 1 \\ 22 & 40 & 1 \end{vmatrix}$$

$$= 2 \times 10 \begin{vmatrix} 4 & 1 & 1 \\ 8 & 2 & 1 \\ 11 & 4 & 1 \end{vmatrix}$$

$$= 20 [4(2-4) - 1(8-11) + 1(32-22)]$$

(65)

$$= 20 [-8 + 3 + 10]$$

$$= 20 \times 5$$

$$= 100$$

$$\Delta_2 = \begin{vmatrix} 100 & 8 & 1 \\ 400 & 16 & 1 \\ 1600 & 22 & 1 \end{vmatrix}$$

$$= 100 \times 2 \begin{vmatrix} 1 & 4 & 1 \\ 4 & 8 & 1 \\ 16 & 11 & 1 \end{vmatrix}$$

$$= 200 [1(8-11) - 4(4-16) + 1(44-128)]$$

$$= 200 [-3 + 48 - 84]$$

$$= 200 \times -39$$

$$= -7800$$

$$\Delta_3 = \begin{vmatrix} 100 & 10 & 8 \\ 400 & 20 & 16 \\ 1600 & 40 & 22 \end{vmatrix}$$

$$= 100 \times 10 \times 2 \begin{vmatrix} 1 & 1 & 4 \\ 4 & 2 & 8 \\ 16 & 4 & 11 \end{vmatrix}$$

$$= 2000 [1(22-32) - 1(44-128) + 4(16-32)]$$

$$= 2000 [-10 + 84 - 64]$$

$$= 2000 \times (10)$$

$$= 20000$$

By Cramer's rule

$$a = \frac{\Delta_1}{\Delta}$$

$$= \frac{100}{-6000} = -\frac{1}{60}$$

$$b = \frac{\Delta_2}{\Delta}$$

$$= \frac{-7800}{-6000}$$

$$= \frac{13}{10}$$

$$c = \frac{\Delta_3}{\Delta}$$

$$= \frac{20000}{-6000}$$

$$= -\frac{20}{6}$$

$$= -\frac{10}{3}$$

$$a = -\frac{1}{60} ; b = \frac{13}{10} ; c = -\frac{10}{3}$$

So the eqn of the path is $y = -\frac{1}{60}x^2 + \frac{13}{10}x - \frac{10}{3}$

when $x = 70$

$$y = -\frac{1}{60}(70)^2 + \frac{13}{10}(70) - \frac{10}{3}$$

$$= -\frac{1}{60} \times 4900 + \frac{13}{10} \times 70 - \frac{10}{3}$$

$$= -\frac{245}{3} + 91 - \frac{10}{3}$$

$$= \frac{-245 + 273 - 10}{3}$$

$$= \frac{18}{3} = 6$$

when $x = 70$, $y = 6$, so the ball went by 6 metres high over the boundary line and it is impossible for a fielder standing even just before the boundary line to jump and catch the ball. Hence the ball went for a Super six and CSK won the match.

H.W

Exercise 1.4

① Solve by Cramer's rule

(i) $5x - 2y + 16 = 0$; $x + 3y - 7 = 0$

(iii) $3x + 3y - z = 11$; $2x - y + 2z = 9$; $4x + 3y + 2z = 25$

① Solve by Cramer's rule,

(ii) $\frac{3}{x} + 2y = 12$; $\frac{2}{x} + 3y = 13$

Solution

Let $\frac{1}{x} = a$

The system of equation becomes

$$3a + 2y = 12$$

$$2a + 3y = 13$$

$$\Delta = \begin{vmatrix} 3 & 2 \\ 2 & 3 \end{vmatrix} = 9 - 4 = 5$$

$$\Delta_1 = \begin{vmatrix} 12 & 2 \\ 13 & 3 \end{vmatrix} = 36 - 26 = 10$$

$$\Delta_2 = \begin{vmatrix} 3 & 12 \\ 2 & 13 \end{vmatrix} = 39 - 24 = 15$$

By Cramer's rule

$$a = \frac{\Delta_1}{\Delta}$$

$$= \frac{10}{5} = 2$$

$$y = \frac{\Delta_2}{\Delta} = \frac{15}{5} = 3$$

$$a = 2 \quad ; \quad y = 3$$

$$\frac{1}{x} = a \Rightarrow x = \frac{1}{a}$$

$$\boxed{\begin{array}{l} x = \frac{1}{2} \\ y = 3 \end{array}}$$

$$\textcircled{1} \text{ (iv) } \frac{3}{x} - \frac{4}{y} - \frac{2}{z} - 1 = 0 \quad ; \quad \frac{1}{x} + \frac{2}{y} + \frac{1}{z} - 2 = 0$$

$$\frac{2}{x} - \frac{5}{y} - \frac{4}{z} + 1 = 0$$

Solution

The system of equations are

$$\frac{3}{x} - \frac{4}{y} - \frac{2}{z} = 1$$

$$\frac{1}{x} + \frac{2}{y} + \frac{1}{z} = 2$$

$$\frac{2}{x} - \frac{5}{y} - \frac{4}{z} = -1$$

$$\text{Let } \frac{1}{x} = a \quad ; \quad \frac{1}{y} = b \quad ; \quad \frac{1}{z} = c$$

$\therefore$ The system of equations becomes

$$3a - 4b - 2c = 1$$

$$a + 2b + c = 2$$

$$2a - 5b - 4c = -1$$

$$\Delta = \begin{vmatrix} 3 & -4 & -2 \\ 1 & 2 & 1 \\ 2 & -5 & -4 \end{vmatrix}$$

$$= 3(-8 + 5) + 4(-4 - 2) - 2(-5 - 4)$$

$$= 3(-3) + 4(-6) - 2(-9)$$

$$= -9 - 24 + 18$$

$$= -15$$

$$\Delta_1 = \begin{vmatrix} 1 & -4 & -2 \\ 2 & 2 & 1 \\ -1 & -5 & -4 \end{vmatrix}$$

$$= 1(-8+5) + 4(-8+1) - 2(-10+2)$$

$$= 1(-3) + 4(-7) - 2(-8)$$

$$= -3 - 28 + 16$$

$$= -15$$

$$\Delta_2 = \begin{vmatrix} 3 & 1 & -2 \\ 1 & 2 & 1 \\ 2 & -1 & -4 \end{vmatrix}$$

$$= 3(-8+1) - 1(-4-2) - 2(-1-4)$$

$$= 3(-7) - 1(-6) - 2(-5)$$

$$= -21 + 6 + 10$$

$$= -5$$

$$\Delta_3 = \begin{vmatrix} 3 & -4 & 1 \\ 1 & 2 & 2 \\ 2 & -5 & -1 \end{vmatrix}$$

$$= 3(-2+10) + 4(-1-4) + 1(-5-4)$$

$$= 3(8) + 4(-5) + 1(-9)$$

$$= 24 - 20 - 9$$

$$= -5$$

By Cramer's rule

$$a = \frac{\Delta_1}{\Delta} = \frac{-15}{-15} = 1$$

$$b = \frac{\Delta_2}{\Delta} = \frac{-5}{-15} = \frac{1}{3}$$

$$c = \frac{\Delta_3}{\Delta} = \frac{-5}{-15} = \frac{1}{3}$$

(70)

$$a = 1 ; \quad b = \frac{1}{3} ; \quad c = \frac{1}{3}$$

$$\frac{1}{x} = a \Rightarrow x = \frac{1}{a} = 1$$

$$\frac{1}{y} = b \Rightarrow y = \frac{1}{b} = \frac{1}{\frac{1}{3}} = 3$$

$$\frac{1}{z} = c \Rightarrow z = \frac{1}{c} = \frac{1}{\frac{1}{3}} = 3$$

$$x = 1 ; \quad y = 3 ; \quad z = 3$$

Ex 1.4

- (2) In a competitive examination, one mark is awarded for every correct answer while $\frac{1}{4}$ mark is deducted for every wrong answer. A student answered 100 questions and got 80 marks. How many questions did he answer correctly? (use Cramer's rule)

Solution

Let x and y be the number of questions answered correctly and incorrectly

By the data $x + y = 100$ — (1)

$$1 \cdot x + \left(-\frac{1}{4}\right)y = 80$$

$$x - \frac{1}{4}y = 80$$

$$4x - y = 320 \text{ — (2)}$$

$$\Delta = \begin{vmatrix} 1 & 1 \\ 4 & -1 \end{vmatrix}$$

$$= -1 - 4$$

$$= -5$$

$$\Delta_1 = \begin{vmatrix} 100 & 1 \\ 320 & -1 \end{vmatrix}$$

$$= -100 - 320$$

$$= -420$$

$$\Delta_2 = \begin{vmatrix} 1 & 100 \\ 4 & 320 \end{vmatrix}$$

$$= 320 - 400$$

$$= -80$$

By Cramer's rule

$$x = \frac{\Delta_1}{\Delta}$$

$$= \frac{-420}{-5}$$

$$= 84$$

$$y = \frac{\Delta_2}{\Delta}$$

$$= \frac{-80}{-5}$$

$$= 16$$

The student has answered 84 questions correctly.

Exercise 1.4 (3)

A chemist has one solution with 50% acid and another solution which is 25% acid. How much each should be mixed to make ~~the~~ 10 litre of a 40% acid solution? (use Cramer's rule)

Solution

Let x litres of 50% acid and y litre of 25% acid solution to be mixed.

By the given data

$$x + y = 10 \quad \text{--- (1)}$$

$$50\% \text{ of } x + 25\% \text{ of } y = 40\% \text{ of } (x + y)$$

$$\frac{50}{100}x + \frac{25}{100}y = \frac{40}{100}(10)$$

$$\frac{1}{2}x + \frac{1}{4}y = 4$$

$$2x + y = 16 \quad \text{--- (2)}$$

$$\Delta = \begin{vmatrix} 1 & 1 \\ 2 & 1 \end{vmatrix}$$

$$= 1 - 2$$

$$= -1$$

$$\Delta_1 = \begin{vmatrix} 10 & 1 \\ 16 & 1 \end{vmatrix}$$

$$= 10 - 16$$

$$= -6$$

$$\Delta_2 = \begin{vmatrix} 1 & 10 \\ 2 & 16 \end{vmatrix}$$

$$= 16 - 20$$

$$= -4$$

By Cramer's rule

$$x = \frac{\Delta_1}{\Delta} = \frac{-6}{-1} = 6$$

$$y = \frac{\Delta_2}{\Delta} = \frac{-4}{-1} = 4$$

i.e, 6 litres of 50% acid and 4 litres of 25% acid solution to be mixed to get 10 litres of 40% acid solution

- ④ A fish tank can be filled in 10 minutes using both pumps A and B simultaneously. However pump B can pump the water in or out at the same rate. If pump B is inadvertently run in reverse, then the tank will be filled in 30 minutes. How long would it take each pump to fill the tank by itself? (use Cramer's rule)

Solution:

Given $\frac{1}{A} + \frac{1}{B} = \frac{1}{10}$

$$\frac{1}{A} - \frac{1}{B} = \frac{1}{30}$$

Let $\frac{1}{A} = x$; $\frac{1}{B} = y$

Then $x + y = \frac{1}{10} \Rightarrow 10x + 10y = 1$

$$x - y = \frac{1}{30} \Rightarrow 30x - 30y = 1$$

$$\Delta = \begin{vmatrix} 10 & 10 \\ 30 & -30 \end{vmatrix}$$

$$= -300 - 300$$

$$= -600$$

$$\Delta_1 = \begin{vmatrix} 1 & 10 \\ 1 & -30 \end{vmatrix} = -30 - 10$$

$$= -40$$

$$\Delta_2 = \begin{vmatrix} 10 & 1 \\ 30 & 1 \end{vmatrix}$$

$$= 10 - 30$$

$$= -20$$

By Cramer's rule

$$x = \frac{\Delta_1}{\Delta} = \frac{-40}{-600}$$

$$= \frac{1}{15}$$

$$y = \frac{\Delta_2}{\Delta} = \frac{-20}{-600}$$

$$= \frac{1}{30}$$

$$x = \frac{1}{15} \Rightarrow A = 15$$

$$y = \frac{1}{30} \Rightarrow B = 30$$

Pump A, B fill the tank by 15 minutes and 30 minutes respectively.

- ⑤ A family of 3 people went out for dinner in a restaurant. The cost of two dosai, three idlies and two vadais is ₹ 150. The cost of two dosai, two idlies and four vadais is ₹ 200. The cost of five dosai, four idlies and two vadais is ₹ 250. The family has ₹ 350 in hand and they ate 3 dosai, six idlies and six vadais will they be able to manage to pay the bill within the amount they had? (use Cramer's rule)

Solution

(15)

Let the cost of one dosai, one idly and

one vadai be ₹ x , y and z respectively.

By the data,

$$2x + 3y + 2z = 150$$

$$2x + 2y + 4z = 200$$

$$5x + 4y + 2z = 250$$

dosai	idly	vadai	Total
2	3	2	150
2	2	4	200
5	4	2	250
3	6	6	?

$$\begin{aligned} \Delta &= \begin{vmatrix} 2 & 3 & 2 \\ 2 & 2 & 4 \\ 5 & 4 & 2 \end{vmatrix} = 2(4-16) - 3(4-20) + 2(8-10) \\ &= 2(-12) - 3(-16) + 2(-2) \\ &= -24 + 48 - 4 \\ &= 20 \end{aligned}$$

$$\Delta_1 = \begin{vmatrix} 150 & 3 & 2 \\ 200 & 2 & 4 \\ 250 & 4 & 2 \end{vmatrix}$$

$$= 50 \times 2 \begin{vmatrix} 3 & 3 & 1 \\ 4 & 2 & 2 \\ 5 & 4 & 1 \end{vmatrix}$$

$$= 100 [3(2-8) - 3(4-10) + 1(16-10)]$$

$$= 100 [-18 + 18 + 6]$$

$$= 600$$

$$\Delta_2 = \begin{vmatrix} 2 & 150 & 2 \\ 2 & 200 & 4 \\ 5 & 250 & 2 \end{vmatrix}$$

$$= 50 \times 2 \begin{vmatrix} 2 & 3 & 1 \\ 2 & 4 & 2 \\ 5 & 5 & 1 \end{vmatrix}$$

$$= 100 [2(4-10) - 3(2-10) + 1(10-20)]$$

$$= 100 [-12 + 24 - 10] = 200$$

(76)

$$\Delta_3 = \begin{vmatrix} 2 & 3 & 150 \\ 2 & 2 & 200 \\ 5 & 4 & 250 \end{vmatrix}$$

$$= 50 \begin{vmatrix} 2 & 3 & 3 \\ 2 & 2 & 4 \\ 5 & 4 & 5 \end{vmatrix}$$

$$= 50 [2(10-16) - 3(10-20) + 3(8-10)]$$

$$= 50 [-12 + 30 - 6]$$

$$= 50 [12]$$

$$= 600$$

$$\begin{array}{l|l|l} x = \frac{\Delta_1}{\Delta} & y = \frac{\Delta_2}{\Delta} & z = \frac{\Delta_3}{\Delta} \\ \hline = \frac{600}{20} & = \frac{200}{20} & = \frac{600}{20} \\ \hline = 30 & = 10 & = 30 \end{array}$$

The cost of one dosai is ₹30, one idly is ₹10
And one vadaai is ₹30

$$\begin{aligned} \text{The cost of 3 dosai, 6 idly, and 6 vadais} \\ &= 3 \times 30 + 6 \times 10 + 6 \times 30 \\ &= 90 + 60 + 180 \\ &= 330 < 350 \end{aligned}$$

yes, they will able to manage to pay the bill

(iii) Gaussian Elimination Method

Example 1.27

Solve the following system of linear equations by Gaussian elimination method:

$$4x + 3y + 6z = 25; \quad x + 5y + 7z = 13; \quad 2x + 9y + z = 1$$

Solution:

Transforming the augmented matrix to echelon form, we get

$$[A|B] = \left[\begin{array}{ccc|c} 4 & 3 & 6 & 25 \\ 1 & 5 & 7 & 13 \\ 2 & 9 & 1 & 1 \end{array} \right]$$

$$\xrightarrow{R_1 \leftrightarrow R_2} \left[\begin{array}{ccc|c} 1 & 5 & 7 & 13 \\ 4 & 3 & 6 & 25 \\ 2 & 9 & 1 & 1 \end{array} \right]$$

$$\xrightarrow{\begin{array}{l} R_2 \rightarrow R_2 - 4R_1 \\ R_3 \rightarrow R_3 - 2R_1 \end{array}} \left[\begin{array}{ccc|c} 1 & 5 & 7 & 13 \\ 0 & -17 & -22 & -27 \\ 0 & -1 & -13 & -25 \end{array} \right]$$

$$\xrightarrow{\begin{array}{l} R_2 \rightarrow R_2 \times (-1) \\ R_3 \rightarrow R_3 \times (-1) \end{array}} \left[\begin{array}{ccc|c} 1 & 5 & 7 & 13 \\ 0 & 17 & 22 & 27 \\ 0 & 1 & 13 & 25 \end{array} \right]$$

$$\xrightarrow{R_3 \rightarrow 17R_3 - R_2} \left[\begin{array}{ccc|c} 1 & 5 & 7 & 13 \\ 0 & 17 & 22 & 27 \\ 0 & 0 & 199 & 398 \end{array} \right]$$

Rewrite the equivalent matrix in $Ax = B$ form

$$\left[\begin{array}{ccc} 1 & 5 & 7 \\ 0 & 17 & 22 \\ 0 & 0 & 199 \end{array} \right] \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 13 \\ 27 \\ 398 \end{bmatrix}$$

(78)

$$x + 5y + 7z = 13 \quad \text{--- (1)}$$

$$17y + 22z = 27 \quad \text{--- (2)}$$

$$199z = 398 \quad \text{--- (3)}$$

$$(3) \Rightarrow z = \frac{398}{199}$$

$$\boxed{z = 2}$$

$$(2) \Rightarrow 17y + 22(2) = 27$$

$$17y + 44 = 27$$

$$17y = 27 - 44$$

$$17y = -17$$

$$\boxed{y = -1}$$

$$(1) \Rightarrow x + 5(-1) + 7(2) = 13$$

$$x - 5 + 14 = 13$$

$$x + 9 = 13$$

$$\boxed{x = 4}$$

$$(x, y, z) = (4, -1, 2)$$

H.W

Exercise 1.5

① Solve the following system of linear equations by Gaussian elimination method.

(i) $2x - 2y + 3z = 2$; $x + 2y - z = 3$; $3x - y + 2z =$

(ii) $2x + 4y + 6z = 22$; $3x + 8y + 5z = 27$;

$-x + y + 2z = 2.$

Example 1.28

The upward speed $v(t)$ of a rocket at time t is approximated by $v(t) = at^2 + bt + c$, $0 \leq t \leq 100$, where a , b and c are constants. It has been found that the speed at times $t=3$, $t=6$ and $t=9$ seconds are respectively, 64, 133 and 208 miles per second respectively. Find the speed at time $t=15$ seconds (use Gaussian elimination method)

Solution

Since $v(3) = 64$; $v(6) = 133$

$v(9) = 208$, we get the following system of linear equations

t	v
3	64
6	133
9	208
15	?

$$a(3)^2 + b(3) + c = 64$$

$$9a + 3b + c = 64 \quad \text{--- (1)}$$

$$a(6)^2 + b(6) + c = 133$$

$$36a + 6b + c = 133 \quad \text{--- (2)}$$

$$a(9)^2 + b(9) + c = 208$$

$$81a + 9b + c = 208 \quad \text{--- (3)}$$

Reducing the above system from augmented matrix to an equivalent row-echelon form by using elementary row operations, we get

$$[A|B] = \left[\begin{array}{ccc|c} 9 & 3 & 1 & 64 \\ 36 & 6 & 1 & 133 \\ 81 & 9 & 1 & 208 \end{array} \right]$$

$$\begin{array}{l} R_2 \rightarrow R_2 - 4R_1 \\ R_3 \rightarrow R_3 - 9R_1 \end{array} \rightarrow \left[\begin{array}{ccc|c} 9 & 3 & 1 & 64 \\ 0 & -6 & -3 & -123 \\ 0 & -18 & -8 & -368 \end{array} \right]$$

$$\begin{array}{l} R_2 \rightarrow R_2 / -3 \\ \longrightarrow \\ R_3 \rightarrow R_3 / -2 \end{array} \left[\begin{array}{ccc|c} 9 & 3 & 1 & 64 \\ 0 & 2 & 1 & 41 \\ 0 & 9 & 4 & 184 \end{array} \right]$$

(80)

$$R_3 \rightarrow 2R_3 - 9R_2 \left[\begin{array}{ccc|c} 9 & 3 & 1 & 64 \\ 0 & 2 & 1 & 41 \\ 0 & 0 & -1 & -1 \end{array} \right]$$

writing the equivalent equations from the row-echelon matrix, we get

$$9a + 3b + c = 64 \quad \text{--- (4)}$$

$$2b + c = 41 \quad \text{--- (5)}$$

$$-c = -1$$

$$\boxed{c = 1}$$

$$\text{(5)} \Rightarrow 2b + 1 = 41$$

$$2b = 40$$

$$\boxed{b = 20}$$

$$\text{(4)} \Rightarrow 9a + 60 + 1 = 64$$

$$9a = 64 - 61$$

$$9a = 3$$

$$\boxed{a = \frac{1}{3}}$$

So, we get $V(t) = \frac{1}{3}t^2 + 20t + 1$

When $t = 15$

$$V(15) = \frac{1}{3}(15)^2 + 20(15) + 1$$

$$= \frac{1}{3} \times 225 + 300 + 1$$

$$= 75 + 300 + 1$$

$$= 376 \text{ miles/second.}$$

Ex 1.5(2)

If ax^2+bx+c is divided by $x+3$, $x-5$ and $x-1$, the remainders are 21, 61 and 9 respectively. Find a , b and c (use Gaussian elimination method)

Solution :-

$$\text{Let } f(x) = ax^2 + bx + c$$

$$\text{By the data } f(-3) = 21 ; f(5) = 61 ; f(1) = 9$$

$\therefore$ The system of equations are

$$(-3)^2 a + (-3)b + c = 21$$

$$9a - 3b + c = 21 \text{ --- (1)}$$

$$(5)^2 a + 5b + c = 61$$

$$25a + 5b + c = 61 \text{ --- (2)}$$

$$(1)^2 a + 1 \cdot b + c = 9$$

$$a + b + c = 9 \text{ --- (3)}$$

The augmented matrix

$$[A|B] = \left[\begin{array}{ccc|c} 9 & -3 & 1 & 21 \\ 25 & 5 & 1 & 61 \\ 1 & 1 & 1 & 9 \end{array} \right]$$

$$\xrightarrow{R_1 \leftrightarrow R_3} \left[\begin{array}{ccc|c} 1 & 1 & 1 & 9 \\ 25 & 5 & 1 & 61 \\ 9 & -3 & 1 & 21 \end{array} \right]$$

$$\begin{array}{l} R_2 \rightarrow R_2 - 25R_1 \\ R_3 \rightarrow R_3 - 9R_1 \end{array} \rightarrow \left[\begin{array}{ccc|c} 1 & 1 & 1 & 9 \\ 0 & -20 & -24 & -164 \\ 0 & -12 & -8 & -60 \end{array} \right]$$

$$\begin{array}{l} R_2 \rightarrow R_2 / -4 \\ R_3 \rightarrow R_3 / 4 \end{array} \rightarrow \left[\begin{array}{ccc|c} 1 & 1 & 1 & 9 \\ 0 & 5 & 6 & 41 \\ 0 & 3 & 2 & 15 \end{array} \right]$$

$$R_3 \rightarrow 5R_3 - 3R_2 \rightarrow \begin{bmatrix} 1 & 1 & 1 & | & 9 \\ 0 & 5 & 6 & | & 41 \\ 0 & 0 & -8 & | & -48 \end{bmatrix}$$

This is in row echelon form.
Rewrite the last equivalent matrix in $Ax=B$ form

~~$a+b+c=9$~~ $a+b+c=9 \quad \text{--- (4)}$

$5b+6c=41 \quad \text{--- (5)}$

$-8c = -48$

$$\boxed{c = 6}$$

$(5) \Rightarrow 5b + 36 = 41$

$5b = 41 - 36$

$5b = 5$

$$\boxed{b = 1}$$

$(4) \Rightarrow a + 1 + 6 = 9$

$$\boxed{a = 2}$$

$$\underline{a = 2 ; b = 1 ; c = 6}$$

- ③ An amount of ₹ 65000 is invested in three bonds at the rates of 6%, 8% and 9% per annum respectively. The total annual income is ₹ 4800. The income from the third bond is ₹ 600 more than that from the second bond. Determine the price of each bond (use Gaussian elimination method)

Soln ∴ Let x , y and z be the amounts invested in the three bonds.

By the given data,

$$x + y + z = 65000 \quad \text{--- (1)}$$

$$6\% \text{ of } x + 8\% \text{ of } y + 9\% \text{ of } z = 4800$$

$$\frac{6}{100}x + \frac{8}{100}y + \frac{9}{100}z = 4800$$

(83)

$$6x + 8y + 9z = 480000 \quad \text{--- (2)}$$

$$9\% \text{ of } z = 600 + 8\% \text{ of } y$$

$$\frac{9}{100}z = 600 + \frac{8}{100}y$$

$$9z = 60000 + 8y$$

$$8y - 9z = -60000 \quad \text{--- (3)}$$

The augmented matrix

$$[A|B] = \left[\begin{array}{ccc|c} 1 & 1 & 1 & 65000 \\ 6 & 8 & 9 & 480000 \\ 0 & 8 & -9 & -60000 \end{array} \right]$$

$$R_2 \rightarrow R_2 - 6R_1 \rightarrow \left[\begin{array}{ccc|c} 1 & 1 & 1 & 65000 \\ 0 & 2 & 3 & 90000 \\ 0 & 8 & -9 & -60000 \end{array} \right]$$

$$R_3 \rightarrow R_3 - 4R_2 \rightarrow \left[\begin{array}{ccc|c} 1 & 1 & 1 & 65000 \\ 0 & 2 & 3 & 90000 \\ 0 & 0 & -21 & -420000 \end{array} \right]$$

This is in row echelon form

Rewrite the last equivalent matrix in $Ax=B$ form

$$x + y + z = 65000 \quad \text{--- (4)}$$

$$2y + 3z = 90000 \quad \text{--- (5)}$$

$$-21z = -420000 \quad \text{--- (6)}$$

$$(6) \Rightarrow z = \frac{-420000}{-21}$$

$$\boxed{z = 20000}$$

$$(5) \Rightarrow 2y + 60000 = 90000$$

$$2y = 30000$$

$$y = 15000$$

(84)

$$\textcircled{1} \Rightarrow x + 15000 + 20000 = 65000$$

$$x = 65000 - 35000$$

$$x = 30000$$

The amount invested in the first bond is ₹ 30000, second bond is ₹ 15000 and the third bond is ₹ 20000

- ④ A boy is walking along the path $y = ax^2 + bx + c$ through the points $(-6, 8)$, $(-2, -12)$ and $(3, 8)$. He wants to meet his friend at $P(7, 60)$. Will he meet his friend? (use Gaussian elimination method)

Soln Eqn of the boy's path is $y = ax^2 + bx + c$
It passes through the point $(-6, 8)$, $(-2, -12)$ and $(3, 8)$

$$\therefore a(-6)^2 + b(-6) + c = 8$$

$$36a - 6b + c = 8 \quad \text{--- (1)}$$

$$a(-2)^2 + b(-2) + c = -12$$

$$4a - 2b + c = -12 \quad \text{--- (2)}$$

$$a(3)^2 + b(3) + c = 8$$

$$9a + 3b + c = 8 \quad \text{--- (3)}$$

The augmented matrix

$$[A|B] = \left[\begin{array}{ccc|c} 36 & -6 & 1 & 8 \\ 4 & -2 & 1 & -12 \\ 9 & 3 & 1 & 8 \end{array} \right]$$

(85)

$$\begin{array}{l} R_2 \rightarrow 9R_2 - R_1 \\ R_3 \rightarrow 4R_3 - R_1 \end{array} \rightarrow \left[\begin{array}{ccc|c} 36 & -6 & 1 & 8 \\ 0 & -12 & 8 & -116 \\ 0 & 18 & 3 & 24 \end{array} \right]$$

$$\begin{array}{l} R_2 \rightarrow R_2/4 \\ R_3 \rightarrow R_3/3 \end{array} \rightarrow \left[\begin{array}{ccc|c} 36 & -6 & 1 & 8 \\ 0 & -3 & 2 & -29 \\ 0 & 6 & 1 & 8 \end{array} \right]$$

$$R_3 \rightarrow R_3 + 2R_2 \rightarrow \left[\begin{array}{ccc|c} 36 & -6 & 1 & 8 \\ 0 & -3 & 2 & -29 \\ 0 & 0 & 5 & -50 \end{array} \right]$$

This is in row echelon form
Rewrite the last equivalent matrix in $Ax = B$ form

$$36a - 6b + c = 8 \quad \text{--- (4)}$$

$$-3b + 2c = -29 \quad \text{--- (5)}$$

$$5c = -50 \quad \text{--- (6)}$$

$$(6) \Rightarrow \boxed{c = -10}$$

$$(5) \Rightarrow -3b - 20 = -29$$

$$-3b = -29 + 20$$

$$-3b = -9$$

$$\boxed{b = 3}$$

$$(4) \Rightarrow 36a - 18 - 10 = 8$$

$$36a - 28 = 8$$

$$36a = 36$$

$$\boxed{a = 1}$$

∴ The eqn is $y = x^2 + 3x - 10$

when $x = 7$

$$\begin{aligned} y &= 7^2 + 3(7) - 10 \\ &= 49 + 21 - 10 \\ &= 70 - 10 \\ &= 60 \end{aligned}$$

The point $(7, 60)$ satisfies this equation.
Hence the boy will meet his friend.

Application of Matrices: Consistency of system of Linear Equations by Rank method.

Rouche - Capelli Theorem

A system of linear equations written in the matrix form as $AX = B$ is consistent if and only if the rank of the coefficient matrix is equal to the rank of the augmented matrix; that is, $P(A) = P([A|B])$

By Rouche - Capelli theorem, we have the following rule

- If there are 'n' unknowns in the system of equations and $P(A) = P([A|B]) = n$, then the system is consistent and has a unique solution.
- In particular, if there are 3 unknowns in the system of equations and $P(A) = P([A|B]) = 2$, then the system has infinitely many solutions and these solutions form a one parameter family.

- In the same manner, if there are 3 unknowns (87) in a system of equations and $P(A) = P([A|B]) = 1$, then the system has infinitely many solutions and these solutions form a two parameter family.
- If $P(A) \neq P([A|B])$, then the system $Ax = B$ is inconsistent and has no solution.

Example 1.29

Test for consistency of the following system of linear equations if possible solve

$$x + 2y - z = 3; \quad 3x - y + 2z = 1; \quad x - 2y + 3z = 3$$

$$x - y + z = -1$$

Solution :

Number of unknowns : 3

The augmented matrix $[A|B] = \left[\begin{array}{ccc|c} 1 & 2 & -1 & 3 \\ 3 & -1 & 2 & 1 \\ 1 & -2 & 3 & 3 \\ 1 & -1 & 1 & -1 \end{array} \right]$

$$\begin{array}{l} R_2 \rightarrow R_2 - 3R_1 \\ R_3 \rightarrow R_3 - R_1 \\ R_4 \rightarrow R_4 - R_1 \end{array} \rightarrow \left[\begin{array}{ccc|c} 1 & 2 & -1 & 3 \\ 0 & -7 & 5 & -8 \\ 0 & -4 & 4 & 0 \\ 0 & -3 & 2 & -4 \end{array} \right]$$

$$R_3 \rightarrow R_3/4 \rightarrow \left[\begin{array}{ccc|c} 1 & 2 & -1 & 3 \\ 0 & -7 & 5 & -8 \\ 0 & -1 & 1 & 0 \\ 0 & -3 & 2 & -4 \end{array} \right]$$

$$\begin{array}{l} R_3 \rightarrow 7R_3 - R_2 \\ R_4 \rightarrow R_4 - 3R_3 \end{array} \rightarrow \left[\begin{array}{ccc|c} 1 & 2 & -1 & 3 \\ 0 & -7 & 5 & -8 \\ 0 & 0 & 2 & 8 \\ 0 & 0 & -1 & -4 \end{array} \right]$$

$$R_4 \rightarrow 2R_4 + R_3 \rightarrow \left[\begin{array}{ccc|c} 1 & 2 & -1 & 3 \\ 0 & -7 & 5 & -8 \\ 0 & 0 & 2 & 8 \\ 0 & 0 & 0 & 0 \end{array} \right]$$

This is in row echelon form

$$\text{Here, } \rho([A|B]) = 3 ; \rho(A) = 3$$

$$\rho([A|B]) = \rho(A) = 3 (\text{no. of unknowns})$$

$\therefore$ By Rouché-Capelli theorem,
The system is consistent and has
unique soln.

To find the soln, Rewrite the last equivalent matrix
in $Ax = B$ form

$$x + 2y - z = 3 \quad \text{--- (1)}$$

$$-7y + 5z = -8 \quad \text{--- (2)}$$

$$2z = 8 \quad \text{--- (3)}$$

$$0 = 0$$

$$\text{(3)} \Rightarrow \boxed{z = 4}$$

$$\text{(2)} \Rightarrow -7y + 20 = -8$$

$$-7y = -28$$

$$\boxed{y = 4}$$

$$\text{(1)} \Rightarrow x + 8 - 4 = 3$$

$$x + 4 = 3$$

$$\boxed{x = -1}$$

$$(x, y, z) = (-1, 4, 4) \checkmark$$

Example 1.30

Test for consistency of the following system of linear equations and if possible solve:

$$4x - 2y + 6z = 8; \quad x + y - 3z = -1; \quad 15x - 3y + 9z = 21$$

Solution :

Number of unknowns : 3

The augmented matrix $[A|B] = \left[\begin{array}{ccc|c} 4 & -2 & 6 & 8 \\ 1 & 1 & -3 & -1 \\ 15 & -3 & 9 & 21 \end{array} \right]$

$$\xrightarrow{R_1 \leftrightarrow R_2} \left[\begin{array}{ccc|c} 1 & 1 & -3 & -1 \\ 4 & -2 & 6 & 8 \\ 15 & -3 & 9 & 21 \end{array} \right]$$

$$\xrightarrow{\begin{array}{l} R_2 \rightarrow R_2/2 \\ R_3 \rightarrow R_3/3 \end{array}} \left[\begin{array}{ccc|c} 1 & 1 & -3 & -1 \\ 2 & -1 & 3 & 4 \\ 5 & -1 & 3 & 7 \end{array} \right]$$

$$\xrightarrow{\begin{array}{l} R_2 \rightarrow R_2 - 2R_1 \\ R_3 \rightarrow R_3 - 5R_1 \end{array}} \left[\begin{array}{ccc|c} 1 & 1 & -3 & -1 \\ 0 & -3 & 9 & 6 \\ 0 & -6 & 18 & 12 \end{array} \right]$$

$$\xrightarrow{R_3 \rightarrow R_3 - 2R_2} \left[\begin{array}{ccc|c} 1 & 1 & -3 & -1 \\ 0 & -3 & 9 & 6 \\ 0 & 0 & 0 & 0 \end{array} \right]$$

$$\xrightarrow{R_2 \rightarrow R_2/3} \left[\begin{array}{ccc|c} 1 & 1 & -3 & -1 \\ 0 & -1 & 3 & 2 \\ 0 & 0 & 0 & 0 \end{array} \right]$$

This is in row echelon form,

Here $\rho([A|B]) = 2$; $\rho(A) = 2$

$\rho([A|B]) = \rho(A) = 2 < 3$ (no. of unknowns).

Rewrite the last equivalent matrix in $Ax = B$ form

(90)

$$x + y + 3z = -1 \quad \text{--- (1)}$$

$$-y + 3z = 2 \quad \text{--- (2)}$$

There are 3 unknowns and 2 equations

$\therefore$ we assign $z = t, t \in \mathbb{R}$

$$(2) \Rightarrow -y + 3t = 2$$

$$-y = 2 - 3t$$

$$y = 3t - 2$$

$$(1) \Rightarrow x + 3t - 2 - 3t = -1$$

$$x - 2 = -1$$

$$x = 1$$

$$x = 1; y = 3t - 2; z = t, t \in \mathbb{R}.$$

The above solution set is a one parameter family of solutions

$\therefore$ The given system is consistent and has infinitely many solutions which form a one parameter family of solutions.

Example 1.31

Test for consistency of the following system of linear equations and if possible solve,

$$x - y + z = -9; 2x - 2y + 2z = -18$$

$$3x - 3y + 3z + 27 = 0.$$



Here number of unknowns : 3

The augmented matrix

$$[A|B] = \left[\begin{array}{ccc|c} 1 & -1 & 1 & -9 \\ 2 & -2 & 2 & -18 \\ 3 & -3 & 3 & -27 \end{array} \right]$$

$$\begin{array}{l} R_2 \rightarrow R_2 - 2R_1 \\ R_3 \rightarrow R_3 - 3R_1 \end{array} \rightarrow \left[\begin{array}{ccc|c} 1 & -1 & 1 & -9 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array} \right]$$

This is in row echelon form

$$\text{Here } \rho([A|B]) = 1; \rho(A) = 1$$

$$\rho([A|B]) = \rho(A) = 1 < 3 (\text{no. of unknowns})$$

The system is consistent and has infinitely many soln.

Rewrite the last Equivalent matrix in $Ax = B$ form

$$x - y + z = -9$$

There are three unknowns and one eqn.

we assign $z = t, y = s, s, t \in \mathbb{R}$.

$$x - s + t = -9$$

$$x = -9 + s - t$$

The soln is $(x = -9 + s - t, y = s, z = t), s, t \in \mathbb{R}$.

The above soln set is a two-parameter family of solutions.

The given system of equations is consistent and has infinitely many solutions which form a two parameter family of solutions.

Example 1.32

Test the consistency of the following system of linear equations $x - y + z = -9$; $2x - y + z = 4$

$$3x - y + z = 6 ; 4x - y + 2z = 7$$

Solution :

The number of unknowns : 3

The augmented matrix $[A|B] = \left[\begin{array}{ccc|c} 1 & -1 & 1 & -9 \\ 2 & -1 & 1 & 4 \\ 3 & -1 & 1 & 6 \\ 4 & -1 & 2 & 7 \end{array} \right]$

$$\begin{array}{l} R_2 \rightarrow R_2 - 2R_1 \\ R_3 \rightarrow R_3 - 3R_1 \\ R_4 \rightarrow R_4 - 4R_1 \end{array} \rightarrow \left[\begin{array}{ccc|c} 1 & -1 & 1 & 9 \\ 0 & 1 & -1 & 22 \\ 0 & 2 & -2 & 33 \\ 0 & 3 & -2 & 43 \end{array} \right]$$

$$\begin{array}{l} R_3 \rightarrow R_3 - 2R_2 \\ R_4 \rightarrow R_4 - 3R_2 \end{array} \rightarrow \left[\begin{array}{ccc|c} 1 & -1 & 1 & 9 \\ 0 & 1 & -1 & 22 \\ 0 & 0 & 0 & -11 \\ 0 & 0 & 1 & -23 \end{array} \right]$$

$$\xrightarrow{R_3 \leftrightarrow R_4} \left[\begin{array}{ccc|c} 1 & -1 & 1 & 9 \\ 0 & 1 & -1 & 22 \\ 0 & 0 & 1 & -23 \\ 0 & 0 & 0 & -11 \end{array} \right]$$

This is row echelon form.

$$\rho([A|B]) = 4 ; \rho(A) = 3$$

$$\rho([A|B]) \neq \rho(A)$$

$\therefore$ The system is inconsistent and has no solution.

If we write the equivalent system of equations using the echelon form, we get.

$$x - y + z = -9$$

$$y - z = 22$$

$$z = -23$$

$$0 = -11$$

The last equation is a contradiction.
So, the given system of equations is inconsistent and has no solution.

H.W Exercise 1.6

① Test for consistency and if possible, solve the following system of equations by rank method:

(i) $x - y + 2z = 2$; $2x + y + 4z = 7$; $4x - y + z = 4$

(ii) $3x + y + z = 2$; $x - 3y + 2z = 1$; $7x - y + 4z = 5$

(iii) $2x + 2y + z = 5$; $x - y + z = 1$; $3x + y + 2z = 4$

(iv) $2x - y + z = 2$; $6x - 3y + 3z = 6$; $4x - 2y + 2z = 4$

Example 1.33

Find the condition on a, b and c so that the following system of linear equations has one parameter family of solutions; $x + y + z = a$;
 $x + 2y + 3z = b$; $3x + 5y + 7z = c$

Solution:-

Here, the number of unknowns : 3

The augmented matrix $[A|B] = \left[\begin{array}{ccc|c} 1 & 1 & 1 & a \\ 1 & 2 & 3 & b \\ 3 & 5 & 7 & c \end{array} \right]$

$$\begin{array}{l} R_2 \rightarrow R_2 - R_1 \\ R_3 \rightarrow R_3 - 3R_1 \end{array} \rightarrow \left[\begin{array}{ccc|c} 1 & 1 & 1 & a \\ 0 & 1 & 2 & b-a \\ 0 & 2 & 4 & c-3a \end{array} \right]$$

$$R_3 \rightarrow R_3 - 2R_2 \rightarrow \left[\begin{array}{ccc|c} 1 & 1 & 1 & a \\ 0 & 1 & 2 & b-a \\ 0 & 0 & 0 & c-a-2b \end{array} \right]$$

(94)

Given the system have one parameter family of solutions, we must have $P([A|B]) = P(A) = 2$.

So, the third row in the echelon form should be a zero row; so $c - 2b - a = 0$

$$\boxed{c = a + 2b}$$

Example 1.34 (X)

Investigate for what values of λ and μ the system of linear equations $x + 2y + z = 7$, $x + y + \lambda z = \mu$, $x + 3y - 5z = 5$ has (i) no soln (ii) a unique soln (iii) an infinite number of solns.

Soln:

Here number of unknowns: 3

The augmented matrix $[A|B] = \left[\begin{array}{ccc|c} 1 & 2 & 1 & 7 \\ 1 & 1 & \lambda & \mu \\ 1 & 3 & -5 & 5 \end{array} \right]$

$$R_2 \leftrightarrow R_3 \rightarrow \left[\begin{array}{ccc|c} 1 & 2 & 1 & 7 \\ 1 & 3 & -5 & 5 \\ 1 & 1 & \lambda & \mu \end{array} \right]$$

$$\begin{array}{l} R_2 \rightarrow R_2 - R_1 \\ R_3 \rightarrow R_3 - R_1 \end{array} \rightarrow \left[\begin{array}{ccc|c} 1 & 2 & 1 & 7 \\ 0 & 1 & -6 & -2 \\ 0 & -1 & \lambda - 1 & \mu - 7 \end{array} \right]$$

$$R_3 \rightarrow R_3 + R_2 \rightarrow \left[\begin{array}{ccc|c} 1 & 2 & 1 & 7 \\ 0 & 1 & -6 & -2 \\ 0 & 0 & \lambda - 7 & \mu - 9 \end{array} \right]$$

This is row echelon form

(i) no soln

(95)

$$\text{If } \lambda - 7 = 0 ; \mu - 9 \neq 0$$

$$\text{i.e., } \lambda = 7 ; \mu \neq 9$$

$$\text{Then } \rho([A|B]) = 3 ; \rho(A) = 2$$

$\rho([A|B]) \neq \rho(A)$. Hence the given system is inconsistent and has no soln.

(ii) a unique soln

$$\text{If } \lambda - 7 \neq 0 ; \mu \in \mathbb{R}$$

$$\lambda \neq 7$$

$$\text{Then } \rho([A|B]) = 3 ; \rho(A) = 3$$

$$\rho([A|B]) = \rho(A) = 3 (\text{no. of unknowns})$$

Hence the given system is consistent and has a unique soln.

(ii) an infinite number of solns

$$\text{If } \lambda - 7 = 0 ; \mu - 9 = 0$$

$$\text{i.e., } \lambda = 7 ; \mu = 9$$

$$\text{Then } \rho([A|B]) = 2 ; \rho(A) = 2$$

$$\rho([A|B]) = \rho(A) = 2 < 3 (\text{no. of unknowns})$$

Hence the given system is consistent and has infinite number of solutions.

Exercise 1.6

- (3) Investigate the values of λ and μ the system of linear equations $2x+3y+5z=9$;
 $7x+3y-5z=8$; $2x+3y+\lambda z=\mu$ have (i) no soln
 (ii) a unique soln (iii) an infinite number of solns
Soln :

Here number of unknowns : 3

The augmented matrix $[A|B] = \left[\begin{array}{ccc|c} 2 & 3 & 5 & 9 \\ 7 & 3 & -5 & 8 \\ 2 & 3 & \lambda & \mu \end{array} \right]$

$$\begin{array}{l} R_2 \rightarrow 2R_2 - 7R_1 \\ R_3 \rightarrow R_3 - R_1 \end{array} \left[\begin{array}{ccc|c} 2 & 3 & 5 & 9 \\ 0 & -15 & -45 & -47 \\ 0 & 0 & \lambda-5 & \mu-9 \end{array} \right]$$

This is in row echelon form

- (i) no soln : If $\lambda-5=0$; $\mu-9 \neq 0$
 i.e., $\lambda=5$; $\mu \neq 9$

Then $\rho([A|B])=3$; $\rho(A)=2$

$\rho([A|B]) \neq \rho(A)$. Hence the given system is inconsistent and has no soln.

- (ii) a unique soln : If $\lambda-5 \neq 0$; $\mu \in \mathbb{R}$
 i.e., $\lambda \neq 5$

Then $\rho([A|B])=3$; $\rho(A)=3$

$\rho([A|B]) = \rho(A) = 3$ (no. of unknowns)

Hence the system is consistent and has unique soln.

(97)

(iii) an infinite number of solutions

$$I + \lambda - 5 = 0 ; \mu - 9 = 0$$

$$\text{i.e., } \lambda = 5 ; \mu = 9$$

$$\rho([A|B]) = 2 ; \rho(A) = 2$$

$$\rho([A|B]) = \rho(A) = 2 < \text{no. of unknowns}$$

$\therefore$ The system is consistent and has infinite number of soln.

② Find the value of k for which the equations

$$kx - 2y + z = 1, x - 2ky + z = -2, x - 2y + kz = 1 \text{ have}$$

(i) no solution (ii) unique solution (iii) infinitely many solution.

Solution :

Here number of unknowns : 3

$$\text{The augmented matrix } [A|B] = \left[\begin{array}{ccc|c} k & -2 & 1 & 1 \\ 1 & -2k & 1 & -2 \\ 1 & -2 & k & 1 \end{array} \right]$$

$$\xrightarrow{R_1 \leftrightarrow R_3} \left[\begin{array}{ccc|c} 1 & -2 & k & 1 \\ 1 & -2k & 1 & -2 \\ k & -2 & 1 & 1 \end{array} \right]$$

$$\xrightarrow{\begin{array}{l} R_2 \rightarrow R_2 - R_1 \\ R_3 \rightarrow R_3 - kR_1 \end{array}} \left[\begin{array}{ccc|c} 1 & -2 & k & 1 \\ 0 & -2k+2 & 1-k & -3 \\ 0 & -2+2k & 1-k^2 & 1-k \end{array} \right]$$

$$R_3 \rightarrow R_3 + R_2 \rightarrow \left[\begin{array}{ccc|c} 1 & -2 & k & 1 \\ 0 & -2k+2 & 1-k & -3 \\ 0 & 0 & 2-k-k^2 & -2-k \end{array} \right]$$

$$\rightarrow \left[\begin{array}{ccc|c} 1 & -2 & k & 1 \\ 0 & -2k+2 & 1-k & -3 \\ 0 & 0 & (k+2)(1-k) & -(k+2) \end{array} \right]$$

This is in row echelon form

$$\begin{aligned} & 2-k-k^2 \\ &= -(k^2+k-2) \\ &= -[(k+2)(k-1)] \\ &= (k+2)(1-k) \end{aligned}$$

(i) No soln

If $k=1$

$$P([A|B]) \neq P(A)$$

The system is inconsistent and has no soln.

(ii) Unique soln

If $k \neq 1$; $k \neq -2$

$$\text{Then } P([A|B]) = 3 ; P(A) = 3$$

$$P([A|B]) = P(A) = 3 \text{ (no. of unknowns)}$$

$\therefore$ The system is consistent and has Unique soln

(iii) Infinitely many soln

If $k = -2$

$$\text{Then } P([A|B]) = 2 ; P(A) = 2$$

$$P([A|B]) = P(A) = 2 < 3 \text{ (no. of unknowns)}$$

$\therefore$ The system is consistent and has infinitely many solution.

Homogeneous System of linear Equations

- If $\rho(A) = \rho([A|0]) = n$ (number of unknowns)
Then a homogeneous system has a unique solution and it is the trivial solution.
Since $\rho(A) = n$, $|A| \neq 0$, so for trivial solution $|A| \neq 0$.
- If $\rho(A) = \rho([A|0]) < n$ (number of unknowns)
Then a homogeneous system has a nontrivial solution. Since $\rho(A) < n$, $|A| = 0$.
So for nontrivial solution $|A| = 0$.

Example 1.35

Solve the following system:

$$x + 2y + 3z = 0 ; 3x + 4y + 4z = 0 ; 7x + 10y + 12z = 0$$

Solution:

Here number of unknowns : 3

The augmented matrix $[A|0] = \left[\begin{array}{ccc|c} 1 & 2 & 3 & 0 \\ 3 & 4 & 4 & 0 \\ 7 & 10 & 12 & 0 \end{array} \right]$

$$\begin{array}{l} R_2 \rightarrow R_2 - 3R_1 \\ R_3 \rightarrow R_3 - 7R_1 \end{array} \rightarrow \left[\begin{array}{ccc|c} 1 & 2 & 3 & 0 \\ 0 & -2 & -5 & 0 \\ 0 & -4 & -9 & 0 \end{array} \right]$$

$$R_3 \rightarrow R_3 - 2R_2 \rightarrow \left[\begin{array}{ccc|c} 1 & 2 & 3 & 0 \\ 0 & -2 & -5 & 0 \\ 0 & 0 & 1 & 0 \end{array} \right]$$

This is in row echelon form

$$\rho([A|0]) = 3 ; \quad \rho(A) = 3$$

$$\rho([A|0]) = \rho(A) = 3 \text{ (no. of unknowns)}$$

$\therefore$ The system is consistent and has a unique soln. Since $x=0$; $y=0$; $z=0$ is always a soln of homogeneous system, the only solution is the trivial solution $x=0, y=0, z=0$

Example 1.36

Solve the system : $x+3y-2z=0$, $2x-y+4z=0$

$$x-11y+14z=0.$$

Solution:

Here no. of unknowns : 3

The augmented matrix $[A|0] = \left[\begin{array}{ccc|c} 1 & 3 & -2 & 0 \\ 2 & -1 & 4 & 0 \\ 1 & -11 & 14 & 0 \end{array} \right]$

$$\begin{array}{l} R_2 \rightarrow R_2 - 2R_1 \\ R_3 \rightarrow R_3 - R_1 \end{array} \rightarrow \left[\begin{array}{ccc|c} 1 & 3 & -2 & 0 \\ 0 & -7 & 8 & 0 \\ 0 & -14 & 16 & 0 \end{array} \right]$$

$$\xrightarrow{R_3 \rightarrow R_3 - 2R_2} \left[\begin{array}{ccc|c} 1 & 3 & -2 & 0 \\ 0 & -7 & 8 & 0 \\ 0 & 0 & 0 & 0 \end{array} \right]$$

This is in row echelon form

$$\rho([A|0]) = 2 ; \quad \rho(A) = 2$$

$$\rho([A|0]) = \rho(A) = 2 < 3 \text{ (no. of unknowns)}$$

∴ The system is consistent and infinitely many solutions (Non trivial solns)

Rewrite the last ~~eqn~~ equivalent matrix in eqn form

$$x + 3y - 2z = 0 \quad \text{--- (1)}$$

$$-7y + 8z = 0 \quad \text{--- (2)}$$

$$0 = 0$$

There are 3 unknowns and 2 equations

Assign $z = t, t \in \mathbb{R}$

$$(2) \Rightarrow -7y + 8t = 0$$

$$-7y = -8t$$

$$y = \frac{8}{7}t$$

$$(1) \Rightarrow x + 3\left(\frac{8}{7}t\right) - 2t = 0$$

$$\cancel{x} = 2t - \frac{24}{7}t$$

$$x = \frac{14t - 24t}{7}$$

$$x = -\frac{10}{7}t$$

The soln is $(x = -\frac{10}{7}t, y = \frac{8}{7}t, z = t), t \in \mathbb{R}$

The given system forms one parameter family of solutions.

Example 1.37

Solve the system: $x+y-2z=0$, $2x-3y+z=0$
 $3x-7y+10z=0$; $6x-9y+10z=0$

Solution :

Here number of unknowns : 3

The augmented matrix $[A|0] = \left[\begin{array}{ccc|c} 1 & 1 & -2 & 0 \\ 2 & -3 & 1 & 0 \\ 3 & -7 & 10 & 0 \\ 6 & -9 & 10 & 0 \end{array} \right]$

$$\begin{array}{l} R_2 \rightarrow R_2 - 2R_1 \\ R_3 \rightarrow R_3 - 3R_1 \\ R_4 \rightarrow R_4 - 6R_1 \end{array} \rightarrow \left[\begin{array}{ccc|c} 1 & 1 & -2 & 0 \\ 0 & -5 & 5 & 0 \\ 0 & -10 & 16 & 0 \\ 0 & -15 & 22 & 0 \end{array} \right]$$

$$\begin{array}{l} R_2 \rightarrow R_2/5 \\ R_3 \rightarrow R_3/2 \end{array} \rightarrow \left[\begin{array}{ccc|c} 1 & 1 & -2 & 0 \\ 0 & -1 & 1 & 0 \\ 0 & -5 & 8 & 0 \\ 0 & -15 & 22 & 0 \end{array} \right]$$

$$\begin{array}{l} R_3 \rightarrow R_3 - 5R_2 \\ R_4 \rightarrow R_4 - 3R_3 \end{array} \rightarrow \left[\begin{array}{ccc|c} 1 & 1 & -2 & 0 \\ 0 & -1 & 1 & 0 \\ 0 & 0 & 3 & 0 \\ 0 & 0 & -2 & 0 \end{array} \right]$$

$$R_4 \rightarrow 3R_4 + 2R_3 \rightarrow \left[\begin{array}{ccc|c} 1 & 1 & -2 & 0 \\ 0 & -1 & 1 & 0 \\ 0 & 0 & 3 & 0 \\ 0 & 0 & 0 & 0 \end{array} \right]$$

This is in row echelon form

$$P([A|0]) = 3; \quad P(A) = 3$$

$$P([A|0]) = P(A) = 3 \text{ (no. of unknowns)}$$

$\therefore$ The system is consistent and has a unique soln.
 Since $x=0$; $y=0$; $z=0$ is always a soln of

homogeneous system, the only soln is the trivial soln $x=0, y=0, z=0$

Example 1.38

Determine the values of λ for which the following system of equations $(3\lambda-8)x+3y+3z=0$
 $3x+(3\lambda-8)y+3z=0$; $3x+3y+(3\lambda-8)z=0$ has a nontrivial solution

Solution

Here number of unknowns: 3

Given the system has non trivial soln.

$$\therefore \rho([A|0]) = \rho(A) < 3$$

The determinant of the coefficient matrix should be '0'.

Hence we get,

$$\begin{vmatrix} 3\lambda-8 & 3 & 3 \\ 3 & 3\lambda-8 & 3 \\ 3 & 3 & 3\lambda-8 \end{vmatrix} = 0$$

$$\begin{vmatrix} 3\lambda-2 & 3\lambda-2 & 3\lambda-2 \\ 3 & 3\lambda-8 & 3 \\ 3 & 3 & 3\lambda-8 \end{vmatrix} = 0 \quad R_1 \rightarrow R_1 + R_2 + R_3$$

$$(3\lambda-2) \begin{vmatrix} 1 & 1 & 1 \\ 3 & 3\lambda-8 & 3 \\ 3 & 3 & 3\lambda-8 \end{vmatrix} = 0$$

$$(3\lambda - 2) \begin{vmatrix} 0 & 0 & 1 \\ 0 & 3\lambda - 11 & 3 \\ 11 - 3\lambda & 0 & 3\lambda - 8 \end{vmatrix} \begin{matrix} C_1 \rightarrow C_1 - C_3 \\ C_2 \rightarrow C_2 - C_1 \end{matrix} = 0 \quad (104)$$

$$(3\lambda - 2) [1(0 - (3\lambda - 11)(11 - 3\lambda))] = 0$$

$$(3\lambda - 2) [(3\lambda - 11)(3\lambda - 11)] = 0$$

$$(3\lambda - 2)(3\lambda - 11)^2 = 0$$

$$3\lambda - 2 = 0; (3\lambda - 11)^2 = 0$$

$$\lambda = \frac{2}{3} \quad 3\lambda - 11 = 0$$

$$\lambda = \frac{11}{3}$$

$$\boxed{\lambda = \frac{2}{3} \text{ or } \lambda = \frac{11}{3}}$$

Example 1.39

By using Gaussian elimination method, balance the chemical reaction eqn:



Solution

We are searching for positive integers

x_1, x_2, x_3 and x_4 such that



Equating the number of carbon atoms

$$5x_1 = x_3 \Rightarrow 5x_1 - x_3 = 0 \quad \text{--- (1)}$$

Equating hydrogen atoms,

$$8x_1 = 2x_4 \Rightarrow 4x_1 - x_4 = 0 \quad \text{--- (2)}$$

Equating Oxygen atoms,

$$2x_2 = 2x_3 + x_4$$

$$2x_2 - 2x_3 - x_4 = 0 \quad \text{--- (3)}$$

Here number of unknowns : 4

The augmented matrix

$$[A|0] = \left[\begin{array}{cccc|c} 5 & 0 & -1 & 0 & 0 \\ 4 & 0 & 0 & -1 & 0 \\ 0 & 2 & -2 & -1 & 0 \end{array} \right]$$

$$\xrightarrow{R_2 \rightarrow 5R_2 - 4R_1} \left[\begin{array}{cccc|c} 5 & 0 & -1 & 0 & 0 \\ 0 & 0 & 4 & -5 & 0 \\ 0 & 2 & -2 & -1 & 0 \end{array} \right]$$

$$\xrightarrow{R_2 \leftrightarrow R_3} \left[\begin{array}{cccc|c} 5 & 0 & -1 & 0 & 0 \\ 0 & 2 & -2 & -1 & 0 \\ 0 & 0 & 4 & -5 & 0 \end{array} \right]$$

This is in row echelon form,

$$\rho([A|0]) = \rho(A) = 3 < 4 \text{ (number of unknowns)}$$

$\therefore$ The system is consistent and has infinitely many solutions

Writing the equations using the echelon form

$$5x_1 - x_3 = 0 \quad \text{--- (4)}$$

$$2x_2 - 2x_3 - x_4 = 0 \quad \text{--- (5)}$$

$$4x_3 - 5x_4 = 0 \quad \text{--- (6)}$$

There are ~~four~~ 4 unknowns and 3 equations,

Assign $x_4 = t$, $t \in \mathbb{R}$.

(106)

$$(6) \Rightarrow 4x_3 - 5t = 0$$

$$4x_3 = 5t$$

$$\boxed{x_3 = \frac{5}{4}t}$$

$$(5) \Rightarrow 2x_2 - 2\left(\frac{5}{4}\right)t - t = 0$$

$$2x_2 = \frac{10}{4}t + t$$

$$2x_2 = \frac{10t + 4t}{4}$$

$$2x_2 = \frac{14t}{4}$$

$$\boxed{x_2 = \frac{7t}{4}}$$

$$(4) \Rightarrow 5x_1 - \frac{5}{4}t = 0$$

$$5x_1 = \frac{5}{4}t$$

$$\boxed{x_1 = \frac{1}{4}t}$$

$$(x_1, x_2, x_3, x_4) = \left(\frac{t}{4}, \frac{7t}{4}, \frac{5t}{4}, t\right), t \in \mathbb{R}.$$

Since x_1, x_2, x_3 and x_4 are positive integers

let us choose $t = 4$

Then we get $x_1 = 1$; $x_2 = 7$; $x_3 = 5$; $x_4 = 4$

So, the balanced eqn is



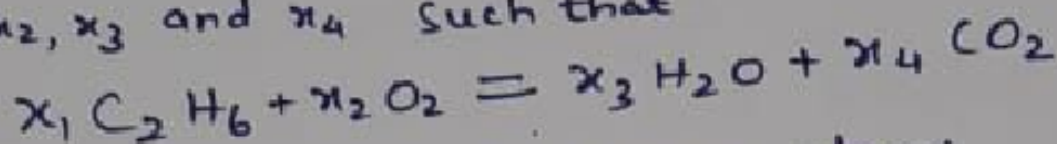
Ex 1.7

(3) By using Gaussian elimination method, balance the chemical reaction equation:



Solution

We are searching the positive integers x_1, x_2, x_3 and x_4 such that



Equating the number of Carbon atoms

$$2x_1 = x_4 \Rightarrow 2x_1 - x_4 = 0$$

Equating Hydrogen atoms

$$6x_1 = 2x_3 \Rightarrow 3x_1 - x_3 = 0$$

Equating Oxygen atoms

$$2x_2 = x_3 + 2x_4 \Rightarrow 2x_2 - x_3 - 2x_4 = 0$$

Here number of unknowns : 4

The augmented matrix $[A|0] = \left[\begin{array}{cccc|c} 2 & 0 & 0 & -1 & 0 \\ 3 & 0 & -1 & 0 & 0 \\ 0 & 2 & -1 & -2 & 0 \end{array} \right]$

$$\xrightarrow{R_2 \rightarrow 2R_2 - 3R_1} \left[\begin{array}{cccc|c} 2 & 0 & 0 & -1 & 0 \\ 0 & 0 & -2 & 3 & 0 \\ 0 & 2 & -1 & -2 & 0 \end{array} \right]$$

$$\xrightarrow{R_2 \leftrightarrow R_3} \left[\begin{array}{cccc|c} 2 & 0 & 0 & -1 & 0 \\ 0 & 2 & -1 & -2 & 0 \\ 0 & 0 & -2 & 3 & 0 \end{array} \right]$$

This is in row echelon form

$$\rho([A|0]) = \rho(A) = 3 < 4 \text{ (no. of unknowns)}$$

$\therefore$ The system is consistent and infinitely many solutions

writing the equations using echelon form

$$2x_1 - x_4 = 0 \quad \text{--- (1)}$$

$$2x_2 - x_3 - 2x_4 = 0 \quad \text{--- (2)}$$

$$-2x_3 + 3x_4 = 0 \quad \text{--- (3)}$$

(108)

There are 4 unknowns and 3 equations

Assign $x_4 = t$, $t \in \mathbb{R}$

$$(3) \Rightarrow -2x_3 + 3t = 0$$

$$-2x_3 = -3t$$

$$\boxed{x_3 = \frac{3t}{2}}$$

$$(2) \Rightarrow 2x_2 - \frac{3t}{2} - 2t = 0$$

$$2x_2 = \frac{3t}{2} + 2t$$

$$2x_2 = \frac{7t}{2}$$

$$\boxed{x_2 = \frac{7t}{4}}$$

$$(1) \Rightarrow 2x_1 - t = 0$$

$$2x_1 = t$$

$$\boxed{x_1 = \frac{t}{2}}$$

$$(x_1, x_2, x_3, x_4) = \left(\frac{t}{2}, \frac{7t}{4}, \frac{3t}{2}, t \right), t \in \mathbb{R}$$

Since x_1, x_2, x_3 and x_4 are positive integers
let us choose $t = 4$

Then we get $x_1 = 2$; $x_2 = 7$; $x_3 = 6$; $x_4 = 4$

So, the balanced eqn is



- ② Determine the values of λ for which the following system of equations $x+y+3z=0$; $4x+3y+\lambda z=0$; $2x+y+2z=0$ has
(i) a unique solution (ii) a non trivial soln.

Solution :

Here number of unknowns: 3

- (i) The given system has a unique soln

$$\therefore \rho(A) = \rho([A|0]) = 3$$

$$\rho(A) = 3 \Rightarrow \begin{vmatrix} 1 & 1 & 3 \\ 4 & 3 & \lambda \\ 2 & 1 & 2 \end{vmatrix} \neq 0$$

$$1(6-\lambda) - 1(8-2\lambda) + 3(4-6) \neq 0$$

$$6 - \lambda - 8 + 2\lambda - 6 \neq 0$$

$$\lambda - 8 \neq 0$$

$$\lambda \neq 8$$

If $\lambda \neq 8$, then the system has a unique soln

- (ii) The given system has a non trivial soln,

$$\therefore \rho(A) = \rho([A|0]) = 2$$

$$\rho(A) = 2 \Rightarrow \begin{vmatrix} 1 & 1 & 3 \\ 4 & 3 & \lambda \\ 2 & 1 & 2 \end{vmatrix} = 0$$

$$\lambda - 8 = 0$$

$$\lambda = 8$$

If $\lambda = 8$, then the system has a nontrivial solution

Example 1.40

If the system of equations $px+by+cz=0$;
 $ax+qy+cz=0$; $ax+by+rz=0$ has a non trivial
 Soln and $p \neq a$; $q \neq b$; $r \neq c$ P.T $\frac{p}{p-a} + \frac{q}{q-b} + \frac{r}{r-c} = 2$

Solution Number of unknowns: 3

Given the system has non trivial solution.

$$\therefore \rho(A) = \rho([A|0]) = 2 < 3$$

$$\rho(A) = 2 \Rightarrow \begin{vmatrix} p & b & c \\ a & q & c \\ a & b & r \end{vmatrix} = 0$$

$$\begin{vmatrix} p & b & c \\ a-p & q-b & 0 \\ a-p & 0 & r-c \end{vmatrix} = 0 \quad \begin{array}{l} R_2 \rightarrow R_2 - R_1 \\ R_3 \rightarrow R_3 - R_1 \end{array}$$

$$\begin{vmatrix} p & b & c \\ -(p-a) & q-b & 0 \\ -(p-a) & 0 & r-c \end{vmatrix} = 0$$

$$(p-a)(q-b)(r-c) \begin{vmatrix} \frac{p}{p-a} & \frac{b}{q-b} & \frac{c}{r-c} \\ -1 & 1 & 0 \\ -1 & 0 & 1 \end{vmatrix} = 0$$

$$\begin{vmatrix} \frac{p}{p-a} & \frac{b}{q-b} & \frac{c}{r-c} \\ -1 & 1 & 0 \\ -1 & 0 & 1 \end{vmatrix} = 0$$

$$\frac{p}{p-a} (1-0) - \frac{b}{q-b} (-1) + \frac{c}{r-c} (0+1) = 0$$

$$\frac{p}{p-a} + \frac{b}{q-b} + \frac{c}{r-c} = 0$$



$$\frac{p}{p-a} + \frac{q-(q-b)}{q-b} + \frac{r-(r-c)}{r-c} = 0$$

$$\frac{p}{p-a} + \frac{q}{q-b} - \frac{q-b}{q-b} + \frac{r}{r-c} - \frac{r-c}{r-c} = 0$$

$$\frac{p}{p-a} + \frac{q}{q-b} - 1 + \frac{r}{r-c} - 1 = 0$$

$$\frac{p}{p-a} + \frac{q}{q-b} + \frac{r}{r-c} = 2$$

How

Exercise 1.7

① Solve the following system of homogeneous equations:

$$(i) \quad 3x + 2y + 7z = 0; \quad 4x - 3y - 2z = 0; \\ 5x + 9y + 23z = 0$$

$$(ii) \quad 2x + 3y - z = 0; \quad x - y - 2z = 0; \\ 3x + y + 3z = 0$$