Padasalai's Telegram Groups!

(தலைப்பிற்கு கீழே உள்ள லிங்கை கிளிக் செய்து குழுவில் இணையவும்!)

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ARTHI TUITION CENTER KATTUPUTHUR - 621207

HIGHER SECONDARY - SECOND YEAR

CLASSIFICATION OF TEXT BOOK PROBLEMS

&

CLASSIFICATION OF CREATIVE QUESTION

DEPARTMENT OF MATHEMATICS

12

: PREPARED BY :

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1. APPLICATIONS OF MATRICES AND DETERMINANTS

Points to remember :

- Adjoint of a square matrix A = Transpose of the cofactor matrix of A.
- $A\,(\text{adj}\,A) = (\text{adj}\,A)\,A = |A|\,I_n$.
- $A^{-1} = \dfrac{1}{|A|}\,\text{adj}\,A$.
- (i) $|A^{-1}| = \dfrac{1}{|A|}$ (ii) $(A^T)^{-1} = (A^{-1})^T$ (iii) $(\lambda A)^{-1} = \dfrac{1}{\lambda}A^{-1}$, Where λ is a non-zero scalar.
- (i) $(AB)^{-1} = B^{-1}A^{-1}$ (ii) $(A^{-1})^{-1} = A$
- If A is a non-singular square matrix of order n, then
  (i) $(adj A)^{-1} = adj(A^{-1}) = \dfrac{1}{|A|}A$
  (ii) $|adj A| = |A|^{n-1}$
  (iii) $adj(adj A) = |A|^{n-2}\,A$
  (iv) $adj(\lambda A) = \lambda^{n-1}\,adj(A)$, λ is a non zero scalar
  (v) $|adj(adj A)| = |A|^{(n-1)^2}$
  (vi) $(adj A)^T = adj(A^T)$
  (vii) $adj(AB) = (adj B)(adj A)$
- (i) $A^{-1} = \pm\dfrac{1}{\sqrt{|adj A|}}\,adj A$ (ii) $A = \pm\dfrac{1}{\sqrt{|adj A|}}\,adj(adj A)$
- (i) A matrix A is orthogonal if $AA^T = A^TA = I$
  (ii) A matrix A is orthogonal if and only if A is non-singular and $A^{-1} = A^T$
- Methods to solve the system of linear equations A X = B
  (i) By matrix inversion method $X = A^{-1}B$, $|A| \neq 0$
  (ii) By Cramer's rule $x = \dfrac{\Delta_1}{\Delta}, y = \dfrac{\Delta_2}{\Delta}, z = \dfrac{\Delta_3}{\Delta}, \Delta \neq 0$.
  (iii) By Gaussian elimination method
- (i) If $\rho(A) = \rho([A|B])$ = number of unknowns, then the system has unique solution.
  (ii) If $\rho(A) = \rho([A|B])$ < number of unknowns, then the system has infinitely many solutions.
  (iii) If $\rho(A) \neq \rho([A|B])$ then the system is inconsistent and has no solution.
- The homogenous system of linear equations AX = O
  (i) has the trivial solution, if $|A| \neq 0$.
  (ii) has a non trivial solution, if $|A| = 0$.

BOOK BACK ONE MARKS

1. If $|adj(adjA)| = |A|^9$, then the order of the square matrix A is

   1) 3 **2) 4**
   3) 2 4) 5

   Solution :

   $$|adj(adjA)| = |A|^{(n-1)^2}$$

   Given $|adj(adjA)| = |A|^9$

   $$|A|^{(n-1)^2} = |A|^9$$
   $$(n-1)^2 = 9$$
   $$n-1 = 3$$
   $$n = 4$$

2. If A is a 3 x 3 non-singular matrix such that $AA^T = A^TA$ and $B = A^{-1}A^T$, then $BB^T =$

   1) A 2) B
   3) I 4) B^T

   Solution :

   Given $B = A^{-1}A^T$

   $$BB^T = (A^{-1}A^T)(A^{-1}A^T)^T$$
   $$= (A^{-1}A^T)(A^T)^T(A^{-1})^T$$
   $$= (A^{-1}A^T)A(A^{-1})^T$$
   $$= (A^{-1}A)A^T(A^T)^{-1}$$
   $$= I.I = I$$

3. If $A = \begin{bmatrix} 3 & 5 \\ 1 & 2 \end{bmatrix}$, B = adj A and C = 3A, then $\frac{|adj\ B|}{|C|} =$

   1) $\frac{1}{3}$ **2) $\frac{1}{9}$**
   3) $\frac{1}{4}$ 4) 1

   Solution :

   $$\frac{|adjB|}{|C|} = \frac{|adj(adjA)|}{|3A|} = \frac{|A|^{(n-1)^2}}{3^2|A|}$$
   $$= \frac{|A|}{9|A|} = \frac{1}{9}$$

4. If $A\begin{bmatrix} 1 & -2 \\ 1 & 4 \end{bmatrix} = \begin{bmatrix} 6 & 0 \\ 0 & 6 \end{bmatrix}$, then A =

   1) $\begin{bmatrix} 1 & -2 \\ 1 & 4 \end{bmatrix}$ 2) $\begin{bmatrix} 1 & 2 \\ -1 & 4 \end{bmatrix}$

   3) $\begin{bmatrix} 4 & 2 \\ -1 & 1 \end{bmatrix}$ 4) $\begin{bmatrix} 4 & -1 \\ 2 & 1 \end{bmatrix}$

   Solution :

   $$AX = B \Rightarrow A = BX^{-1}$$

   Where $X = \begin{bmatrix} 1 & -2 \\ 1 & 4 \end{bmatrix}$

   $$X^{-1} = \frac{1}{6}\begin{bmatrix} 4 & 2 \\ -1 & 1 \end{bmatrix}$$
   $$A = 6\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\frac{1}{6}\begin{bmatrix} 4 & 2 \\ -1 & 1 \end{bmatrix}$$
   $$A = \begin{bmatrix} 4 & 2 \\ -1 & 1 \end{bmatrix}$$

5. If $A = \begin{bmatrix} 7 & 3 \\ 4 & 2 \end{bmatrix}$, then $9I - A =$

   1) A^{-1} 2) $\frac{A^{-1}}{2}$
   3) $3A^{-1}$ **4) $2A^{-1}$**

   Solution :

   $$9I-A = \begin{bmatrix} 9 & 0 \\ 0 & 9 \end{bmatrix} - \begin{bmatrix} 7 & 3 \\ 4 & 2 \end{bmatrix}$$
   $$= \begin{bmatrix} 2 & -3 \\ -4 & 7 \end{bmatrix}$$
   $$A^{-1} = \frac{1}{2}\begin{bmatrix} 2 & -3 \\ -4 & 7 \end{bmatrix}$$
   $$2A^{-1} = \begin{bmatrix} 2 & -3 \\ -4 & 7 \end{bmatrix}$$

6. If $A = \begin{bmatrix} 2 & 0 \\ 1 & 5 \end{bmatrix}$ and $B = \begin{bmatrix} 1 & 4 \\ 2 & 0 \end{bmatrix}$ then $|adj\ (AB)| =$

1) –40 **2) –80**
3) –60 4) –20

Solution :

$$AB = \begin{bmatrix} 2+0 & 8+0 \\ 1+10 & 4+0 \end{bmatrix} = \begin{bmatrix} 2 & 8 \\ 11 & 4 \end{bmatrix}$$

$$adj(AB) = \begin{bmatrix} 4 & -8 \\ -11 & 2 \end{bmatrix}$$

$$|adj(AB)| = 8 - 88 = -80$$

7. If $P = \begin{bmatrix} 1 & x & 0 \\ 1 & 3 & 0 \\ 2 & 4 & -2 \end{bmatrix}$ is the adjoint of 3x3 matrix A and $|A| = 4$, then x is

1) 15 2) 12
3) 14 **4) 11**

Solution :

$$|adjA| = |A|^{n-1}$$

$$\begin{vmatrix} 1 & x & 0 \\ 1 & 3 & 0 \\ 2 & 4 & -2 \end{vmatrix} = 4^2$$

$$-6 + 2x = 16$$

$$x = 11$$

8. If $A = \begin{bmatrix} 3 & 1 & -1 \\ 2 & -2 & 0 \\ 1 & 2 & -1 \end{bmatrix}$ and $A^{-1} = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$ then the value of a_{23} is

1) 0 2) –2
3) –3 **4) –1**

Solution :

$$|A| = 3(2-0) - 1(-2-0) - 1(4+2)$$

$$= 6 + 2 - 6$$

$$|A| = 2$$

$$a_{23} = \frac{1}{|A|} \text{cofactors of } a_{32}$$

$$= \frac{1}{2}(-1)\begin{vmatrix} 3 & -1 \\ 2 & 0 \end{vmatrix}$$

$$= -\frac{1}{2}(0+2) = -1$$

9. If A,B and C are invertible matrices of some order, then which one of the following is not true?

1) adj A = |A| A^{-1}
2) adj (AB) = (adj A) (adj B)
3) det A^{-1} = (det A)$^{-1}$
4) $(ABC)^{-1} = C^{-1}B^{-1}A^{-1}$

Solution :

Result : adj(AB) = (adj A)(adj B)

10. If $(AB)^{-1} = \begin{bmatrix} 12 & -17 \\ -19 & 27 \end{bmatrix}$ and $A^{-1} = \begin{bmatrix} 1 & -1 \\ -2 & 3 \end{bmatrix}$, then $B^{-1} =$

1) $\begin{bmatrix} 2 & -5 \\ -3 & 8 \end{bmatrix}$ 2) $\begin{bmatrix} 8 & 5 \\ 3 & 2 \end{bmatrix}$
3) $\begin{bmatrix} 3 & 1 \\ 2 & 1 \end{bmatrix}$ 4) $\begin{bmatrix} 8 & -5 \\ -3 & 2 \end{bmatrix}$

Solution :

since $(AB)^{-1} = B^{-1}A^{-1}$

$$\begin{bmatrix} 12 & -17 \\ -19 & 27 \end{bmatrix} = B^{-1}A^{-1}$$

$$B^{-1}\begin{bmatrix} 1 & -1 \\ -2 & 3 \end{bmatrix} = \begin{bmatrix} 12 & -17 \\ -19 & 27 \end{bmatrix}$$

$$B^{-1}X = \begin{bmatrix} 12 & -17 \\ -19 & 27 \end{bmatrix} \text{ where } X = \begin{bmatrix} 1 & -1 \\ -2 & 3 \end{bmatrix}$$

$$X^{-1} = \frac{1}{1}\begin{bmatrix} 3 & 1 \\ 2 & 1 \end{bmatrix}$$

$$B^{-1} = \begin{bmatrix} 12 & -17 \\ -19 & 27 \end{bmatrix} X^{-1}$$

$$= \begin{bmatrix} 12 & -17 \\ -19 & 27 \end{bmatrix}\begin{bmatrix} 3 & 1 \\ 2 & 1 \end{bmatrix} = \begin{bmatrix} 2 & -5 \\ -3 & 8 \end{bmatrix}$$

11. If A^TA^{-1} is symmetric, then $A^2 =$

1) A^{-1} 2) $(A^T)^2$

3) A^T 4) $(A^{-1})^2$

Solution :

A is symmetric then $A = A^T$

A^TA^{-1} is symmetric $\Rightarrow A^TA^{-1} = (A^TA^{-1})^T$

$A^TA^{-1} = (A^{-1})^T(A^T)^T$

$A^TA^{-1} = (A^{-1})^T A$

Pre-multiply by A^T on both sides

$A^TA^TA^{-1} = A^T(A^{-1})^T A$

Post- multiply by A on both sides

$(A^T)^2A^{-1}A = A^T(A^{-1})^T AA$

$(A^T)^2 I = (A^{-1}A)^T A^2$

$(A^T)^2 = A^2$

12. If A is a non-singular matrix such that $A^{-1} = \begin{bmatrix} 5 & 3 \\ -2 & -1 \end{bmatrix}$, then $(A^T)^{-1} =$

1) $\begin{bmatrix} -5 & 3 \\ 2 & 1 \end{bmatrix}$ 2) $\begin{bmatrix} 5 & 3 \\ -2 & -1 \end{bmatrix}$

3) $\begin{bmatrix} -1 & -3 \\ 2 & 5 \end{bmatrix}$ 4) $\begin{bmatrix} 5 & -2 \\ 3 & -1 \end{bmatrix}$

Solution :

since $(A^T)^{-1} = (A^{-1})^T$

$$(A^T)^{-1} = \begin{bmatrix} 5 & 3 \\ -2 & -1 \end{bmatrix}^T$$

$$= \begin{bmatrix} 5 & -2 \\ 3 & -1 \end{bmatrix}$$

13. If $A = \begin{bmatrix} \frac{3}{5} & \frac{4}{5} \\ x & \frac{3}{5} \end{bmatrix}$ and $A^T = A^{-1}$, then the value of x is

1) $\frac{-4}{5}$ 2) $\frac{-3}{5}$

3) $\frac{3}{5}$ 4) $\frac{4}{5}$

Solution :

since $A^{-1} = A^T$

$AA^T = A^TA = I$ (orthogonal)

$$\frac{1}{5}\begin{bmatrix} 3 & 4 \\ 5x & 3 \end{bmatrix}\frac{1}{5}\begin{bmatrix} 3 & 5x \\ 4 & 3 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

$$\frac{1}{25}\begin{bmatrix} 25 & 15x+12 \\ 15x+12 & 25x^2+9 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}.$$

$$\frac{1}{25}[15x+12] = 0$$

$$[15x+12] = 0$$

$$x = \frac{-4}{5}$$

14. If $A = \begin{bmatrix} 1 & \tan\frac{\theta}{2} \\ -\tan\frac{\theta}{2} & 1 \end{bmatrix}$ and AB = I, then B =

1) $\left(\cos^2\frac{\theta}{2}\right)A$ 2) $\left(\cos^2\frac{\theta}{2}\right)A^T$

3) $(\cos^2\theta)I$ 4) $\left(\sin^2\frac{\theta}{2}\right)A$

Solution :

AB = I

$B = A^{-1}$

$$B = \frac{1}{1+\tan^2\frac{\theta}{2}}\begin{bmatrix} 1 & -\tan\frac{\theta}{2} \\ \tan\frac{\theta}{2} & 1 \end{bmatrix}$$

$$B = \frac{1}{\sec^2\frac{\theta}{2}}A^T$$

$$B = \cos^2\frac{\theta}{2}A^T$$

15. If A= $\begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix}$ and A(adjA) = $\begin{bmatrix} k & 0 \\ 0 & k \end{bmatrix}$, then k =

1) 0 2) sin θ
3) cos θ **4) 1**

Solution :

We know that $A(adjA) = (adjA)A = |A|I$

$\Rightarrow |A| = k$

$k = \begin{vmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{vmatrix}$

$k = \cos^2\theta + \sin^2\theta = 1$

16. If A= $\begin{bmatrix} 2 & 3 \\ 5 & -2 \end{bmatrix}$ be such that λA^{-1} = A, then λ is

1) 17 2) 14
3) 19 4) 21

Solution :

$\lambda A^{-1} = A$

$\lambda \frac{1}{|A|} adjA = A$

$\lambda \frac{1}{-19} \begin{bmatrix} -2 & -3 \\ -5 & 2 \end{bmatrix} = \begin{bmatrix} 2 & 3 \\ 5 & -2 \end{bmatrix}$

$\frac{\lambda}{19} \begin{bmatrix} 2 & 3 \\ 5 & -2 \end{bmatrix} = \begin{bmatrix} 2 & 3 \\ 5 & -2 \end{bmatrix}$

$\frac{\lambda}{19} = 1 \Rightarrow \lambda = 19$

17. If adjA = $\begin{bmatrix} 2 & 3 \\ 4 & -1 \end{bmatrix}$ and adjB = $\begin{bmatrix} 1 & -2 \\ -3 & 1 \end{bmatrix}$ then adj(AB) is

1) $\begin{bmatrix} -7 & -1 \\ 7 & -9 \end{bmatrix}$ **2) $\begin{bmatrix} -6 & 5 \\ -2 & -10 \end{bmatrix}$**

3) $\begin{bmatrix} -7 & 7 \\ -1 & -9 \end{bmatrix}$ 4) $\begin{bmatrix} -6 & -2 \\ 5 & -10 \end{bmatrix}$

Solution :

$adj(AB) = (adjB)(adjA)$

$= \begin{bmatrix} 1 & -2 \\ -3 & 1 \end{bmatrix} \begin{bmatrix} 2 & 3 \\ 4 & -1 \end{bmatrix} = \begin{bmatrix} -6 & 5 \\ -2 & -10 \end{bmatrix}$

18. The rank of the matrix $\begin{bmatrix} 1 & 2 & 3 & 4 \\ 2 & 4 & 6 & 8 \\ -1 & -2 & -3 & -4 \end{bmatrix}$ is

1) 1 2) 2
3) 4 4) 3

Solution :

$\begin{bmatrix} 1 & 2 & 3 & 4 \\ 2 & 4 & 6 & 8 \\ -1 & -2 & -3 & -4 \end{bmatrix}$

$\xrightarrow[R_3 \to R_3 + R_1]{R_2 \to R_2 - 2R_1} \begin{bmatrix} 1 & 2 & 3 & 4 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$

Rank is 1

19. If $x^a y^b = e^m, x^c y^d = e^n$, $\Delta_1 = \begin{vmatrix} m & b \\ n & d \end{vmatrix}$, $\Delta_2 = \begin{vmatrix} a & m \\ c & n \end{vmatrix}$, $\Delta_3 = \begin{vmatrix} a & b \\ c & d \end{vmatrix}$, then the values of x and y are respectively.

1) $e^{(\Delta_2/\Delta_1)}, e^{(\Delta_3/\Delta_1)}$
2) log (Δ₁/Δ₃), log (Δ₂ / Δ₃)
3) log (Δ₂ / Δ₁), log (Δ₃ / Δ₁)
4) $e^{(\Delta_1/\Delta_3)}, e^{(\Delta_2/\Delta_3)}$

Solution :

$x^a y^b = e^m, x^c y^d = e^n$

Taking log on both sides

$a \log x + b \log y = m$

$c \log x + d \log y = n$

$\Delta_3 = \begin{vmatrix} a & b \\ c & d \end{vmatrix} = ad - bc \neq 0$

$\log x = \frac{\Delta_1}{\Delta_3}$ $\log y = \frac{\Delta_2}{\Delta_3}$

$x = e^{\frac{\Delta_1}{\Delta_3}}$ $y = e^{\frac{\Delta_2}{\Delta_3}}$

20. Which of the following is/are correct?

(i) Adjoint of a symmetric matrix is also a symmetric matrix.

(ii) Adjoint of a diagonal matrix is also a diagonal matrix.

(iii) If A is a square matrix of order n and λ is a scalar, then $adj(\lambda A) = \lambda^n adj(A)$

(iv) $A(adj\ A) = (adj\ A)A = |A|I$

1) only (i) 2) (ii) and (iii)

3) (iii) and (iv) **4) (i), (ii) and (iv)**

Solution :

Result: $adj(\lambda A) = \lambda^{n-1} adj(A)$ is true.

21. If $\rho(A) = \rho([A|B])$, then the system AX = B of linear equations is

1) consistent and has a unique solution

2) consistent

3) consistent and has infinitely many solution

4) inconsistent

Solution :

Ans (2) consistent.

22. If $0 \le \theta \le \pi$ and the system of equations $x + (\sin\theta)y - (\cos\theta)z = 0$, $(\cos\theta)x - y + z = 0$, $(\sin\theta)x + y - z = 0$ has a non-trivial solution then θ is

1) $\frac{2\pi}{3}$ 2) $\frac{3\pi}{4}$

3) $\frac{5\pi}{6}$ **4) $\frac{\pi}{4}$**

Solution :

$$A = \begin{bmatrix} 1 & \sin\theta & -\cos\theta \\ \cos\theta & -1 & 1 \\ \sin\theta & 1 & -1 \end{bmatrix}$$

The system has non-trivial solution if $|A| = 0$

$$\begin{vmatrix} 1 & \sin\theta & -\cos\theta \\ \cos\theta & -1 & 1 \\ \sin\theta & 1 & -1 \end{vmatrix} = 0$$

$$1(0) - \sin\theta(-\cos\theta - \sin\theta) - \cos\theta(\cos\theta + \sin\theta) = 0$$

$$\sin\theta\cos\theta + \sin^2\theta - \cos^2\theta - \cos\theta\sin\theta = 0$$

$$\sin^2\theta = \cos^2\theta$$

$$\theta = \frac{\pi}{4}$$

23. The augmented matrix of a system of linear equations is $\begin{bmatrix} 1 & 2 & 7 & 3 \\ 0 & 1 & 4 & 6 \\ 0 & 0 & \lambda-7 & \mu+5 \end{bmatrix}$. The system has infinitely many solutions if

1) $\lambda = 7, \mu \ne -5$ 2) $\lambda = -7, \mu = 5$

3) $\lambda \ne 7, \mu \ne -5$ **4) $\lambda = 7, \mu = -5$**

Solution :

When $\lambda = 7, \mu = -5$

$$[A\,|\,B] = \begin{bmatrix} 1 & 2 & 7 & 3 \\ 0 & 1 & 4 & 6 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$$\rho(A) = \rho[A\,|\,B] = 2 < 3$$

The system is consistent and has infinitely many solutions

24. Let $A = \begin{bmatrix} 2 & -1 & 1 \\ -1 & 2 & -1 \\ 1 & -1 & 2 \end{bmatrix}$ and $4B = \begin{bmatrix} 3 & 1 & -1 \\ 1 & 3 & x \\ -1 & 1 & 3 \end{bmatrix}$. If B is the inverse of A, then the value of x is

1) 2 2) 4

3) 3 **4) 1**

Solution :

Since B is the inverse of A

AB = I

$$\begin{bmatrix} 2 & -1 & 1 \\ -1 & 2 & -1 \\ 1 & -1 & 2 \end{bmatrix} \frac{1}{4} \begin{bmatrix} 3 & 1 & -1 \\ 1 & 3 & x \\ -1 & 1 & 3 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$\frac{1}{4} a_{13} = 0$$

$$\frac{1}{4}[-2 - x + 3] = 0$$

$$\frac{1}{4}[1 - x] = 0$$

$$x = 1$$

25. If $A = \begin{bmatrix} 3 & -3 & 4 \\ 2 & -3 & 4 \\ 0 & -1 & 1 \end{bmatrix}$, then adj(adjA) is

1) $\begin{bmatrix} 3 & -3 & 4 \\ 2 & -3 & 4 \\ 0 & -1 & 1 \end{bmatrix}$ 2) $\begin{bmatrix} 6 & -6 & 8 \\ 4 & -6 & 8 \\ 0 & -2 & 2 \end{bmatrix}$

3) $\begin{bmatrix} -3 & 3 & -4 \\ -2 & 3 & -4 \\ 0 & 1 & -1 \end{bmatrix}$ 4) $\begin{bmatrix} 3 & -3 & 4 \\ 0 & -1 & 1 \\ 2 & -3 & 4 \end{bmatrix}$

Solution :

$|A| = 3(1) + 3(2) + 4(-2)$

$= 3 + 6 - 8$

$|A| = 1$

$adj(adjA) = |A|^{n-2} A$

$= 1.A = A$

BOOK SUMS ***(Exercise and Examples) :***

1. If $A = \begin{bmatrix} 8 & -6 & 2 \\ -6 & 7 & -4 \\ 2 & -4 & 3 \end{bmatrix}$, verify that A(adj A) = (adj A)A = $|A|I_3$.

2. If $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ is non-singular, find A^{-1}.

3. Find the inverse of the matrix $\begin{bmatrix} 2 & -1 & 3 \\ -5 & 3 & 1 \\ -3 & 2 & 3 \end{bmatrix}$.

4. If A is a non-singular matrix of odd order, prove that |adj A| is positive.

5. Find a matrix A if adj(A) = $\begin{bmatrix} 7 & 7 & -7 \\ -1 & 11 & 7 \\ 11 & 5 & 7 \end{bmatrix}$.

6. If adj A = $\begin{bmatrix} -1 & 2 & 2 \\ 1 & 1 & 2 \\ 2 & 2 & 1 \end{bmatrix}$, find A^{-1}.

7. If A is symmetric, prove that adj A is also symmetric.

8. Verify the property $(A^T)^{-1} = (A^{-1})^T$ with $A = \begin{bmatrix} 2 & 9 \\ 1 & 7 \end{bmatrix}$

9. Verify $(AB)^{-1} = B^{-1}A^{-1}$ with $A = \begin{bmatrix} 0 & -3 \\ 1 & 4 \end{bmatrix}$, $B = \begin{bmatrix} -2 & -3 \\ 0 & -1 \end{bmatrix}$

10. If $A = \begin{bmatrix} 4 & 3 \\ 2 & 5 \end{bmatrix}$, find x and y such that $A^2 + xA + yI_2 = O_2$. Hence, find A^{-1}.

11. Prove that $\begin{bmatrix}\cos\theta & -\sin\theta\\ \sin\theta & \cos\theta\end{bmatrix}$ is orthogonal.

12. If $A=\frac{1}{7}\begin{bmatrix}6 & -3 & a\\ b & -2 & 6\\ 2 & c & 3\end{bmatrix}$ is orthogonal, find a, b and c and hence A^{-1}.

13. Find the adjoint of the following

(i) $\begin{bmatrix}-3 & 4\\ 6 & 2\end{bmatrix}$ (ii) $\begin{bmatrix}2 & 3 & 1\\ 3 & 4 & 1\\ 3 & 7 & 2\end{bmatrix}$ (iii) $\frac{1}{3}\begin{bmatrix}2 & 2 & 1\\ -2 & 1 & 2\\ 1 & -2 & 2\end{bmatrix}$

14. Find the inverse (if it exists) of the following

(i) $\begin{bmatrix}-2 & 4\\ 1 & -3\end{bmatrix}$ (ii) $\begin{bmatrix}5 & 1 & 1\\ 1 & 5 & 1\\ 1 & 1 & 5\end{bmatrix}$ (iii) $\begin{bmatrix}2 & 3 & 1\\ 3 & 4 & 1\\ 3 & 7 & 2\end{bmatrix}$

15. If $F(\alpha)=\begin{bmatrix}\cos\alpha & 0 & \sin\alpha\\ 0 & 1 & 0\\ -\sin\alpha & 0 & \cos\alpha\end{bmatrix}$, show that $[F(\alpha)]^{-1}=F(-\alpha)$

16. If $A=\begin{bmatrix}5 & 3\\ -1 & -2\end{bmatrix}$, show that $A^2-3A-7I_2=O_2$. Hence find A^{-1}.

17. If $A=\frac{1}{9}\begin{bmatrix}-8 & 1 & 4\\ 4 & 4 & 7\\ 1 & -8 & 4\end{bmatrix}$, prove that $A^{-1}=A^T$.

18. If $A=\begin{bmatrix}8 & -4\\ -5 & 3\end{bmatrix}$, verify that $A(\text{adj } A)=(\text{adj } A)\,A=|A|\,I_2$.

19. If $A=\begin{bmatrix}3 & 2\\ 7 & 5\end{bmatrix}$ and $B=\begin{bmatrix}-1 & -3\\ 5 & 2\end{bmatrix}$, verify that $(AB)^{-1}=B^{-1}A^{-1}$.

20. If $\text{adj}(A)=\begin{bmatrix}2 & -4 & 2\\ -3 & 12 & -7\\ -2 & 0 & 2\end{bmatrix}$, find A.

21. If $\text{adj}(A)=\begin{bmatrix}0 & -2 & 0\\ 6 & 2 & -6\\ -3 & 0 & 6\end{bmatrix}$, find A^{-1}.

22. Find adj(adj(A)) if $\text{adj } A=\begin{bmatrix}1 & 0 & 1\\ 0 & 2 & 0\\ -1 & 0 & 1\end{bmatrix}$

23. $A=\begin{bmatrix}1 & \tan x\\ -\tan x & 1\end{bmatrix}$, show that $A^T A^{-1}=\begin{bmatrix}\cos 2x & -\sin 2x\\ \sin 2x & \cos 2x\end{bmatrix}$

24. Find the matrix A for which $A\begin{bmatrix} 5 & 3 \\ -1 & -2 \end{bmatrix} = \begin{bmatrix} 14 & 7 \\ 7 & 7 \end{bmatrix}$.

25. Given $A = \begin{bmatrix} 1 & -1 \\ 2 & 0 \end{bmatrix}$, $B = \begin{bmatrix} 3 & -2 \\ 1 & 1 \end{bmatrix}$ and $C = \begin{bmatrix} 1 & 1 \\ 2 & 2 \end{bmatrix}$, find a matrix X such that AXB = C.

26. If $A = \begin{bmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{bmatrix}$, show that $A^{-1} = \frac{1}{2}(A^2 - 3I)$

27. Decrypt the received encoded message $\begin{bmatrix} 2 & -3 \end{bmatrix}$ $\begin{bmatrix} 20 & 4 \end{bmatrix}$ with the encryption matrix $\begin{bmatrix} -1 & -1 \\ 2 & 1 \end{bmatrix}$ and the decryption matrix as its inverse, where the system of codes are described by the numbers 1-26 to the letters A – Z respectively, and the number 0 to a blank space.

28. Reduce the matrix $\begin{bmatrix} 3 & -1 & 2 \\ -6 & 2 & 4 \\ -3 & 1 & 2 \end{bmatrix}$ to a row-echelon form.

29. Reduce the matrix $\begin{bmatrix} 0 & 3 & 1 & 6 \\ -1 & 0 & 2 & 5 \\ 4 & 2 & 0 & 0 \end{bmatrix}$ to a row-echelon form.

30. Find the rank of each of the following matrices (i) $\begin{bmatrix} 3 & 2 & 5 \\ 1 & 1 & 2 \\ 3 & 3 & 6 \end{bmatrix}$ (ii) $\begin{bmatrix} 4 & 3 & 1 & -2 \\ -3 & -1 & -2 & 4 \\ 6 & 7 & -1 & 2 \end{bmatrix}$

31. Find the rank of the following matrices which are in row-echelon form

(i) $\begin{bmatrix} 2 & 0 & -7 \\ 0 & 3 & 1 \\ 0 & 0 & 1 \end{bmatrix}$ (ii) $\begin{bmatrix} -2 & 2 & -1 \\ 0 & 5 & 1 \\ 0 & 0 & 0 \end{bmatrix}$ (iii) $\begin{bmatrix} 6 & 0 & -9 \\ 0 & 2 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$

32. Find the rank of the matrix $\begin{bmatrix} 1 & 2 & 3 \\ 2 & 1 & 4 \\ 3 & 0 & 5 \end{bmatrix}$ by reducing it to a row-echelon form.

33. Find the rank of the matrix $\begin{bmatrix} 2 & -2 & 4 & 3 \\ -3 & 4 & -2 & -1 \\ 6 & 2 & -1 & 7 \end{bmatrix}$ by reducing it to an echelon form.

34. Show that the matrix $\begin{bmatrix} 3 & 1 & 4 \\ 2 & 0 & -1 \\ 5 & 2 & 1 \end{bmatrix}$ is non-singular and reduce it to the identity matrix by elementary row transformations.

35. Find the inverse of the non-singular matrix $A=\begin{bmatrix}0 & 5\\ -1 & 6\end{bmatrix}$, by Gauss-Jordan method.

36. Find the inverse of $A=\begin{bmatrix}2 & 1 & 1\\ 3 & 2 & 1\\ 2 & 1 & 2\end{bmatrix}$, by Gauss-Jordan method.

37. Find the rank of the following matrices by minor method:

(i) $\begin{bmatrix}2 & -4\\ -1 & 2\end{bmatrix}$ (ii) $\begin{bmatrix}-1 & 3\\ 4 & -7\\ 3 & -4\end{bmatrix}$ (iii) $\begin{bmatrix}1 & -2 & -1 & 0\\ 3 & -6 & -3 & 1\end{bmatrix}$

(iv) $\begin{bmatrix}1 & -2 & 3\\ 2 & 4 & -6\\ 5 & 1 & -1\end{bmatrix}$ (v) $\begin{bmatrix}0 & 1 & 2 & 1\\ 0 & 2 & 4 & 3\\ 8 & 1 & 0 & 2\end{bmatrix}$

38. Find the rank of the following matrices by row reduction method:

(i) $\begin{bmatrix}1 & 1 & 1 & 3\\ 2 & -1 & 3 & 4\\ 5 & -1 & 7 & 11\end{bmatrix}$ (ii) $\begin{bmatrix}1 & 2 & -1\\ 3 & -1 & 2\\ 1 & -2 & 3\\ 1 & -1 & 1\end{bmatrix}$ (iii) $\begin{bmatrix}3 & -8 & 5 & 2\\ 2 & -5 & 1 & 4\\ -1 & 2 & 3 & -2\end{bmatrix}$

39. Find the inverse of each of the following by Gauss-Jordan method:

(i) $\begin{bmatrix}2 & -1\\ 5 & -2\end{bmatrix}$ (ii) $\begin{bmatrix}1 & -1 & 0\\ 1 & 0 & -1\\ 6 & -2 & -3\end{bmatrix}$ (iii) $\begin{bmatrix}1 & 2 & 3\\ 2 & 5 & 3\\ 1 & 0 & 8\end{bmatrix}$

40. Solve the following system of linear equations, using matrix inversion method.
$5x+2y=3,\ 3x+2y=5.$

41. Solve the following system of equations, using matrix inversion method.
$2x_1+3x_2+3x_3=5, x_1-2x_2+x_3=-4, 3x_1-x_2-2x_3=3.$

42. If $A=\begin{bmatrix}-4 & 4 & 4\\ -7 & 1 & 3\\ 5 & -3 & -1\end{bmatrix}$ and $B=\begin{bmatrix}1 & -1 & 1\\ 1 & -2 & -2\\ 2 & 1 & 3\end{bmatrix}$, find the products AB and BA and hence solve the system of equations $x-y+z=4, x-2y-2z=9, 2x+y+3z=1.$

43. Solve the following system of linear equations by matrix inversion method.

(i) $2x+5y=-2, x+2y=-3$

(ii) $2x-y=8, 3x+2y=-2$

(iii) $2x+3y-z=9, x+y+z=9, 3x-y-z=-1$

(iv) $x+y+z-2=0, 6x-4y+5z-31=0, 5x+2y+2z=13$

44. If $A=\begin{bmatrix}-5 & 1 & 3\\ 7 & 1 & -5\\ 1 & -1 & 1\end{bmatrix}$ and $B=\begin{bmatrix}1 & 1 & 2\\ 3 & 2 & 1\\ 2 & 1 & 3\end{bmatrix}$, find the products AB and BA and hence solve the system of equations $x+y+2z=1,\ 3x+2y+z=7,\ 2x+y+3z=2.$

45. A man is appointed in a job with a monthly salary of certain amount and a fixed amount of annual increment. If his salary was Rs.19,800 per month at the end of the first month after 3 years of service and Rs.23,400 per month at the end of the first month after 9 years of service, find his starting salary and his annual increment. (Use matrix inversion method to solve the problem.)

46. 4 men and 4 women can finish a piece of work jointly in 3 days while 2 men and 5 women can finish the same work jointly in 4 days. Find the time taken by one man alone and that of one woman alone to finish the same work by using matrix inversion method.

47. The prices of three commodities A, B and C are Rs. x, y and z per units respectively. A person P purchases 4 units of B and sells two units of A and 5 units of C . Person Q purchases 2 units of C and sells 3 units of A and one unit of B . Person R purchases one unit of A and sells 3 unit of B and one unit of C . In the process P, Q and R earn Rs.15,000, Rs.1,000 and Rs.4,000 respectively. Find the prices per unit of A, B and C . (Use matrix inversion method to solve the problem.)

48. Solve, by Cramer's rule, the system of equations $x_1 - x_2 = 3$, $2x_1 + 3x_2 + 4x_3 = 17$, $x_2 + 2x_3 = 7$.

49. In a T20 match, Chennai Super Kings needed just 6 runs to win with 1 ball left to go in the last over. The last ball was bowled and the batsman at the crease hit it high up. The ball traversed along a path in a vertical plane and the equation of the path is $y = ax^2+bx+c$ with respect to a xy -coordinate system in the vertical plane and the ball traversed through the points (10,8), (20,16), (40,22) can you conclude that Chennai Super Kings won the match?

Justify your answer. (All distances are measured in metres and the meeting point of the plane of the path with the farthest boundary line is (70,0).)

50. Solve the following systems of linear equations by Cramer's rule:

(i) $5x - 2y + 16 = 0,\ x + 3y - 7 = 0$

(ii) $\frac{3}{x} + 2y = 12,\ \frac{2}{x} + 3y = 13$

(iii) $3x + 3y - z = 11,\ 2x - y + 2z = 9,\ 4x + 3y + 2z = 25$

(iv) $\frac{3}{x} - \frac{4}{y} - \frac{2}{z} - 1 = 0,\ \frac{1}{x} + \frac{2}{y} + \frac{1}{z} - 2 = 0,\ \frac{2}{x} - \frac{5}{y} - \frac{4}{z} + 1 = 0$

51. In a competitive examination, one mark is awarded for every correct answer while $\frac{1}{4}$ mark is deducted for every wrong answer. A student answered 100 questions and got 80 marks. How many questions did he answer correctly ? (Use Cramer's rule to solve the problem).

52. A chemist has one solution which is 50% acid and another solution which is 25% acid. How much each should be mixed to make 10 litres of a 40% acid solution ?

(Use Cramer's rule to solve the problem).

53. A fish tank can be filled in 10 minutes using both pumps A and B simultaneously. However, pump B can pump water in or out at the same rate. If pump B is inadvertently run in reverse, then the tank will be filled in 30 minutes. How long would it take each pump to fill the tank by itself ?

(Use Cramer's rule to solve the problem).

54. A family of 3 people went out for dinner in a restaurant. The cost of two dosai, three idlies and two vadais is Rs.150. The cost of the two dosai, two idlies and four vadais is Rs.200. The cost of five dosai, four idlies and two vadais is Rs.250. The family has Rs.350 in hand and they ate 3 dosai and six idlies and six vadais. Will they be able to manage to pay the bill within the amount they had ?

55. Solve the following system of linear equations, by Gaussian elimination method.

$4x+3y+6z=25,\ x+5y+7z=13,\ 2x+9y+z=1.$

56. The upward speed $v(t)$ of a rocket at time t is approximated by $v(t)=at^2+bt+c, 0\le t\le 100$ where a, b and c are constants. It has been found that the speed at times $t=3$, $t=6$ and $t=9$ seconds are respectively 64, 133 and 208 miles per second respectively. Find the speed at time $t=15$ seconds. (Use Gaussian elimination method.)

57. Solve the following systems of linear equations by Gaussian elimination method:

(i) $2x-2y+3z=2, x+2y-z=3, 3x-y+2z=1$

(ii) $2x+4y+6z=22, 3x+8y+5z=27, -x+y+2z=2$

58. If ax^2+bx+c is divided by $x+3$, $x-5$ and $x-1$, the remainders are 21, 61 and 9 respectively. Find a, b and c. (Use Gaussian elimination method.)

59. An amount of Rs.65,000 is invested in three bonds at the rates of 6%, 8% and 9% per annum respectively. The total annual income is Rs.4,800. The income from the third bond is Rs.600 more than that from the second bond. Determine the price of each bond.

(Use Gaussian elimination method.)

60. A boy is walking along the path $y=ax^2+bx+c$ through the points (–6, 8), (–2, –12) and (3, 8).

He wants to meet his friend at $P(7, 60)$. Will he meet his friend? (Use Gaussian elimination method.)

61. Test for consistency of the following system of linear equations and if possible solve :

$x+2y-z=3, 3x-y+2z=1, x-2y+3z=3, x-y+z+1=0.$

62. Test for consistency of the following system of linear equations and if possible solve :

$4x-2y+6z=8, x+y-3z=-1, 15x-3y+9z=21$

63. Test for consistency of the following system of linear equations and if possible solve :

$x-y+z=-9, 2x-2y+2z=-18, 3x-3y+3z+27=0.$

64. Test the consistency of the following system of linear equations

$x-y+z=-9, 2x-y+z=4, 3x-y+z=6, 4x-y+2z=7.$

65. Find the condition on a, b and c so that the following system of linear equations has one parameter family of solutions : $x+y+z=a, x+2y+3z=b, 3x+5y+7z=c.$

66. Investigate for what values of λ and μ the system of linear equations.

$x+2y+z=7, x+y+\lambda z=\mu, x+3y-5z=5$ has (i) no solution (ii) a unique solution (iii) an infinite number of solutions.

67. Test for consistency and if possible, solve the following systems of equations by rank method.

(i) $x-y+2z=2, 2x+y+4z=7, 4x-y+z=4$

(ii) $3x+y+z=2, x-3y+2z=1, 7x-y+4z=5$

(iii) $2x+2y+z=5, x-y+z=1, 3x+y+2z=4$

(iv) $2x-y+z=2, 6x-3y+3z=6, 4x-2y+2z=4$

68. Find the value of k for which the equations $kx-2y+z=-1, x-2ky+z=-2, x-2y+kz=1$ have

(i) no solution (ii) unique solution (iii) infinitely many solution

69. Investigate the values of λ and μ the system of linear equations $2x + 3y + 5z = 9$, $7x + 3y - 5z = 8$, $2x + 3y + \lambda z = \mu$, have

(i) no solution (ii) a unique solution (iii) an infinite number of solutions

70. Solve the following system : $x+2y+3z=0, 3x+4y+4z=0, 7x+10y+12z=0.$

71. Solve the following system : $x+3y-2z=0, 2x-y+4z=0, x-11y+14z=0.$

72. Solve the system : $x+y-2z=0, 2x-3y+z=0, 3x-7y+10z=0, 6x-9y+10z=0.$

73. Determine the values of λ for which the following system of equations :
$(3\lambda-8)x+3y+3z=0, 3x+(3\lambda-8)y+3z=0, 3x+3y+(3\lambda-8)z=0.$

74. By using Gaussian elimination method, balance the chemical reaction equation :
$C_5H_8 + O_2 \rightarrow CO_2 + H_2O$.

75. If the system of equations $px+by+cz=0, ax+qy+cz=0, ax+by+rz=0$ has a non-trivial solution and $p \neq a, q \neq b, r \neq c$ prove that $\frac{p}{p-a}+\frac{q}{q-b}+\frac{r}{r-c}=2.$

76. Solve the following system of homogenous equations.
(i) $3x + 2y + 7z = 0$, $4x - 3y - 2z = 0$, $5x + 9y + 23z = 0$
(ii) $2x + 3y - z = 0$, $x - y - 2z = 0$, $3x + y + 3z = 0$

77. Determine the values of λ for which the following system of equations $x + y + 3z = 0$, $4x+3y+\lambda z= 0$, $2x + y + 2z = 0$ has (i) a unique solution (ii) a non-trivial solution

78. By using Gaussian elimination method, balance the chemical reaction equation.
$C_2H_6 + O_2 \rightarrow H_2O + CO_2$

Additional Questions :

79. Prove that for every square matrix A of order n, $A\,(\text{adj } A) = (\text{adj } A)\,A = |A|\, I_n$.
80. Prove that if a square matrix has an inverse, then it is unique.
81. Prove that for every square matrix A of order n, then A^{-1} exists if and only if A is non-singular.
82. Prove that if A is non-singular, then

(i) $|A^{-1}|=\frac{1}{|A|}$ (ii) $(A^T)^{-1} = (A^{-1})^T$ (iii) $(\lambda A)^{-1} = \frac{1}{\lambda}A^{-1}$, Where λ is a non-zero scalar.

83. Prove that if A is a non-singular square matrix of order n , then

(i) $(adjA)^{-1} = adj(A^{-1}) = \frac{1}{|A|}A$

(ii) $|adj A|=|A|^{n-1}$

(iii) $adj(adj A)=|A|^{n-2} A$

(iv) $adj(\lambda A)=\lambda^{n-1} adj(A)$, λ is a non zero scalar

(v) $|adj(adj A)|=|A|^{(n-1)^2}$

(vi) $(adj A)^T = adj(A^T)$

84. Prove that if A and B are any two non-singular square matrices of order n, then adj(AB) = (adj B) (adj A)
85. State and prove left cancellation law.
86. State and prove right cancellation law.
87. State and prove reversal law for inverses.
88. State and prove law of double inverse.

2. COMPLEX NUMBERS

Points to Remember

- Rectangular form of a complex number is $x+iy$ (or) $(x+yi)$, where x and y are real numbers.
- Two complex numbers $z_1 = x_1 + iy_1$ and $z_2 = x_2 + iy_2$ are said to be equal if and only if $\text{Re}(z_1) = \text{Re}(z_2)$ and $\text{Im}(z_1) = \text{Im}(z_2)$. That is $x_1 = x_2$ and $y_1 = y_2$
- The conjugate of the complex number $x+iy$ is defined as the complex number $x-iy$.
- Properties of complex conjugates.
  1. $\overline{z_1+z_2} = \overline{z_1} + \overline{z_2}$
  2. $\overline{z_1-z_2} = \overline{z_1} - \overline{z_2}$
  3. $\overline{z_1 z_2} = \overline{z_1}\,\overline{z_2}$
  4. $\overline{\left(\dfrac{z_1}{z_2}\right)} = \dfrac{\overline{z_1}}{\overline{z_2}}, z_2 \neq 0$
  5. $\text{Re}(z) = \dfrac{z+\bar{z}}{2}$
  6. $\text{Im}(z) = \dfrac{z-\bar{z}}{2i}$
  7. $\overline{(z^n)} = (\bar{z})^n$, where n is an integer
  8. z is real if and only if $z = \bar{z}$
  9. z is purely imaginary if and only if $z = -\bar{z}$
  10. $\bar{\bar{z}} = z$
- If $z = x + iy$, then $\sqrt{x^2+y^2}$ is called modulus of z. It is denoted by $|z|$.
- Properties of Modulus of a complex number.
  1. $|z| = |\bar{z}|$
  2. $|z_1 + z_2| \leq |z_1| + |z_2|$ (Triangle inequality)
  3. $|z_1 z_2| = |z_1||z_2|$
  4. $|z_1 - z_2| \geq \big||z_1| - |z_2|\big|$
  5. $\left|\dfrac{z_1}{z_2}\right| = \dfrac{|z_1|}{|z_2|}, z_2 \neq 0$
  6. $|z^n| = |z|^n$, where n is an integer
  7. $\text{Re}(z) \leq |z|$
  8. $\text{Im}(z) \leq |z|$

- Formula for finding square root of a complex number.

  $\sqrt{a+ib} = \pm\left(\sqrt{\frac{|z|+a}{2}} + i\frac{b}{|b|}\sqrt{\frac{|z|-a}{2}}\right)$, where $z = a+ib$ and $b \neq 0$

- Let r and θ be polar coordinates of the point $P(x, y)$ that corresponds to a non-zero complex number $z = x+iy$. The polar from or trigonometric form of a complex number P is $z = r(\cos\theta + i\sin\theta)$

- **Properties of polar form**

  1. If $z = r(\cos\theta + i\sin\theta)$, then $z^{-1} = \frac{1}{r}(\cos\theta - i\sin\theta)$
  2. If $z_1 = r_1(\cos\theta_1 + i\sin\theta_1)$ and $z_2 = r_2(\cos\theta_2 + i\sin\theta_2)$, then $z_1 z_2 = r_1 r_2(\cos(\theta_1+\theta_2) + i\sin(\theta_1+\theta_2))$
  3. If $z_1 = r_1(\cos\theta_1 + i\sin\theta_1)$ and $z_2 = r_2(\cos\theta_2 + i\sin\theta_2)$, then $\frac{z_1}{z_2} = \frac{r_1}{r_2}(\cos(\theta_1-\theta_2) + i\sin(\theta_1-\theta_2))$

- **De Moivre's Theorem**

  1. Given any complex number $\cos\theta + i\sin\theta$ and any integer n, $(\cos\theta + i\sin\theta)^n = \cos n\theta + i\sin n\theta$
  2. If x is rational, then $\cos x\theta + i\sin x\theta$ is one of the values of $(\cos\theta + i\sin\theta)^x$

- The n^{th} roots of complex number $z = r(\cos\theta + i\sin\theta)$ are

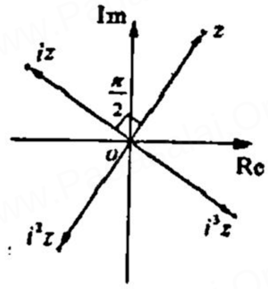
  $z^{1/n} = r^{1/n}\left(\cos\left(\frac{\theta+2k\pi}{n}\right) + i\sin\left(\frac{\theta+2k\pi}{n}\right)\right)$, k = 0, 1, 2, 3, ….. n − 1

Powers of imaginary unit i

$i^0 = 1,\ i^1 = i$	$i^2 = -1$	$i^3 = i^2 i = -i$	$i^4 = i^2 i^2 = 1$
$(i)^{-1} = \frac{1}{i} = \frac{i}{(i)^2} = -i$	$(i)^{-2} = -1$	$(i)^{-3} = i$	$(i)^{-4} = 1 = i^4$

- We note that, for any integer n, i^n has only four possible values: they correspond to values of n when divided by 4 leave the remainders 0, 1, 2, and 3. That is when the integer $n \leq -4$ or $n \geq 4$, using division algorithm, n can be written as $n = 4q + k$, $0 \leq k < 4$, k and q are integers and we write $(i)^n = (i)^{4q+k} = (i)^{4q}(i)^k = ((i)^4)^q (i)^k = (1)^q (i)^k = (i)^k$

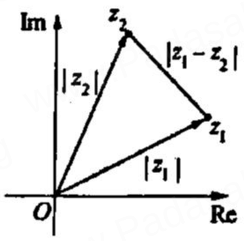
- In general, multiplication of a complex number z by i successively gives a 90° counter clockwise rotation successively about the origin.



- Complex numbers obey the laws of indices

  (i) $z^m z^n = z^{m+n}$ (ii) $\frac{z^m}{z^n} = z^{m-n}, z \neq 0$ (iii) $(z^m)^n = z^{mn}$ (iv) $(z_1z_2)^m = z_1^m z_2^m$

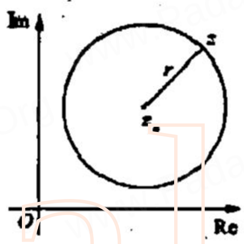
- The distance between the two points z_1 and z_2 in complex plane is $|z_1 - z_2|$



- To find the lower bound and upper bound use $||z_1| - |z_2|| \leq |z_1 + z_2| \leq |z_1| + |z_2|$
- $|z - z_0| = r$ is the complex form of the equation of a circle.

  (i) $|z - z_0| < r$ represents the points interior of the circle

  (ii) $|z - z_0| > r$ represents the points exterior of the circle



- If $z = 0$, the argument θ is undefined; and so it is understood that $z \neq 0$ whenever polar coordinate are used
- If the complex number $z = x + iy$ has polar coordinates (r, θ), its conjugate $\bar{z} = x - iy$ has polar coordinates $(r, -\theta)$
- Principal value of θ or principal argument of z and is denoted by Arg z.

  $-\pi < \text{Arg}(z) \leq \pi$ (or) $-\pi < \theta \leq \pi$

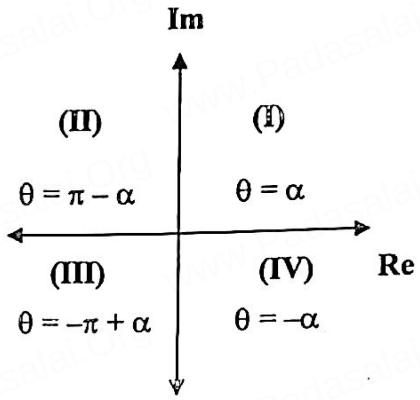
- $\arg z = \text{Arg } z + 2n\pi, n \in \mathbb{Z}$.

z	*1*	*i*	*-1*	*-i*
Arg (z)	*0*	$\frac{\pi}{2}$	π	$-\frac{\pi}{2}$
arg z	$2n\pi$	$2n\pi + \frac{\pi}{2}$	$2n\pi + \pi$	$2n\pi - \frac{\pi}{2}$

- **General rule for determining the argument θ**

Let $z = x + iy$, here $x, y \in R$

$$\alpha = \tan^{-1}\frac{|y|}{|x|}$$



- Some of the properties of arguments are

  (i) $\arg(z_1 z_2) = \arg z_1 + \arg z_2$

  (ii) $\arg\left(\frac{z_1}{z_2}\right) = \arg z_1 - \arg z_2$

  (iii) $\arg z^n = n \arg z$

  (iv) The alternate form of $\cos\theta + i\sin\theta$ is $\cos(2k\pi+\theta)+i\sin(2k\pi+\theta), k\in Z$.

- Euler's form of the complex number

  $e^{i\theta} = \cos\theta + i\sin\theta$

- $(\cos\theta - i\sin\theta)^n = \cos n\theta - i\sin n\theta$

  $(\cos\theta + i\sin\theta)^{-n} = \cos n\theta - i\sin n\theta$

  $(\cos\theta - i\sin\theta)^{-n} = \cos n\theta + i\sin n\theta$

  $\sin\theta + i\cos\theta = i(\cos\theta - i\sin\theta)$

- The n^{th} roots of unity $1, \omega, \omega^2, \ldots\ldots, \omega^{n-1}$ are in geometric progression with common ratio ω.
- The sum of all the n^{th} roots of unity is $1 + \omega + \omega^2 + \ldots\ldots + \omega^{n-1} = 0$
- The product of all the n^{th} roots of unity is $1.\ \omega.\ \omega^2 \ldots\ \omega^{n-1} = (-1)^{n-1}$
- All the n roots of n^{th} roots unity lie on the circumference of a circle whose centre is at the origin and radius equal to 1 and these roots divide the circle into n equal parts and form a polygon of n sides.
- In this chapter the letter ω is used for n^{th} roots of unity. Therefore the value of ω is depending on n as shown in following table.

Value of n	2	3	4	5	…	k
Value of ω	$e^{i\frac{2\pi}{2}}$	$e^{i\frac{2\pi}{3}}$	$e^{i\frac{2\pi}{4}}$	$e^{i\frac{2\pi}{5}}$	…	$e^{i\frac{2\pi}{k}}$

BOOK BACK ONE MARKS

1. $i^n + i^{n+1} + i^{n+2} + i^{n+3}$ is

   1) 0 2) 1
   3) –1 4) i

   Solution :

   $i^n + i^{n+1} + i^{n+2} + i^{n+3}$
   $= i^n (1 + i^1 + i^2 + i^3)$
   $= i^n (1 + i - 1 - i)$
   $= i^n (0)$
   $= 0$

2. The value of $\sum_{n=1}^{13}(i^n + i^{n+1})$ is

   1) 1 + i 2) i
   3) 1 4) 0

   Solution :

   $= \sum_{n=1}^{13}(i^n + i^{n+1})$

   $= \sum_{n=1}^{13} i^n\left(1+\frac{1}{i}\right) = \sum_{n=1}^{13} i^n\left(1+\frac{i}{i^2}\right)$

   $= (1-i)\sum_{n=1}^{13} i^n$

   $= (1-i)(i + i^2 + i^3 + \ldots + i^{13})$

   $= (1-i)\, i = i - i^2 = 1 + i$

3. The area of the triangle formed by the complex numbers z, iz and $z + iz$ in the Argand's diagram is

   1) $\frac{1}{2}|z|^2$ 2) $|z|^2$
   3) $\frac{3}{2}|z|^2$ 4) $2|z|^2$

   Solution :

   AB = $|A - B| = |z - iz|$

   'D' is mid point of AB = $\frac{A+B}{2} = \frac{z+iz}{2}$

   CD = $\left|z + iz - \left(\frac{z+iz}{2}\right)\right| = \left|\frac{z+iz}{2}\right|$

   Area of the $\Delta^{le} = \frac{1}{2}$ AB x CD

   $= \frac{1}{2}|z - iz|\left|\frac{z+iz}{2}\right|$

   $= \frac{1}{4}|z^2 - i^2 z^2|$

   $= \frac{1}{4} \times 2\,|z|^2$

   $= \frac{1}{2}|z|^2$

4. The conjugate of a complex number is $\frac{1}{i-2}$. Then, the complex number is

   1) $\frac{1}{i+2}$ 2) $\frac{-1}{i+2}$
   3) $\frac{-1}{i-2}$ 4) $\frac{1}{i-2}$

   Solution :

   Let $\bar{z} = \frac{1}{i-2}$ ⇒ $\bar{\bar{z}} = \overline{\left(\frac{1}{i-2}\right)}$

   $\bar{\bar{z}} = z = \frac{1}{-2-i} = \frac{-1}{i+2}$

5. If $z = \frac{(\sqrt{3}+i)^3(3i+4)^2}{(8+6i)^2}$, then $|z|$ is equal to

   1) 0 2) 1
   3) 2 4) 3

   Solution :

   $|z| = \frac{|\sqrt{3}+i|^3|3i+4|^2}{|8+6i|^2}$

   $= \frac{(\sqrt{3+1})^3(\sqrt{9+16})^2}{(\sqrt{64+36})^2}$

   $= \frac{8 \times 25}{100} = 2$

6. If z is a non zero complex number, such that $2iz^2 = \bar{z}$ then $|z|$ is

1) $\frac{1}{2}$ 2) 1
3) 2 4) 3

Solution :

$2iz^2 = \bar{z}$

$2iz\bar{z} = \bar{z}$

$2iz = 1$

$z = \frac{1}{2i} \times \frac{i}{i} = \frac{i}{2i^2} = -\frac{i}{2}$

$|z| = \sqrt{\frac{1}{4}} = \frac{1}{2}$

7. If $|z - 2 + i| \le 2$, then the greatest value of $|z|$ is

1) $\sqrt{3} - 2$ 2) $\sqrt{3} + 2$
3) $\sqrt{5} - 2$ 4) $\sqrt{5} + 2$

Solution :

$|z - 2 + i| \ge |z| - |2 - i|$

$2 \ge |z| - \sqrt{5}$

$\sqrt{5} + 2 \ge |z|$

$|z| \le \sqrt{5} + 2$

8. If $\left|z - \frac{3}{z}\right| = 2$, then the least value of $|z|$ is

1) 1 2) 2
3) 3 4) 5

Solution :

$|z| = \left|z - \frac{3}{z} + \frac{3}{z}\right|$

$\le \left|z - \frac{3}{z}\right| + \left|\frac{3}{z}\right|$

$|z| - 2 - \frac{3}{|z|} \le 0$

$|z|^2 - 2|z| - 3 \le 0$

$|z|$ is 3 or 1

The least value of $|z|$ is 1

9. If $|z| = 1$, then the value of $\frac{1+z}{1+\bar{z}}$ is

1) z 2) $\bar{z}$
3) $\frac{1}{z}$ 4) 1

Solution :

$\frac{1+z}{1+\bar{z}} = \frac{1+z}{1+\frac{1}{z}} = \frac{1+z}{\frac{1+z}{z}} = z$

$[\because |z| = 1 \Rightarrow z\bar{z} = 1 \Rightarrow \bar{z} = \frac{1}{z}]$

10. The solution of the equation $|z| - z = 1 + 2i$ is

1) $\frac{3}{2} - 2i$ 2) $-\frac{3}{2} + 2i$
3) $2 - \frac{3}{2}i$ 4) $2 + \frac{3}{2}i$

Solution :

Let $z = |z| - 1 - 2i$ .... (1)

$|z|^2 = z\bar{z} = (|z| - 1 - 2i)(|z| - 1 + 2i)$

$= |z|^2 + |z|(-i + 2i - 1 - 2i) +$
$(-1 - 2i)(-1 + 2i)$

$|z|^2 = |z|^2 - 2|z| + 5$

$2|z| = 5 \Rightarrow |z| = \frac{5}{2}$

From (1)

$z = \frac{5}{2} - 1 - 2i$

$z = \frac{3}{2} - 2i$

11. If $|z_1| = 1$, $|z_2| = 2$, $|z_3| = 3$ and $|9z_1z_2 + 4z_1z_3 + z_2z_3| = 12$, then the value of $|z_1 + z_2 + z_3|$ is

1) 1 2) 2
3) 3 4) 4

Solution :

We know that

$|z_1 + z_2 + z_3| = \frac{|9z_1z_2 + 4z_1z_3 + z_2z_3|}{|z_1||z_2||z_3|}$

$|z_1 + z_2 + z_3| = \frac{12}{1 \times 2 \times 3} = 2$

12. If z is a complex number such that $z \in C \setminus R$ and $z + \frac{1}{z} \in R$, then $|z|$ is

1) 0 **2) 1**
3) 2 4) 3

Solution :

only when $|z| = 1$

we get $z + \frac{1}{z} \in R$

$$Re\left(\frac{(x-1)+iy}{(x+1)+iy} \times \frac{(x+1)-iy}{(x+1)-iy}\right) = 0$$

$$\left(\frac{(x-1)(x+1)+y^2}{(x+1)^2+y^2}\right) = 0$$

$$x^2 - 1 + y^2 = 0$$

$$x^2 + y^2 = 1$$

$$|z|^2 = 1$$

$$|z| = 1$$

13. z_1, z_2 and z_3 are complex numbers such that $z_1 + z_2 + z_3 = 0$ and $|z_1| = |z_2| = |z_3| = 1$ then $z_1^2 + z_2^2 + z_3^2$ is

1) 3 2) 2
3) 1 **4) 0**

Solution :

$$z_1 = \frac{1}{\overline{z_1}}, \; z_2 = \frac{1}{\overline{z_2}}, \; z_3 = \frac{1}{\overline{z_3}}$$

$$(z_1 + z_2 + z_3)^2$$

$$= z_1^2 + z_2^2 + z_3^2 + 2z_1z_2 + 2z_2z_3 + 2z_3z_1$$

$$0 = z_1^2 + z_2^2 + z_3^2 + 2\left(\frac{\overline{z_3} + \overline{z_1} + \overline{z_2}}{\overline{z_1}\,\overline{z_3}\,\overline{z_2}}\right)$$

$$0 = z_1^2 + z_2^2 + z_3^2 + 2\left(\frac{\overline{z_1 + z_2 + z_3}}{\overline{z_1\, z_2\, z_3}}\right)$$

$$0 = z_1^2 + z_2^2 + z_3^2 + 2(0)$$

$$0 = z_1^2 + z_2^2 + z_3^2$$

14. If $\frac{z-1}{z+1}$ is purely imaginary, then $|z|$ is

1) $\frac{1}{2}$ **2) 1**
3) 2 4) 3

Solution :

$\frac{z-1}{z+1}$ is purely imaginary

ie. $Re\left(\frac{z-1}{z+1}\right) = 0$

$$Re\left(\frac{x+iy-1}{x+iy+1}\right) = 0$$

15. If $z = x+iy$ is a complex number such that $|z+2| = |z-2|$, then the locus of z is

1) real axis **2) imaginary axis**
3) ellipse 4) circle

Solution :

Let $z = x + iy$

$$|z+2| = |z-2|$$

$$|x+iy+2| = |x+iy-2|$$

$$\sqrt{(x+2)^2 + y^2} = \sqrt{(x-2)^2 + y^2}$$

$$x^2 + 4x + 4 + y^2 = x^2 - 4x + 4 + y^2$$

$$8x = 0 \Rightarrow x = 0$$

i.e. imaginary axis

16. The principal argument of $\frac{3}{-1+i}$ is

1) $\frac{-5\pi}{6}$ 2) $\frac{-2\pi}{3}$
3) $\frac{-3\pi}{4}$ 4) $\frac{-\pi}{2}$

Solution :

$$\frac{3}{-1+i} = \frac{3}{-1+i} \times \frac{-1-i}{-1-i} = \frac{-3(1+i)}{2}$$

$$= \frac{1}{2}(-3-3i)$$

$\left(\frac{-3}{2}, \frac{-3}{2}\right)$ lies in III quadrant

θ is also lies in III quadrant

$\therefore \; \theta = -\pi + \alpha$

$$= -\pi + \tan^{-1}\left|\frac{y}{x}\right|$$

$$= -\pi + \tan^{-1}\left(\frac{1}{1}\right)$$

$$= -\pi + \tan^{-1}(1)$$

$$= -\pi + \frac{\pi}{4}$$

$$\theta = \frac{-3\pi}{4}$$

17. The principal argument of $(\sin 40° + i\cos 40°)^5$ is

1) **−110°** 2) −70°
3) 70° 4) 110°

Solution :

$$(\sin 40° + i\cos 40°)^5$$
$$= [i(\cos 40° + i^3 \sin 40°)]^5$$
$$= i^5[\cos 40° - i\sin 40°]^5$$
$$= i[\cos 200 - i\sin 200]$$
$$= (\cos 90 + i\sin 90)\ (\cos 200 - i\sin 200)$$
$$= e^{i90}.\ e^{-i200} = e^{i(90-200)} = e^{-i110}$$
$$= \cos(-110°) + i\sin(-110°)$$

18. If $(1 + i)(1 + 2i)(1 + 3i) \ldots (1 + ni) = x + iy$, then $2.5.10\ldots(1 + n^2)$ is

1) 1 2) i
3) $x^2 + y^2$ 4) $1 + n^2$

Solution :

$(1+i)(1+2i)(1+3i) \ldots (1+ni) = x + iy$,
Take modulus on both sides,

$$|1+i|\,|1+2i|\,|1+3i| \ldots |1+ni| = |x+iy|$$

$$2.5.10\ldots(1+n^2) = x^2 + y^2$$

19. If $\omega \neq 1$ is a cubic root of unity and $(1 + \omega)^7 = A + B\omega$, then (A, B) equals

1) (1, 0) 2) (−1, 1)
3) (0, 1) **4) (1, 1)**

Solution :

$$(1+\omega)^7 = A + B\omega$$
$$(1+\omega)^6.(1+\omega) = A + B\omega$$
$$(-\omega^2)^6(1+\omega) = A + B\omega$$
$$\omega^{12}(1+\omega) = A + B\omega$$
$$1 + \omega = A + B\omega$$
$$A = 1, B = 1$$

20. The principal argument of the complex number $\dfrac{(1+i\sqrt{3})^2}{4i(1-i\sqrt{3})}$ is

1) $\dfrac{2\pi}{3}$ 2) $\dfrac{\pi}{6}$
3) $\dfrac{5\pi}{6}$ **4) $\dfrac{\pi}{2}$**

Solution :

$$\arg\left(\frac{(1+i\sqrt{3})^2}{4i(1-i\sqrt{3})}\right)$$
$$= \arg(1+i\sqrt{3})^2 - \arg 4i - \arg(1-i\sqrt{3})$$
$$= 2\arg(1+i\sqrt{3}) - \arg(0+4i) - \arg(1-i\sqrt{3})$$
$$= 2\tan^{-1}\left(\frac{\sqrt{3}}{1}\right) - \tan^{-1}\left(\frac{4}{0}\right) - \tan^{-1}\left(\frac{-\sqrt{3}}{1}\right)$$
$$= 2\left(\frac{\pi}{3}\right) - \frac{\pi}{2} - \left(\frac{-\pi}{3}\right) = 2\frac{\pi}{3} + \frac{\pi}{3} - \frac{\pi}{2}$$
$$= \pi - \frac{\pi}{2} = \frac{\pi}{2}$$

21. If α and β are the roots of $x^2 + x + 1 = 0$, then $\alpha^{2020} + \beta^{2020}$ is

1) −2 **2) −1**
3) 1 4) 2

Solution :

$$x^2 + x + 1 = 0$$
$$x = \frac{-1 \pm \sqrt{1-4}}{2(1)} = \frac{-1 \pm \sqrt{-3}}{2} = \frac{-1 \pm i\sqrt{3}}{2}$$
$$\alpha = \frac{-1+i\sqrt{3}}{2},\ \beta = \frac{-1-i\sqrt{3}}{2}$$
$$\alpha = \omega,\ \beta = \omega^2$$
$$\alpha^{2020} + \beta^{2020} = \omega^{2020} + (\omega^2)^{2020}$$
$$= \omega^{2020} + (\omega^{2020})^2$$
$$= (\omega^{2019}\omega^1) + (\omega^{2019}\omega^1)^2$$
$$= \omega + \omega^2 = -1$$
$$= -1$$

22. The product of all four values of $\left(\cos\frac{\pi}{3}+i\sin\frac{\pi}{3}\right)^{\frac{3}{4}}$ is

1) −2 2) −1
3) 1 4) 2

Solution :

$$\left(\cos\frac{\pi}{3}+i\sin\frac{\pi}{3}\right)^{\frac{3}{4}} = \text{cis}\left(2k\pi+\frac{\pi}{3}\right)^{\frac{3}{4}}$$

$$= \text{cis}\left(\frac{6k\pi+\pi}{3}\right)^{\frac{3}{4}}$$

$$= \text{cis}\left(\frac{6k\pi+\pi}{4}\right), k = 0, 1, 2, 3$$

$k = 0 \Rightarrow \text{cis}\frac{\pi}{4}$; $k = 2 \Rightarrow \text{cis}\frac{13\pi}{4}$

$k = 1 \Rightarrow \text{cis}\frac{7\pi}{4}$; $k = 3 \Rightarrow \text{cis}\frac{19\pi}{4}$

$$\text{product} = \text{cis}\left(\frac{\pi}{4}+\frac{7\pi}{4}+\frac{13\pi}{4}+\frac{19\pi}{4}\right)$$

$$= \text{cis}\left(\frac{40\pi}{4}\right) = \text{cis}10\pi$$

$$= \cos 10\pi + i\sin 10\pi = 1 + 0i$$

$$= 1$$

23. If $\omega \neq 1$ is a cubic root of unity and $\begin{vmatrix} 1 & 1 & 1 \\ 1 & -\omega^2-1 & \omega^2 \\ 1 & \omega^2 & \omega^7 \end{vmatrix} = 3k$, then k is equal to

1) 1 2) −1
3) $\sqrt{3}\,i$ 4) $-\sqrt{3}\,i$

Solution :

$$\begin{vmatrix} 1 & 1 & 1 \\ 1 & -\omega^2-1 & \omega^2 \\ 1 & \omega^2 & \omega^7 \end{vmatrix} = 3k$$

$$\begin{vmatrix} 1 & 1 & 1 \\ 1 & \omega & \omega^2 \\ 1 & \omega^2 & \omega \end{vmatrix} = 3k$$

$$\Rightarrow (\omega^2-\omega^4) - 1(\omega-\omega^2) + 1(\omega^2-\omega) = 3k$$

$$\omega^2-\omega-\omega+\omega^2+\omega^2-\omega = 3k$$

$$3\omega^2-3\omega = 3k \Rightarrow 3(\omega^2-\omega) = 3k$$

$$k = \omega^2-\omega$$

$$= \frac{1}{2}\left[-1-i\sqrt{3}+1-i\sqrt{3}\right]$$

$$= \frac{1}{2}\left[-2i\sqrt{3}\right]$$

$$k = -i\sqrt{3}$$

$$= -\sqrt{3}i$$

24. The value of $\left(\frac{1+\sqrt{3}i}{1-\sqrt{3}i}\right)^{10}$ is

1) $\text{cis}\frac{2\pi}{3}$ 2) $\text{cis}\frac{4\pi}{3}$
3) $-\text{cis}\frac{2\pi}{3}$ 4) $-\text{cis}\frac{4\pi}{3}$

Solution :

Let $1+\sqrt{3}i = r(\cos\theta+i\sin\theta)$

$$r = \sqrt{1+3} = 2$$

$(1, \sqrt{3})$ lies in I quadrant, θ is also lies in I quadrant

∴ θ = α

$$\theta = \tan^{-1}\left(\frac{\sqrt{3}}{1}\right) \Rightarrow \theta = \frac{\pi}{3}$$

$$1+\sqrt{3}i = 2\,\text{cis}\frac{\pi}{3} = 2\,e^{i\frac{\pi}{3}}$$

Replace is 'i' by '−i'

$$1-\sqrt{3}i = 2\,\text{cis}\left(\frac{-\pi}{3}\right) = 2\,e^{-i\frac{\pi}{3}}$$

$$\left(\frac{1+\sqrt{3}i}{1-\sqrt{3}i}\right)^{10} = \left(\frac{2e^{i\frac{\pi}{3}}}{2e^{-i\frac{\pi}{3}}}\right)^{10}$$

$$= \left(e^{i\frac{\pi}{3}}.e^{i\frac{\pi}{3}}\right)^{10}$$

$$= \left(e^{i2\frac{\pi}{3}}\right)^{10} = e^{i20\frac{\pi}{3}}$$

$$= \text{cis}\ 20\frac{\pi}{3}$$

$$= \text{cis}\left(6\pi + \frac{2\pi}{3}\right)$$

$$= \text{cis}\,\frac{2\pi}{3}$$

25. If $\omega = \text{cis}\,\frac{2\pi}{3}$, then the number of distinct root of $\begin{vmatrix} z+1 & \omega & \omega^2 \\ \omega & z+\omega^2 & 1 \\ \omega^2 & 1 & z+\omega \end{vmatrix} = 0$

1) 1 2) 2
3) 3 4) 4

Solution :

$$\begin{vmatrix} z+1 & \omega & \omega^2 \\ \omega & z+\omega^2 & 1 \\ \omega^2 & 1 & z+\omega \end{vmatrix} = 0$$

$$[(z+1)((z+\omega^2)(z+\omega)-1)-1]$$

$$-\omega(z\omega+\omega^2-\omega^2)+\omega^2(\omega-z\omega^2-\omega^4)=0$$

$$\Rightarrow (z+1)(z^2+z\omega+z\omega^2+\omega^2)-z\omega^2+$$

$$\omega^3-z\omega^4-\omega^6=0$$

$$\Rightarrow z^2+z^2\omega+z^2\omega^2+z^2+z\omega+z\omega^2$$

$$-z\omega^2+1-z\omega-1=0$$

$$\Rightarrow z^2(1+\omega+\omega^2+z)=0$$

$$\Rightarrow z^3=0$$

$$\Rightarrow z=0,0,0$$

The number of distinct roots is 1

BOOK SUMS ***(Exercise and Examples) :***

1. Simplify the following

(i) i^7 (ii) i^{1729} (iii) $i^{-1924}+i^{2018}$ (iv) $\sum_{n=1}^{102} i^n$ (v) $i\, i^2\, i^3 \ldots i^{40}$

2. Simplify the following

(i) $i^{1947}+i^{1950}$ (ii) $i^{1948}-i^{-1869}$ (iii) $\sum_{n=1}^{12} i^n$

(iv) $i^{59}+\frac{1}{i^{59}}$ (v) $i\, i^2\, i^3 \ldots i^{2000}$ (vi) $\sum_{n=1}^{10} i^{n+50}$

3. Find the value of the real numbers x and y, if the complex numbers $(2+i)x+(1-i)y+2i-3$ and $x+(-1+2i)\,y+1+i$ are equal.

4. Evaluate the following if $z = 5-2i$ and $w=-1+3i$

(i) $z+w$ (ii) $z-i\,w$ (iii) $2z+3w$

(iv) $z\,w$ (v) $z^2+2zw+w^2$ (vi) $(z+w)^2$

5. Given the complex number $z=2+3i$, represent the complex numbers in Argand diagram.

(i) z, iz and $z+iz$ (ii) z, $-iz$ and $z-iz$

6. Find the values of the real numbers x and y, if the complex numbers $(3-i)x-(2-i)y+2i+5$ and $2x+(-1+2i)y+3+2i$ are equal.

7. If $z_1 = 1-3i$, $z_2=-4i$ and $z_3=5$, show that

(i) $(z_1+z_2)+z_3=z_1+(z_2+z_3)$ (ii) $(z_1z_2)z_3=z_1(z_2z_3)$

8. If $z_1 = 3, z_2 = -7i$ and $z_3 = 5+4i$, show that

 (i) $z_1(z_2+z_3) = z_1z_2 + z_1z_3$ (ii) $(z_1+z_2)z_3 = z_1z_3 + z_2z_3$

9. If $z_1 = 2+5i$, $z_2 = -3-4i$ and $z_3 = 1+i$, find the additive and multiplicative inverse of z_1, z_2 and z_3.

10. Write $\frac{3+4i}{5-12i}$ in the $x+iy$ form, hence find its real and imaginary parts.

11. Simplify : $\left(\frac{1+i}{1-i}\right)^3 - \left(\frac{1-i}{1+i}\right)^3$

12. If $\frac{z+3}{z-5i} = \frac{1+4i}{2}$, find the complex number z.

13. If $z_1 = 3-2i$ and $z_2 = 6+4i$ find $\frac{z_1}{z_2}$.

14. Find z^{-1}, if $z = (2+3i)(1-i)$.

15. Show that (i) $(2+i\sqrt{3})^{10} + (2-i\sqrt{3})^{10}$ is real and (ii) $\left(\frac{19+9i}{5-3i}\right)^{15} - \left(\frac{8+i}{1+2i}\right)^{15}$ is purely imaginary.

16. Write the following in the rectangular form :

 (i) $\overline{(5+9i)+(2-4i)}$ (ii) $\frac{10-5i}{6+2i}$ (iii) $\overline{3i} + \frac{1}{2-i}$

17. If $z = x + iy$, find the following in rectangular form.

 (i) Re $\left(\frac{1}{z}\right)$ (ii) Re $(i\bar{z})$ (iii) Im $(3z + 4\bar{z} - 4i)$

18. If $z_1 = 2 - i$ and $z_2 = -4 + 3i$, find the inverse of z_1z_2 and $\frac{z_1}{z_2}$

19. The complex numbers u, v and w are related by $\frac{1}{u} = \frac{1}{v} + \frac{1}{w}$. If $v = 3-4i$ and $w = 4+3i$,

 find u is rectangular form.

20. Prove the following properties :

 (i) z is real if and only if $z = \bar{z}$ (ii) Re (z) = $\frac{z+\bar{z}}{2}$ and Im$(z) = \frac{z-\bar{z}}{2i}$

21. Find the least value of the positive integer n for which $(\sqrt{3}+i)^n$

 (i) real (ii) purely imaginary

22. Show that (i) $(2+i\sqrt{3})^{10} - (2-i\sqrt{3})^{10}$ is purely imaginary (ii) $\left(\frac{19-7i}{9+i}\right)^{12} + \left(\frac{20-5i}{7-6i}\right)^{12}$ is real

23. If $z_1 = 3+4i$, $z_2 = 5-12i$ and $z_3 = 6+8i$, find $|z_1|, |z_2|, |z_3|, |z_1+z_2|, |z_2-z_3|$, and $|z_1+z_3|$.

24. Find the following :

 (i) $\left|\frac{2+i}{-1+2i}\right|$ (ii) $|\overline{(1+i)}(2+3i)(4i-3)|$ (iii) $\left|\frac{i(2+i)^3}{(1+i)^2}\right|$

25. Which one of the points i, $-2+i$ and 3 is farthest from the origin?

26. If z_1, z_2 and z_3 are three complex numbers such that $|z_1|=|z_2|=|z_3|=|z_1+z_2+z_3|=1$, find the value of $\left|\frac{1}{z_1}+\frac{1}{z_2}+\frac{1}{z_3}\right|$

27. If $|z|=2$ show that $3 \le |z+3+4i| \le 7$

28. Show that the points 1, $\frac{-1}{2}+i\frac{\sqrt{3}}{2}$ and $\frac{-1}{2}-i\frac{\sqrt{3}}{2}$ are the vertices of an equilateral triangle.

29. If z_1, z_2 and z_3 be complex numbers such that $|z_1|=|z_2|=|z_3|=r>0$ and $z_1+z_2+z_3 \neq 0$. Prove that $\left|\frac{z_1z_2+z_2z_3+z_3z_1}{z_1+z_2+z_3}\right|=r$

30. Show that the equation $z^2=\bar{z}$ has four solutions.

31. Find the square root of $6-8i$

32. Find the modulus of the following complex numbers

(i) $\frac{2i}{3+4i}$ (ii) $\frac{2-i}{1+i}+\frac{1-2i}{1-i}$ (iii) $(1-i)^{10}$ (iv) $2i(3-4i)(4-3i)$

33. For any two complex numbers z_1 and z_2, such that $|z_1|=|z_2|=1$ and $z_1z_2 \neq -1$, then show that $\frac{z_1+z_2}{1+z_1z_2}$ is a real number.

34. Which one of the points $10-8i$, $11+6i$ is closest to $1+i$

35. If $|z|=3$, show that $7 \le |z+6-8i| \le 13$.

36. If $|z|=1$, show that $2 \le |z^2-3| \le 4$.

37. If $\left|z-\frac{2}{z}\right|=2$, show that the greatest and least value of $|z|$ are $\sqrt{3}+1$ and $\sqrt{3}-1$ respectively.

38. If z_1, z_2 and z_3 are three complex numbers such that $|z_1|=1, |z_2|=2, |z_3|=3$ and $|z_1+z_2+z_3|=1$, show that $|9z_1z_2+4z_1z_3+z_2z_3|=6$.

39. If the area of the triangle formed by the vertices z, iz and $z+iz$ is 50 square units, find the value of $|z|$.

40. Show that the equation $z^3+2\bar{z}=0$ has five solutions.

41. Find the square roots of (i) $4+3i$ (ii) $-6+8i$ (iii) $-5-12i$

42. Given the complex number $z=3+2i$, represent the complex numbers z, iz and $z+iz$ in one Argand diagram. Show that these complex numbers form the vertices of an isosceles right triangle.

43. Show that $|3z-5+i|=4$ represents a circle, and find its centre and radius.

44. Show that $|z+2-i|<2$ represents interior points of a circle. Find its centre and radius.

45. Obtain the Cartesian form of the locus of z in each of the following cases :

(i) $|z|=|z-i|$ (ii) $|2z-3-i|=3$

46. If $z = x + iy$ is a complex number such that $\left|\frac{z-4i}{z+4i}\right| = 1$ show that the locus of z is real axis.

47. If $z = x + iy$ is a complex number such that $\text{Im}\left(\frac{2z+1}{iz+1}\right) = 0$, show that the locus of z is $2x^2 + 2y^2 + x - 2y = 0$

48. Obtain the Cartesian form of the locus of $z = x + iy$ in each of the following cases :

(i) $[\text{Re}(iz)]^2 = 3$ (ii) $\text{Im}[(1-i)z+1] = 0$ (iii) $|z+i| = |z-1|$ (iv) $\bar{z} = z^{-1}$

49. Show that the following equations represent a circle and find its centre and radius.

(i) $|z-2-i| = 3$ (ii) $|2z+2-4i| = 2$ (iii) $|3z-6+12i| = 8$

50. Obtain the Cartesian equation for the locus of $z = x + iy$ in each of the following cases :

(i) $|z-4| = 16$ (ii) $|z-4|^2 - |z-1|^2 = 16$

51. Find the modulus and principal argument of the following complex numbers.

(i) $\sqrt{3}+i$ (ii) $-\sqrt{3}+i$ (iii) $-\sqrt{3}-i$ (iv) $\sqrt{3}-i$

52. Represent the complex number (i) $-1-i$ (ii) $1+i\sqrt{3}$ in polar form

53. Find the principal argument $Arg z$, when $z = \frac{-2}{1+i\sqrt{3}}$

54. Find the product $\frac{3}{2}\left(\cos\frac{\pi}{3} + i\sin\frac{\pi}{3}\right) \cdot 6\left(\cos\frac{5\pi}{6} + i\sin\frac{5\pi}{6}\right)$ in rectangular form.

55. Find the quotient $\dfrac{2\left(\cos\frac{9\pi}{4} + i\sin\frac{9\pi}{4}\right)}{4\left(\cos\left(\frac{-3\pi}{2}\right) + i\sin\left(\frac{-3\pi}{2}\right)\right)}$ in rectangular form.

56. If $z = x + iy$ and $\arg\left(\frac{z-1}{z+1}\right) = \frac{\pi}{2}$, then show that $x^2 + y^2 = 1$.

57. Write in polar form of the following complex numbers

(i) $2 + i2\sqrt{3}$ (ii) $3 - i\sqrt{3}$ (iii) $-2-i2$ (iv) $\dfrac{i-1}{\cos\frac{\pi}{3} + i\sin\frac{\pi}{3}}$

58. Find the rectangular form of the complex numbers

(i) $\left(\cos\frac{\pi}{6} + i\sin\frac{\pi}{6}\right)\left(\cos\frac{\pi}{12} + i\sin\frac{\pi}{12}\right)$ (ii) $\dfrac{\cos\frac{\pi}{6} - i\sin\frac{\pi}{6}}{2\left(\cos\frac{\pi}{3} + i\sin\frac{\pi}{3}\right)}$

59. If $(x_1 + iy_1)(x_2 + iy_2)(x_3 + iy_3)\ldots(x_n + iy_n) = a + ib$, show that

(i) $(x_1^2 + y_1^2)(x_2^2 + y_2^2)(x_3^2 + y_3^3)\ldots(x_n^2 + y_n^2) = (a^2 + b^2)$

(ii) $\sum_{r=1}^{n} \tan^{-1}\left(\frac{y_r}{x_r}\right) = \tan^{-1}\left(\frac{b}{a}\right) + 2k\pi, k \in Z$

60. If $\frac{1+z}{1-z} = \cos 2\theta + i \sin 2\theta$, show that $z = i \tan \theta$

61. If $\cos \alpha + \cos \beta + \cos \gamma = \sin \alpha + \sin \beta + \sin \gamma = 0$, then show that
(i) $\cos 3\alpha + \cos 3\beta + \cos 3\gamma = 3 \cos (\alpha + \beta + \gamma)$ and
(ii) $\sin 3\alpha + \sin 3\beta + \sin 3\gamma = 3 \sin (\alpha + \beta + \gamma)$.

62. If $z = x+iy$ and $\arg\left(\frac{z-i}{z+2}\right) = \frac{\pi}{4}$, then show that $x^2+y^2+3x-3y+2=0$.

63. If $z=(\cos\theta+i\sin\theta)$, show that $z^n+\frac{1}{z^n}=2\cos n\theta$ and $z^n-\frac{1}{z^n}=2i\sin n\theta$.

64. Simplify : $\left(\sin\frac{\pi}{6}+i\cos\frac{\pi}{6}\right)^{18}$.

65. Simplify : $\left(\frac{1+\cos 2\theta+i\sin 2\theta}{1+\cos 2\theta-i\sin 2\theta}\right)^{30}$.

66. Simplify : (i) $(1+i)^{18}$ (ii) $\left(-\sqrt{3}+3i\right)^{31}$

67. Find the cube roots of unity.

68. Find the fourth roots of unity.

69. Solve the equation $z^3+8i=0$, where $z \in C$

70. Find all cube roots of $\sqrt{3}+i$

71. Suppose z_1, z_2 and z_3 are the vertices of an equilateral triangle inscribed in the circle $|z|=2$. If $z_1 = 1+i\sqrt{3}$, then find z_2 and z_3.

72. If $\omega \neq 1$ is a cube root of unity, then show that $\frac{a+b\omega+c\omega^2}{b+c\omega+a\omega^2}+\frac{a+b\omega+c\omega^2}{c+a\omega+b\omega^2}=-1$

73. Show that $\left(\frac{\sqrt{3}}{2}+\frac{i}{2}\right)^5+\left(\frac{\sqrt{3}}{2}-\frac{i}{2}\right)^5=-\sqrt{3}$

74. Find the value of $\left(\frac{1+\sin\frac{\pi}{10}+i\cos\frac{\pi}{10}}{1+\sin\frac{\pi}{10}-i\cos\frac{\pi}{10}}\right)^{10}$

75. $2\cos\alpha = x+\frac{1}{x}$ and $2\cos\beta = y+\frac{1}{y}$, show that

(i) $\frac{x}{y}+\frac{y}{x}=2\cos(\alpha-\beta)$ (ii) $xy-\frac{1}{xy}=2i\sin(\alpha+\beta)$

(iii) $\frac{x^m}{y^n}-\frac{y^n}{x^m}=2i\sin(m\alpha-n\beta)$ (iv) $x^m y^n+\frac{1}{x^m y^n}=2\cos(m\alpha+n\beta)$

76. Solve the equation $z^3+27=0$.

77. If $\omega \neq 1$ is a cube root of unity, show that the roots of the equation $(z-1)^3+8=0$ are $-1, 1-2\omega, 1-2\omega^2$.

78. Find the value of $\sum_{k=1}^{8}\left(\cos\frac{2k\pi}{9}+i\sin\frac{2k\pi}{9}\right)$

79. If $\omega \neq 1$ is a cube root of unity, show that

(i) $(1-\omega+\omega^2)^6+(1+\omega-\omega^2)^6=128$

(ii) $(1+\omega)(1+\omega^2)(1+\omega^4)(1+\omega^8)\ldots(1+\omega^{2^{11}})=1$

80. If $z=2-2i$, find the rotation of z by θ radians in the counter clockwise direction about the origin when (i) $\theta=\frac{\pi}{3}$ (ii) $\theta=\frac{2\pi}{3}$ (iii) $\theta=\frac{3\pi}{2}$

81. Prove that the values of $\sqrt[4]{-1}$ are $\pm\frac{1}{\sqrt{2}}(1\pm i)$

Additional Questions :

82. State and prove commutative property under addition.

83. Prove that the multiplicative inverse of a non-zero complex number $z=x+iy$ is $\frac{x}{x^2+y^2}+i\frac{-y}{x^2+y^2}$.

84. Write the algebraic properties of complex numbers under addition.

85. Write the algebraic properties of complex numbers under multiplication.

86. Write the properties of complex conjugates.

87. For any two complex numbers z_1 and z_2, prove that $\overline{z_1+z_2}=\overline{z_1}+\overline{z_2}$

88. Prove that $\overline{z_1 z_2}=\overline{z_1}\,\overline{z_2}$

89. Prove that z is purely imaginary if and only if $z=-\overline{z}$

90. Write the properties of modulus of a complex number.

91. State and prove triangle inequality.

92. For any two complex numbers z_1 and z_2. Prove that $|z_1 z_2|=|z_1||z_2|$

93. Define circle.

94. Write the properties of arguments in polar from of a complex number.

95. If $z=r(\cos\theta+i\sin\theta)$, then prove that $z^{-1}=\frac{1}{r}(\cos\theta-i\sin\theta)$

96. If $z_1=r_1(\cos\theta_1+i\sin\theta_1)$ and $z_2=r_2(\cos\theta_2+i\sin\theta_2)$ then prove that $z_1 z_2=r_1 r_2(\cos(\theta_1+\theta_2)+i\sin(\theta_1+\theta_2))$.

97. If $z_1=r_1(\cos\theta_1+i\sin\theta_1)$ and $z_2=r_2(\cos\theta_2+i\sin\theta_2)$ then prove that

$$\frac{z_1}{z_2}=\frac{r_1}{r_2}\left[\cos(\theta_1-\theta_2)+i\sin(\theta_1-\theta_2)\right]$$

98. Find the sixth roots of unity.

3. THEORY OF EQUATIONS

Points to Remember

- For the quadratic equation $ax^2+bx+c=0$,

  i) $\Delta=b^2-4ac>0$ iff the roots are real and distinct.

  ii) $\Delta=b^2-4ac<0$ iff the equation has no real roots.

  iii) $\Delta=b^2-4ac=0$ iff the roots are real and equal.

Fundamental theorem of algebra :

- Every polynomial equation of degree n has at least one root in C.

Complex conjugate root theorem

- If a complex number z_0 is a root of a polynomial equation with real co-efficients, then complex conjugate $\overline{z_0}$ is also a root.
- If $p+\sqrt{q}$ is a root of a quadratic equation then $p-\sqrt{q}$ is also a root of the same equation where p, q are rational and $\sqrt{q}$ is irrational.
- If $\sqrt{p}+\sqrt{q}$ is a root of a polynomial equation then $\sqrt{p}-\sqrt{q}$, $-\sqrt{p}+\sqrt{q}$ and $-\sqrt{p}-\sqrt{q}$ are also roots of the same equation.
- If the sum of the co-efficients in $P(x)=0$ is P(1). Then 1 is a root of $P(x)=0$.
- If the sum of the co-efficients of odd powers = sum of the co-efficients of even powers, then -1 is a root of $P(x)=0$.

Rational root theorem

- Let $a_n x^n+.....+a_1x+a_0$ with $a_n\neq 0$, $a_0\neq 0$ be a polynomial with integer co-efficients. If $\frac{p}{q}$ with $(p,q)=1$ is a root of the polynomial, then p is a factor of a_0 and q is a factor of a_n

Reciprocal polynomial

- A polynomial $P(x)$ of degree n is said to be a reciprocal polynomial if one of the conditions is true

  (i) $P(x)=x^nP\left(\frac{1}{x}\right)$ (ii) $P(x)=-x^nP\left(\frac{1}{x}\right)$

- A change of sign in the co-efficients is said to occur at the j^{th} power of x in $P(x)$ if the co-efficient of x^{j+1} and the co-efficient of x^j (or) co-efficient of x^{j-1} and co-efficient of x^j are of different signs.
- **Vieta's formula for quadratic equation :**

  If α, β are the roots of $ax^2+bx+c=0$ then $\alpha+\beta=-\frac{b}{a}$ and $\alpha\beta=\frac{c}{a}$.

  Also, x^2-x(sum of the roots) + product of the roots = 0

- **Vieta's formula for polynomial of degree 3.**
  Co-efficient of $x^2 = -(\alpha+\beta+\gamma)$ where α, β, γ are its roots
  Co-efficient of $x = \alpha\beta+\beta\gamma+\gamma\alpha$ and constant term $= -\alpha\beta\gamma$
  Also, $x^3-(\alpha+\beta+\gamma)x^2+(\alpha\beta+\beta\gamma+\gamma\alpha)x-\alpha\beta\gamma=0$
- **Vieta's formula for polynomial equation of degree n > 3.**
  Co-efficient of $x^{n-1} = \Sigma_1 = -\Sigma\alpha_1$
  Co-efficient of $x^{n-2} = \Sigma_2 = \Sigma\alpha_1\alpha_2$
  Co-efficient of $x^{n-3} = \Sigma_3 = -\Sigma\alpha_1\alpha_2\alpha_3$
  Co-efficient of $x = \Sigma_{n-1} = (-1)^{n-1}\Sigma\alpha_1\alpha_2\ldots\alpha_{n-1}$
  Co-efficient of x^0 = constant term $= \Sigma_n = (-1)^n\alpha_1\alpha_2\ldots\alpha_n$
  A polynomial equation $a_nx^n+a_{n-1}x^{n-1}+\ldots+a_1x+a_0=0$ $(a_n\neq 0)$ is a reciprocal equation iff one of the following statements is true.
  i) $a_n=a_0, a_{n-1}=a_1, a_{n-2}=a_2, \ldots$
  ii) $a_n=-a_0, a_{n-1}=-a_1, a_{n-2}=-a_2, \ldots$

Descartes rule

- If p is the number of positive zeros of a polynomial $P(x)$ with real co-efficients and s is the number of sign changes in Co-efficient of $P(x)$, then $s-p$ is a non negative even integer.

Bounds for the number of real and imaginary roots

- Let m denote the number of sign changes in coefficients of $P(x)$ of degree n and $P(x)$ has atmost m positive zeros.
- Let k denote the number of sign changes in coefficients of $P(-x)$ of degree n and $P(x)$ has atmost k negative zeros.
- Then $P(x)$ has atleast (m+k) real roots and atleast n-(m+k) imaginary roots.

BOOK BACK ONE MARKS

1. A zero of $x^3 + 64$ is

1) 0 2) 4
3) 4*i* 4) –4

Solution :

$x^3+64=0$

$x^3 = -64$

$x^3 = (-4)^3$

$x = -4$

2. If *f* and *g* are polynomials of degrees *m* and *n* respectively, and if $h(x) = (f \circ g)(x)$, then the degree of *h* is

1) *mn* 2) $m + n$
3) m^n 4) n^m

Solution :

Let $f(x) = x^m$, $g(x) = x^n$

$h(x) = (fog)(x)$

$= f(g(x))$

$= f(x^n)$

$= (x^n)^m$

$= x^{mn}$

The degree of $h(x)$ is mn.

3. A polynomial equation in *x* of degree *n* always has

1) *n* distinct roots 2) *n* real roots
3) *n* imaginary roots 4) at most one root.

Solution :

Every polynomial equation of degree n has at least one root in C.

∴ n imaginary roots

4. If α, β and γ are the roots of $x^3 + px^2 + qx + r$, then $\sum \frac{1}{\alpha}$ is

1) $-\frac{q}{r}$ 2) $-\frac{p}{r}$
3) $\frac{q}{r}$ 4) $-\frac{q}{p}$

Solution :

$\alpha\beta + \beta\gamma + \gamma\alpha = \frac{c}{a} = \frac{q}{1} = q$

$\alpha\beta\gamma = \frac{-d}{a} = \frac{-r}{1} = -r$

$\sum \frac{1}{\alpha} = \frac{1}{\alpha} + \frac{1}{\beta} + \frac{1}{\gamma}$

$= \frac{\beta\gamma + \beta\gamma + \alpha\beta}{\alpha\beta\gamma} = \frac{q}{-r}$

$= -\frac{q}{r}$

(Book answer is wrong)

5. According to the rational root theorem, which number is not possible rational root of $4x^7 + 2x^4 - 10x^3 - 5$?

1) –1 2) $\frac{5}{4}$
3) $\frac{4}{5}$ 4) 5

Solution :

The given polynomial equation is $4x^7 + 2x^4 - 10x^3 - 5 = 0$

$a_n = 4$, $a_0 = -5$

If $\frac{p}{q}$ is a root of the polynomial, then as $(p,q) = 1$, p must divide –5 and q must divide 4.

The possible values of p are ±1, ±5 and the possible values of q are ±1, ±2, ±4.

Using these p and q, we can form only fractions $\pm1, \pm\frac{1}{2}, \pm\frac{1}{4}, \pm\frac{5}{1}, \pm\frac{5}{2}, \pm\frac{5}{4}$

Hence $\frac{4}{5}$ is not a possible root of the equation

6. The polynomial x^3-kx^2+9x has three real zeros if and only if, k satisfies

1) $|k|\le 6$ 2) $k=0$

3) $|k|>6$ 4) $|k|\ge 6$

Solution :

$P(x) = x^3-kx^2+9x$

$= x(x^2-kx+9)$

Clearly, $P(x)$ has one root as zero which real. The other roots are determined by the factor x^2-kx+9

This factor will give real roots if $b^2-4ac\ge0$

For real roots $b^2-4ac\ge0$

$(-k)^2-4(1)(9)\ge0$

$\Rightarrow k^2\ge36$

$\Rightarrow |k|\ge6$

7. The number of real numbers in $[0, 2\pi]$ satisfying $\sin^4 x - 2\sin^2 x+1$ is

1) 2 2) 4

3) 1 4) ∞

Solution :

The given equation is

$\Rightarrow \sin^4 x-2\sin^2 x+1=0$

$\Rightarrow (\sin^2 x-1)^2=0$

$\Rightarrow \sin^2 x-1=0$

$\Rightarrow \sin^2 x=1$

$\Rightarrow \frac{1-\cos 2x}{2}=1$

$\Rightarrow 1-\cos 2x = 2$

$\Rightarrow -\cos 2x = 2-1=1$

$\Rightarrow \cos 2x=-1$

$\Rightarrow 2x = (2n+1)\pi,\ n\in z$

$\Rightarrow x = (2n+1)\frac{\pi}{2}$

$x = \frac{\pi}{2},\frac{3\pi}{2}\in[0,2\pi]$

Hence two real numbers are there in the interval $[0,2\pi]$ satisfying the given equation.

8. If $x^3+12x^2+10ax+1999$ definitely has a positive zero, if and only if

1) $a\ge0$ 2) $a>0$

3) $a<0$ 4) $a\le0$

Solution :

The equation $x^3+12x^2+10ax+1999$ has a positive root if it has at least one change of sign. So, a must be negative.

$\therefore a<0$

9. The polynomial x^3+2x+3 has

1) **one negative and two imaginary zeros**

2) one positive and two imaginary zeros

3) three real zeros

4) no zeros

Solution :

$p(x)=x^3+2x+3$

$p(x)$ has no sign change

$p(-x) = (-x)^3+2(-x)+3$

$= -x^3-2x+3$

$p(-x)$ has only one sign change and atmost one negative root.

$p(x)$ has no positive root and at most one negative root. Degree of $p(x)$ is 3

$\Rightarrow$ imaginary roots $= 3-1=2$,

The polynomial has one negative and two imaginary roots.

10. The number of positive zeros of the polynomial $\sum_{j=0}^{n} {}^nC_r(-1)^r x^r$ is

1) 0 2) n

3) $<n$ 4) r

Solution :

$\sum_{j=0}^{n} {}^nC_r(-1)^r x^r = {}^nC_0(-1)^0x^0+{}^nC_1(-1)^1x^1$

$+{}^nC_2(-1)^2x^2+\dots+{}^nC_n(-1)^nx^n$

$=1-nx^1+{}^nC_2x^2-{}^nC_3x^3+\dots+x^n$

Since its degree is n and it has n changes of sign, the number of positive roots are n.

BOOK SUMS ***(Exercise and Examples) :***

1. If α and β are the roots of the quadratic equation $17x^2+43x-73=0$, construct a quadratic equation whose roots are $\alpha+2$ and $\beta+2$.
2. If α and β are the roots of the quadratic equation $2x^2-7x+13=0$, construct a quadratic equation whose roots are α^2 and β^2.
3. If α, β and γ are the roots of the equation $x^3+px^2+qx+r=0$, find the value of $\sum\frac{1}{\beta\gamma}$ in terms of the coefficients.
4. Find the sum of the squares of the roots of $ax^4+bx^3+cx^2+dx+e=0,\ a\neq 0$.
5. Find the condition that the roots of cubic equation $x^3+ax^2+bx+c=0$ are in the ratio $p:q:r$.
6. Form the equation whose roots are the squares of the roots of the cubic equation $x^3+ax^2+bx+c=0$.
7. If p is real, discuss the nature of the roots of the equation $4x^2+4px+p+2=0$, in terms of p.
8. If the sides of a cubic box are increased by 1, 2, 3 units respectively to form a cuboid, then the volume is increased by 52 cubic units. Find the volume of the cuboid.
9. Construct a cubic equation with roots

   (i) 1, 2 and 3 (ii) 1, 1 and -2 (iii) 2, $\frac{1}{2}$ and 1.
10. If α, β and γ are the roots of the cubic equation $x^3+2x^2+3x+4=0$ form a cubic equation whose roots are

    (i) $2\alpha, 2\beta, 2\gamma$ (ii) $\frac{1}{\alpha},\frac{1}{\beta},\frac{1}{\gamma}$ (iii) $-\alpha,-\beta,-\gamma$
11. Solve the equation $3x^3-16x^2+23x-6=0$ if the product of two roots is 1.
12. Find the sum of squares of roots of the equation $2x^4-8x^3+6x^2-3=0$
13. Solve the equation $x^3-9x^2+14x+24=0$ if it is given that two of its roots are in the ratio 3 : 2.
14. If α, β and γ are the roots of the polynomial equation $ax^3+bx^2+cx+d=0$, find the value of $\sum\frac{\alpha}{\beta\gamma}$ in terms of the coefficients.
15. If α, β, γ and δ are the roots of the polynomial equation $2x^4+5x^3-7x^2+8=0$, find a quadratic equation with integer coefficients whose roots are $\alpha+\beta+\gamma+\delta$ and $\alpha\beta\gamma\delta$.
16. If p and q are the roots of the equation $\ell x^2+nx+n=0$, show that $\sqrt{\frac{p}{q}}+\sqrt{\frac{q}{p}}+\sqrt{\frac{n}{\ell}}=0$.
17. If the equations $x^2+px+q=0$ and $x^2+p'x+q'=0$ have a common root, show that it must be equal to $\frac{pq'-p'q}{q-q'}$ or $\frac{q-q'}{p'-p}$.
18. Formulate into a mathematical problem to find a number such that when its cube root is added to it, the result is 6.
19. A 12 metre tall tree was broken into two parts. It was found that the height of the part which was left standing was the cube root of the length of the part that was cut away. Formulate this into a mathematical problem to find the height of the part which was cut away.

20. Find the monic polynomial equation of minimum degree with real coefficients having $2-\sqrt{3}\,i$ as a root.
21. Find a polynomial equation of minimum degree with rational coefficients, having $2-\sqrt{3}$ as a root.
22. Form a polynomial equation with integer coefficients with $\sqrt{\frac{\sqrt{2}}{\sqrt{3}}}$ as a root.
23. Show that the equation $2x^2-6x+7=0$ cannot be satisfied by any real values of x.
24. If $x^2+2(k+2)x+9k=0$ has equal roots, find k.
25. Show that, if p, q, r are rational, the roots of the equation $x^2-2px+p^2-q^2+2qr-r^2=0$ are rational.
26. Prove that a line cannot intersect a circle at more than two points.
27. If k is real, discuss the nature of the roots of the polynomial equation $2x^2+kx+k=0$, in terms of k.
28. Find a polynomial equation of minimum degree with rational coefficients, having $2+\sqrt{3}i$ as a root.
29. Find a polynomial equation of minimum degree with rational coefficients, having $2i+3$ as a root.
30. Find a polynomial equation of minimum degree with rational coefficients, having $\sqrt{5}-\sqrt{3}$ as a root.
31. Prove that a straight line and parabola cannot intersect at more than two points.
32. If $2+i$ and $3-\sqrt{2}$ are roots of the equation $x^6-13x^5+62x^4-126x^3+65x^2+127x-140=0$, find all roots.
33. Solve the equation $x^4-9x^2+20=0$.
34. Solve the equation $x^3-3x^2-33x+35=0$.
35. Solve the equation $2x^3+11x^2-9x-18=0$.
36. Obtain the condition that the roots of $x^3+px^2+qx+r=0$ are in A.P.
37. Find the condition that the roots of $ax^3+bx^2+cx+d=0$ are in geometric progression. Assume $a, b, c, d\neq 0$
38. If the roots of $x^3+px^2+qx+r=0$ are in H.P., prove that $9pqr=27r^2+2q^3$. Assume p, q, r $\neq$ 0.
39. It is known that the roots of the equation $x^3-6x^2-4x+24=0$ are in arithmetic progression. Find its roots.
40. Solve the cubic equation : $2x^3-x^2-18x+9=0$ if sum of two of its roots vanishes.
41. Solve the equation $9x^3-36x^2+44x-16=0$ if the roots form an arithmetic progression.
42. Solve the equation $3x^3-26x^2+52x-24=0$ if its roots form a geometric progression.
43. Determine k and solve the equation $2x^3-6x^2+3x+k=0$ if one of its roots is twice the sum of the other two roots.
44. Find all zeros of the polynomial $x^6-3x^5-5x^4+22x^3-39x^2-39x+135$, if it is known that $1+2i$ and $\sqrt{3}$ are two of its zeros.
45. Solve the cubic equations : (i) $2x^3-9x^2+10x=3$, (ii) $8x^3-2x^2-7x+3=0$
46. Solve the equation : $x^4-14x^2+45=0$
47. Solve the equation $(x-2)(x-7)(x-3)(x+2)+19=0$.

48. Solve the equation $(2x-3)(6x-1)(3x-2)(x-2)-5=0$.

49. Solve : (i) $(x-5)(x-7)(x+6)(x+4)=504$ (ii) $(x-4)(x-7)(x-2)(x+1)=16$

50. Solve : $(2x-1)(x+3)(x-2)(2x+3)+20=0$

51. Solve the equation $x^3-5x^2-4x+20=0$.

52. Find the roots of $2x^3+3x^2+2x+3=0$.

53. Solve the equation $7x^3-43x^2=43x-7$.

54. Solve the following equation : $x^4-10x^3+26x^2-10x+1=0$

55. Find solution, if any, of the equation $2\cos^2 x-9\cos x+4=0$

56. Solve the following equations (i) $\sin^2 x-5\sin x+4=0$ (ii) $12x^3+8x=29x^2-4$

57. Examine for the rational roots of (i) $2x^3-x^2-1=0$ (ii) $x^8-3x+1=0$

58. Solve : $8x^{\frac{3}{2n}}-8x^{\frac{-3}{2n}}=63$

59. Solve : $2\sqrt{\frac{x}{a}}+3\sqrt{\frac{a}{x}}=\frac{b}{a}+\frac{6a}{b}$

60. Solve the equations (i) $6x^4-35x^3+62x^2-35x+6=0$ (ii) $x^4+3x^3-3x-1=0$

61. Find all real numbers satisfying $4^x-3(2^{x+2})+2^5=0$.

62. Solve the equation $6x^4-5x^3-38x^2-5x+6=0$ if it is known that $\frac{1}{3}$ is a solution.

63. Show that the polynomial $9x^9+2x^5-x^4-7x^2+2$ has at least six imaginary roots.

64. Discuss the nature of the roots of the following polynomials:

(i) $x^{2018}+1947x^{1950}+15x^8+26x^6+2019$ (ii) $x^5-19x^4+2x^3+5x^2+11$

65. Discuss the maximum possible number of positive and negative roots of the polynomial equation $9x^9-4x^8+4x^7-3x^6+2x^5+x^3+7x^2+7x+2=0$.

66. Discuss the maximum possible number of positive and negative zeros of the polynomial equations x^2-5x+6 and $x^2-5x+16$.Also draw rough sketch of the graphs.

67. Show that the equation $x^9-5x^5+4x^4+2x^2+1=0$ has atleast 6 imaginary solutions.

68. Determine the number of positive and negative roots of the equation $x^9-5x^8-14x^7=0$.

69. Find the exact number of real zeros and imaginary of the polynomial $x^9+9x^7+7x^5+5x^3+3x$.

Additional Questions :

70. Define : fundamental theorem of Algebra.

71. State and prove complex conjugate root theorem.

72. If p and q be rational numbers such that $\sqrt{q}$ is irrational and $p+\sqrt{q}$ it is a root of a quadratic equation with rational coefficients, then prove that $p-\sqrt{q}$ is also a root of the same equation.

73. Define : Rational root theorem.

74. Define : Reciprocal polynomial.

75. Prove that a polynomial equation $a_nx^n+a_{n-1}x^{n-1}+.....+a_1x+a_0=0$ $(a_n\neq 0)$ is a reciprocal equation iff one of the following statements is true.

i) $a_n=a_0, a_{n-1}=a_1, a_{n-2}=a_2,.....$ ii) $a_n=-a_0, a_{n-1}=-a_1, a_{n-2}=-a_2,.....$

76. Define : Descartes rule

4. INVERSE TRIGONOMETRIC FUNCTIONS

Points to Remember

- **Inverse Trigonometric Functions**

Inverse sine function	*Inverse cosine function*	*Inverse tangent function*	*Inverse cosecant function*	*Inverse secant function*	*Inverse cot function*
Domain $[-1, 1]$	Domain $[-1, 1]$	Domain $\mathbb{R}$	Domain $(-\infty, -1] \cup [1, \infty)$	Domain $(-\infty, -1] \cup [1, \infty)$	Domain $\mathbb{R}$
Range $\left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$	Range $[0, \pi]$	Range $\left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$	Range $\left[-\frac{\pi}{2}, \frac{\pi}{2}\right] - \{0\}$	Range $[0, \pi] - \left\{\frac{\pi}{2}\right\}$	Range $(0, \pi)$
not a periodic function	not a periodic function	not a periodic function	not a periodic function	not a periodic function	not a periodic function
odd function	neither even nor odd function	odd function	odd function	neither even nor odd function	neither even nor odd function
strictly increasing function	strictly decreasing function	strictly increasing function	strictly decreasing function with respect to its domain.	strictly decreasing function with respect to its domain.	strictly decreasing function
one to one function	one to one function	one to one function	one to one function	one to one function	one to one function

- **Properties of inverse Trigonometric Functions.**

Property I

(i) $\sin^{-1}(\sin\theta) = \theta$, if $\theta \in \left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$

(ii) $\cos^{-1}(\cos\theta) = \theta$, if $\theta \in [0, \pi]$

(iii) $\tan^{-1}(\tan\theta) = \theta$, if $\theta \in \left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$

(iv) $\operatorname{cosec}^{-1}(\operatorname{cosec}\theta) = \theta$, if $\theta \in \left[-\frac{\pi}{2}, \frac{\pi}{2}\right] \setminus \{0\}$

(v) $\sec^{-1}(\sec\theta) = \theta$, if $\theta \in [0, \pi] \setminus \left\{\frac{\pi}{2}\right\}$

(vi) $\cot^{-1}(\cot\theta) = \theta$, if $\theta \in (0, \pi)$

Property II

(i) $\sin(\sin^{-1}x) = x$, if $x \in [-1, 1]$

(ii) $\cos(\cos^{-1}x) = x$, if $x \in [-1, 1]$

(iii) $\tan(\tan^{-1}x) = x$, if $x \in \mathbb{R}$

(iv) $\operatorname{cosec}(\operatorname{cosec}^{-1}x) = x$, if $\in \mathbb{R} \setminus (1,1)$

(v) $\sec(\sec^{-1}x) = x$, if $x \in \mathbb{R} \setminus (-1,1)$

(vi) $\cot(\cot^{-1}x) = x$, if $x \in \mathbb{R}$

Property III (Reciprocal inverse identities)

(i) $\sin^{-1}\left(\frac{1}{x}\right) = \operatorname{cosec} x$, if $x \in R \setminus (-1,1)$

(ii) $\cos^{-1}\left(\frac{1}{x}\right) = \sec x$, if $x \in R \setminus (-1,1)$

(iii) $\tan^{-1}\left(\frac{1}{x}\right) = \begin{cases} \cot^{-1} x & if\ x > 0 \\ -\pi + \cot^{-1} x & if\ x < 0 \end{cases}$

Property-IV (Reflection identities)

(i) $\sin^{-1}(-x) = -\sin^{-1} x$, if $x \in [-1, 1]$

(ii) $\tan^{-1}(-x) = -\tan^{-1} x$, if $x \in R$.

(iii) $\operatorname{cosec}^{-1}(-x) = -\operatorname{cosec}^{-1} x$, if $|x| \geq 1$ or $x \in R \setminus (-1, 1)$

(iv) $\cos^{-1}(-x) = \pi - \cos^{-1} x$, if $x \in [-1, 1]$

(v) $\sec^{-1}(-x) = \pi - \sec^{-1} x$, if $|x| \geq 1$ or $x \in R \setminus (-1, 1)$

(vi) $\cot^{-1}(-x) = \pi - \cot^{-1} x$, if $x \in R$.

Property-V (cofunction inverse identities)

(i) $\sin^{-1} x + \cos^{-1} x = \frac{\pi}{2}$, $x \in [-1, 1]$

(ii) $\tan^{-1} x + \cot^{-1} x = \frac{\pi}{2}$, $x \in R$

(iii) $\operatorname{cosec}^{-1} x + \sec^{-1} x = \frac{\pi}{2}$, $x \in R \setminus (-1, 1)$ or $|x| \geq 1$

Property-VI

(i) $\sin^{-1} x + \sin^{-1} y = \sin^{-1}\left(x\sqrt{1-y^2} + y\sqrt{1-x^2}\right)$ where either $x^2 + y^2 \leq 1$ or $xy < 0$.

(ii) $\sin^{-1} x - \sin^{-1} y = \sin^{-1}\left(x\sqrt{1-y^2} - y\sqrt{1-x^2}\right)$ where either $x^2 + y^2 \leq 1$ or $xy > 0$.

(iii) $\cos^{-1} x + \cos^{-1} y = \cos^{-1}\left(xy - \sqrt{1-x^2}\sqrt{1-y^2}\right)$, if $x + y \geq 0$.

(iv) $\cos^{-1} x - \cos^{-1} y = \cos^{-1}\left(xy + \sqrt{1-x^2}\sqrt{1-y^2}\right)$, if $x \leq y$

(v) $\tan^{-1} x + \tan^{-1} y = \tan^{-1}\left(\frac{x+y}{1-xy}\right)$, if $xy < 1$.

(vi) $\tan^{-1} x - \tan^{-1} y = \tan^{-1}\left(\frac{x-y}{1+xy}\right)$, if $xy > -1$.

Property-VII

(i) $2\tan^{-1}x = \tan^{-1}\left(\frac{2x}{1-x^2}\right)$, $|x| < 1$

(ii) $2\tan^{-1}x = \cos^{-1}\left(\frac{1-x^2}{1+x^2}\right)$, $x \geq 0$

(iii) $2\tan^{-1}x = \sin^{-1}\left(\frac{2x}{1+x^2}\right)$, $|x| \leq 1$

Property-VIII

(i) $\sin^{-1}\left(2x\sqrt{1-x^2}\right) = 2\sin^{-1}x$, if $|x| \leq \frac{1}{\sqrt{2}}$ or $-\frac{1}{\sqrt{2}} \leq x \leq \frac{1}{\sqrt{2}}$.

(ii) $\sin^{-1}\left(2x\sqrt{1-x^2}\right) = 2\cos^{-1}x$, if $\frac{1}{\sqrt{2}} \leq x \leq 1$.

Property-IX

(i) $\sin^{-1}x = \cos^{-1}\sqrt{1-x^2}$, if $0 \leq x \leq 1$.

(ii) $\sin^{-1}x = -\cos^{-1}\sqrt{1-x^2}$, if $-1 \leq x < 0$

(iii) $\sin^{-1}x = \tan^{-1}\left(\frac{x}{\sqrt{1-x^2}}\right)$, if $-1 < x < 1$.

(iv) $\cos^{-1}x = \sin^{-1}\sqrt{1-x^2}$, if $0 \leq x \leq 1$.

(v) $\cos^{-1}x = \pi - \sin^{-1}\sqrt{1-x^2}$, if $-1 \leq x < 0$.

(vi) $\tan^{-1}x = \sin^{-1}\left(\frac{x}{\sqrt{1+x^2}}\right) = \cos^{-1}\left(\frac{1}{\sqrt{1+x^2}}\right)$, if $x > 0$.

Property-X

(i) $3\sin^{-1}x = \sin^{-1}\left(3x-4x^3\right)$, $x \in \left[-\frac{1}{2}, \frac{1}{2}\right]$

(ii) $3\cos^{-1}x = \cos^{-1}\left(4x^3-3x\right)$, $x \in \left[\frac{1}{2}, 1\right]$

6. If $\sin^{-1}x + \sin^{-1}y + \sin^{-1}z = \frac{3\pi}{2}$, the value of $x^{2017} + y^{2018} + z^{2019} - \frac{9}{x^{101}+y^{101}+z^{101}}$ is

1) 0 2) 1
3) 2 4) 3

Solution :

Given

$\sin^{-1}x + \sin^{-1}y + \sin^{-1}z = \frac{3\pi}{2}$

w.k.t $\sin^{-1}x$ attains its maximum value at $x = 1$ and its maximum value is $\frac{\pi}{2}$.

Hence the above equation has only one solution $x = y = z = 1$.

Hence,

$x^{2017} + x^{2018} + x^{2019} - \frac{9}{x^{101}+y^{101}+z^{101}}$

$= 1^{2017} + 1^{2018} + 1^{2019} - \frac{9}{1^{101}+1^{101}+1^{101}}$

$= 1 + 1 + 1 - \frac{9}{1+1+1}$

$= 3 - \frac{9}{3}$

$= 3 - 3 = 0$

7. If $\cot^{-1}x = \frac{2\pi}{5}$ for some $x \in R$, the value of $\tan^{-1}x$ is

1) $-\frac{\pi}{10}$ 2) $\frac{\pi}{5}$
3) $\frac{\pi}{10}$ 4) $-\frac{\pi}{5}$

Solution :

$\cot^{-1}x = \frac{2\pi}{5}$ $\left(\therefore \tan^{-1}x + \cot^{-1}x = \frac{\pi}{2}\right)$

$\frac{\pi}{2} - \tan^{-1}x = \frac{2\pi}{5}$

$\frac{\pi}{2} - \frac{2\pi}{5} = \tan^{-1}x$

$\tan^{-1}x = \frac{5\pi - 4\pi}{10} = \frac{\pi}{10}$

8. The domain of the function defined by $f(x) = \sin^{-1}\sqrt{x-1}$ is

1) [1, 2] 2) [–1, 1]
3) [0, 1] 4) [–1, 0]

Solution :

Given $f(x) = \sin^{-1}\sqrt{x-1}$

First of all the expression $\sqrt{x-1}$ must be real and it is real only when

$x - 1 \geq 0 \Rightarrow 0 \leq x - 1$ -------- (1)

The domain of $\sin^{-1}x$ is $-1 \leq x \leq 1$

(i.e) $-1 \leq \sqrt{x-1} \leq 1$

hence $\sqrt{x-1} \leq 1$ (or) $x - 1 \leq 1$ -------- (2)

Combining both the inequalities (1) and (2) we get,

$0 \leq x - 1 \leq 1 \Rightarrow 1 \leq x \leq 2$

9. If $x = \frac{1}{5}$, the value of $\cos(\cos^{-1}x + 2\sin^{-1}x)$ is

1) $-\sqrt{\frac{24}{25}}$ 2) $\sqrt{\frac{24}{25}}$
3) $\frac{1}{5}$ 4) $-\frac{1}{5}$

Solution :

Given $x = \frac{1}{5}$

$\cos(\cos^{-1}x + 2\sin^{-1}x)$

$= \cos(\cos^{-1}x + \sin^{-1}x + \sin^{-1}x)$

$= \cos\left(\frac{\pi}{2} + \sin^{-1}x\right)$

$\left(\because \cos\left(\frac{\pi}{2} + \theta\right) = -\sin\theta\right)$

$= -\sin(\sin^{-1}x)$

$= -x$

$= \frac{-1}{5}$

10. $\tan^{-1}\left(\frac{1}{4}\right)+\tan^{-1}\left(\frac{2}{9}\right)$ is equal to

1) $\frac{1}{2}\cos^{-1}\left(\frac{3}{5}\right)$ 2) $\frac{1}{2}\sin^{-1}\left(\frac{3}{5}\right)$

3) $\frac{1}{2}\tan^{-1}\left(\frac{3}{5}\right)$ 4) $\tan^{-1}\left(\frac{1}{2}\right)$

Solution :

w.k.t $\tan^{-1}A+\tan^{-1}B=\tan^{-1}\left(\frac{A+B}{1-AB}\right)$

$\tan^{-1}\left(\frac{1}{4}\right)+\tan^{-1}\left(\frac{2}{9}\right)$

$= \tan^{-1}\left[\frac{\frac{1}{4}+\frac{2}{9}}{1-\left(\frac{1}{4}x\frac{2}{9}\right)}\right]$

$= \tan^{-1}\left[\frac{\frac{9+8}{36}}{\frac{36-2}{36}}\right]$

$= \tan^{-1}\left[\frac{17}{34}\right] = \tan^{-1}\left[\frac{1}{2}\right]$

11. If the function $f(x)=\sin^{-1}(x^2-3)$, then x belongs to

1) $[-1,1]$

2) $[\sqrt{2}, 2]$

3) $[-2,-\sqrt{2}]\cup[\sqrt{2},2]$

4) $[-2,-\sqrt{2}]\cap[\sqrt{2},2]$

Solution :

$f(x)=\sin^{-1}(x^2-3)$ then $-1\le x^2-3\le 1$

$\Rightarrow -1+3\le x^2\le 1+3$

$\Rightarrow 2\le x^2\le 4$

$\Rightarrow \pm\sqrt{2}\le x\le \pm 2 \Rightarrow \sqrt{2}\le x\le 2$ and

$\Rightarrow -2\le x\le -\sqrt{2}$

Hence $x\in[-2,-\sqrt{2}]\cup[\sqrt{2},2]$

12. If $\cot^{-1}2$ and $\cot^{-1}3$ are two angles of a triangle, then the third angle is

1) $\frac{\pi}{4}$ 2) $\frac{3\pi}{4}$

3) $\frac{\pi}{6}$ 4) $\frac{\pi}{3}$

Solution :

$\cot^{-1}2+\cot^{-1}3=\tan^{-1}\frac{1}{2}+\tan^{-1}\frac{1}{3}$

$= \tan^{-1}\left(\frac{\frac{1}{2}+\frac{1}{3}}{1-\left(\frac{1}{2}\right)\left(\frac{1}{3}\right)}\right)$

$= \tan^{-1}\left(\frac{\frac{3+2}{6}}{\frac{6-1}{6}}\right)$

$= \tan^{-1}\left(\frac{5}{5}\right)=\tan^{-1}(1)=\frac{\pi}{4}$

The sum of two angles $\frac{\pi}{4}$

Sum of three angles in triangle $=\pi$

Hence the third angle $=\pi-\frac{\pi}{4}$

$=\frac{3\pi}{4}$

13. $\sin^{-1}\left(\tan\frac{\pi}{4}\right)-\sin^{-1}\left(\sqrt{\frac{3}{x}}\right)=\frac{\pi}{6}$. Then x is a root of the equation

1) $x^2-x-6=0$ 2) $x^2-x-12=0$

3) $x^2+x-12=0$ 4) $x^2+x-6=0$

Solution :

$\sin^{-1}\left(\tan\frac{\pi}{4}\right)-\sin^{-1}\left(\sqrt{\frac{3}{x}}\right)=\frac{\pi}{6}$

$\sin^{-1}(1)-\sin^{-1}\left(\sqrt{\frac{3}{x}}\right)=\frac{\pi}{6}$

$\frac{\pi}{2}-\sin^{-1}\left(\sqrt{\frac{3}{x}}\right)=\frac{\pi}{6}$

$$\frac{\pi}{2}-\frac{\pi}{6}=\sin^{-1}\left(\sqrt{\frac{3}{x}}\right)$$

$$\sin^{-1}\left(\sqrt{\frac{3}{x}}\right)=\frac{\pi}{3}$$

$$\sqrt{\frac{3}{x}}=\sin\frac{\pi}{3}$$

$$\sqrt{\frac{3}{x}}=\frac{\sqrt{3}}{2}$$

squaring on both sides $\frac{3}{x}=\frac{3}{4}$

$x=4$

Moreover, $x=4$ is a root of the equation

$x^2-x-12=0$

14. $\sin^{-1}\left(2\cos^2 x-1\right)+\cos^{-1}\left(1-2\sin^2 x\right)=$

1) $\frac{\pi}{2}$ 2) $\frac{\pi}{3}$

3) $\frac{\pi}{4}$ 4) $\frac{\pi}{6}$

Solution :

$\sin^{-1}\left(2\cos^2 x-1\right)+\cos^{-1}\left(1-2\sin^2 x\right)$

$=\sin^{-1}(\cos 2x)+\cos^{-1}(\cos 2x)=\frac{\pi}{2}$

[since $\sin^{-1}x+\cos^{-1}x=\frac{\pi}{2}$]

15. If $\cot^{-1}\left(\sqrt{\sin\alpha}\right)+\tan^{-1}\left(\sqrt{\sin\alpha}\right)=u$, then $\cos 2u$ is equal to

1) $\tan^2\alpha$ 2) 0

3) –1 4) $\tan 2\alpha$

Solution :

w.k.t $\tan^{-1}x+\cot^{-1}x=\frac{\pi}{2}$

Given

$\cot^{-1}\left(\sqrt{\sin\alpha}\right)+\tan^{-1}\left(\sqrt{\sin\alpha}\right)=u$

$u=\frac{\pi}{2}$

$\cos 2u=\cos\left(2\times\frac{\pi}{2}\right)$

$=\cos\pi$

$=-1$

16. If $|x|\le 1$, then $2\tan^{-1}x-\sin^{-1}\frac{2x}{1+x^2}$ is equal to

1) $\tan^{-1}x$ 2) $\sin^{-1}x$

3) 0 4) π

Solution :

W.K.T, $2\tan^{-1}x=\sin^{-1}\left(\frac{2x}{1+x^2}\right)$

$\therefore 2\tan^{-1}x-\sin^{-1}\left(\frac{2x}{1+x^2}\right)$

$=2\tan^{-1}x-2\tan^{-1}x$

$=0$

17. The equation $\tan^{-1}x-\cot^{-1}x=\tan^{-1}\left(\frac{1}{\sqrt{3}}\right)$ has

1) no solution

2) unique solution

3) two solutions

4) infinite number of solutions

Solution :

$\tan^{-1}x-\cot^{-1}x=\tan^{-1}\left(\frac{1}{\sqrt{3}}\right)$

$\tan^{-1}x-\left(\frac{\pi}{2}-\tan^{-1}x\right)=\frac{\pi}{6}$

$\tan^{-1}x-\frac{\pi}{2}+\tan^{-1}x=\frac{\pi}{6}$

$2\tan^{-1}x=\frac{\pi}{6}+\frac{\pi}{2}$

$=\frac{2\pi}{3}$

$\tan^{-1}x=\frac{\pi}{3}$

$x=\tan\frac{\pi}{3}$

$x=\sqrt{3}$

Hence the given equation has only one solution (or) unique solution.

18. If $\sin^{-1} x + \cot^{-1}\left(\frac{1}{2}\right) = \frac{\pi}{2}$, then x is equal to

1) $\frac{1}{2}$ 2) $\frac{1}{\sqrt{5}}$

3) $\frac{2}{\sqrt{5}}$ 4) $\frac{\sqrt{3}}{2}$

Solution :

Let $\cot^{-1}\left(\frac{1}{2}\right) = \alpha$ .... (1)

$\cot \alpha = \frac{1}{2}$

$\tan \alpha = 2$

$\sec^2 \alpha = 1 + \tan^2 \alpha = 1 + 2^2 = 5$

$\sec \alpha = \sqrt{5}$

$\cos \alpha = \frac{1}{\sqrt{5}}$

$\alpha = \cos^{-1} \frac{1}{\sqrt{5}}$ .... (2)

From (1) and (2), we get

$\cot^{-1} \frac{1}{2} = \cos^{-1} \frac{1}{\sqrt{5}}$

Given

$\sin^{-1} x + \cot^{-1}\left(\frac{1}{2}\right) = \frac{\pi}{2}$

$\sin^{-1} x + \cos^{-1} \frac{1}{\sqrt{5}} = \frac{\pi}{2}$

$x = \frac{1}{\sqrt{5}}$

19. If $\sin^{-1} \frac{x}{5} + \operatorname{cosec}^{-1} \frac{5}{4} = \frac{\pi}{2}$, then the value of x is

1) 4 2) 5

3) 2 **4) 3**

Solution :

$\sin^{-1} \frac{x}{5} + \operatorname{cosec}^{-1} \frac{5}{4} = \frac{\pi}{2}$

$\sin^{-1} \frac{x}{5} + \sin^{-1} \frac{4}{5} = \frac{\pi}{2}$

$\sin^{-1} \frac{x}{5} = \frac{\pi}{2} - \sin^{-1} \frac{4}{5}$

$\frac{x}{5} = \sin\left(\frac{\pi}{2} - \sin^{-1} \frac{4}{5}\right)$

$= \cos\left(\sin^{-1} \frac{4}{5}\right)$

$= \cos\left(\cos^{-1} \frac{3}{5}\right)$

(since $\sin\theta = \frac{4}{5} = \frac{opp}{hyp}$

$\cos\theta = \frac{adj}{hyp} = \frac{\sqrt{5^2+4^2}}{5} = \frac{3}{5}$)

$\Rightarrow \frac{x}{5} = \frac{3}{5}$

$\Rightarrow x = 3$

20. $\sin(\tan^{-1} x)$, $|x| < 1$ is equal to

1) $\frac{x}{\sqrt{1-x^2}}$ 2) $\frac{1}{\sqrt{1-x^2}}$

3) $\frac{1}{\sqrt{1+x^2}}$ **4) $\frac{x}{\sqrt{1+x^2}}$**

Solution :

Let $\tan^{-1} x = \theta$

$\tan\theta = x$

$1 + \tan^2\theta = 1 + x^2$

$\sec^2\theta = 1 + x^2$

$\sec\theta = \sqrt{1+x^2}$

$\cos\theta = \frac{1}{\sqrt{1+x^2}}$

Now, $\tan\theta = \frac{\sin\theta}{\cos\theta}$

$\sin\theta = \tan\theta \times \cos\theta$

$= x \cdot \frac{1}{\sqrt{1+x^2}}$

$\sin(\tan^{-1} x) = \frac{x}{\sqrt{1+x^2}}$

BOOK SUMS ***(Exercise and Examples) :***

1. Find the principal value of $\sin^{-1}\left(-\frac{1}{2}\right)$ (in radians and degrees).
2. Find the principal value of $\sin^{-1}(2)$, if it exists.
3. Find the principal value of

   (i) $\sin^{-1}\left(\frac{1}{\sqrt{2}}\right)$ (ii) $\sin^{-1}\left(\sin\left(-\frac{\pi}{3}\right)\right)$ (iii) $\sin^{-1}\left(\sin\left(\frac{5\pi}{6}\right)\right)$

4. Find the domain of $\sin^{-1}(2-3x^2)$.
5. Find all the values of x such that

   (i) $-10\pi \le x \le 10\pi$ and $\sin x = 0$ (ii) $-8\pi \le x \le 8\pi$ and $\sin x = -1$

6. Find the period and amplitude of

   (i) $y = \sin 7x$ (ii) $y = -\sin\left(\frac{1}{3}x\right)$ (iii) $y = 4\sin(-2x)$.

7. Sketch the graph of $y = \sin\left(\frac{1}{3}x\right)$ for $0 \le x < 6\pi$.
8. Find the value of (i) $\sin^{-1}\left(\sin\left(\frac{2\pi}{3}\right)\right)$ (ii) $\sin^{-1}\left(\sin\left(\frac{5\pi}{4}\right)\right)$.
9. For what value of x does $\sin x = \sin^{-1} x$?
10. Find the domain of the following

    (i) $f(x) = \sin^{-1}\left(\frac{x^2+1}{2x}\right)$ (ii) $g(x) = 2\sin^{-1}(2x-1) - \frac{\pi}{4}$

11. Find the value of $\sin^{-1}\left(\sin\frac{5\pi}{9}\cos\frac{\pi}{9} + \cos\frac{5\pi}{9}\sin\frac{\pi}{9}\right)$
12. Find the principal value of $\cos^{-1}\left(\frac{\sqrt{3}}{2}\right)$
13. Find (i) $\cos^{-1}\left(-\frac{1}{\sqrt{2}}\right)$ (ii) $\cos^{-1}\left(\cos\left(-\frac{\pi}{3}\right)\right)$ (iii) $\cos^{-1}\left(\cos\left(\frac{7\pi}{6}\right)\right)$
14. Find the domain of $\cos^{-1}\left(\frac{2+\sin x}{3}\right)$
15. Find all values of x such that

    (i) $-6\pi \le x \le 6\pi$ and $\cos x = 0$ (ii) $-5\pi \le x \le 5\pi$ and $\cos x = 1$

16. State the reason for $\cos^{-1}\left[\cos\left(-\frac{\pi}{6}\right)\right] \ne \frac{-\pi}{6}$.
17. Is $\cos^{-1}(-x) = \pi - \cos^{-1}(x)$ true? Justify your answer.
18. Find the principal value of $\cos^{-1}\left(\frac{1}{2}\right)$.

19. Find the value of

(i) $2\cos^{-1}\left(\frac{1}{2}\right)+\sin^{-1}\left(\frac{1}{2}\right)$ (ii) $\cos^{-1}\left(\frac{1}{2}\right)+\sin^{-1}(-1)$ (iii) $\cos^{-1}\left(\cos\frac{\pi}{7}\cos\frac{\pi}{17}-\sin\frac{\pi}{7}\sin\frac{\pi}{17}\right)$

20. Find the domain of (i) $f(x)=\sin^{-1}\left(\frac{|x|-2}{3}\right)+\cos^{-1}\left(\frac{1-|x|}{4}\right)$ (ii) $g(x)=\sin^{-1}x+\cos^{-1}x$

21. For what value of x, the inequality $\frac{\pi}{2}<\cos^{-1}(3x-1)<\pi$ holds?

22. Find the value of

(i) $\cos\left(\cos^{-1}\left(\frac{4}{5}\right)+\sin^{-1}\left(\frac{4}{5}\right)\right)$ (ii) $\cos^{-1}\left(\cos\left(\frac{4\pi}{3}\right)\right)+\cos^{-1}\left(\cos\left(\frac{5\pi}{4}\right)\right)$

23. Find the principal value of $\tan^{-1}(\sqrt{3})$.

24. Find (i) $\tan^{-1}(-\sqrt{3})$ (ii) $\tan^{-1}\left(\tan\frac{3\pi}{5}\right)$ (iii) $\tan(\tan^{-1}(2019))$

25. Find the value of $\tan^{-1}(-1)+\cos^{-1}\left(\frac{1}{2}\right)+\sin^{-1}\left(-\frac{1}{2}\right)$

26. Prove that $\tan(\sin^{-1}x)=\frac{x}{\sqrt{1-x^2}}, -1<x<1$

27. Find the domain of the following functions :

(i) $\tan^{-1}\left(\sqrt{9-x^2}\right)$ (ii) $\frac{1}{2}\tan^{-1}(1-x^2)-\frac{\pi}{4}$.

28. Find the value of (i) $\tan^{-1}\left(\tan\frac{5\pi}{4}\right)$ (ii) $\tan^{-1}\left(\tan\left(-\frac{\pi}{6}\right)\right)$

29. Find the value of

(i) $\tan\left(\tan^{-1}\left(\frac{7\pi}{4}\right)\right)$ (ii) $\tan(\tan^{-1}(1947))$ (iii) $\tan(\tan^{-1}(-0.2021))$

30. Find the value of (i) $\tan\left(\cos^{-1}\left(\frac{1}{2}\right)-\sin^{-1}\left(-\frac{1}{2}\right)\right)$ (ii) $\sin\left(\tan^{-1}\left(\frac{1}{2}\right)-\cos^{-1}\left(\frac{4}{5}\right)\right)$

(iii) $\cos\left(\sin^{-1}\left(\frac{4}{5}\right)-\tan^{-1}\left(\frac{3}{4}\right)\right)$

31. Find the principal value of

(i) $\operatorname{cosec}^{-1}(-1)$ (ii) $\sec^{-1}(-2)$.

32. Find the value of $\sec^{-1}\left(-\frac{2\sqrt{3}}{3}\right)$

33. If $\cot^{-1}\left(\frac{1}{7}\right)=\theta$, find the value of $\cos\theta$

34. Show that $\cot^{-1}\left(\frac{1}{\sqrt{x^2-1}}\right)=\sec^{-1}x, |x|>1$.

35. Find the principal value of

(i) $\sec^{-1}\left(\frac{2}{\sqrt{3}}\right)$ (ii) $\cot^{-1}\left(\sqrt{3}\right)$ (iii) $\operatorname{cosec}^{-1}\left(-\sqrt{2}\right)$

36. Find the value of

(i) $\tan^{-1}\left(\sqrt{3}\right)-\sec^{-1}(-2)$ (ii) $\sin^{-1}(-1)+\cos^{-1}\left(\frac{1}{2}\right)+\cot^{-1}(2)$

(iii) $\cot^{-1}(1)+\sin^{-1}\left(-\frac{\sqrt{3}}{2}\right)-\sec^{-1}\left(-\sqrt{2}\right)$

37. Prove that $\frac{\pi}{2}\le\sin^{-1}x+2\cos^{-1}x\le\frac{3\pi}{2}$.

38. Simplify :

(i) $\cos^{-1}\left(\cos\left(\frac{13\pi}{3}\right)\right)$ (ii) $\tan^{-1}\left(\tan\left(\frac{3\pi}{4}\right)\right)$ (iii) $\sec^{-1}\left(\sec\left(\frac{5\pi}{3}\right)\right)$ (iv) $\sin^{-1}(\sin 10)$

39. Find the value of

(i) $\sin\left[\frac{\pi}{3}-\sin^{-1}\left(-\frac{1}{2}\right)\right]$ (ii) $\cos\left[\frac{1}{2}\cos^{-1}\left(\frac{1}{8}\right)\right]$ (iii) $\tan\left[\frac{1}{2}\sin^{-1}\left(\frac{2a}{1+a^2}\right)+\frac{1}{2}\cos^{-1}\left(\frac{1-a^2}{1+a^2}\right)\right]$

40. Prove that $\tan\left(\sin^{-1}x\right)=\frac{x}{\sqrt{1-x^2}}$ for $|x|<1$.

41. Evaluate $\sin\left[\sin^{-1}\left(\frac{3}{5}\right)+\sec^{-1}\left(\frac{5}{4}\right)\right]$

42. Prove that (i) $\tan^{-1}\frac{1}{2}+\tan^{-1}\frac{1}{3}=\frac{\pi}{4}$ (ii) $2\tan^{-1}\frac{1}{2}+\tan^{-1}\frac{1}{7}=\tan^{-1}\frac{31}{17}$

43. If $\cos^{-1}x+\cos^{-1}y+\cos^{-1}z=\pi$ and $0<x,y,z<1$, show that $x^2+y^2+z^2+2xyz=1$

44. If $a_1, a_2, a_3, \ldots a_n$ is an arithmetic progression with common difference d, prove that

$$\tan\left[\tan^{-1}\left(\frac{d}{1+a_1a_2}\right)+\tan^{-1}\left(\frac{d}{1+a_2a_3}\right)+\ldots+\tan^{-1}\left(\frac{d}{1+a_na_{n-1}}\right)\right]=\frac{a_n-a_1}{1+a_1a_n}$$

45. Solve $\tan^{-1}\left(\frac{1-x}{1+x}\right)=\frac{1}{2}\tan^{-1}x$ for $x>0$.

46. Solve $\sin^{-1}x>\cos^{-1}x$

47. Show that $\cot\left(\sin^{-1}x\right)=\frac{\sqrt{1-x^2}}{x}, -1\le x\le 1$ and $x\ne 0$

48. Solve $\tan^{-1}2x+\tan^{-1}3x=\frac{\pi}{4}$, if $6x^2<1$.

49. Solve $\tan^{-1}\left(\frac{x-1}{x-2}\right)+\tan^{-1}\left(\frac{x+1}{x+2}\right)=\frac{\pi}{4}$

50. Solve $\cos\left(\sin^{-1}\left(\frac{x}{\sqrt{1+x^2}}\right)\right)=\sin\left\{\cot^{-1}\left(\frac{3}{4}\right)\right\}$

51. Find the value, if it exists. If not, give the reason for non-existence.

(i) $\sin^{-1}(\cos\pi)$ (ii) $\tan^{-1}\left(\sin\left(-\frac{5\pi}{2}\right)\right)$ (iii) $\sin^{-1}[\sin 5]$

52. Find the value of the expression in terms of x, with the help of a reference triangle.

(i) $\sin(\cos^{-1}(1-x))$ (ii) $\cos(\tan^{-1}(3x-1))$ (iii) $\tan\left(\sin^{-1}\left(x+\frac{1}{2}\right)\right)$

53. Find the value of

(i) $\sin^{-1}\left(\cos\left(\sin^{-1}\left(\frac{\sqrt{3}}{2}\right)\right)\right)$ (ii) $\cot\left(\sin^{-1}\frac{3}{5}+\sin^{-1}\frac{4}{5}\right)$ (iii) $\tan\left(\sin^{-1}\frac{3}{5}+\cot^{-1}\frac{3}{2}\right)$

54. Prove that

(i) $\tan^{-1}\frac{2}{11}+\tan^{-1}\frac{7}{24}=\tan^{-1}\frac{1}{2}$ (ii) $\sin^{-1}\frac{3}{5}-\cos^{-1}\frac{12}{13}=\sin^{-1}\frac{16}{65}$

55. Prove that $\tan^{-1}x+\tan^{-1}y+\tan^{-1}z=\tan^{-1}\left[\frac{x+y+z-xyz}{1-xy-yz-zx}\right]$

56. If $\tan^{-1}x+\tan^{-1}y+\tan^{-1}z=\pi$, show that $x+y+z=xyz$

57. Prove that $\tan^{-1}x+\tan^{-1}\frac{2x}{1-x^2}=\tan^{-1}\frac{3x-x^3}{1-3x^2}, |x|<\frac{1}{\sqrt{3}}$

58. Simplify: $\tan^{-1}\frac{x}{y}-\tan^{-1}\frac{x-y}{x+y}$

59. Solve:

(i) $\sin^{-1}\frac{5}{x}+\sin^{-1}\frac{12}{x}=\frac{\pi}{2}$ (ii) $2\tan^{-1}x=\cos^{-1}\frac{1-a^2}{1+a^2}-\cos^{-1}\frac{1-b^2}{1+b^2}, a>0, b>0$

(iii) $2\tan^{-1}(\cos x)=\tan^{-1}(2\operatorname{cosec} x)$ (iv) $\cot^{-1}x-\cot^{-1}(x+2)=\frac{\pi}{12}, x>0$

60. Find the number of solution of the equation $\tan^{-1}(x-1)+\tan^{-1}x+\tan^{-1}(x+1)=\tan^{-1}(3x)$

Additional Questions :

61. Define : Periodic function with examples.

62. Define : Odd and Even functions.

63. Prove the following

(i) $\sin^{-1}(\sin\theta)=\theta$, if $\theta\in\left[-\frac{\pi}{2},\frac{\pi}{2}\right]$ (ii) $\cos^{-1}(\cos\theta)=\theta$, if $\theta\in[0,\pi]$

(iii) $\tan^{-1}(\tan\theta)=\theta$, if $\theta\in\left(-\frac{\pi}{2},\frac{\pi}{2}\right)$ (iv) $\operatorname{cosec}^{-1}(\operatorname{cosec}\theta)=\theta$, if $\theta\in\left[-\frac{\pi}{2},\frac{\pi}{2}\right]\setminus\{0\}$

(v) $\sec^{-1}(\sec\theta)=\theta$, if $\theta\in[0,\pi]\setminus\left\{\frac{\pi}{2}\right\}$ (vi) $\cot^{-1}(\cot\theta)=\theta$, if $\theta\in(0,\pi)$

64. Prove the following results

(i) $\sin(\sin^{-1}x) = x$, if $x \in [-1, 1]$

(ii) $\cos(\cos^{-1}x) = x$, if $x \in [-1, 1]$

(iii) $\tan(\tan^{-1}x) = x$, if $x \in \mathbb{R}$

(iv) $\operatorname{cosec}(\operatorname{cosec}^{-1}x) = x$, if $x \in \mathbb{R} \setminus (-1,1)$

(v) $\sec(\sec^{-1}x) = x$, if $x \in \mathbb{R} \setminus (-1,1)$

(vi) $\cot(\cot^{-1}x) = x$, if $x \in \mathbb{R}$

65. Prove the following reciprocal inverse identities :

(i) $\sin^{-1}\left(\frac{1}{x}\right) = \operatorname{cosec} x$, if $x \in \mathbb{R} \setminus (-1,1)$

(ii) $\cos^{-1}\left(\frac{1}{x}\right) = \sec x$, if $x \in \mathbb{R} \setminus (-1,1)$

(iii) $\tan^{-1}\left(\frac{1}{x}\right) = \begin{cases} \cot^{-1}x & if\ x > 0 \\ -\pi + \cot^{-1}x & if\ x < 0 \end{cases}$

66. Prove the following reflection identities :

(i) $\sin^{-1}(-x) = -\sin^{-1}x$, if $x \in [-1, 1]$

(ii) $\tan^{-1}(-x) = -\tan^{-1}x$, if $x \in \mathbb{R}$.

(iii) $\operatorname{cosec}^{-1}(-x) = -\operatorname{cosec}^{-1}x$, if $|x| \geq 1$ or $x \in \mathbb{R} \setminus (-1, 1)$

(iv) $\cos^{-1}(-x) = \pi - \cos^{-1}x$, if $x \in [-1, 1]$

(v) $\sec^{-1}(-x) = \pi - \sec^{-1}x$, if $|x| \geq 1$ or $x \in \mathbb{R} \setminus (-1, 1)$

(vi) $\cot^{-1}(-x) = \pi - \cot^{-1}x$, if $x \in \mathbb{R}$.

67. Prove the following cofunction inverse identities :

(i) $\sin^{-1}x + \cos^{-1}x = \frac{\pi}{2}$, $x \in [-1, 1]$

(ii) $\tan^{-1}x + \cot^{-1}x = \frac{\pi}{2}$, $x \in \mathbb{R}$

(iii) $\operatorname{cosec}^{-1}x + \sec^{-1}x = \frac{\pi}{2}$, $x \in \mathbb{R} \setminus (-1, 1)$ or $|x| \geq 1$

68. Prove the following results

(i) $\sin^{-1}x + \sin^{-1}y = \sin^{-1}\left(x\sqrt{1-y^2} + y\sqrt{1-x^2}\right)$ where either $x^2 + y^2 \leq 1$ or $xy < 0$.

(ii) $\sin^{-1}x - \sin^{-1}y = \sin^{-1}\left(x\sqrt{1-y^2} - y\sqrt{1-x^2}\right)$ where either $x^2 + y^2 \leq 1$ or $xy > 0$.

(iii) $\cos^{-1}x + \cos^{-1}y = \cos^{-1}\left(xy - \sqrt{1-x^2}\sqrt{1-y^2}\right)$, if $x + y \geq 0$.

(iv) $\cos^{-1}x - \cos^{-1}y = \cos^{-1}\left(xy + \sqrt{1-x^2}\sqrt{1-y^2}\right)$, if $x \leq y$

(v) $\tan^{-1}x + \tan^{-1}y = \tan^{-1}\left(\frac{x+y}{1-xy}\right)$, if $xy < 1$.

(vi) $\tan^{-1}x - \tan^{-1}y = \tan^{-1}\left(\frac{x-y}{1+xy}\right)$, if $xy > -1$.

69. Prove the following results

(i) $2\tan^{-1}x = \tan^{-1}\left(\frac{2x}{1-x^2}\right)$, $|x| < 1$

(ii) $2\tan^{-1}x = \cos^{-1}\left(\frac{1-x^2}{1+x^2}\right)$, $x \geq 0$

(iii) $2\tan^{-1}x = \sin^{-1}\left(\frac{2x}{1+x^2}\right)$, $|x| \leq 1$

70. Prove the following results

(i) $\sin^{-1}\left(2x\sqrt{1-x^2}\right) = 2\sin^{-1}x$, if $|x| \le \frac{1}{\sqrt{2}}$ or $-\frac{1}{\sqrt{2}} \le x \le \frac{1}{\sqrt{2}}$.

(ii) $\sin^{-1}\left(2x\sqrt{1-x^2}\right) = 2\cos^{-1}x$, if $\frac{1}{\sqrt{2}} \le x \le 1$.

71. Prove the following results

(i) $\sin^{-1}x = \cos^{-1}\sqrt{1-x^2}$, if $0 \le x \le 1$.

(ii) $\sin^{-1}x = -\cos^{-1}\sqrt{1-x^2}$, if $-1 \le x < 0$.

(iii) $\sin^{-1}x = \tan^{-1}\left(\frac{x}{\sqrt{1-x^2}}\right)$, if $-1 < x < 1$.

(iv) $\cos^{-1}x = \sin^{-1}\sqrt{1-x^2}$, if $0 \le x \le 1$.

(v) $\cos^{-1}x = \pi - \sin^{-1}\sqrt{1-x^2}$, if $-1 \le x < 0$.

(vi) $\tan^{-1}x = \sin^{-1}\left(\frac{x}{\sqrt{1+x^2}}\right) = \cos^{-1}\left(\frac{1}{\sqrt{1+x^2}}\right)$, if $x > 0$.

72. Prove the following results

(i) $3\sin^{-1}x = \sin^{-1}\left(3x-4x^3\right), x \in \left[-\frac{1}{2},\frac{1}{2}\right]$

(ii) $3\cos^{-1}x = \cos^{-1}\left(4x^3-3x\right), x \in \left[\frac{1}{2},1\right]$

73. Sketch the graph of $y = \sin x$ in $[0, 2\pi]$

74. Sketch the graph of $y = \sin^{-1}x$ in $y \in \left[-\frac{\pi}{2},\frac{\pi}{2}\right]$

75. Sketch the graph of $y = \cos x$ in $[0, 2\pi]$

76. Sketch the graph of $y = \cos^{-1}x$ in $y \in [0, \pi]$

77. Sketch the graph of $y = \tan x$ in $y \in \left(-\frac{\pi}{2},\frac{\pi}{2}\right)$

78. Sketch the graph of $y = \tan^{-1}x$ in $y \in \left(-\frac{\pi}{2},\frac{\pi}{2}\right)$

79. Sketch the graph of $y = \operatorname{cosec} x$ in $(0, 2\pi) \setminus \{\pi\}$

80. Sketch the graph of $y = \operatorname{cosec}^{-1}x$ in $y \in \left[-\frac{\pi}{2},\frac{\pi}{2}\right] \setminus \{0\}$

81. Sketch the graph of $y = \sec x$ in $[0, \pi] \setminus \left\{\frac{\pi}{2}\right\}$

82. Sketch the graph of $y = \sec^{-1}x$ in $y \in [0, \pi] \setminus \{0\}$

83. Sketch the graph of $y = \cot x$ in $(0, \pi)$

84. Sketch the graph of $y = \cot^{-1}x$ in $y \in (0, \pi)$

5. TWO DIMENSIONAL ANALYTICAL GEOMETRY-II

Points to Remembers :

- Equation of the circle in a standard form is $(x-h)^2+(y-k)^2=r^2$
  (i) centre (h,k) (ii) radius ' r '
- Equation of a circle in general form is $x^2+y^2+2gx+2fy+c=0$
  (i) centre (–g, –f) (ii) radius = $\sqrt{g^2+f^2-c}$
- The circle through the intersection of the line $\ell x+my+n=0$ and the circle $x^2+y^2+2gx+2fy+c=0$ is $x^2+y^2+2gx+2fy+c+\lambda(\ell x+my+n)=0, \lambda \in R^1$
- Equation of a circle with (x_1,y_1) and (x_2,y_2) as extremities of one of the diameters is $(x-x_1)(x-x_2)+(y-y_1)(y-y_2)=0$
- Equation of tangent at (x_1,y_1) on circle $x^2+y^2+2gx+2fy+c=0$ is $xx_1+yy_1+g(x+x_1)+f(y+y_1)+c=0$
- Equation of normal at (x_1,y_1) on circle $x^2+y^2+2gx+2fy+c=0$ is $yx_1-xy_1+g(y-y_1)-f(x-x_1)=0$

Table - 1

Tangent and normal

Curve	*Equation*	*Equation of tangent*	*Equation of normal*
circle	$x^2+y^2=a^2$	i) **Cartesian form** $xx_1+yy_1=a^2$ ii) **Parametric form** $x\cos\theta+y\sin\theta=a$	i) **Cartesian form** $xy_1-yx_1=0$ ii) **Parametric form** $x\sin\theta-y\cos\theta=0$
Parabola	$y^2=4ax$	i) **Cartesian form** $yy_1=2a(x+x_1)$ ii) **Parametric form** $yt=x+at^2$	i) **Cartesian form** $xy_1+2y=2ay_1+x_1y_1$ ii) **Parametric form** $y+xt=at^3+2at$
Ellipse	$\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$	i) **Cartesian form** $\frac{xx_1}{a^2}+\frac{yy_1}{b^2}=1$ ii) **Parametric form** $\frac{x\cos\theta}{a}+\frac{y\sin\theta}{b}=1$	i) **Cartesian form** $\frac{a^2x}{x_1}+\frac{b^2y}{y_1}=a^2-b^2$ ii) **Parametric form** $\frac{ax}{\cos\theta}-\frac{by}{\sin\theta}=a^2-b^2$
Hyperbola	$\frac{x^2}{a^2}-\frac{y^2}{b^2}=1$	i) **Cartesian form** $\frac{xx_1}{a^2}-\frac{yy_1}{b^2}=1$ ii) **Parametric form** $\frac{x\sec\theta}{a}-\frac{y\tan\theta}{b}=1$	i) **Cartesian form** $\frac{a^2x}{x_1}+\frac{b^2y}{y_1}=a^2+b^2$ ii) **Parametric form** $\frac{ax}{\sec\theta}+\frac{by}{\tan\theta}=a^2+b^2$

Table – 2

Condition for the sine $y = mx + c$ to be a tangent to the conics

Curve	*Equation*	*Condition to be tangent*	*Point of contact*	*Equation of tangent*
circle	$x^2 + y^2 = a^2$	$c^2 = a^2(1+m^2)$	$\left(\frac{\mp am}{\sqrt{1+m^2}}, \frac{\pm a}{\sqrt{1+m^2}}\right)$	$y = mx \pm a\sqrt{1+m^2}$
Parabola	$y^2 = 4ax$	$c = \frac{a}{m}$	$\left(\frac{a}{m^2}, \frac{2a}{m}\right)$	$y = mx + \frac{a}{m}$
Ellipse	$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$	$c^2 = a^2m^2 + b^2$	$\left(\frac{-a^2m}{c}, \frac{b^2}{c}\right)$	$y = mx \pm \sqrt{a^2m^2 + b^2}$
Hyperbola	$\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$	$c^2 = a^2m^2 - b^2$	$\left(\frac{-a^2m}{c}, \frac{-b^2}{c}\right)$	$y = mx \pm \sqrt{a^2m^2 - b^2}$

Table – 3

Parametric forms

Conic	*Parametric equations*	*Parameter*	*Range of parameter*	*Any point on the conic*
Circle	$x = a\cos\theta$ $y = a\sin\theta$	θ	$0 \le \theta \le 2\pi$	'θ' or $(a\cos\theta, a\sin\theta)$
Parabola	$x = at^2$ $y = 2at$	t	$-\infty < t < \infty$	't' or $(at^2, 2at)$
Ellipse	$x = a\cos\theta$ $y = b\sin\theta$	θ	$0 \le \theta \le 2\pi$	'θ' or $(a\cos\theta, b\sin\theta)$
Hyperbola	$x = a\sec\theta$ $y = b\tan\theta$	θ	$-\pi \le \theta \le \pi$ except $\theta = \pm\frac{\pi}{2}$	'θ' or $(a\sec\theta, b\tan\theta)$

Identifying the conic from the general equation of conic $Ax^2 + Bxy + Cy^2 + Dx + Ey + F = 0$

The graph of the second degree equation is one of a circle, parabola, an ellipse, a hyperbola, a point, an empty set, a single line or a pair of lines. When,

- $A = C = 1$, $B = 0$, $D = -2h$, $E = -2k$, $F = h^2 + k^2 - r^2$ the general equation reduces to $(x-h)^2 + (y-k)^2 = r^2$, which is a circle.
- $B = 0$ and either A or $C = 0$, the general equation yields a parabola under study, at this level.
- $A \ne C$ and A and C are of the same sign the general equation yields an ellipse.
- $A \ne C$ and A and C are of opposite signs the general equation yields a hyperbola.
- $A = C$ and $B = D = E = F = 0$, the general equation yields a point $x^2 + y^2 = 0$.
- $A = C = F$ and $B = D = E = 0$, the general equation yields an empty set $x^2 + y^2 + 1 = 0$, as there is no real solution.
- $A \ne 0$ or $C \ne 0$ and others are zeros, the general equation yield coordinate axes.
- $A = -C$ and rests are zero, the general equation yields a pair of lines $x^2 - y^2 = 0$.

BOOK BACK ONE MARKS

1. The equation of the circle passing through (1, 5) and (4, 1) and touching y-axis is $x^2+y^2-5x-6y+9+\lambda(4x+3y-19)=0$ where λ is equal to

(1) $0, -\frac{40}{9}$ (2) 0

(3) $\frac{40}{9}$ (4) $\frac{-40}{9}$

Solution :

$$x^2+y^2-5x+4\lambda x+3\lambda y-6y-19\lambda+9=0$$

$$x^2+y^2+2x\left(2\lambda-\frac{5}{2}\right)+2y\left(\frac{3\lambda}{2}-3\right)+9-19\lambda=0$$

centre – y axis distance = radius

$$2\lambda-\frac{5}{2}=\sqrt{\left(2\lambda-\frac{5}{2}\right)^2+\left(\frac{3\lambda}{2}-3\right)^2+19\lambda-9}$$

$$\left(2\lambda-\frac{5}{2}\right)^2=\left(2\lambda-\frac{5}{2}\right)^2+$$

$$\left(\frac{3\lambda}{2}-3\right)^2+19\lambda-9$$

$$0=\frac{9\lambda^2}{4}+9-9\lambda+19\lambda-9$$

$$\frac{9\lambda^2}{4}+10\lambda=0$$

$$\lambda\left(\frac{9\lambda+40}{4}\right)=0$$

$\lambda=0$

$9\lambda=-40$

$\lambda=\frac{-40}{9}$

2. The eccentricity of the hyperbola whose latus rectum is 8 and conjugate axis is equal to half the distance between the foci is

(1) $\frac{4}{3}$ (2) $\frac{4}{\sqrt{3}}$

(3) $\frac{2}{\sqrt{3}}$ (4) $\frac{3}{2}$

Solution :

$\frac{2b^2}{a}=8$

$b^2=\frac{8a}{2}$

$b^2=4a$

$2b=\frac{1}{2}(2ae)$

$2b=ae$

$a^2e^2=4b^2$

$=4(4a)$ $(\because b^2=4a)$

$=16a$

$e^2=\frac{16a}{a^2}=\frac{16}{a}$

$b^2=a^2e^2-a^2$

$4a=16a-a^2$

$a^2-16a+4a=0$

$a^2-12a=0$

$a(a-12)=0$

$a=12$

$e^2=\frac{16}{12}=\frac{4}{3}$

$e=\frac{2}{\sqrt{3}}$

3. The circle $x^2+y^2=4x+8y+5$ intersects the line $3x-4y=m$ at two distinct points if

(1) $15<m<65$ (2) $35<m<85$

(3) $-85<m<-35$ **(4) $-35<m<15$**

Solution :

$x^2+y^2-4x-8y-5=0$

centre $=(-g,-f)$

$=(2,4)$

radius $=\sqrt{4+16+5}$

$=5$

distance $=\left|\frac{3x_1-4y_1-m}{\sqrt{(3)^2+(-4)^2}}\right|<5$

$$\left|\frac{3(2)-4(4)-m}{\sqrt{9+16}}\right| < 5$$

$|6-16-m| < 25$

$|10+m| < 25$

$-25 < 10+m < 25$

$-35 < m < 15$

4. The length of the diameter of the circle which touches the x -axis at the point (1, 0) and passes through the point (2, 3).

(1) $\frac{6}{5}$ (2) $\frac{5}{3}$

(3) $\frac{10}{3}$ (4) $\frac{3}{5}$

Solution : $P(x,y)$

centre $(1,a)$

radius $= \sqrt{(x_2-x_1)^2+(y_2-y_1)^2}$

$a = \sqrt{(x-1)^2+(y-a)^2}$

If passes (2, 3)

$a^2 = (2-1)^2+(3-a)^2$

$a^2 = 1+9-6a+a^2$

$6a = 10$

$a = \frac{10}{6}$

$a = \frac{5}{3}$

Diameter $= 2a = \frac{10}{3}$

5. The radius of the circle $3x^2+by^2+4bx-6by+b^2=0$ is

(1) 1 (2) 3

(3) $\sqrt{10}$ (4) $\sqrt{11}$

Solution :

Co-efficient of x^2 = co-efficient of y^2.

$b = 3$

$3x^2+3y^2+12x-18y+9=0$

$x^2+y^2+4x-6y+3=0$

$g=2,\ f=-3,\ c=3$

$r = \sqrt{g^2+f^2-c}$

$= \sqrt{4+9-3}$

$r = \sqrt{10}$

6. The centre of the circle inscribed in a square formed by the lines $x^2-8x-12=0$ and $y^2-14y+45=0$ is

(1) (4, 7) (2) (7, 4)

(3) (9, 4) (4) (4,9)

Solution :

$x^2-8x-12=0$

$x^2-8x+16-16-12=0$

$(x-4)^2 = 28$

$x-4 = \pm 2\sqrt{7}$

$x = 4\pm 2\sqrt{7}$

$y^2-14y+45 = 0$

$(y-9)(y-5) = 0$

$y = 5, 9$

$(4-2\sqrt{7},5)$ $(4+2\sqrt{7},9)$

centre $= \left(\frac{4+2\sqrt{7}+4-2\sqrt{7}}{2}, \frac{5+9}{2}\right)$

$= (4, 7)$

7. The equation of the normal to the circle $x^2+y^2-2x-2y+1=0$ which is parallel to the line $2x+4y=3$ is

(1) $x+2y=3$ (2) $x+2y+3=0$

(3) $2x+4y+3=0$ (4) $x-2y+3=0$

Solutions :

$x^2+y^2-2x-2y+1=0$

centre $= (-g,-f) = (1, 1)$

normal equation

$2x+4y=k$

It passes (1, 1)

$2+4 = k$

$k = 6$

$2x+4y = 6$

$x+2y = 3$

8. If $P(x,y)$ be any point on $16x^2+25y^2=400$ with foci F_1 (3, 0) and F_2 (–3, 0) then PF_1+PF_2 is

(1) 8 (2) 6
(3) 10 (4) 12

Solution :

$16x^2+25y^2=400$

$\frac{x^2}{25}+\frac{y^2}{16}=1$

$a^2=25$

$a=5$

$F_1P+F_2P=2a=2(5)=10$

9. The radius of the circle passing through the point (6 , 2) two of whose diameters are $x+y=6$ and $x+2y=4$ is

(1) 10 (2) $2\sqrt{5}$
(3) 6 (4) 4

Solution :

$x+y=6$ $x-2=6$

$x+2y=4$ $x=8$

$-y=2$ centre (8, –2)

$y=-2$

radius $=\sqrt{(6-8)^2+(2+2)^2}$

$=\sqrt{(-2)^2+(4)^2}$

$=\sqrt{4+16}$

$=\sqrt{20}$

$=2\sqrt{5}$

10. The area of quadrilateral formed with foci of the hyperbolas $\frac{x^2}{a^2}-\frac{y^2}{b^2}=1$ and $\frac{x^2}{a^2}-\frac{y^2}{b^2}=-1$ is

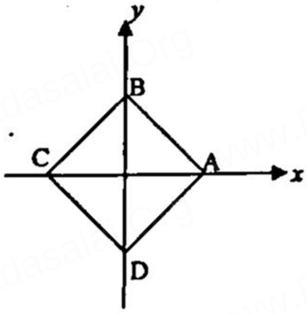
(1) $4(a^2+b^2)$ (2) $2(a^2+b^2)$
(3) a^2+b^2 (4) $\frac{1}{2}(a^2+b^2)$

Solution :

$\frac{x^2}{a^2}-\frac{y^2}{b^2}=1$

A(ae, 0), C(–ae, 0)

$e=\sqrt{\frac{a^2+b^2}{a^2}}$



$\frac{y^2}{b^2}-\frac{x^2}{a^2}=1$

B(0, be), D(0, –be)

$e=\sqrt{\frac{a^2+b^2}{b^2}}$

Area of quadrilateral ABCD

$=\frac{1}{2}d_1 \times d_2$

$=\frac{1}{2}2ae \times 2\,be$

$=\frac{a\sqrt{a^2+b^2}}{a} \times \frac{2b\sqrt{a^2+b^2}}{b}=2(a^2+b^2)$

11. If the normals of the parabola $y^2=4x$ drawn at the end points of its latus rectum are tangents to the circle $(x-3)^2+(y+2)^2=r^2$, then the value of r^2 is

(1) 2 (2) 3
(3) 1 (4) 4

Solution :

$y^2=4x$

$y^2=4ax$

$a=1$

Length of latus rectum 4a = 4

End Points are (1, 2), (1, –2)

($\because$ (a, 2a), (a, –2a))

Tangent equation at(1, 2)

$yy_1=4\left(\frac{x+x_1}{2}\right)$

$yy_1=2(x+x_1)$

$y=(x+1)$

$x-y+1=0$

Normal equation

$x+y+k=0$

It passes (1, 2)

$1+2+k=0$

$k=-3$

Normal equation $x+y-3=0$, centre : (3, –2)

radius $=\left|\frac{3-2-3}{\sqrt{1^2+1^2}}\right|=\frac{2}{\sqrt{2}}=\sqrt{2}$

$r=\sqrt{2}$ $r^2=2$

12. If $x+y=k$ is a normal to the parabola $y^2=12x$, then the value of k is

(1) 3 (2) -1

(3) 1 (4) 9

Solution :

$y^2 = 12x$

a $= 3$

y $= -x+k$

m $= -1$

c $= k$

condition for the normal equation of $y^2 = 4ax$

$k = -2am - am^3$

$k = -2(3)(-1) - 3(-1)^3$

$k = 6+3$

$k = 9$

13. The ellipse $E_1 : \frac{x^2}{9}+\frac{y^2}{4}=1$ is inscribed in a rectangle R whose sides are parallel to the coordinate axes. Another ellipse E_2 passing through the point (0, 4) circumscribes the rectangle R. The eccentricity of the ellipse is

(1) $\frac{\sqrt{2}}{2}$ (2) $\frac{\sqrt{3}}{2}$

(3) $\frac{1}{2}$ (4) $\frac{3}{4}$

Solution :

Equation of ellipse

$\frac{x^2}{b^2}+\frac{y^2}{a^2}=1$

It passes (0, 4)

$\frac{(0)^2}{b^2}+\frac{(4)^2}{a^2}=1$

$\frac{16}{a^2} = 1$

$a^2 = 16$

$\frac{x^2}{b^2}+\frac{y^2}{16} = 1$

It passes (3, 2)

$\frac{9}{b^2}+\frac{4}{16} = 1$

$\frac{9}{b^2} = 1-\frac{1}{4}$

$\frac{9}{b^2} = \frac{3}{4}$

$b^2 = \frac{36}{3}$

$b^2 = 12$

$e = \sqrt{\frac{a^2-b^2}{a^2}}$

$e = \sqrt{\frac{4}{16}} = \frac{1}{2}$

14. Tangents are drawn to the hyperbola $\frac{x^2}{9}-\frac{y^2}{4}=1$ parallel to the straight line $2x-y=1$. One of the points of contact of tangents on the hyperbola is

(1) $\left(\frac{9}{2\sqrt{2}},\frac{-1}{\sqrt{2}}\right)$ (2) $\left(\frac{-9}{2\sqrt{2}},\frac{1}{\sqrt{2}}\right)$

(3) $\left(\frac{9}{2\sqrt{2}},\frac{1}{\sqrt{2}}\right)$ (4) $\left(3\sqrt{3},-2\sqrt{2}\right)$

Solution :

$a^2 = 9,\quad b^2 = 4 \quad m = 2$

$c = \pm\sqrt{a^2m^2-b^2}$

$= \pm\sqrt{9\times4-4} = \pm\sqrt{32}$

Take $c = -4\sqrt{2}$

Point of contact $= \left(\frac{-a^2m}{c},\frac{-b^2}{c}\right)$

$= \left(\frac{-9\times2}{-4\sqrt{2}},\frac{-4}{-4\sqrt{2}}\right)$

$= \left(\frac{9}{2\sqrt{2}},\frac{1}{\sqrt{2}}\right)$

15. The equation of the circle passing through the foci of the ellipse $\frac{x^2}{16}+\frac{y^2}{9}=1$ having centre at (0, 3) is

(1) $x^2+y^2-6y-7=0$

(2) $x^2+y^2-6y+7=0$

(3) $x^2+y^2-6y-5=0$

(4) $x^2+y^2-6y+5=0$

Solutions :

$a^2 = 16,\quad b^2 = 9$

$e = \sqrt{\frac{a^2-b^2}{a^2}} = \sqrt{\frac{16-9}{16}} = \frac{\sqrt{7}}{4}$

Foci $(\pm ae, 0) = (\pm\sqrt{7}, 0)$

equation of the circle with centre (0, 3) is

$(x-0)^2+(y-3)^2=r^2$ .... (1)

It passes $(\sqrt{7},0)$

$(\sqrt{7})^2+3^2=r^2$

$r^2=16$

$(1)\Rightarrow x^2+(y-3)^2=16$

$x^2+y^2-6y-7=0$

16. Let C be the circle with centre at (1, 1)and radius = 1. If T is the circle centered at (0, y) passing through the origin and touching the circle C externally, then the radius of T is equal to

(1) $\frac{\sqrt{3}}{\sqrt{2}}$ (2) $\frac{\sqrt{3}}{2}$

(3) $\frac{1}{2}$ (4) $\frac{1}{4}$

Solution :

$r_1=1$ let $r_2=y$

$c_1c_2=r_1+r_2$

c_1 (1, 1)

c_2 (0, y)

$\sqrt{(1-0)^2+(1-y)^2}=1+y$

$1+1^2+y^2-2y=(1+y)^2$

$1+1+y^2-2y=1+y^2+2y$

$1=2y+2y$

$4y=1$

$y=\frac{1}{4}$

17. Consider an ellipse whose centre is of the origin and its major axis is along x-axis. If its eccentricity is $\frac{3}{5}$ and the distance between its foci is 6, then the area of the quadrilateral inscribed in the ellipse with diagonals as major and minor axis of the ellipse is

(1) 8 (2) 32

(3) 80 (4) 40

Solution :

$2ae=6$

$ae=3$

$e=\frac{3}{5}$

$a\left(\frac{3}{5}\right)=3$

$a=5$

$b^2=a^2-a^2e^2=25-9$

$b^2=16$

$b=4$

Area of the quadrilateral $=\frac{1}{2}d_1\times d_2$

$=\frac{1}{2}(2a)(2b)$

$=5\times2\times4$

$=40$ sq.units

18. Area of the greatest rectangle inscribed in the ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$ is

(1) $2ab$ (2) ab

(3) $\sqrt{ab}$ (4) $\frac{a}{b}$

Solution :

$x=a\cos\theta$

$y=b\sin\theta$

Area $=\ell\times b$

$=2a\cos\theta\times2b\sin\theta$

$A=2ab\sin2\theta$

$A'=4ab\cos2\theta$

$A'=0$

$\cos2\theta=0$

$\theta=45°$

$A=2ab\sin2(45)$

$A=2ab$

19. An ellipse has OB as semi minor axes, F and F′ its foci and the angle FBF′ is a right angle. Then the eccentricity of the ellipse is

(1) $\frac{1}{\sqrt{2}}$ (2) $\frac{1}{2}$

(3) $\frac{1}{4}$ (4) $\frac{1}{\sqrt{3}}$

Solution :

$FF'^2=BF^2+BF'^2$

$(2ae)^2=\left(\sqrt{(ae-0)^2+(0-b)^2}\right)^2+\left(\sqrt{(-ae-0)^2+(0-b)^2}\right)^2$

$4a^2e^2=a^2e^2+b^2+a^2e^2+b^2$

$4a^2e^2-2a^2e^2=2b^2$

$2b^2=2a^2e^2$

$b^2=a^2e^2$

$a^2 - a^2e^2 = a^2e^2$

$a^2 = 2a^2e^2$

$e^2 = \frac{1}{2}$

$e = \frac{1}{\sqrt{2}}$

20. The eccentricity of the ellipse $(x-3)^2+(y-4)^2=\frac{y^2}{9}$ is

(1) $\frac{\sqrt{3}}{2}$ (2) $\frac{1}{3}$

(3) $\frac{1}{3\sqrt{2}}$ (4) $\frac{1}{\sqrt{3}}$

Solution :

$FP^2 = e^2PM^2$

$(x-3)^2 + (y-4)^2 = \left(\frac{1}{3}\right)^2\left(\frac{0+y+0}{\sqrt{0+1}}\right)^2$

$= \left(\frac{1}{3}\right)^2\left(\frac{\ell x+my+n}{\sqrt{\ell^2+m^2}}\right)^2$

$e = \frac{1}{3}$

21. If the two tangents drawn from a point P to the parabola $y^2 = 4x$ are at right angles then the locus of P is

(1) $2x+1=0$ (2) $x=-1$

(3) $2x-1=0$ (4) $x=1$

Solution :

$y^2 = 4x$

length of latus rectum

$4a = 4$

$a = 1$

equation of directrix $x = -a$

$x = -1$

22. The circle passing through (1,–2) and touching the axis of x at (3, 0)passing through the point

(1) (–5, 2) (2) (2,–5)

(3) (5,–2) (4) (–2, 5)

Solution :

$(x-3)^2+(y-k)^2=k^2$

It passes $(1,-2)$

$(1-3)^2+(-2-k)^2=k^2$

$4+4+4k+k^2 = k^2$

$4k = -8$

$k = -2$

$(x-3)^2+(y+2)^2=4$

lies on $(5,-2)$

$2^2+0^2=4$

$4=4$

23. The locus of a point whose distance from $(-2,0)$ is $\frac{2}{3}$ times its distance from the line $x=\frac{-9}{2}$ is

(1) a parabola (2) a hyperbola

(3) an ellipse (4) a circle

Solution :

$FP = \frac{2}{3}PM$

$\frac{FP}{PM} = \frac{2}{3} < 1$

$e < 1$ it is an ellipse

24. The values of m for which the line $y=mx+2\sqrt{5}$ touches the hyperbola $16x^2-9y^2=144$ are the roots of $x^2-(a+b)x-4=0$, then the value of $(a+b)$ is

(1) 2 (2) 4

(3) 0 (4) -2

Solution :

$\frac{x^2}{9}-\frac{y^2}{16}=1$ $y = mx+2\sqrt{5}$

$a^2=9,$ $b^2=16$ $c = 2\sqrt{5}$

condition $c^2=a^2m^2-b^2$

$(2\sqrt{5})^2 = 9m^2-16$

$20 = 9m^2-16$

$9m^2 = 36$

$m^2 = 4$

$m = \pm 2$

Take the values of m roots

$a = 2$ $b = -2$

product $= -4$

$a+b = 2-2 = 0$

25. If the coordinates at one end of a diameter of the circle $x^2+y^2-8x-4y+c=0$ are (11, 2), the coordinates of the other end are

(1) (–5, 2) (2) (2, –5)

(3) (5, –2) (4) (–2, 5)

Solution :

$$\left(\frac{11+x}{2},\frac{2+y}{2}\right)=(4,2)$$

$$\frac{11+x}{2}=4 \qquad \frac{2+y}{2}=2$$

$$x=8-11 \qquad 2+y=4$$

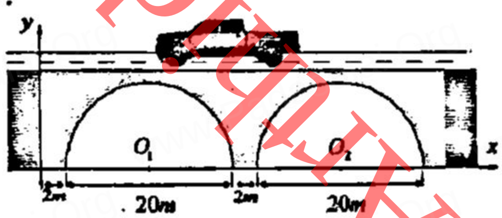
$$x=-3 \qquad y=2$$

(Book answer is wrong)

Correct answer is (–3, 2)

BOOK SUMS ***(Exercise and Examples) :***

1. Find the general equation of a circle with centre (–3, –4) and radius 3 units.
2. Find the equation of the circle described on the chord $3x+y+5=0$ of the circle $x^2+y^2=16$ as diameter.
3. Determine whether $x+y-1=0$ is the equation of a diameter of the circle $x^2+y^2-6x+4y+c=0$ for all possible values of c.
4. Find the general equation of the circle whose diameter is the line segment joining the points (–4, –2) and (1, 1).
5. Examine the position of the point (2, 3) with respect to the circle $x^2+y^2-6x-8y+12=0$.
6. The line $3x+4y-12=0$ meets the coordinate axes at A and B. Find the equation of the circle drawn on AB as diameter.
7. A line $3x+4y+10=0$ cuts a chord of length 6 units on a circle with centre of the circle (2, 1). Find the equation of the circle in general form.
8. A circle of radius 3 units touches both the axes. Find the equations of all possible circles formed in the general form.
9. Find the centre and radius of the circle $3x^2+(a+1)y^2+6x-9y+a+4=0$.
10. Find the equation of the circle passing through the points (1, 1), (2, –1) and (3, 2).
11. Find the equations of the tangent and normal to the circle $x^2+y^2=25$ at $P(-3, 4)$.
12. If $y=4x+c$ is a tangent to the circle $x^2+y^2=9$, find c.
13. A road bridge over an irrigation canal have two semi circular vents each with a span of $20m$ and the supporting pillars of width $2m$. Use figure to write the equations that represent the semi-verticular vents.



14. Obtain the equation of the circles with radius 5 cm and touching x-axis at the origin in general form.
15. Find the equation of the circle with centre (2, –1) and passing through the point (3, 6) in standard form.
16. Find the equation of circles that touch both the axes and pass through (–4, –2) in general form.
17. Find the equation of the circle with centre (2, 3) and passing through the intersection of the lines $3x-2y-1=0$ and $4x+y-27=0$
18. Obtain the equation of the circle for which (3, 4) and (2, –7) are the ends of a diameter.

19. Find the equation of the circle through the points (1, 0), (–1, 0) and (0, 1).
20. A circle of area 9π square units has two of its diameters along the lines $x+y=5$ and $x-y=1$. Find the equation of the circle.
21. If $y=2\sqrt{2}x+c$ is a tangent to the circle $x^2+y^2=16$, find the value of c.
22. Find the equation of the tangent and normal to the circle $x^2+y^2-6x+6y-8=0$ at (2, 2).
23. Determine whether the points (–2, 1), (0, 0) and (–4, –3) lie outside, on or inside the circle $x^2+y^2-5x+2y-5=0$
24. Find centre and radius of the following circles.
(i) $x^2+(y+2)^2=0$ (ii) $x^2+y^2+6x-4y+4=0$
(iii) $x^2+y^2-x+2y-3=0$ (iv) $2x^2+2y^2-6x+4y+2=0$
25. If the equation $3x^2+(3-p)xy+qy^2-2px=8pq$ represents a circle, find p and q. Also determine the centre and radius of the circle.
26. Find the length of Latus rectum of the parabola $y^2=4ax$.
27. Find the length of Latus rectum of the ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$
28. Find the equation of the parabola with focus $(-\sqrt{2},0)$ and directrix $x=\sqrt{2}$.
29. Find the equation of the parabola whose vertex is (5, –2) and focus (2, –2).
30. Find the equation of the parabola with vertex (–1, –2), axis parallel to y-axis and passing through (3, 6).
31. Find the vertex, focus, directrix, and length of the latus rectum of the parabola $x^2-4x-5y-1=0$.
32. Find the equation of the ellipse with foci (±2, 0), vertices (±3, 0).
33. Find the equation of the ellipse whose eccentricity is $\frac{1}{2}$, one of the foci is (2, 3) and a directrix is $x=7$. Also find the length of the major and minor axes of the ellipse.
34. Find the foci, vertices and length of major and minor axis of the conic $4x^2+36y^2+40x-288y+532=0$
35. For the ellipse $4x^2+y^2+24x-2y+21=0$, find the centre, vertices, and the foci. Also prove that the length of latus rectum is 2.
36. Find the equation of the hyperbola with vertices (0, ±4) and foci (0, ±6).
37. Find the vertices, foci for the hyperbola $9x^2-16y^2=144$
38. Find the centre, foci, and eccentricity of the hyperbola $11x^2-25y^2-44x+50y-256=0$
39. The orbit of Halley's Comet is an ellipse 36.18 astronomical units long and by 9.12 astronomical units wide. Find its eccentricity.
40. Find the equation of the parabola in each of the cases given below:
(i) focus (4, 0) and directrix $x=-4$.
(ii) passes through (2, –3) and symmetric about y-axis.
(iii) vertex (1, –2) and focus (4, –2). (iv) end points of latus rectum (4, –8) and (4, 8)

53. Find the equations of tangents to the hyperbola $\frac{x^2}{16}-\frac{y^2}{64}=1$ which are parallel to $10x-3y+9=0$.

54. Show that the line $x-y+4=0$ is a tangent to the ellipse $x^2+3y^2=12$. Also find the coordinates of the point of contact.

55. Find the equation of the tangent to the parabola $y^2=16x$ perpendicular to $2x+2y+3=0$.

56. Find the equation of the tangent at $t=2$ to the parabola $y^2=8x$. (Hint: use parametric form)

57. Find the equations of the tangent and normal to hyperbola $12x^2-9y^2=108$ at $\theta=\frac{\pi}{3}$. (Hint: use parametric form)

58. Prove that the point of intersection of the tangents at 't_1' and 't_2' on the parabola $y^2=4ax$ is $[at_1t_2, a(t_1+t_2)]$

59. If the normal at the point 't_1' on the parabola $y^2=4ax$ meets the parabola again at the point 't_2', then prove that $t_2=-\left(t_1+\frac{2}{t_1}\right)$.

60. A semielliptical archway over a one-way road has a height of $3m$ and a width of $12m$. The truck has a width of $3m$ and a height of $2.7\,m$. Will the truck clear the opening of the archway?

61. The maximum and minimum distances of the Earth from the Sun respectively are 152×10^6 km and 94.5×10^6 km. The Sun is at one focus of the elliptical orbit. Find the distance from the Sun to the other focus.

62. A concrete bridge is designed as a parabolic arch. The road over bridge is $40m$ long and the maximum height of the arch is $15m$. Write the equation of the parabolic arch.

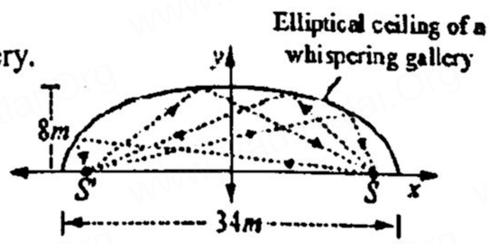
63. The parabolic communication antenna has a focus at $2m$ distance from the vertex of the antenna. Find the width of the antenna $3m$ from the vertex.

64. The equation $y=\frac{1}{32}x^2$ models cross sections of parabolic mirrors that are used for solar energy. There is a heating tube located at the focus of each parabola; how high is this tube located above the vertex of the parabola?

65. A search light has a parabolic reflector (has a cross section that forms a 'bowl'). The parabolic bowl is $40cm$ wide from rim to rim and $30cm$ deep. The bulb is located at the focus.
(1) What is the equation of the parabola used for reflector?
(2) How far from the vertex is the bulb to be placed so that the maximum distance covered?

66. An equation of the elliptical part of an optical lens system is $\frac{x^2}{16}+\frac{y^2}{9}=1$. The parabolic part of the system has a focus in common with the right focus of the ellipse. The vertex of the parabola is at the origin and the parabola opens to the right. Determine the equation of the parabola.

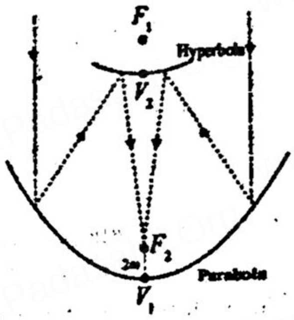
67. A room $34m$ long is constructed to be a whispering gallery. The room has an elliptical ceiling, as show in figure. If the maximum height of the ceiling is $8m$, determine where the foci are located.



68. If the equation of the ellipse is $\frac{(x-11)^2}{484}+\frac{y^2}{64}=1$ (x and y are measured in centimeters) where to the nearest centimeter, should the patient's kidney stone be placed so that the reflected sound hits the kidney stone?

69. Two coast guard stations are located 600 km apart at points $A(0, 0)$ and $B(0, 600)$. A distress signal from a ship at P is received at slightly different times by two stations. It is determined that the ship is 200 km farther from station A than it is from station B. Determine the equation of hyperbola that passes through the location of the ship.

70. Certain telescopes contain both parabolic mirror and a hyperbolic mirror. In the telescope shown in figure the parabola and hyperbola share focus F_1 which is $14m$ above the vertex of the parabola. The hyperbola's second focus F_2 is $2m$ above the parabola's vertex. The vertex of the hyperbolic mirror is $1m$ below F_1. Position a coordinate system with the origin at the centre of the hyperbola and with the foci on the y-axis. Then find the equation of the hyperbola.



71. A bridge has a parabolic arch that is $10m$ high in the centre and $30m$ wide at the bottom. Find the height of the arch $6m$ from the centre, on either sides.

72. A tunnel through a mountain for a four lane highway is to have a elliptical opening. The total width of the highway (not the opening) is to be $16m$, and the height at the edge of the road must be sufficient for a truck $4m$ high to clear if the highest point of the opening is to be $5m$ approximately. How wide must the opening be?

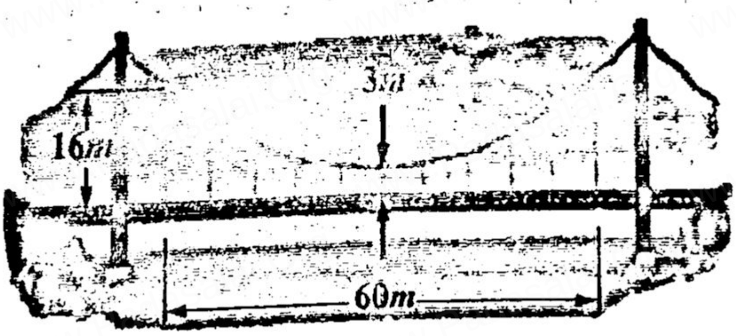
73. At a water fountain, water attains a maximum height of $4m$ at horizontal distance of $0.5\ m$ from its origin. If the path of water is a parabola, find the height of water at a horizontal distance of $0.75m$ from the point of origin.

74. An engineer designs a satellite dish with a parabolic cross section. The dish is $5m$ wide at the opening, and the focus is placed $1.2\ m$ from the vertex

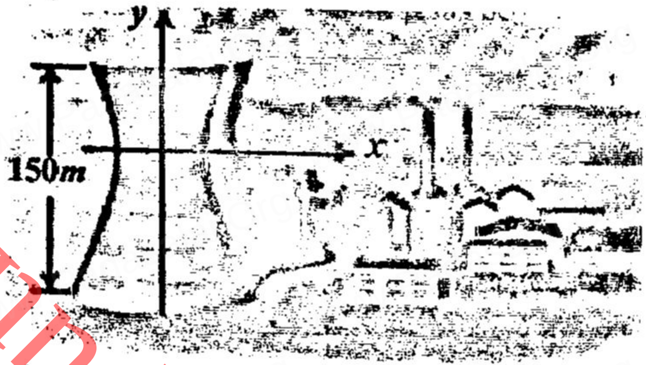
(a) Position a coordinate system with the origin at the vertex and the x-axis on the parabola's axis of symmetry and find an equation of the parabola.

(b) Find the depth of the satellite dish at the vertex.

75. Parabolic cable of a $60m$ portion of the roadbed of a suspension bridge are positioned as shown below. Vertical Cables are to be spaced every $6m$ along this portion of the roadbed. Calculate the lengths of first two of these vertical cables from the vertex.



76. Cross section of a Nuclear cooling tower is in the shape of a hyperbola with equation $\frac{x^2}{30^2}-\frac{y^2}{44^2}=1$. The tower is 150*m* tall and the distance from the top of the tower to the centre of the hyperbola is half the distance from the base of the tower to the centre of the hyperbola. Find the diameter of the top and base of the tower.



77. A rod of length 1.2 *m* moves with its ends always touching the coordinate axes. The locus of a point *P* on the rod, which is 0.3 *m* from the end in contact with *x* -axis is an ellipse. Find the eccentricity.
78. Assume that water issuing from the end of a horizontal pipe, 7.5 *m* above the ground, describes a parabolic path. The vertex of the parabolic path is at the end of the pipe. At a position 2.5 *m* below the line of the pipe, the flow of water has curved outward 3*m* beyond the vertical line through the end of the pipe. How far beyond this vertical line will the water strike the ground?
79. On lighting a rocket cracker it gets projected in a parabolic path and reaches a maximum height of 4*m* when it is 6*m* away from the point of projection. Finally it reaches the ground 12*m* away from the starting point. Find the angle of projection.
80. Points *A* and *B* are 10*km* apart and it is determined from the sound of an explosion heard at those points at different times that the location of the explosion is 6 *km* closer to *A* than *B* . Show that the location of the explosion is restricted to a particular curve and find an equation of it.

Additional Questions :

81. Prove that the circle passing through the points of intersection (real or imaginary) of the line $\ell x + my + n = 0$ and the circle $x^2 + y^2 + 2gx + 2fy + c = 0$ is the circle of the form $x^2 + y^2 + 2gx + 2fy + c + \lambda(\ell x + my + n) = 0, \lambda \in R'$.
82. Prove that the equation of a circle with (x_1, y_1) and (x_2, y_2) as extremities of one of the diameters of the circle is $(x - x_1)(x - x_2) + (y - y_1)(y - y_2) = 0$.
83. Prove that the position of a point $P(x_1, y_1)$ with respect to a given circle $x^2 + y^2 + 2gx + 2fy + c = 0$ in the plane containing the circle is outside or on or inside the circle according as

$$x_1^2+y_1^2+2gx_1+2fy_1+c \text{ is} \begin{cases} >0 & \text{or,} \\ =0 & \text{or,} \\ <0 \end{cases}$$

84. Prove that from any point outside the circle $x^2+y^2=a^2$ two tangents can be drawn.
85. Prove that the sum of the focal distances of any point on the ellipse is equal to length of the major axis.
86. Prove that three normals can be drawn to a parabola $y^2 = 4ax$ from a given point, one of which is always real.

6. APPLICATIONS OF VECTOR ALGEBRA

Points to remember :

- For a given set of three vectors $\vec{a}, \vec{b}$ and $\vec{c}$, the scalar $(\vec{a} \times \vec{b}).\vec{c}$ is called a scalar triple product of $\vec{a}, \vec{b}, \vec{c}$.
- The volume of the parallelopiped formed by using the three vectors $\vec{a}, \vec{b}$ and $\vec{c}$, as co-terminus edges is given by $|(\vec{a} \times \vec{b}).\vec{c}|$.
- The scalar triple product of three non-zero vectors is zero if and only if the three vectors are coplanar.
- Three vectors $\vec{a}, \vec{b}, \vec{c}$ are coplanar, if and only if there exist scalars r, s, t ∈ R such that atleast one of them is non-zero and $r\vec{a}+s\vec{b}+t\vec{c}=\vec{0}$
- If $\vec{a}, \vec{b}, \vec{c}$ and $\vec{p}, \vec{q}, \vec{r}$ are any two systems of three vectors, and if $\vec{p}=x_1\vec{a}+y_1\vec{b}+z_1\vec{c}$, $\vec{q}=x_2\vec{a}+y_2\vec{b}+z_2\vec{c}$ and $\vec{r}=x_3\vec{a}+y_3\vec{b}+z_3\vec{c}$ then $[\vec{p},\vec{q},\vec{r}] = \begin{vmatrix} x_1 & y_1 & z_1 \\ x_2 & y_2 & z_2 \\ x_3 & y_3 & z_3 \end{vmatrix} [\vec{a},\vec{b},\vec{c}]$
- For a given set of three vectors $\vec{a}, \vec{b}, \vec{c}$ the vector $\vec{a} \times (\vec{b} \times \vec{c})$ is called vector triple product.
- For any three vectors $\vec{a}, \vec{b}, \vec{c}$ we have $\vec{a} \times (\vec{b} \times \vec{c}) = (\vec{a}.\vec{c})\vec{b} - (\vec{a}.\vec{b})\vec{c}$.
- Parametric form of the vector equation of a straight line that passes through a given point with position vector $\vec{a}$ and parallel to a given vector $\vec{b}$ is $\vec{r}=\vec{a}+t\vec{b}$, where t ∈ R.
- Cartesian equations of a straight line that passes through the point (x_1, y_1, z_1) and parallel to a vector with direction ratios b_1, b_2, b_3 are $\frac{x-x_1}{b_1}=\frac{y-y_1}{b_2}=\frac{z-z_1}{b_3}$.
- Any point on the line $\frac{x-x_1}{b_1}=\frac{y-y_1}{b_2}=\frac{z-z_1}{b_3}$ is of the form $(x_1+tb_1, y_1+tb_2, z_1+tb_3)$, t ∈ R.
- Parametric form of vector equation of a straight line that passes through two given points with position vectors $\vec{a}$ and $\vec{b}$ is $\vec{r}=\vec{a}+t(\vec{b}-\vec{a})$, t ∈ R.
- Cartesian equations of a line that passes through two given points (x_1, y_1, z_1) and (x_2, y_2, z_2) are $\frac{x-x_1}{x_2-x_1}=\frac{y-y_1}{y_2-y_1}=\frac{z-z_1}{z_2-z_1}$.
- If θ is the acute angle between two straight lines $\vec{r}=\vec{a}+s\vec{b}$ and $\vec{r}=\vec{c}+t\vec{d}$, then $\theta = \cos^{-1}\left(\frac{|\vec{b}.\vec{d}|}{|\vec{b}||\vec{d}|}\right)$
- Two lines are said to be coplanar if they lie in the same plane.
- Two lines in space are called skew lines if they are not parallel and do not intersect
- The shortest distance between the two skew lines is the length of the line segment perpendicular to both the skew lines.
- The shortest distance between the two skew lines $\vec{r}=\vec{a}+s\vec{b}$ and $\vec{r}=\vec{c}+t\vec{d}$ is

$\delta = \dfrac{|(\vec{c}-\vec{a}).(\vec{b}\times\vec{d})|}{|\vec{b}\times\vec{d}|}$, where $|\vec{b}\times\vec{d}| \neq 0$.

- Two straight lines $\vec{r} = \vec{a} + s\vec{b}$ and $\vec{r} = \vec{c} + t\vec{d}$ intersect each other if $(\vec{c}-\vec{a}).(\vec{b}\times\vec{d}) = 0$
- The shortest distance between the two parallel lines $\vec{r} = \vec{a} + s\vec{b}$ and $\vec{r} = \vec{c} + t\vec{b}$ is $d = \dfrac{|(\vec{c}-\vec{a})\times\vec{b}|}{|\vec{b}|}$, where $|\vec{b}| \neq 0$
- If two lines $\dfrac{x-x_1}{b_1} = \dfrac{y-y_1}{b_2} = \dfrac{z-z_1}{b_3}$ and $\dfrac{x-x_2}{d_1} = \dfrac{y-y_2}{d_2} = \dfrac{z-z_2}{d_3}$ intersect, then
$$\begin{vmatrix} x_2-x_1 & y_2-y_1 & z_2-z_1 \\ b_1 & b_2 & b_3 \\ d_1 & d_2 & d_3 \end{vmatrix} = 0$$
- A straight line which is perpendicular to a plane is called a normal to the plane.
- The equation of the plane at a distance p from the origin and perpendicular to the unit normal vector $\hat{d}$ is $\vec{r}.\hat{d} = p$ (normal form)
- Cartesian equation of the plane in normal form is $\ell x + my + nz = p$.
- Vector form of the equation of a plane passing through a point with position vector $\vec{a}$ and perpendicular to $\vec{n}$ is $(\vec{r}-\vec{a}).\vec{n} = 0$
- Cartesian equation of a plane normal to a vector with direction ratios a,b,c and passing through a given point (x_1, y_1, z_1) is $a(x-x_1)+b(y-y_1)+c(z-z_1)=0$
- Intercept form of the equation of the plane $\vec{r}.\vec{n} = q$, having intercepts a, b, c on the x, y, z axes respectively is $\dfrac{x}{a}+\dfrac{y}{b}+\dfrac{z}{c}=1$.
- Parametric form of vector equation of the plane passing through three given non-collinear points is $\vec{r} = \vec{a} + s(\vec{b}-\vec{a}) + t(\vec{c}-\vec{a})$
- Cartesian equation of the plane passing through three non-collinear points is
$$\begin{vmatrix} x-x_1 & y-y_1 & z-z_1 \\ x_2-x_1 & y_2-y_1 & z_2-z_1 \\ x_3-x_1 & y_3-y_1 & z_3-z_1 \end{vmatrix} = 0.$$
- A straight line will lie on a plane if every point on the line, lie in the plane and the normal to the plane is perpendicular to the line.
- The two given non-parallel lines $\vec{r} = \vec{a} + s\vec{b}$ and $\vec{r} = \vec{c} + t\vec{d}$ are coplanar if $(\vec{c}-\vec{a}).(\vec{b}\times\vec{d}) = 0$
- Two lines $\dfrac{x-x_1}{b_1} = \dfrac{y-y_1}{b_2} = \dfrac{z-z_1}{b_3}$ and $\dfrac{x-x_2}{d_1} = \dfrac{y-y_2}{d_2} = \dfrac{z-z_2}{d_3}$ are coplanar if
$$\begin{vmatrix} x_2-x_1 & y_2-y_1 & z_2-z_1 \\ b_1 & b_2 & b_3 \\ d_1 & d_2 & d_3 \end{vmatrix} = 0$$
- Non-parametric form of vector equation of the plane containing the two coplanar lines $\vec{r} = \vec{a} + s\vec{b}$ and $\vec{r} = \vec{c} + t\vec{d}$ is $(\vec{r}-\vec{a}).(\vec{b}\times\vec{d}) = 0$ or $(\vec{r}-\vec{c}).(\vec{b}\times\vec{d}) = 0$.

- The acute angle θ between the two planes $\vec{r}.\vec{n}_1 = p_1$ and $\vec{r}.\vec{n}_2 = p_2$ is $\theta = \cos^{-1}\left(\frac{|\vec{n}_1.\vec{n}_2|}{|\vec{n}_1||\vec{n}_2|}\right)$
- If θ is the acute angle between the line $\vec{r} = \vec{a} + t\vec{b}$ and the plane $\vec{r}.\vec{n} = p$, then $\theta = \sin^{-1}\left(\frac{|\vec{b}.\vec{n}|}{|\vec{b}||\vec{n}|}\right)$
- The perpendicular distance from a point with position vector $\vec{u}$ to the plane $\vec{r}.\vec{n} = p$ is given by $\delta = \frac{|\vec{u}.\vec{n} - p|}{|\vec{n}|}$
- The perpendicular distance from a point (x_1, y_1, z_1) to the plane $ax + by + cz = p$ is $\delta = \frac{|ax_1 + by_1 + cz_1 - p|}{\sqrt{a^2 + b^2 + c^2}}$.
- The perpendicular distance from the origin to the plane $ax + by + cz + d = 0$ is given by $\delta = \frac{|d|}{\sqrt{a^2 + b^2 + c^2}}$
- The distance between two parallel planes $ax+by+cz+d_1 = 0$ and $ax+by+cz+d_2 = 0$ is given by $\frac{|d_1 - d_2|}{\sqrt{a^2 + b^2 + c^2}}$
- The vector equation of a plane which passes through the line of intersection of the planes $\vec{r}.\vec{n}_1 = d_1$ and $\vec{r}.\vec{n}_2 = d_2$ is given by $(\vec{r}.\vec{n}_1 - d_1) + \lambda(\vec{r}.\vec{n}_2 - d_2) = 0$, Where $\lambda \in R$.
- The equation of a plane passing through the line of intersection of the planes $a_1x + b_1y + c_1z = d_1$ and $a_2x + b_2y + c_2z = d_2$ is given by $(a_1x + b_1y + c_1z - d_1) + \lambda(a_2x + b_2y + c_2z - d_2) = 0$.
- The position vector of the point of intersection of the line $\vec{r} = \vec{a} + t\vec{b}$ and the plane $\vec{r}.\vec{n} = p$ is $\vec{u} = \vec{a} + \left(\frac{p - (\vec{a}.\vec{n})}{\vec{b}.\vec{n}}\right)\vec{b}$, where $\vec{b}.\vec{n} \neq \vec{0}$
- If $\vec{v}$ is the position vector of the image of $\vec{u}$ in the plane $\vec{r}.\vec{n} = p$, then $\vec{v} = \vec{u} + \frac{2[p - (\vec{u}.\vec{n})]}{|\vec{n}|^2}\vec{n}$

BOOK BACK ONE MARKS

1. If $\vec{a}$ and $\vec{b}$ are parallel vectors, then $[\vec{a}, \vec{c}, \vec{b}]$ is equal to

(1) 2 (2) –1

(3) 1 **(4) 0**

Solution :

$\vec{a} \parallel \vec{b} \Rightarrow \vec{a} = \lambda \vec{b}$

$[\vec{a}\,\vec{b}\,\vec{c}] = [\lambda\vec{b}\;\vec{b}\;\vec{c}]$

$= \lambda[\vec{b}\;\vec{b}\;\vec{c}]$

$= 0$

2. If a vector $\vec{\alpha}$ lies in the plane of $\vec{\beta}$ and $\vec{\gamma}$, then

(1) $[\vec{\alpha}, \vec{\beta}, \vec{\gamma}] = 1$ (2) $[\vec{\alpha}, \vec{\beta}, \vec{\gamma}] = -1$

(3) $[\vec{\alpha}, \vec{\beta}, \vec{\gamma}] = 0$ (4) $[\vec{\alpha}, \vec{\beta}, \vec{\gamma}] = 2$

Solution :

Since $\vec{\alpha}$ lies in the plane of $\vec{\beta}$ and $\vec{\gamma}$

$\vec{\alpha} \cdot [\vec{\beta} \times \vec{\gamma}] = 0$

$[\vec{\alpha}\,\vec{\beta}\,\vec{\gamma}] = 0$

3. If $\vec{a}.\vec{b} = \vec{b}.\vec{c} = \vec{c}.\vec{a} = 0$ then the value of $[\vec{a}, \vec{b}, \vec{c}]$ is

(1) $|\vec{a}|\,|\vec{b}|\,|\vec{c}|$ (2) $\frac{1}{3}|\vec{a}|\,|\vec{b}|\,|\vec{c}|$

(3) 1 (4) –1

Solution :

$\vec{a}.\vec{b} = 0 \Rightarrow \vec{a} \perp \vec{b}$

$\vec{b}.\vec{c} = 0 \Rightarrow \vec{b} \perp \vec{c}$

$\vec{a}.\vec{c} = 0 \Rightarrow \vec{a} \perp \vec{c}$

$\vec{a}, \vec{b}, \vec{c}$ are mutually $\perp$ vectors

$[\vec{a}, \vec{b}, \vec{c}] = |\vec{a}|\,|\vec{b}|\,|\vec{c}|$

4. If $\vec{a}, \vec{b}, \vec{c}$ are three unit vectors such that $\vec{a}$ is perpendicular to $\vec{b}$, and is parallel to $\vec{c}$ then $\vec{a} \times (\vec{b} \times \vec{c})$ is equal to

(1) $\vec{a}$ **(2) $\vec{b}$**

(3) $\vec{c}$ (4) $\vec{0}$

Solution :

$\vec{a} \times (\vec{b} \times \vec{c}) = (\vec{a}.\vec{c})\vec{b} - (\vec{a}.\vec{b})\vec{c}$

$= (\vec{a}.\vec{c})\vec{b} - 0$

$= \vec{b}$

Hint : $\vec{a} \perp \vec{b} \Rightarrow \vec{a}\vec{b} = 0$

$\vec{a} \parallel \vec{c}$

$\vec{a}.\vec{c} = |\vec{a}|\,|\vec{c}| \cos\theta$

$\vec{a}.\vec{c} = 1$

5. If $[\vec{a}, \vec{b}, \vec{c}] = 1$ then the value of $\frac{\vec{a}.(\vec{b}\times\vec{c})}{(\vec{c}\times\vec{a}).\vec{b}} + \frac{\vec{b}.(\vec{c}\times\vec{a})}{(\vec{a}\times\vec{b}).\vec{c}} + \frac{\vec{c}.(\vec{a}\times\vec{b})}{(\vec{c}\times\vec{b}).\vec{a}}$ is

(1) 1 (2) –1

(3) 2 (4) 3

Solution :

$\frac{\vec{a}.(\vec{b}\times\vec{c})}{(\vec{c}\times\vec{a}).\vec{b}} + \frac{\vec{b}.(\vec{c}\times\vec{a})}{(\vec{a}\times\vec{b}).\vec{c}} + \frac{\vec{c}.(\vec{a}\times\vec{b})}{(\vec{c}\times\vec{b}).\vec{a}}$

$= \frac{[\vec{a}\vec{b}\vec{c}]}{[\vec{a}\vec{b}\vec{c}]} + \frac{[\vec{a}\vec{b}\vec{c}]}{[\vec{a}\vec{b}\vec{c}]} + \frac{[\vec{a}\vec{b}\vec{c}]}{-[\vec{a}\vec{b}\vec{c}]}$

$= 1 + 1 - 1$

$= 1$

Book Answer is wrong

6. The volume of the parallelepiped with its edges represented by the vectors $\hat{i}+\hat{j}, \hat{i}+2\hat{j}, \hat{i}+\hat{j}+\pi\hat{k}$ is

(1) $\frac{\pi}{2}$ (2) $\frac{\pi}{3}$

(3) π (4) $\frac{\pi}{4}$

Solution :

$$\text{volume} = \begin{vmatrix} 1 & 1 & 0 \\ 1 & 2 & 0 \\ 1 & 1 & \pi \end{vmatrix}$$

$$= 1(2\pi - 0) - 1(\pi - 0) + 0(1-2)$$

$$= 2\pi - \pi + 0 = \pi$$

7. If $\vec{a}$ and $\vec{b}$ are unit vectors such that $[\vec{a}, \vec{b}, \vec{a} \times \vec{b}] = \frac{\pi}{4}$, then the angle between $\vec{a}$ and $\vec{b}$ is

(1) $\frac{\pi}{6}$ (2) $\frac{\pi}{4}$

(3) $\frac{\pi}{3}$ (4) $\frac{\pi}{2}$

Solution :

$$[\vec{a}, \vec{b}, \vec{a} \times \vec{b}] = \frac{\pi}{4}$$

$$\vec{a}.[\vec{b} \times (\vec{a} \times \vec{b})] = \frac{\pi}{4}$$

$$\vec{a}.[(\vec{b}.\vec{b})\vec{a} - (\vec{b}.\vec{a})\vec{b}] = \frac{\pi}{4}$$

$$\vec{a}.(\vec{a} - \vec{b}(\vec{a}.\vec{b})) = \frac{\pi}{4}$$

$$\vec{a}.\vec{a} - (\vec{a}.\vec{b})(\vec{a}.\vec{b}) = \frac{\pi}{4}$$

$$1 - (|\vec{a}||\vec{b}|\cos\theta)^2 = \frac{\pi}{4}$$

$$1 - (1.1\cos\theta)^2 = \frac{\pi}{4}$$

$$1 - \cos^2\theta = \frac{\pi}{4}$$

$$\sin^2\theta = \frac{\pi}{4}$$

$$\sin^2\theta = \frac{1}{4} \text{ (Replacing } \frac{\pi}{4} \text{ by } \frac{1}{4})$$

$$\sin\theta = \frac{1}{2}$$

$$\theta = \frac{\pi}{6}$$

8. If $\vec{a} = \hat{i} + \hat{j} + \hat{k}$, $\vec{b} = \hat{i} + \hat{j}$, $\vec{c} = \hat{i}$ and $(\vec{a} \times \vec{b}) \times \vec{c} = \lambda\vec{a} + \mu\vec{b}$, then the value of $\lambda + \mu$ is

(1) 0 (2) 1

(3) 6 (4) 3

Solution :

$$(\vec{a} \times \vec{b}) \times \vec{c} = \lambda\vec{a} + \mu\vec{b}$$

$$(\vec{a}.\vec{c})\vec{b} - (\vec{b}.\vec{c})\vec{a} = \lambda\vec{a} + \mu\vec{b}$$

$$\vec{b} - \vec{a} = \lambda\vec{a} + \mu\vec{b}$$

$$-\vec{a} + \vec{b} = \lambda\vec{a} + \mu\vec{b}$$

$$\lambda = -1 \quad \mu = 1$$

$$\lambda + \mu = -1 + 1 = 0$$

9. If $\vec{a}, \vec{b}, \vec{c}$ are non-coplanar, non-zero vectors such that $[\vec{a}, \vec{b}, \vec{c}] = 3$, then $\{[\vec{a} \times \vec{b}, \vec{b} \times \vec{c}, \vec{c} \times \vec{a}]\}^2$ is equal to

(1) 81 (2) 9

(3) 27 (4) 18

Solution :

$$\{[\vec{a} \times \vec{b}\ \vec{b} \times \vec{c}\ \vec{c} \times \vec{a}]\}^2 = [[\vec{a}\ \vec{b}\ \vec{c}]^2]^2$$

$$= (3^2)^2$$

$$= (9)^2$$

$$= 81$$

10. If $\vec{a}, \vec{b}, \vec{c}$ are three non-coplanar vectors such that $\vec{a} \times (\vec{b} \times \vec{c}) = \frac{\vec{b}+\vec{c}}{\sqrt{2}}$, then the angle between $\vec{a}$ and $\vec{b}$ is

(1) $\frac{\pi}{2}$ (2) $\frac{3\pi}{4}$

(3) $\frac{\pi}{4}$ (4) π

Solution :

$$\vec{a} \times (\vec{b} \times \vec{c}) = \frac{\vec{b}+\vec{c}}{\sqrt{2}}$$

$$(\vec{a}.\vec{c})\vec{b} - (\vec{a}.\vec{b})\vec{c} = \frac{1}{\sqrt{2}}(\vec{b}) + \frac{1}{\sqrt{2}}(\vec{c})$$

$$-(\vec{a}.\vec{b}) = \frac{1}{\sqrt{2}}$$

$$\vec{a}.\vec{b} = -\frac{1}{\sqrt{2}}$$

$$\cos\theta = \cos\left(\pi - \frac{\pi}{4}\right)$$

$$\theta = \frac{3\pi}{4} \ (\theta \text{ is obtuse})$$

11. If the volume of the parallelopiped with $\vec{a} \times \vec{b}, \vec{b} \times \vec{c}, \vec{c} \times \vec{a}$ as coterminous edges is 8 cubic units, then the volume of the parallelopiped with $(\vec{a} \times \vec{b}) \times (\vec{b} \times \vec{c})$, $(\vec{b} \times \vec{c}) \times (\vec{c} \times \vec{a})$ and $(\vec{c} \times \vec{a}) \times (\vec{a} \times \vec{b})$ as coterminous edges is,

(1) 8 cubic units (2) 512 cubic units

(3) 64 cubic units (4) 24 cubic units

Solution :

$$[(\vec{a} \times \vec{b}) \times (\vec{b} \times \vec{c}), (\vec{b} \times \vec{c}) \times (\vec{c} \times \vec{a}), (\vec{c} \times \vec{a}) \times (\vec{a} \times \vec{b})]$$

$$= [\vec{a} \times \vec{b} \ \ \vec{b} \times \vec{c} \ \ \vec{c} \times \vec{a}]^2$$

$$= (8)^2$$

$$= 64$$

12. Consider the vectors $\vec{a}, \vec{b}, \vec{c}, \vec{d}$ such that $(\vec{a} \times \vec{b}) \times (\vec{c} \times \vec{d}) = \vec{0}$. Let P_1 and P_2 be the planes determined by the pairs of vectors $\vec{a}, \vec{b}$ and $\vec{c}, \vec{d}$ respectively. Then the angle between P_1 and P_2 is

(1) 0° (2) 45°

(3) 60° (4) 90°

Solution :

$$(\vec{a} \times \vec{b}) \times (\vec{c} \times \vec{d}) = \vec{0}$$

$$(\vec{a} \times \vec{b}) \parallel (\vec{c} \times \vec{d})$$

$$\theta = 0°$$

13. If $\vec{a} \times (\vec{b} \times \vec{c}) = (\vec{a} \times \vec{b}) \times \vec{c}$, where $\vec{a}, \vec{b}, \vec{c}$ are any three vectors such that $\vec{b}.\vec{c} \neq 0$ and $\vec{a}.\vec{b} \neq 0$, then $\vec{a}$ and $\vec{c}$ are

(1) perpendicular

(2) parallel

(3) inclined at an angle $\frac{\pi}{3}$

(4) inclined at an angle $\frac{\pi}{6}$

Solution :

$$\vec{a} \times (\vec{b} \times \vec{c}) = (\vec{a} \times \vec{b}) \times \vec{c}$$

$$(\vec{a}.\vec{c})\vec{b} - (\vec{a}.\vec{b})\vec{c} = (\vec{a}.\vec{c})\vec{b} - (\vec{b}.\vec{c})\vec{a}$$

$$(\vec{a}.\vec{b})\vec{c} = (\vec{b}.\vec{c})\vec{a}$$

$$\vec{c} = \left(\frac{\vec{b}.\vec{c}}{\vec{a}.\vec{b}}\right)\vec{a}$$

$$\vec{c} = \lambda \vec{a}$$

$$\vec{c} \parallel \vec{a}$$

14. If $\vec{a}=2\hat{i}+3\hat{j}-\hat{k}$, $\vec{b}=\hat{i}+2\hat{j}-5\hat{k}$, $\vec{c}=3\hat{i}+5\hat{j}-\hat{k}$, then a vector perpendicular to $\vec{a}$ and lies in the plane containing $\vec{b}$ and $\vec{c}$ is

(1) $-17\hat{i}+21\hat{j}-97\hat{k}$

(2) $17\hat{i}+21\hat{j}-123\hat{k}$

(3) $-17\hat{i}-21\hat{j}+97\hat{k}$

(4) $-17\hat{i}-21\hat{j}-97\hat{k}$

Solution :

$$\vec{b}\times\vec{c}=\begin{vmatrix}\hat{i} & \hat{j} & \hat{k}\\ 1 & 2 & -5\\ 3 & 5 & -1\end{vmatrix}$$

$$=\hat{i}(-2+25)-\hat{j}(-1+15)+\hat{k}(5-6)$$

$$\vec{b}\times\vec{c}=23\hat{i}-14\hat{j}-\hat{k}$$

$$\vec{a}\times(\vec{b}\times\vec{c})=\begin{vmatrix}\hat{i} & \hat{j} & \hat{k}\\ 2 & 3 & -1\\ 23 & -14 & -1\end{vmatrix}$$

$$=\hat{i}(-3-14)-\hat{j}(-2+23)+\hat{k}(-28-69)$$

$$=-17\hat{i}-21\hat{j}-97\hat{k}$$

15. The angle between the lines $\frac{x-2}{3}=\frac{y+1}{-2}, z=2$ and $\frac{x-1}{1}=\frac{2y+3}{3}=\frac{z+5}{2}$ is

(1) $\frac{\pi}{6}$ (2) $\frac{\pi}{4}$

(3) $\frac{\pi}{3}$ (4) $\frac{\pi}{2}$

Solution :

$$\vec{b}=3\hat{i}-2\hat{j}$$

$$\vec{c}=\hat{i}+\frac{3}{2}\hat{j}+2\hat{k}$$

$$\vec{b}.\vec{c}=3-2\left(\frac{3}{2}\right)+0$$

$$\vec{b}.\vec{c}=0 \Rightarrow \vec{b}\perp\vec{c}$$

$$\theta=90^\circ \text{ (or) } \frac{\pi}{2}$$

16. If the line $\frac{x-2}{3}=\frac{y-1}{-5}=\frac{z+2}{2}$ lies in the plane $x+3y-\alpha z+\beta=0$, then (α,β) is

(1) $(-5,5)$ (2) $(-6,7)$

(3) $(5,-5)$ (4) $(6,-7)$

Solution :

$$\vec{b}=3\hat{i}-5\hat{j}+2\hat{k}$$

$$\vec{n}=\hat{i}+3\hat{j}-\alpha\hat{k}$$

$$\hat{b}.\hat{n}=0$$

$$3-15-2\alpha=0$$

$$-2\alpha=12$$

$$\alpha=-6$$

Also $(2,1,-2)$ lies on plane $x+3y-\alpha z+\beta=0$

$$2+3-12+\beta=0$$

$$-7+\beta=0$$

$$\beta=7$$

$$(\alpha,\beta)=(-6,7)$$

17. The angle between the line $\vec{r}=(\hat{i}+2\hat{j}-3\hat{k})+t(2\hat{i}+\hat{j}-2\hat{k})$ and the plane $\vec{r}.(\hat{i}+\hat{j})+4=0$ is

(1) 0° (2) 30°

(3) 45° (4) 90°

Solution :

$$\vec{b}=2\hat{i}+\hat{j}-2\hat{k}$$

$$\vec{n}=\hat{i}+\hat{j}$$

$$|\vec{b}|=\sqrt{4+1+4}=\sqrt{9}=3$$

$$|\vec{n}|=\sqrt{1+1}$$

$$|\vec{n}|=\sqrt{2}$$

$$\vec{b}.\vec{n}=2+1=3$$

$\sin\theta = \left(\frac{|\vec{b}.\vec{n}|}{|\vec{b}||\vec{n}|}\right) = \frac{3}{3\sqrt{2}}$

$\sin\theta = \frac{1}{\sqrt{2}}$

$\theta = \sin^{-1}\left(\frac{1}{\sqrt{2}}\right)$

$\theta = 45°$

18. The coordinates of the point where the line $\vec{r} = (6\hat{i} - \hat{j} - 3\hat{k}) + t(-\hat{i} + 4\hat{k})$ meets the plane $\vec{r}.(\hat{i} + \hat{j} - \hat{k}) = 3$ are

(1) (2,1,0) (2) (7,–1,–7)

(3) (1,2,–6) **(4) (5,–1,1)**

Solution :

$\frac{x-6}{-1} = \frac{y+1}{0} = \frac{z+3}{4} = \lambda$

Point of this form $(-\lambda + 6, -1, 4\lambda - 3)$

This lies on the plane $x + y - z - 3 = 0$

$-\lambda + 6 - 1 - 4\lambda + 3 - 3 = 0$

$-5\lambda = -5$

$\lambda = 1$

Point of intersection (5, –1, 1)

19. Distance from the origin to the plane $3x - 6y + 2z + 7 = 0$ is

(1) 0 **(2) 1**

(3) 2 (4) 3

Solution :

$\text{Distance} = \frac{7}{\sqrt{(3)^2 + (-6)^2 + (2)^2}}$

$= \frac{7}{\sqrt{9+36+4}} = \frac{7}{7} = 1$

20. The distance between the planes $x + 2y + 3z + 7 = 0$ and $2x + 4y + 6z + 7 = 0$ is

(1) $\frac{\sqrt{7}}{2\sqrt{2}}$ (2) $\frac{7}{2}$

(3) $\frac{\sqrt{7}}{2}$ (4) $\frac{7}{2\sqrt{2}}$

Solution :

The planes are

$x + 2y + 3z + 7 = 0$

$x + 2y + 3z + \frac{7}{2} = 0$

$\text{Distance} = \frac{\left|7 - \frac{7}{2}\right|}{\sqrt{(1)^2 + (2)^2 + (3)^2}} = \frac{\frac{7}{2}}{\sqrt{1+4+9}}$

$= \frac{7}{2\sqrt{14}} = \frac{\sqrt{7}\sqrt{7}}{2\sqrt{2}\sqrt{7}}$

$= \frac{\sqrt{7}}{2\sqrt{2}}$

21. If the direction cosines of a line are $\frac{1}{c}, \frac{1}{c}, \frac{1}{c}$ then

(1) $c = \pm 3$ **(2) $c = \pm\sqrt{3}$**

(3) $c > 0$ (4) $0 < c < 1$

Solution :

DC'S of a line $\frac{1}{c}, \frac{1}{c}, \frac{1}{c}$ then

$(\cos\alpha, \cos\beta, \cos\gamma) = \left(\frac{1}{c}, \frac{1}{c}, \frac{1}{c}\right)$

$\cos^2\alpha + \cos^2\beta + \cos^2\gamma = 1$

$\frac{1}{c^2} + \frac{1}{c^2} + \frac{1}{c^2} = 1$

$\frac{3}{c^2} = 1$

$c^2 = 3$

$c = \pm\sqrt{3}$

22. The vector equation $\vec{r} = (\hat{i} - 2\hat{j} - \hat{k}) + t(6\hat{j} - \hat{k})$ represents a straight line passing through the points

(1) (0, 6, –1) and (1, –2, –1)

(2) (0, 6, –1) and (–1, –4, –2)

(3) (1, –2, –1) and (1, 4, –2)

(4) (1, –2, –1) and (0, –6, 1)

Solution :

$\vec{a} = \hat{i} - 2\hat{j} - \hat{k}$

Cartesian form

$\frac{x-1}{0} = \frac{y+2}{6} = \frac{z+1}{-1} = \lambda$

Any point of this form $(1, 6\lambda - 2, -\lambda - 1)$

$\lambda = 0$ then (1, –2, –1)

$\lambda = 1$ then (1, 4, –2)

23. If the distance of the point (1,1,1) from the origin is half of its distance from the plane $x + y + z + k = 0$, then the values of k are

(1) ±3 (2) ±6

(3) –3, 9 (4) 3, –9

Solution :

Distance from (1, 1, 1) to (0, 0, 0)

$= \frac{1}{2}\left|\frac{x_1 + y_1 + z_1 + k}{\sqrt{(1)^2 + (1)^2 + (1)^2}}\right|$

$\sqrt{(1)^2 + (1)^2 + (1)^2} = \frac{1}{2}\left|\frac{3+k}{\sqrt{3}}\right|$

$2\sqrt{3}\sqrt{3} = |3+k|$

$|3+k| = 6$

$3 + k = \pm 6$

$k = -3 + 6$ $k = -3 - 6$

$k = 3$ $k = -9$

24. If the planes $\vec{r}.(2\hat{i} - \lambda\hat{j} + \hat{k}) = 3$ and $\vec{r}.(4\hat{i} + \hat{j} - \mu\hat{k}) = 5$ are parallel, then the value of λ and μ are

(1) $\frac{1}{2}, -2$ (2) $\frac{-1}{2}, 2$

(3) $-\frac{1}{2}, -2$ (4) $\frac{1}{2}, 2$

Solution :

S $\vec{n_1} = \vec{n_2}$

S $(2\hat{i} - \lambda\hat{j} + \hat{k}) = 4\hat{i} + \hat{j} - \mu\hat{k}$

S $= 2$

$4\hat{i} - 2\lambda\hat{j} + 2\hat{k} = 4\hat{i} + \hat{j} - \mu\hat{k}$

$-2\lambda = 1$ $-\mu = 2$

$\lambda = -\frac{1}{2}$ $\mu = -2$

25. If the length of the perpendicular from the origin to the plane $2x + 3y + \lambda z = 1, \lambda > 0$ is $\frac{1}{5}$, then the value of λ is

(1) $2\sqrt{3}$ (2) $3\sqrt{2}$

(3) 0 (4) 1

Solution :

Distance $= \frac{1}{5}$

$\left|\frac{2x_1 + 3y_1 + \lambda z_1 - 1}{\sqrt{2^2 + 3^2 + \lambda^2}}\right| = \frac{1}{5}$

$\left|\frac{-1}{\sqrt{4 + 9 + \lambda^2}}\right| = \frac{1}{5}$

$5^2 = 13 + \lambda^2$

$\lambda^2 = 25 - 13$

$\lambda^2 = 12$

$\lambda = \pm\sqrt{12}$

$\lambda = \pm 2\sqrt{3}$

BOOK SUMS ***(Exercise and Examples) :***

1. With usual notations, in any triangle ABC, prove the following by vector method.
   (i) $a^2 = b^2 + c^2 - 2bc \cos A$ (ii) $b^2 = c^2 + a^2 - 2ca \cos B$ (iii) $c^2 = a^2 + b^2 - 2ab \cos C$
2. With usual notations, in any triangle ABC, prove the following by vector method.
   (i) $a = b\cos C + c\cos B$ (ii) $b = c\cos A + a\cos C$ (iii) $c = a\cos B + b\cos A$
3. By vector method, prove that $\cos(\alpha + \beta) = \cos\alpha \cos\beta - \sin\alpha \sin\beta$.
4. With usual notations, in any triangle ABC, prove by vector method that $\frac{a}{\sin A} = \frac{b}{\sin B} = \frac{c}{\sin C}$
5. Prove by vector method that $\sin(\alpha - \beta) = \sin\alpha \cos\beta - \cos\alpha \sin\beta$
6. If D is the midpoint of the side BC of a triangle ABC, show by vector method that
   $|\overline{AB}|^2 + |\overline{AC}|^2 = 2\left(|\overline{AD}|^2 + |\overline{BD}|^2\right)$
7. Prove by vector method that the perpendiculars (altitudes) from the vertices to the opposite sides of a triangle are concurrent.
8. In triangle ABC, the points D, E, F are the midpoints of the sides BC, CA and AB respectively. Using vector method, show that the area of ΔDEF is equal to $\frac{1}{4}$ (area of ΔABC).
9. A particle acted upon by constant forces $2\hat{i} + 5\hat{j} + 6\hat{k}$ and $-\hat{i} - 2\hat{j} - \hat{k}$ is displaced from the point $(4, -3, -2)$ to the point $(6, 1, -3)$. Find the total work done by the forces.
10. A particle is acted upon by the forces $3\hat{i} - 2\hat{j} + 2\hat{k}$ and $2\hat{i} + \hat{j} - \hat{k}$ is displaced from the point $(1, 3, -1)$ to the point $(4, -1, \lambda)$. If the work done by the forces is 16 units, find the value of λ.
11. Find the magnitude and the direction cosines of the torque about the point $(2, 0, -1)$ of a force $2\hat{i} + \hat{j} - \hat{k}$, whose line of action passes through the origin.
12. Prove by vector method that if a line is drawn from the centre of a circle to the midpoint of a chord, then the line is perpendicular to the chord.
13. Prove by vector method that the median to the base of an isosceles triangle is perpendicular to the base.
14. Prove by vector method that an angle in a semi-circle is a right angle.
15. Prove by vector method that the diagonals of a rhombus bisect each other at right angles.
16. Using vector method, prove that if the diagonals of a parallelogram are equal, then it is a rectangle.
17. Prove by vector method that the area of the quadrilateral $ABCD$ having diagonals AC and BD is $\frac{1}{2}|\overline{AC} \times \overline{BD}|$
18. Prove by vector method that the parallelograms on the same base and between the same parallels are equal in area.
19. If G is the centroid of a ΔABC, prove that
    (area of ΔGAB) = (area of ΔGBC) = (area of ΔGCA) = $\frac{1}{3}$(area of ΔABC).
20. Using vector method, prove that $\cos(\alpha - \beta) = \cos\alpha\cos\beta + \sin\alpha\sin\beta$.
21. Prove by vector method that $\sin(\alpha + \beta) = \sin\alpha\cos\beta + \cos\alpha\sin\beta$.
22. A particle acted on by constant forces $8\hat{i} + 2\hat{j} - 6\hat{k}$ and $6\hat{i} + 2\hat{j} - 2\hat{k}$ is displaced from the point $(1, 2, 3)$ to the point $(5, 4, 1)$. Find the total work done by the forces.

23. Forces of magnitudes $5\sqrt{2}$ and $10\sqrt{2}$ units acting in the directions $3\hat{i}+4\hat{j}+5\hat{k}$ and $10\hat{i}+6\hat{j}-8\hat{k}$ respectively, act on a particle which is displaced from the point with position vector $4\hat{i}-3\hat{j}-2\hat{k}$ to the point with position vector $6\hat{i}+\hat{j}-3\hat{k}$. Find the work done by the forces.

24. Find the magnitude and direction cosines of the torque of a force represented by $3\hat{i}+4\hat{j}-5\hat{k}$ about the point with position vector $2\hat{i}-3\hat{j}+4\hat{k}$ acting through a point whose position vector is $4\hat{i}+2\hat{j}-3\hat{k}$.

25. Find the torque of the resultant of the three forces represented by $-3\hat{i}+6\hat{j}-3\hat{k}$, $4\hat{i}-10\hat{j}+12\hat{k}$ and $4\hat{i}+7\hat{j}$ acting at the point with position vector $8\hat{i}-6\hat{j}-4\hat{k}$, about the point with position vector $18\hat{i}+3\hat{j}-9\hat{k}$.

26. If $\vec{a}=-3\hat{i}-\hat{j}+5\hat{k}$, $\vec{b}=\hat{i}-2\hat{j}+\hat{k}$, $\vec{c}=4\hat{j}-5\hat{k}$, find $\vec{a}.(\vec{b}\times\vec{c})$

27. Find the volume of the parallelepiped whose coterminus edges are given by the vectors $2\hat{i}-3\hat{j}+4\hat{k}$, $\hat{i}+2\hat{j}-\hat{k}$ and $3\hat{i}-\hat{j}+2\hat{k}$

28. Show that the vectors $\hat{i}+2\hat{j}-3\hat{k}$, $2\hat{i}-\hat{j}+2\hat{k}$ and $3\hat{i}+\hat{j}-\hat{k}$ are coplanar.

29. If $2\hat{i}-\hat{j}+3\hat{k}$, $3\hat{i}+2\hat{j}+\hat{k}$, $\hat{i}+m\hat{j}+4\hat{k}$ are coplanar, find the value of m.

30. Show that the four points (6,–7,0), (16,–19,–4), (0,3,–6), (2,–5,10) lie on a same plane.

31. If the vectors $\vec{a},\vec{b},\vec{c}$ are coplanar, then prove that the vectors $\vec{a}+\vec{b},\vec{b}+\vec{c},\vec{c}+\vec{a}$ are also coplanar.

32. If $\vec{a},\vec{b},\vec{c}$ are three vectors, prove that $[\vec{a}+\vec{c},\vec{a}+\vec{b},\vec{a}+\vec{b}+\vec{c}]=-[\vec{a},\vec{b},\vec{c}]$

33. If $\vec{a}=\hat{i}-2\hat{j}+3\hat{k}$, $\vec{b}=2\hat{i}+\hat{j}-2\hat{k}$, $\vec{c}=3\hat{i}+2\hat{j}+\hat{k}$ find $\vec{a}.(\vec{b}\times\vec{c})$

34. Find the volume of the parallelepiped whose coterminous edges are represented by the vectors $-6\hat{i}+14\hat{j}+10\hat{k}$, $14\hat{i}-10\hat{j}-6\hat{k}$ and $2\hat{i}+4\hat{j}-2\hat{k}$

35. The volume of the parallelepiped whose coterminus edges are $7\hat{i}+\lambda\hat{j}-3\hat{k}$, $\hat{i}+2\hat{j}-\hat{k}$, $-3\hat{i}+7\hat{j}+5\hat{k}$ is 90 cubic units. Find the value of λ.

36. If $\vec{a},\vec{b},\vec{c}$ are three non-coplanar vectors represented by concurrent edges of a parallelepiped of volume 4 cubic units, find the value of $(\vec{a}+\vec{b}).(\vec{b}\times\vec{c})+(\vec{b}+\vec{c}).(\vec{c}\times\vec{a})+(\vec{c}+\vec{a}).(\vec{a}\times\vec{b})$.

37. Find the altitude of a parallelepiped determined by the vectors $\vec{a}=-2\hat{i}+5\hat{j}+3\hat{k}$, $\vec{b}=\hat{i}+3\hat{j}-2\hat{k}$ and $\vec{c}=-3\hat{i}+\hat{j}+4\hat{k}$ if the base is taken as the parallelogram determined by $\vec{b}$ and $\vec{c}$.

38. Determine whether the three vectors $2\hat{i}+3\hat{j}+\hat{k}$, $\hat{i}-2\hat{j}+2\hat{k}$ and $3\hat{i}+\hat{j}+3\hat{k}$ are coplanar.

39. Let $\vec{a}=\hat{i}+\hat{j}+\hat{k}$, $\vec{b}=\hat{i}$ and $\vec{c}=c_1\hat{i}+c_2\hat{j}+c_3\hat{k}$. If $c_1=1$ and $c_2=2$, find c_3 such that $\vec{a}$, $\vec{b}$ and $\vec{c}$ are coplanar.

40. If $\vec{a}=\hat{i}-\hat{k}$, $\vec{b}=x\hat{i}+\hat{j}+(1-x)\hat{k}$, $\vec{c}=y\hat{i}+x\hat{j}+(1+x-y)\hat{k}$, show that $[\vec{a},\vec{b},\vec{c}]$ depends on neither x nor y.

41. If the vectors $a\hat{i}+a\hat{j}+c\hat{k}$, $\hat{i}+\hat{k}$ and $c\hat{i}+c\hat{j}+b\hat{k}$ are coplanar, prove that c is the geometric mean of a and b.

42. Let $\vec{a},\vec{b},\vec{c}$ be three non-zero vectors such that $\vec{c}$ is a unit vector perpendicular to both $\vec{a}$ and $\vec{b}$. If the angle between $\vec{a}$ and $\vec{b}$ is $\frac{\pi}{6}$, show that $[\vec{a},\vec{b},\vec{c}]^2=\frac{1}{4}|\vec{a}|^2|\vec{b}|^2$.

43. Prove that $[\vec{a}\times\vec{b}, \vec{b}\times\vec{c}, \vec{c}\times\vec{a}]=[\vec{a},\vec{b},\vec{c}]^2$

44. Prove that $(\vec{a}.(\vec{b}\times\vec{c}))\vec{a} = (\vec{a}\times\vec{b})\times(\vec{a}\times\vec{c})$

45. For any four vectors $\vec{a},\vec{b},\vec{c},\vec{d}$, we have
$(\vec{a}\times\vec{b})\times(\vec{c}\times\vec{d}) = [\vec{a},\vec{b},\vec{d}]\,\vec{c} - [\vec{a},\vec{b},\vec{c}]\,\vec{d} = [\vec{a},\vec{c},\vec{d}]\,\vec{b} - [\vec{b},\vec{c},\vec{d}]\,\vec{a}$

46. If $\vec{a}=-2\hat{i}+3\hat{j}-2\hat{k}$, $\vec{b}=3\hat{i}-\hat{j}+3\hat{k}$, $\vec{c}=2\hat{i}-5\hat{j}+\hat{k}$, find $(\vec{a}\times\vec{b})\times\vec{c}$ and $\vec{a}\times(\vec{b}\times\vec{c})$. State whether they are equal.

47. If $\vec{a}=\hat{i}-\hat{j}$, $\vec{b}=\hat{i}-\hat{j}-4\hat{k}$, $\vec{c}=3\hat{j}-\hat{k}$ and $\vec{d}=2\hat{i}+5\hat{j}+\hat{k}$, verify that
(i) $(\vec{a}\times\vec{b})\times(\vec{c}\times\vec{d})=[\vec{a},\vec{b},\vec{d}]\,\vec{c}-[\vec{a},\vec{b},\vec{c}]\,\vec{d}$
(ii) $(\vec{a}\times\vec{b})\times(\vec{c}\times\vec{d})=[\vec{a},\vec{c},\vec{d}]\,\vec{b}-[\vec{b},\vec{c},\vec{d}]\,\vec{a}$

48. If $\vec{a}=\hat{i}-2\hat{j}+3\hat{k}$, $\vec{b}=2\hat{i}+\hat{j}-2\hat{k}$, $\vec{c}=3\hat{i}+2\hat{j}+\hat{k}$, find (i) $(\vec{a}\times\vec{b})\times\vec{c}$ (ii) $\vec{a}\times(\vec{b}\times\vec{c})$

49. For any vector $\vec{a}$, prove that $\hat{i}\times(\vec{a}\times\hat{i})+\hat{j}\times(\vec{a}\times\hat{j})+\hat{k}\times(\vec{a}\times\hat{k})=2\vec{a}$

50. Prove that $[\vec{a}-\vec{b}, \vec{b}-\vec{c}, \vec{c}-\vec{a}]=0$

51. If $\vec{a}=2\hat{i}+3\hat{j}-\hat{k}$, $\vec{b}=3\hat{i}+5\hat{j}+2\hat{k}$, $\vec{c}=-\hat{i}-2\hat{j}+3\hat{k}$ verify that
(i) $(\vec{a}\times\vec{b})\times\vec{c}=(\vec{a}.\vec{c})\vec{b}-(\vec{b}.\vec{c})\vec{a}$ (ii) $\vec{a}\times(\vec{b}\times\vec{c})=(\vec{a}.\vec{c})\vec{b}-(\vec{a}.\vec{b})\vec{c}$

52. $\vec{a}=2\hat{i}+3\hat{j}-\hat{k}$, $\vec{b}=-\hat{i}+2\hat{j}-4\hat{k}$, $\vec{c}=\hat{i}+\hat{j}+\hat{k}$ then find the value of $(\vec{a}\times\vec{b}).(\vec{a}\times\vec{c})$

53. If $\vec{a},\vec{b},\vec{c},\vec{d}$ are coplanar vectors, show that $(\vec{a}\times\vec{b})\times(\vec{c}\times\vec{d})=\vec{0}$

54. If $\vec{a}=\hat{i}+2\hat{j}+3\hat{k}$, $\vec{b}=2\hat{i}-\hat{j}+\hat{k}$, $\vec{c}=3\hat{i}+2\hat{j}+\hat{k}$ and $\vec{a}\times(\vec{b}\times\vec{c})=\ell\vec{a}+m\vec{b}+n\vec{c}$, find the values of ℓ,m,n.

55. If $\hat{a},\hat{b},\hat{c}$ are three unit vectors such that $\hat{b}$ and $\hat{c}$ are non-parallel and $\hat{a}\times(\hat{b}\times\hat{c})=\frac{1}{2}\hat{b}$, find the angle between $\hat{a}$ and $\vec{c}$.

56. A straight line passes through the point (1, 2, –3) and parallel to $4\hat{i}+5\hat{j}-7\hat{k}$. Find (i) vector equation in parametric form (ii) vector equation in non-parametric form (iii) Cartesian equations of the straight line.

57. The vector equation in parametric form of a line is $\vec{r}=(3\hat{i}-2\hat{j}+6\hat{k})+t(2\hat{i}-\hat{j}+3\hat{k})$. Find (i) the direction cosines of the straight line (ii) vector equation in non-parametric form of the line (iii) Cartesian equations of the line.

58. Find the vector equation in parametric form and Cartesian equations of the line passing through (–4, 2, –3) and is parallel to the line $\frac{-x-2}{4}=\frac{y+3}{-2}=\frac{2z-6}{3}$.

59. Find the vector equation in parametric form and Cartesian equations of a straight line passing through the points (–5, 7, –4) and (13, –5, 2). Find the point where the straight line crosses the xy-plane.

60. Find the angles between the straight line $\frac{x+3}{2}=\frac{y-1}{2}=-z$ with coordinate axes.

61. Find the acute angle between the lines $\vec{r}=(\hat{i}+2\hat{j}+4\hat{k})+t(2\hat{i}+2\hat{j}+\hat{k})$ and the straight line passing through the points (5,1, 4) and (9, 2,12).

62. Find the acute angle between the straight lines $\frac{x+4}{2}=\frac{y}{1}=\frac{z+1}{-2}$ and $\frac{x-1}{4}=\frac{y+1}{-4}=\frac{z-2}{2}$ and state whether they are parallel or perpendicular.
63. Show that the straight line passing through the points A(6,7,5) and B(8,10,6) is perpendicular to the straight line passing through the points C(10, 2, –5) and D(8, 3, –4).
64. Show that the lines $\frac{x-1}{4}=\frac{2-y}{6}=\frac{z-4}{12}$ and $\frac{x-3}{-2}=\frac{y-3}{3}=\frac{5-z}{6}$ are parallel.
65. Find the non-parametric form of vector equation and Cartesian equations of the straight line passing through the point with position vector $4\hat{i}+3\hat{j}-7\hat{k}$ and parallel to the vector $2\hat{i}-6\hat{j}+7\hat{k}$.
66. Find the parametric form of vector equation and Cartesian equations of the straight line passing through the point (–2, 3, 4) and parallel to the straight line $\frac{x-1}{-4}=\frac{y+3}{5}=\frac{8-z}{6}$.
67. Find the points where the straight line passes through (6,7, 4) and (8, 4,9) cuts the *xz* and *yz* planes.
68. Find the direction cosines of the straight line passing through the points (5,6,7) and (7,9,13). Also, find the parametric form of vector equation and Cartesian equations of the straight line passing through two given points.
69. Find the acute angle between the following lines.
(i) $\vec{r}=(4\hat{i}-\hat{j})+t(\hat{i}+2\hat{j}-2\hat{k})$, $\vec{r}=(\hat{i}-2\hat{j}+4\hat{k})+s(-\hat{i}-2\hat{j}+2\hat{k})$
(ii) $\frac{x+4}{3}=\frac{y-7}{4}=\frac{z+5}{5}$, $\vec{r}=4\hat{k}+t(2\hat{i}+\hat{j}+\hat{k})$
(iii) $2x=3y=-z$ and $6x=-y=-4z$
70. The vertices of ΔABC are $A(7,2,1)$, B(6,0,3) and C(4, 2, 4). Find $\angle ABC$.
71. If the straight line joining the points (2,1,4) and (a – 1, 4, – 1) is parallel to the line joining the points (0, 2, *b* – 1) and (5, 3, –2), find the values of *a* and *b*.
72. If the straight lines $\frac{x-5}{5m+2}=\frac{2-y}{5}=\frac{1-z}{-1}$ and $x=\frac{2y+1}{4m}=\frac{1-z}{-3}$ are perpendicular to each other, find the value of m.
73. Show that the points (2, 3, 4), (–1, 4, 5) and (8, 1, 2) are collinear.
74. Find the point of intersection of the lines $\frac{x-1}{2}=\frac{y-2}{3}=\frac{z-3}{4}$ and $\frac{x-4}{5}=\frac{y-1}{2}=z$.
75. Find the parametric form of vector equation of a straight line passing through the point of intersection of the straight lines $\vec{r}=(\hat{i}+3\hat{j}-\hat{k})+t(2\hat{i}+3\hat{j}+2\hat{k})$ and $\frac{x-2}{1}=\frac{y-4}{2}=\frac{z+3}{4}$, and perpendicular to both straight lines.
76. Determine whether the pair of straight lines $\vec{r}=(2\hat{i}+6\hat{j}+3\hat{k})+t(2\hat{i}+3\hat{j}+4\hat{k})$, $\vec{r}=(2\hat{j}-3\hat{k})+s(\hat{i}+2\hat{j}+3\hat{k})$ are parallel. Find the shortest distance between them.
77. Find the shortest distance between the two given straight lines $\vec{r}=(2\hat{i}+3\hat{j}+4\hat{k})+t(-2\hat{i}+\hat{j}-2\hat{k})$ and $\frac{x-3}{2}=\frac{y}{-1}=\frac{z+2}{2}$
78. Find the coordinates of the foot of the perpendicular drawn from the point (–1, 2, 3) to the straight line $\vec{r}=(\hat{i}-4\hat{j}+3\hat{k})+t(2\hat{i}+3\hat{j}+\hat{k})$. Also, find the shortest distance from the given point to the straight line.

79. Find the parametric form of vector equation and Cartesian equations of a straight line passing through (5, 2, 8) and is perpendicular to the straight lines $\vec{r}=(\hat{i}+\hat{j}-\hat{k})+s(2\hat{i}-2\hat{j}+\hat{k})$ and $\vec{r}=(2\hat{i}-\hat{j}-3\hat{k})+t(\hat{i}+2\hat{j}+2\hat{k})$.
80. Show that the lines $\vec{r}=(6\hat{i}+\hat{j}+2\hat{k})+s(\hat{i}+2\hat{j}-3\hat{k})$ and $\vec{r}=(3\hat{i}+2\hat{j}-2\hat{k})+t(2\hat{i}+4\hat{j}-5\hat{k})$ are skew lines and hence find the shortest distance between them.
81. If the two lines $\frac{x-1}{2}=\frac{y+1}{3}=\frac{z-1}{4}$ and $\frac{x-3}{1}=\frac{y-m}{2}=z$ intersect at a point, find the value of m .
82. Show that the lines $\frac{x-3}{3}=\frac{y-3}{-1},z-1=0$, and $\frac{x-6}{2}=\frac{z-1}{3},y-2=0$, intersect. Also find the point of intersection.
83. Show that the straight lines $x+1=2y=-12z$ and $x=y+2=6z-6$ are skew and hence find the shortest distance between them.
84. Find the parametric form of vector equation of the straight line passing through (–1, 2, 1) and parallel to the straight line $\vec{r}=(2\hat{i}+3\hat{j}-\hat{k})+t(\hat{i}-2\hat{j}+\hat{k})$ and hence find the shortest distance between the lines.
85. Find the foot of the perpendicular drawn from the point (5, 4, 2) to the line $\frac{x+1}{2}=\frac{y-3}{3}=\frac{z-1}{-1}$. Also, find the equation of the perpendicular.
86. Find the vector and Cartesian form of the equations of a plane which is at a distance of 12 units from the origin and perpendicular to $6\hat{i}+2\hat{j}-3\hat{k}$
87. If the Cartesian equation of a plane is $3x-4y+3z=-8$, find the vector equation of the plane in the standard form.
88. Find the direction cosines of the normal to the plane and length of the perpendicular from the origin to the plane $\vec{r}.(3\hat{i}-4\hat{j}+12\hat{k})=5$.
89. Find the vector and Cartesian equations of the plane passing through the point with position vector $4\hat{i}+2\hat{j}-3\hat{k}$ and normal to vector $2\hat{i}-\hat{j}+\hat{k}$.
90. A variable plane moves in such a way that the sum of the reciprocals of its intercepts on the coordinate axes is a constant. Show that the plane passes through a fixed point.
91. Find a parametric form of vector equation of a plane which is at a distance of 7 units from the origin having 3, –4, 5 as direction ratios of a normal to it.
92. Find the direction cosines of the normal to the plane $12x+3y-4z=65$. Also, find the non-parametric form of vector equation of a plane and the length of the perpendicular to the plane from the origin.
93. Find the vector and Cartesian equations of the plane passing through the point with position vector $2\hat{i}+6\hat{j}+3\hat{k}$ and normal to the vector $\hat{i}+3\hat{j}+5\hat{k}$.
94. A plane passes through the point (–1, 1, 2) and the normal to the plane of magnitude $3\sqrt{3}$ makes equal acute angles with the coordinate axes. Find the equation of the plane.
95. Find the intercepts cut off by the plane $\vec{r}.(6\hat{i}+4\hat{j}-3\hat{k})=12$ on the coordinate axes.
96. If a plane meets the coordinate axes at A, B, C such that the centriod of the triangle ABC is the point (u, v, w), find the equation of the plane.

97. Find the non-parametric form of vector equation, and Cartesian equation of the plane passing through the point (0, 1, –5) and parallel to the straight lines $\vec{r}=(\hat{i}+2\hat{j}-4\hat{k})+s(2\hat{i}+3\hat{j}+6\hat{k})$ and $\vec{r}=(\hat{i}-3\hat{j}+5\hat{k})+t(\hat{i}+\hat{j}-\hat{k})$.

98. Find the vector parametric, vector non-parametric and Cartesian form of the equation of the plane passing through the points (–1, 2, 0), (2, 2, –1) and parallel to the straight line $\frac{x-1}{1}=\frac{2y+1}{2}=\frac{z+1}{-1}$.

99. Find the non-parametric form of vector equation, and Cartesian equation of the plane passing through the point (2,3,6) and parallel to the straight lines $\frac{x-1}{2}=\frac{y+1}{3}=\frac{z-3}{1}$ and $\frac{x+3}{2}=\frac{y-3}{-5}=\frac{z+1}{-3}$

100. Find the parametric form of vector equation, and Cartesian equations of the plane passing through the points (2, 2, 1), (9, 3, 6) and perpendicular to the plane $2x+6y+6z=9$.

101. Find parametric form of vector equation and Cartesian equations of the plane passing through the points (2, 2, 1), (1, –2, 3) and parallel to the straight line passing through the points (2, 1, –3) and (–1, 5, –8).

102. Find the non-parametric form of vector equation and cartesian equation of the plane passing through the point (1, –2, 4) and perpendicular to the plane $x+2y-3z=11$ and parallel to the line $\frac{x+7}{3}=\frac{y+3}{-1}=\frac{z}{1}$.

103. Find the parametric form of vector equation, and Cartesian equations of the plane containing the line $\vec{r}=(\hat{i}-\hat{j}+3\hat{k})+t(2\hat{i}-\hat{j}+4\hat{k})$ and perpendicular to plane $\vec{r}\cdot(\hat{i}+2\hat{j}+\hat{k})=8$.

104. Find the parametric vector, non-parametric vector and Cartesian form of the equations of the plane passing through the three non collinear points (3, 6, –2), (–1, –2, 6) and (6, –4, –2).

105. Find the non-parametric form of vector equation, and Cartesian equations of the plane $\vec{r}=(6\hat{i}-\hat{j}+\hat{k})+s(-\hat{i}+2\hat{j}+\hat{k})+t(-5\hat{i}-4\hat{j}-5\hat{k})$

106. Verify whether the line $\frac{x-3}{-4}=\frac{y-4}{-7}=\frac{z+3}{12}$ lies in the plane $5x-y+z=8$.

107. Show that the lines $\vec{r}=(-\hat{i}-3\hat{j}-5\hat{k})+s(3\hat{i}+5\hat{j}+7\hat{k})$ and $\vec{r}=(2\hat{i}+4\hat{j}+6\hat{k})+t(\hat{i}+4\hat{j}+7\hat{k})$ are coplanar. Also, find the non-parametric form of vector equation of the plane containing these lines.

108. Show that the straight lines $\vec{r}=(5\hat{i}+7\hat{j}-3\hat{k})+s(4\hat{i}+4\hat{j}-5\hat{k})$ and $\vec{r}=(8\hat{i}+4\hat{j}+5\hat{k})+t(7\hat{i}+\hat{j}+3\hat{k})$ are coplanar. Find the vector equation of the plane in which they lie.

109. Show that the lines $\frac{x-2}{1}=\frac{y-3}{1}=\frac{z-4}{3}$ and $\frac{x-1}{-3}=\frac{y-4}{2}=\frac{z-5}{1}$ are coplanar. Also, find the plane containing these lines.

110. If the straight lines $\frac{x-1}{1}=\frac{y-2}{2}=\frac{z-3}{m^2}$ and $\frac{x-3}{1}=\frac{y-2}{m^2}=\frac{z-1}{2}$ are coplanar, find the distinct real values of m.

111. If the straight lines $\frac{x-1}{2}=\frac{y+1}{\lambda}=\frac{z}{2}$ and $\frac{x+1}{5}=\frac{y+1}{2}=\frac{z}{\lambda}$ are coplanar, find λ and equations of the planes containing these two lines.

112. Find the acute angle between the planes $\vec{r}\cdot(2\hat{i}+2\hat{j}+2\hat{k})=11$ and $4x-2y+2z=15$

113.Find the angle between the straight line $\vec{r}=(2\hat{i}+3\hat{j}+\hat{k})+t(\hat{i}-\hat{j}+\hat{k})$ and the plane $2x-y+z=5$.

114.Find the distance of a point (2, 5, –3) from the plane $\vec{r}.(6\hat{i}-3\hat{j}+2\hat{k})=5$..

115.Find the distance of the point (5, –5, 10) from the point of intersection of a straight line passing through the points $A(4, 1, 2)$ and $B(7, 5, 4)$ with the plane $x-y+z=5$.

116.Find the distance between the parallel planes $x+2y-2z+1=0$ and $2x+4y-4z+5=0$.

117.Find the distance between the planes $\vec{r}.(2\hat{i}-\hat{j}-2\hat{k})=6$ and $\vec{r}.(6\hat{i}-3\hat{j}-6\hat{k})=27$

118.Find the equation of the plane passing through the intersection of the planes $\vec{r}.(\hat{i}+\hat{j}+\hat{k})+1=0$ and $\vec{r}.(2\hat{i}-3\hat{j}+5\hat{k})=2$ and the point (–1, 2, 1).

119.Find the equation of the plane passing through the intersection of the planes $2x+3y-z+7=0$ and $x+y-2z+5=0$ and is perpendicular to the plane $x+y-3z-5=0$

120.Find the image of the point whose position vector is $\hat{i}+2\hat{j}+3\hat{k}$ in the plane $\vec{r}.(\hat{i}+2\hat{j}+4\hat{k})=38$.

121.Find the coordinates of the point where the straight line $\vec{r}=(2\hat{i}-\hat{j}+2\hat{k})+t(3\hat{i}+4\hat{j}+2\hat{k})$ intersects the plane $x-y+z-5=0$

122.Find the equation of the plane passing through the line of intersection of the planes $\vec{r}.(2\hat{i}-7\hat{j}+4\hat{k})=3$ and $3x-5y+4z+11=0$, and the point (–2, 1, 3).

123.Find the equation of the plane passing through the line of intersection of the planes $x+2y+3z=2$ and $x-y+z=3$, and at a distance $\frac{2}{\sqrt{3}}$ from the point (3, 1, –1).

124.Find the angle between the line $\vec{r}=(2\hat{i}-\hat{j}+\hat{k})+t(\hat{i}+2\hat{j}-2\hat{k})$ and the plane $\vec{r}.(6\hat{i}+3\hat{j}+2\hat{k})=8$

125.Find the angle between the plane $\vec{r}.(\hat{i}+\hat{j}-2\hat{k})=3$ and $2x-2y+z=2$.

126.Find the equation of the plane which passes through the point (3, 4, –1) and is parallel to the plane $2x-3y+5z+7=0$. Also, find the distance between the two planes.

127.Find the length of the perpendicular from the point (1, –2, 3) to the plane $x-y+z=5$.

128.Find the point of intersection of the line $x-1=\frac{y}{2}=z+1$ with the plane $2x-y+2z=2$. Also, find the angle between the line and the plane.

129.Find the coordinates of the foot of the perpendicular and length of the perpendicular from the point (4, 3, 2) to the plane $x+2y+3z=2$.

Additional Questions :

130.Show that, the scalar triple product of three non-zero vectors is zero if, and only if, the three vectors are coplanar.

131.Prove that three vectors $\vec{a},\vec{b},\vec{c}$ are coplanar if and only if there exist scalars r, s, t ∈ R. Such that atleast one of them is non-zero and $r\vec{a}+s\vec{b}+t\vec{c}=\vec{0}$.

132.State and prove Jacobi's identity.

133.State and prove Lagrange's identity.

134.A point on the straight line and the direction of the straight line are given, find (a) parametric form of vector equation (b) non-parametric of vector equation (c) Cartesian equation.

135.Prove that the shortest distance between the two parallel lines $\vec{r}=\vec{a}+s\vec{b}$ and $\vec{r}=\vec{c}+t\vec{b}$ is given by $d=\frac{|(\vec{c}-\vec{a})\times\vec{b}|}{|\vec{b}|}$, where $|\vec{b}|\neq 0$.

136.Show that the shortest distance between the two skew lines $\vec{r}=\vec{a}+s\vec{b}$ and $\vec{r}=\vec{c}+t\vec{d}$ is given by, $\delta=\frac{|(\vec{c}-\vec{a}).(\vec{b}\times\vec{d})|}{|\vec{b}\times\vec{d}|}$, where $|\vec{b}\times\vec{d}|\neq 0$.

137.Prove that the equation of the plane of a distance p from the origin and perpendicular to the unit normal vector $\hat{d}$ is $\vec{r}.\vec{d}=p$.

138.Derive intercept form of the equation of a plane.

139.Show that if three non-collinear points with position vectors $\vec{a},\vec{b},\vec{c}$ are given, then the vector equation of the plane passing through the given points in parametric form is $\vec{r}=\vec{a}+s(\vec{b}-\vec{a})+t(\vec{c}-\vec{a})$, where $\vec{b}\neq\vec{0}$, $\vec{c}\neq\vec{0}$ and $s, t \in R$.

140.Prove that the perpendicular distance from a point with position vector $\vec{u}$ to the plane $\vec{r}.\vec{n}=p$ is given by $\delta=\frac{|\vec{u}.\vec{n}-p|}{|\vec{n}|}$.

141.Show that the distance between two parallel planes $ax+by+cz+d_1=0$ and $ax+by+cz+d_2=0$ is given by $\frac{|d_1-d_2|}{\sqrt{a^2+b^2+c^2}}$

142.Prove that the vector equation of a plane which passes through the line of intersection of the planes $\vec{r}.\vec{n_1}=d_1$ and $\vec{r}.\vec{n_2}=d_2$ is given by $(\vec{r}.\vec{n}_1-d_1)+\lambda(\vec{r}.\vec{n}_2-d_2)=0$, where $\lambda\in R$.

143.Show that the position vector of the point of intersection of the straight line $\vec{r}=\vec{a}+t\vec{b}$ and the plane $\vec{r}.\vec{n}=p$ is $\vec{u}=\vec{a}+\left(\frac{p-(\vec{a}.\vec{n})}{\vec{b}.\vec{n}}\right)\vec{b}$. Provided $\vec{b}.\vec{n}\neq 0$.