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Business Mathematics and Statistics

12th Std

VOLUME - I

Based on the New Syllabus and
New Textbook for the year 2019-20

Salient Features

- Prepared as per the New Textbook for the year 2019 - 20.
- Exhaustive Additional Questions & Answers in all chapters.

SURA PUBLICATIONS
Chennai

2019-20 Edition

ISBN : 978-81-8449-808-0

Code No : SG 99

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PREFACE

The woods are lovely, dark and deep.
But I have promises to keep, and
miles to go before I sleep

\- Robert Frost

Respected Principals, Correspondents, Head Masters / Head Mistresses, Teachers, and dear Students.

From the bottom of our heart, we at SURA Publications sincerely thank you for the support and patronage that you have extended to us for more than a decade.

It is in our sincerest effort we take the pride of releasing **SURA'S Business Mathematics and Statistics** for +2 Standard – Edition 2019. This guide has been authored and edited by qualified teachers having teaching experience for over a decade in their respective subject fields. This Guide has been reviewed by reputed Professors who are currently serving as Head of the Department in esteemed Universities and Colleges.

With due respect to Teachers, I would like to mention that this guide will serve as a teaching companion to qualified teachers. Also, this guide will be an excellent learning companion to students with exhaustive exercises, additional problems and 1 marks as per new model in addition to precise answers for exercise problems.

In complete cognizance of the dedicated role of Teachers, I completely believe that our students will learn the subject effectively with this guide and prove their excellence in Board Examinations.

I once again sincerely thank the Teachers, Parents and Students for supporting and valuing our efforts.

God Bless all.

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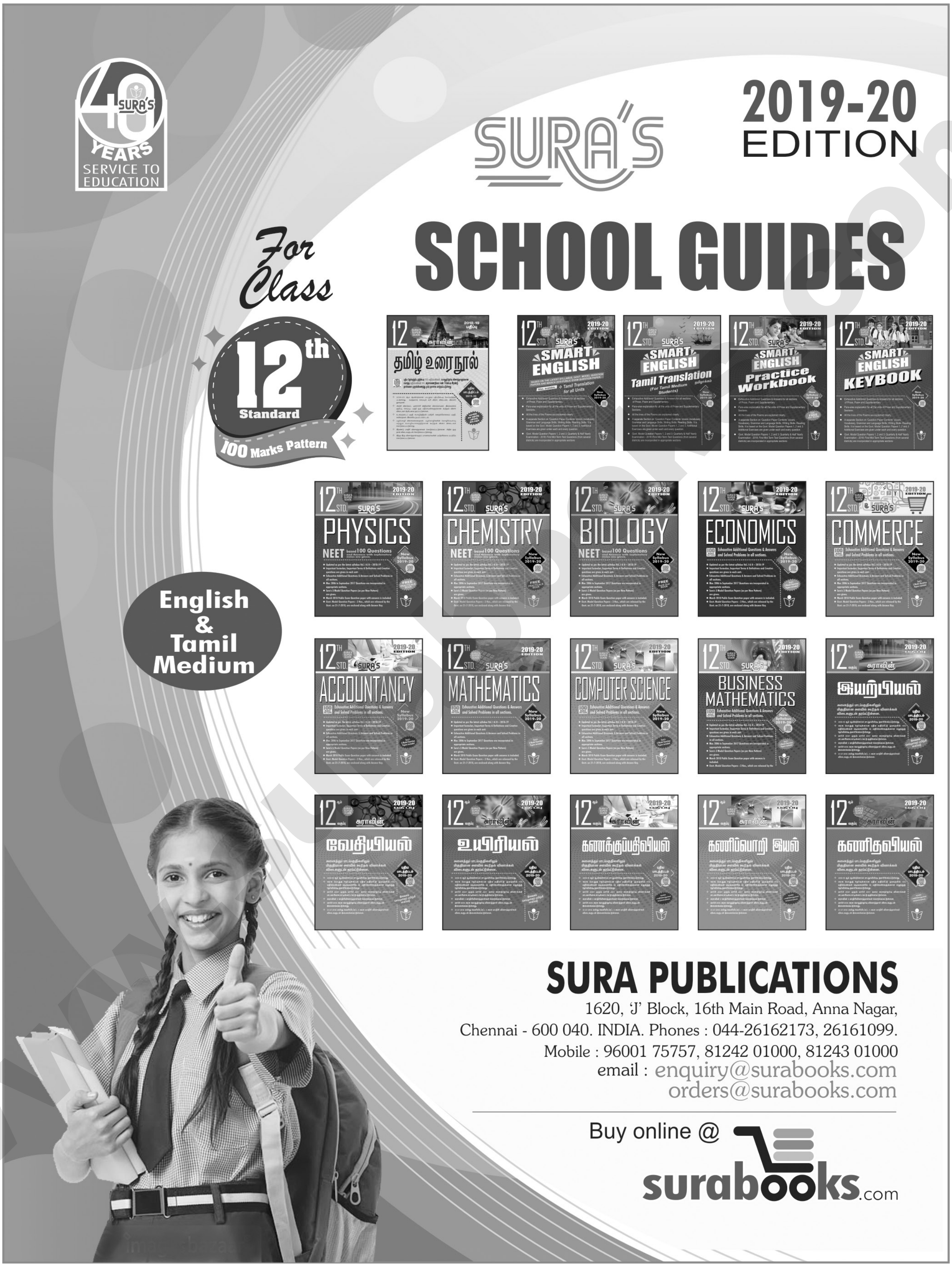
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Chapter 1

APPLICATIONS OF MATRICES AND DETERMINANTS

CHAPTER SNAPSHOT

Rank of a matrix :-

The rank of a matrix A is the order of the largest non-zero minor of A and is denoted by ρ (A).

(i) $\rho(A) \geq 0$.

(ii) If A is a matrix of order $m \times n$, then $\rho(A) \leq \min\{m, n\}$.

(iii) Rank of a zero matrix is 0.

(iv) The rank of a non - singular matrix of order $n \times n$ *is "n"*.

Elementary transformations :

(i) Interchange any two rows (or columns)

$$R_i \leftrightarrow R_j \ (C_i \leftrightarrow C_j)$$

(ii) Multiplication of each element of a row (or column) by any non-zero scalar *k*.

$$R_i \rightarrow k R_i \text{ (or } C_i \rightarrow k C_i)$$

(iii) Addition to the elements of any row (or column) the same scalar multiples of corresponding elements of any other row (or column).

$$R_i \rightarrow R_i + k R_j \text{ (or } C_i \rightarrow C_i + k C_j)$$

Equivalent matrices:

Two matrices A and B are said to be equivalent if one is obtained from the another by applying a finite number of elementary transformations.

$A \cong b$

Echelon form :

A matrix A of order $m \times n$ is said to be in echelon form if

(i) Every row of a which has all its entries 0 occurs below every row which has a non - zero entry.

(ii) The number of zeros before the first non - zero element in a row is less than the number of such rows in the next row.

Transition matrix :

The transition probabilities P_{jk} satisfy $P_{jk} > 0$ and $\sum\limits_{k} P_{jk} = 1$ for all j

FORMULAE TO REMEMBER

1. Linear equations can be written in matrix form AX = B, then the solution is $X = A^{-1}B$, provided $|A| \neq 0$.
2. Consistency of non homogeneous linear equations by rank method.

(i) If $\rho([A,B]) = \rho(A)$, then the equations are consistent.

(ii) If $\rho([A,B]) = \rho(A) = n$, where n is the number of variables then the equations are consistent and have unique solution.

(iii) If $\rho([A,B]) = \rho(A) < n$, then the equations are consistent and have infinitely many solutions.

(iv) If $\rho([A,B]) \neq \rho(A)$, then the equations are inconsistent and has no solution.

3. Solving non-homogeneous linear equations by Cramer's rule.

If $a_1x + b_1y + c_1z = d_1$, $a_2x + b_2y + c_2z = d_2$, $a_3x + b_3y + c_3z = d_3$

Then $\Delta = \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} \neq 0$, $\Delta x = \begin{vmatrix} d_1 & b_1 & c_1 \\ d_2 & b_2 & c_2 \\ d_3 & b_3 & c_3 \end{vmatrix}$,

$\Delta y = \begin{vmatrix} a_1 & d_1 & c_1 \\ a_2 & d_2 & c_2 \\ a_3 & d_3 & c_3 \end{vmatrix}$, $\Delta z = \begin{vmatrix} a_1 & b_1 & d_1 \\ a_2 & b_2 & d_2 \\ a_3 & b_3 & d_3 \end{vmatrix}$

Then $x = \dfrac{\Delta x}{\Delta}$, $y = \dfrac{\Delta y}{\Delta}$ and $z = \dfrac{\Delta z}{\Delta}$

TEXTUAL QUESTIONS

EXERCISE 1.1

1. Find the rank of each of the following matrices.

(i) $\begin{pmatrix} 5 & 6 \\ 7 & 8 \end{pmatrix}$

Solution:

Let A $= \begin{pmatrix} 5 & 6 \\ 7 & 8 \end{pmatrix}$

Order of A is 2 × 2

$\therefore \rho(A) \leq 2$ [Since minimum of (2, 2) is 2]

Consider the second order minor,

$$\begin{vmatrix} 5 & 6 \\ 7 & 8 \end{vmatrix} = 40 - 42 = -2 \neq 0.$$

There is a minor of order 2, which is not zero

$\therefore \rho(A) = 2$

(ii) $\begin{pmatrix} 1 & -1 \\ 3 & -6 \end{pmatrix}$

Solution:

Let A $= \begin{pmatrix} 1 & -1 \\ 3 & -6 \end{pmatrix}$

Order of A is 2 × 2

$\therefore \rho(A) \leq 2$ [Since minimum of (2, 2) is 2]

Consider the second order minor,

$$\begin{vmatrix} 1 & -1 \\ 3 & -6 \end{vmatrix} = -6 - (-3) = -6 + 3 = -3 \neq 0.$$

There is a minor of order 2, which is not zero

$\therefore \rho(A) = 2.$

(iii) $\begin{pmatrix} 1 & 4 \\ 2 & 8 \end{pmatrix}$

Solution:

Let A $= \begin{pmatrix} 1 & 4 \\ 2 & 8 \end{pmatrix}$

Order of A is 2 × 2 [Since minimum of (2,2) is 2]

Consider the second order minor $\begin{vmatrix} 1 & 4 \\ 2 & 8 \end{vmatrix}$

$= 8 - 8$

$= 0.$

Since the second order minor vanishes, $\rho(A) \neq 2$

Consider a first order minor $|1| \neq 0$

There is a minor of order 1, which is not zero

$\therefore \rho(A) = 1.$

(iv) $\begin{pmatrix} 2 & -1 & 1 \\ 3 & 1 & -5 \\ 1 & 1 & 1 \end{pmatrix}$

Solution:

$$\text{Let } A = \begin{pmatrix} 2 & -1 & 1 \\ 3 & 1 & -5 \\ 1 & 1 & 1 \end{pmatrix}$$

The order of A is 3×3

$\therefore \rho(A) \leq 3$ [Since minimum of (3, 3) is 3]

Let us transform the matrix A to an echelon form

Matrix A	Elementary Transformation
$A = \begin{pmatrix} 2 & -1 & 1 \\ 3 & 1 & -5 \\ 1 & 1 & 1 \end{pmatrix}$	
$\sim \begin{pmatrix} 1 & 1 & 1 \\ 3 & 1 & -5 \\ 2 & -1 & 1 \end{pmatrix}$	$R_1 \leftrightarrow R_3$
$\sim \begin{pmatrix} 1 & 1 & 1 \\ 0 & -2 & -8 \\ 0 & -3 & -1 \end{pmatrix}$	$R_2 \to R_2 - 3R_1$ $R_3 \to R_3 - 2R_1$
$\sim \begin{pmatrix} 1 & 1 & 1 \\ 0 & -1 & -4 \\ 0 & -3 & -1 \end{pmatrix}$	$R_2 \to R_2 \div 2$
$\sim \begin{pmatrix} 1 & 1 & 1 \\ 0 & -1 & -4 \\ 0 & 0 & 11 \end{pmatrix}$	$R_3 \to R_3 - 3R_2$

This matrix is in echelon from and number of non-zero rows is 3.

$\therefore \rho(A) = 3.$

(v) $\begin{pmatrix} -1 & 2 & -2 \\ 4 & -3 & 4 \\ -2 & 4 & -4 \end{pmatrix}$

Solution:

$$\text{Let } A = \begin{pmatrix} -1 & 2 & -2 \\ 4 & -3 & 4 \\ -2 & 4 & -4 \end{pmatrix}$$

The order of A is 3×3

$\therefore \rho(A) \leq 3$ [Since minimum of (3, 3) is 3]

Let us transform the matrix to an echelon form.

Matrix A	Elementary Transformation
$A = \begin{pmatrix} -1 & 2 & -2 \\ 4 & -3 & 4 \\ -2 & 4 & -4 \end{pmatrix}$	
$\sim \begin{pmatrix} 1 & -2 & 2 \\ 4 & -3 & 4 \\ -2 & 4 & -4 \end{pmatrix}$	$R_1 \to R_1(-1)$
$\sim \begin{pmatrix} 1 & -2 & 2 \\ 0 & 5 & -4 \\ -2 & 4 & -4 \end{pmatrix}$	$R_2 \to R_2 - 4R_1$
$\sim \begin{pmatrix} 1 & -2 & 2 \\ 0 & 5 & -4 \\ 0 & 0 & 0 \end{pmatrix}$	$R_3 \to R_3 + 2R_1$

The matrix is in echelon form and the number of non-zero rows is 2.

$\therefore \rho(A) = 2.$

(vi) $\begin{pmatrix} 1 & 2 & -1 & 3 \\ 2 & 4 & 1 & -2 \\ 3 & 6 & 3 & -7 \end{pmatrix}$

Solution:

$$\text{Let } A = \begin{pmatrix} 1 & 2 & -1 & 3 \\ 2 & 4 & 1 & -2 \\ 3 & 6 & 3 & -7 \end{pmatrix}$$

The order of A is 3×4

$\therefore \rho(A) \leq 3$ [Since minimum of (3, 3) is 3]

Let us transform the matrix to an echelon form.

Matrix A	Elementary Transformation
$A = \begin{pmatrix} 1 & 2 & -1 & 3 \\ 2 & 4 & 1 & -2 \\ 3 & 6 & 3 & -7 \end{pmatrix}$	
$\sim \begin{pmatrix} 1 & 2 & -1 & 3 \\ 0 & 0 & 3 & -8 \\ 3 & 6 & 3 & -7 \end{pmatrix}$	$R_2 \to R_2 - 2R_1$
$\sim \begin{pmatrix} 1 & 2 & -1 & 3 \\ 0 & 0 & 3 & -8 \\ 0 & 0 & 6 & -16 \end{pmatrix}$	$R_3 \to R_2 - 3R_1$
$\sim \begin{pmatrix} 1 & 2 & -1 & 3 \\ 0 & 0 & 3 & -18 \\ 0 & 0 & 0 & 0 \end{pmatrix}$	$R_3 \to R_3 - 2R_2$

The matrix is in echelon form and the number of non- zero rows is 2.

$$\therefore \rho(A) = 2.$$

(vii) $\begin{pmatrix} \mathbf{3} & \mathbf{1} & \mathbf{-5} & \mathbf{-1} \\ \mathbf{1} & \mathbf{-2} & \mathbf{1} & \mathbf{-5} \\ \mathbf{1} & \mathbf{5} & \mathbf{-7} & \mathbf{2} \end{pmatrix}$

Solution:

$$A = \begin{pmatrix} 3 & 1 & -5 & -1 \\ 1 & -2 & 1 & -5 \\ 1 & 5 & -7 & 2 \end{pmatrix}$$

The order of A is 3×4

$\therefore \rho(A) \le 3$ [Since minimum of (3, 4) is 3]

Let us transform the matrix A to an echelon form.

Matrix A	Elementary Transformation
$A = \begin{pmatrix} 3 & 1 & -5 & -1 \\ 1 & -2 & 1 & -5 \\ 1 & 5 & -7 & 2 \end{pmatrix}$	
$\sim \begin{pmatrix} 1 & 5 & -7 & 2 \\ 1 & -2 & 1 & -5 \\ 3 & 1 & -5 & -1 \end{pmatrix}$	$R_1 \leftrightarrow R_3$
$\sim \begin{pmatrix} 1 & 5 & -7 & 2 \\ 0 & -7 & 8 & -7 \\ 3 & 1 & -5 & -1 \end{pmatrix}$	$R_2 \to R_2 - R_1$
$\sim \begin{pmatrix} 1 & 5 & -7 & 2 \\ 0 & -7 & 8 & -7 \\ 0 & -14 & 16 & -7 \end{pmatrix}$	$R_3 \to R_3 - 3R_1$
$\sim \begin{pmatrix} 1 & 5 & -7 & 2 \\ 0 & -7 & 8 & -7 \\ 0 & 0 & 0 & 7 \end{pmatrix}$	$R_3 \to R_3 - 2R_2$

The matrix is in echelon form and the number of non-zero matrix is 3.

$$\therefore \rho(A) = 3.$$

(viii) $\begin{pmatrix} \mathbf{1} & \mathbf{-2} & \mathbf{3} & \mathbf{4} \\ \mathbf{-2} & \mathbf{4} & \mathbf{-1} & \mathbf{-3} \\ \mathbf{-1} & \mathbf{2} & \mathbf{7} & \mathbf{6} \end{pmatrix}$

Solution:

$$A = \begin{pmatrix} 1 & -2 & 3 & 4 \\ -2 & 4 & -1 & -3 \\ -1 & 2 & 7 & 6 \end{pmatrix}$$

The order of A is 3×4

$$\therefore \rho(A) \le \text{minimum of } (3, 4)$$

$$\rho(A) \le 3$$

Let us transform the matrix A to an echelon form.

Matrix A	Elementary Transformation
$A = \begin{pmatrix} 1 & -2 & 3 & 4 \\ -2 & 4 & -1 & -3 \\ -1 & 2 & 7 & 6 \end{pmatrix}$	
$\sim \begin{pmatrix} 1 & -2 & 3 & 4 \\ 0 & 0 & 5 & 5 \\ -1 & 2 & 7 & 6 \end{pmatrix}$	$R_2 \to R_2 + 2R_1$
$\sim \begin{pmatrix} 1 & -2 & 3 & 4 \\ 0 & 0 & 5 & 5 \\ 0 & 0 & 10 & 10 \end{pmatrix}$	$R_3 \to R_3 + R_1$
$\sim \begin{pmatrix} 1 & -2 & 3 & 4 \\ 0 & 0 & 5 & 5 \\ 0 & 0 & 0 & 0 \end{pmatrix}$	$R_3 \to R_3 - 2R_2$

The matrix is in echelon form and the number of non- zero rows is 2.

$\therefore \rho(A) = 2.$

2. If $A = \begin{pmatrix} 1 & 1 & -1 \\ 2 & -3 & 4 \\ 3 & -2 & 3 \end{pmatrix}$ **and** $B = \begin{pmatrix} 1 & -2 & 3 \\ -2 & 4 & -6 \\ 5 & 1 & -1 \end{pmatrix}$, **then find the rank of AB and the rank of BA.**

Solution:

Given $A = \begin{pmatrix} 1 & 1 & -1 \\ 2 & -3 & 4 \\ 3 & -2 & 3 \end{pmatrix}$ and $B = \begin{pmatrix} 1 & -2 & 3 \\ -2 & 4 & -6 \\ 5 & 1 & -1 \end{pmatrix}$

$$AB = \begin{pmatrix} 1 & 1 & -1 \\ 2 & -3 & 4 \\ 3 & -2 & 3 \end{pmatrix}\begin{pmatrix} 1 & -2 & 3 \\ -2 & 4 & -6 \\ 5 & 1 & -1 \end{pmatrix}$$

$$= \begin{pmatrix} 1-2+5 & -2+4-1 & 3-6+1 \\ 2+6+20 & -4-12+4 & 6+18-4 \\ 3+4+15 & -6-8+3 & 9+12-3 \end{pmatrix}$$

$$= \begin{pmatrix} -6 & 1 & -2 \\ 28 & -12 & 20 \\ 22 & -11 & 18 \end{pmatrix} = \begin{pmatrix} -6 & 1 & -2 \\ 28 & -12 & 20 \\ 22 & -11 & 18 \end{pmatrix}$$

Matrix (AB)	Elementary Transformation
$AB = \begin{pmatrix} -6 & 1 & -2 \\ 28 & -12 & 20 \\ 22 & -11 & 18 \end{pmatrix}$	
$\sim \begin{pmatrix} 1 & -6 & -2 \\ -12 & 28 & 20 \\ -11 & 22 & 18 \end{pmatrix}$	$C_1 \leftrightarrow C_2$
$\sim \begin{pmatrix} 1 & -6 & -2 \\ 0 & -44 & -4 \\ -11 & 22 & 18 \end{pmatrix}$	$R_2 \to R_2 + 12R_1$
$\sim \begin{pmatrix} 1 & -6 & -2 \\ 0 & -44 & -4 \\ 0 & -44 & -4 \end{pmatrix}$	$R_3 \to R_3 + 11R_1$
$\sim \begin{pmatrix} 1 & -6 & -2 \\ 0 & -44 & -4 \\ 0 & 0 & 0 \end{pmatrix}$	$R_3 \to R_3 - R_2$

The matrix is in echelon form and the number of non-zero rows is 2.

$\therefore \rho(AB) = 2.$

$$\text{Now, } BA = \begin{pmatrix} 1 & -2 & 3 \\ -2 & 4 & -6 \\ 5 & 1 & -1 \end{pmatrix}\begin{pmatrix} 1 & 1 & -1 \\ 2 & -3 & 4 \\ 3 & -2 & 3 \end{pmatrix}$$

$$= \begin{pmatrix} 1-4+9 & 1+6-6 & -1-8+9 \\ -2+8-18 & -2-12+12 & 2+16-18 \\ 5+2-3 & 5-3+2 & -5+4-3 \end{pmatrix}$$

$$= \begin{pmatrix} 6 & 1 & 0 \\ -12 & -2 & 0 \\ 4 & 4 & -4 \end{pmatrix}$$

Matrix (BA)	Elementary Transformation
$BA = \begin{pmatrix} 6 & 1 & 0 \\ -12 & -2 & 0 \\ 4 & 4 & -4 \end{pmatrix}$	
$\sim \begin{pmatrix} 1 & 6 & 0 \\ -2 & -12 & 0 \\ 4 & 4 & -4 \end{pmatrix}$	$C_1 \leftrightarrow C_2$
$\sim \begin{pmatrix} 1 & 6 & 0 \\ 0 & 0 & 0 \\ 4 & 4 & -4 \end{pmatrix}$	$R_2 \to R_2 + 2R_1$
$\sim \begin{pmatrix} 1 & 6 & 0 \\ 0 & 0 & 0 \\ 0 & -20 & -4 \end{pmatrix}$	$R_3 \to R_3 - 4R_1$

The number of non-zero rows is 2.

$\therefore \rho(BA) = 2.$

3. Solve the following system of equations by rank method

$x + y + z = 9,\ 2x + 5y + 7z = 52,\ 2x - y - z = -6$

Solution:

Given non-homogeneous equations are

$$x + y + z = 9$$
$$2x + 5y + 7z = 52$$
$$2x - y - z = -6$$

The matrix equation corresponding to the given system is

$$\begin{pmatrix} 1 & 1 & 1 \\ 2 & 5 & 7 \\ 2 & -1 & -1 \end{pmatrix}\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 9 \\ 52 \\ -6 \end{pmatrix}$$

$$AX = B$$

Augmented matrix [A, B]	Elementary Transformation
$\begin{pmatrix} 1 & 1 & 1 & 9 \\ 2 & 5 & 7 & 52 \\ 2 & -1 & -1 & -6 \end{pmatrix}$	
$\sim \begin{pmatrix} 1 & 1 & 1 & 9 \\ 0 & 3 & 5 & 34 \\ 2 & -1 & -1 & -6 \end{pmatrix}$	$R_2 \to R_2 - 2R_1$
$\sim \begin{pmatrix} 1 & 1 & 1 & 9 \\ 0 & 3 & 5 & 34 \\ 0 & -3 & -3 & -24 \end{pmatrix}$	$R_3 \to R_3 - 2R_1$
$\sim \begin{pmatrix} 1 & 1 & 1 & 9 \\ 0 & 3 & 5 & 34 \\ 0 & 0 & 2 & 10 \end{pmatrix}$	$R_3 \to R_3 + R_2$

The last equivalent matrix is in echelon form and it has three non-zero rows.

$\therefore \rho(A) = 3.$

$\rho([A,B]) = 3 =$ Number of unknowns.

$\therefore$ The given system is consistent and has unique solution.

To find the solution, let us rewrite the above echelon from into the matrix form.

$$\begin{pmatrix} 1 & 1 & 1 \\ 0 & 3 & 5 \\ 0 & 0 & 2 \end{pmatrix}\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 9 \\ 34 \\ 10 \end{pmatrix}$$

$\Rightarrow \quad x + y + z = 9$ ----- (1)

$3y + 5z = 34$ ----- (2)

$2z = 10$ ----- (3)

From (3), $2z = 10 \Rightarrow z = \frac{10}{2} = 5$

Substituting $z = 5$ in (2) we get,

$3y + 5(5) = 34$

$\Rightarrow \quad 3y + 25 = 34$

$\Rightarrow \quad 3y = 34 - 25$

$\Rightarrow \quad 3y = 9$

$\Rightarrow \quad y = \frac{\cancel{9}^{3}}{\cancel{3}} = 3.$

Substituting $y = 3$ and $z = 5$ in (1) we get,

$x + 3 + 5 = 9$

$\Rightarrow \quad x + 8 = 9$

$\Rightarrow \quad x = 9 - 8$

$\Rightarrow \quad x = 1.$

$\therefore$ Solution set is $\{1, 3, 5\}$

4. Show that the equations $5x + 3y + 7z = 4$, $3x + 26y + 2z = 9$, $7x + 2y + 10z = 5$ are consistent and solve them by rank method.

Solution:

Given non-homogeneous equations are

$5x + 3y + 7z = 4$

$3x + 26y + 2z = 9$

$7x + 2y + 10z = 5$

The matrix equation corresponding to the given system is

$$\begin{pmatrix} 5 & 3 & 7 \\ 3 & 26 & 2 \\ 7 & 2 & 10 \end{pmatrix}\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 4 \\ 9 \\ 5 \end{pmatrix}$$

$$AX = B$$

Augmented matrix $[A, B] = \begin{pmatrix} 5 & 3 & 7 & 4 \\ 3 & 26 & 2 & 9 \\ 7 & 2 & 10 & 5 \end{pmatrix}$

Augmented matrix [A, B]	Elementary Transformation
$\begin{pmatrix} 5 & 3 & 7 & 4 \\ 3 & 26 & 2 & 9 \\ 7 & 2 & 10 & 5 \end{pmatrix}$	
$\sim \begin{pmatrix} 3 & 26 & 2 & 9 \\ 5 & 3 & 7 & 4 \\ 7 & 2 & 10 & 5 \end{pmatrix}$	$R_1 \leftrightarrow R_2$
$\sim \begin{pmatrix} 1 & \frac{26}{3} & \frac{2}{3} & 3 \\ 5 & 3 & 7 & 4 \\ 7 & 2 & 10 & 5 \end{pmatrix}$	$R_1 \to R_1 \div 3$

$\sim \begin{pmatrix} 1 & \frac{26}{3} & \frac{2}{3} & 3 \\ 0 & \frac{-121}{3} & \frac{11}{3} & -11 \\ 7 & 2 & 10 & 5 \end{pmatrix}$	$R_2 \to R_2 - 5R_1$
$\sim \begin{pmatrix} 1 & \frac{26}{3} & \frac{2}{3} & 3 \\ 0 & \frac{-121}{3} & \frac{11}{3} & -11 \\ 0 & \frac{-176}{3} & \frac{16}{3} & -16 \end{pmatrix}$	$R_3 \to R_3 - 7R_1$
$\sim \begin{pmatrix} 1 & \frac{26}{3} & \frac{2}{3} & 3 \\ 0 & \frac{-11}{3} & \frac{1}{3} & -1 \\ 0 & \frac{-11}{3} & \frac{1}{3} & -1 \end{pmatrix}$	$R_2 \to R_2 \div 11$ $R_3 \to R_3 \div 16$
$\sim \begin{pmatrix} 1 & \frac{26}{3} & \frac{2}{3} & 3 \\ 0 & \frac{-11}{3} & \frac{1}{3} & -1 \\ 0 & 0 & 0 & 0 \end{pmatrix}$	$R_3 \to R_3 - R_2$

Here $\rho(A) = \rho(A, B) = 2 <$ Number of unknowns.

$\therefore$ The system is consistent with infinitely many solutions let us rewrite the above echelon form into matrix form.

$$\begin{pmatrix} 1 & \frac{26}{3} & \frac{2}{3} \\ 0 & \frac{-11}{3} & \frac{1}{3} \\ 0 & 0 & 0 \end{pmatrix}\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 3 \\ -1 \\ 0 \end{pmatrix}$$

$$x + \frac{26}{3}y + \frac{2}{3}z = 3 \qquad \text{----- (1)}$$

$$\frac{-11}{3}y + \frac{1}{3}z = -1 \qquad \text{----- (2)}$$

let $z = k$ where $k \in R$

$$(2) \Rightarrow \frac{-11}{3}y + \frac{k}{3} = -1$$

$$\Rightarrow \frac{-11}{\cancel{3}}y = -1 - \frac{k}{3} = \frac{-3-k}{\cancel{3}}$$

$$\Rightarrow -11y = -3 - k \qquad 11y = 3 + k$$

$$\Rightarrow y = \frac{1}{11}(3 + k)$$

Substituting $y = \frac{1}{11}(3 + k)$ and $z = k$ in (1) we get,

$$x + \frac{26}{3}\left(\frac{3+k}{11}\right) + \frac{2}{3}k = 3$$

$$x = -\frac{26}{3}\left(\frac{3+k}{11}\right) - \frac{2k}{3} + 3$$

$$= \frac{78-26k}{33} - \frac{2k}{3} + 3$$

$$= -\frac{78-26k-22k+99}{33}$$

$$= \frac{21-48k}{33} = \frac{3(7-16k)}{33}$$

$$= \frac{1}{11}(7 - 16k)$$

$\therefore$ Solution set is $\{\frac{1}{11}(7 - 16k), \frac{1}{11}(3 + k), k\}$ $k \in R$.

Hence, for different values of k, we get infinitely many solutions.

5. Show that the following system of equations have unique solution: $x + y + z = 3, x + 2y + 3z = 4, x + 4y + 9z = 6$ by rank method.

Solution:

Given non-homogeneous equations are

$$x + y + z = 3$$
$$x + 2y + 3z = 4$$
$$x + 4y + 9z = 6$$

The matrix equation corresponding to the given system is

$$\begin{pmatrix} 1 & 1 & 1 \\ 1 & 2 & 3 \\ 1 & 4 & 9 \end{pmatrix}\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 3 \\ 4 \\ 6 \end{pmatrix}$$

$$A\,X = B$$

Augmented matrix [A, B]	Elementary Transformation
$\begin{pmatrix} 1 & 1 & 1 & 3 \\ 1 & 2 & 3 & 4 \\ 1 & 4 & 9 & 6 \end{pmatrix}$	
$\sim \begin{pmatrix} 1 & 1 & 1 & 3 \\ 0 & 1 & 2 & 1 \\ 0 & 3 & 8 & 3 \end{pmatrix}$	$R_2 \to R_2 - R_1$ $R_3 \to R_3 - R_1$
$\sim \begin{pmatrix} 1 & 1 & 1 & 3 \\ 0 & 1 & 2 & 1 \\ 0 & 0 & 2 & 0 \end{pmatrix}$	$R_3 \to R_3 - 3R_2$

Clearly the last equivalent matrix is in echelon form and it has three non-zero rows.

$\therefore$ $\rho(A) = 3$ and $\rho([A, B]) = 3$

$\Rightarrow \rho(A) = \rho([A, B]) = 3 =$ Number of unknowns.

$\therefore$ The given system is consistent and has unique solution.

To find the solution, let us rewrite the above echelon form into the matrix form.

$$\begin{pmatrix} 1 & 1 & 1 \\ 0 & 1 & 2 \\ 0 & 0 & 2 \end{pmatrix}\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 3 \\ 1 \\ 0 \end{pmatrix}$$

$$\begin{aligned} \Rightarrow \quad x + y + z &= 3 && \text{----- (1)} \\ y + 2z &= 1 && \text{----- (2)} \\ 2z &= 0 && \text{----- (3)} \end{aligned}$$

$(3) \Rightarrow 2z = 0 \quad \Rightarrow z = \dfrac{0}{2} = 0$

$(2) \Rightarrow y + 2(0) = 1 \Rightarrow y + 0 = 1 \Rightarrow y = 1 - 0 = 1$

$$\begin{aligned} (1) \Rightarrow \quad x + 1 + 0 &= 3 \\ \Rightarrow \quad x + 1 &= 3 \\ \Rightarrow \quad x &= 3 - 1 \\ \Rightarrow \quad x &= 2. \end{aligned}$$

$\therefore$ Solution set {2, 1, 0}

6. For what values of the parameter λ, will the following equations fail to have unique solution: $3x - y + \lambda z = 1$, $2x + y + z = 2$, $x + 2y - \lambda z = -1$ by rank method.

Solution:

Given non-homogeneous equations are

$$\begin{aligned} 3x - y + \lambda z &= 1 \\ 2x + y + z &= 2 \\ x + 2y - \lambda z &= -1 \end{aligned}$$

The matrix equation corresponding to the given system is

$$\begin{pmatrix} 3 & -1 & \lambda \\ 2 & 1 & 1 \\ 1 & 2 & -\lambda \end{pmatrix}\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 1 \\ 2 \\ -1 \end{pmatrix}$$

Augmented matrix [A, B]	Elementary Transformation
$\begin{pmatrix} 3 & -1 & \lambda & 1 \\ 2 & 1 & 1 & 2 \\ 1 & 2 & -\lambda & -1 \end{pmatrix}$	
$\sim \begin{pmatrix} 1 & 2 & -\lambda & -1 \\ 2 & 1 & 1 & 2 \\ 3 & -1 & \lambda & 1 \end{pmatrix}$	$R_1 \leftrightarrow R_3$
$\sim \begin{pmatrix} 1 & 2 & -\lambda & -1 \\ 0 & -3 & 1+2\lambda & 4 \\ 3 & -1 & \lambda & 1 \end{pmatrix}$	$R_2 \to R_2 - 2R_1$
$\sim \begin{pmatrix} 1 & 2 & -\lambda & -1 \\ 0 & -3 & 1+2\lambda & 4 \\ 0 & -7 & 4\lambda & 4 \end{pmatrix}$	$R_3 \to R_3 - 3R_1$
$\sim \begin{pmatrix} 1 & 2 & -\lambda & -1 \\ 0 & -1 & \frac{1+2\lambda}{3} & \frac{4}{3} \\ 0 & -1 & \frac{4\lambda}{7} & \frac{4}{7} \end{pmatrix}$	$R_2 \to R_2 \div 3$ $R_3 \to R_3 \div 7$
$\sim \begin{pmatrix} 1 & 2 & -\lambda & -1 \\ 0 & -1 & \frac{1+2\lambda}{3} & \frac{4}{3} \\ 0 & 0 & \frac{-7-2\lambda}{21} & \frac{-16}{21} \end{pmatrix}$	$R_3 \to R_3 - R_2$

Since

$$\frac{4\lambda}{7} - \frac{1+2\lambda}{3} = \frac{12\lambda - 7 - 14\lambda}{21} = \frac{-7-2\lambda}{21}$$

$$\text{and } \frac{4}{7} - \frac{4}{3} = \frac{12-28}{21} = \frac{-16}{21}$$

Since the system is fail to have unique solution either it can have infinitely many solution or it may be inconsistent.

$\therefore$ This can happen only when $\dfrac{-7-2\lambda}{21} = 0$.

$\Rightarrow \quad -7 - 2\lambda = 0$

$\Rightarrow \quad -7 = 2\lambda$

$\Rightarrow \quad \lambda = \dfrac{-7}{2}$.

7. The price of three commodities X,Y and Z are x,y and z respectively Mr. Anand Purchases 6 units of Z and sells 2 units of X and 3 units of Y. Mr. Amar Purchases a unit of Y and sells 3 units of X and 2units of Z. Mr. Amit Purchases a unit of X and sells 3 units of Y and a unit of Z. In the process they earn ₹ 5,000/-, ₹ 2,000/- and ₹ 5,500/- respectively Find the prices per unit of three commodities by rank method.

Solution:

Given that the price of commodities X, Y and Z are x, y and z respectively.

By the given data,

Transaction	x	y	z	Earning
Mr. Anand	+2	+3	–6	Rs. 5000
Mr. Amar	+3	–1	+2	Rs. 2000
Mr. Amit	–1	+3	+1	Rs. 5500

Here, purchasing is taken as negative symbol and selling is taken as positive symbol.

Thus, the non-homogeneous equations are

$$2x + 3y - 6z = 5000$$
$$3x - y + 2z = 2000$$
$$-x + 3y + z = 5500$$

The matrix equation corresponding to the given system is

$$\begin{pmatrix} 2 & 3 & -6 \\ 3 & -1 & 2 \\ -1 & 3 & 1 \end{pmatrix}\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 5000 \\ 2000 \\ 5500 \end{pmatrix}$$

$$A\,X = B$$

Augmented matrix [A, B]	Elementary Transformation
$\begin{pmatrix} 2 & 3 & -6 & 5000 \\ 3 & -1 & 2 & 2000 \\ -1 & 3 & 1 & 5500 \end{pmatrix}$	
$\sim \begin{pmatrix} -1 & 3 & 1 & 5500 \\ 3 & -1 & 2 & 2000 \\ 2 & 3 & -6 & 5000 \end{pmatrix}$	$R_1 \leftrightarrow R_3$
$\sim \begin{pmatrix} 1 & -3 & -1 & -5000 \\ 3 & -1 & 2 & 2000 \\ 2 & 3 & -6 & 5500 \end{pmatrix}$	$R_1 \to R_1(-1)$
$\sim \begin{pmatrix} 1 & -3 & -1 & -5500 \\ 0 & 8 & 5 & 18500 \\ 0 & 9 & -4 & 16000 \end{pmatrix}$	$R_2 \to R_2 - 3R_1$ $R_3 \to R_3 - 2R_1$
$\sim \begin{pmatrix} 1 & -3 & -1 & -5500 \\ 0 & 1 & \frac{5}{8} & \frac{18500}{8} \\ 0 & 1 & \frac{-4}{9} & \frac{16000}{9} \end{pmatrix}$	$R_2 \to R_2 \div 8$ $R_3 \to R_3 \div 9$
$\sim \begin{pmatrix} 1 & -3 & -1 & -5500 \\ 0 & 1 & \frac{5}{8} & \frac{18500}{8} \\ 0 & 0 & \frac{-4}{9} - \frac{5}{8} & \frac{16000}{9} - \frac{18500}{8} \end{pmatrix}$	$R_3 \to R_3 - R_2$
$\sim \begin{pmatrix} 1 & -3 & -1 & -5500 \\ 0 & 1 & \frac{5}{8} & \frac{18500}{8} \\ 0 & 0 & \frac{-77}{72} & \frac{-38500}{72} \end{pmatrix}$	

$$\frac{-4}{9}\frac{-5}{8} = \frac{-32-45}{72} = \frac{-77}{72}$$

$$\frac{16000}{9}-\frac{18500}{8} = \frac{128000-166500}{72} = \frac{38500}{72}$$

Clearly the last equivalent matrix is in echelon form and it has three non-zero rows.

$\therefore\ \rho(A) = \rho([A, B]) = 3 =$ Number of unknowns.

$\therefore$ The given system is consistent and has unique solution.

To find the solution, let us rewrite the above echelon form into the matrix form.

$$\begin{pmatrix} 1 & -3 & -1 \\ 0 & 1 & \frac{5}{8} \\ 0 & 0 & \frac{-77}{72} \end{pmatrix}\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} -5000 \\ \frac{18500}{8} \\ \frac{-38500}{72} \end{pmatrix}$$

$\Rightarrow\quad x - 3y - z = -5500 \quad$ ----- (1)

$y + \frac{5}{8}z = \frac{18500}{8} \quad$ ----- (2)

$\frac{-77}{72}z = -\frac{38500}{72} \quad$ ----- (3)

(3) $\Rightarrow \quad \frac{-77z}{\cancel{72}} = \frac{-38500}{\cancel{72}}$

$\Rightarrow\quad z = \frac{-38500}{-77}$

$\Rightarrow\quad z = 500.$

(2) $\Rightarrow y + \frac{5}{8}(500) = \frac{18500}{8}$

$y = \frac{18500}{8} - \frac{2500}{8}$

$= \frac{\cancel{16000}^{2000}}{\cancel{8}}$

$\Rightarrow\quad y = 2000$

(1)$\Rightarrow x - 3(2000) - 500 = -5500$

$\Rightarrow x - 6000 - 500 = -5500$

$\Rightarrow\quad x - 6500 = -5500$

$\Rightarrow\quad x = -5500 + 6500$

$\Rightarrow\quad x = 1000$

Hence, the prices per unit of three commodities are ₹1000, ₹ 2000 and ₹ 500 respectively.

8. An amount of ₹ 5,000/- is to be deposited in three different bonds bearing 6%, 7% and 8% per year respectively. Total annual income is ₹358/-. If the income from first two investments is ₹ 70/- more than the income from the third, then find the amount of investment in each bond by rank method.

Solution:

Let the amount of investment in each bond be ₹ x, ₹ y and ₹ z respectively.

Given $x + y + z = 5000 \quad$ ----- (1)

Also $\frac{6x}{100}+\frac{7y}{100}+\frac{8z}{100} = 358$

$$\boxed{\because \text{Interest} = \frac{PNR}{100} = \frac{x\times1\times6}{100} = \frac{6x}{100}}$$

$\Rightarrow\quad \frac{6x+7y+8z}{100} = 358$

$\Rightarrow\quad 6x + 7y + 8z = 35800 \quad$ ----- (2)

Given that $\frac{6x}{100}+\frac{7y}{100} = 70 + \frac{8z}{100}$

$\Rightarrow\quad \frac{6x+7y}{\cancel{100}} = \frac{7000+8z}{\cancel{100}}$

$\Rightarrow\quad 6x + 7y = 700 + 8z$

$\Rightarrow\quad 6x + 7y - 8z = 7000 \quad$ ----- (3)

The matrix equation corresponding to the given system is.

$$\begin{pmatrix} 1 & 1 & 1 \\ 6 & 7 & 8 \\ 6 & 7 & -8 \end{pmatrix}\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 5000 \\ 35800 \\ 7000 \end{pmatrix}$$

Augmented matrix [A, B]	Elementary Transformation
$\begin{pmatrix} 1 & 1 & 1 & 5000 \\ 6 & 7 & 8 & 35800 \\ 6 & 7 & -8 & 7000 \end{pmatrix}$	
$\sim\begin{pmatrix} 1 & 1 & 1 & 5000 \\ 0 & 1 & 2 & 5800 \\ 0 & 1 & -14 & -23000 \end{pmatrix}$	$R_2 \to R_2 - 6R_1$ $R_3 \to R_3 - 6R_1$
$\sim\begin{pmatrix} 1 & 1 & 1 & 5000 \\ 0 & 1 & 2 & 5800 \\ 0 & 0 & -16 & -28800 \end{pmatrix}$	$R_3 \to R_3 - R_2$

The last equivalent matrix is in echelon form and $\rho(A) = \rho([A, B]) = 3$ = Number of unknowns.

Thus, the given system is consistent with unique solution. To find the solution, let us rewrite the above echelon form into the matrix form.

$$\begin{pmatrix} 1 & 1 & 1 \\ 0 & 1 & 2 \\ 0 & 0 & -16 \end{pmatrix}\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 5000 \\ 5800 \\ -28800 \end{pmatrix}$$

$\Rightarrow \quad x + y + z = 5000$ ----- (1)

$\Rightarrow \quad y + 2z = 5800$ ----- (2)

$\Rightarrow \quad -16z = -28800$ ----- (3)

$(3) \Rightarrow \quad -16z = -28800$

$\Rightarrow \quad z = -\dfrac{28800}{-16} = 1800$

Substituting $z = 1800$ in (2) we get,

$y + 2(1800) = 5800$

$\Rightarrow \quad y + 3600 = 5800$

$\Rightarrow \quad y = 5800 - 3600$

$\Rightarrow \quad y = 2200$

Substituting $y = 2200$ and $z = 1800$ in (1) we get,

$x + 2200 + 1800 = 5000$

$\Rightarrow \quad x + 4000 = 5000$

$\Rightarrow \quad x = 5000 - 4000$

$\Rightarrow \quad x = 1000$

Hence, the amount of investment in each bond is Rs. 1000, Rs.2200 and Rs. 1800 respectively.

EXERCISE 1.2

1. Solve the following equations by using Cramer's rule.

(i) $2x + 3y = 7$, $\quad 3x + 5y = 9$

Solution:

$$\Delta = \begin{vmatrix} 2 & 3 \\ 3 & 5 \end{vmatrix} = 10 - 9 = 1 \neq 0.$$

Since $\Delta \neq 0$, we can apply Cramer's rule and the system is consistent with unique solution.

$$\Delta x = \begin{vmatrix} 7 & 3 \\ 9 & 5 \end{vmatrix} = 7(5) - 9(3)$$

$= 35 - 27 = 8$

$$\Delta y = \begin{vmatrix} 2 & 7 \\ 3 & 9 \end{vmatrix} = 2(9) - 3(7)$$

$= 18 - 21 = -3$

$\therefore \quad x = \dfrac{\Delta x}{\Delta} = \dfrac{8}{1} = 8$

$y = \dfrac{\Delta y}{\Delta} = \dfrac{-3}{1} = -3$

$\therefore$ Solution set is $\{8, -3\}$

(ii) $5x + 3y = 17$; $\quad 3x + 7y = 31$

Solution:

$$\Delta = \begin{vmatrix} 5 & 3 \\ 3 & 7 \end{vmatrix} = 5(7) - 3(3)$$

$= 35 - 9 = 26$

Since $\Delta \neq 0$, we can apply Cramer's rule and the system is consistent with unique solution.

$$\Delta x = \begin{vmatrix} 17 & 3 \\ 31 & 7 \end{vmatrix} = 17(7) - 31(3)$$

$= 119 - 93 = 26$

$$\Delta y = \begin{vmatrix} 5 & 17 \\ 3 & 31 \end{vmatrix} = 5(31) - 17(3)$$

$= 155 - 51 = 104.$

$x = \dfrac{\Delta x}{\Delta} = \dfrac{\cancel{26}}{\cancel{26}} = 1$

$y = \dfrac{\Delta y}{\Delta} = \dfrac{\overset{4}{\cancel{104}}}{\cancel{26}} = 4$

$\therefore$ Solution set is $\{1, 4\}$

(iii) $2x + y - z = 3$, $x + y + z = 1$, $x - 2y - 3z = 4$

Solution:

$$\Delta = \begin{vmatrix} 2 & 1 & -1 \\ 1 & 1 & 1 \\ 1 & -2 & -3 \end{vmatrix} = 2\begin{vmatrix} 1 & 1 \\ -2 & -3 \end{vmatrix} - 1\begin{vmatrix} 1 & 1 \\ 1 & -3 \end{vmatrix} - 1\begin{vmatrix} 1 & 1 \\ 1 & -2 \end{vmatrix}$$

$= 2(-3+2) - 1(-3-1) - 1(-2-1)$

$= 2(-1) - 1(-4) - 1(-3)$

$= -2 + 4 + 3 = 5.$

Since $\Delta \neq 0$, we can apply Cramer's rule and the system is consistent with unique solution.

$$x = \begin{vmatrix} 3 & 1 & -1 \\ 1 & 1 & 1 \\ 4 & -2 & -3 \end{vmatrix} = 3\begin{vmatrix} 1 & 1 \\ -2 & -3 \end{vmatrix} - 1\begin{vmatrix} 1 & 1 \\ 4 & -3 \end{vmatrix} - 1\begin{vmatrix} 1 & 1 \\ 4 & -2 \end{vmatrix}$$

$= 3(-3+2) - 1(-3-4) - 1(-2-4)$

$= 3(-1) - 1(-7) - 1(-6)$

$= -3 + 7 + 6 = 10.$

$$\Delta y = \begin{vmatrix} 2 & 3 & -1 \\ 1 & 1 & 1 \\ 1 & 4 & -3 \end{vmatrix} = 2\begin{vmatrix} 1 & 1 \\ 4 & -3 \end{vmatrix} - 3\begin{vmatrix} 1 & 1 \\ 1 & -3 \end{vmatrix} - 1\begin{vmatrix} 1 & 1 \\ 1 & 4 \end{vmatrix}$$

$= 2(-3-4) - 3(-3-1) - 1(4-1)$

$= 2(-7) - 3(-4) - 1(3)$

$= -14 + 12 - 3 = -5$

$$\Delta z = \begin{vmatrix} 2 & 1 & 3 \\ 1 & 1 & 1 \\ 1 & -2 & 4 \end{vmatrix}$$

$$= 2\begin{vmatrix} 1 & 1 \\ -2 & 4 \end{vmatrix} - 1\begin{vmatrix} 1 & 1 \\ 1 & 4 \end{vmatrix} + 3\begin{vmatrix} 1 & 1 \\ 1 & -2 \end{vmatrix}$$

$= 2(4+2) - 1(4-1) + 3(-2-1)$

$= 2(6) - 1(3) + 3(-3)$

$= 12 - 3 - 9$

$= 0$

$$\therefore x = \frac{\Delta x}{\Delta} = \frac{\overset{2}{\cancel{10}}}{\cancel{5}} = 2$$

$$y = \frac{\Delta y}{\Delta} = \frac{\cancel{-5}}{\cancel{5}} = -1$$

$$z = \frac{\Delta z}{\Delta} = \frac{0}{5} = 0$$

$\therefore$ Solution set is $\{2, -1, 0\}$

(iv) $x + y + z = 6, 2x + 3y - z = 5, 6x - 2y - 3z = -7.$

Solution:

$$\Delta = \begin{vmatrix} 1 & 1 & 1 \\ 2 & 3 & -1 \\ 6 & -2 & -3 \end{vmatrix}$$

$$= 1\begin{vmatrix} 3 & -1 \\ -2 & -3 \end{vmatrix} - 1\begin{vmatrix} 2 & -1 \\ 6 & -3 \end{vmatrix} + 1\begin{vmatrix} 2 & 3 \\ 6 & -2 \end{vmatrix}$$

$= 1(-9-2) - 1(-6+6) + 1(-4-18)$

$= 1(-11) - 1(0) + 1(-22)$

$= -11 - 22 = -33 \neq 0$

Since $\Delta \neq 0$, Cramer's rule can be applied and the system is consistent with unique solution.

$$\Delta x = \begin{vmatrix} 6 & 1 & 1 \\ 5 & 3 & -1 \\ -7 & -2 & -3 \end{vmatrix}$$

$$= 6\begin{vmatrix} 3 & -1 \\ -2 & -3 \end{vmatrix} - 1\begin{vmatrix} 5 & -1 \\ -7 & -3 \end{vmatrix} + 1\begin{vmatrix} 5 & 3 \\ -7 & -2 \end{vmatrix}$$

$= 6(-9-2) - 1(-15-7) + 1(-10+21)$

$= 6(-11) - 1(-22) + 1(11)$

$= -66 + 22 + 11 = -33$

$$\Delta y = \begin{vmatrix} 1 & 6 & 1 \\ 2 & 5 & -1 \\ 6 & -7 & -3 \end{vmatrix}$$

$$= 1\begin{vmatrix} 5 & -1 \\ -7 & -3 \end{vmatrix} - 6\begin{vmatrix} 2 & -1 \\ 6 & -3 \end{vmatrix} + 1\begin{vmatrix} 2 & 5 \\ 6 & -7 \end{vmatrix}$$

$= 1(-15-7) - 6(-6+6) + 1(-14-30)$

$= 1(-22) - 6(0) + 1(-44)$

$= -22 - 44 = -66$

$$\Delta z = \begin{vmatrix} 1 & 1 & 6 \\ 2 & 3 & 5 \\ 6 & -2 & -7 \end{vmatrix}$$

$$= 1\begin{vmatrix} 3 & 5 \\ -2 & -7 \end{vmatrix} - 1\begin{vmatrix} 2 & 5 \\ 6 & -7 \end{vmatrix} + 6\begin{vmatrix} 2 & 3 \\ 6 & -2 \end{vmatrix}$$

$= 1(-21+10) - 1(-14-30) + 6(-4-18)$

$= 1(-11) - 1(-44) + 6(-22)$

$= -11 + 44 - 132 = -99$

$$x = \frac{\Delta x}{\Delta} = \frac{\cancel{-33}}{\cancel{-33}} = 1$$

$$y = \frac{\Delta y}{\Delta} = \frac{\overset{2}{\cancel{-66}}}{\cancel{-33}} = 2$$

$$z = \frac{\Delta z}{\Delta} = \frac{\overset{3}{\cancel{-99}}}{\cancel{-33}} = 3$$

$\therefore$ Solution set is $\{1, 2, 3\}$

(v) $x + 4y + 3z = 2,\ 2x - 6y + 6z = -3,$
$5x - 2y + 3z = -5$

Solution:

$$\Delta = \begin{vmatrix} 1 & 4 & 3 \\ 2 & -6 & 6 \\ 5 & -2 & 3 \end{vmatrix}$$

$$= 1\begin{vmatrix} -6 & 6 \\ -2 & 3 \end{vmatrix} - 4\begin{vmatrix} 2 & 6 \\ 5 & 3 \end{vmatrix} + 3\begin{vmatrix} 2 & -6 \\ 5 & -2 \end{vmatrix}$$

$= 1(-18 + 12) - 4(6 - 30) + 3(-4 + 30)$
$= 1(-6) - 4(-24) + 3(26)$

$= -6 + 96 + 78 = 168 \neq 0.$

Since $\Delta \neq 0$, the system is consistent with unique solution and Cramer's rule can be applied.

$$\Delta x = \begin{vmatrix} 2 & 4 & 3 \\ -3 & -6 & 6 \\ -5 & -2 & 3 \end{vmatrix}$$

$$= 2\begin{vmatrix} -6 & 6 \\ -2 & 3 \end{vmatrix} - 4\begin{vmatrix} -3 & 6 \\ -5 & 3 \end{vmatrix} + 3\begin{vmatrix} -3 & -6 \\ -5 & -2 \end{vmatrix}$$

$= 2(-18 + 12) - 4(-9 + 30) + 3(6 - 30)$

$= 2(-6) - 4(21) + 3(-24)$

$= -12 - 84 - 72 = -168$

$$\Delta y = \begin{vmatrix} 1 & 2 & 3 \\ 2 & -3 & 6 \\ 5 & -5 & 3 \end{vmatrix} = 1\begin{vmatrix} -3 & 6 \\ -5 & 3 \end{vmatrix} - 2\begin{vmatrix} 2 & 6 \\ 5 & 3 \end{vmatrix} + 3\begin{vmatrix} 2 & -3 \\ 5 & -5 \end{vmatrix}$$

$= 1(-9 + 30) - 2(6 - 30) + 3(-10 + 15)$
$= 1(21) - 2(-24) + 3(5)$
$= 21 + 48 + 15 = 84$

$$\Delta z = \begin{vmatrix} 1 & 4 & 2 \\ 2 & -6 & -3 \\ 5 & -2 & -5 \end{vmatrix}$$

$$= 1\begin{vmatrix} -6 & -3 \\ -2 & -5 \end{vmatrix} - 4\begin{vmatrix} 2 & -3 \\ 5 & -5 \end{vmatrix} + 2\begin{vmatrix} 2 & -6 \\ 5 & -2 \end{vmatrix}$$

$= 1(30 - 6) - 4(-10 + 15) + 2(-4 + 30)$

$= 24 - 4(5) + 2(26)$

$= 24 - 20 + 52 = 56$

$$x = \frac{\Delta x}{\Delta} = \frac{\cancel{-168}}{\cancel{168}} = -1$$

$$y = \frac{\Delta y}{\Delta} = \frac{\cancel{84}}{\underset{2}{\cancel{168}}} = \frac{1}{2}$$

$$z = \frac{\Delta z}{\Delta} = \frac{\cancel{56}}{\underset{3}{\cancel{168}}} = \frac{1}{3}$$

Solution set is $\{-1, \frac{1}{2}, \frac{1}{3}\}$

2. A commodity was produced by using 3 units of labour and 2 units of capital, the total cost is ₹62. If the commodity had been produced by using 4 units of labour and one unit of capital, the cost is ₹ 56. What is the cost per unit of labour and capital? (Use determinant method).

Solution:

Let Rs. x represents the cost per unit of labour and Rs. y represents the cost per unit of capital

Given $\quad 3x + 2y = 62$

$\quad 4x + y = 56$

$$\Delta = \begin{vmatrix} 3 & 2 \\ 4 & 1 \end{vmatrix} = 3(1) - 4(2) = 3 - 8 = -5$$

Since $\Delta \neq 0$, the system is consistent with unique solution and Cramer's rule can be applied.

$$\Delta x = \begin{vmatrix} 62 & 2 \\ 56 & 1 \end{vmatrix} = 62(1) - 56(2) = 62 - 112 = -50$$

$$\Delta y = \begin{vmatrix} 3 & 62 \\ 4 & 56 \end{vmatrix} = 3(56) - 4(62)$$

$= 168 - 248 = -80$

$$\therefore \quad x = \frac{\Delta x}{\Delta} = \frac{\overset{10}{\cancel{-50}}}{\cancel{-5}} = 10$$

$$y = \frac{\Delta y}{\Delta} = \frac{\overset{16}{\cancel{-80}}}{\cancel{-5}} = 16.$$

$\therefore$ Cost per unit of labour is Rs. 10 and the cost per unit of capital is Rs. 16.

3. A total of ₹ 8,600 was invested in two accounts. One account earned $4\frac{3}{4}$ % annual interest and the other earned $6\frac{1}{2}$ % annual interest. If the total interest for one year was ₹ 431.25, how much was invested in each account? (Use determinant method).

Solution:

Let the amount invested in the two accounts be Rs x and Rs. y respectively

By the given data, $x + y = 8600$ ----- (1)

$$4\frac{3}{4}\times\frac{x}{100}+6\frac{1}{2}\times\frac{y}{100} = 431.25$$

$$\left[\therefore \text{ interest} = \frac{PNR}{100}\right]$$

$$\Rightarrow \frac{19x}{400}+\frac{13y}{200} = 431.25$$

$$\Rightarrow \frac{19x+26y}{400} = 431.25$$

$$19x + 26y = 172500 \quad \text{----- (2)}$$

$$\Delta = \begin{vmatrix} 1 & 1 \\ 19 & 26 \end{vmatrix} = 1(26) - 1(19) = 26 - 19 = 7$$

$$\Delta x = \begin{vmatrix} 8600 & 1 \\ 172500 & 26 \end{vmatrix} = 8600(26) - 1(172500) = 223600 - 172500 = 51100$$

$$\Delta y = \begin{vmatrix} 1 & 8600 \\ 19 & 172500 \end{vmatrix} = 1(172500) - 19(8600) = 172500 - 163400 = 9100$$

$$\therefore \quad x = \frac{\Delta x}{\Delta} = \frac{51100}{7} = 7300$$

$$y = \frac{\Delta y}{\Delta} = \frac{9100}{7} = 1300$$

$\therefore$ Investment in the interest of $4\frac{3}{4}$% account is Rs. 7300 and investment in the rate of $6\frac{1}{2}$% account is Rs. 1300.

4. At marina two types of games viz., Horse riding and Quad Bikes riding are available on hourly rent. Keren and Benita spent ₹ 780 and ₹ 560 during the month of May.

Name	Number of Hours		Total amount spent (in ₹)
	Horse Riding	Quad Bike Riding	
Keren	3	4	780
Benita	2	3	560

Find the hourly charges for the two games (rides). (Use determinant method).

Solution:

Let the hourly charge for horse riding be Rs. x and the hourly charge for quad bike be Rs. y from the given data, $3x + 4y = 780$

$$2x + 3y = 560$$

$$\Delta = \begin{vmatrix} 3 & 4 \\ 2 & 3 \end{vmatrix} = 3(3) - 2(4) = 9-8 = 1 \neq 0$$

Since $\Delta \neq 0$, the system is consistent with unique solution and Cramer's rule can be applied.

$$\Delta x = \begin{vmatrix} 780 & 4 \\ 560 & 3 \end{vmatrix} = 780(3) - 4(560) = 2340 - 2240 = 100$$

$$\Delta y = \begin{vmatrix} 3 & 780 \\ 2 & 560 \end{vmatrix} = 3(560) - 2(780) = 1680 - 1560 = 120$$

$$\therefore \quad x = \frac{\Delta x}{\Delta} = \frac{100}{1} = 100$$

$$y = \frac{\Delta y}{\Delta} = \frac{120}{1} = 120$$

$\therefore$ Hourly charges for the two rides are Rs.100 and Rs.120 respectively.

5. In a market survey three commodities A, B and C were considered. In finding out the index number some fixed weights were assigned to the three varieties in each of the commodities. The table below provides the information regarding the consumption of three commodities according to the three varieties and also the total weight received by the commodity

Commodity Variety	Variety			Total Weight
	I	II	III	
A	1	2	3	11
B	2	4	5	21
C	3	5	6	27

Find the weights assigned to the three varieties by using Cramer's Rule.

Solution:

Let the weight assigned to the three varieties be Rs. x, Rs. y and Rs. z respectively.

By the given data,

$$x + 2y + 3z = 11$$
$$2x + 4y + 5z = 21$$
$$3x + 5y + 6z = 27$$

$$\Delta = \begin{vmatrix} 1 & 2 & 3 \\ 2 & 4 & 5 \\ 3 & 5 & 6 \end{vmatrix} = 1\begin{vmatrix} 4 & 5 \\ 5 & 6 \end{vmatrix} - 2\begin{vmatrix} 2 & 5 \\ 3 & 6 \end{vmatrix} + 3\begin{vmatrix} 2 & 4 \\ 3 & 5 \end{vmatrix}$$

$= 1(24 - 25) - 2(12 - 15) + 3(10 - 12)$

$= 1(-1) - 2(-3) + 3(-2)$

$= -1 + 6 - 6 = -1 \neq 0.$

Since $\Delta \neq 0$, the system is consistent with unique solution and Cramer's rule can be applied.

$$\Delta x = \begin{vmatrix} 11 & 2 & 3 \\ 21 & 4 & 5 \\ 27 & 5 & 6 \end{vmatrix}$$

$$= 11\begin{vmatrix} 4 & 5 \\ 5 & 6 \end{vmatrix} - 2\begin{vmatrix} 21 & 5 \\ 27 & 6 \end{vmatrix} + 3\begin{vmatrix} 21 & 4 \\ 27 & 5 \end{vmatrix}$$

$= 11(24 - 25) - 2(126 - 135) + 3(105 - 108)$

$= 11(-1) - 2(-9) + 3(-3)$

$= -11 + 18 - 9$

$= -2$

$$\Delta y = \begin{vmatrix} 1 & 11 & 3 \\ 2 & 21 & 5 \\ 3 & 27 & 6 \end{vmatrix}$$

$$= \begin{vmatrix} 21 & 5 \\ 27 & 6 \end{vmatrix} - 11\begin{vmatrix} 2 & 5 \\ 3 & 6 \end{vmatrix} + 3\begin{vmatrix} 2 & 21 \\ 3 & 27 \end{vmatrix}$$

$= 1(126 - 135) - 11(12 - 15) + 3(54 - 63)$

$= -9 - 11(-3) + 3(-9)$

$= -9 + 33 - 27$

$= -3$

$$\Delta z = \begin{vmatrix} 1 & 2 & 11 \\ 2 & 4 & 21 \\ 3 & 5 & 27 \end{vmatrix} = 1\begin{vmatrix} 4 & 21 \\ 5 & 27 \end{vmatrix} - 2\begin{vmatrix} 2 & 21 \\ 3 & 27 \end{vmatrix} + 11\begin{vmatrix} 2 & 4 \\ 3 & 5 \end{vmatrix}$$

$= 1(108 - 105) - 2(54 - 63) + 11(10 - 12)$

$= 1(3) - 2(-9) + 11(-2)$

$= 3 + 18 - 22$

$= -1$

$$x = \frac{\Delta x}{\Delta} = \frac{-2}{-1} = 2$$

$$y = \frac{\Delta y}{\Delta} = \frac{-3}{-1} = 3$$

$$\text{and } z = \frac{\Delta z}{\Delta} = \frac{-1}{-1} = 1$$

Hence, the weights assigned to the three varieties are 2, 3 and 1 respectively.

6. A total of ₹ 8,500 was invested in three interest earning accounts. The interest rates were 2%, 3% and 6% if the total simple interest for one year was ₹ 380 and the amount invested at 6% was equal to the sum of the amounts in the other two accounts, then how much was invested in each account? (Use Cramer's rule).

Solution:

Let the amount invested in the rate of 2%, 3% and 6% be Rs. x, Rs. y and Rs. z respectively.

By the given data,

$$x + y + z = 8500 \quad \text{----- (1)}$$

$$\frac{2x}{100} + \frac{3y}{100} + \frac{6z}{100} = 380$$

$$\Rightarrow \frac{2x + 3y + 6z}{100} = 380$$

$$\boxed{\because \text{Interest} = \frac{PNR}{100} = \frac{x \times 1 \times 2}{100} = \frac{2x}{100}}$$

$$\Rightarrow 2x + 3y + 6z = 38000 \quad \text{----- (2)}$$

Also, $z = x + y$

$$x + y - z = 0 \quad \text{----- (3)}$$

$$\Delta = \begin{vmatrix} 1 & 1 & 1 \\ 2 & 3 & 6 \\ 1 & 1 & -1 \end{vmatrix}$$

$$= 1\begin{vmatrix}3 & 6\\1 & -1\end{vmatrix} - 1\begin{vmatrix}2 & 6\\1 & -1\end{vmatrix} + 1\begin{vmatrix}2 & 3\\1 & 1\end{vmatrix}$$

$= 1(-3-6) - 1(-2-6) + 1(2-3)$

$= 1(-9) - 1(-8) + 1(-1)$

$= -9 + 8 - 1 = -2 \neq 0$

Since $\Delta \neq 0$, Cramer's rule can be applied and the system is consistent with unique solution.

$$\Delta x = \begin{vmatrix}8500 & 1 & 1\\38000 & 3 & 6\\0 & 1 & -1\end{vmatrix}$$

$$= 8500\begin{vmatrix}3 & 6\\1 & -1\end{vmatrix} - 1\begin{vmatrix}38000 & 6\\1 & -1\end{vmatrix} + 1\begin{vmatrix}38000 & 3\\0 & 1\end{vmatrix}$$

$= 8500(-3-6) - 1(-38000 \quad -0) + 1(38000-0)$

$= 8500(-9) - 1(-38000) + 1(38000)$

$= -76500 + 38000 + 38000$

$= -500$

$$\Delta y = \begin{vmatrix}1 & 8500 & 1\\2 & 38000 & 6\\1 & 0 & -1\end{vmatrix}$$

$$= 1\begin{vmatrix}38000 & 6\\0 & -1\end{vmatrix} - 8500\begin{vmatrix}2 & 6\\1 & -1\end{vmatrix} + 1\begin{vmatrix}2 & 38000\\1 & 0\end{vmatrix}$$

$= 1(-38000-0) - 8500 \quad (-2-6) + 1(0-38000)$

$= -38000 - 8500(-8) - 38000$

$= -38000 + 68000 - 38000$

$= -8000$

$$\Delta z = \begin{vmatrix}1 & 1 & 8500\\2 & 3 & 38000\\1 & 1 & 0\end{vmatrix}$$

$$= 1\begin{vmatrix}3 & 38000\\1 & 0\end{vmatrix} - 1\begin{vmatrix}2 & 38000\\1 & 0\end{vmatrix} + 8500\begin{vmatrix}2 & 3\\1 & 1\end{vmatrix}$$

$= 1(0-38000) - 1(0-38000) \quad +85000(2-3)$

$= -38000 + 38000 + 8500 \quad (-1)$

$= -8500$

$$\therefore x = \frac{\Delta x}{\Delta} = \frac{\overset{250}{\cancel{-500}}}{\cancel{-2}} = +250$$

$$y = \frac{\Delta y}{\Delta} = \frac{\overset{4000}{\cancel{-8000}}}{\cancel{-2}} = 4000$$

$$z = \frac{\Delta z}{\Delta} = \frac{\overset{4250}{\cancel{-8500}}}{\cancel{-2}} = 4250$$

Hence, the amount invested in the three accounts are Rs. 250, Rs. 4000 and Rs. 4250 respectively.

EXERCISE 1.3

1. **The subscription department of a magazine sends out a letter to a large mailing list inviting subscriptions for the magazine. Some of the people receiving this letter already subscribe to the magazine while others do not. From this mailing list, 45% of those who already subscribe will subscribe again while 30% of those who do not now subscribe will subscribe. On the last letter, it was found that 40% of those receiving it ordered a subscription. What percent of those receiving the current letter can be expected to order a subscription?**

Solution:

Transition probability matrix

$$T = \begin{array}{c} \\ A \\ B \end{array}\begin{array}{c} \begin{array}{cc} A & B \end{array} \\ \begin{pmatrix} \cdot 45 & \cdot 55 \\ \cdot 30 & \cdot 70 \end{pmatrix}\end{array}$$

Where A represents the percentage of subscribers and B represents the percentage of non-subscribers.

By the given data, 40% received the order of subscription ⇒ 60% are non-subscribers.

$A = 40\% = \cdot 40$

and $B = 60\% = \cdot 60$

$$\therefore \begin{array}{c} \begin{array}{cc} A & B \end{array} \\ \begin{pmatrix} \cdot 40 & \cdot 60 \end{pmatrix}\end{array} \begin{array}{c} \\ A \\ B \end{array}\begin{array}{c} \begin{array}{cc} A & B \end{array} \\ \begin{pmatrix} \cdot 45 & \cdot 55 \\ \cdot 30 & \cdot 70 \end{pmatrix}\end{array} =$$

$$\begin{pmatrix}(\cdot 40)(\cdot 45) + (\cdot 60)(\cdot 30) & (\cdot 40)(.55) + (\cdot 60)(.70)\end{pmatrix}$$

$$= \begin{pmatrix}\cdot 18 + \cdot 18 & \cdot 22 + \cdot 42\end{pmatrix}$$

$= (\cdot 36 \quad \cdot 64)$

⇒ 36 % of those receiving the current letter can be expected to order a subscription.

2. A new transit system has just gone into operation in Chennai. Of those who use the transit system this year, 30% will switch over to using metro train next year and 70% will continue to use the transit system. Of those who use metro train this year, 70% will continue to use metro train next year and 30% will switch over to the transit system. Suppose the population of Chennai city remains constant and that 60% of the commuters use the transit system and 40% of the commuters use metro train this year.

(i) What percent of commuters will be using the transit system after one year?

(ii) What percent of commuters will be using the transit system in the long run?

Solution:

Transition probability matrix

$$T = \begin{array}{c} \\ A \\ B \end{array}\begin{array}{c} \begin{array}{cc} A & B \end{array} \\ \begin{pmatrix} \cdot 70 & \cdot 30 \\ \cdot 30 & \cdot 70 \end{pmatrix} \end{array}$$

Where A represents the percentage of people using transit system and B represents the percentage of people using metro train.

By the given data

$$A = 60\% = \cdot 60$$

$$\text{and } B = 40\% = \cdot 4$$

$$\begin{pmatrix} \cdot 60 & \cdot 40 \end{pmatrix}\begin{pmatrix} \cdot 70 & \cdot 30 \\ \cdot 30 & \cdot 70 \end{pmatrix}$$

$$= \begin{pmatrix} (\cdot 6)(\cdot 7)+(\cdot 4)(\cdot 3) & (\cdot 6)(\cdot 3)+(\cdot 4)(\cdot 7) \end{pmatrix}$$

$$= \begin{pmatrix} \cdot 42+\cdot 12 & \cdot 18+\cdot 28 \end{pmatrix}$$

$$= \begin{pmatrix} \cdot 54 & \cdot 46 \end{pmatrix}$$

$$\therefore \quad A = 54\% \text{ and } B = 46\%$$

(i) The percent of Commuters using the transit system after one year is 54% and the percent of commuters using the metro train after one year is 46%

(ii) Equilibrium will be reached in the long run. At equilibrium we must have

$$\begin{pmatrix} A & B \end{pmatrix} T = \begin{pmatrix} A & B \end{pmatrix}$$

where $A + B = 1$

$$\Rightarrow \begin{pmatrix} A & B \end{pmatrix}\begin{pmatrix} \cdot 7 & \cdot 3 \\ \cdot 3 & \cdot 7 \end{pmatrix} = \begin{pmatrix} A & B \end{pmatrix}$$

$$\Rightarrow \begin{pmatrix} \cdot 7A+\cdot 3B & \cdot 3A+\cdot 7B \end{pmatrix} = \begin{pmatrix} A & B \end{pmatrix}$$

Equaling the entries on both sides, we get

$$\cdot 7A + \cdot 3B = A$$

$$\Rightarrow \cdot 7A + \cdot 3(1-A) = A \quad [\because A + B = 1 \Rightarrow B = 1 - A]$$

$$\Rightarrow \cdot 7A + \cdot 3 - \cdot 3A = A$$

$$\Rightarrow \cdot 3 = A - \cdot 7A + \cdot 3A$$

$$\Rightarrow \cdot 3 = A(1 - \cdot 7 + \cdot 3)$$

$$\Rightarrow \cdot 3 = A(\cdot 3 + \cdot 3)$$

$$\Rightarrow \cdot 3 = A(\cdot 6)$$

$$\Rightarrow A = \frac{\cdot 3}{\cdot 6} = \frac{1}{2} = \cdot 50$$

$\therefore$ The percent of commuters using the transit system in the long run is 50%.

3. Two types of soaps A and B are in the market. Their present market shares are 15% for A and 85% for B. Of those who bought A the previous year, 65% continue to buy it again while 35% switch over to B. Of those who bought B the previous year, 55% buy it again and 45% switch over to A. Find their market shares after one year and when is the equilibrium reached?

Solution:

Transition probability matrix

$$T = \begin{array}{c} \\ A \\ B \end{array}\begin{array}{c} \begin{array}{cc} A & B \end{array} \\ \begin{pmatrix} \cdot 65 & \cdot 35 \\ \cdot 45 & \cdot 55 \end{pmatrix} \end{array}$$

Where A represents the percent of people those who bought soap A and B represents the percent of people those who bought soap B.

By the given data,

$$A = 15\% = \cdot 15$$

$$\text{and } B = 85\% = \cdot 85$$

Percentage after one year is

$$\begin{pmatrix} \cdot 15 & \cdot 85 \end{pmatrix}\begin{pmatrix} \cdot 65 & \cdot 35 \\ \cdot 45 & \cdot 55 \end{pmatrix}$$

$$= \begin{pmatrix} (\cdot 15)(\cdot 65)+(\cdot 85)(\cdot 45) & \cdot 15(\cdot 35)+\cdot 85(\cdot 55) \end{pmatrix}$$

$$= \begin{pmatrix} \cdot 0975+\cdot 3825 & \cdot 0525+\cdot 4675 \end{pmatrix}$$

$$= \begin{pmatrix} \cdot 48 & \cdot 52 \end{pmatrix}$$

Hence, market share after one year is 48% and 52%

At equilibrium,

$$(A \quad B)\,T = (A \quad B)$$

$$(A \quad B)\begin{pmatrix} \cdot65 & \cdot35 \\ \cdot45 & \cdot55 \end{pmatrix} = (A \quad B)$$

$$(\cdot65A + \cdot45B \quad \cdot35A + \cdot55B) = (A \quad B)$$

Equating the corresponding entries on both sides we get,

$$\cdot65A + \cdot45B = A$$

$$\Rightarrow \cdot65A + \cdot45(1-A) = A$$

[Since A+B = 1 ⇒ B= 1–A]

$$\Rightarrow \cdot65A + \cdot45 - \cdot45A = A$$

$$\Rightarrow \cdot45 = A - \cdot65A + \cdot45A$$

$$\Rightarrow \cdot45 = A(1 - \cdot65 + \cdot45)$$

$$\Rightarrow \cdot45 = A(\cdot35 + \cdot45)$$

$$\Rightarrow \cdot45 = A(\cdot8)$$

$$\Rightarrow A = \frac{\cdot45}{\cdot8} = \cdot5625 = 56.25\%$$

$$\therefore B = 1 - A = 1 - \cdot5625 = \cdot4375 = 43.75\%$$

∴ Equilibrium is reached when A = 56.25% and B = 43.75%

4. Two products A and B currently share the market with shares 50% and 50% each respectively. Each week some brand switching takes place. Of those who bought A the previous week, 60% buy it again whereas 40% switch over to B. Of those who bought B the previous week, 80% buy it again where as 20% switch over to A. Find their shares after one week and after two weeks. If the price war continues, when is the equilibrium reached?

Solution:

Transition probability matrix

$$T = \begin{array}{c} \\ A \\ B \end{array}\begin{array}{c} \begin{array}{cc} A & B \end{array} \\ \begin{pmatrix} \cdot6 & \cdot4 \\ \cdot2 & \cdot8 \end{pmatrix} \end{array}$$

By the given data

$$A = 50\% = \cdot5$$

$$B = 50\% = \cdot5$$

Shares after one week

$$(\cdot5 \quad \cdot5)\begin{pmatrix} \cdot6 & \cdot4 \\ \cdot2 & \cdot8 \end{pmatrix}$$

$$= ((\cdot5)(\cdot6) + (\cdot5)(\cdot2) \quad \cdot5(\cdot4) + \cdot5(\cdot8))$$

$$= (\cdot30 + \cdot10 \quad \cdot20 + \cdot40)$$

$$= (\cdot40 \quad \cdot60)$$

∴ Shares after one week for products A and B are 40% and 60% respectively.

Shares after two weeks

$$(\cdot4 \quad \cdot6)\begin{pmatrix} \cdot6 & \cdot4 \\ \cdot2 & \cdot8 \end{pmatrix}$$

$$= ((\cdot4)(\cdot6) + (\cdot6)(\cdot2) \quad (\cdot4)(\cdot4) + \cdot6(0\cdot8))$$

$$= (\cdot24 + \cdot12 \quad \cdot16 + \cdot48)$$

$$= (\cdot36 \quad \cdot64)$$

∴ Shares after two week for products A and B are 36% and 64% respectively.

At equilibrium, we must have

$$(A \quad B)\,T = (A \quad B)$$

where $A + B = 1$

$$(A \quad B)\begin{pmatrix} \cdot6 & \cdot4 \\ \cdot2 & \cdot8 \end{pmatrix} = (A \quad B)$$

$$\Rightarrow (\cdot6A + \cdot2B \quad \cdot4A + \cdot8B) = (A \quad B)$$

Equating the corresponding entries on both sides we get,

$$\cdot6A + \cdot2B = A$$

$$\Rightarrow \cdot6A + \cdot2(1 - A) = A$$

$$\Rightarrow \cdot6A + \cdot2 - \cdot2A = A$$

$$\Rightarrow \cdot2 = A - \cdot6A + \cdot2A$$

$$\Rightarrow \cdot2 = A(1 - \cdot6 + \cdot2)$$

$$\Rightarrow \cdot2 = A(\cdot4 + \cdot2)$$

$$\Rightarrow \cdot2 = A(\cdot6)$$

$$\Rightarrow A = \frac{\cdot2}{\cdot6} = \cdot33 \Rightarrow A = 33\%$$

and B = 1 – A = 1 – ·33 = ·67 ⇒ B = 67%

∴ Equilibrium is reached when A = 33% and B = 67%

EXERCISE 1.4

Choose the correct answer

1. If A=(1 2 3), then the rank of AA^T is

(a) 0 (b) 2 (c) 3 (d) 1

[Ans: (d) 1]

Hint:

$$AA^T = \begin{pmatrix}1 & 2 & 3\end{pmatrix}\begin{pmatrix}1\\2\\3\end{pmatrix} = (1+4+9) = 14$$

$\therefore$ Rank of AA^T is 1

2. The rank of $m \times n$ matrix whose elements are unity is

(a) 0 (b) 1 (c) m (d) n

[Ans: (b) 1]

Hint:

$$A = \begin{pmatrix}1 & 1 & 1 & 1 & 1 & --- & 1\\ 1 & 1 & 1 & 1 & 1 & --- & 1\\ - & - & - & - & - & --- & -\\ - & - & - & - & - & --- & -\\ 1 & 1 & 1 & 1 & 1 & 1 & 1\end{pmatrix}$$

m rows and n columns applying the elementary transformation.

$R_2 \to R_2 - R_1, R_3 \to R_1$ ----- $R_m \to R_m - R_1$ we get

$$A = \begin{pmatrix}1 & 1 & 1 & --- & --- & 1\\ 0 & 0 & 0 & --- & --- & 0\\ 0 & 0 & 0 & --- & --- & 0\\ - & - & - & --- & --- & -\\ - & - & - & - & - & -\\ 0 & 0 & 0 & 0 & 0 & 0\end{pmatrix}$$

$\therefore$ Rank is 1

3. If $T = \begin{array}{c c} & \begin{array}{cc} A & B \end{array} \\ \begin{array}{c} A \\ B \end{array} & \begin{pmatrix}0{\cdot}4 & 0{\cdot}6\\ 0{\cdot}2 & 0{\cdot}8\end{pmatrix}\end{array}$ **is a transition probability matrix, then at equilibrium A is equal to**

(a) $\frac{1}{4}$ (b) $\frac{1}{5}$ (c) $\frac{1}{6}$ (d) $\frac{1}{8}$

[Ans: (a) $\frac{1}{4}$]

Hint: At equilibrium,

$$\begin{pmatrix}A & B\end{pmatrix}\begin{pmatrix}0{\cdot}4 & 0{\cdot}6\\ 0{\cdot}2 & 0{\cdot}8\end{pmatrix} = \begin{pmatrix}A & B\end{pmatrix}$$

Where $A + B = 1$

$$\begin{pmatrix}0{\cdot}4A+0{\cdot}2B & 0{\cdot}6A+0{\cdot}8B\end{pmatrix} = \begin{pmatrix}A & B\end{pmatrix}$$

$0{\cdot}4A + 0{\cdot}2B = A$

$0{\cdot}4A + 0{\cdot}2(1 - A) = A$

[since $A+B = 1 \Rightarrow B = 1-A$]

$\Rightarrow 0{\cdot}4A + 0{\cdot}2 - 0{\cdot}2A = A$

$\Rightarrow 0{\cdot}2 = A - 0{\cdot}4A + 0{\cdot}2A$

$\Rightarrow 0{\cdot}2 = A(1 - 0{\cdot}4 + 0{\cdot}2)$

$= A(0{\cdot}8)$

$\Rightarrow A = \frac{0\cdot 2}{0\cdot 8} = \frac{1}{4}$

4. If $A = \begin{pmatrix}2 & 0\\ 0 & 8\end{pmatrix}$ **then $\rho(A)$ is**

(a) 0 (b) 1 (c) 2 (d) n

[Ans: (c) 2]

Hint:

If $A = \begin{pmatrix}2 & 0\\ 0 & 8\end{pmatrix}$

$|A| = \begin{vmatrix}2 & 0\\ 0 & 8\end{vmatrix} = 16 - 0 = 16 \neq 0.$

Since the second order determinant does not vanish. $\rho(A) = 2$

5. The rank of the matrix $\begin{pmatrix}1 & 1 & 1\\ 1 & 2 & 3\\ 1 & 4 & 9\end{pmatrix}$

(a) 0 (b) 1 (c) 2 (d) 3

[Ans: (d) 3]

Hint:

Let $A = \begin{pmatrix}1 & 1 & 1\\ 1 & 2 & 3\\ 1 & 4 & 9\end{pmatrix}$

$\sim \begin{pmatrix}1 & 1 & 1\\ 0 & 2 & 2\\ 0 & 3 & 8\end{pmatrix}$ $R_2 \to R_2 - R_1$, $R_3 \to R_3 - R_1$

$= \begin{pmatrix}1 & 1 & 1\\ 0 & 1 & 2\\ 0 & 0 & 2\end{pmatrix}$ $R_3 \to R_3 - 3R_2$

Since there are 3 non-zero rows, $\rho(A) = 3$

$\therefore \rho(A) = 3$

6. The rank of the unit matrix of order n is

(a) $n-1$ (b) n (c) $n+1$ (d) n^2

[Ans: (b) n]

Hint:

The rank of the unit matrix of order n
[Since the rank of a non-singular matrix of order $n \times n$ is 'n']

7. If $\rho(A) = r$ then which of the following is correct?

(a) all the minors of order r which does not vanish.
(b) A has at least one minor of order r which does not vanish.
(c) A has at least one $(r+1)$ order minor which vanishes.
(d) all $(r+1)$ and higher order minors should not vanish.

[Ans: (b) A has at least one minor of order r which does not vanish.]

8. If A = $\begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix}$ then the rank of AA^T is

(a) 0 (b) 1 (c) 2 (d) 3

[Ans: (b) 1]

Hint:

$$\text{If } A = \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix} \text{ then } A^T = (1\ 2\ 3)$$

$$AA^T = \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix}(1\ 2\ 3) = \begin{pmatrix} 1 & 2 & 3 \\ 2 & 4 & 6 \\ 3 & 6 & 9 \end{pmatrix}$$

$$\therefore \sim \begin{pmatrix} 1 & 2 & 3 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

$\therefore$ Rank of AA^T is 1

9. If the rank of the matrix $\begin{pmatrix} \lambda & -1 & 0 \\ 0 & \lambda & -1 \\ -1 & 0 & \lambda \end{pmatrix}$ is 2. Then λ is

(a) 1 (b) 2
(c) 3 (d) only real number

[Ans: (a) 1]

Hint:

$$\text{Rank of } \begin{pmatrix} \lambda & -1 & 0 \\ 0 & \lambda & -1 \\ -1 & 0 & \lambda \end{pmatrix} = 2$$

Since the rank is 2,

$$\begin{vmatrix} \lambda & -1 & 0 \\ 0 & \lambda & -1 \\ -1 & 0 & \lambda \end{vmatrix} = 0$$

$$\Rightarrow \lambda \begin{vmatrix} \lambda & -1 \\ 0 & \lambda \end{vmatrix} + 1 \begin{vmatrix} 0 & -1 \\ -1 & \lambda \end{vmatrix} + 0 \begin{vmatrix} 0 & \lambda \\ -1 & 0 \end{vmatrix} = 0$$

$$\lambda(\lambda^2) + 1(0-1) + 0 = 0$$

$$\lambda^3 - 1 = 0 \Rightarrow \lambda^3 = 1$$

$$\lambda = 1$$

10. The rank of the diagonal matrix $\begin{pmatrix} 1 & & & & & \\ & 2 & & & & \\ & & -3 & & & \\ & & & 0 & & \\ & & & & 0 & \\ & & & & & 0 \end{pmatrix}$

(a) 0 (b) 2 (c) 3 (d) 5

[Ans: (c) 3]

Hint:

Since there are 3 non-zero rows, rank is 3

11. If T = $\begin{array}{c} \\ A \\ B \end{array}\begin{array}{c} \begin{array}{cc} A & B \end{array} \\ \begin{pmatrix} 0\cdot7 & 0\cdot3 \\ 0\cdot6 & x \end{pmatrix} \end{array}$ is a transition probability matrix, then the value of x is

(a) 0.2 (b) 0.3 (c) 0.4 (d) 0.7

[Ans: (c) 0·4]

Hint:

$$T = \begin{array}{c} \\ A \\ B \end{array}\begin{array}{c} \begin{array}{cc} A & B \end{array} \\ \begin{pmatrix} 0\cdot7 & 0\cdot3 \\ 0\cdot6 & x \end{pmatrix} \end{array}$$

Since this is a transition probability matrix,

$$0\cdot6 + x = 1 \Rightarrow x = 1 - 0\cdot6 = 0\cdot4$$

12. Which of the following is not an elementary transformation?

(a) $R_i \leftrightarrow R_j$ (b) $R_i \to 2R_i + 2C_j$
(c) $R_i \to 2R_i - 4R_j$ (d) $C_i \to C_i + 5C_j$

[Ans: (b) $R_i \to 2R_i + 2C_j$]

Hint:

$R_i \to 2R_i + 2C_j$ is not an elementary transformation since it includes rows and columns.

13. If ρ(A) = ρ(A, B) then the system is

(a) Consistent and has infinitely many solutions

(b) Consistent and has a unique solution

(c) Consistent (d) inconsistent

[Ans: (c) Consistent]

Hint:

If ρ(A)=ρ(A, B) then the system can have a unique solution or infinitely many solutions.

∴ The system is consistent.

14. If ρ(A) = ρ(A, B)= the number of unknowns, then the system is

(a) Consistent and has infinitely many solutions

(b) Consistent and has a unique solution

(c) inconsistent (d) consistent

[Ans: (b) Consistent and has a unique solution]

Hint:

If ρ(A)=ρ(A, B) = Number of unknowns then the system is consistent and has a unique solution.

15. If ρ(A) ≠ ρ(A, B), then the system is

(a) Consistent and has infinitely many solutions

(b) Consistent and has a unique solution

(c) inconsistent (d) consistent

[Ans: (c) inconsistent]

Hint:

If ρ(A) ≠ ρ(A, B), then the system is inconsistent.

16. In a transition probability matrix, all the entries are greater than or equal to

(a) 2 (b) 1 (c) 0 (d) 3

[Ans: (c) 0]

Hint:

$[\because 0 < p < 1]$

17. If the number of variables in a non-homogeneous system AX = B is *n*, then the system possesses a unique solution only when

(a) $\rho(A) = \rho(A, B) > n$

(b) $\rho(A) = \rho(A, B) = n$

(c) $\rho(A) = \rho(A, B) < n$ (d) none of these

[Ans: (b) ρ(A) = ρ(A, B) = *n*]

18. The system of equations 4*x*+6*y* = 5, 6*x* + 9*y* = 7 has

(a) a unique solution (b) no solution

(c) infinitely many solutions (d) none of these

[Ans: (b) no solution]

Hint:

Given
$$4x + 6y = 5$$
$$6x + 9y = 7$$

$$[A, B] = \begin{pmatrix} 4 & 6 & 5 \\ 6 & 9 & 7 \end{pmatrix}$$

$$\sim \begin{pmatrix} 1 & \frac{6}{4} & \frac{5}{4} \\ 6 & 9 & 7 \end{pmatrix} R_1 \to R_1 \div 4$$

$$\sim \begin{pmatrix} 1 & \frac{3}{2} & \frac{5}{4} \\ 0 & 0 & \frac{-1}{2} \end{pmatrix} R_2 \to R_2 - 6R_1$$

Here ρ(A) = 1 and ρ(A, B) = 2

Since ρ(A) ≠ ρ(A, B), the system has no solution

19. For the system of equations *x* + 2*y* + 3*z* = 1, 2*x* + *y* + 3*z* = 2 5*x* + 5*y* + 9*z* = 4

(a) there is only one solution

(b) there exists infinitely many solutions

(c) there is no solution (d) None of these

[Ans: (a) there is only one solution.]

Hint:

Given $x + 2y + 3z = 1$, $2x + y + 3z = 2$,

$5x + 5y + 9z = 4$

$$[A, B] = \begin{pmatrix} 1 & 2 & 3 & 1 \\ 2 & 1 & 3 & 2 \\ 5 & 5 & 9 & 4 \end{pmatrix}$$

$$\sim \begin{pmatrix} 1 & 2 & 3 & 1 \\ 0 & -3 & -3 & 0 \\ 0 & -5 & -6 & -1 \end{pmatrix} \begin{matrix} R_2 \to R_2 - 2R_1 \\ R_3 \to R_3 - 3R_1 \end{matrix}$$

$$\sim \begin{pmatrix} 1 & 2 & 3 & 1 \\ 0 & -1 & -1 & 0 \\ 0 & -5 & -6 & -1 \end{pmatrix} R_2 \to R_2 \div 5$$

$$\sim \begin{pmatrix} 1 & 2 & 3 & 1 \\ 0 & -1 & -1 & 0 \\ 0 & 0 & -1 & -1 \end{pmatrix} R_3 \to R_3 - 5R_2$$

Here $\rho(A) = 3$ and $\rho(A, B) = 3$

$\therefore$ $\rho(A) = \rho(A, B) = 3$ = Number of unknowns.

20. If $|A| \neq 0$, then A is

(a) non- singular matrix (b) singular matrix

(c) zero matrix (d) none of these

[Ans: (a) non- singular matrix]

21. The system of linear equations $x + y + z = 2$, $2x + y - z = 3$, $3x + 2y + k = 4$ has unique solution, if k is not equal to

(a) 4 (b) 0 (c) –4 (d) 1

[Ans: (b) 0]

Hint:

Given $x + y + z = 2$, $2x + y - z = 3$, $3x + 2y + k = 4$

$$[A, B] = \begin{pmatrix} 1 & 1 & 1 & 2 \\ 2 & 1 & -1 & 3 \\ 3 & 2 & k & 4 \end{pmatrix} \sim \begin{pmatrix} 1 & 1 & 1 & 2 \\ 0 & -1 & -3 & -1 \\ 0 & -1 & k-3 & -2 \end{pmatrix}$$

$R_2 \to R_2 - 2R_1$
$R_3 \to R_3 - 3R_1$

$$\sim \begin{pmatrix} 1 & 1 & 1 & 2 \\ 0 & -1 & -3 & -1 \\ 0 & 0 & k & -1 \end{pmatrix} R_3 \to R_3 - R_2$$

For unique solution $\rho(A) = \rho(A, B) = 3$

This can happen only when $k \neq 0$

22. Cramer's rule is applicable only to get an unique solution when

(a) $\Delta_z \neq 0$ (b) $\Delta_x \neq 0$

(c) $\Delta \neq 0$ (d) $\Delta_y \neq 0$

[Ans: (c) $\Delta \neq 0$]

23. If $\frac{a_1}{x} + \frac{b_1}{y} = c_1$, $\frac{a_2}{x} + \frac{b_2}{y} = c_2$, $\Delta_1 = \begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix}$;

$\Delta_2 = \begin{vmatrix} b_1 & c_1 \\ b_2 & c_2 \end{vmatrix}$; $\Delta_3 = \begin{vmatrix} c_1 & a_1 \\ c_2 & a_2 \end{vmatrix}$ then (x, y) is

(a) $\left(\frac{\Delta_2}{\Delta_1}, \frac{\Delta_3}{\Delta_1}\right)$ (b) $\left(\frac{\Delta_3}{\Delta_1}, \frac{\Delta_2}{\Delta_1}\right)$

(c) $\left(\frac{\Delta_1}{\Delta_2}, \frac{\Delta_1}{\Delta_3}\right)$ (d) $\left(\frac{-\Delta_1}{\Delta_2}, \frac{-\Delta_1}{\Delta_3}\right)$

[Ans: (d) $\left(\frac{-\Delta_1}{\Delta_2}, \frac{-\Delta_1}{\Delta_3}\right)$]

Hint:

Given $\frac{a_1}{x} + \frac{b_1}{y} = c_1$ $\frac{a_2}{x} + \frac{b_2}{y} = c_2$

and $\Delta_1 = \begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix}$; $\Delta_2 = \begin{vmatrix} b_1 & c_1 \\ b_2 & c_2 \end{vmatrix}$; $\Delta_3 = \begin{vmatrix} c_1 & a_1 \\ c_2 & a_2 \end{vmatrix}$

$\Rightarrow$ $\Delta_2 = -\begin{vmatrix} c_1 & b_1 \\ c_2 & b_2 \end{vmatrix}$ $[\therefore C_1 \leftrightarrow C_2]$

$\Rightarrow$ $-\Delta_2 = \begin{vmatrix} c_1 & b_1 \\ c_2 & b_2 \end{vmatrix}$

and $\Delta_3 = \begin{vmatrix} c_1 & a_1 \\ c_2 & a_2 \end{vmatrix} = -\begin{vmatrix} a_1 & c_1 \\ a_2 & c_2 \end{vmatrix}$

$C_1 \leftrightarrow C_2$

$\therefore$ $\frac{1}{x} = \frac{-\Delta_2}{\Delta_1}$ and $\frac{1}{y} = \frac{-\Delta_3}{+\Delta_1}$

$\Rightarrow$ $x = \frac{-\Delta_1}{\Delta_2}$ and $y = \frac{-\Delta_1}{\Delta_3}$

24. $|A_{n\times n}| = 3$ $|\text{adj}A| = 243$ then the value n is

(a) 4 (b) 5 (c) 6 (d) 7

[Ans: (c) 6]

Hint:

Given $|A_{n\times n}| = 3$, $|\text{adj } A| = 243$.

If A is a square matrix of order n, then

$$\begin{aligned} |\text{adj } A| &= |A|^{n-1} \\ 243 &= 3^{n-1} \\ 3^5 &= 3^{n-1} \\ 5 &= n-1 \\ n &= 5 + 1 = 6 \end{aligned}$$

25. Rank of a null matrix is

(a) 0 (b) –1 (c) ∞ (d) 1

[Ans: (a) 0]

Miscellaneous problems

1. Find the rank of the matrix $A = \begin{pmatrix} 1 & -3 & 4 & 7 \\ 9 & 1 & 2 & 0 \end{pmatrix}$

Solution:

$$\text{Given } A = \begin{pmatrix} 1 & -3 & 4 & 7 \\ 9 & 1 & 2 & 0 \end{pmatrix}$$

$$\sim \begin{pmatrix} 1 & -3 & 4 & 0 \\ 0 & 28 & -34 & -63 \end{pmatrix} \quad R_2 \to R_2 - 9R_1$$

$$\sim \begin{pmatrix} 1 & -3 & 4 & 0 \\ 0 & 0 & \frac{10}{3} & -63 \end{pmatrix} \quad R_2 \to R_2 + \frac{28}{3}.R_1$$

The last equivalent matrix is in echelon form and there are 2 non-zero rows.

$\therefore \quad \rho(A) = 2.$

2. Find the rank of the matrix $A = \begin{pmatrix} -2 & 1 & 3 & 4 \\ 0 & 1 & 1 & 2 \\ 1 & 3 & 4 & 7 \end{pmatrix}$

Solution:

$$\text{Given} \quad A = \begin{pmatrix} -2 & 1 & 3 & 4 \\ 0 & 1 & 1 & 2 \\ 1 & 3 & 4 & 7 \end{pmatrix}$$

$$\sim \begin{pmatrix} 1 & 3 & 4 & 7 \\ 0 & 1 & 1 & 2 \\ -2 & 1 & 3 & 4 \end{pmatrix} \quad R_1 \leftrightarrow R_3$$

$$\sim \begin{pmatrix} 1 & 3 & 4 & 7 \\ 0 & 1 & 1 & 2 \\ 0 & 7 & 11 & 18 \end{pmatrix} \quad R_3 \to R_3 + 2R_1$$

$$\sim \begin{pmatrix} 1 & 3 & 4 & 7 \\ 0 & 1 & 1 & 2 \\ 0 & 0 & 4 & 4 \end{pmatrix} \quad R_3 \to R_3 - 7R_2$$

The last equivalent matrix is in echelon form and there are 3 non - zero rows.

$\therefore \quad \rho(A) = 3.$

3. Find the rank of the matrix $A = \begin{pmatrix} 4 & 5 & 2 & 2 \\ 3 & 2 & 1 & 6 \\ 4 & 4 & 8 & 0 \end{pmatrix}$

Solution:

$$\text{Given} \quad A = \begin{pmatrix} 4 & 5 & 2 & 2 \\ 3 & 2 & 1 & 6 \\ 4 & 4 & 8 & 0 \end{pmatrix}$$

$$\sim \begin{pmatrix} 4 & 5 & 2 & 2 \\ 3 & 2 & 1 & 6 \\ 1 & 1 & 2 & 0 \end{pmatrix} \quad R_3 \to R_3 \div 4$$

$$\sim \begin{pmatrix} 1 & 1 & 2 & 0 \\ 3 & 2 & 1 & 6 \\ 4 & 5 & 2 & 2 \end{pmatrix} \quad R_1 \leftrightarrow R_3$$

$$\sim \begin{pmatrix} 1 & 1 & 2 & 0 \\ 0 & -1 & -5 & 6 \\ 0 & 1 & -6 & 2 \end{pmatrix} \quad \begin{matrix} R_2 \to R_2 - 3R_1 \\ R_3 \to R_3 - 4R_1 \end{matrix}$$

$$\sim \begin{pmatrix} 1 & 1 & 2 & 0 \\ 0 & -1 & -5 & 6 \\ 0 & 0 & -11 & 8 \end{pmatrix} \quad R_3 \to R_3 + R_2$$

The last equivalent matrix is in echelon form and there are 3 non-zero rows.

$\therefore \quad \rho(A) = 3.$

4. Examine the consistency of the system of equations:

$$\boldsymbol{x + y + z = 7, x + 2y + 3z = 18, y + 2z = 6}.$$

Solution:

Given non homogeneous equations are $x + y + z = 7$, $x + 2y + 3z = 18$, $y + 2z = 6$.

Augmented matrix [A, B]	Elementary Transformation
$\begin{pmatrix} 1 & 1 & 1 & 7 \\ 1 & 2 & 3 & 18 \\ 0 & 1 & 2 & 6 \end{pmatrix}$	
$\sim \begin{pmatrix} 1 & 1 & 1 & 7 \\ 0 & 1 & 2 & 11 \\ 0 & 1 & 2 & 6 \end{pmatrix}$	$R_2 \to R_2 - R_1$
$\sim \begin{pmatrix} 1 & 1 & 1 & 7 \\ 0 & 1 & 2 & 11 \\ 0 & 0 & 0 & -5 \end{pmatrix}$	$R_3 \to R_3 - R_2$

Here $\rho(A) = 2$ and $\rho(A, B) = 3$

Since $\rho(A) \neq \rho(A, B)$, the given system is inconsistent and has no solution.

5. Find k if the equations $2x + 3y - z = 5, 3x - y + 4z = 2, x + 7y - 6z = k$ are consistent.

Solution:

Given non-homogeneous equations are

$2x + 3y - z = 5, 3x - y + 4z = 2, x + 7y - 6z = k$

Augmented matrix [A, B]	Elementary Transformation
$\begin{pmatrix} 2 & 3 & -1 & 5 \\ 3 & -1 & 4 & 2 \\ 1 & 7 & -6 & k \end{pmatrix}$	
$\sim \begin{pmatrix} 1 & 7 & -6 & k \\ 3 & -1 & 4 & 2 \\ 2 & 3 & -1 & 5 \end{pmatrix}$	$R_1 \leftrightarrow R_3$
$\sim \begin{pmatrix} 1 & 7 & -6 & k \\ 0 & -22 & 22 & 2-3k \\ 0 & -11 & 11 & 5-2k \end{pmatrix}$	$R_2 \to R_2 - 3R_1$ $R_3 \to R_3 - 2R_1$
$\sim \begin{pmatrix} 1 & 7 & -6 & k \\ 0 & -22 & 22 & 2-3k \\ 0 & 0 & 0 & 2(5-2k)-(2-3k) \end{pmatrix}$	$R_3 \to 2R_3 - R_2$
$\sim \begin{pmatrix} 1 & 7 & -6 & k \\ 0 & -22 & 22 & 2-3k \\ 0 & 0 & 0 & 10-4k-2+3k \end{pmatrix}$	
$\sim \begin{pmatrix} 1 & 7 & -6 & k \\ 0 & -22 & 22 & 2-3k \\ 0 & 0 & 0 & 8-k \end{pmatrix}$	

Here $\rho(A) = 2$

Since the given system is consistent, $\rho(A, B)$ must be equal to 2.

This can happen only when

$8 - k = 0 \Rightarrow k = 8$

6. Find k if the equations $x + y + z = 1, 3x - y - z = 4, x + 5y + 5z = k$ are inconsistent.

Solution:

$x + y + z = 1, 3x - y - z = 4, x + 5y + 5z = k$

Augmented matrix [A, B]	Elementary Transformation
$\begin{pmatrix} 1 & 1 & 1 & 1 \\ 3 & -1 & -1 & 4 \\ 1 & 5 & 5 & k \end{pmatrix}$	
$\sim \begin{pmatrix} 1 & 1 & 1 & 1 \\ 0 & -4 & -4 & 1 \\ 0 & 4 & 4 & k-1 \end{pmatrix}$	$R_2 \to R_2 - 3R_1$ $R_3 \to R_3 - R_1$
$\sim \begin{pmatrix} 1 & 1 & 1 & 1 \\ 0 & -4 & -4 & 1 \\ 0 & 0 & 0 & k \end{pmatrix}$	$R_3 \to R_3 + R_2$

Here clearly $\rho(A) = 2$.

Since the given system is inconsistent, $\rho(A) \neq \rho(A, B)$

This can happen only when $k \neq 0$.

$\therefore$ k can take any value other than zero.

7. Solve the equations $x + 2y + z = 7, 2x - y + 2z = 4, x + y - 2z = -1$ by using Cramer's rule.

Solution:

$$\Delta = \begin{vmatrix} 1 & 2 & 1 \\ 2 & -1 & 2 \\ 1 & 1 & -2 \end{vmatrix}$$

$$= 1\begin{vmatrix} -1 & 2 \\ 1 & -2 \end{vmatrix} - 2\begin{vmatrix} 2 & 2 \\ 1 & -2 \end{vmatrix} + 1\begin{vmatrix} 2 & -1 \\ 1 & 1 \end{vmatrix}$$

$= 1(2 -2) - 2(-4 -2) + 1(2 + 1)$

$= 1 (0) - 2 (-6) + 1(3)$

$= 12 + 3 = 15 \neq 0.$

Since $\Delta \neq 0$, Cramer's rule can be applied and the system is consistent with unique solution.

$$\Delta x = \begin{vmatrix} 7 & 2 & 1 \\ 4 & -1 & 2 \\ -1 & 1 & -2 \end{vmatrix}$$

$$= 7\begin{vmatrix} -1 & 2 \\ 1 & -2 \end{vmatrix} - 2\begin{vmatrix} 4 & 2 \\ -1 & -2 \end{vmatrix} + 1\begin{vmatrix} 4 & -1 \\ -1 & 1 \end{vmatrix}$$

$= 7 (2 -2) -2 (-8 + 2) + 1 (4 - 1)$

$= 7 (0) - 2(-6) + 1(3)$

$= 12 + 3 = 15.$

$$\Delta y = \begin{vmatrix} 1 & 7 & 1 \\ 2 & 4 & 2 \\ 1 & -1 & -2 \end{vmatrix}$$

$$= 1\begin{vmatrix} 4 & 2 \\ -1 & -2 \end{vmatrix} - 7\begin{vmatrix} 2 & 2 \\ 1 & -2 \end{vmatrix} + 1\begin{vmatrix} 2 & 4 \\ 1 & -1 \end{vmatrix}$$

$= 1 (- 8 + 2) -7(-4 -2) + 1(-2 -4)$

$= 1\,(-6) - 7\,(-6) + 1\,(-6)$

$= -6 + 42 - 6 = 30$

$$\Delta z = \begin{vmatrix} 1 & 2 & 7 \\ 2 & -1 & 4 \\ 1 & 1 & -1 \end{vmatrix}$$

$$= 1\begin{vmatrix} -1 & 4 \\ 1 & -1 \end{vmatrix} - 2\begin{vmatrix} 2 & 4 \\ 1 & -1 \end{vmatrix} + 7\begin{vmatrix} 2 & -1 \\ 1 & 1 \end{vmatrix}$$

$= 1\,(1 - 4) - 2(-2 - 4) + 7(2 + 1)$

$= 1\,(-3) - 2\,(-6) + 7\,(3)$

$= -3 + 12 + 21 = 30$

$$\therefore \quad x = \frac{\Delta x}{\Delta} = \frac{\cancel{15}}{\cancel{15}} = 1$$

$$y = \frac{\Delta y}{\Delta} = \frac{\overset{2}{\cancel{30}}}{\cancel{15}} = 2$$

$$z = \frac{\Delta z}{\Delta} = \frac{\overset{2}{\cancel{30}}}{\cancel{15}} = 2$$

$\therefore$ Solution set is {1, 2, 2}

8. The cost of 2kg. of wheat and 1kg. of sugar is ₹100. The cost of 1kg. of wheat and 1kg. of rice is ₹80. The cost of 3kg. of wheat, 2kg. of sugar and 1kg of rice is ₹220. Find the cost of each per kg., Using Cramer's rule.

Solution:

Let the cost of 1kg of wheat be Rs. x, 1kg of sugar be Rs. y and 1kg of rice be Rs. z.

By the given data,

$$2x + y = 100$$
$$x + z = 80$$
$$3x + 2y + z = 220$$

$$\Delta = \begin{vmatrix} 2 & 1 & 0 \\ 1 & 0 & 1 \\ 3 & 2 & 1 \end{vmatrix} = 2\begin{vmatrix} 0 & 1 \\ 2 & 1 \end{vmatrix} - 1\begin{vmatrix} 1 & 1 \\ 3 & 1 \end{vmatrix} + 0$$

$= 2\,(0 - 2) - 1\,(1 - 3) + 0$

$= 2(-2) - 1(-2)$

$= -4 + 2 = -2$

$$\Delta x = \begin{vmatrix} 100 & 1 & 0 \\ 80 & 0 & 1 \\ 220 & 2 & 1 \end{vmatrix} = 100\begin{vmatrix} 0 & 1 \\ 2 & 1 \end{vmatrix} - 1\begin{vmatrix} 80 & 1 \\ 220 & 1 \end{vmatrix} + 0$$

$= 100(0 - 2) - 1\,(80 - 220)$

$= 100(-2) - 1(-140)$

$= -200 + 140 = -60.$

$$\Delta y = \begin{vmatrix} 2 & 100 & 0 \\ 1 & 80 & 1 \\ 3 & 220 & 1 \end{vmatrix} = 2\begin{vmatrix} 80 & 1 \\ 220 & 1 \end{vmatrix} - 100\begin{vmatrix} 1 & 1 \\ 3 & 1 \end{vmatrix} + 0$$

$= 2\,(80 - 220) - 100\,(1 - 3)$

$= 2\,(-140) - 100\,(-2)$

$= -280 + 200 = -80.$

$$\Delta z = \begin{vmatrix} 2 & 1 & 100 \\ 1 & 0 & 80 \\ 3 & 2 & 220 \end{vmatrix}$$

$$= 2\begin{vmatrix} 0 & 80 \\ 2 & 220 \end{vmatrix} - 1\begin{vmatrix} 1 & 80 \\ 3 & 220 \end{vmatrix} + 100\begin{vmatrix} 1 & 0 \\ 3 & 2 \end{vmatrix}$$

$= 2(0 - 160) - 1(220 - 240) + 100(2 - 0)$

$= 2(-160) - 1(-20) + 100(2)$

$= -320 + 20 + 200$

$= -100$

$$x = \frac{\Delta x}{\Delta} = \frac{-60}{-2} = 30$$

$$y = \frac{\Delta y}{\Delta} = \frac{-80}{-2} = 40$$

$$z = \frac{\Delta z}{\Delta} = \frac{-100}{-2} = 50$$

$\therefore$ The cost of 1kg of wheat is ₹. 30

The cost of 1kg sugar is ₹. 40 and

The cost of 1 kg of rice is ₹. 50.

9. A salesman has the following record of sales during three months for three items A,B and C, which have different rates of commission.

Months	Sales of units			Total commission drawn (in ₹)
	A	B	C	
January	90	100	20	800
February	130	50	40	900
March	60	100	30	850

Find out the rate of commission on the items A,B and C by using Cramer's rule.

Solution:

Let the rate of commission on the items A, B and C be x, y and z respectively.

By the given data, the non-homogeneous equations are

$$90x + 100y + 20z = 800$$
$$\Rightarrow 9x + 10y + 2z = 80$$
$$130x + 50y + 40z = 900$$
$$\Rightarrow 13x + 5y + 4z = 90$$
$$60x + 100y + 30z = 850$$
$$\Rightarrow 6x + 10y + 3z = 85$$

$$\Delta = \begin{vmatrix} 9 & 10 & 2 \\ 13 & 5 & 4 \\ 6 & 10 & 3 \end{vmatrix}$$

$$= 9\begin{vmatrix} 5 & 4 \\ 10 & 3 \end{vmatrix} - 10\begin{vmatrix} 13 & 4 \\ 6 & 3 \end{vmatrix} + 2\begin{vmatrix} 13 & 5 \\ 6 & 10 \end{vmatrix}$$

$$= 9(15 - 40) - 10(39 - 24) + 2(130 - 30)$$
$$= 9(-25) - 10(15) + 2(100)$$
$$= -225 - 150 + 200$$
$$= -175$$

Since $\Delta \neq 0$, Cramer's rule can be applied and the system is consistent with unique solution.

$$\Delta x = \begin{vmatrix} 80 & 10 & 2 \\ 90 & 5 & 4 \\ 85 & 10 & 3 \end{vmatrix}$$

$$= 80\begin{vmatrix} 5 & 4 \\ 10 & 3 \end{vmatrix} - 10\begin{vmatrix} 90 & 4 \\ 85 & 3 \end{vmatrix} + 2\begin{vmatrix} 90 & 5 \\ 85 & 10 \end{vmatrix}$$

$$= 80(15 - 40) - 10(270 - 340) + 2(900 - 425)$$
$$= 80(-25) - 10(-70) + 2(475)$$
$$= -2000 + 700 + 950$$
$$= -350$$

$$\Delta y = \begin{vmatrix} 9 & 80 & 2 \\ 13 & 90 & 4 \\ 6 & 85 & 3 \end{vmatrix}$$

$$= 9\begin{vmatrix} 90 & 4 \\ 85 & 3 \end{vmatrix} - 80\begin{vmatrix} 13 & 4 \\ 6 & 3 \end{vmatrix} + 2\begin{vmatrix} 13 & 90 \\ 6 & 85 \end{vmatrix}$$

$$= 9(270 - 340) - 80(39 - 24) + 2(1105 - 540)$$
$$= 9(-70) - 80(15) + 2(565)$$
$$= -630 - 1200 + 1130$$
$$= -700$$

$$\Delta z = \begin{vmatrix} 9 & 10 & 80 \\ 13 & 5 & 90 \\ 6 & 10 & 85 \end{vmatrix}$$

$$= 9\begin{vmatrix} 5 & 90 \\ 10 & 85 \end{vmatrix} - 10\begin{vmatrix} 13 & 90 \\ 6 & 85 \end{vmatrix} + 80\begin{vmatrix} 13 & 5 \\ 6 & 10 \end{vmatrix}$$

$$= 9(425 - 900) - 10(1105 - 540) + 80(130 - 30)$$
$$= 9(-475) - 10(565) + 80(100)$$
$$= -4275 - 5650 + 8000$$
$$= -1925$$

$$x = \frac{\Delta x}{\Delta} = \frac{\overset{2}{\cancel{-350}}}{\cancel{-175}} = 2$$

$$y = \frac{\Delta y}{\Delta} = \frac{\overset{4}{\cancel{-700}}}{\cancel{-175}} = 4$$

$$z = \frac{\Delta z}{\Delta} = \frac{\overset{11}{\cancel{-1925}}}{\cancel{-175}} = 11$$

$\therefore$ The rate of commission on the items A, B and C are 2%, 4% and 11%

10. The subscription department of a magazine sends out a letter to a large mailing list inviting subscriptions for the magazine. Some of the people receiving this letter already subscribe to the magazine while others do not. From this mailing list, 60% of those who already subscribe will subscribe again while 25% of those who do not now subscribe will subscribe. On the last letter it was found that 40% of those receiving it ordered a subscription. What percent of those receiving the current letter can be expected to order a subscription?

Solution:

Let A represents the percent of people who subscribe the magazine and B represents the percent of people who do not subscribe the magazine.

Given 60% of people subscribe again implies 40% of people do not subscribe. And 25% of people are going to subscribe implies 75% of people are not going to subscribe.

∴ Transition probability matrix

$$T = \begin{array}{c} \\ A \\ B \end{array}\begin{array}{c} \begin{matrix} A & B \end{matrix} \\ \begin{pmatrix} \cdot 6 & \cdot 4 \\ \cdot 025 & \cdot 75 \end{pmatrix} \end{array}$$

Also, it is given that 40 % of those received the order of subscription implies 60% are not going to receive the order.

$$\therefore (\cdot 4 \quad \cdot 6)\begin{pmatrix} \cdot 6 & \cdot 4 \\ \cdot 25 & \cdot 75 \end{pmatrix}$$

$$= ((\cdot 4)(\cdot 6) + (\cdot 6)(\cdot 25) \quad (\cdot 4)(\cdot 4) + (\cdot 6)(\cdot 75))$$

$$= (\cdot 24 + \cdot 15 \quad \cdot 16 + \cdot 45) = (\cdot 39 \quad \cdot 61)$$

∴ 39% of people who received the current letter can be expected to order a subscription.

Practice Problems

I. Choose the correct answer :

1. **If the minor of a_{23} = the co-factor of a_{23} in $|a_{ij}|$ then the minor of a_{23} is.**

(a) 1 (b) 2 (c) 0 (d) 3

[Ans: (c) 0]

2. **If AB = BA = |A| I then the matrix B is the.**

(a) inverse of A (b) Transpose of A
(c) Adjoint of A (d) 2A

[Ans: (c) Adjoint of A]

3. **If A is a square matrix of order 3, then |adj A| is.**

(a) $|A|^2$ (b) $|A|$ (c) $|A|^3$ (d) $|A|^4$

[Ans: (a) $|A|^2$]

4. **If |A| = 0, then |adj A| is.**

(a) 0 (b) 1 (c) –1 (d) ± 1

[Ans: (a) 0]

5. **For what value of k, the matrix A = $\begin{pmatrix} 2 & k \\ 3 & 5 \end{pmatrix}$ has no inverse?**

(a) $\frac{3}{10}$ (b) $\frac{10}{3}$ (c) 3 (d) 10

[Ans: (b) $\frac{10}{3}$]

6. **The rank of an $n \times n$ matrix each of whose elements is 2 is**

(a) 1 (b) 2 (c) n (d) n^2

[Ans: (a) 1]

7. **The value of $\begin{vmatrix} 5^2 & 5^3 & 5^4 \\ 5^3 & 5^4 & 5^5 \\ 5^4 & 5^5 & 5^6 \end{vmatrix}$ is.**

(a) 5^2 (b) 0 (c) 5^{13} (d) 5^9

[Ans: (b) 0]

8. **If $\begin{vmatrix} 2x & 5 \\ 8 & x \end{vmatrix} = \begin{vmatrix} 6 & -2 \\ 7 & 3 \end{vmatrix}$ then x = .**

(a) 3 (b) ± 3 (c) ± 6 (d) 6

[Ans: (c) ± 6]

9. **If A is a singular matrix, then Adj A is.**

(a) non-singular (b) singular
(c) symmetric (d) not defined

[Ans: (b) Singular]

10. **If A, B are two $n \times n$ non-singular matrices, then.**

(a) AB is non-singular (b) AB is singular
(c) $(AB)^{-1} = A^{-1} B^{-1}$
(d) $(AB)^{-1}$ does not exit

[Ans: (a) AB is non-singular]

II. Fill in the blanks :

1. **If A = $\begin{bmatrix} a & 0 & 0 \\ 0 & a & 0 \\ 0 & 0 & a \end{bmatrix}$ then the value of |adj A| is ______**

[Ans: a^6]

2. **For any 2 × 2 matrix, if A (adj A) = $\begin{vmatrix} 10 & 0 \\ 0 & 10 \end{vmatrix}$ then |A| is ________.**

[Ans: 10]

3. **If A is a square matrix of order n, then |Adj A| = ______.** [Ans: $|A|^{n-1}$]

4. **If A is a matrix of order 3 and |A| = 8 then |adj A| = _______.** [Ans: 64]

5. **If A is a square matrix such that A^2 = I, then A^{-1} =________.** [Ans: A]

6. **The system of equation $x + y + z = 2$, $3x - y + 2z = 6$ and $3x + y - z = -18$ has ________ solution**

[Ans: unique]

7. The number of solutions of the system of equations $2x + y - z = 7$, $x - 3y + 2z = 1$, $x + 4y - 3z = 5$ is ____.

[Ans: 0]

8. The system of linear equations $x + y + z = 2$, $2x + y - z = 3$, $3x + 2y + kz = 4$ has a unique solution if k is________.

[Ans: Not equal to 0]

9. The value of λ for which the system of equations $x + y + z = 5$, $x + 2y + 3z = 9$, $x + 3y + \lambda z = \mu$ is ______.

[Ans: $\lambda \neq 5$]

10. A set of values of the variable $x_1, x_2, \ldots x_n$ satisfying all the equations simultaneously is called _______ of the system

[Ans: Solution]

III. Match the following :

1.	Rank of a matrix	i.	1
2.	If A is a matrix of order $m \times n$, then $\rho(A) \leq$	ii.	non-zero row
3.	Rank of a zero matrix is	iii.	n
4.	Rank of a non-singular matrix of order $n \times n$ is	iv.	unique solution
5.	If A is of rank 2, then adj A is of rank	v.	inconsistent
6.	A row having at least one non-zero element is	vi.	infinitely many solutions
7.	For the system of equations AX = B, the solution is $X = A^{-1}B$ provided	vii.	$\leq \min\{m, n\}$
8.	If $\rho(A, B) = \rho(A) < n$ then the system has	viii.	0
9.	If $\rho(A, B) = \rho(A) = n$, then the system has	xi.	$\|A\| \neq 0$
10.	If $\rho(A, B) \neq \rho(A)$ then the system is	x.	≥ 0

[Ans: 1 - x, 2 - vii, 3 - viii, 4 - iii, 5 - i, 6 - ii, 7 - ix, 8 - vi, 9 - iv, 10 - v]

IV. Choose the odd one out:

1. The system of non-homogeneous equations will have.

(a) unique solution

(b) Infinitely many solutions

(c) No solution (d) Trivial solution

[Ans: (d) Trivial solution]

2. Rank of a 2×2 matrix may be

(a) 0 (b) 1 (c) 2 (d) 3

[Ans: (d) 3]

3. The transition probabilities P_{jk} satisfy

(a) $P_{jk} > 0$

(b) $\sum_{k}^{1} P_{jk} = 1$ for all j

(c) $P_{jk} \leq 0$ (d) $P_{jk} > 1$

[Ans: (c) $P_{jk} \leq 0$]

4. Which is one correct?

(a) $R_1 \to R_1 + R_2$, (b) $C_1 \to C_1 - 2C_2$,

(c) $R_3 \leftrightarrow R_1$, (d) $R_1 \leftrightarrow C_1$

[Ans: (d) $R_1 \leftrightarrow C_1$]

5. If $|A| = 0$, then

(a) A is a singular matrix

(b) System has either no solution or infinitely many solutions

(c) No solution

(d) non-singular matrix

[Ans: (d) non-singular matrix]

V. Which is the following is not correct in the given statement:

1.

(a) $|A| = |A^T|$ where $A = [a_{ij}]_{3\times 3}$

(b) $|KA| = K^3|A|$ where $A = [a_{ij}]_{3\times 3}$

(c) If is a non-singular matrix, then $|A| \neq 0$

(d) $\begin{vmatrix} a+b & c+d \\ e+f & g+h \end{vmatrix} = \begin{vmatrix} a & c \\ e & g \end{vmatrix} + \begin{vmatrix} b & d \\ f & h \end{vmatrix}$

[Ans: (d) $\begin{vmatrix} a+b & c+d \\ e+f & g+h \end{vmatrix} = \begin{vmatrix} a & c \\ e & g \end{vmatrix} + \begin{vmatrix} b & d \\ f & h \end{vmatrix}$]

2. For the matrix A = $[a_{ij}]_{3\times3}$

(a) Order of minor is less than the order of |A|.

(b) Minor of an element can never be equal to co-factor of the same element.

(c) Value of a determinant is obtained by multiplying elements of a row or column by corresponding factors

(d) Order of minor and co-factors of elements of A is same.

[Ans: (b) Minor of an element can never be equal to co-factor of the same element.]

3. If A is an invertible matrix, then which of the following is not true?

(a) $(A^2)^{-1} = (A^{-1})^2$ (b) $|A^{-1}| = |A|^{-1}$

(c) $(A^T)^{-1} = (A^{-1})^T$ (d) $|A| \neq 0$

[Ans: (a) $(A^2)^{-1} = (A^{-1})^2$]

4.

(a) If three planes intersect at a point, then the system has unique solution.

(b) if three planes intersect along a line then the system has infinitely many solutions lying on this line.

(c) If two planes intersect at a point then the system has unique solution.

(d) If three planes are parallel and distinct and there is no point in common, then the system has no solutions

[Ans: (c) If two planes intersect at a point then the system has unique solution.]

5. Solution of the system of equations $x + 2y = 7$ and $3x + 6y = 21$ is

(a) $x = 5, y = 1$ (b) $x = 3, y = 2$

(c) $x = 0, y = 1$ (d) $x = -3, y = 5$

[Ans: (c) $x = 0, y = 1$]

Additional Question

2 Mark Questions:

1. Find the rank of the matrix $\begin{bmatrix} 7 & -1 \\ 2 & 1 \end{bmatrix}$

Solution:

$$\text{Let } A = \begin{bmatrix} 7 & -1 \\ 2 & 1 \end{bmatrix}$$

The order of A is 2×2

$$\rho(A) \leq \min(2, 2)$$

$$\Rightarrow \quad \rho(A) \leq 2.$$

$$\begin{vmatrix} 7 & -1 \\ 2 & 1 \end{vmatrix} = 7 - (-2) = 7 + 2 = 9 \neq 0$$

The highest order of non-vanishing minor of A is 2

$$\therefore \quad \rho(A) = 2$$

2. Find the rank of the matrix $\begin{pmatrix} 2 & -4 \\ -1 & 2 \end{pmatrix}$

Solution:

$$\text{Let } A = \begin{pmatrix} 2 & -4 \\ -1 & 2 \end{pmatrix}$$

The order of A is 2×2

$$\rho(A) \leq \min(2, 2)$$

$$\Rightarrow \quad \rho(A) \leq 2.$$

$$\begin{vmatrix} 2 & -4 \\ -1 & 2 \end{vmatrix} = 4 - 4 = 0.$$

Since the second order minor vanishes, $\rho(A) \neq 2$.

We have to try for atleast one non-zero first order minor.

ie. atleast one non-zero element of A.

This is possible because A has non-zero element.

$$\therefore \quad \rho(A) = 1.$$

3. Solve $x + 2y = 3$ and $x + y = 2$ using Cramer's rule.

Solution:

$$\Delta = \begin{vmatrix} 1 & 2 \\ 1 & 1 \end{vmatrix} = (1)(1) - (1)(2) = 1 - 2 = -1$$

Since $\Delta \neq 0$, Cramer's rule can be applied and the system is consistent with unique solution.

$$\Delta x = \begin{vmatrix} 3 & 2 \\ 2 & 1 \end{vmatrix} = 3 - 4 = -1$$

$$\Delta y = \begin{vmatrix} 1 & 3 \\ 1 & 2 \end{vmatrix} = 2 - 3 = -1$$

$$x = \frac{\Delta x}{\Delta} = \frac{-1}{-1} = 1$$

$$y = \frac{\Delta y}{\Delta} = \frac{-1}{-1} = 1$$

$\therefore$ solution set is {1, 1}

4. Solve: $x + 2y = 3$ and $2x + 4y = 6$ using rank method.

Solution:

The non-homogeneous equations are

$x + 2y = 3, \quad 2x + 4y = 6$

Augmented matrix [A, B]	Elementary Transformation
$\begin{pmatrix} 1 & 2 & 3 \\ 2 & 4 & 6 \end{pmatrix}$	
$\sim \begin{pmatrix} 1 & 2 & 3 \\ 0 & 0 & 0 \end{pmatrix}$	$R_2 \to R_2 - 2R_1$

Here $\rho(A) = 1$ and $\rho([A, B]) = 1$

Since $\rho(A) = \rho([A, B]) = 1 <$ Number of unknowns, the given system is consistent with infinitely many solutions.

To find the solution, let us rewrite the above echelon form into the matrix form, we get

$$\begin{pmatrix} 1 & 2 \\ 0 & 0 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 3 \\ 0 \end{pmatrix}$$

$$\Rightarrow \quad x + 2y = 3 \quad \text{----- (1)}$$

$$\text{let } y = k, \quad k \in R$$

$$(1) \Rightarrow \quad x + 2k = 3 \quad \Rightarrow \quad x = 3 - 2k$$

$\therefore$ Solution set is $\{3 - 2k, k\}, k \in R$.

For different values of k, we get infinite number of solutions.

5. Show that the equations $x + y + z = 6$, $x + 2y + 3z = 14$ and $x + 4y + 7z = 30$ are consistent.

Solution:

Given non-homogeneous equations are

$x + y + z = 6, x + 2y + 3z = 14, x + 4y + 7z = 30.$

Augmented matrix	Elementary Transformation
$\begin{pmatrix} 1 & 2 & 1 & 6 \\ 1 & 2 & 3 & 14 \\ 1 & 4 & 7 & 30 \end{pmatrix}$	
$\sim \begin{pmatrix} 1 & 1 & 1 & 6 \\ 0 & 1 & 2 & 8 \\ 0 & 3 & 6 & 24 \end{pmatrix}$	$R_2 \to R_2 - R_1$ $R_3 \to R_3 - R_1$
$\sim \begin{pmatrix} 1 & 1 & 1 & 6 \\ 0 & 1 & 2 & 8 \\ 0 & 0 & 0 & 0 \end{pmatrix}$	$R_3 \to R_3 - 3R_2$

Here $\rho(A) = 2$ and $\rho(A, B) = 2$

$\therefore \quad \rho(A) = \rho(A, B) = 2 <$ Number of unknowns.

$\therefore \quad$ The given system is consistent.

6. Solve : $2x + 3y = 4$ and $4x + 6y = 8$ using Cramer's rule.

Solution:

$$\Delta = \begin{vmatrix} 2 & 3 \\ 4 & 6 \end{vmatrix} = 12 - 12 = 0$$

$$\Delta x = \begin{vmatrix} 4 & 3 \\ 8 & 6 \end{vmatrix} = 24 - 24 = 0$$

$$\Delta y = \begin{vmatrix} 2 & 4 \\ 4 & 8 \end{vmatrix} = 16 - 16 = 0$$

$$\therefore \quad \Delta = \Delta x = \Delta y = 0$$

$\therefore$ The system is consistent with infinite number of solutions.

$$\text{let } y = k, \quad k \in R$$

$$\therefore 2x + 3k = 4 \quad \Rightarrow 2x = 4 - 3k$$

$$\Rightarrow x = \frac{1}{2}(4 - 3k), k \in R$$

$\therefore$ Solution set is $\left\{\frac{4 - 3k}{2}, K\right\}, k \in R.$

7. If A and B are non-singular matrices, prove that AB is non-singular.

Solution:

Since A and B are non-singular,

$$|A| \neq 0, \qquad |B| \neq 0$$

$$\text{Consider } |AB| = |A| \cdot |B|$$

$\neq 0$ since $|A| \neq 0$ and $|B| \neq 0$.

$$\Rightarrow \quad |AB| \neq 0$$

$\therefore$ AB is non-singular.

8. For what value of x, the matrix

$$A = \begin{vmatrix} 1 & -2 & 3 \\ 1 & 2 & 1 \\ x & 2 & -3 \end{vmatrix} \text{ is singular?}$$

Solution:

The matrix A is singular, if

$$\begin{vmatrix} 1 & -2 & 3 \\ 1 & 2 & 1 \\ x & 2 & -3 \end{vmatrix} = 0$$

$$1\begin{vmatrix} 2 & 1 \\ 2 & -3 \end{vmatrix} + 2\begin{vmatrix} 1 & 1 \\ x & -3 \end{vmatrix} + 3\begin{vmatrix} 1 & 2 \\ x & 2 \end{vmatrix} = 0$$

$\Rightarrow 1\,(-6-2) + 2\,(-3-x) + 3\,(2-2x) = 0$

$\Rightarrow 1\,(-8) - 6 - 2x + 6 - 6x = 0$

$\Rightarrow \quad -8 - 2x - 6x = 0$

$\Rightarrow \quad -8 - 8x = 0$

$\Rightarrow \quad -8 = 8x$

$\Rightarrow \quad x = \dfrac{-8}{8} = -1$

9. If $\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 1 \\ -1 \\ 0 \end{pmatrix}$ **find x, y and z**

Solution:

Given $\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 1 \\ -1 \\ 0 \end{pmatrix}$

$$\Rightarrow \begin{pmatrix} x+0+0 \\ 0+y+0 \\ 0+0+z \end{pmatrix} = \begin{pmatrix} 1 \\ -1 \\ 0 \end{pmatrix}$$

$$\Rightarrow \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 1 \\ -1 \\ 0 \end{pmatrix}$$

$\Rightarrow \quad x = 1, \quad y = -1, \quad z = 0$

10. Two newspapers A and B are published in a city .Their market shares are 15% for A and 85% for B of those who bought A the previous year, 65% continue to buy it again while 35% switch over to B. Of those who bought B the previous year, 55% buy it again and 45% switch over to A. Find their market shares after one year.

Solution:

Transition probability matrix

$$T = \begin{array}{c c} & \begin{array}{cc} A & B \end{array} \\ \begin{array}{c} A \\ B \end{array} & \begin{pmatrix} \cdot 65 & \cdot 35 \\ \cdot 45 & \cdot 55 \end{pmatrix} \end{array}$$

Given present market shares are 15% for A and 85% for B

$\therefore$ Market shares after one year

$= \begin{pmatrix} \cdot 15 & \cdot 85 \end{pmatrix}\begin{pmatrix} \cdot 65 & \cdot 35 \\ \cdot 45 & \cdot 55 \end{pmatrix}$

$= \begin{pmatrix} (\cdot 15)(\cdot 65)+(\cdot 85)(\cdot 45) & \cdot 15\times\cdot 35+\cdot 85\times\cdot 55 \end{pmatrix}$

$= \begin{pmatrix} \cdot 0975+0\cdot 3825 & \cdot 0525+\cdot 4675 \end{pmatrix}$

$= \begin{pmatrix} 0\cdot 48 & 0\cdot 52 \end{pmatrix}$

$\therefore$ Market shares after one year for A is 48% and for B is 52%

3 Mark questions:

1. Find the rank of the matrix

$$A = \begin{pmatrix} 2 & 4 & 5 \\ 4 & 8 & 10 \\ -6 & -12 & -15 \end{pmatrix}$$

Solution:

The order of A is 3×3

$\therefore \quad \rho(A) \le \min(3, 3)$

$\Rightarrow \quad \rho(A) \le 3$

Matrix	Elementary Transformation
$\begin{pmatrix} 2 & 4 & 5 \\ 4 & 8 & 10 \\ -6 & -12 & -15 \end{pmatrix}$	
$\sim \begin{pmatrix} 1 & 1 & 1 \\ 2 & 2 & 2 \\ -3 & -3 & -3 \end{pmatrix}$	$C_1 \to C_1 \div 2$ $C_2 \to C_2 \div 4$ and $C_3 \to C_3 \div 5$
$\sim \begin{pmatrix} 1 & 1 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$	$R_2 \to R_2 - 2R_1$ $R_3 \to R_3 + 3R_1$

The last equivalent matrix is in echelon form and it has one non-zero row.

$\therefore \quad \rho(A) = 1$

2. Find the rank of the matrix $A = \begin{pmatrix} 1 & 2 & -4 & 5 \\ 2 & -1 & 3 & 6 \\ 8 & 1 & 9 & 7 \end{pmatrix}$

Solution:

The order of A is 3×4

$\therefore \quad \rho(A) \le \min(3, 4)$

$\rho(A) \le 3$

Consider the third order minor,

$$\begin{vmatrix} 1 & 2 & -4 \\ 2 & -1 & 3 \\ 8 & 1 & 9 \end{vmatrix} = 1\begin{vmatrix} -1 & 3 \\ 1 & 9 \end{vmatrix} - 2\begin{vmatrix} 2 & 3 \\ 8 & 9 \end{vmatrix} - 4\begin{vmatrix} 2 & -1 \\ 8 & 1 \end{vmatrix}$$

$= 1(-9-3) - 2(18-24) - 4(2+8)$

$= 1(-12) - 2(-6) - 4(10)$

$= -12 + 12 - 40$

$= -40 \neq 0.$

There is a minor of order 3, which is not zero

$\therefore \quad \rho(A) = 3.$

3. Show that the equations $2x - y + z = 7$, $3x + y - 5z = 13$, $x + y + z = 5$ are consistent and have a unique solution.

Solution:

The non-homogeneous equation are

$2x - y + z = 7,\ 3x + y - 5z = 13,\ x + y + z = 5$

Augmented matrix [A, B]	Elementary Transformation
$\begin{pmatrix} 2 & -1 & 1 & 7 \\ 3 & 1 & -5 & 13 \\ 1 & 1 & 1 & 5 \end{pmatrix}$	
$\sim \begin{pmatrix} 1 & 1 & 1 & 5 \\ 3 & 1 & -5 & 13 \\ 2 & -1 & 1 & 7 \end{pmatrix}$	$R_1 \leftrightarrow R_3$
$\sim \begin{pmatrix} 1 & 1 & 1 & 5 \\ 0 & -2 & -8 & -2 \\ 0 & -3 & -1 & -3 \end{pmatrix}$	$R_2 \to R_2 - 3R_1$ $R_3 \to R_3 - 2R_1$
$\sim \begin{pmatrix} 1 & 1 & 1 & 5 \\ 0 & -2 & -8 & -2 \\ 0 & 0 & 11 & 0 \end{pmatrix}$	$R_3 \to R_3 - \frac{3}{2} R_2$

Clearly $\rho(A) = 3$ and $\rho(A, B) = 3$

$\Rightarrow \rho(A) = \rho(A, B) = 3 =$ Number of unknowns

$\therefore$ The given system is consistent and has unique solution.

4. Show that the equations $x + 2y = 3$, $y - z = 2$, $x + y + z = 1$ are consistent and have infinite sets of solution.

Solution:

Given non-homogeneous equations are

$x + 2y = 3, \qquad y - z = 2, \qquad x + y + z = 1$

Augmented matrix [A, B]	Elementary Transformation
$\begin{pmatrix} 1 & 2 & 0 & 3 \\ 0 & 1 & -1 & 2 \\ 1 & 1 & 1 & 1 \end{pmatrix}$	
$\sim \begin{pmatrix} 1 & 2 & 0 & 3 \\ 0 & 1 & -1 & 2 \\ 0 & -1 & 1 & -2 \end{pmatrix}$	$R_3 \to R_3 + R_1$
$\sim \begin{pmatrix} 1 & 2 & 0 & 3 \\ 0 & 1 & -1 & 2 \\ 0 & 0 & 0 & 0 \end{pmatrix}$	$R_3 \to R_3 + R_2$

Obviously, $\rho(A) = 2$ and $\rho(A, B) = 2$

Hence $\rho(A) = \rho(A, B) = 2 <$ Number of unknowns.

$\therefore$ The system is consistent and has infinite number of solutions.

5. Show that the equations $x - 3y + 4z = 3$, $2x - 5y + 7z = 6$, $3x - 8y + 11z = 1$ are inconsistent.

Solution:

Given non-homogeneous equations are

$x - 3y + 4z = 3,\ 2x - 5y + 7z = 6,\ 3x - 8y + 11z = 1$

Augmented matrix [A, B]	Elementary Transformation
$\begin{pmatrix} 1 & -3 & 4 & 3 \\ 2 & -5 & 7 & 6 \\ 3 & -8 & 11 & 1 \end{pmatrix}$	
$\sim \begin{pmatrix} 1 & -3 & 4 & 3 \\ 0 & 1 & -1 & 0 \\ 0 & 1 & -1 & -8 \end{pmatrix}$	$R_2 \to R_2 - 2R_1$ $R_3 \to R_3 - 3R_1$
$\sim \begin{pmatrix} 1 & -3 & 4 & 3 \\ 0 & 1 & -1 & 0 \\ 0 & 0 & -0 & -8 \end{pmatrix}$	$R_3 \to R_3 - R_2$

Clearly $\rho(A) = 2$ and $\rho(A, B) = 3$

$$\rho(A, B) \neq \rho(A)$$

Hence, the given system is inconsistent and has no solution.

6. Solve : $2x - 3y - 1 = 0$, $5x + 2y - 12 = 0$ by Cramer's rule.

Solution:

The non-homogeneous equations are

$2x - 3y - 1 = 0$, $\quad 5x + 2y - 12 = 0$

$$\Delta = \begin{vmatrix} 2 & -3 \\ 5 & 2 \end{vmatrix} = 4 + 15 = 19 \neq 0$$

Since $\Delta \neq 0$, Cramer's rule can be applied and the system is consistent with unique solution.

$$\Delta x = \begin{vmatrix} 1 & -3 \\ 12 & 2 \end{vmatrix} = 2 + 36 = 38$$

$$\Delta y = \begin{vmatrix} 2 & 1 \\ 5 & 12 \end{vmatrix} = 24 - 5 = 19$$

$$x = \frac{\Delta x}{\Delta} = \frac{38}{19} = 1$$

$$y = \frac{\Delta y}{\Delta} = \frac{19}{19} = 1$$

$\therefore$ Solution set is {2, 1}

7. If $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 2 \\ -1 \\ 3 \end{bmatrix}$ find x, y and z.

Solution: Given

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 2 \\ -1 \\ 3 \end{pmatrix}$$

$$\Rightarrow \begin{pmatrix} x+0+0 \\ 0+0+z \\ 0+y+0 \end{pmatrix} = \begin{pmatrix} 2 \\ -1 \\ 3 \end{pmatrix}$$

$$\Rightarrow \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 2 \\ -1 \\ 3 \end{pmatrix}$$

$\Rightarrow x = 2, \quad z = -1$ and $\quad y = 3$

$\therefore$ Solution set is {2, 3, –1}

8. If $A = \begin{pmatrix} 2 & 4 \\ 4 & 3 \end{pmatrix}$, $X = \begin{pmatrix} n \\ 1 \end{pmatrix}$ $B = \begin{pmatrix} 8 \\ 11 \end{pmatrix}$ and AX = B then find n.

Solution:

Given AX = B

$$\begin{pmatrix} 2 & 4 \\ 4 & 3 \end{pmatrix}\begin{pmatrix} n \\ 1 \end{pmatrix} = \begin{pmatrix} 8 \\ 11 \end{pmatrix}$$

$$\Rightarrow \begin{pmatrix} 2n+4 \\ 4n+3 \end{pmatrix} = \begin{pmatrix} 8 \\ 11 \end{pmatrix}$$

Equating the corresponding entries on both sides, we get

$$2n + 4 = 8$$
$$2n = 8 - 4$$
$$2n = 4$$
$$n = \frac{4}{2}$$
$$2 = 2.$$

9. Solve: $2x + 3y = 5$, $6x + 5y = 11$

Solution:

Given non-homogeneous equations are

$2x + 3y = 5$, $\quad 6x + 5y = 11$

$$\Delta = \begin{vmatrix} 2 & 3 \\ 6 & 5 \end{vmatrix} = 10 - 18 = -8$$

Since $\Delta \neq 0$, Cramer's rule can be applied and the system is consistent with unique solution.

$$\Delta x = \begin{vmatrix} 5 & 3 \\ 11 & 5 \end{vmatrix} = 25 - 33 = -8$$

$$\Delta y = \begin{vmatrix} 2 & 5 \\ 6 & 11 \end{vmatrix} = 22 - 30 = -8$$

$$\therefore \quad x = \frac{\Delta x}{\Delta} = \frac{-8}{-8} = 1$$

$$y = \frac{\Delta y}{\Delta} = \frac{-8}{-8} = 1$$

$\therefore$ Solution set is {1, 1}

10. Two products A and B currently share the market with shares 60% and 40% each respectively. Each week some brand switching latees place. Of those who bought A the previous week 70% buy it again whereas 30% switch over to B. Of those who bought B the previous week, 80% buy it again whereas 20% switch over to A. Find their shares after one week and after two weeks.

Solution:

Transition probability matrix

$$T = \begin{array}{c} \\ A \\ B \end{array}\begin{array}{c} \begin{array}{cc} A & B \end{array} \\ \begin{pmatrix} 0\cdot 7 & 0\cdot 3 \\ 0\cdot 2 & 0\cdot 8 \end{pmatrix} \end{array}$$

Shares after one week

$$(\cdot 6 \quad \cdot 4)\begin{pmatrix} \cdot 7 & \cdot 3 \\ \cdot 2 & \cdot 8 \end{pmatrix}$$

$$= (\cdot 6\times\cdot 7+\cdot 4\times\cdot 2 \quad \cdot 6\times\cdot 3+\cdot 4\times\cdot 8)$$

$$= (\cdot 42+\cdot 08 \quad \cdot 18+\cdot 32) = (\cdot 50 \quad \cdot 50)$$

$\Rightarrow$ A = 50% and B = 50%

Shares after two weeks $(\cdot 5 \quad \cdot 5)\begin{pmatrix} \cdot 7 & \cdot 3 \\ \cdot 2 & \cdot 8 \end{pmatrix}$

$$= (\cdot 5\times\cdot 7+\cdot 5\times\cdot 2 \quad \cdot 5\times\cdot 3+\cdot 5\times\cdot 8)$$

$$= (\cdot 35+\cdot 10 \quad \cdot 15+\cdot 40) = (\cdot 45 \quad \cdot 55)$$

A = 45% and B = 55%

5 Mark Questions:

1. The sum of three numbers is 6. If we multiply the third number by 2 and add the first number to the result we get 7. By adding second and third numbers to three times the first number we get 12. Find the numbers using rank method.

Solution:

Let the three numbers be x, y and z respectively

Given
$$x+y+z = 6$$
$$x+2z = 7$$
$$3x+y+z = 12$$

Augmented matrix [A, B]	Elementary Transformation
$\begin{pmatrix} 1 & 1 & 1 & 6 \\ 1 & 0 & 2 & 7 \\ 3 & 1 & 1 & 12 \end{pmatrix}$	
$\sim \begin{pmatrix} 1 & 1 & 1 & 6 \\ 0 & -1 & 1 & 1 \\ 0 & -2 & -2 & -6 \end{pmatrix}$	$R_2 \to R_2 - R_1$ $R_3 \to R_3 - 3R_1$
$\sim \begin{pmatrix} 1 & 1 & 1 & 6 \\ 0 & -1 & 1 & 1 \\ 0 & 0 & -4 & -8 \end{pmatrix}$	$R_3 \to R_3 - 2R_2$

The last equivalent matrix is in echelon form

$\rho(A) = 3$ and $\rho(A, B) = 3$

$\therefore$ $\rho(A) = \rho(A, B) = 3$ = Number of unknowns

$\therefore$ The system is consistent and has unique solution.

To find the solutions, let us rewrite the echelon form into matrix form.

$$\begin{pmatrix} 1 & 1 & 1 \\ 0 & -1 & 1 \\ 0 & 0 & -4 \end{pmatrix}\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 6 \\ 1 \\ -8 \end{pmatrix}$$

$$x+y+z = 6 \quad \text{----- (1)}$$
$$-y+z = 1 \quad \text{----- (2)}$$
$$-4z = -8 \quad \text{----- (3)}$$

From (3), $-4z = -8 \Rightarrow z = \frac{-8}{-4} = 2$

Substituting $z = 2$ in (2) we get

$-y+2 = 1 \Rightarrow -y = 1-2 \Rightarrow -y = -1$

$\Rightarrow \quad y = 1$

Substituting $y = 1$ and $z = 2$ in (1) we get,

$x+1+2 = 6 \Rightarrow x+3 = 6 \Rightarrow x = 6-3$

$\Rightarrow \quad x = 3$

Hence, the numbers are 3, 1, 2.

2. A mixture is to be made of three foods A, B, C. The three foods A, B, C contain nutrients P, Q, R as shown below

Ounces per pound of Nutrient			
Food	**P**	**Q**	**R**
A	1	2	5
B	3	1	1
C	4	2	1

How to form a mixture which will have 8 ounces of P, 5 ounces of Q and 7 ounces of R? (Cramer's rule).

Solution:

Let x pounds of food A, y pounds of food B and z pounds of food C be needed to form the mixture.

Given
$$x + 3y + 4z = 8$$
$$2x + y + 2z = 5$$
$$5x + y + z = 7$$

$$\Delta = \begin{vmatrix} 1 & 3 & 4 \\ 2 & 1 & 2 \\ 5 & 1 & 1 \end{vmatrix}$$

$$= 1\begin{vmatrix} 1 & 2 \\ 1 & 1 \end{vmatrix} - 3\begin{vmatrix} 2 & 2 \\ 5 & 1 \end{vmatrix} + 4\begin{vmatrix} 2 & 1 \\ 5 & 1 \end{vmatrix}$$

$$= 1(1-2) - 3(2-10) + 4(2-5)$$
$$= 1(-1) - 3(-8) + 4(-3)$$
$$= -1 + 24 - 12 = 11$$

Since $\Delta \neq 0$, Cramer's rule can be applied and the system has unique solution.

$$\Delta x = \begin{vmatrix} 8 & 3 & 4 \\ 5 & 1 & 2 \\ 7 & 1 & 1 \end{vmatrix}$$

$$= 8\begin{vmatrix} 1 & 2 \\ 1 & 1 \end{vmatrix} - 3\begin{vmatrix} 5 & 2 \\ 7 & 1 \end{vmatrix} + 4\begin{vmatrix} 5 & 1 \\ 7 & 1 \end{vmatrix}$$

$$= 8(1-2) - 3(5-14) + 4(5-7)$$
$$= 8(-1) - 3(-9) + 4(-2)$$
$$= -8 + 27 - 8 = 11$$

$$\Delta y = \begin{vmatrix} 1 & 8 & 4 \\ 2 & 5 & 2 \\ 5 & 7 & 1 \end{vmatrix} = 1\begin{vmatrix} 5 & 2 \\ 7 & 1 \end{vmatrix} - 8\begin{vmatrix} 2 & 2 \\ 5 & 1 \end{vmatrix} + 4\begin{vmatrix} 2 & 5 \\ 5 & 7 \end{vmatrix}$$

$$= 1(5-14) - 8(2-10) + 4(14-25)$$
$$= 1(-9) - 8(-8) + 4(-11)$$
$$= -9 + 64 - 44 = 11$$

$$\Delta z = \begin{vmatrix} 1 & 3 & 8 \\ 2 & 1 & 5 \\ 5 & 1 & 7 \end{vmatrix} = 1\begin{vmatrix} 1 & 5 \\ 1 & 7 \end{vmatrix} - 3\begin{vmatrix} 2 & 5 \\ 5 & 7 \end{vmatrix} + 8\begin{vmatrix} 2 & 1 \\ 5 & 1 \end{vmatrix}$$

$$= 1(7-5) - 3(14-25) + 8(2-5)$$
$$= 1(2) - 3(-11) + 8(-3)$$
$$= 2 + 33 - 24$$
$$= 11$$

$$\therefore \quad x = \frac{\Delta x}{\Delta} = \frac{11}{11} = 1$$
$$y = \frac{\Delta y}{\Delta} = \frac{11}{11} = 1$$
$$z = \frac{\Delta z}{\Delta} = \frac{11}{11} = 1$$

Hence, the mixture is formed by mixing one pound of each of the foods A, B and C.

3. For what values of k, the system of equations $kx + y + z = 1, x + ky + z = 1, x + y + kz = 1$ have
(i) Unique solution
(ii) More than one solution
(iii) no solution

Solution:

The given non-homogeneous equations can be written as

$$\begin{pmatrix} k & 1 & 1 \\ 1 & k & 1 \\ 1 & 1 & k \end{pmatrix}\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$$

Augmented matrix [A, B]	Elementary Transformation
$\begin{pmatrix} k & 1 & 1 & 1 \\ 1 & k & 1 & 1 \\ 1 & 1 & k & 1 \end{pmatrix}$	
$\sim \begin{pmatrix} 1 & 1 & k & 1 \\ 1 & k & 1 & 1 \\ k & 1 & 1 & 1 \end{pmatrix}$	$R_1 \leftrightarrow R_3$
$\sim \begin{pmatrix} 1 & 1 & k & 1 \\ 0 & k-1 & 1-k & 0 \\ 0 & 1-k & 1-k^2 & 1-k \end{pmatrix}$	$R_2 \to R_2 - R_1$ $R_3 \to R_3 - kR_1$

$\sim\begin{pmatrix}1 & 1 & k & 1\\0 & k-1 & 1-k & 0\\0 & 0 & 2-k-k^2 & 1-k\end{pmatrix}$	$R_3 \to R_3 + R_2$
$\sim\begin{pmatrix}1 & 1 & k & 1\\0 & k-1 & 1-k & 0\\0 & 0 & k^2+k-2 & k-1\end{pmatrix}$	$R_1 \to R_3(-1)$
$\sim\begin{pmatrix}1 & 1 & k & 1\\0 & k-1 & 1-k & 0\\0 & 0 & (k-1)(k+2) & k-1\end{pmatrix}$	

Case (i)

When $k \neq 1$, and $k \neq -2$

$\rho(A) = \rho(A, B) = 3$ = Number of unknowns.

$\therefore$ The system has unique solution.

Case (ii)

When $k = 1$

$$[A, B] \sim \begin{pmatrix}1 & 1 & 1 & 1\\0 & 0 & 0 & 0\\0 & 0 & 0 & 0\end{pmatrix}$$

$\rho(A) = \rho(A, B) = 1$ < Number of unknowns.

$\therefore$ The system is consistent and has infinitely many solutions.

Case (iii)

When $k = -2$

$$[A, B] \sim \begin{pmatrix}1 & 1 & -2 & 1\\0 & -3 & 3 & 0\\0 & 0 & 0 & -3\end{pmatrix}$$

$\rho(A) = 2,\ \rho(A, B) = 3$

$\Rightarrow \rho(A) \neq \rho(A, B)$

$\therefore$ The system is inconsistent and has no solution.

4. Using determinants, find the quadratic defined by $f(x) = ax^2 + bx + c$ if $f(1) = 0$, $f(2) = -2$ and $f(3) = -6$.

Solution:

Given $f(x) = ax^2 + bx + c$

$f(1) = 0 \Rightarrow a(1)^2 + b(1) + c = 0 \Rightarrow a + b + c = 0$ -----(1)

$f(2) = -2 \Rightarrow a(2^2) + b(2) + c = -2 \Rightarrow 4a + 2b + c = -2$ -----(2)

$f(3) = -6 \Rightarrow a(3^2) + b(3) + c = -6 \Rightarrow 9a + 3b + c = -6$ -----(3)

Now
$$\Delta = \begin{vmatrix}1 & 1 & 1\\4 & 2 & 1\\9 & 3 & 1\end{vmatrix}$$
$$= 1(2-3) - 1(4-9) + 1(12-18)$$
$$= -1 + 5 - 6 = -2 \neq 0.$$

Since $\Delta \neq 0$, Cramer's rule can be applied and the system has unique solution.

$$\Delta a = \begin{vmatrix}0 & 1 & 1\\-2 & 2 & 1\\-6 & 3 & 1\end{vmatrix}$$
$$= 0 - 1(-2+6) + 1(-6+12)$$
$$= -4 + 6 = 2$$

$$\Delta b = \begin{vmatrix}1 & 0 & 1\\4 & -2 & 1\\9 & -6 & 1\end{vmatrix}$$
$$= 1(-2+6) + 0 + 1(-24+18)$$
$$= 4 - 6 = -2$$

$$\Delta c = \begin{vmatrix}1 & 1 & 0\\4 & 2 & -2\\9 & 3 & -6\end{vmatrix}$$
$$= 1(-12+6) - 1(-24+18) + 0$$
$$= -6 + 6 = 0$$

$$a = \frac{\Delta a}{\Delta} = \frac{\cancel{2}}{\cancel{-2}} = -1$$

$$b = \frac{\Delta b}{\Delta} = \frac{\cancel{-2}}{\cancel{-2}} = 1$$

$$c = \frac{\Delta c}{\Delta} = \frac{0}{-2} = 0$$

$f(n) = (-1)x^2 + 1(x) + 0 \Rightarrow f(x) = x^2 + x.$

5. A new transit system has just gone into operation in a city. Of those who use the transit system this year, 10% will switch over to using their own car next year and 90% will continue to use the transit system. Of those who use their cars this year, 80% will continue to use their cars next year and 20% will switch over to the transit system. Suppose the population of the city remains constant and that 50% of the commuters use the transit system and 50% of the commuters use their own car this year,

(i) What percent of commuters will be using the transit system after one year?

(ii) What percent of commuters will be using the transit system in the long run?

Solution:

Let A represents the percent of commuters who use the transit system and B represents the percent of commuters who use their own car.

Transition probability matrix

$$T = \begin{array}{c c} & \begin{array}{cc} A & B \end{array} \\ \begin{array}{c} A \\ B \end{array} & \begin{pmatrix} \cdot 9 & \cdot 1 \\ \cdot 2 & \cdot 8 \end{pmatrix} \end{array}$$

Given 50% of commuters use the transit system and 50% of the commuters use their own car this year.

(i) Percentage of commuters after one year

$$\begin{pmatrix} \cdot 5 & \cdot 5 \end{pmatrix}\begin{pmatrix} \cdot 9 & \cdot 1 \\ \cdot 2 & \cdot 8 \end{pmatrix}$$

$= \begin{pmatrix} \cdot 5\times\cdot 9+\cdot 5\times\cdot 2 & \cdot 5\times\cdot 1+\cdot 5\times\cdot 8 \end{pmatrix}$

$= \begin{pmatrix} \cdot 45+\cdot 10 & \cdot 05+\cdot 40 \end{pmatrix}$

$= \begin{pmatrix} \cdot 55 & \cdot 45 \end{pmatrix}$

A = 55% and B = 45%

(ii) Equilibrium will be reached in the long run at equilibrium, we must have

$$\begin{pmatrix} A & B \end{pmatrix}T = \begin{pmatrix} A & B \end{pmatrix} \text{ where } A + B = 1$$

$$\Rightarrow \begin{pmatrix} A & B \end{pmatrix}\begin{pmatrix} \cdot 9 & \cdot 1 \\ \cdot 2 & \cdot 8 \end{pmatrix} = \begin{pmatrix} A & B \end{pmatrix}$$

$$\begin{pmatrix} \cdot 9A+\cdot 2B & \cdot 1A+\cdot 8B \end{pmatrix} = \begin{pmatrix} A & B \end{pmatrix}$$

Equating the corresponding entries on both sides we get,

$\cdot 9A + \cdot 2B = A \Rightarrow \cdot 9A + \cdot 2(1 - A) = A$

[Since $A + B = 1 \Rightarrow B = 1 - A$]

$$\Rightarrow \quad \cdot 9A + \cdot 2 - \cdot 2A = A$$

$$\Rightarrow \quad \cdot 2 = A - \cdot 9A + \cdot 2A$$

$$\Rightarrow \quad \cdot 2 = A(1 - \cdot 9 + \cdot 2)$$

$$\Rightarrow \quad \cdot 2 = A(\cdot 3)$$

$$\Rightarrow \quad A = \frac{\cdot 2}{\cdot 3} = 0.666 \Rightarrow A = 67\%$$

∴ 67% of the commuters will be using the transit system in the long run.

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Chapter 2

INTEGRAL CALCULUS- I

FORMULAE TO REMEMBER

(i) Integration is the reverse process of differentiation

(ii) $\int k\, f(x)\, dx = k \int f(x)\, dx$ where k is a constant.

(iii) $\int [\, f(x) \pm g(x)]\, dx = \int f(x) dx \pm \int g(x)\, dx$

(iv) The following are the four principal methods of integration

(i) Integration by decomposition

(ii) Integration by Parts

(iii) Integration by Substitution

(iv) Integration by successive reduction

First fundamental theorem of integral calculus :

If $f(x)$ is a continuous function and $F(x) = \int_a^x f(t) dt$, then $F'(x) = f(x)$.

Second fundamental theorem of integral calculus :

$$\int_a^b f(x) dx = F(b) - F(a)$$

(i) $\int_a^b f(x) dx$ is a definite constant, whereas $\int_a^x f(t)\, dt$ is a function of the variable x

Indefinite integral :-

An integral function which is expressed without limits, and so containing an containing an arbitrary constant.

Proper definite integral :-

An integral function which has both the limits. a and b are finite.

Improper definite integral :-

An integral function, in which the limits either a or b or both are infinite.

Gamma function :-

For $n > 0$, $\int_0^\infty x^{n-1} e^{-x}\, dx$ and is denoted by $\Gamma(n)$

1) $\int x^n dx = \frac{x^{n+1}}{n+1} + c, n \neq -1$

2) $\int (ax+b)^n dx = \frac{(ax+b)^{n+1}}{a(n+1)} + c,\ n \neq -1$

3) $\int \frac{1}{x} dx = \log|x| + c$

4) $\int \frac{1}{ax+b} dx = \frac{1}{a} \log|ax+b| + c$

5) $\int e^x dx = e^x + c$

6) $\int e^{ax+b} dx = \frac{1}{\;} e^{ax+b} + c$

7) $\int a^x dx = \frac{a^x}{\log a} + c,\ a > 0$ and $a \neq 1$

8) $\int a^{mx+n} dx = \frac{1}{m \log a} a^{mx+n} + c,\ a > 0$ and $a \neq 1$

9) $\int \sin x\, dx = -\cos x + c$

10) $\int \sin(ax+b)\, dx = -\frac{1}{a} \cos(ax+b) + c$

11) $\int \cos\ dx = \sin x + c$

12) $\int \cos(ax+b)\, dx = \frac{1}{a} \sin(ax+b) + c$

13) $\int \sec^2 x\, dx = \tan x + c$

14) $\int \sec^2(ax+b)\, dx = \frac{1}{a} \tan(ax+b) + c$

15) $\int \operatorname{cosec}^2 x\, dx = -\cot x + c$

16) $\int \operatorname{cosec}^2(ax+b)\, dx = -\frac{1}{a} \cot(ax+b) + c$

17) $\int u dv = uv - \int v du$ where u and v are two differentiable functions of x [Integration by parts].

18) The code word used in the above formula is

I $\rightarrow$ Inverse trigonometric function
L $\rightarrow$ Logarithmic function
A $\rightarrow$ Algebraic function
T $\rightarrow$ Trigonometric function
E $\rightarrow$ Exponential function

19) Bernoulli's formula :

$\int u dv = uv - u' v_1 + u'' v_2 - u''' v_3 + ...$

When $u' u'' u'''$.... are the successive derivatives of u and $v_1 v_2 v_3$ .... are the repeated integrals of v.

20) $\int [f(x)]^n f'(x)\, dx = \dfrac{f(x)^{n+1}}{n+1} + c,\ n \neq -1$

21) $\int \dfrac{f'(x)}{f(x)}\, dx = \log |f(x)| + c$

22) $\int \dfrac{f'(x)}{\sqrt{f(x)}}\, dx = 2\sqrt{f(x)} + c$

23) $\int e^x [f(x) + f'(x)]\, dx = e^x f(x) + c$

24) $\int e^{ax} [a f(x) + f'(x)]\, dx = e^{ax} f(x) + c$

25) $\int \dfrac{dx}{a^2 - x^2} = \dfrac{1}{2a} \log \left|\dfrac{a+x}{a-x}\right| + c$

26) $\int \dfrac{dx}{x^2 - a^2} = \dfrac{1}{2a} \log \left|\dfrac{x-a}{x+a}\right| + c$

27) $\int \dfrac{dx}{\sqrt{x^2 - a^2}} = \log |x + \int \sqrt{x^2 - a^2}\ | + c$

28) $\int \dfrac{dx}{\sqrt{x^2 + a^2}} = \log |x + \int \sqrt{x^2 + a^2}\ | + c$

29) $\int \sqrt{x^2 - a^2}\, dx = \dfrac{x}{2} \int \sqrt{x^2 - a^2} - \dfrac{a^2}{2} \log |x + \int \sqrt{x^2 - a^2}\ | + c$

30) $\int \sqrt{x^2 + a^2}\, dx = \dfrac{x}{2} \int \sqrt{x^2 + a^2} + \dfrac{a^2}{2} \log |x + \int \sqrt{x^2 + a^2}\ | + c$

Properties of definite integral

1) $\int_a^b f(x)dx = \int_a^b f(t)dt$

2) $\int_a^b f(x)dx = -\int_b^a f(x)dx$

3) $\int_a^b [f(x) \pm g(x)]\, dx = \int_a^b f(x)dx \pm \int_a^b g(x)dx$

4) $\int_a^b f(x)dx = \int_a^c f(x)dx + \int_c^b f(x)dx$

5) $\int_o^b f(x)dx = \int_o^a f(a-x)dx$

6) $\int_{-a}^a f(x)dx = 2\int_o^a f(x)dx$ if $f(x)$ is an even function

7) $\int_{-a}^a f(x)dx = 0$ if $f(x)$ is an odd function

8) $\int_a^b f(x)dx = \int_a^b f(a+b-x)dx$

TEXTUAL QUESTIONS

EXERCISE 2.1

Integrate the following with respect to x

1. $\sqrt{3x+5}$

Sol. $\int \sqrt{3x+5}\, dx$

$= \int (3x+5)^{1/2} dx$

$= \dfrac{(3x+5)^{1/2+1}}{3\left(\frac{1}{2}+1\right)} + c$

$\left[\because \int (ax+b)^n\, dx = \dfrac{(ax+b)^{n+1}}{a(n+1)} + c\right]$

$= \dfrac{(3x+5)^{3/2}}{3\left(\frac{3}{2}\right)} + c = \dfrac{(3x+5)^{3/2}}{\frac{9}{2}} + c$

$= \dfrac{2}{9}(3x+5)^{3/2} + c$

2. $\left(9x^2 - \dfrac{4}{x^2}\right)^2$

Sol. $\int \left(9x^2 - \dfrac{4}{x^2}\right)^2 dx$

$= \int \left[(9x^2)^2 - 2(9\cancel{x^2})\left(\dfrac{4}{\cancel{x^2}}\right) + \left(\dfrac{4}{x^2}\right)^2\right] dx$

$[\because (a-b)^2 = a^2 - 2ab + b^2]$

$= \int \left(81x^4 - 72 + \dfrac{16}{x^4}\right) dx$

$= 81\dfrac{x^{4+1}}{4+1} - 72x + 16\dfrac{x^{-4+1}}{-4+1} + c$

$\left[\because \dfrac{16}{x^4} = 16x^{-4}\right]$

$= 81\dfrac{x^5}{5} - 72x + 16\dfrac{x^{-3}}{-3} + c$

$= \dfrac{81}{5}x^5 - 72x - \dfrac{16}{3x^3} + c$

3. $(3+x)(2-5x)$

Sol. $\int (3+x)(2-5x)\,dx$

$= \int (6 - 15x + 2x - 5x^2)\,dx$

$= \int (6 - 13x - 5x^2)\,dx$

$= 6x - \dfrac{13x^2}{2} - \dfrac{5x^3}{3} + c$

4. $\sqrt{x}\,(x^3 - 2x + 3)$

Sol. $\int \sqrt{x}\,(x^3 - 2x + 3)\,dx$

$= \int x^{\frac{1}{2}}(x^3 - 2x + 3)\,dx$

$= \int \left(x^{3+\frac{1}{2}} - 2x^{1+\frac{1}{2}} + 3x^{\frac{1}{2}} \right) dx$

$= \dfrac{x^{\frac{7}{2}+1}}{\frac{7}{2}+1} - \dfrac{2x^{\frac{3}{2}+1}}{\frac{3}{2}+1} + \dfrac{3x^{\frac{1}{2}+1}}{\frac{1}{2}+1} + c$

$= \dfrac{x^{\frac{9}{2}}}{\frac{9}{2}} - 2\dfrac{x^{\frac{5}{2}}}{\frac{5}{2}} + 3\dfrac{x^{\frac{3}{2}}}{\frac{3}{2}} + c$

$= \dfrac{2}{9}x^{\frac{9}{2}} - 2\times\dfrac{2}{5}x^{\frac{5}{2}} + \cancel{3}\times\dfrac{2}{\cancel{3}}x^{\frac{3}{2}} + c$

$= \dfrac{2}{9}x^{\frac{9}{2}} - \dfrac{4}{5}x^{\frac{5}{2}} + 2x^{\frac{3}{2}} + c$

5. $\dfrac{8x+13}{\sqrt{4x+7}}$

Sol. $\int \dfrac{8x+13}{\sqrt{4x+7}}\,dx$

$= \int \dfrac{8x+14-1}{\sqrt{4x+7}}\,dx$

$= \int \dfrac{2(4x+7)-1}{\sqrt{4x+7}}\,dx$

$= 2\int \dfrac{(4x+7)}{\sqrt{4x+7}}\,dx - \int \dfrac{1}{\sqrt{4x+7}}\,dx$

$= 2\int \sqrt{4x+7}\,dx - \int \dfrac{1}{\sqrt{4x+7}}\,dx$

$= 2\int (4x+7)^{\frac{1}{2}}\,dx - \int (4x+7)^{-\frac{1}{2}}\,dx$

$= 2\,\dfrac{(4x+7)^{\frac{1}{2}+1}}{4\left(\frac{1}{2}+1\right)} - \dfrac{(4x+7)^{-\frac{1}{2}+1}}{4\left(\frac{-1}{2}+1\right)} + c$

$= 2\dfrac{(4x+7)^{\frac{3}{2}}}{4\left(\frac{3}{2}\right)} - \dfrac{(4x+7)^{\frac{1}{2}}}{4\left(\frac{1}{2}\right)} + c$

$= \cancel{2}\dfrac{(4x+7)^{\frac{3}{2}}}{\cancel{6}_3} - \dfrac{(4x+7)^{\frac{1}{2}}}{2} + c$

$= \dfrac{(4x+7)^{\frac{3}{2}}}{3} - \dfrac{(4x+7)^{\frac{1}{2}}}{2} + c$

6. $\dfrac{1}{\sqrt{x+1}+\sqrt{x-1}}$

Sol. $\int \dfrac{1}{\sqrt{x+1}+\sqrt{x-1}}\,dx$

= Multiplying and dividing the conjugate of the denominator we get

$= \int \dfrac{\sqrt{x+1}-\sqrt{x-1}\,dx}{\left(\sqrt{x+1}+\sqrt{x-1}\right)\left(\sqrt{x+1}-\sqrt{x-1}\right)}$

$= \int \dfrac{\sqrt{x+1}-\sqrt{x-1}}{(x+1)-(x-1)}\,dx$

[$\because (a+b)(a-b) = a^2-b^2$]

$= \int \dfrac{\sqrt{x+1}-\sqrt{x-1}}{\cancel{x}+1-\cancel{x}+1}\,dx$

$= \int \dfrac{\sqrt{x+1}-\sqrt{x-1}}{2}\,dx$

$= \dfrac{1}{2}\int \left((x+1)^{½} - (x-1)^{½}\right)dx$

$= \dfrac{1}{2}\left[\dfrac{(x+1)^{\frac{3}{2}}}{\frac{3}{2}} - \dfrac{(x-1)^{\frac{3}{2}}}{\frac{3}{2}} \right] + c$

$= \dfrac{1}{\cancel{2}\times\frac{3}{\cancel{2}}}\left[(x+1)^{\frac{3}{2}} - (x-1)^{\frac{3}{2}}\right] + c$

$= \dfrac{1}{3}\left[(x+1)^{\frac{3}{2}} - (x-1)^{\frac{3}{2}}\right] + c$

7. If $f(x) = x + b$, $f(1) = 5$ and $f(2) = 13$, then find $f(x)$

Sol.

Given $f'(x) = x + b$, $f(1) = 5$ and $f(2) = 13$

$f'(x) = x + b$

$\Rightarrow \int f'(x)\,dx = \int (x+b)\,dx$

[$\therefore$ Integration is the reverse process of differentiation]

$$\Rightarrow f(0) = \frac{x^2}{2} + bx + c \qquad \text{---(1)}$$

Given $f(1) = 5$

$$\Rightarrow 5 = \frac{1^2}{2} + b(1) + c$$

$$\Rightarrow 5 = \frac{1}{2} + b(1) + c \Rightarrow 5 - \frac{1}{2} = b + c$$

$$\Rightarrow \frac{10-1}{2} = b + c \Rightarrow b + c = \frac{9}{2}$$

$$\Rightarrow 2b + 2c = 9 \qquad \text{---(2)}$$

Also $f(2) = 13 \Rightarrow 13 = \frac{2^2}{2} + b(2) + c$

$$\Rightarrow 13 = 2 + 2b + c$$

$$\Rightarrow 13 - 2 = 2b + c$$

$$\Rightarrow 2b + c = 11 \qquad \text{---(3)}$$

$$(2) - (3) \rightarrow 2b + 2c = 9$$

$$-2b + -c = -11$$

$$\boxed{c = -2}$$

Substituting $c = -2$ in (3) we get

$$2b - 2 = 11 \Rightarrow 2b = 11 + 2 \Rightarrow 2b = 13$$

$$\Rightarrow \boxed{b = \frac{13}{2}}$$

Substituting $b = \frac{13}{2}$, $c = -2$ in (1) we get,

$$f(x) = \frac{x^2}{2} + \frac{13}{2} x - 2$$

8. If $f(x) = 8x^3 - 2x$ and $f(2)=8$, then find $f(x)$.

Sol.

Given $f'(x) = 8x^3 - 2x,\ f(2) = 8$

$$f'(x) = 8x^3 - 2x$$

$$\Rightarrow \int f'(x)\,dx = \int (8x^3 - 2x)\,dx$$

$$\Rightarrow f(x) = \frac{\overset{2}{\cancel{8}} x^4}{\cancel{4}} - \frac{\cancel{2} x^2}{\cancel{2}} + c$$

$$\Rightarrow f(x) = 2x^4 - x^2 + c \qquad \text{---(1)}$$

Given $f(2) = 8$

$$\Rightarrow 8 = 2(2^4) - 2^2 + c$$

$$\Rightarrow 8 = 32 - 4 + c$$

$$\Rightarrow 8 = 32 - 4 + c$$

$$\Rightarrow 8 - 28 = c$$

$$\Rightarrow c = -20$$

Substituting c $= -20$ in (1) we get.

$$f(x) = 2x^4 - x^2 - 20$$

EXERCISE 2.2

Integrate the following with respect to x.

1. $\left(\sqrt{2x} - \frac{1}{\sqrt{2x}}\right)^2$

Sol. $\int \left(\sqrt{2x} - \frac{1}{\sqrt{2x}}\right)^2 dx$

$$= \int \left[\left(\sqrt{2x}\right)^2 - 2\left(\cancel{\sqrt{2x}}\right)\left(\frac{1}{\cancel{\sqrt{2x}}}\right) + \left(\frac{1}{\sqrt{2x}}\right)^2 \right] dx$$

[∵ $(a-b)^2 = a^2 - 2ab + b^2$]

$$= \int \left(2x - 2 + \frac{1}{2x}\right) dx$$

$$= 2\frac{x^2}{2} - 2x + \frac{1}{2} \log|x| + c$$

$$= x^2 - 2x + \frac{1}{2} \log|x| + c$$

2. $\frac{x^4 - x^2 + 2}{x - 1}$

Sol. $\int \frac{x^4 - x^2 + 2}{x - 1} dx$

$$= \int \left(x^3 + x^2 + \frac{2}{x-1}\right) dx$$

$$= \frac{x^4}{4} + \frac{x^3}{3} + 2 \log|x-1| + c$$

$$\begin{array}{r|l} & x^3 + x^2 \\ \hline x-1 & x^4 - x^2 + 2 \\ & {}^{(-)}x^4 - {}^{(+)}x^3 \\ \hline & x^3 - x^2 \\ & {}^{(-)}x^3 - {}^{(+)}x^2 \\ \hline & 2 \\ \hline \end{array}$$

3. $\int \frac{x^3}{x+2} dx$

Sol. $\int \frac{x^3}{x+2} dx$

$$= \int \left(x^2 - 2x + 4 - \frac{8}{x+2}\right) dx$$

$$= \frac{x^3}{3} - \frac{\cancel{2} x^2}{\cancel{2}} + 4x - 8 \log|x+2| + c$$

$$= \frac{x^3}{3} - x^2 + 4x - 8 \log|x+2| + c$$

$$\left[\because \int \frac{1}{x}dx = \log|x| + c\right]$$

$$\begin{array}{r|l} & x^2 - 2x + 4 \\ \hline x+2 & x^3 \\ & \overset{(-)}{}x^3 \overset{(-)}{+} 2x^2 \\ \hline & -2x^2 \\ & \overset{(+)}{-}2x^2 \overset{(+)}{-} 4x \\ \hline & 4x \\ & \overset{(-)}{}4x \overset{(-)}{+} 8 \\ \hline & -8 \end{array}$$

4. $\dfrac{x^3 + 3x^2 - 7x + 11}{x+5}$

Sol. $\displaystyle\int \frac{x^3 + 3x^2 - 7x + 11}{x+5}dx$

$$= \int\left(x^2 - 2x + 3 - \frac{4}{x+5}\right)dx$$

$$= \frac{x^3}{3} - \frac{\cancel{2}x^2}{\cancel{2}} + 3x - 4\log|x+5| + c$$

$$= \frac{x^3}{3} - x^2 + 3x - 4\log|x+5| + c$$

$$\begin{array}{r|l} & x^2 - 2x + 3 \\ \hline x+5 & x^3 + 3x^2 - 7x + 11 \\ & \overset{(-)}{}x^3 \overset{(-)}{+} 5x^2 \\ \hline & -2x^2 - 7x \\ & (\pm)\, 2x^2\, (\pm)\, 10x \\ \hline & 3x + 11 \\ & \overset{(-)}{}3x \overset{(-)}{+} 15 \\ \hline & -4 \end{array}$$

5. $\dfrac{3x+2}{(x-2)(x-3)}$

Sol. $\displaystyle\int \frac{(3x+2)dx}{(x-2)(x-3)}$

$$= \int\left(\frac{-8}{x-2} + \frac{11}{x-3}\right)dx$$

$$= -8\log|x-2| + 11\log|x-3| + c$$

$$= -11\log|x-3| - 8\log|x-2| + c$$

$$\frac{3x+2}{(x-2)(x-3)} = \frac{A}{x-2} + \frac{B}{x-3}$$

$\Rightarrow 3x + 2 = A(x-3) + B(x-2)$

Put $x = 3$

$9 + 2 = B(1) \Rightarrow B = 11$

Put $x = 2$

$8 = A(-1) \Rightarrow A = -8$

$$\frac{3x+2}{(x-2)(x-3)} = \frac{-8}{x-2} + \frac{11}{x-3}$$

6. $\dfrac{4x^2 + 2x + 6}{(x+1)^2(x-3)}$

Sol. $\displaystyle\int \frac{4x^2 + 2x + 6}{(x+1)^2(x-3)}dx$

$$= \int\left(\frac{A}{x+1} + \frac{B}{(x+1)^2} + \frac{C}{x-3}\right)dx$$

$$= \int\left(\frac{1}{x+1} + \frac{-2}{(x+1)^2} + \frac{3}{x-3}\right)dx$$

$$= \log|x+1| - 2\int (x+1)^{-2}dx + 3\log|x-3| + c$$

$$= \log|x+1| - 2\frac{(x+1)^{-2+1}}{-2+1} + 3\log|x-3| + c$$

$$= \log|x+1| + 2(x+1)^{-1} + 3\log|x-3| + c$$

$$= \log|x+1| + \frac{2}{x+1} + 3\log|x-3| + c$$

$$\frac{4x^2 + 2x + 6}{(x+1)^2(x-3)} = \frac{A}{x+1} + \frac{B}{(x+1)^2} + \frac{C}{x-3}$$

$4x^2 + 2x + 6 = A(x+1)(x-3) + B(x-3) + C(x+1)^2$

$\Rightarrow$ Putting $x = -1$, $4 - 2 + 6 = B(-4)$

$\Rightarrow 8 = B(-4) \Rightarrow \boxed{B = -2}$

Putting $x = 3$

$36 + 6 + 6 = C(16)$

$\Rightarrow 48 = 16C \Rightarrow \boxed{C = 3}$

Putting $x = 0$,

$6 = -3A - 3B + C$

$\Rightarrow 6 = -3A + 6 + 3$

$\Rightarrow 3A = 3 \Rightarrow \boxed{A = 1}$

7. $\dfrac{3x^2 2x+5}{(x-1)(x^2+5)}$

Sol. $\int \dfrac{3x^2-2x+5}{(x-1)(x^2+5)}\,dx$

$= \int\left(\dfrac{A}{x-1}+\dfrac{Bx+c}{x^2+5}\right)dx$

$= \int\left(\dfrac{1}{x-1}+\dfrac{2x+0}{x^2+5}\right)dx$

$= \int \dfrac{1}{x-1}\,dx + \int \dfrac{2x}{x^2+5}\,dx$

$= \log|x-1| + \log|x^2+5| + c$

$\left[\because \int \dfrac{f^1(x)}{f(x)}\,dx = \log|f(x)| + c\right]$

$= \log|(x^2+5)(x-1)| + c$

$[\because \log m + \log n = \log mn]$

$= \log|x^3 - x^2 + 5x - 5| + c$

$\dfrac{3x^2-2x+5}{(x-1)(x^2+5)} = \dfrac{A}{(x-1)} + \dfrac{Bx+c}{(x^2+5)}$

$\Rightarrow 3x^2 - 2x+5 = A(x^2+5) + (Bx+C)(x-1)$

Putting $x = 1$,

$3-2+5 = A(1+5)$

$\Rightarrow 6 = A(6) \Rightarrow \boxed{A=1}$

Putting $x = 0$,

$5 = 5A - C$

$\Rightarrow 5 = 5 - C$ $[\because A = 1]$

$\Rightarrow C = 5-5 \Rightarrow \boxed{C=0}$

Putting $x = -1$,

$3+2+5 = A(6) + (C-B)(-2)$

$\Rightarrow 10 = 6A + 2B - 2C$

$\Rightarrow 10 = 6 + 2B + 0$

$\Rightarrow 10 - 6 = 2B \Rightarrow 4 = 2B$

$\Rightarrow \boxed{B=2}$

8. **If $f'(x) = \dfrac{1}{x}$ and $f(1) = \dfrac{\pi}{4}$, then find $f(x)$.**

Sol.

Given $f'(x) = \dfrac{1}{x}$

$\Rightarrow \int f'(x)\,dx = \int \dfrac{1}{x}\,dx$

$\Rightarrow f(x) = \log|x| + c$ ---(1)

Also, $f(1) = \dfrac{\pi}{4}$

$\Rightarrow \dfrac{\pi}{4} = \log|1| + c$

$\Rightarrow \dfrac{\pi}{4} = c$ $[\because \log 1 = 0]$

Substituting $c = \dfrac{\pi}{4}$ in (1) we get,

$f(x) = \log|x| + \dfrac{\pi}{4}$

EXERCISE 2.3

Integrate the following with respect to x

1. $e^{x\log a} + e^{a\log a} - e^{n\log x}$

Sol.

$\int(e^{x\log a} + e^{a\log a} - e^{n\log x})\,dx$

$= \int\left(e^{\log a^x} + e^{\log a^a} - e^{\log x^n}\right)dx$

$[\because m\log n = \log n^m]$

$= \int(a^x + a^a - x^n)\,dx$ $[\because e^{\log x} = x]$

$= \left[\dfrac{a^x}{\log a}\right] + a^a(x) - \dfrac{x^{n+1}}{n+1} + c$

$\left[\because \int a^x = \dfrac{a^x}{\log a}\right]$

2. $\dfrac{a^x - e^{x\log b}}{e^{x\log a}\ b^x}$

Sol. $\int \dfrac{a^x - e^{x\log b}}{e^{x\log a}\ b^x}\,dx$

$= \int \dfrac{a^x - e^{\log b^x}}{e^{\log a^x}\ b^x}\,dx$ $[\because m\log n = \log n^m]$

$= \int \dfrac{a^x - b^x}{a^x . b^x}\,dx$ $[\because e^{\log x} = x]$

$= \int \dfrac{a^x}{a^x b^x}\,dx - \int \dfrac{b^x}{a^x b^x}\,dx$

$= \int \dfrac{1}{b^x}\,dx - \int \dfrac{1}{a^x}\,dx$

$= \int b^{-x}\,dx - \int a^{-x}\,dx$ $\left[\because \int a^{-x}\,dx = \dfrac{a^{-x}}{-\log a} + c\right]$

$$= \frac{b^{-x}}{-\log b} - \frac{a^{-x}}{-\log a} + c$$

$$= -\frac{b^{-x}}{\log b} + \frac{a^{-x}}{\log a} + c$$

$$= \frac{-1}{b^x \log b} + \frac{1}{\log a \,.\, a^x} + c$$

$$= \frac{1}{a^x \log a} - \frac{-1}{b^x \log b} + c$$

3. $(e^x + 1)^2 e^x$

Sol.

$$\int (e^x+1)^2 e^x \, dx$$

$$= \int [(e^x)^2 + 2(e^x)(1) + 1^2]\, e^x \, dx \qquad [\because (a+b)^2 = a^2 + 2ab + b^2]$$

$$= \int (e^{2x} + 2e^x + 1)\, e^x \, dx$$

$$= \int (e^{3x} + 2e^{2x} + e^x)\, dx \qquad [a^m \,.\, a^n = a^{m+n}]$$

$$= \frac{e^{3x}}{3} + \cancel{2}\frac{e^{2x}}{\cancel{2}} + e^x + c$$

$$= \frac{e^{3x}}{3} + e^{2x} + e^x + c$$

4. $\dfrac{e^{3x} - e^{-3x}}{e^x}$

Sol.

$$\int \frac{e^{3x} - e^{-3x}}{e^x} dx$$

$$= \int \frac{e^{3x}}{e^x} dx - \int \frac{e^{-3x}}{e^x} dx$$

$$= \int e^{3x-x} dx - \int e^{-3x-x} dx \qquad \left[\because \frac{a^m}{a^m} = a^{m-n}\right]$$

$$= \int e^{2x} dx - \int e^{-4x} dx$$

$$= \frac{e^{2x}}{2} - \frac{e^{-4x}}{-4} + c$$

$$= \frac{e^{2x}}{2} + \frac{e^{-4x}}{4} + c$$

5. $\dfrac{e^{3x} + e^{5x}}{e^x + e^{-x}}$

Sol.

$$\int \frac{e^{3x} + e^{5x}}{e^x + e^{-x}} dx$$

$$= \int \left(\frac{e^{3x} + e^{5x}}{e^x + \frac{1}{e^x}} \right) dx = \int \frac{e^{3x} + e^{5x}}{\frac{e^{2x}+1}{e^x}} dx$$

$$= \int \frac{e^x\left(e^{3x} + e^{5x}\right)}{e^{2x}+1} dx$$

$$= \int e^x . \frac{e^{3x}\cancel{\left(1+e^{2x}\right)}}{\cancel{\left(e^{2x}+1\right)}} dx \qquad [\text{Taking } e^{3x} \text{ common}]$$

$$= \int e^{4x} dx = \frac{e^{4x}}{4} + c$$

6. $\left(1 - \dfrac{1}{x^2}\right) . e^{\left(x+\frac{1}{x}\right)}$

Sol.

Let $I = \int \left(1 - \frac{1}{x^2}\right) . e^{\left(x+\frac{1}{x}\right)} dx$

put $x + \frac{1}{x} = t$

on differentiating we get, $\left(1 - \frac{1}{x^2}\right) dx = dt$

$$\therefore \quad I = \int e^t \, dt = e^t + c$$

$$= e^{x+\frac{1}{x}} + c \qquad \left[\because t = x + \frac{1}{x}\right]$$

7. $\dfrac{1}{x(\log x)^2}$

Sol.

Let I $= \int \frac{1}{x(\log x)^2} dx$

put $\log x = t$ on differentiating we get,

$$\frac{1}{x} dx = dt$$

$$\therefore \quad I = \int \frac{1}{t^2} dt \qquad [\because t = \log x]$$

$$I = \int t^{-2} dt$$

$$= \frac{t^{-2+1}}{-2+1} + c$$

$$= \frac{t^{-1}}{-1} + c \Rightarrow \frac{-1}{t} + c$$

$$= \frac{-1}{\log|x|} + c \qquad [\because t = \log|x|]$$

8. If $f'(x) = e^x$ and $f(0) = 2$, then find $f(x)$

Sol.

Given $f'(x) = e^x$

$\Rightarrow \int f'(x)\,dx = \int e^x\,dx$ [Taking integration both sides]

$\Rightarrow f(x) = e^x + c$ ... (1)

Also, $f(0) = 2$

$\Rightarrow 2 = e^0 + c$

$\Rightarrow 2 = 1 + c$

$\Rightarrow 2 - 1 = c$

$\Rightarrow c = 1$

Substituting $c = 1$ in (1) we get,

$f(x) = e^x + 1$

EXERCISE 2.4

Integrate the following with respect to x.

1. $2\cos x - 3\sin x + 4\sec^2 x - 5\,\text{cosec}^2 x$

Sol.

$\int (2\cos x - 3\sin x + 4\sec^2 x - 5\,\text{cosec}^2 x)\,dx$

$= 2\int\cos x\,dx - 3\int\sin x\,dx + 4\int\sec^2 x\,dx - 5\int\text{cosec}^2 x\,dx$

$= 2(\sin x) - 3(-\cos x) + 4\tan x - 5(-\cot x) + c$

$= 2\sin x + 3\cos x + 4\tan x + 5\cot x + c$

2. $\sin^3 x$

Sol.

$\int \sin^3 x\,dx$

We know that $\sin 3x = 3\sin x - 4\sin^3 x$

$\Rightarrow 4\sin^3 x = 3\sin x - \sin 3x$

$\Rightarrow \sin^3 x = \frac{1}{4}(3\sin x - \sin 3x)$

$\therefore \int \sin^3 x\,dx = \frac{1}{4}\int(3\sin x - \cancel{3}\sin 3x)\,dx$

$= \frac{3}{4}\int \sin x\,dx - \frac{1}{4}\int \sin 3x\,dx$

$= \frac{3}{4}(-\cos x) - \frac{1}{4}\left(-\frac{\cos 3x}{3}\right) + c$ $\left[\because \int \sin ax\,dx = \frac{-1}{a}\cos ax + c\right]$

$= -\frac{3}{4}\cos x + \left(\frac{\cos 3x}{12}\right) + c$

3. $\dfrac{\cos 2x + 2\sin^2 x}{\cos^2 x}$

Sol. $\int \frac{\cos 2x + 2\sin^2 x}{\cos^2 x}\,dx$

$= \int \frac{\cos^2 x - \sin^2 x + 2\sin^2 x}{\cos^2 x}\,dx$ $[\because \cos 2x = \cos^2 x - \sin^2 x]$

$= \int \frac{\cos^2 x + \sin^2 x}{\cos^2 x}\,dx$

$= \int \frac{1}{\cos^2 x}\,dx$ $[\because \sin^2 x + \cos^2 x = 1]$

$= \int \sec^2 x\,dx$

$= \tan x + c$

4. $\dfrac{1}{\sin^2 x\cos^2 x}$

Sol. $\int \frac{1}{\sin^2 x\cos^2 x}\,dx$

$= \int \frac{(\sin^2 x + \cos^2 x)}{\sin^2 x\cos^2 x}\,dx$ $[\because 1 = \sin^2 x + \cos^2 x]$

$= \int \frac{\cancel{\sin^2 x}}{\cancel{\sin^2 x}\cos^2 x}\,dx + \int \frac{\cancel{\cos^2 x}}{\sin^2 x\cancel{\cos^2 x}}\,dx$

$= \int \frac{1}{\cos^2 x}\,dx + \int \frac{dx}{\sin^2 x}$

$= \int \sec^2 x\,dx + \int \text{cosec}^2 x\,dx$

$= \tan x - \cot x + c$

5. $\sqrt{1 - \sin 2x}$

Sol. $\int \sqrt{1 - \sin 2x}\,dx$

$= \int \sqrt{\sin^2 x + \cos^2 x - 2\sin x\cos x}\,dx$ $[\because 1 = \sin^2 x + \cos^2 x \sin 2x = 2\sin x\cos x]$

$= \int \sqrt{(\sin x - \cos x)^2}\,dx$ $[\because (a-b)^2 = a^2 - 2ab + b^2]$

$= \int (\sin x - \cos x)\,dx$

$= -\cos x - \sin x + c$

$= -(\cos x + \sin x) + c$

EXERCISE 2.5

Integrate the following with respect to x.

1. $x e^{-x}$

Sol.

$\int x e^{-x}\,dx$

Let $u = x$ and $dv = e^{-x}\,dx$

$du = 1\,dx$, $v = \frac{e^{-x}}{-1} = -e^{-x}$

Using integration by parts,

$\int u\,dv = uv - \int v\,du$

$\int xe^{-x}\,dx = x(-e^{-x}) - \int(-e^{-x})\,dx$

$= -xe^{-x} + \int e^{-x}\,dx$

$$= -xe^{-x} + \frac{e^{-x}}{-1} + c$$

$$= -xe^{-x} - e^{-x} + c$$

$$= -e^{-x}(x+1) + c$$

1 L A T E

Algebraic function

$\therefore u = x$

2. x^3e^{3x}

Sol.

Let I $= \int x^3 \, e^{3x} \, dx$

Let $u = x^3$; $dv = = e^{3x}\, dx$

Using Bernoulli's formula,

$$\int udv = uv - u^{1}v_1 + u^{11}v_2 - u^{111}v_3$$

$$\therefore \int x^3 e^{3x}\, dx$$

$$= x^3\frac{e^{3x}}{3} - 3x^2\left(\frac{e^{3x}}{9}\right) + 6x\left(\frac{e^{3x}}{27}\right) - 6\left(\frac{e^{3x}}{81}\right) + c$$

$$= \frac{x^3e^{3x}}{3} - \frac{x^2e^{3x}}{3} + \frac{2xe^{3x}}{9} - \frac{2e^{3x}}{27} + c$$

$$= e^{3x}\left[\frac{x^3}{3} - \frac{x^2}{3} + \frac{2x}{9} - \frac{2}{27}\right] + c$$

1 L A T E

Algebraic function

Successive derivatives	Repeated Integrals
$u = x^3$	$dv = e^{3x}dx$
$u^1 = 3x^3$	$v = \frac{e^{3x}}{3}$
$u^{11} = 6x$	$v_1 = \frac{e^{3x}}{9}$
$u^{111} = 6$	$v_2 = \frac{e^{3x}}{27}$
	$v_3 = \frac{e^{3x}}{81}$

3. **log*x***

Sol.

Let I $= \int \log x \, dx$

Let $u = \log x$; $dv = dx$

$$du = \frac{1}{x}dx \; ; v = x$$

Using integration by parts,

$$\int udv = uv - \int vdu$$

$$\Rightarrow \int \log x\, dx = x\log x - \int x\frac{1}{x}dx$$

$$= x\log x - \int dx$$

$$= x\log x - x + c$$

$$= x(\log x - 1) + c$$

1 L A T E

logarithmic function

4. ***x* log*x***

Sol.

Let I $= \int x \log x \, dx$

Let $u = \log x$; $dv = x\, dx$

$$du = \frac{1}{x}\, dx \; ; v = \frac{x^2}{2}$$

Using integration by parts,

$$\int udv = uv - \int vdu$$

$$\Rightarrow \int x\log x\, dx = \log x\left(\frac{x^2}{2}\right) - \int \frac{x^2}{2}\,\frac{1}{x}\, dx$$

$$= \frac{x^2}{2}\log x - \frac{1}{2}\int x\, dx$$

$$= \frac{x^2}{2}\log x - \frac{1}{2}\,\frac{x^2}{2} + c$$

$$= \frac{x^2}{2}\log x - \frac{x^2}{4} + c$$

$$= \frac{x^2}{2}\left(\log x - \frac{1}{2}\right) + c$$

1 L A T E

logarithmic function

5. $x^n \log x$

Sol.

Let I $= \int x^n \log x\, dx$

Let $u = \log x$; $dv = x^n dx$

$$du = \frac{1}{x}\, dx \; ; v = \frac{x^{n+1}}{n+1}$$

using integration by parts we get,

$\int udv = uv - \int vdu$

$\Rightarrow \int x^n \log x\, dx$

$= \frac{x^{n+1}}{n+1} \log x - \int \frac{x^{n+1}}{n+1} \frac{1}{x}\, dx$

$= \frac{x^{n+1}}{n+1} \log x - \frac{1}{n+1} \int x^{n+1-1}\, dx$

$= \frac{x^{n+1}}{n+1} \log x - \frac{1}{n+1} \int x^n\, dx$

$= \frac{x^{n+1}}{n+1} \log x - \frac{1}{n+1} \frac{x^{n+1}}{n+1} + c$

$= \frac{x^{n+1}}{n+1} \left(\log x - \frac{1}{n+1}\right) + c$

1 L A T E
↳ logarithmic function

6. $x^5 e^{x^2}$

Sol.

Let I $= \int x^5 e^{x^2}\, dx$

$= \int x^4 . x\; e^{x^2}\, dx$

$= \int (x^2)^2 x\; e^{x^2}\, dx$

put $t = x^2$

$\Rightarrow dt = 2x\, dx$

$\Rightarrow \frac{dt}{2} = x\, dx$

substituting there values in I we get,

$I = \int t^2 e^t \frac{dt}{2}$

$= \frac{1}{2} \int t^2 e^t\, dt$

using Bernoullis theorem

$\int udv = uv - u^1 v_1 + u^{11} v_2$

$\therefore \quad I = \frac{1}{2} \int t^2 e^t\, dt$

$= \frac{1}{2} [t^2 e^t - 2\, te^t + 2\, e^t] + c$

$= \frac{1}{2} e^t [t^2 - 2t + 2] + c$

$I = \frac{e^{x^2}}{2} (x^4 - 2x^2 + 2] + c \quad [\because t = x^2]$

1 L A T E
↳ Algebraic function

Let $u = t^2$; $dv = e^t$

Successive derivatives	Repeated Integrals
$u = t^2$	$dv = et$
$u^1 = 2t$	$v = e^t$
$u^{11} = 2$	$v_1 = e^t$
	$v_2 = e^t$

EXERCISE 2.6

Integrate the following with respect to x

1. $\frac{2x+5}{x^2+5x-7}$

Sol.

Let I $= \int \frac{2x+5}{x^2+5x-7}\, dx$

Let $t = x^2 + 5x - 7$

$\Rightarrow dt = (2x+5)\, dt$

$\therefore \quad I = \int \frac{dt}{t}$

$= \log |t| + c$

$= \log |x^2 + 5x - 7| + c$

$[\because t = x^2 + 5x - 7]$

2. $\frac{e^{3\log x}}{x^4+1}$

Sol.

Let I $= \int \frac{e^{3\log x}}{x^4+1}\, dx$

$= \int \frac{e^{\log x^3}}{x^4+1}\, dx$

$[\because m \log n = \log n^m]$

$= \int \frac{x^3}{x^4+1}\, dx \quad [\because e^{\log x} = x]$

put $t = x^4 + 1$

$\Rightarrow dt = 4x^3\, dt$

$\Rightarrow \dfrac{dt}{4} = x^3\, dx$

$\therefore \quad I = \int \dfrac{dt/4}{t}$

$= \dfrac{1}{4}\int \dfrac{dt}{t} = \dfrac{1}{4}\log|t| + c$

$= \dfrac{1}{4}\log|x^4+1| + c \quad [\because t = x^4+1]$

3. $\dfrac{e^{2x}}{e^{2x}-2}$

Sol.

Let $I = \int \dfrac{e^{2x}}{e^{2x}-2}\, dx$

put $t = e^{2x} - 2$

$\Rightarrow dt = 2e^{2x}\, dx$

$\Rightarrow \dfrac{dt}{2} = e^{2x}\, dx$

$\therefore \quad I = \int \dfrac{dt/2}{t} = \dfrac{\log|t|}{2} + c$

$= \dfrac{\log|e^{2x}-2|}{2} + c$

4. $\dfrac{(\log x)^3}{x}$

Sol.

Let $I = \int \dfrac{(\log x)^3}{x}\, dx$

put $t = \log x$

$\Rightarrow dt = \dfrac{1}{x}\, dx$

$\therefore \quad I = \int t^3\, dt$

$= \dfrac{t^4}{4} + c$

$= \dfrac{(\log x)^4}{4} + c \quad [\because t = \log x]$

5. $\dfrac{6x+7}{\sqrt{3x^2+7x-1}}$

Sol.

Let $I = \int \dfrac{6x+7}{\sqrt{3x^2+7x-1}}\, dx$

Let $t = 3x^2 + 7x - 1$

$\Rightarrow dt = (6x+7)\, dx$

$\therefore \quad I = \int \dfrac{dt}{\sqrt{t}} = \int t^{\frac{1}{2}}\, dt$

$= \dfrac{t^{-\frac{1}{2}+1}}{-\frac{1}{2}+1} + c = \dfrac{t^{\frac{1}{2}}}{\frac{1}{2}} + c$

$= 2\sqrt{t} + c$

$= 2\sqrt{3x^2+7x-1} + c$

$[\because t = 3x^2 + 7x - 1]$

6. $(4x+2)\sqrt{x^2+x+1}$

Sol.

Let $I = \int (4x+2)\sqrt{x^2+x+1}\, dx$

put $t = x^2 + x + 1$

$dt = (2x+1)\, dx$

$\therefore \quad I = 2\int (2x+1)\sqrt{x^2+x+1}\, dx$

$= 2\int \sqrt{t}\, dt$

$= 2\int t^{1/2}\, dt = 2\dfrac{t^{\frac{1}{2}+1}}{\frac{1}{2}+1} + c$

$= 2\dfrac{t^{3/2}}{3/2} + c$

$= \dfrac{4}{3}t^{3/2} + c$

$= \dfrac{4}{3}(x^2+x+1)^{3/2} + c \quad [\because t = x^2 + x + 1]$

7. $x^8(1+x^9)^5$

Sol.

Let $I = \int x^8(1+x^9)^5\, dx$

put $t = 1 + x^9$

$\Rightarrow dt = 0 + 9x^8\, dx$

$\Rightarrow dt = 9x^8\, dx$

$\Rightarrow \dfrac{dt}{9} = x^8\, dx$

$\therefore \quad I = \int \dfrac{t^5}{9}\, dt = \dfrac{1}{9}\int t^5 dt$

$$= \frac{1}{9}\left[\frac{t^6}{6}\right] + c = \frac{1}{54}\, t^6 + c$$

$$= \frac{1}{54}\,(1 + x^9)^6 + c$$

8. $\dfrac{x^{e-1} + e^{x-1}}{x^e + e^x}$

Sol.

Let I $= \int \dfrac{x^{e-1} + e^{x-1}}{x^e + e^x}\, dx$

put $t = x^e + e^x$

$\Rightarrow dt = (e\,x^{e-1} + e^x)\, dx$

$dt = e\,(x^{e-1} + e^{x-1})\, dx$

$\Rightarrow \dfrac{dt}{e} = (x^{e-1} + e^{x-1})\, dx$

$\therefore$ I $= \int \dfrac{dt}{e(t)} = \dfrac{1}{e}\int \dfrac{dt}{t}$

$= \dfrac{1}{e} \log|t| + c$

$= \dfrac{1}{e} \log|x^e + e^x| + c$ $\quad [\because t = x^e + e^x]$

9. $\dfrac{1}{x \log x}$

Sol.

Let I $= \int \dfrac{1}{x\log x}\, dx$

put $\log x = t$

$\Rightarrow \dfrac{1}{x}\, dx = dt$

$\therefore$ I $= \int \dfrac{1}{t}\, dt = \log|t| + c$

$= \log|\log x| + c$ $\quad [\because t = \log x]$

10. $\dfrac{x}{2x^4 - 3x^2 - 2}$

Sol. Let I $= \int \dfrac{x}{2x^4 - 3x^2 - 2}\, dx$

$= \int \dfrac{x}{(x^2-2)(2x^2+1)}\, dx$

putting $x^2 = t$, we get

$2x\, dx = dt \Rightarrow x dx = \dfrac{dt}{2}$

$\therefore$ I $= \dfrac{1}{2}\int \dfrac{dt}{(t-2)(2t+1)}$

$= \dfrac{1}{2}\int \left(\dfrac{A}{t-2} + \dfrac{B}{2t+1}\right) dt$

$= \dfrac{1}{2}\int \left(\dfrac{1/5}{t-2} + \dfrac{-2/5}{2t+1}\right) dt$

$= \dfrac{1}{2}\left[\dfrac{1}{5}\log|t-2| - \dfrac{2}{5}\dfrac{\log|2t+1|}{2}\right] + c$

$= \dfrac{1}{10}\,[\log(t-2) - \log|2t+1|] + c$

$= \dfrac{1}{10}\log\left|\dfrac{t-2}{2t+1}\right| + c$

$= \dfrac{1}{10}\log\left|\dfrac{x^2-2}{2x^2+1}\right| + c$ $\quad [\because t = x^2]$

Factorising
$2x^4 - 3x^2 - 2$

-4, -3

$\dfrac{-4}{2}$ $\quad \dfrac{1}{2}$

$\dfrac{-2}{1}$ $\quad \dfrac{1}{2}$

$(x^2 - 2)$ $\quad (2x^2 + 1)$

$\dfrac{1}{(t-2)(2t+1)} = \dfrac{A}{t-2} + \dfrac{B}{2t+1}$

$\Rightarrow 1 = A(2t+1) + B(t-2)$

put $t = 2$

$\Rightarrow 1 = 5A \Rightarrow$ $\boxed{A = \dfrac{1}{5}}$

put $t = 0$

$\Rightarrow 1 = A - 2B$

$\Rightarrow 1 = \dfrac{1}{5} - 2B$

$\Rightarrow 2B = \dfrac{1}{5} - 1 = \dfrac{1-5}{5}$

$\Rightarrow 2B = -\dfrac{4}{5}$

$\Rightarrow B = -\dfrac{2}{5}$

11. $e^x (1 + x) \log (xe^x)$

Sol. Let $I = \int e^x (1 + x) \log (xe^x)\, dx$

Put $t = x e^x$

$$\Rightarrow dt = (x \cdot e^x + e^x (1))\, dx$$

$$= e^x (x + 1)\, dx$$

$$\therefore \quad I = \int \log t \cdot dt$$

Let $u = \log t$; $dv = dt$

$du = \frac{1}{t}\, dt$; $v = t$

$\therefore$ Using integration by parts we get,

$$I = \int u dv = uv - \int v du$$

$$= t \log t - \int t^{\frac{1}{x(x^2}}\, dt$$

$$= t \log t - \int dt = t \log t - t + c$$

$$= t \log t - t + c$$

$$= xe^x \log (xe^x) - (xe^x) + c \qquad [\because t = xe^x]$$

$$= xe^x (\log (xe^x) - 1) + c$$

1 L A T E └──→ logarithmic function

12. $\dfrac{1}{x(x^2+1)}$

Sol. Let $I = \int \frac{1}{x(x^2+1)}\, dx$

$$= \int \left(\frac{A}{x} + \frac{Bx+C}{x^2+1}\right) dx$$

$$= \int \left(\frac{1}{x} + \frac{-x+0}{x^2+1}\right) dx$$

$$= \int \frac{1}{x}\, dx - \int \frac{x}{x^2+1}\, dx$$

$$= \log |x| - \frac{1}{2}\int \frac{2x\, dx}{x^2+1}$$

(multiplying and dividing by 2 in the second integral)

$$= \log |x| - \frac{1}{2} \log |x^2 + 1| + C$$

$$[\because t = x^2 + 1 \Rightarrow dt = 2x\, dx$$

$$\therefore \int \frac{2x\, dx}{x^2+1} = \int \frac{dt}{t} = \log |t| = \log |x^2 + 1|]$$

$$\frac{1}{x(x^2+1)} = \frac{A}{x} + \frac{Bx+C}{x^2+1}$$

$\Rightarrow$ $1 = A (x^2 + 1) + (Bx + C)(x)$

Put $x = 0 \Rightarrow 1 = A$

Put $x = 1$

$\Rightarrow$ $1 = 2A + B + C$

$\Rightarrow$ $1 = 2 + B + C$

$\Rightarrow$ $B + C = -1$

Put $x = -1$

$1 = 2A - 1(-B + C)$

$1 = 2A + B - C \Rightarrow 1 = 2 + B - C$

$B - C = -1$

$\Rightarrow$ $B + C = -1$

$2B = -2 \Rightarrow B = -1$

$\therefore$ $-1 + C = -1$

$\Rightarrow$ $C = -1 + 1$ $\quad C = 0$

13. $e^x \left[\dfrac{1}{x^2} - \dfrac{2}{x^3}\right]$

Sol. Let $I = \int e^x \left[\frac{1}{x^2} - \frac{2}{x^3}\right] dx$

Let $f(x) = \frac{1}{x^2} = x^{-2}$

$$\Rightarrow f'(x) = -2x^{-2-1}$$

$$= -2x^{-3} = \frac{-2}{x^3}$$

$$\Rightarrow f'(x) = \frac{-2}{x^3}$$

$$\therefore \quad I = \int e^x [f(x) + f'(x)]\, dx$$

$$= e^x f(x) + c$$

$$= e^x \frac{1}{x^2} + c$$

$$= \frac{e^x}{x^2} + c$$

14. $e^x \left[\dfrac{x-1}{(x+1)^3}\right]$

Sol. Let $I = \int e^x \left[\frac{x-1}{(x+1)^3}\right] dx$

$$= \int e^x \left[\frac{x-1+1-1}{(x+1)^3}\right] dx$$

[Adding and subtracting 1 in the numerator]

$$= \int e^x \left[\frac{x+1-2}{(x+1)^3}\right] dx$$

$$= \int e^x \left(\frac{x+1}{(x+1)^3} - \frac{2}{(x+1)^3}\right) dx$$

$$= \int e^x \left(\frac{1}{(x+1)^2} - \frac{2}{(x+1)^3}\right) dx$$

Let $f(x) = \frac{1}{(x+1)^2} = (x+1)^{-2}$

$\Rightarrow f'(x) = -2(x+1)^{-2-1}$

$= -2(x+1)^{-3'}$

$= \frac{-2}{(x+1)^3}$

$\therefore$ I $= \int e^x [f(x) + f'(x)]\, dx$

$= e^x f(x) + c$

$= e^x \frac{1}{(x+1)^2} + c$

$= \frac{e^x}{(x+1)^2} + c$

15. $e^{3x}\left[\frac{3x-1}{9x^2}\right]$

Sol. Let I $= \int e^{3x}\left[\frac{3x-1}{9x^2}\right] dx$

$= \int e^{3x}\left(\frac{3x}{9x^2} - \frac{1}{9x^2}\right) dx$

$= \int e^{3x}\left[\frac{3}{9x} - \frac{1}{9x^2}\right] dx$

Let $f(x) = \frac{1}{9x}$ and $a = 3$

$\Rightarrow f(x) = \frac{1}{9}\cdot x^{-1}$

$\Rightarrow f'(x) = \frac{1}{9}\cdot(-1)\, x^{-1-1}$

$\Rightarrow f'(x) = \frac{-1}{9}x^{-2} = \frac{-1}{9x^2}$

$\therefore$ I $= \int e^{3x}(3.f(x) + f'(x))\, dx$

$= e^{3x} f(x) + c$

$[\because \int e^{ax}[a f(x) + f'(x)]\, dx = e^{ax} f(x) + c]$

$= e^{3x}\frac{1}{9x} + c = \frac{e^{3x}}{9x} + c$

EXERCISE 2.7

Integrate the following with respect to x

1. $\frac{1}{9-16x^2}$

Sol. $\int \frac{1}{9-16x^2}\, dx$

$= \frac{1}{16}\int \frac{1}{\frac{9}{16} - x^2}\, dx$

$= \frac{1}{16}\int \frac{1}{\left(\frac{3}{4}\right)^2 - x^2}\, dx$

$= \frac{1}{16}\left[\frac{1}{2\left(\frac{3}{4}\right)}\log\left|\frac{\frac{3}{4}+x}{\frac{3}{4}-x}\right|\right] + c$

$[\because \int \frac{dx}{x^2-a^2} = \frac{1}{2a}\log\left|\frac{x-a}{x+a}\right| + c]$

$= \frac{1}{16} \times \frac{2}{3}\log\left|\frac{(3+4x)/4}{(3-4x)/4}\right| + c$

$= \frac{1}{24}\log\left|\frac{3+4x}{3-4x}\right| + c$

2. $\frac{1}{9-8x-x^2}$

Sol. Let I $= \int \frac{dx}{9-8x-x^2}$

$= -\int \frac{dx}{x^2+8x-9}$

$= -\int \frac{dx}{x^2+8x+16-16-9}$

$\frac{1}{2}$ [co-effective of x]^2

$\Rightarrow \left[\frac{1}{2}(8)\right]^2 = 4^2$

= 16 Adding and subtracting 16

$$= -\int \frac{dx}{(x+4)^2 - 25}$$

$$= -\int \frac{dx}{(x+4)^2 - 5^2}$$

$$= \int \frac{dx}{5^2 - (x+4)^2}$$

$$[\because \int \frac{dx}{x^2 - a^2} = \frac{1}{2a} \log \left|\frac{x-a}{x+a}\right| + c]$$

$$= \frac{1}{2(5)} \log \left|\frac{5+x+4}{5-x-4}\right| + c$$

$$= \frac{1}{10} \log \left|\frac{9+x}{1-x}\right| + c$$

3. $\mathbf{\dfrac{1}{2x^2 - 9}}$

Sol. $\int \frac{1}{2x^2 - 9}\, dx$

$$= \frac{1}{2} \int \frac{dx}{x^2 - \frac{9}{2}}$$

$$= \frac{1}{2} \int \frac{dx}{x^2 - \left(\frac{3}{\sqrt{2}}\right)^2}$$

$$= \frac{1}{2} \times \frac{1}{2\left(\frac{3}{\sqrt{2}}\right)} \log \left|\frac{x - \frac{3}{\sqrt{2}}}{x + \frac{3}{\sqrt{2}}}\right| + c$$

$$[\because \int \frac{dx}{x^2 - a^2} = \frac{1}{2a} \log \left|\frac{x-a}{x+a}\right| + c]$$

$$= \frac{1}{\frac{12}{\sqrt{2}}} \log \left|\frac{\sqrt{2}x - 3}{\sqrt{2}x + 3}\right| + c$$

$$= \frac{1}{6 \times \frac{\sqrt{2} \times \cancel{\sqrt{2}}}{\cancel{\sqrt{2}}}} \log \left|\frac{\sqrt{2}x - 3}{\sqrt{2}x + 3}\right| + c$$

$$= \frac{1}{6 \times \sqrt{2}} \log \left|\frac{\sqrt{2}x - 3}{\sqrt{2}x + 3}\right| + c$$

4. $\mathbf{\dfrac{1}{x^2 - x - 2}}$

Sol. $\int \frac{dx}{x^2 - x - 2}$

$$= \int \frac{dx}{x^2 - x + \frac{1}{4} - \frac{1}{4} - 2}$$

Adding and subtracting $[\frac{1}{2}$ co - effective of $x]^2$ $= \left[\frac{1}{2}(-1)\right]^2 = \frac{1}{4}$

$$= \int \frac{dx}{\left(x^2 - \frac{1}{2}\right)^2 - \frac{9}{4}}$$

$$= \int \frac{dx}{\left(x - \frac{1}{2}\right)^2 - \left(\frac{3}{2}\right)^2}$$

$$[\because \int \frac{dx}{x^2 - a^2} = \frac{1}{2a} \log \left|\frac{x-a}{x+a}\right| + c]$$

$$= \frac{1}{2\left(\frac{3}{2}\right)} \log \left|\frac{x - \frac{1}{2} - \frac{3}{2}}{x - \frac{1}{2} + \frac{3}{2}}\right| + c$$

$$= \frac{1}{3} \log \left|\frac{x-2}{x+1}\right| + c$$

5. $\mathbf{\dfrac{1}{x^2 + 3x + 2}}$

Sol. $\int \frac{dx}{x^2 + 3x + 2}$

$$= \int \frac{dx}{x^2 + 3x + \frac{9}{4} - \frac{9}{4} + 2}$$

$\left[\frac{1}{2}(3)\right]^2 = \frac{9}{4}$

Adding and subtracting $\frac{9}{4}$

$$= \int \frac{dx}{\left(x + \frac{3}{2}\right)^2 - \frac{1}{4}}$$

$$= \int \frac{dx}{\left(x+\frac{3}{2}\right)^2 - \left(\frac{1}{2}\right)^2}$$

$$= \frac{1}{2\times\frac{1}{2}} \log \left|\frac{x+\frac{3}{2}-\frac{1}{2}}{x+\frac{3}{2}+\frac{1}{2}}\right| + c$$

$$\left[\because \int \frac{dx}{x^2-a^2} = \frac{1}{2a}\log\left|\frac{x-a}{x+a}\right| + c\right]$$

$$= \log\left|\frac{x+1}{x+2}\right| + c$$

6. $\dfrac{1}{2x^2+6x-8}$

Sol. $\displaystyle\int \frac{dx}{2x^2+6x-8}$

$$= \frac{1}{2}\int \frac{dx}{x^2+3x-4}$$

$$= \frac{1}{2}\int \frac{dx}{x^2+3x+\frac{9}{4}-\frac{9}{4}-4}$$

$\left[\frac{1}{2}(3)\right]^2 = \frac{9}{4}$

Adding and subtracting $\frac{9}{4}$

$$= \frac{1}{2}\int \frac{dx}{\left(x+\frac{3}{2}\right)^2 - \frac{25}{4}}$$

$$= \frac{1}{2}\int \frac{dx}{\left(x+\frac{3}{2}\right)^2 - \left(\frac{5}{2}\right)^2}$$

$$= \frac{1}{2} \times \frac{1}{\cancel{2}\times\frac{5}{\cancel{2}}} \log\left|\frac{x+\frac{3}{2}-\frac{5}{2}}{x+\frac{3}{2}+\frac{5}{2}}\right| + c$$

$$\left[\because \int \frac{dx}{x^2-a^2} = \frac{1}{2a}\log\left|\frac{x-a}{x+a}\right| + c\right]$$

$$= \frac{1}{10}\log\left|\frac{x-1}{x+4}\right| + c$$

7. $\dfrac{e^x}{e^{2x}-9}$

Sol. Let I $= \displaystyle\int \frac{e^x dx}{e^{2x}-9}\, dx$

$$= \int \frac{e^x dx}{\left(e^x\right)^2 - 9}$$

Put $e^x = t \Rightarrow e^x dx = dt$

$$\therefore \text{ I } = \int \frac{dt}{t^2-9} = \int \frac{dt}{t^2-3^2}$$

$$= \frac{1}{2\times 3}\log\left|\frac{t-3}{t+3}\right| + c$$

$$\left[\because \int \frac{dx}{x^2-a^2} = \frac{1}{2a}\log\left|\frac{x-a}{x+a}\right| + c\right]$$

$$= \frac{1}{6}\log\left|\frac{e^x-3}{e^x+3}\right| + c$$

8. $\dfrac{1}{\sqrt{9x^2-7}}$

Sol. $\displaystyle\int \frac{dx}{\sqrt{9x^2-7}}$

$$= \int \frac{dx}{\sqrt{9\left(x^2-\frac{7}{9}\right)}}$$

$$= \frac{1}{3}\int \frac{dx}{\sqrt{x^2 - \left(\frac{\sqrt{7}}{3}\right)^2}}$$

$$\left[\because \int \frac{dx}{\sqrt{x^2+a^2}} = \log\left|x+\sqrt{x^2+a^2}\right| + c\right]$$

$$= \frac{1}{3}\log\left|x+\sqrt{x^2-\frac{7}{9}}\right| + c$$

$$= \frac{1}{3}\log\left|x+\sqrt{\frac{9x^2-7}{3}}\right| + c$$

$$= \frac{1}{3}\log\left|3x+\sqrt{9x^2-7}\right| + c$$

9. $\dfrac{1}{\sqrt{x^2+6x+13}}$

Sol. $\displaystyle\int \frac{dx}{\sqrt{x^2+6x+13}}$

$$= \int \frac{dx}{\sqrt{x^2+6x+9-9+13}}$$

$$= \int \frac{dx}{\sqrt{(x+3)^2+4}} = \int \frac{dx}{\sqrt{(x+3)^2+2^2}}$$

$$\left[\because \int \frac{dx}{\sqrt{x^2+a^2}} = \log\left|x+\sqrt{x^2+a^2}\right| + c\right]$$

$$= \log\left|x+3+\sqrt{x^2+6x+13}\right| + c$$

$[\frac{1}{2}$ [co-effecient of x]]^2

$= \left[\frac{1}{2}(6)\right]^2 = 3^2 = 9$

Adding and subtracting 9

10. $\dfrac{1}{\sqrt{x^2-3x+2}}$

Sol. $\displaystyle\int \frac{dx}{\sqrt{x^2-3x+2}}$

$$= \int \frac{dx}{\sqrt{x^2-3x+\frac{9}{4}-\frac{9}{4}+2}}$$

$\left[\frac{1}{2}(3)\right]^2 = \frac{9}{4}$

Adding and subtracting $\frac{9}{4}$

$$= \int \frac{dx}{\sqrt{\left(x-\frac{3}{2}\right)^2-\frac{1}{4}}}$$

$$= \int \frac{dx}{\sqrt{\left(x-\frac{3}{2}\right)^2-\left(\frac{1}{2}\right)^2}}$$

$$\left[\because \int \frac{dx}{\sqrt{x^2+a^2}} = \log\left|x+\sqrt{x^2+a^2}\right| + c\right]$$

$$= \log\left|\left(x-\frac{3}{2}\right)+\sqrt{x^2-3x+2}\right| + c$$

11. $\dfrac{x^3}{\sqrt{x^8-1}}$

Sol. $\displaystyle\int \frac{x^3dx}{\sqrt{x^8-1}}$

$$= \int \frac{x^3dx}{\sqrt{\left(x^4\right)^2-1}}$$

Put $x^4 = t \Rightarrow 4x^3\ dx = dt \Rightarrow x^3\ dx = \dfrac{dt}{4}$

$$= \frac{1}{4}\int \frac{dt}{\sqrt{t^2-1}}$$

$$= \frac{1}{4}\int \frac{dt}{\sqrt{t^2-1^2}}$$

$$= \frac{1}{4}\log\left|t+\sqrt{t^2-1}\right| + c$$

$$\left[\because \int \frac{dx}{\sqrt{x^2+a^2}} = \log\left|x+\sqrt{x^2+a^2}\right| + c\right]$$

$$= \frac{1}{4}\log\left|x^4+\sqrt{x^8-1}\right| + c$$

12. $\sqrt{1+x+x^2}$

Sol. $\displaystyle\int \sqrt{x^2+x+1}\ dx$

$$= \int \sqrt{x^2+x+\frac{1}{4}-\frac{1}{4}+1}\ dx$$

Adding and subtracting

$\frac{1}{2}$ [co - effective of x]^2

$= \left[\frac{1}{2}(-1)\right]^2 = \frac{1}{4}$

$$= \int \sqrt{\left(x+\frac{1}{2}\right)^2 + \frac{3}{4}}\ dx$$

$$= \int \sqrt{\left(x+\frac{1}{2}\right)^2 + \left(\frac{\sqrt{3}}{2}\right)^2}\ dx$$

$$\left[\because \int \sqrt{x^2+a^2}\ dx = \frac{x}{2}\sqrt{x^2+a^2} + \frac{a^2}{2}\log\left|x+\sqrt{x^2+a^2}\right| + c\right]$$

$$= \frac{1}{2}\left(x+\frac{1}{2}\right)\sqrt{x^2+x+1} + \frac{3}{4(2)} \log\left|\left(x+\frac{1}{2}\right)+\sqrt{x^2+x+1}\right| + c$$

$$= \frac{x+\frac{1}{2}}{2}\sqrt{x^2+x+1} + \frac{3}{8} \log\left|\left(x+\frac{1}{2}\right)+\sqrt{x^2+x+1}\right| + c$$

13. $\sqrt{x^2-2}$

Sol. $\int \sqrt{x^2-2}\ dx$

$$= \int \sqrt{x^2-(\sqrt{2})^2}\ dx$$

$$= \frac{x}{2}\sqrt{x^2-2} - \frac{2}{2}\log\left|x+\sqrt{x^2-2}\right| + c$$

$$\left[\because \int \sqrt{x^2-a^2}\ dx = \frac{x}{2}\sqrt{x^2-a^2} - \frac{a^2}{2}\log\left|x+\sqrt{x^2-a^2}\right| + c\right]$$

$$= \frac{x}{2}\sqrt{x^2-2} - \log\left|x+\sqrt{x^2-2}\right| + c$$

14. $\sqrt{4x^2-5}$

Sol. $\int \sqrt{4x^2-5}\ dx$

$$= \int \sqrt{4\left(x^2-\frac{5}{4}\right)}\ dx$$

$$= 2\int \sqrt{x^2-\left(\frac{\sqrt{5}}{2}\right)^2}\ dx$$

$$= 2\left[\frac{x}{2}\sqrt{x^2-\frac{5}{4}} - \frac{5}{4(2)}\log\left|x+\sqrt{x^2-\frac{5}{4}}\right|\right] + c$$

$$= \frac{x}{2}\sqrt{4x^2-5} - \frac{5}{4}\log\left|2x+\sqrt{4x^2-5}\right| + c$$

$$= \frac{1}{4}\left[2x\sqrt{4x^2-5} - 5\log\left|2x+\sqrt{4x^2-5}\right|\right] + c$$

15. $\sqrt{2x^2+4x+1}$

Sol. $\int \sqrt{2x^2+4x+1}\ dx$

$$= \int \sqrt{2\left(x^2+2x+\frac{1}{2}\right)}\ dx$$

$$= \sqrt{2}\int \sqrt{x^2+2x+1-1+\frac{1}{2}}\ dx$$

$\left[\frac{1}{2}(2)\right]^2 = 1^2 = 1$
Adding and subtracting 1

$$= \sqrt{2}\int \sqrt{(x+1)^2-\frac{1}{2}}\ dx$$

$$= \sqrt{2}\int \sqrt{(x+1)^2-\left(\frac{1}{\sqrt{2}}\right)^2}\ dx$$

$$\left[\because \int \sqrt{x^2-a^2}\ dx = \frac{x}{2}\sqrt{x^2-a^2} - \frac{a^2}{2}\log\left|x+\sqrt{x^2-a^2}\right| + c\right]$$

$$= \sqrt{2}\left[\frac{x+1}{2}\sqrt{x^2+2x+\frac{1}{2}} - \frac{1}{2(2)}\log\left|(x+1)+\sqrt{x^2+2x+\frac{1}{2}}\right|\right] + c$$

$$= \frac{x+1}{\sqrt{2}}\ \frac{\sqrt{2x^2+4x+1}}{\sqrt{2}} - \frac{\sqrt{2}}{4}\log\left|\sqrt{2}(x+1)+\sqrt{2x^2+4x+1}\right| + c$$

$$= \frac{x+1}{\sqrt{2}}\sqrt{2x^2+4x+1} - \frac{\sqrt{2}}{4}\log\left|\sqrt{2}(x+1)+\sqrt{2x^2+4x+1}\right| + c$$

16. $\dfrac{1}{x+\sqrt{x^2-1}}$

Sol. $\displaystyle\int \frac{dx}{x+\sqrt{x^2-1}}$

Multiply and divide by the conjugate of the denominator we get,

$$\int \frac{1}{x+\sqrt{x^2-1}} \times \frac{x-\sqrt{x^2-1}}{x-\sqrt{x^2-1}}\, dx$$

$$= \int \frac{\left(x-\sqrt{x^2-1}\right)\, dx}{x^2-\left(x^2-1\right)}$$

$[\because (a+b)(a-b) = a^2-b^2]$

$$= \int \frac{\left(x-\sqrt{x^2-1}\right)\, dx}{x^2-x^2-1}$$

$$= \int \left(x-\sqrt{x^2-1}\right) dx$$

$$= \int x\, dx - \int \sqrt{x^2-1}\, dx$$

$$= \frac{x^2}{2} - \left[\frac{x}{2}\sqrt{x^2-1} - \frac{1}{2}\log\left|x+\sqrt{x^2-1}\right| + c\right]$$

$$\left[\because \int \sqrt{x^2-a^2}\, dx = \frac{x}{2}\sqrt{x^2-a^2} - \frac{a^2}{2}\log\left|x+\sqrt{x^2-a^2}\right| + c\right]$$

$$= \frac{x^2}{2} - \frac{x}{2}\sqrt{x^2-1} + \frac{1}{2}\log\left|x+\sqrt{x^2-1}\right| + c$$

EXERCISE 2.8

Using second fundamental theorem. evaluate the following

1. $\displaystyle\int_0^1 e^{2x} dx$

Sol. $\displaystyle\int_0^1 e^{2x} dx$

$$= \left[\frac{e^{2x}}{2}\right]_0^1 = \frac{1}{2}\left[e^{2(1)} - e^{2(0)}\right]$$

$$= \frac{1}{2}\left[e^2 - e^0\right] = \frac{1}{2}\left[e^2 - 1\right]$$

2. $\displaystyle\int_0^{\frac{1}{4}} \sqrt{1-4x}\, dx$

Sol. $\displaystyle\int_0^{\frac{1}{4}} \sqrt{1-4x}\, dx = \int_0^{\frac{1}{4}} (1-4x)^{\frac{1}{2}}\, dx$

$$= \left[\frac{(1-4x)^{\frac{1}{2}+1}}{-4\left(\frac{1}{2}+1\right)}\right]_0^{\frac{1}{4}} = \left[\frac{(1-4x)^{\frac{3}{2}}}{-4\left(\frac{3}{2}\right)}\right]_0^{\frac{1}{4}}$$

$$= \left[\frac{(1-4x)^{\frac{3}{2}}}{-6}\right]_0^{\frac{1}{4}}$$

$$= \frac{-1}{6}\left[\left(1-4\left(\frac{1}{4}\right)\right)^{\frac{3}{2}} - (1-4(0))^{\frac{3}{2}}\right]$$

$$= \frac{-1}{6}\left[0^{\frac{3}{2}} - 1^{\frac{3}{2}}\right]$$

$$= \frac{-1}{6}(0-1) = \frac{1}{6}$$

3. $\displaystyle\int_1^2 \frac{x\, dx}{x^2+1}$

Sol. Let I $= \displaystyle\int_1^2 \frac{x\, dx}{x^2+1}$

put $t = x^2 + 1$

$\Rightarrow dt = 2x\, dx$

$\Rightarrow \dfrac{dt}{2} = x\, dx$

$\therefore$ I $= \dfrac{1}{2}\displaystyle\int_2^5 \frac{dt}{t}$

When $x = 1$, $t = 1^2 + 1 = 2$
When $x = 2$, $t = 2^2 + 1 = 5$

$$= \frac{1}{2}\left[\log|t|\right]_2^5$$

$$= \frac{1}{2}\left[\log 5 - \log 2\right]$$

$$= \frac{1}{2}\log\left(\frac{5}{2}\right)$$

4. $\displaystyle\int_0^3 \frac{e^x}{1+e^x}\, dx$

Sol. Let I $= \displaystyle\int_0^3 \frac{e^x}{1+e^x}\, dx$

put $t = 1 + e^x$

$\Rightarrow dt = e^x\, dx$

When $x = 0$, $t = 1 + e^0 = 1 + 1 = 2$

When $x = 3$, $t = 1 + e^3$

$\therefore I = \int_2^{1+e^3} \frac{dt}{t} = [\log t]_2^{1+e^3}$

$= \log(1 + e^3) - \log 2$

$= \log\left(\frac{1+e^3}{2}\right)$

5. $\int_0^1 x e^{x^2} dx$

Sol. Let $I = \int_0^1 x e^{x^2} dx$

put $t = x^2$

$\Rightarrow dt = 2x\, dx$

$\Rightarrow \frac{dt}{2} = x\, dx$

When $x = 0$, $t = 0^2 = 0$

When $x = 1$, $t = 1^2 = 1$

$\therefore I = \int_0^1 e^t \frac{dt}{2} = \frac{1}{2}\int_0^1 e^t dt$

$= \frac{1}{2}\left[e^t\right]_0^1 = \frac{1}{2}(e^1 - e^0)$

$= \frac{1}{2}(e - 1)$

6. $\int_1^e \frac{dx}{x(1+\log x)^3}$

Sol. Let $I = \int_1^e \frac{dx}{x(1+\log x)^3}$

Let $t = 1 + \log x$

$\Rightarrow dt = \frac{1}{x}\, dx$

When $x = 1$, $t = 1 + \log 1 = 1 + 0 = 1$

When $x = e$, $t = 1 + \log e = 1 + 1 = 2$

$\therefore I = \int_1^2 \frac{dt}{t^3} = \int_1^2 t^{-3}\, dt$

$= \left[\frac{t^{-3+1}}{-3+1}\right]_1^2$

$= \left[\frac{t^{-2}}{-2}\right]_1^2 = \left[-\frac{1}{2t^2}\right]_1^2$

$= -\frac{1}{2}\left[\frac{1}{2^2} - \frac{1}{1^2}\right]$

$= -\frac{1}{2}\left[\frac{1}{4} - 1\right] = -\frac{1}{2}\left(\frac{-3}{4}\right) = \frac{3}{8}$

7. $\int_{-1}^1 \frac{(2x+3)\, dx}{x^2+3x+7}$

Sol. Let $t = x^2 + 3x + 7$

$\Rightarrow dt = (2x + 3)\, dx$

When $x = -1$, $t = (-1)^2 + 3(-1) + 7 = 1 - 3 + 7 = 5$

When $x = 1$, $t = 1^2 + 3(1) + 7 = 11$

$\therefore I = \int_5^{11} \frac{dt}{t} = (\log t)_5^{11}$

$= \log 11 - \log 5$

$= \log\left(\frac{11}{5}\right)$

8. $\int_0^{\frac{\pi}{2}} \sqrt{1+\cos x}\; dx$

Sol. Let $I = \int_0^{\frac{\pi}{2}} \sqrt{1+\cos x}\; dx$

We know that $\cos 2x = 2\cos^2 x - 1$

$\Rightarrow 1 + \cos 2x = 2\cos^2 x$

$\Rightarrow 1 + \cos x = 2\cos^2 \frac{x}{2}$

$\therefore I = \int_0^{\frac{\pi}{2}} \sqrt{2\cos^2\frac{x}{2}}\; dx$

$= \sqrt{2}\int_0^{\frac{\pi}{2}} \cos\frac{x}{2}\; dx$

$= \sqrt{2}\left(\frac{\sin\frac{x}{2}}{\frac{1}{2}}\right)_0^{\frac{\pi}{2}}$

$= 2\sqrt{2}\left(\sin\frac{\pi}{2(2)} - \sin\frac{0}{2}\right)$

$= 2\sqrt{2}\left(\sin\frac{\pi}{4} - \sin 0\right)$

$= 2\sqrt{2}\left[\frac{1}{\sqrt{2}} - 0\right]$

$= \frac{2\cancel{\sqrt{2}}}{\cancel{\sqrt{2}}} = 2$

9. $\int_1^2 \frac{x-1}{x^2}\,dx$

Sol. Let $I = \int_1^2 \frac{x-1}{x^2}\,dx$

$= \int_1^2 \left(\frac{x}{x^2} - \frac{1}{x^2}\right) dx$

$= \int_1^2 \left(\frac{1}{x} - x^{-2}\right) dx$

$= \left(\log x - \frac{x^{-2+1}}{-2+1}\right)_1^2$

$= \left(\log x - \frac{x^{-1}}{-1}\right)_1^2 = \left(\log x - \frac{1}{x}\right)_1^2$

$= \left(\log 2 + \frac{1}{2}\right) - \left(\log 1 + \frac{1}{1}\right)$

$= \log 2 + \frac{1}{2} - (0+1)$

$= \log 2 + \frac{1}{2} - 1$

$= \log 2 - \frac{1}{2}$

$= \frac{2\log 2 - 1}{2}$

$= \frac{1}{2}(2\log 2 - 1)$

II. Evaluate the following

1. $\int_1^4 f(x)\,dx$ where $f(x) = \begin{cases} 4x+3, & 1 \le x \le 2 \\ 3x+5, & 2 < x \le 4 \end{cases}$

Sol. $\int_1^4 f(x)\,dx = \int_1^2 f(x)\,dx + \int_2^4 f(x)\,dx$

$= \int_1^2 (4x+3)\,dx + \int_2^4 (3x+5)\,dx$

$= \left(\frac{\cancel{4}^{2}x^2}{\cancel{2}} + 3x\right)_1^2 + \left(\frac{3x^2}{2} + 5x\right)_2^4$

$= (2x^2 + 3x)_1^2 + \left(\frac{3x^2}{2} + 5x\right)_2^4$

$= [2(2^2) + 3(2)] - [2(1)^2 + 3(1)]$

$+ \left[\frac{3(4^2)}{2} + 5(4)\right] - \left[\frac{3(2^2)}{2} + 5(2)\right]$

$= (8+6) - (2+3) + (24+20) - (6+10)$

$= 14 - 5 + 44 - 16$

$= 58 - 21 = 37$

2. $\int_0^2 f(x)dx$ where $f(x) = \begin{cases} 3-2x-x^2, & x \le 1 \\ x^2+2x-3, & 1 < x \le 2 \end{cases}$

Sol. $\int_0^2 f(x)\,dx = \int_0^1 f(x)\,dx + \int_1^2 f(x)\,dx$

$= \int_0^1 (3-2x-x^2)\,dx + \int_1^2 (x^2+2x-3)\,dx$

$= \left(3x - \frac{\cancel{2}x^2}{\cancel{2}} - \frac{x^3}{3}\right)_0^1 + \left(\frac{x^3}{3} + \frac{\cancel{2}x^2}{\cancel{2}} - 3x\right)_1^2$

$= \left(3 - 1 - \frac{1}{3}\right) - (0) + \left(\frac{8}{3} + 4 - 6\right) - \left(\frac{1}{3} + 1 - 3\right)$

$= \left(\cancel{2} - \frac{1}{3}\right) + \left(\frac{8}{3} - \cancel{2}\right) - \left(\frac{1}{3} - 2\right)$

$= -\frac{1}{3} + \frac{8}{3} - \frac{1}{3} + 2 = \frac{-1+8-1}{3} + 2$

$= \frac{6}{3} + 2 = 2 + 2 = 4$

3. $\int_{-1}^1 f(x)dx$ where $f(x) = \begin{cases} x, & x \ge 0 \\ -x, & x < 0 \end{cases}$

Sol. $\int_{-1}^1 f(x)\,dx = \int_{-1}^0 f(x)\,dx + \int_0^1 f(x)\,dx$

$= \int_{-1}^0 -x\,dx + \int_0^1 x\,dx$

$= \int_0^{-1} x\,dx + \int_0^1 x\,dx$

$\left[\because \int_a^b f(x)\,dx = -\int_b^a f(x)\,dx\right]$

$= \left[\frac{x^2}{2}\right]_0^{-1} + \left[\frac{x^2}{2}\right]_0^1$

$= \frac{1}{2}[(-1)^2 - 0)] + \frac{1}{2}(1^2 - 0)$

$$= \frac{1}{2}(1) + \frac{1}{2}(1)$$

$$= \frac{1}{2} + \frac{1}{2} = 1$$

$$= 1$$

4. $f(x) = \begin{cases} cx, & 0 < x < 1 \\ 0, & \text{otherwise} \end{cases}$ **Find 'c' if** $\int_0^1 f(x)\,dx$

Sol. Given $f(x) = \begin{cases} cx, 0 < x < 1 \\ 0, \text{otherwise} \end{cases}$

Also, given $\int_0^1 f(x)\,dx = 2$

$$\Rightarrow \int_0^1 cx\,dx = 2$$

$$\Rightarrow c\left[\frac{x^2}{2}\right]_0^1 = 2$$

$$\Rightarrow \frac{c}{2}[1^2 - 0^2] = 2$$

$$\Rightarrow \frac{c}{2}(1-0) = 2$$

$$\Rightarrow \frac{c}{2} = 2$$

$$\Rightarrow c = 4$$

EXERCISE 2.9

Evaluate the following using properties of definite integrals

1. $\int_{-\frac{\pi}{4}}^{\frac{\pi}{4}} x^3 \cos^3 x\,dx$

Sol. Let $f(x) = x^3 \cos^3 x$

$f(-x) = (-x)^3 [\cos(-x)]^3$

$= -x^3 (\cos x)^3$

[Since $\cos x$ is an even function]

$= -f(x)$

$\therefore f(-x) = -f(x) \Rightarrow f(x)$ is an odd function

By the property, $\int_{-a}^{a} f(x)\,dx = 0$ if $f(x)$ is an odd function

$$\Rightarrow \int_{-\frac{\pi}{4}}^{\frac{\pi}{4}} x^3 \cos^3 x\,dx = 0$$

2. $\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \sin^2\theta\,d\theta$

Sol. Let $f(\theta) = \sin^2\theta$

$f(-\theta) = [\sin(-\theta)]^2$

$= [-\sin\theta]^2$

[Since $\sin\theta$ is an odd function]

$= \sin^2\theta$

$= f(\theta)$

$\therefore f(-\theta) = f(\theta) \Rightarrow f(\theta)$ is an even function

$\therefore$ By the property, $\int_{-a}^{a} f(x)\,dx$

$= 2\int_0^a f(x)\,dx$ if $f(x)$ is an even function

$$\Rightarrow \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \sin^2\theta\,d\theta = 2\int_0^{\frac{\pi}{2}} \sin^2\theta\,d\theta$$

$$= 2\int_0^{\frac{\pi}{2}} \frac{(1-\cos 2\theta)}{2}\,d\theta$$

$[\because \cos 2\theta = 1 - 2\sin^2\theta$
$\Rightarrow 2\sin^2\theta = 1 - \cos 2\theta$
$\Rightarrow \sin^2\theta = \frac{1-\cos 2\theta}{2}]$

$$= \int_0^{\frac{\pi}{2}} (1-\cos 2\theta)\,d\theta = \left[\theta - \frac{\sin 2\theta}{2}\right]_0^{\frac{\pi}{2}}$$

$$= \left(\frac{\pi}{2} - \frac{\sin\left(2\times\frac{\pi}{2}\right)}{2}\right) - \left(0 - \frac{\sin(2)(0)}{2}\right)$$

$$= \frac{\pi}{2} - \frac{\sin\pi}{2} - 0 + 0$$

$[\because \sin 0 = 0$ and $\sin\pi = 0$]

$$= \frac{\pi}{2} - \frac{0}{2}$$

$$= \frac{\pi}{2}$$

3. $\int_{-1}^{1} \log\left(\frac{2-x}{2+x}\right) dx$

Sol. Let $f(x) = \log\left(\frac{2-x}{2+x}\right)$

$= \log(2-x) - \log(2+x)$

$f(-x) = \log(2-(-x)) - \log(2-x)$

$= \log(2+x) - \log(2-x)$

$= -[\log(2-x) - \log(2+x)]$

$= -f(x)$

$\therefore f(-x) = -f(x) \Rightarrow f(x)$ is an odd function.

$\therefore$ By the property, $\int_{-a}^{a} f(x)\,dx = 0$ if $f(x)$ is an odd function

$$\Rightarrow \int_{-1}^{1} \log\left(\frac{2-x}{2+x}\right)dx = 0$$

4. $\displaystyle\int_0^{\frac{\pi}{2}} \frac{\sin^7 x}{\sin^7 x + \cos^7 x}dx$

Sol. Let I $= \displaystyle\int_0^{\frac{\pi}{2}} \frac{\sin^7 x}{\sin^7 x + \cos^7 x}dx$ --- (1)

By the property, $\int_0^a f(x)\,dx = \int_0^a f(a-x)\,dx$

$$I = \int_0^{\frac{\pi}{2}} \frac{\sin^7\left(\frac{\pi}{2}-x\right)}{\sin^7\left(\frac{\pi}{2}-x\right)+\cos^7\left(\frac{\pi}{2}-x\right)}dx$$

$$I = \int_0^{\frac{\pi}{2}} \frac{\cos^7 x\,dx}{\cos^7 x + \sin^7 x} \quad \text{---(2)}$$

$[\because \sin\left(\frac{\pi}{2}-x\right) = \cos x$ and $\cos\left(\frac{\pi}{2}-x\right) = \sin x]$

Adding (1) and (2) we get,

$$2I = \int_0^{\frac{\pi}{2}}\left(\frac{\sin^7 x}{\sin^7 x+\cos^7 x} + \frac{\cos^7 x}{\cos^7 x + \sin^7 x}\right)dx$$

$$= \int_0^{\frac{\pi}{2}} \frac{\cancel{\sin^7 x + \cos^7 x}}{\cancel{\sin^7 x + \cos^7 x}}dx$$

$$= \int_0^{\frac{\pi}{2}} dx = [x]_0^{\frac{\pi}{2}} = \frac{\pi}{2} - 0 = \frac{\pi}{2}$$

$$\therefore 2I = \frac{\pi}{2}$$

$$\Rightarrow I = \frac{\pi}{4}$$

5. $\displaystyle\int_0^1 \log\left(\frac{1}{x}-1\right)dx$

Sol. Let I $= \displaystyle\int_0^1 \log\left(\frac{1}{x}-1\right)dx$

$$I = \int_0^1 \log\left(\frac{1-x}{x}\right)dx \quad \text{---(1)}$$

By the property, $\int_0^a f(x)\,dx = \int_0^a f(a-x)\,dx$

$$I = \int_0^1 \log\left(\frac{1-(1-x)}{1-x}\right)dx$$

$$I = \int_0^1 \log\left(\frac{1-1+x}{1-x}\right)dx$$

$$= \int_0^1 \log\left(\frac{x}{1-x}\right)dx \quad \text{---(2)}$$

Adding (1) and (2) we get,

$$2I = \int_0^1 \left[\log\left(\frac{1-x}{x}\right) + \log\left(\frac{x}{1-x}\right)\right]dx$$

$$= \int_0^1 \left[\log\left(\frac{\cancel{1-x}}{\cancel{x}} \times \frac{\cancel{x}}{\cancel{1-x}}\right)\right]dx$$

$[\because \log m + \log n = \log mn]$

$$= \int_0^1 \log 1\,dx = 0 \qquad [\because \log 1 = 0]$$

$2I = 0$

$\Rightarrow I = 0$

6. $\displaystyle\int_0^1 \log \frac{x}{(1-x)^{\frac{3}{4}}}dx$

Sol. Let I $= \displaystyle\int_0^1 \frac{x}{(1-x)^{\frac{3}{4}}}dx$

$$= -\int_0^1 \frac{-x}{(1-x)^{\frac{3}{4}}}dx$$

[Multiply and divide by –1]

$$= -\int_0^1 \frac{1-x-1}{(1-x)^{\frac{3}{4}}}dx$$

[Adding and subtracting 1 in the numerator]

$$= \int_1^0 \frac{1-x-1}{(1-x)^{\frac{3}{4}}}dx$$

$$[\because \int_a^b f(x)\,dx = -\int_b^a f(x)\,dx\,]$$

$$= \int_1^0 \left(\frac{1-x}{(1-x)^{\frac{3}{4}}} - \frac{1}{(1-x)^{\frac{3}{4}}}\right)dx$$

$$= \int_1^0 \left((1-x)^{1-\frac{3}{4}} - (1-x)^{-\frac{3}{4}}\right)dx$$

$$= \int_1^0 (1-x)^{\frac{1}{4}}\,dx - \int_1^0 (1-x)^{-\frac{3}{4}}\,dx$$

$$= \left[\frac{(1-x)^{\frac{1}{4}+1}}{-1\left(\frac{1}{4}+1\right)} - \frac{(1-x)^{-\frac{3}{4}+1}}{-1\left(-\frac{3}{4}+1\right)}\right]_1^0$$

$$= \left[-\frac{(1-x)^{\frac{5}{4}}}{\frac{5}{4}} + \frac{(1-x)^{\frac{1}{4}}}{\frac{1}{4}}\right]_1^0$$

$$= \left[-\frac{4}{5}(1-x)^{\frac{5}{4}} + 4(1-x)^{\frac{1}{4}}\right]_1^0$$

$$= -\frac{4}{5}\left(1^{\frac{5}{4}}\right) + 4\left(1^{\frac{1}{4}}\right) - 0$$

$$= -\frac{4}{5}(1) + 4(1) = -\frac{4}{5} + 4$$

$$= \frac{-4+20}{5} = \frac{16}{5}$$

EXERCISE 2.10

1. Evaluate the following

(i) $\Gamma(4)$

Sol. Gamma integral $\Gamma(n+1) = n!$ where n is a positive integer.

$\therefore \Gamma(4) = (4-1)! = 3! = 3 \times 2 = 6$

(ii) $\Gamma\left(\frac{9}{2}\right)$

Sol. We know $\Gamma\left(\frac{1}{2}\right) = \sqrt{\pi}$

$$\Gamma\left(\frac{9}{2}\right) = \frac{7}{2}\Gamma\left(\frac{7}{2}\right)$$

$$= \frac{7}{2}\Gamma\left(\frac{5}{2}\right)$$

$$= \frac{7}{2}\times\frac{5}{2}\times\Gamma\left(\frac{3}{2}\right)$$

$$= \frac{7}{2}\times\frac{5}{2}\times\left(\frac{3}{2}\right)\times\Gamma\left(\frac{1}{2}\right)$$

$$= \frac{7}{2}\times\frac{5}{2}\times\frac{3}{2}\times\frac{1}{2}\sqrt{\pi}$$

$$= \frac{105}{16}\sqrt{\pi}$$

(iii) $\int_0^\infty e^{-mx}\,x^6 dx$

Sol. Let $I = \int_0^\infty e^{-mx}\,x^6 dx$

Gamma Integral $\int_0^\infty x^n e^{-ax}\,dx = \frac{n!}{a^{n+1}}$

Here $n = 6$, and $a = m$

$\therefore\ I = \int_0^\infty e^{-mx}\,x^6\,dx$

$$= \frac{6!}{m^{6+1}} = \frac{6!}{m^7}$$

(iv) $\int_0^\infty e^{-4x}\,x^4\,dx$

Sol. Let $I = \int_0^\infty e^{-4x}\,x^4\,dx$

Here $n = 4$ and $a = 4$

$\therefore\ I = \frac{4!}{4^{4+1}} = \frac{4!}{4^5}$

$$[\because \int_0^\infty x^n e^{-ax}dx = \frac{n!}{a^{n+1}}]$$

$$= \frac{\cancel{4}\times3\times\cancel{2}\times1}{\cancel{4}\times4\times4\times4\times\cancel{4}^{2}}$$

$$= \frac{3}{128}$$

(v) $\int_0^\infty e^{-\frac{x}{2}}x^5 dx$

Sol. Let $I = \int_0^\infty e^{-\frac{x}{2}}x^5 dx$

Here $n = 5$ and $a = \frac{1}{2}$

$$\left[\because \int_0^\infty x^n e^{-ax}dx = \frac{n!}{a^{n+1}}\right]$$

$$\therefore \text{ I} = \frac{5!}{\left(\frac{1}{2}\right)^{5+1}} = \frac{5!}{\left(\frac{1}{2}\right)^{6}} = 2^6(5!)$$

2. **If** $f(x)\begin{cases} x^2e^{-2x}, x\geq 0 \\ 0, \text{ other wise}\end{cases}$ **then evaluate** $\int_0^\infty f(x)\,dx.$

Sol. Given $f(x) = \begin{cases} x^2e^{-2x}, x\geq 0 \\ 0, \text{ other wise}\end{cases}$

$$\therefore \int_0^\infty f(x)\,dx = \int_0^\infty x^2e^{-2x}dx$$

Here $n = 2$ and $a = 2$

$$\therefore \int_0^\infty f(x)\,dx = \frac{2!}{2^{2+1}} = \frac{2!}{2^3}$$

$$\left[\because \int_0^\infty x^n e^{-ax}dx = \frac{n!}{a^{n+1}}\right]$$

$$= \frac{\cancel{2}\times 1}{2\times 2\times \cancel{2}} = \frac{1}{4}$$

$$\therefore \int_0^\infty f(x)\,dx = \frac{1}{4}$$

EXERCISE 2.11

Evaluate the following integrals as the limit of the sum

(1) $\int_0^1 (x+4)dx$

Sol. $$\int_a^b f(x)dx = \lim_{\substack{x\to\infty \\ h\to 0}} \sum_{r=1}^n h\,f(a+rh)$$

Here $a = 0, b = 1$

$$h = \frac{b-a}{n} = \frac{1-0}{n} = \frac{1}{n}$$

and $f(x) = x+4$

$$f(a+rh) = f\left(0+\frac{r}{n}\right) = f\left(\frac{r}{n}\right)$$

$$= \frac{r}{n}+4$$

$$\therefore \int_0^1 (x+4)dx = \lim_{n\to\infty}\sum_{r=1}^n \frac{1}{n}\left(\frac{r}{n}+4\right)$$

$$= \lim_{n\to\infty}\sum_{r=1}^n \left(\frac{r}{n^2}+\frac{4}{n}\right)$$

$$= \lim_{n\to\infty}\left(\frac{1}{n^2}\sum_{r=1}^n r+\frac{4}{n}\sum_{r=1}^n 1\right)$$

$$= \lim_{n\to\infty}\left(\frac{1}{n^2}\cdot\frac{n(n+1)}{2}+\frac{4}{\cancel{n}}\cdot(\cancel{n})\right)$$

$$\left(\sum_{r=1}^n 1 = n\right)\sum_{r=1}^n r = \frac{n(n+1)}{2}$$

$$= \lim_{n\to\infty}\left(\frac{1}{\cancel{n^2}}\cdot\frac{\cancel{n^2}\left(1+\frac{1}{n}\right)}{2}+4\right)$$

$$= \lim_{n\to\infty}\frac{1+\frac{1}{n}}{2}+4$$

$$= \frac{1+0}{2}+4$$

$$\left[\because \text{when } n\to\infty, \frac{1}{n}\to 0\right]$$

$$= \frac{1}{2}+4 = \frac{1+8}{2} = \frac{9}{2}$$

(2) $\int_1^3 x\,dx$

Sol. $$\int_a^b f(x)dx = \lim_{\substack{n\to\infty \\ h\to 0}} \sum_{r=1}^n h.f(a+rh)$$

Here $a = 1,\quad b = 3$

$$h = \frac{b-a}{n} = \frac{3-1}{n} = \frac{2}{n}$$

and $f(x) = x$

Now, $f(a+rh) = f\left(1+r\cdot\frac{2}{n}\right) = f\left(1+\frac{2r}{n}\right)$

$$\therefore \int_1^3 x\,dx = \lim_{n\to\infty}\sum_{r=1}^n \frac{2}{n}\left(1+\frac{2r}{n}\right)$$

$$= \lim_{n\to\infty}\sum_{r=1}^n \left(\frac{2}{n}+\frac{4r}{n^2}\right)$$

$$= \lim_{n\to\infty}\left(\frac{2}{n}\cdot\sum_{r=1}^n 1+\frac{4}{n^2}\cdot\sum_{r=1}^n r\right)$$

$$\left(\because \sum_{r=1}^n 1 = n\right)$$

$$= \lim_{n\to\infty}\left(\frac{2}{\cancel{n}}\cdot\cancel{n}+\frac{\overset{2}{\cancel{4}}}{n^2}\cdot\frac{n(n+1)}{\cancel{2}}\right)$$

$$\left[\sum_{r=1}^{n} r = \frac{n(n+1)}{2}\right]$$

$$= \lim_{n\to\infty}\left(2+\frac{\cancel{4}^{2}}{\cancel{n^2}}\cdot \cancel{n^2}\frac{\left(1+\frac{1}{n}\right)}{\cancel{2}}\right)$$

$$= 2+\lim_{n\to\infty}\left(1+\frac{1}{n}\right)^2$$

$$= 2+\lim_{n\to\infty}\left(2+\frac{2}{n}\right)$$

$$= 2+2+\lim_{n\to\infty}\frac{2}{n}$$

$$\left[\because \text{ when } n\to\infty, \frac{1}{n}\to 0\right]$$

$$= 2+2+0$$

$$= 4$$

(3) $\int_1^3 (2x+3)dx$

Sol. $\int_a^b f(x)dx = \lim_{\substack{n\to\infty\\h\to 0}}\sum_{r=1}^{n} h.f(a+rh)$

Here $a = 1$, $b = 3$

$$h = \frac{b-a}{n} = \frac{3-1}{n} = \frac{2}{n}$$

and $f(x) = 2x + 3$

$$\therefore\ f(a+rh) = f\left(1+r\cdot\frac{2}{n}\right) = f\left(1+\frac{2r}{n}\right)$$

$$= 2\left(1+\frac{2r}{n}\right)+3$$

$$= 2+\frac{4r}{n}+3 = 5+\frac{4r}{n}$$

$$\therefore \int_1^3 (2x+3)dx = \lim_{n\to\infty}\sum_{r=1}^{n}\frac{2}{n}\left(5+\frac{4r}{n}\right)$$

$$= \lim_{n\to\infty}\sum_{r=1}^{n}\left(\frac{10}{n}+\frac{8r}{n^2}\right)$$

$$= \lim_{n\to\infty}\left(\frac{10}{n}\cdot\sum_{r=1}^{n}1+\frac{8}{n^2}\cdot\sum_{r=1}^{n}r\right)$$

$$= \lim_{n\to\infty}\left(\frac{10}{\cancel{n}}\times\cancel{n}+\frac{8}{n^2}\cdot\frac{n(n+1)}{2}\right)$$

$$\left[\because \Sigma 1 = n\ \&\ \sum_{r=1}^{n} r = \frac{n(n+1)}{2}\right]$$

$$= 10+\lim_{n\to\infty}\frac{8}{\cancel{n^2}}\cdot\frac{\cancel{n^2}\left(1+\frac{1}{n}\right)}{2}$$

$$= 10+4\cdot\lim_{n\to\infty}\left(1+\frac{1}{n}\right)$$

$$= 10 + 4\ (1 + 0)$$

$$\left[\because \text{ when } n\to\infty, \frac{1}{n}\to 0\right]$$

$$= 10 + 4 = 14$$

(4) $\int_0^1 x^2\, dx$

Sol. $\int_a^b f(x)dx = \lim_{\substack{n\to\infty\\h\to 0}}\sum_{r=1}^{n} h.f(a+rh)$

Here $a = 0$, $b = 1$

$$h = \frac{b-a}{n} = \frac{1-0}{n} = \frac{1}{n}$$

and $f(x) = x^2$

$$\text{Now, } f(a+rh) = f\left(0+r\cdot\frac{1}{n}\right) = f\left(\frac{r}{n}\right)$$

$$= \left(\frac{r}{n}\right)^2 = \frac{r^2}{n^2}$$

$$\therefore \int_0^1 x^2dx = \lim_{n\to\infty}\sum_{r=1}^{n}\cdot\frac{1}{n}\left(\frac{r^2}{n^2}\right)$$

$$= \lim_{n\to\infty}\frac{1}{n^3}\cdot\sum_{r=1}^{n}\cdot r^2$$

$$= \lim_{n\to\infty}\frac{1}{n^3}\cdot\frac{n(n+1)(2n+1)}{6}$$

$$\left[\because \sum_{r=1}^{n} r^2 = \frac{n(n+1)(2n+1)}{6}\right]$$

$$\lim_{n\to\infty}\cdot\frac{1}{\cancel{n^3}}\cdot\frac{\cancel{n^3}\left(1+\frac{1}{n}\right)\left(2+\frac{1}{n}\right)}{6}$$

[Taking n common from each bracket]

$$= \frac{1}{6}\cdot\lim_{n\to\infty}\left(1+\frac{1}{n}\right)\left(2+\frac{1}{n}\right)$$

$$= \frac{1}{6}(1+0)(2+0)$$

$$[\because \text{ when } n \to \infty, \frac{1}{n} \to 0]$$

$$= \frac{1}{6}(1)(2) = \frac{\cancel{2}}{\cancel{6}_3}$$

$$= \frac{1}{3}$$

EXERCISE 2.12

Choose the correct answer:

1. $\int \frac{1}{x^3} dx$ is

(a) $\frac{-3}{x^2}+c$ (b) $\frac{-1}{2x^2}+c$

(c) $\frac{-1}{3x^2}+c$ (d) $\frac{-2}{x^2}+c$

[Ans: (b) $\frac{-1}{2x^2}+c$]

Hint: $= \int x^{-3} dx = \frac{x^{-3+1}}{-3+1} = \frac{x^{-2}}{-2} = \frac{-1}{2x^2}+c$

2. $\int 2^x dx$ is

(a) $2^x \log 2 + c$ (b) $2^x + c$

(c) $\frac{2^x}{\log 2}+c$ (d) $\frac{\log 2}{2^x}+c$

Hint: $= \frac{2^x}{\log 2}+c$ **[Ans: (c) $\frac{2^x}{\log 2}+c$]**

3. $\int \frac{\sin 2x}{2\sin x} dx$ is

(a) $\sin x + c$ (b) $\frac{1}{2}\sin x + c$

(b) $\cos x + c$ (d) $\frac{1}{2}\cos x + c$

[Ans: (a) $\sin x + c$]

Hint: $$= \int \frac{\cancel{2\sin x}\cos x}{\cancel{2\sin x}} dx$$

$$= \int \cos x\, dx$$

$$= \sin x + c$$

4. $\int \frac{\sin 5x - \sin x}{\cos 3x} dx$ is

(a) $-\cos 2x + c$ (b) $-\cos 2x + c$

(c) $-\frac{1}{4}\cos 2x + c$ (d) $-4\cos 2x + c$

[Ans: (a) $-\cos 2x + c$]

Hint: $\int \frac{\sin 5x - \sin x}{\cos 3x} dx$

$$= \int \frac{2\cos\left(\frac{5x+x}{2}\right)\cdot \sin\left(\frac{5x-x}{2}\right)\cdot dx}{\cos 3x}$$

$$[\text{since } \sin\alpha - \sin\beta = 2\cos\frac{(\alpha+\beta)}{2}\cdot\sin\frac{(\alpha-\beta)}{2}]$$

$$= 2\int \frac{\cancel{\cos 3x}.\sin 2x}{\cancel{\cos 3x}} dx$$

$$= 2\int \sin 2x\, dx$$

$$= \cancel{2}\left(-\frac{\cos 2x}{\cancel{2}}\right)+c$$

$$= -\cos 2x + c$$

5. $\int \frac{\log x}{x} dx, x > 0$ is

(a) $\frac{1}{2}(\log x)^2 + c$ (b) $-\frac{1}{2}(\log x)^2$

(c) $\frac{2}{x^2}+c$ (d) $\frac{2}{x^2}+c$

[Ans: (a) $\frac{1}{2}(\log x)^2 + c$]

Hint: $$= \int \frac{\log x}{x} dx, x > 0$$

$$\text{Put } \log x = t \Rightarrow \frac{1}{x} dx = dt$$

$$\therefore \quad I = \int t \cdot dt = \frac{t^2}{2}+c$$

$$= \frac{1}{2}(\log x)^2 + c$$

6. $\int \frac{e^x}{\sqrt{1+e^x}} dx$ is

(a) $\frac{e^x}{\sqrt{1+e^x}}+c$ (b) $2\sqrt{1+e^x}+c$

(c) $\sqrt{1+e^x}+c$ (d) $e^x\sqrt{1+e^x}+c$

[Ans: (b) $2\sqrt{1+e^x}+c$]

Hint: $I = \int \frac{e^x dx}{\sqrt{1+e^x}}$

$$\text{Put } 1 + e^x = t \Rightarrow e^x dx = dt$$

$$\therefore \quad I = \int \frac{dt}{\sqrt{t}} = \int t^{\frac{-1}{2}} dt = \frac{t^{-\frac{1}{2}+1}}{-\frac{1}{2}+1}$$

$$= \frac{t^{\frac{1}{2}}}{\frac{1}{2}} = 2\sqrt{t} + c$$

$$= 2\sqrt{1+e^x} + c$$

7. $\int \sqrt{e^x}\,dx$ **is**

(a) $\sqrt{e^x} + c$ (b) $2\sqrt{e^x} + c$

(c) $\frac{1}{2}\sqrt{e^x} + c$ (d) $\frac{1}{2\sqrt{e^x}} + c$

[Ans: (b) $2\sqrt{e^x} + c$]

Hint: $\int \sqrt{e^x}\,dx = \int (e^x)^{\frac{1}{2}}dx = \int e^{\frac{x}{2}}dx$

$$= \frac{e^{\frac{x}{2}}}{\frac{1}{2}} + c$$

$$= 2e^{\frac{x}{2}} + c = 2e^{x \times \frac{1}{2}} + c$$

$$= 2\sqrt{e^x} + c$$

8. $\int e^{2x}[2x^2 + 2x]dx$

(a) $e^{2x}x^2 + c$ (b) $xe^{2x} + c$

(c) $2x^2e^2 + c$ (d) $\frac{x^2e^x}{2} + c$

[Ans: (a) $e^{2x} \cdot x^2 + c$]

Hint: $\int e^{2x}[2x^2 + 2x]dx$

Let $f(x) = x^2 \Rightarrow f'(x) = 2x$

We know $\int e^{ax}(a \cdot f(x) + f'(x))\,dx = e^{ax} \cdot f(x) + c$

Here $a = 2$

$\int e^{2x}(2 \cdot f(x) + f'(x))\,dx = e^{2x} \cdot f(x) + c$

$= e^{2x} \cdot x^2 + c$

9. $\int \frac{e^x}{e^x + 1}dx$ **is**

(a) $\log\left|\frac{e^x}{e^x+1}\right| + c$ (b) $\log\left|\frac{e^x+1}{e^x}\right| + c$

(c) $\log|e^x| + c$ (d) $\log|e^x+1| + c$

[Ans: (a) $\log|e^x + 1| + c$]

Hint: $I = \int \frac{e^x}{e^x+1}dx$

Put $t = e^x + 1 \Rightarrow dt = e^x \cdot dx$

$I = \int \frac{dt}{t} = \log|t| + c$

$= \log|e^x + 1| + c$

10. $\int\left[\frac{9}{x-3} - \frac{1}{x+1}\right]dx$ **is**

(a) $\log|x-3| - \log|x+1| + c$

(b) $\log|x-3| + \log|x+1| + c$

(c) $9\log|x-3| - \log|x+1| + c$

(d) $9\log|x-3| + \log|x+1| + c$

[Ans: (c) $9\log|x-3| - \log|x+1| + c$]

Hint: $I = \int\left[\frac{9}{x-3} - \frac{1}{x+1}\right]dx$

$= 9 \cdot \int \frac{1}{x-3}dx - \int \frac{dx}{x+1}$

$= 9 \cdot \log|x-3| - \log|x+1|\; c$

11. $\int \frac{2x^3}{4+x^4}dx$ **is**

(a) $\log|4 + x^4| + c$ (b) $\frac{1}{2}\log|4+x^4| + c$

(c) $\frac{1}{4}\log|4+x^4| + c$ (d) $\log\left|\frac{2x^3}{4+x^4}\right| + c$

[Ans: (b) $\frac{1}{2}\log|4+x^4| + c$]

Hint: $I = \int \frac{2x^3}{4+x^4}dx$

Put $t = 4 + x^4$

$\Rightarrow dt = 0 + 4x^3\,dx$

$\frac{dt}{2} = 2 \cdot x^3\,dx$

$I = \frac{1}{2}\int \frac{dt}{t} = \frac{1}{2}\log|t| + c$

$= \frac{1}{2}\log|4+x^4| + c$

12. $\int \frac{dx}{\sqrt{x^2-36}}$ **is**

(a) $\sqrt{x^2-36} + c$

(b) $\log\left|x + \sqrt{x^2-36}\right| + c$

(c) $\log\left|x - \sqrt{x^2-36}\right| + c$

(d) $\log\left|x^2 + \sqrt{x^2-36}\right| + c$

[Ans: (b) $\log\left|x + \sqrt{x^2-36}\right| + c$]

Hint: $\int \frac{dx}{\sqrt{x^2-36}}$

$= \int \frac{dx}{\sqrt{x^2-6^2}}$

$= \log\left|x+\sqrt{x^2-36}\right|+c$

13. $\int \frac{2x+3}{\sqrt{x^2+3x+2}}dx$

(a) $\sqrt{x^2+3x+2}+c$

(b) $2\sqrt{x^2+3x+2}+c$

(c) $\log(x^2+3x+2)+c$

(d) $\frac{2}{3}\left(x^2+3x+2\right)^{\frac{3}{2}}+c$

[Ans: (b) $2\sqrt{x^2+3x+2}+c$]

Hint: $I = \int \frac{(2x+3)}{\sqrt{x^2+3x+2}}dx$

Put $t = x^2+3x+2$

$dt = (2x+3)dx$

$\therefore$ $I = \int \frac{dt}{\sqrt{t}} = \int t^{-\frac{1}{2}}dt$

$= \frac{t^{-\frac{1}{2}+1}}{-\frac{1}{2}+1}+c = \frac{t^{\frac{1}{2}}}{\frac{1}{2}}+c$

$= 2\sqrt{t}+c$

$= 2\sqrt{x^2+3x+2}f+c$

14. $\int_0^1 (2x+1)dx$ is

(a) 1 (b) 2 (c) 3 (d) 4

[Ans: (b) 2]

Hint: $\int_0^1 (2x+1)dx = \left[2\frac{x^2}{2}+x\right]_0^1$

$= \left[x^2+x\right]_0^1$

$= (1+1)-(0+0)$

$= 2-0 = 2$

15. $\int_2^4 \frac{dx}{x}$ is

(a) log 4 (b) 0

(c) log 2 (d) log 8

[Ans: (c) log 2]

Hint: $\int_2^4 \frac{dx}{x} = \left[\log x\right]_2^4 = \log 4 - \log 2$

$= \log\frac{4}{2} = \log 2$

16. $\int_0^\infty e^{-2x}dx$ is

(a) 0 (b) 1 (c) 2 (d) $\frac{1}{2}$

[Ans: (d) $\frac{1}{2}$]

Hint: $\int_0^\infty e^{-2x}dx = \left[\frac{e^{-2x}}{-2}\right]_0^\infty$

$= -\frac{1}{2}\left[e^{-\infty}-e^0\right]$

$= -\frac{1}{2}(0-1) = -\frac{1}{2}(-1)$

$= \frac{1}{2}$

17. $\int_{-1}^1 x^3 e^{x^4} dx$ is

(a) 1 (b) $2\int_0^1 x^3 e^{x^4}dx$

(c) 0 (d) e^{x^4}

[Ans: (c) 0]

Hint: $\int_{-1}^1 x^3 e^{x^4}dx$

Let $f(x) = x^3 e^{x^4}$

$f(-x) = (-x)^3 e^{(-x)^4}$

$= -x^3 e^{x^4} = -f(x)$

Since $f(x)$ is an odd function, $\int_{-1}^1 x^3 e^{x^4}dx = 0$

18. If $f(x)$ is a continuous function and $a < c < b$, then $\int_a^c f(x)dx + \int_c^b f(x)dx$ is

(a) $\int_a^b f(x)dx - \int_a^c f(x)dx$

(b) $\int_a^c f(x)\,dx - \int_a^b f(x)\,dx$

(c) $\int_a^b f(x)\,dx$ (d) 0

[Ans: (c) $\int_a^b f(x)\,dx$]

19. The value of $\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \cos x\,dx$ is

(a) 0 (b) 2 (c) 1 (d) 4

[Ans: (b) 2]

Hint: The value of $\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \cos x\,dx$

$$= 2\int_0^{\frac{\pi}{2}} \cos x\,dx$$

[$\because$ cos x is an even function]

$$= 2(\sin x)_0^{\frac{\pi}{2}}$$

$$= 2\left(\sin\frac{\pi}{2} - \sin 0\right)$$

$$= 2\,(1-0) = 2$$

20. $\int_0^1 \sqrt{x^4(1-x)^2}\,dx$ is

(a) $\frac{1}{12}$ (b) $\frac{-7}{12}$ (c) $\frac{7}{12}$ (d) $\frac{-1}{12}$

[Ans: (a) $\frac{1}{12}$]

Hint: $\int_0^1 \sqrt{x^4(1-x)^2}\,dx = \int_0^1 x^2(1-x)\,dx$

$$= \int_0^1 (x^2 - x^3)\,dx$$

$$= \left[\frac{x^3}{3} - \frac{x^4}{4}\right]_0^1 = \left(\frac{1}{3} - \frac{1}{4}\right) - 0$$

$$= \frac{4-3}{12} = \frac{1}{12}$$

21. If $\int_0^1 f(x)\,dx = 1$, $\int_0^1 x\,f(x)\,dx = a$, $\int_0^1 x^2 f(x)\,dx = a^2$, then $\int_0^1 (a-x)^2 f(x)\,dx$ is

(a) $4a^2$ (b) 0 (c) $2a^2$ (d) 1

[Ans: (b) 0]

Hint: If $\int_0^1 f(x)\,dx = 1$, $\int_0^1 x\,f(x)\,dx = a$,

$\int_0^1 x^2 f(x)\,dx = a^2$, then $\int_0^1 (a-x)^2 f(x)\,dx$

$$= \int_0^1 \left(a^2 + x^2 - 2ax\right) f(x)\,dx$$

$$= \int_0^1 a^2 f(x)\,dx + \int_0^1 x^2 f(x)\,dx - \int_0^1 2ax\,f(x)\,dx$$

$$= a^2\int_0^1 f(x)\,dx + a^2 - 2a\int_0^1 x\,f(x)\,dx$$

$$= a^2\,(1) + a^2 - 2a(a)$$

$$= 2a^2 - 2a^2 = 0$$

22. The value of $\int_2^3 (5-x)\,dx - \int_2^3 f(x)\,dx$ is

(a) 1 (b) 0 (c) −1 (d) 5

[Ans: (b) 0]

Hint: $\int_2^3 f(5-x)\,dx - \int_2^3 f(x)\,dx$

$$= \int_2^3 f(5-x)\,dx - \int_2^3 f(2+3-x)\,dx$$

$$\left[\because \int_a^b f(x)\,dx = \int_a^b f(a+b-x)\,dx\right]$$

$$= \int_2^3 f(5-x)\,dx - \int_2^3 f(5-x)\,dx$$

$$= 0$$

23. $\int_0^4 \left(\sqrt{x} + \frac{1}{\sqrt{x}}\right) dx$ is

(a) $\frac{20}{3}$ (b) $\frac{21}{3}$ (c) $\frac{28}{3}$ (d) $\frac{1}{3}$

[Ans: (c) $\frac{28}{3}$]

Hint: $\int_0^4 \left(\sqrt{x}+\frac{1}{\sqrt{x}}\right)dx$

$$= \int_0^4 \left(x^{\frac{1}{2}}+x^{-\frac{1}{2}}\right)dx$$

$$= \left[\frac{x^{\frac{3}{2}}}{\frac{3}{2}}+\frac{x^{\frac{1}{2}}}{\frac{1}{2}}\right]_0^4$$

$$= \left[\frac{2}{3}\left(x^{\frac{3}{2}}\right)+2\sqrt{x}\right]_0^4$$

$$= \left[\frac{2}{3}x\sqrt{x}+2\sqrt{x}\right]_0^4$$

$$= \left(\frac{2}{3}(4)\sqrt{4}+2\sqrt{4}\right)-0$$

$$= \frac{8}{3}(2)+2(2)$$

$$= \frac{16}{3}+4 = \frac{16+12}{3} = \frac{28}{3}$$

24. $\int_0^{\frac{\pi}{3}} \tan x\, dx$ **is**

(a) log 2 (b) 0

(c) $\log\sqrt{2}$ (d) 2 log 2

[Ans: (a) log 2]

Hint: $I = \int_0^{\frac{\pi}{3}} \tan x\, dx = \int_0^{\frac{\pi}{3}} \frac{\sin x}{\cos x} dx$

Put $\cos x = t \Rightarrow -\sin x\, dx = dt$

$\Rightarrow \sin x\, dx = -dt$

$\therefore I = -\int_1^{\frac{1}{2}} \frac{dt}{t}$ [$\because$ when $x = 0$, $t = \cos 0 = 1$ when $x = \frac{\pi}{3}$, $t = \cos\frac{\pi}{3} = \frac{1}{2}$]

$$= \int_{\frac{1}{2}}^{1} \frac{dt}{t} = [\log(t)]_{\frac{1}{2}}^{1}$$

$$= \log 1 - \log\frac{1}{2}$$

$$= \log 1 - (\log 1 - \log 2)$$

$$= 0 - 0 + \log 2 = \log 2$$

25. Using the factorial representation of the gamma function, which of the following is the solution for the gamma function $\Gamma(n)$ when $n = 8$

(a) 5040 (b) 5400

(c) 4500 (d) 5540

[Ans: (a) 5040]

Hint: $\Gamma(n)$ when $n = 8$ is 7!

$$= 7 \times 6 \times 5 \times 4 \times 3 \times 2 \times 1$$

$$= 5040$$

26. $\Gamma(n)$ is

(a) $(n-1)!$ (b) $n!$

(c) $n\,\Gamma(n)$ (d) $(n-1)\,\Gamma(n)$

[Ans: (b) $(n-1)!$]

27. $\Gamma(1)$ is

(a) 0 (b) 1 (c) n (d) $n!$

Hint: $\Gamma(n) = (n-1)!$

$\Rightarrow \Gamma(1) = (1-1)! = 0! = 1$

[Ans: (b) 1]

28. If $n > 0$, then $\Gamma(n)$ is

(a) $\int_0^1 e^{-x}x^{n-1}dx$ (b) $\int_0^1 e^{-x}x^{n}dx$

(c) $\int_0^\infty e^{-x}x^{-n}dx$ (d) $\int_0^\infty e^{-x}x^{n-1}dx$

[Ans: (c) $\int_0^1 e^{-x}x^{n-1}dx$]

29. $\Gamma\left(\frac{3}{2}\right)$

(a) $\sqrt{\pi}$ (b) $\frac{\sqrt{\pi}}{2}$ (c) $2\sqrt{\pi}$ (d) $\frac{3}{2}$

Hint: $= \frac{1}{2}\times\sqrt{\pi} = \frac{\sqrt{\pi}}{2}$ **[Ans: (c) $\frac{\sqrt{\pi}}{2}$]**

30. $\int_0^\infty x^4 e^{-x}\, dx$ **is**

(a) 12 (b) 4 (c) 4! (d) 64

[Ans: (c) 4!]

Hint: $\int_0^\infty x^4 e^{-x}\, dx$

Here $a = 1, n = 4$

$$[\because \int_0^\infty x^n e^{-ax}\, dx = \frac{n!}{a^{n+1}}]$$

$$= \frac{4!}{(1)^5} = 4!$$

Miscellaneous problems

Evaluate the following integrals

1. $\displaystyle\int \frac{1}{\sqrt{x+2}-\sqrt{x+3}}\,dx$

Sol. $\displaystyle I = \int \frac{1}{\sqrt{x+2}-\sqrt{x+3}}\,dx$

Multiply and divide with the conjugate of the denominator,

We get

$$I = \int \frac{\sqrt{x+2}+\sqrt{x+3}}{\left(\sqrt{x+2}-\sqrt{x+3}\right)\left(\sqrt{x+2}+\sqrt{x+3}\right)}$$

$$= \int \frac{\sqrt{x+2}+\sqrt{x+3}}{(x+2)-(x+3)}\,dx$$

$[\because (a+b)(a-b) = a^2 - b^2]$

$$= \int \frac{\sqrt{x+2}+\sqrt{x+3}}{\cancel{x}+2-\cancel{x}-3}\,dx$$

$$= -\int \left(\sqrt{x+2}+\sqrt{x+3}\right)dx$$

$$= \left[\frac{(x+2)^{\frac{1}{2}+1}}{\frac{1}{2}+1}+\frac{(x+3)^{\frac{1}{2}+1}}{\frac{1}{2}+1}\right]+c$$

$$= -\frac{2}{3}\left[(x+2)^{\frac{3}{2}}+(x+3)^{\frac{3}{2}}\right]+c$$

2. $\displaystyle\int \frac{dx}{2-3x-2x^2}$

Sol.

$$I = \int \frac{dx}{2-3x-2x^2}$$

$$= -\frac{1}{2}\int \frac{dx}{x^2+\frac{3}{2}x-1}$$

$$= -\frac{1}{2}\int \frac{dx}{x^2+\frac{3}{2}x+\frac{9}{16}-\frac{9}{16}-1}$$

$$\boxed{\left[\frac{1}{2}\left(\frac{3}{2}\right)\right]^2 = \left(\frac{3}{4}\right)^2 \quad \text{Adding \& subtracting} = \frac{9}{16}}$$

$$= -\frac{1}{2}\int \frac{dx}{\left(x+\frac{3}{4}\right)^2-\frac{25}{16}}$$

$$= \frac{1}{2}\int \frac{dx}{\frac{25}{16}-\left(x+\frac{3}{4}\right)^2}$$

$$= \frac{1}{2}\int \frac{dx}{\left(\frac{5}{4}\right)^2-\left(x+\frac{3}{4}\right)^2}$$

$$\left[\because \int \frac{dx}{a^2-x^2} = \frac{1}{2a}\log\left(\frac{a+x}{a-x}\right)+c\right]$$

$$= \frac{1}{2}\left[\frac{1}{2\times\frac{5}{4}}\log\left|\frac{\frac{5}{4}+x+\frac{3}{4}}{\frac{5}{4}-x-\frac{3}{4}}\right|\right]+c$$

$$= \frac{1}{\cancel{2}}\left[\frac{1}{\frac{5}{\cancel{2}}}\log\left|\frac{2+x}{\frac{1}{2}-x}\right|\right]+c$$

$$= \frac{1}{5}\log\left|\frac{2+x}{1-2x}\right|+c$$

3. $\displaystyle\int \frac{dx}{e^x+6+5e^{-x}}$

Sol.

$$I = \int \frac{dx}{e^x+6+\frac{5}{e^x}}$$

$$= \int \frac{e^x\,dx}{e^{2x}+6e^x+5}$$

Put $e^x = t \Rightarrow e^x dx = dt$

$$\Rightarrow I = \int \frac{dt}{t^2+6t+5}$$

$$I = \int \frac{dt}{(t+5)(t+1)}$$

$$= \int \frac{A}{t+5}dt+\int\frac{B}{t+1}dt \quad \text{-----(1)}$$

$$\frac{1}{(t+5)(t+1)} = \frac{A}{t+5}+\frac{B}{t+1}$$

$\Rightarrow$ 1 = A(t+1) + B(t+5)

Put t = – 1

$\Rightarrow$ 1 = B(–1 + 5)

$$\Rightarrow \quad 1 = B(4)$$

$$\Rightarrow \quad \boxed{B = \frac{1}{4}}$$

Put $t = -5$

$$\Rightarrow \quad 1 = A(-5+1)$$

$$1 = A(-4)$$

$$\Rightarrow \quad \boxed{A = -\frac{1}{4}}$$

From (1),
$$I = \int \frac{-\frac{1}{4}dt}{t+5} + \int \frac{\frac{1}{4}dt}{t+1}$$

$$= -\frac{1}{4}\log|t+5| + \frac{1}{4}\log|t+1| + c$$

$$= \frac{1}{4}\left[\log|t+1| - \log|t+5|\right] + c$$

$$= \frac{1}{4}\log\left|\frac{t+1}{t+5}\right| + c$$

$$= \frac{1}{4}\log\left|\frac{e^x+1}{e^x+5}\right| + c$$

$[\because t = e^x]$

4. $\int \sqrt{2x^2-3}\,dx$

Sol.

$$I = \int \sqrt{2x^2-3}\,dx$$

$$= \int \sqrt{2\left(x^2 - \frac{3}{2}\right)}\,dx$$

$$= \sqrt{2}\int \sqrt{x^2 - \left(\frac{\sqrt{3}}{\sqrt{2}}\right)^2}\,dx$$

$$\left[\because \int \sqrt{x^2-a^2}\,dx = \frac{x}{2}\sqrt{x^2-a^2} - \frac{a^2}{2}\log\left|x+\sqrt{x^2-a^2}\right| + c\right]$$

$$= \sqrt{2}\left[\frac{x}{2}\sqrt{x^2-\frac{3}{2}} - \frac{3}{2(2)}\log\left|x+\sqrt{x^2-\frac{3}{2}}\right|\right] + c$$

$$= \sqrt{2}\left[\frac{x}{2}\frac{\sqrt{2x^2-3}}{\sqrt{2}} - \frac{3}{4}\log\left|\sqrt{2}x+\sqrt{2x^2-3}\right|\right] + c$$

$$= \frac{x}{2}\sqrt{2x^3-3} - \frac{3\sqrt{2}}{4}\log\left|\sqrt{2}x+\sqrt{2x^2-3}\right| + c$$

5. $\int \sqrt{9x^2+12x+3}\,dx$

Sol.

Let
$$I = \int \sqrt{9x^2+12x+3}\,dx$$

$$= \int \sqrt{9\left(x^2+\frac{12}{9}x+\frac{3}{9}\right)}\,dx$$

$$= 3\int \sqrt{x^2+\frac{4}{3}x+\frac{1}{3}}\,dx$$

$$= 3\int \sqrt{x^2+\frac{4}{3}x+\frac{4}{9}-\frac{4}{9}+\frac{1}{3a}}\,dx$$

$$\left[\frac{1}{2}\left(\frac{4}{3}\right)\right]^2 = \left(\frac{2}{3}\right)^2 = \frac{4}{9}$$

Adding & subtracting $\frac{4}{9}$

$$= 3\int \sqrt{\left(x+\frac{2}{3}\right)^2 - \left(\frac{1}{3}\right)^2}\,dx$$

$$= 3\int \sqrt{\left(x+\frac{2}{3}\right)^2 - \left(\frac{1}{3}\right)^2}\,dx$$

$$\left[\because \int \sqrt{x^2-a^2}\,dx = \frac{x}{2}\sqrt{x^2-a^2} - \frac{a^2}{2}\log\left|x+\sqrt{x^2-a^2}\right| + c\right]$$

$$= 3\left[\frac{\left(x+\frac{2}{3}\right)}{2}\sqrt{x^2+\frac{12x}{9}+\frac{3}{9}} - \frac{1}{9(2)}\log\left|x+\frac{2}{3}+\sqrt{x^2+\frac{12x}{9}+\frac{3}{9}}\right|\right] + c$$

$$= 3\left[\frac{3x+2}{6}\frac{\sqrt{9x^2+12x+3}}{3} - \frac{1}{18}\log\left|3x+2+\sqrt{9x^2+12x+3}\right|\right] + c$$

$$= \frac{3x+2}{6}\sqrt{9x^2+12x+3} - \frac{1}{6}\log\left|3x+2+\sqrt{9x^2+12x+3}\right| + c$$

6. $\int (x+1)^2 \log x\,dx$

Sol.

Let
$$I = \int (x+1)^2 \log x\,dx$$

Let $u = \log x; \quad dv = (x+1)^2\,dx$

$du = \frac{1}{x}dx; \quad v = \frac{(x+1)^3}{3}$

∴ Using integration by parts,

I L A T E
logarithmic function

$$I = \int u\,dv = uv - \int v\,du$$

$$I = \int (x+1)^2 \log x\,dx$$

$$= \frac{(x+1)^3}{3}\log x - \int \frac{(x+1)^3}{3}\cdot\frac{1}{x}dx$$

$$= \frac{(x+1)^3}{3}\log x - \frac{1}{3}\int \frac{x^3+3x^2+3x+1}{x}dx$$

$$= \frac{(x+1)^3}{3}\log x - \frac{1}{3}\int\left(x^2+3x+3+\frac{1}{x}\right)dx$$

$$= \frac{(x+1)^3}{3}\log x - \frac{1}{3}\left[\frac{x^3}{3}+\frac{3x^2}{2}+3x+\log|x|\right]+c$$

$$= \frac{1}{3}\left[(x+1)^3\log x - \frac{x^3}{3}-\frac{3x^2}{2}-3x-\log|x|\right]+c$$

7. $\int \log\left(x-\sqrt{x^2-1}\right)dx$

Sol.

Let $I = \int \log\left(x-\sqrt{x^2-1}\right)dx$

Let $u = \log\left(x-\sqrt{x^2-1}\right);$

$dv = dx$

and $v = x$

$$du = \frac{1}{x-\sqrt{x^2-1}}\left[\frac{d}{dx}\left(x-\sqrt{x^2-1}\right)\right]dx$$

$$= \frac{1}{x-\sqrt{x^2-1}}\left(1-\frac{1}{\cancel{2}}\left(x^2-1\right)^{\frac{1}{2}-1}(\cancel{2}x)\right)dx$$

$$= \frac{1}{x-\sqrt{x^2-1}}\left(1-\frac{x}{\sqrt{x^2-1}}\right)dx$$

$$= \frac{1}{x-\sqrt{x^2-1}}\left(\frac{\sqrt{x^2-1}-x}{\sqrt{x^2-1}}\right)dx$$

$$= \frac{1}{x-\cancel{\sqrt{x^2-1}}}\left(\frac{x-\cancel{\sqrt{x^2-1}}}{\sqrt{x^2-1}}\right)dx$$

$$du = \frac{-1}{\sqrt{x^2-1}}dx$$

∴ Using integration by parts,

$$I = \int u\,dv = uv - \int v\,du$$

$$= \int \log\left(x-\sqrt{x^2-1}\right)dx$$

$$= x\log\left(x-\sqrt{x^2-1}\right) - \int x\cdot\left(\frac{-1}{\sqrt{x^2-1}}dx\right)$$

$$= x\log\left(x-\sqrt{x^2-1}\right) + \int\frac{x}{\sqrt{x^2-1}}dx$$

$$I = x\log\left(x-\sqrt{x^2-1}\right) + I_1 \quad \text{-----(1)}$$

Consider

$$I_1 = \int\frac{x}{\sqrt{x^2-1}}dx$$

Put $t = x^2-1$

$dt = 2x\,dx \Rightarrow \frac{dt}{2} = x\,dx$

$$I_1 = \frac{1}{2}\int\frac{dt}{\sqrt{t}} = \frac{1}{2}\int t^{-\frac{1}{2}}dt$$

$$= \frac{1}{2}\left(\frac{t^{-\frac{1}{2}+1}}{-\frac{1}{2}+1}\right) = \frac{\cancel{1}}{\cancel{2}}\cdot\frac{t^{\frac{1}{2}}}{\frac{\cancel{1}}{\cancel{2}}}$$

$$= \sqrt{t} = \sqrt{x^2-1}$$

$[\because t = x^2-1]$

Substituting I_1 in (1) we get,

$$I = x\log\left(x-\sqrt{x^2-1}\right)+\sqrt{x^2-1}+c$$

8. $\int_0^1 \sqrt{x(x-1)}\,dx$

Sol.

Let $I = \int_0^1 \sqrt{x(x-1)}\,dx$

$$= \int_0^1 \sqrt{x^2-x}\,dx$$

$$= \int_0^1 \sqrt{x^2-x+\frac{1}{4}-\frac{1}{4}}\,dx$$

$$= \int_0^1 \sqrt{\left(x-\frac{1}{2}\right)^2 - \left(\frac{1}{2}\right)^2}\, dx$$

$$\left[\because \int \sqrt{x^2-a^2}\,dx = \frac{x}{2}\sqrt{x^2-a^2} - \frac{a^2}{2}\log\left|x+\sqrt{x^2-a^2}\right|\right]$$

$$= \left[\frac{x-\frac{1}{2}}{2}\sqrt{x^2-x} - \frac{1}{8}\log\left|x-\frac{1}{2}+\sqrt{x^2-x}\right|\right]_0^1$$

$$= \left[\frac{1-\frac{1}{2}}{2}\sqrt{0} - \frac{1}{8}\log\left|1-\frac{1}{2}+0\right|\right] - \left[0-\frac{1}{8}\log\left|-\frac{1}{2}\right|\right]$$

$$= -\frac{1}{8}\log\left|\frac{1}{2}\right| + \frac{1}{8}\log\left|\frac{1}{2}\right|$$

$$= 0.$$

9. $\displaystyle\int_{-1}^{1} x^2 e^{-2x}\, dx$

Sol.

Let	$I = \int_{-1}^{1} x^2 e^{-2x}\,dx$
$u = x^2$	$dv = e^{-2x}$
$u^1 = 2x$	$v = \frac{e^{-2x}}{-2} = \frac{-e^{-2x}}{2}$
$u^{11} = 2$	$v_1 = +\frac{e^{-2x}}{4}$
	$v_2 = -\frac{e^{-2x}}{8}$

Using Bernoulli's formula,

$$I = uv - u^1v_1 + u^{11}v_2$$

$$= \left[x^2\left(\frac{-e^{-2x}}{2}\right) - 2x\left(\frac{e^{-2x}}{4}\right) + 2\left(\frac{-e^{-2x}}{8}\right)\right]_{-1}^{1}$$

$$= \left(e^{-2x}\left[\frac{-x^2}{2} - \frac{x}{2} - \frac{1}{4}\right]\right)_{-1}^{1}$$

$$= \left[e^{-2}\left(-\frac{1}{2}-\frac{1}{2}-\frac{1}{4}\right)\right] - \left[e^{2}\left(-\cancel{\frac{1}{2}}+\cancel{\frac{1}{2}}-\frac{1}{4}\right)\right]$$

$$= e^{-2}\left(-\frac{5}{4}\right) - e^{2}\left(-\frac{1}{4}\right)$$

$$= \frac{-5}{4}e^{-2} + \frac{1}{4}e^{2}$$

$$= \frac{1}{4}\left(e^2 - \frac{5}{e^2}\right) = \frac{1}{4}\left(\frac{e^4-5}{e^2}\right)$$

10. $\displaystyle\int_0^3 \frac{x\,dx}{\sqrt{x+1}+\sqrt{5x+1}}$

Sol. Let $\displaystyle I = \int_0^3 \frac{x\,dx}{\sqrt{x+1}+\sqrt{5x+1}}$

Multiply and divide with the conjugate of the denominator

We get

$$I = \int_0^3 \frac{x\left(\sqrt{x+1}-\sqrt{5x+1}\right)dx}{\left(\sqrt{x+1}+\sqrt{5x+1}\right)\left(\sqrt{x+1}-\sqrt{5x+1}\right)}$$

$$= \int_0^3 \frac{x\left(\sqrt{x+1}-\sqrt{5x+1}\right)dx}{(x+1)-(5x+1)}$$

$[\because (a+b)(a-b) = a^2 - b^2]$

$$= \int_0^3 \frac{x\left(\sqrt{x+1}-\sqrt{5x+1}\right)dx}{x+\cancel{1}-5x-\cancel{1}}$$

$$= \int_0^3 \frac{\cancel{x}\left(\sqrt{x+1}-\sqrt{5x+1}\right)dx}{-4\cancel{x}}$$

$$= -\frac{1}{4}\int_0^3 \left(\sqrt{x+1}-\sqrt{5x+1}\right)dx$$

$$= -\frac{1}{4}\left[\frac{(x+1)^{\frac{3}{2}}}{\frac{3}{2}} - \frac{(5x+1)^{\frac{3}{2}}}{5\left(\frac{3}{2}\right)}\right]_0^3$$

$$= -\frac{1}{4}\left[\frac{2}{3}(x+1)^{\frac{3}{2}} - \frac{2}{15}(5x+1)^{\frac{3}{2}}\right]_0^3$$

$$= -\frac{1}{2}\left[\frac{1}{3}(x+1)^{\frac{3}{2}} - \frac{1}{15}(5x+1)^{\frac{3}{2}}\right]_0^3$$

$$= -\frac{1}{2}\left\{\left[\frac{1}{3}(4)^{\frac{3}{2}} - \frac{1}{15}(16)^{\frac{3}{2}}\right] - \left[\frac{1}{3}(1)^{\frac{3}{2}} - \frac{1}{15}(1)^{\frac{3}{2}}\right]\right\}$$

$$= -\frac{1}{2}\left[\left(\frac{1}{3}(4)\sqrt{4} - \frac{1}{15}16\sqrt{16}\right) - \left(\frac{1}{3}-\frac{1}{15}\right)\right]$$

$$= -\frac{1}{2}\left[\left(\frac{8}{3}-\frac{64}{15}\right) - \frac{1}{3} + \frac{1}{15}\right]$$

$$= -\frac{1}{2}\left[\frac{8}{3}-\frac{64}{15}-\frac{1}{3}+\frac{1}{15}\right]$$

$$= -\frac{1}{2}\left[\frac{8}{3}-\frac{1}{3}-\frac{64}{15}+\frac{1}{15}\right]$$

$$= -\frac{1}{2}\left[\frac{7}{3}-\frac{63}{15}\right] = -\frac{1}{2}\left[\frac{105-189}{45}\right]$$

$$= -\frac{1}{2}\left[\frac{-84}{45}\right] = \frac{42}{45} = \frac{14}{15}$$

$$\therefore \quad I = \frac{14}{15}$$

Additional problems

I. Choose the correct answer :

1. $\int (x-1)e^{-x}\,dx =$ ________ $+ c$

(a) $-xe^{x}$ (b) xe^{x} (c) $-xe^{-x}$ (d) xe^{-x}

[Ans: (c) $-xe^{-x}$]

Hint:

Let $u = x-1$ $\quad dv = e^{-x}\,dx$

$u^{1} = 1$ $\quad v = -e^{-x}$

$u^{11} = 0$ $\quad v_1 = e^{-x}$

$$I = -\int 2^t\,dt = \frac{2^t}{\log e^2} + c = \frac{-2^{\frac{1}{x}}}{\log e^2} + c$$

2. If $\int \frac{2^{\frac{1}{x}}}{x^2}\,dx = k\,2^{\frac{1}{x}} + c$, then k is

(a) $-\frac{1}{\log_e 2}$ (b) $-\log_e 2$

(c) -1 (d) $\frac{1}{2}$

[Ans: (a) $-\frac{1}{\log_e 2}$]

Hint:

put $\quad t = \frac{1}{x} = x^{-1}$

$$\frac{dt}{dx} = (-1)x^{-2} = \frac{-1}{x^2}$$

$$dt = \frac{-1}{x^2}\,dx \Rightarrow -dt = \frac{1}{x^2}\,dx$$

$$I = -\int 2^t\,dt = \frac{2^t}{\log_e 2} + c$$

$$= \frac{-2^{\frac{1}{x}}}{\log_e 2} + c$$

3. $\int |x|^3\,dx =$ ________ $+ c$

(a) $\frac{-x^4}{4}$ (b) $\frac{|x|^4}{4}$

(c) $\frac{x^4}{4}$ (d) none of these

[Ans: (d) none of these]

4. $\int \frac{\cos\sqrt{x}}{\sqrt{x}}\,dx =$ ________ $+ c$

(a) $2\cos\sqrt{x}$ (b) $\sqrt{\frac{\cos x}{x}}$

(c) $\sin\sqrt{x}$ (d) $2\sin\sqrt{x}$

[Ans: (d) $2\sin\sqrt{x}$]

Hint:

put $\quad t = x^{\frac{1}{2}}$

$$\frac{dt}{dx} = \frac{1}{2}x^{\frac{1}{2}-1} = \frac{1}{2\sqrt{x}}$$

$$2\,dt = \frac{1}{\sqrt{x}}\,dx$$

$$I = 2\int \cos t\,dt = 2\sin t = 2\sin\sqrt{x}$$

5. $\int \frac{2}{\left(e^x+e^{-x}\right)^2}\,dx =$ ________ $+ c$

(a) $\frac{-e^{-x}}{e^x+e^{-x}}$ (b) $-\frac{1}{e^x+e^{-x}}$

(c) $\frac{-1}{\left(e^x+1\right)^2}$ (d) $\frac{1}{e^x-e^{-x}}$

[Ans: (a) $\frac{-e^{-x}}{e^x+e^{-x}}$]

Hint:

$$I = \int \frac{2}{\left(e^x+\frac{1}{e^x}\right)^2}\,dx$$

$$= \int \frac{2e^{2x}}{\left(e^{2x}+1\right)^2}\,dx$$

put $\quad t = e^{2x}+1$

$$\frac{dt}{dx} = 2e^{2x} \Rightarrow dt = 2e^{2x}\,dx$$

$$I = \int \frac{dt}{t^2} = \int t^{-2}\,dt$$

$$= \frac{t^{-2+1}}{-2+1} = -\frac{1}{t}$$

$$= \frac{-1}{e^{2x}+1} = \frac{-1}{e^x\left(e^x+e^{-x}\right)}$$

$$= \frac{-e^{-x}}{e^x+e^{-x}}$$

6. $\int e^x (1 - \cot x + \cot^2 x)\, dx =$ _____ $+ c$

(a) $e^x \cot x$ (b) $-e^x \cot x$

(c) $e^x \operatorname{cosec} x$ (d) $-e^x \operatorname{cosec} x$

[Ans: (b) $-e^x \cot x$]

7. $\int e^x f(x) + f'(x)\, dx =$ _____ $+ c$

(a) $e^x f(x)$ (b) $e^x + f(x)$

(c) $2e^x f(x)$ (d) $e^x - f(x)$

[Ans: (a) $e^x f(x)$]

8. If $\int x \sin x\, dx = -x \cos x + \alpha$ then $\alpha =$ _____ $+ c$

(a) $\sin x$ (b) $\cos x$

(c) C (d) none of these

[Ans: (a) $\sin x$]

Hint:

Let $u = x$ $\quad dv = \sin x\, dx$

$u^1 = 1$ $\quad v = -\cos x$

$v_1 = -\sin x$

$I = uv - u^1 v_1$

$= x(-\cos x) - 1(-\sin x) + c$

$= -x \cos x + \sin x + c$

9. If $\int \frac{1}{(x+2)(x^2+1)} dx = a \log|1 + x^2| + b \tan^{-1} x + \frac{1}{5} \log|x + 2| + c$ then

(a) $a = -\frac{1}{10}, b = \frac{-2}{5}$ (b) $a = \frac{1}{10}, b = \frac{-2}{5}$

(c) $a = -\frac{1}{10}, b = \frac{2}{5}$ (d) $a = \frac{1}{10}, b = \frac{2}{5}$

[Ans: (c) $a = -\frac{1}{10}, b = \frac{2}{5}$]

Hint:

$$\frac{1}{(x+2)(x^2+1)} = \frac{A}{x+2} + \frac{Bx+C}{x^2+1}$$

$$= \frac{A(x^2+1) + (Bx+C)(x+2)}{(x+2)(x^2+1)}$$

$1 = A(x^2 + 1) + (Bx + C)(x + 2)$

put $x = -2$, $1 = A(4 + 1) \Rightarrow A = \frac{1}{5}$

put $x = 0$, $1 = A + 2C \Rightarrow 2C = \frac{4}{5}$

$\Rightarrow C = \frac{2}{5}$

put $x = 1$, $1 = 2A + 3B + 3C$

$\Rightarrow 1 = \frac{2}{5} + 3B + \frac{6}{5}$

$\Rightarrow \frac{-3}{5} = 3B \Rightarrow B = \frac{-1}{5}$

$$I = \int \frac{\frac{1}{5}}{x+2} dx + \int \frac{\frac{-1}{5}x + \frac{2}{5}}{x^2+1} dx$$

$$= \frac{1}{5}\log|x+2| + \frac{2}{5}\int \frac{1}{x^2+1} dx - \frac{1}{5}\int \frac{x}{x^2+1}$$

$$= \frac{1}{5}\log|x+2| + \frac{2}{5}\tan^{-1} x - \frac{1}{10}\log|x^2+1| + c$$

10. $\int 3^{x+2}\, dx =$ _____ $+ c$

(a) $\frac{3^x}{\log 3}$ (b) $\frac{9(3^x)}{\log 3}$

(c) $\frac{3.3^x}{\log 3}$ (d) $\frac{3^x}{9 \log 3}$

[Ans: (b) $9 \cdot \frac{3^x}{\log 3}$]

Hint:

$\int 3^{x+2}\, dx = \int 3^x \times 3^2 dx = 9 \int 3^x\, dx = 9 \frac{3^x}{\log 3}$

11. $\int \left(\frac{x}{m} + \frac{m}{x}\right) dx =$ _____ $+ c$

(a) $\frac{x^2}{2m} + m \log|x|$ (b) $\frac{x}{m^2} + m \log|x|$

(c) $-\frac{1}{mx^2} + m \log|x|$ (d) $\frac{1}{m} - \frac{m}{x^2}$

Hint:

$$\int \left(\frac{x}{m} + \frac{m}{x}\right) dx = \int \frac{x}{m} dx + \int \frac{m}{x} dx$$

$$= \frac{1}{m}\frac{x^2}{2} + m \log|x|$$

[Ans: (a) $\frac{x^2}{2m} + m \log|x|$]

12. The anti-derivative of $f(x) = \sqrt{x} + \frac{1}{\sqrt{x}}$ is _____ $+ c$

(a) $\frac{2}{3}x^{\frac{3}{2}} + \frac{2}{x^{\frac{1}{2}}}$ (b) $\frac{3}{2}x^{\frac{3}{2}} + 2x^{\frac{1}{2}}$

(c) $\frac{2}{3}x^{\frac{3}{2}} + 2x^{\frac{1}{2}}$ (d) none

[Ans: (c) $\frac{2}{3}x^{\frac{3}{2}} + 2x^{\frac{1}{2}}$]

Hint:

$$\int \left(\sqrt{x} + \frac{1}{\sqrt{x}}\right) dx = \int x^{\frac{1}{2}} dx + \int x^{\frac{-1}{2}} dx$$

$$= \frac{x^{\frac{3}{2}}}{\frac{3}{2}} + \frac{x^{\frac{1}{2}}}{\frac{1}{2}} + c = \frac{2}{3}x^{\frac{3}{2}} + 2x^{\frac{1}{2}} + c$$

13. $\int a^{3x+2}\, dx =$ ______ $+ c$

(a) $a3^{x+2}$ (b) $\dfrac{a^{3x+2}}{3}$

(c) $\dfrac{a^{3x+2}}{3\log a}$ (d) $3 \log a\, (a^{3x+2})$

Hint:

$$\int a^{3x+2}\, dx = \int a^{3x}\, a^2\, dx = a^2 \int a^{3x}\, dx = a^2 \frac{a^{3x}}{3\log a}$$

$$= \frac{a^{3x+2}}{3\log a}$$

[Ans: (c) $\dfrac{a^{3x+2}}{3\log a}$]

14. $\int e^{3\log x}\,(x^4+1)^{-1}\, dx =$ ______ $+ c$

(a) $\log |x^4+1|$ (b) $4 \log |x^4+1|$

(c) $-4 \log |x^4+1|$ (d) $\dfrac{1}{4} \log |x^4+1|$

Hint:

$$\int e^{3\log x}(x^4+1)^{-1}\, dx = \int e^{\log x^3}(x^4+1)^{-1}\, dx$$

$$= \int \frac{x^3}{x^4+1} dx$$

put $t = x^4 + 1$

$dt = 4x^3\, dx$

$\dfrac{dt}{4} = x^3\, dx$

$$= \int \frac{dt}{4t} = \frac{1}{4}\log|t| + c$$

$$= \frac{1}{4}\log|x^4+1| + c$$

[Ans: (d) $\dfrac{1}{4}\log|x^4+1|$]

15. $\displaystyle\int \frac{dx}{9x^2-4} =$ ______ + c

(a) $\log\left|\dfrac{3x-2}{3x+2}\right|$ (b) $12\log\left|\dfrac{3x-2}{3x+2}\right|$

(c) $\dfrac{1}{12}\log\left|\dfrac{3x-2}{3x+2}\right|$ (d) $\dfrac{1}{6}\log\left|\dfrac{3x-2}{3x+2}\right|$

[Ans: (c) $\dfrac{1}{12}\log\left|\dfrac{3x-2}{3x+2}\right|$]

Hint:

$$\int \frac{1}{(3x)^2 - 2^2} dx = \frac{1}{2(2)}\frac{1}{3}\log\left|\frac{3x-2}{32+2}\right| + c$$

$$= \frac{1}{12}\log\left|\frac{3x-2}{3x+2}\right| + c$$

16. $\displaystyle\int_1^e \log x\, dx =$ ______ $+ c$

(a) 1 (b) $e-1$ (c) $e+1$ (d) 0

[Ans: (a) 1]

Hint:

$$\int_1^e \log x\, dx = [x\log x - x]_1^e + c$$

$$= \text{e log e} - \text{e} - \log 1 + 1 + \text{c}$$

$$= \cancel{e} - \cancel{e} - 0 + 1 + c = 1$$

17. $\displaystyle\int_0^{\frac{\pi}{2}} x\sin x\, dx =$

(a) $\dfrac{\pi}{4}$ (b) $\dfrac{\pi}{2}$ (c) π (d) 1

[Ans: (d) 1]

Hint:

$u = x$ $dv = \sin x\, dx$

$u^1 = 1$ $v = -\cos x$

$v_1 = -\sin x$

$$I = uv - u^1 v_1$$

$$= [-x\cos x + \sin x]_0^{\frac{\pi}{2}}$$

$$= -\frac{\pi}{2}(0) + 1 - 0 = 1$$

18. $\displaystyle\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \sin x\, dx =$

(a) 1 (b) 2 (c) -1 (d) -2

[Ans: (b) 2]

Hint:

$$\int_{\frac{-\pi}{2}}^{\frac{\pi}{2}} x\sin x\, dx = 2\int_0^{\frac{\pi}{2}} x\sin x\, dx$$

$u = x$ $dv = \sin x\, dx$

$u^1 = 1$ $v = -\cos x$

$v_1 = -\sin x$

$$I = uv - u^1 v_1$$

$$= 2[-x\cos x + \sin x]_0^{\frac{\pi}{2}}$$

$$= 2\left[-\frac{\pi}{2}(0) + 1 - 0\right] = 2$$

19. The value of $\displaystyle\int_0^{\frac{\pi}{2}} \cos x\, e^{\sin x}\, dx =$

(a) 1 (b) $e-1$ (c) 0 (d) -1

[Ans: (b) $e-1$]

Hint:

$$t = \sin x$$
$$\frac{dt}{dx} = \cos x$$
$$dt = \cos x\, dx$$
$$I = \int_0^1 e^t dt = \left[e^t\right]_0^1 = e - 1$$

20. The value of the integral $\int_0^{\frac{\pi}{2}} \frac{\sqrt{\cos x}}{\sqrt{\cos x} + \sqrt{\sin x}} dx$ =

(a) 0 (b) $\frac{\pi}{2}$
(c) $\frac{\pi}{4}$ (d) none of these

[Ans: (c) $\frac{\pi}{4}$]

II. Fill in the blanks:

1. $\int \frac{x^5 - x^4}{x^3 - x^2} dx$ = ________ + c.

(a) 1 (b) $\frac{x}{2}$ (c) $\frac{x^3}{3}$ (d) $\frac{x^2}{2}$

[Ans: (c) $\frac{x^3}{3}$]

Hint:

$$= \int \frac{x^4(x-1)}{x^2(x-1)} dx = \int x^2 dx = \frac{x^3}{3} + c$$

2. $\int \frac{2}{1 - \cos 2x} dx$ = ________ + c.

(a) – cot x (b) cot x
(c) sec x (d) tan x

[Ans: (a) – cot x]

Hint:

$$\int \frac{2}{1-\cos 2x} dx = \int \frac{2}{2\cos^2 x} dx$$
$$= \int \operatorname{cosec}^2 x\, dx = -\cot x + c$$

3. $\int e^x a^x dx$ = ________ + c.

(a) $\frac{a^x}{\log a}$ (b) $\frac{(ae)^x}{\log(ae)}$
(c) $\frac{a^x}{\log(ae)}$ (d) $\frac{e^x}{\log a}$

[Ans: (b) $\frac{(ae)^x}{\log(ae)}$]

Hint:

$$\int e^x a^2 dx = \int (ea)^x dx = \frac{(ea)^x}{\log ea}$$

4. $\int \frac{e^{\log\sqrt{x}}}{x} dx$ = ________ + c.

(a) $2\sqrt{x}$ (b) $\sqrt{x}$ (c) $\frac{1}{\sqrt{x}}$ (d) $\frac{1}{x}$

[Ans: (a) $2\sqrt{x}$]

Hint:

$$= \int \frac{\sqrt{x}}{x} dx = \int x^{\frac{-1}{2}} dx = \frac{x^{\frac{-1}{2}+1}}{\frac{-1}{2}+1} = 2x^{\frac{1}{2}} = 2\sqrt{x}$$

5. $\int (1-x)\sqrt{x}\, dx$ = ______ + c.

(a) $\frac{2}{3}x^{\frac{3}{2}} - \frac{5}{2}$ (b) $x^{\frac{3}{2}} - \frac{2}{5}x^{\frac{5}{2}}$
(c) $\frac{3}{2}x^{\frac{2}{3}} - \frac{5}{2}x^{\frac{2}{5}}$ (d) $\frac{2}{3}x^{\frac{3}{2}} - \frac{2}{5}x^{\frac{5}{2}}$

[Ans: (d) $\frac{2}{3}x^{\frac{3}{2}} - \frac{2}{5}x^{\frac{5}{2}}$]

Hint:

$$= \int\left(\sqrt{x} - x\sqrt{x}\right)dx = \int\left(x^{\frac{1}{2}} - x^{\frac{3}{2}}\right)dx$$
$$= \frac{x^{\frac{3}{2}}}{\frac{3}{2}} - \frac{x^{\frac{5}{2}}}{\frac{5}{2}} + c = \frac{2}{3}x^{\frac{3}{2}} - \frac{2}{5}x^{\frac{5}{2}} + c$$

6. $\int \frac{1}{1+\sin x} dx$ = _____ + c.

(a) tan x – sex x (b) sec x – tan x
(c) – tan x – sec x (d) tan x + sec x

[Ans: (a) tan x – sec x]

Hint:

$$\int \frac{1}{1+\sin x} dx = \int \frac{1-\sin x}{1-\sin^2 x} dx$$
$$= \int \frac{1-\sin x}{\cos^2 x} dx$$
$$= \int [\sec^2 x - \tan x \sec x]\, dx$$
$$= \tan x - \sec x$$

7. If $f'(x) = x - \frac{1}{x^2}$ and $f(1) = \frac{1}{2}$, then $f(x)$ = ________.

(a) $\frac{x^2}{2} + \frac{1}{x} + c$ (b) $\frac{x^2}{x} - \frac{1}{x} - 1$
(c) $\frac{x^2}{2} + \frac{1}{x} - 1$ (d) $\frac{x^2}{2} - \frac{1}{x} + 1$

[Ans: (c) $\frac{x^2}{2} + \frac{1}{x} - 1$]

Hint:

$$\int x-\frac{1}{x^2}dx = \int\left(x-x^{-2}\right)dx$$
$$= \frac{x^2}{2}-\frac{x^{-1}}{-1}+c$$
$$f(1) = \frac{1}{2}\Rightarrow\frac{1}{2}+1+c$$
$$f(x) = \frac{x^2}{2}+\frac{1}{x}-1$$

8. $\int \sec^2 (7-4x)\, dx =$ ______ + c.

(a) $-\frac{1}{4}\tan(7-4x)$ (b) $-\frac{1}{7}\tan(7-4x)$

(c) $\frac{1}{4}\tan(7-4x)$ (d) $\frac{1}{7}\tan(7-4x)$

[Ans: (a) $-\frac{1}{4}\tan(7-4x)$]

Hint:

$$= \int \sec^2(-4x+7)\, dx = \frac{\tan(-4x+7)}{-4}$$
$$= \frac{-1}{4}\tan(7-4x)$$

9. $\int \frac{e^x+1}{e^x+x}dx =$ ______ + c.

(a) $\log|e^x+1|$ (b) $\log|e^x+x|$

(c) $\log|e^x-1|$ (d) $\log|e^x-x|$

[Ans: (b) $\log|e^x+x|$]

Hint:

put $t = e^x + x$

$$\frac{dt}{dx} = e^x+1$$
$$\Rightarrow dt = (e^x+1)\, dx$$
$$I = \int\frac{dt}{t} = \log|t|+c$$
$$= \log|e^x+x|+c$$

10. $\int x\cos x\, dx =$ ______ + c.

(a) $x\sin x - \cos x$ (b) $-x\sin x+\cos x$

(c) $-x\sin x - \cos x$ (d) $x\sin x + \cos x$

[Ans: (d) $x\sin x + \cos x$]

Hint:

$u = x$ $dv = \cos x\, dx$

$u^1 = 1$ $v = \sin x$

$v_1 = -\cos x$

$I = uv - u^1v_1 = x\sin x + \cos x$

11. $\int e^x(\sin x + \cos x)\, dx =$ ______ + c.

(a) $e^x\sin x$ (b) $e^{-x}\sin x$

(c) sin x (d) $e^x\cos x$

[Ans: (a) $e^x\sin x$]

Hint:

Let $f(x) = \sin x \Rightarrow f'(x) = \cos x$

$\therefore \int e^x(f(x)+f'(x))dx = e^x f(x)+c$

$\therefore$ $I = e^x\sin x + c$

12. $\int_4^9 \frac{1}{\sqrt{2}}dx =$

(a) 0 (b) 1

(c) 2 (d) 4 **[Ans: (c) 2]**

Hint:

$$= \int_4^9 x^{\frac{-1}{2}}dx$$
$$= \left[\frac{x^{\frac{1}{2}}}{\frac{1}{2}}\right]_4^9 = 2\left[x^{\frac{1}{2}}\right]_4^9 = 2[3-2]$$
$$= 2$$

13. $\int_0^1 \frac{1}{2x-3}dx =$ ________.

(a) $\frac{1}{2}\log 3$ (b) $-\frac{1}{2}\log 3$

(c) $\frac{1}{2}\log 1$ (d) 0

[Ans: (b) $-\frac{1}{2}\log 3$]

Hint:

$$= \int_4^9 x^{\frac{-1}{2}}dx = \frac{1}{2}[\log|2x-3|]_0^1$$
$$= \frac{1}{2}[\log 1-\log(-3)]$$
$$= \frac{1}{2}[0-\log 3] = -\frac{\log 3}{2}$$

14. $\int_1^4 f(x)dx$, where $f(x) = \begin{cases} 7x+3 & if \quad 1 \le x \le 3 \\ 8x & if \quad 3 \le x \le 4 \end{cases}$ is __________.

(a) 58 (b) 60

(c) 62 (d) 52 **[Ans: (c) 62]**

Hint:

$$= \int_1^3 (7x+3)dx + \int_8^4 8x\,dx$$

$$= \left[7\frac{x^2}{2}+3x\right]_1^3 + \left[\frac{\overset{4}{\cancel{8}}x^2}{\cancel{2}}\right]_3^4$$

$$= 7 \times \frac{9}{2} + 9 - \left[\frac{7}{2}+3\right] + [64 - 36]$$

$$= \frac{63}{2} + 9 - \frac{13}{2} + 28 = 62$$

15. $\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \sin^{71} x\,dx =$ ________.

(a) 1 (b) 0

(c) –1 (d) 2π **[Ans: (b) 0]**

Hint:

$$\sin(-x) = (-)^{71} \sin^{71} x$$
$$= -\sin^{71} x$$
$$\therefore \int_{\frac{-\pi}{2}}^{\frac{\pi}{2}} \sin^{71} x\,dx = 0$$

III. Match the following:

1.	$\int e^{-t}\,dt$	a.	Improper definite integral
2.	$\int_0^1 e^{-t}\,dt$	b.	$\Gamma(n)$
3.	$\int_0^\infty e^{-t}\,dt$	c.	$\frac{n!}{a^{n+1}}$
4.	For $n > 0, \int_0^\infty x^{n-1} e^{-x}\,dx$	d.	Indefinite integral
5.	$\int_0^\infty x^n e^{-ax}\,dx$ where n is a positive integer.	e.	$n\,\Gamma(n)$, $n>0$
6.	$\Gamma(n)$	f.	$\sqrt{\pi}$
7.	$\Gamma(1)$	g.	$n!$ where n is a positive integer
8.	$\Gamma(n+1)$	h.	proper definite integer
9.	$\Gamma\left(\frac{1}{2}\right)$	i.	$(n-1)\,\Gamma(n-1)$, $n > 1$
10.	$\Gamma(n+1)$	j.	1

[Ans: 1 - d, 2 - h, 3 - a, 4 - b, 5 - c, 6 - i, 7 - j, 8 - g, 9 - f, 10 - e]

IV. Identify the Incorrect statement:

1.

(i) $\Gamma(7) = 6!$ (ii) $\Gamma(6) = 5!$

(iii) $\Gamma\left(\frac{1}{2}\right) = \sqrt{\pi}$ (iv) $\Gamma(7) = 7!$

[Ans: (iv) Γ (7) = 7!]

Hint:

$\Gamma(n) = (n-1)\,\Gamma(n-1)$

2.

(i) $\sum_{r=1}^{n} r = n\frac{(n+1)}{2}$

(ii) $\sum_{r=1}^{n} r^3 = n\frac{(n+1)(2n+1)}{6}$

(iii) $\sum_{r=1}^{n} r^3 = \left[n\frac{(n+1)}{2}\right]^3$

(iv) $\sum_{r=1}^{n} r^3 = \left[n\frac{(n+1)}{2}\right]^2$

[Ans: (iii) $\sum_{r=1}^{n} r^3 = \left[n\frac{(n+1)}{2}\right]^3$]

3.

(i) $\int \frac{1}{x}dx = \log|x| + c$

(ii) $\int \frac{1}{x}dx = \log x$

(iii) $\int x\,dx = \frac{x^2}{2} + c$

(iv) $\int e^x\,dx = e^x + c$ **[Ans: (ii) $\int \frac{1}{x}dx = \log x$]**

4.

(i) $\int udv = uv - \int vdu$ (ii) $\int udv = uv - u^1v_1$

(iii) $\int udv = uv - u^1v_1 + u^{11}v_2 - ...$

(iv) $\int udv = uv - u^1v_1 + u^{11}v_2 - u^{111}v_3 + ...$

Where u and v are functions of x

[Ans: (ii) $\int udv = uv - u^1v_1$]

5.

(i) $\int_{-a}^{a} f(x)dx = 2\int_0^a f(x)dx$

(ii) $\int_{-a}^{a} f(x)dx = 0$ if $f(x)$ is an odd function

(iii) $\int_{-a}^{a} f(x)dx = 2\int_0^a f(x)dx$ if $f(x)$ is an even function

(iv) $\int_a^b f(x)dx = \int_a^b f(t)dt$

[Ans: (i) $\int_{-a}^{a} f(x)dx = 2\int_0^a f(x)dx$]

6. **If $f(x)$ is an anti-derivative of $f(x)$, then**

(i) $\int_a^b f(x)dx = F(b) - F(a)$

(ii) $\int_a^b f(x)dx = [(F(a) - F(b))]$

(iii) $\int_a^b f(x)dx = -F(a) - F(b)$

(iv) $\int_a^b f(x)dx = [F(x)]_a^b$

[Ans: (iii) $\int_a^b f(x)dx = -F(a) - F(b)$]

7.

(i) $\int_0^a fx\,dx = \int_0^a f(a-x)dx$

(ii) $\int_a^b f(x)dx = \int_a^c f(x)dx + \int_c^b f(x)dx$ if $a < c < b$

(iii) $\int_a^b f(x)dx = -\int_a^b f(x)dx$

(iv) $\int_a^b f(x)dx = -\int_b^a f(x)dx$

[Ans: (iii) $\int_a^b f(x)dx = -\int_a^b f(x)dx$]

8.

(i) $\int_a^b f(x)dx = F(x) + c$

(ii) $\int_a^b f(x)dx = F(b) - F(a)$

(iii) $\int_a^b f(x)dx = [F(x)]_a^b$

(iv) $\int_a^b f(x)dx = [F(x)]_b^a$

[Ans: (iv) $\int_a^b f(x)dx = [F(x)]_b^a$]

9.

(i) $\int e^x [f(x) + f'(x)]\,dx = e^x f(x) + c$

(ii) $\int e^{ax} [a f(x) + f'(x)]\,dx = e^{ax} f(x) + c$

(iii) $\int e^{ax} [a f(x) + f'(x)]\,dx = ae^{ax} f(x) + c$

(iv) $\int \frac{f'(x)}{f(x)} dx = \log |f(x)| + c$

[Ans: (iii) $\int e^{ax} [a f(x) + f'(x)]\,dx = ae^{ax} f(x) + c$]

10.

(i) $\int \sec^2 x\,dx = \tan x + c$

(ii) $\int \text{cosec}^2 x\,dx = \cot x + c$

(iii) $\int \text{cosec}^2 x\,dx = -\cot x + c$

(iv) $\int \text{cosec}^2 (ax+b)\,dx = -\frac{1}{a}\cot(ax+b) + c$

[Ans: (ii) $\int \text{cosec}^2 x\,dx = \cot x + c$]

Additional Question

2 Mark Questions

1. **Evaluate $\int a^{3\log_a x} dx$**

Sol.

We know that $a^{\log_a x} = x$

$$\therefore \int a^{3\log_a x}\,dx = \int a^{\log_a x^3}\,dx = \int x^3\,dx$$

$$= \frac{x^4}{4} + c$$

2. **Evaluate $\int \frac{2\cos^2 x - \cos 2x}{\sin^2 x}$**

Sol.

$$\therefore \int \frac{2\cos^2 x - \cos 2x}{\sin^2 x}$$

$$= \int \frac{2\cos^2 x - \left(2\cos^2 x - 1\right)}{\sin^2 x} dx$$

$$= \int \frac{\cancel{2\cos^2 x} - \cancel{2\cos^2 x} + 1}{\sin^2 x} dx$$

$$= \int \frac{1}{\sin^2 x} dx$$

$$= \int \operatorname{cosec}^2 x \, dx$$

$$= -\cot x + c$$

3. Evaluate $\int \tan^2 x \, dx$

Sol.

$$\int \tan^2 x \, dx = \int (\sec^2 x - 1)\, dx \quad [\because 1 + \tan^2 x = \sec^2 x]$$

$$= \int \sec^2 x \, dx - \int 1 \cdot dx$$

$$= \tan x - x + c.$$

4. Evaluate $\int \frac{2 + 3\cos x}{\sin^2 x} dx$

Sol.

$$\int \frac{2+3\cos x}{\sin^2 x} dx = \int \frac{2}{\sin^2 x} dx + \int \frac{3\cos x}{\sin^2 x} dx$$

$$= 2\int \frac{1}{\sin^2 x} dx + 3\int \frac{\cos x}{\sin x} \cdot \frac{1}{\sin x} dx$$

$$= 2\int \operatorname{cosec}^2 x \, dx + 3\int \cot x \operatorname{cosec} x \, dx$$

$$= -2\cot x - 3\operatorname{cosec} x + c$$

5. Evaluate $\int \frac{2^x + 3^x}{5^x} dx$

Sol.

$$\int \frac{2^x + 3^x}{5^x} dx = \int \frac{2^x}{5^x} dx + \int \frac{3^x}{5^x} dx$$

$$= \int \left(\frac{2}{5}\right)^x dx + \int \left(\frac{3}{5}\right)^x dx$$

$$= \frac{\left(\frac{2}{5}\right)^x}{\log_e\left(\frac{2}{5}\right)} + \frac{\left(\frac{3}{5}\right)^x}{\log_e\left(\frac{3}{5}\right)} + c$$

$$\left[\because \int a^x \, dx = \frac{a^x}{\log_e a} + c\right]$$

6. If $f'(x) = 8x^3 - 2x^2$, $f(2) = 1$, find f(x)

Sol.

Given $f'(x) = 8x^3 - 2x^2$

$$\therefore \int f'(x)\, dx = \int (8x^3 - 2x^2)\, dx$$

$$\Rightarrow f(x) = \frac{8x^4}{4} - \frac{2x^3}{3} + c$$

$$\Rightarrow f(x) = 2x^4 - \frac{2x^3}{3} + c \quad \text{----- (1)}$$

Also, $f(2) = 1$

$$\Rightarrow 1 = 2(2^4) - \frac{2(2^3)}{3} + c$$

$$\Rightarrow 1 = 32 - \frac{16}{3} + c$$

$$\Rightarrow 1 - 32 + \frac{16}{3} = c \quad \Rightarrow -31 + \frac{16}{3} = c$$

$$\Rightarrow \frac{-93+16}{3} = c$$

$$\Rightarrow c = \frac{-77}{3}$$

$$\therefore (1) \to f(x) = 2x^4 - \frac{2x^3}{3} - \frac{77}{3}$$

7. Evaluate $\int x\sqrt{x+2}\, dx$

Sol.

$$\int x\sqrt{x+2}\, dx = \int (x+2-2)\sqrt{x+2}\, dx$$

[Adding & subtracting 2]

$$= \int (x+2)\sqrt{x+2}\, dx - \int 2\sqrt{x+2}\, dx$$

$$= \int (x+2)^{\frac{3}{2}} dx - 2\int (x+2)^{\frac{1}{2}} dx$$

$$= \frac{(x+2)^{\frac{3}{2}+1}}{\frac{3}{2}+1} - 2\frac{(x+2)^{\frac{1}{2}+1}}{\frac{1}{2}+1} + c$$

$$= \frac{(x+2)^{\frac{5}{2}}}{\frac{5}{2}} - 2\frac{(x+2)^{\frac{3}{2}}}{\frac{3}{2}} + c$$

$$= \frac{2}{5}(x+2)^{\frac{5}{2}} - \frac{4}{3}(x+2)^{\frac{3}{2}} + c$$

8. If $\int_0^1 \left(3x^2 + 2x + k\right) dx = 0$, find k.

Sol.

Given

$$\Rightarrow \int_0^1 \left(3x^2 + 2x + k\right) dx = 0$$

$$\Rightarrow \left[\frac{3x^3}{3} + \frac{2x^2}{2} + kx\right]_0^1 = 0$$

$$\Rightarrow \therefore \left[x^3 + x^2 + kx\right]_0^1 = 0$$

$\Rightarrow [1 + 1 + k(1)] - (0) = 0$

$\Rightarrow \quad 2 + k = 0$

$\Rightarrow \quad k = -2$

9. **Evaluate** $\int_0^{\pi} \sin^3 x\,dx$

Sol.

$$\int_0^{\pi} \sin^3 x\,dx = \int_0^{\pi} \frac{3\sin x - \sin 3x}{4}dx$$

$[\because \sin 3x = 3\sin x - 4\sin^3 x]$

$$= \frac{1}{4}\int_0^{\pi}(3\sin x - \sin 3x)dx$$

$$= \frac{1}{4}\left[-3\cos x + \frac{1}{3}\cos 3x\right]_0^{\pi}$$

$$= \frac{1}{4}\left[\left(-3\cos\pi + \frac{1}{3}\cos 3\pi\right) - \left(-3\cos 0 + \frac{1}{3}\cos 3(0)\right)\right]$$

$$= \frac{1}{4}\left[\left(3 - \frac{1}{3}\right) - \left(-3 + \frac{1}{3}\right)\right]$$

$[\because \cos\pi = -1, \cos 3\pi = -1 \text{ and } \cos 0 = 1]$

$$= \frac{1}{4}\left[\frac{8}{3} - \left(\frac{-8}{3}\right)\right]$$

$$= \frac{1}{4}\left(\frac{8}{3} + \frac{8}{3}\right) = \frac{1}{\cancel{4}}\left(\frac{\cancel{16}^{\,4}}{3}\right) = \frac{4}{3}$$

10. **If** $\int_0^{a} 3x^2\,dx = 8$**, find the value of** a**.**

Sol.

Given $\int_0^{a} 3x^2\,dx = 8$

$\Rightarrow \left[\cancel{3}\cdot\frac{x^3}{\cancel{3}}\right]_0^a = 8$

$\Rightarrow [x^3]_0^a = 8$

$\Rightarrow a^3 - 0 = 8$

$\Rightarrow a^3 = 8$

$\Rightarrow a^3 = 2^3$

$\Rightarrow a = 2$

$\therefore a = 2$

3 Mark Questions

1. **Evaluate** $\int \frac{x^4 + x^2 + 1}{x^2 - x + 1}dx$

Sol.

$$\int \frac{(x^2+1)^2 - x^2}{x^2 - x + 1}dx = \int \frac{(x^2+1)^2 - x^2}{x^2 - x + 1}dx$$

$$= \int \frac{(x^2+1+x)\cancel{(x^2+1-x)}}{\cancel{x^2-x+1}}dx$$

$[\because a^2 - b^2 = (a+b)(a-b)]$

$$= \int (x^2 + 1 + x)dx$$

$$= \frac{x^3}{3} + x + \frac{x^2}{2} + c$$

2. **Evaluate** $\int \frac{\cos 2x - \cos 2\alpha}{\cos x - \cos\alpha}dx$

Sol.

$$\int \frac{\cos 2x - \cos 2\alpha}{\cos x - \cos\alpha}dx$$

$$= \int \frac{(2\cos^2 x - 1) - (2\cos^2\alpha - 1)}{\cos x - \cos\alpha}dx$$

$$= \int \frac{2\cos^2 x - \cancel{1} - 2\cos^2\alpha + \cancel{1}}{\cos x - \cos\alpha}dx$$

$$= 2\int \frac{\cos^2 x - \cos^2\alpha}{\cos x - \cos\alpha}dx$$

$$= 2\int \frac{(\cos x + \cos\alpha)\cancel{(\cos x - \cos\alpha)}}{\cancel{\cos x - \cos\alpha}}dx$$

$= 2\int (\cos x + \cos\alpha)\,dx$

$= 2\,[\sin x + \cos\alpha \cdot x] + c$

$= 2\sin x + 2x\cos\alpha + c$

3. **Evaluate** $\int \frac{(a^x + b^x)^2}{a^x b^x}dx$

Sol.

$$\int \frac{(a^x + b^x)^2}{a^x b^x}dx = \int \frac{a^{2x} + b^{2x} + 2a^x b^x}{a^x b^x}dx$$

$[\because (a+b)^2 = a^2 + 2ab + b^2]$

$$= \int \left(\frac{a^{2x}}{a^x b^x} + \frac{b^{2x}}{a^x b^x} + \frac{2a^x b^x}{a^x b^x}\right)dx$$

$$=\int\left(\frac{a^x}{b^x}+\frac{b^x}{a^x}+2\right)dx$$

$$=\int\left(\left(\frac{a}{b}\right)^x+\left(\frac{b}{a}\right)^x+2\right)dx$$

$$=\frac{\left(\frac{a}{b}\right)^x}{\log_e\left(\frac{a}{b}\right)}+\frac{\left(\frac{b}{a}\right)^2}{\log_e\left(\frac{b}{a}\right)}+2x+c$$

4. If $f'(x) = 3x^2 - \dfrac{2}{x^3}$ and $f(1) = 0$, find $f(x)$

Sol. Given

$$f'(x) = 3x^2-\frac{2}{x^3}$$

We know $f(x) = \int f'(x)\,dx$

$$= \int\left(3x^2-\frac{2}{x^3}\right)dx$$

$$f(x) = 3\left(\frac{x^3}{3}\right)-2\left(\frac{x^{-2}}{-2}\right)+c$$

$$f(x) = x^3+\frac{1}{x^2}+c \qquad \text{......(1)}$$

Also $f(1) = 0$

$$0 = 1^3+\frac{1}{1^2}+c$$

$$0 = 1+1+c$$

$$c = -2$$

$$\therefore \quad f(x) = x^3+\frac{1}{x^2}-2$$

5. Evaluate $\displaystyle\int\frac{8^{1+x}+4^{1-x}}{2^x}dx$

Sol. $\displaystyle\int\frac{8^{1+x}+4^{1-x}}{2^x}dx$

$$= \int\frac{\left(2^3\right)^{1+x}+\left(2^2\right)^{1-x}}{2^x}dx$$

$$= \int\frac{2^{3+3x}+2^{2-2x}}{2^x}dx$$

$$= \int\frac{2^{3+3x}}{2^x}dx+\int\frac{2^{2-2x}}{2^x}dx$$

$$= \int 2^{3+3x-x}\,dx+\int 2^{2-2x-x}\,dx$$

$$= \int 2^{2x+3}\,dx+\int 2^{2-3x}\,dx$$

$$= \frac{2^{2x+3}}{2\log 2}+\frac{2^{2-3x}}{-3\log 2}+c$$

$$= \frac{2^{2x+3-1}}{\log 2}-\frac{2^{2-3x}}{3\log 2}+c$$

$$= \frac{2^{2x+2}}{\log 2}-\frac{2^{2-3x}}{3\log 2}+c$$

6. Evaluate $\displaystyle\int\frac{\sec^2 x}{3+\tan x}dx$

Sol.

Let $\quad I = \displaystyle\int\frac{\sec^2 x}{3+\tan x}dx$

Put $\quad 3+\tan x = t$

$\Rightarrow 0+\sec^2 x\,dx = dt$

$\Rightarrow \quad \sec^2 x\,dx = dt$

$\therefore \quad I = \displaystyle\int\frac{dt}{t}$

$= \log|t|+c$

$= \log|3+\tan x|+c$

$[\because t = 3+\tan x]$

7. Evaluate $\int \sin^3 x\cos x\,dx$

Sol.

Let $\quad I = \int\sin^3 x\cos x\,dx$

Put $\quad t = \sin x$

$\Rightarrow \quad dt = \cos x\,dx$

$\therefore \quad I = \displaystyle\int t^3.\,dt = \frac{t^4}{4}+c = \frac{\sin^4 x}{4}+c$

8. Evaluate $\displaystyle\int\frac{1}{\sqrt{16x^2+25}}dx$

Sol. $\displaystyle\int\frac{1}{\sqrt{16x^2+25}}dx$

$$= \int\frac{1}{\sqrt{16\left(x^2+\frac{25}{16}\right)}}dx$$

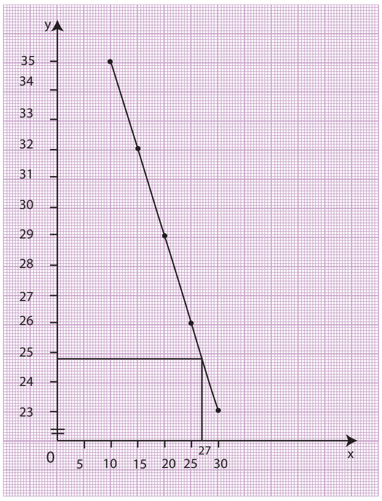
$$= \frac{1}{4}\int\frac{dx}{\sqrt{x^2+\left(\frac{5}{4}\right)^2}}$$

$$= \frac{1}{4}\log\left|x+\sqrt{x^2+\left(\frac{5}{4}\right)^2}\right|+c$$

$$= \frac{1}{4}\log\left|\frac{4x+\sqrt{16x^2+25}}{4}\right|+c$$

2. Using graphic method, find the value of y when $x = 27$.

x	10	15	20	25	30
y	35	32	29	26	23



From the graph, it is clear that when $x = 27$, the value of y is 24.8

3. Find y when $x = 0.2$ given that

x	0	1	2	3	4
y	176	185	194	202	212

Sol.

Since $x = 0.2$ lies at the beginning of the table, use Newton's foward interpolation formula

$$\Rightarrow y = y_0 + \frac{n}{n!}\Delta y_0 + \frac{n(n-1)}{2!}\Delta^2 y_0 + \frac{n(n-1)(n-2)}{3!}\Delta^3(y_0)$$

Here $h = 1, x_0 = 0, x = 0.2$

$\Rightarrow x_0 + nh = 0.2 \Rightarrow 0 + n(1) = 0.2 \Rightarrow n = 0.2$

The forward difference table is

x	y	Δy	$\Delta^2 y$	$\Delta^3 y$	$\Delta^4 y$
0	176				
1	185	9			
2	194	9	0		
3	202	8	–1	–1	
4	212	10	2	3	4

$$\therefore y = 176 + \frac{0.2}{1!}(9) + \frac{(0.2)(0.2-1)}{2!}(0)$$

$$+ \frac{(0.2)(0.2-1)(0.2-2)}{3!}(-1)$$

$$+ \frac{(0.2)(0.2-1)(0.2-2)(0.3-3)}{4!}(4)$$

$$= 176 + 1.8 - 0.048 - 0.1344 = 177.6176$$

$\therefore$ Hence when $x = 0.2$, $y = 177.6176$.

4. If $y_{75} = 2459$, $y_{80} = 2018$, $y_{85} = 1180$, and $y_{90} = 402$, find y_{82}.

Sol.

We can write the given data as follows:

x	75	80	85	90
y	2459	2018	1180	402

Since 82 lies at the beginning of the table, we can use Newton's forward interpolation formula

$$\therefore y = y_0 + \frac{n}{1!}\Delta y_0 + \frac{n(n-1)}{2!}\Delta^2 y_0 + \frac{n(n-1)(n-2)}{3!}\Delta^3 y_0$$

Also $x_0 + nh = 82 \Rightarrow 75 + n(5) = 82 \Rightarrow 5n = 82 - 75 = 7$

$$\Rightarrow n = \frac{7}{5} = 1.4$$

The difference table is

x	y	Δy	$\Delta^2 y$	$\Delta^3 y$
75	2459			
80	2018	–441		
85	1180	–838	–397	
90	402	–778	60	457

$$\therefore y = 2459 + \frac{1.4}{1!}(-441)$$

$$+ \frac{(1.4)(1.4-1)}{2!}(-397)$$

$$+ \frac{(1.4)(1.4-1)(1.4-2)}{3!}(457)$$

$$= 2459 - 617.4 - 111.6 - 25.592$$

$y = 1704.408$ when $x = 82$.

5. Find the number of men getting wages between ₹ 30 and ₹ 35 from the following table.

Wages (x)	20 – 30	30 – 40	40 – 50	50 – 60
No. of men (y)	9	30	35	42

Sol.

Let us calculate the number of men whose wages is less than ₹ 35 by using Newton's forward interpolation formula

$x_0 + nh = x \Rightarrow 30 + n\,(10) = 35$

$\Rightarrow 10n = 35 - 30 = 5$

$\Rightarrow n = \frac{5}{10} = 0.5$

The difference table is

	x	y	Δy	$\Delta^2 y$	$\Delta^3 y$
under	30	9			
under	40	39	30		
under	50	74	35	5	
under	60	116	42	7	2

$$\therefore y = y_0 + \frac{n}{1!}\Delta y_0 + \frac{n(n-1)}{2!}\Delta^2 y_0 + \frac{n(n-1)(n-2)}{3!}\Delta^3 y_0$$

$$\therefore y(35) = 9 + \frac{0.5}{1!}(30) + \frac{(0.5)(0.5-1)}{2}(5) + \frac{(0.5)(0.5-1)(0.5-2)}{6}(2)$$

$$= 9 + 15 - 0.6 + 0.1 = 24 \text{ (approximately)}$$

$\therefore$ Number of men getting wages between ₹ 30 and ₹ 35 is $y(35) - y(30) = 24 - 9 = 15$.

6. Estimate the population for the year 1995.

year (x)	1961	1971	1981	1991	2001
population in thousands (y)	46	66	81	93	101

Sol. Since 1995 lies at the table of the table, use Newton's backward interpolation formula.

Also, $x_n + nh = x \Rightarrow 2001 + n\,(10) = 1995$

$\Rightarrow 10n = 1995 - 2001 \Rightarrow n = \frac{-6}{10} \Rightarrow n = -0.6$

$$y = y_n + \frac{n}{1!}\nabla y_n + \frac{n(n+1)}{2!}\nabla y_n^2 + \frac{n(n+1)(n+2)}{3!}\nabla^3 y_n.$$

The difference table is

x	y	∇y	$\nabla^2 y$	$\nabla^3 y$	$\nabla^4 y$
1961	46				
1971	66	20			
1981	81	15	–5		
1991	93	12	–3	2	
2001	101	8	–4	–1	–3

$$\therefore y = 101 + \frac{(-0.6)}{1!}(8) + \frac{(-0.6)(-0.6+1)}{2!}(-4) + \frac{(-0.6)(-0.6+1)(-0.6+2)}{3!}(-1) + \frac{(-0.6)(-0.6+1)(-0.6+2)(-0.6+3)}{4!}(-3)$$

$$= 101 - (0.6)\,8 + \frac{(-0.6)(0.4)}{2}(-4) + \frac{(-0.6)(0.4)(1.4)}{6}(-1) + \frac{(-0.6)(0.4)(1.4)(2.4)(-3)}{24}$$

$$= 96.\,8368$$

Hence, population for the year 1995 is 96.837 thousands.

7. Using Lagrange's formula, find the value of y when $x = 42$ from the following table.

x	40	50	60	70
y	31	73	124	159

Sol.

By data, we have

$x_0 = 40, x_1 = 50, x_2 = 60, \quad x_3 = 70$

$y_0 = 31, y_1 = 73, y_2 = 124, y_3 = 159.$

Using Lagrange's formula, we get

$$y = y_0\frac{(x-x_1)(x-x_2)(x-x_3)}{(x_0-x_1)(x_0-x_2)(x_0-x_3)} + y_1\frac{(x-x_0)(x-x_2)(x-x_3)}{(x_1-x_0)(x_1-x_2)(x_1-x_3)} + y_2\frac{(x-x_0)(x-x_1)(x-x_3)}{(x_2-x_0)(x_2-x_1)(x_2-x_3)}$$

$$+ y_3 \frac{(x-x_0)(x-x_1)(x-x_2)}{(x_3-x_0)(x_3-x_1)(x_3-x_2)}$$

$$\therefore \quad y(42) = 31 \frac{(-8)(-18)(-28)}{(-10)(-20)(-30)} + 73 \frac{(2)(-18)(-28)}{(10)(-10)(-20)}$$

$$+ 124 \frac{(2)(-8)(-28)}{(20)(10)(-10)}$$

$$+ 59 \frac{(2)(-8)(-28)}{(30)(20)(10)}$$

$= 20.832 + 36.792 - 27.776 + 7.632$

$y = 37.48$

8. Using Lagrange's formula and $y(x)$ from the following table.

x	6	7	10	12
y	13	14	15	17

Sol.

Given

$x_0 = 6, \quad x_1 = 7, \quad x_2 = 10, \quad x_3 = 12$

$y_0 = 13, \quad y_1 = 14, \quad y_2 = 15, \quad y_3 = 17$

Using Lagrange's formula,

$$y(11) = 13 \frac{(4)(1)(-1)}{(-1)(-4)(-6)} + 14 \frac{(5)(1)(-1)}{(1)(-3)(-5)}$$

$$+ 5 \frac{(5)(4)(-1)}{(4)(3)(-2)} + 17 \frac{(5)(4)(1)}{(6)(5)(2)}$$

$= 2.1666 - 4.6666 + 12.5 + 5.6666$

$= 15.6666$

$\therefore y(x) = 15.6666$

PRACTICE 5 MARK QUESTIONS

1. Estimate the production for 1962 and 1965 from the following data.

year	1961	1962	1963	1964	1965	1966	1967
Production in tonnes	200	–	260	306	–	390	430

Sol.

Since five values of $f(x)$ are given, we assume that polynomial is of degree four.

Hence, fifth order differences are zeros.

$\therefore \Delta^5 (y_k) = 0$

$\Rightarrow (E-1)^5 y_k = 0$ (1)

Putting $k = 0$ in (1) we get

$(E-1)^5 y_0 = 0 \Rightarrow (E^5 - 5E^4 + 10E^3 - 10E^2 + 5E - 1) y_0 = 0$

$\Rightarrow y_5 - 5y_4 + 10y_3 - 10y_2 + 5y_1 - y_0 = 0$

$\Rightarrow 390 - 5y_4 + 10y_3 - 10y_2 + 5y_1 - y_0 = 0$

$\Rightarrow y_1 - y_4 = -130$ (2)

Putting $k = 1$ in (1) we get,

$(E-1)^5 y_1 = 0$

$\Rightarrow (E^5 - 5E^4 + 10E^3 - 10E^2 + 5E - 1) y_1 = 0$

$\Rightarrow y_6 - 5y_5 + 10y_4 - 10y_3 + 5y_2 - y_1 = 0$

$\Rightarrow 430 - 5(390) + 10y_4 - 10(306) + 5(260) - y_1 = 0$

$\Rightarrow 10y_4 - y_1 = 3280$ (3)

$(2) + (3) \rightarrow \cancel{y_1} - y_4 + 10y_4 - \cancel{y_1} = -130 + 3280$

$\Rightarrow 9y_4 = 3150$

$\Rightarrow y_4 = 350$

Substituting $y_4 = 350$ in (2) we get,

$y_1 - 350 = -130 \Rightarrow y_1 = -130 + 350$

$\Rightarrow y_1 = 220$

$\therefore$ The productions for 1962 and 1965 are 220 tonnes and 350 tonnes respectively.

2. From the following data, calculate the value of $e^{1.75}$

x	1.7	1.8	1.9	2.0	2.1
e^x	5.474	6.050	6.686	7.389	8.166

Sol. Since $e^{1.75}$ lies at the beginning of the table, we can use Newton's forward interpolation formula

$\therefore x_0 + nh = x \Rightarrow 1.7 + n(0.1) = 1.75$

$\Rightarrow n(0.1) = 1.75 - 1.7 = 0.05$

$$\Rightarrow n = \frac{0.05}{0.1} = 0.5$$

$$y_x = y_0 + \frac{n}{1!}\Delta y_0 + \frac{n(n-1)}{2!}\Delta^2 y_0 + \frac{n(n-1)(n-2)}{3!}\Delta^3 y_0 + \ldots\ldots$$

The difference table is

x	y	Δy	$\Delta^2 y$	$\Delta^3 y$	$\Delta^4 y$
1.7	5.474				
1.8	6.050	0.576			
1.9	6.686	0.636	0.060		
2.0	7.389	0.703	0.067	0.007	
2.1	8.166	0.777	0.074	0.007	0

$$\therefore \quad y(e^{1.75}) = 5.474 + \frac{0.5}{1!}(0.576) + \frac{(0.5)(0.5-1)}{2!}(0.06)$$

$$+ \frac{(0.5)(0.5-1)(0.5-2)}{3!}(0.007)$$

$$y(e^{1.75}) = 5.474 + 0.288 - 0.0075 + 0.0004375$$
$$= 5.7549375$$

3. From the data, find the number of students whose height is between 80 cm and 90 cm

Height in cm (x)	40–60	60–80	80 – 100	100–120	120–140
No. of. students (y) :	250	120	100	70	50

Sol.

Let us calculate the number of students whose height is less than 90 cm using Newton's forward interpolation formula.

$x_0 + nh = x \Rightarrow 60 + n(20) = 90 \Rightarrow 20n = 30$

$\Rightarrow n = \frac{3}{2} = 1.5$

$$y_x = y + \frac{n}{1!}\Delta y_0 + \frac{n(n-1)}{2!}\Delta^2 y_0 + \frac{n(n-1)(n-2)}{3!}\Delta^3 y_0 + \ldots$$

The difference table is as follows:

x	y	Δy	$\Delta^2 y$	$\Delta^3 y$	$\Delta^4 y$
Below 60	250				
Below 80	370	120			
Below 100	470	100	–20		
Below 120	540	70	–30	–10	
Below 140	590	50	–20	10	20

$$\therefore y(90) = 250 + (1.5)(120) + \frac{(1.5)(1.5-1)}{2!}(-20) + \frac{(1.5)(1.5-1)(1.5-2)}{3!}(-10) + \frac{(1.5)(1.5-1)(1.5-2)(1.5-3)}{4!}(20)$$

$= 250 + 180 - 7.5 + 0.625 + 0.46875$

$= 423.59 = 424$ (app)

$\therefore$ Number of students whose height is between 80 cm and 90 cm is $y(90) - y(80)$

$= 424 - 370$

$= 54.$

4. From the following table, estimate the premium for a policy maturing at the age of 58.

Age (x)	40	45	50	55	60
Premium (y)	114.84	96.16	83.32	74.48	68.48

Sol.

Using Newton's backward interpolation formula, we can find y when $x = 58$.

$\therefore x_n + nh = x \Rightarrow 60 + n(5) = 58$

$\Rightarrow 5n = 58 - 60 = -2$

$\Rightarrow n = \frac{-2}{5} = -0.4$

$$\text{and } y(58) = y_n + \frac{n}{1!}\nabla y_n + \frac{n(n+1)}{2!}\nabla^2 y_n + \frac{n(n+1)(n+2)}{3!}\nabla^3 y_n + \ldots$$

The difference table is

x	y	∇y	$\nabla^2 y$	$\nabla^3 y$	$\nabla^4 y$
40	114.84				
45	96.16	–18.68			
50	83.32	–12.84	5.84		
55	74.48	–8.84	4.00	–1.84	
60	63.48	–6.00	2.84	–1.16	0.68

$$\therefore y(58) = 68.48 + \frac{(-0.4)}{1!}(-6) + \frac{(-0.4)(-0.4+1)}{2!}(2.84)$$

$$+\frac{(-0.4)(-0.4+1)(-0.4+2)}{3!}(-1.16)$$

$$+\frac{(-0.4)(-0.4+1)(-0.4+2)(-0.4+3)}{3!}(0.68)$$

$= 68.48 + (0.4)(6)$

$$+\frac{(-0.4)(0.6)}{2}(2.84)$$

$$+\frac{(-0.4)(0.6)(1.6)}{6}(-1.16)$$

$$+\frac{(-0.4)(0.6)(1.6)(2.6)}{24}(0.68)$$

$= 68.48 + 2.4 - 0.3408 + 0.07424 - 0.028288$

$= 70.5851052$

$\Rightarrow y(58) = 70.59$

$\therefore$ Hence, premium for a policy maluting at the age of 58 is 70.59.

5. Using Lagrange's formula find the value of y when $x = 4$ from the following table.

x	0	3	5	6	8
y	276	460	414	343	110

Sol.

Given $x_0 = 0$, $x_1 = 3$, $x_2 = 5$, $x_3 = 6$, $x_4 = 8$

$y_0 = 276$, $y_1 = 460$, $y_2 = 414$, $y_3 = 343$, $y_4 = 110$

Lagrange's formula is

$$y = \frac{(x-x_1)(x-x_2)(x-x_3)(x-x_4)}{(x_0-x_1)(x_0-x_2)(x_0-x_3)(x_0-x_4)}y_0$$

$$+\frac{(x-x_0)(x-x_2)(x-x_3)(x-x_4)}{(x_1-x_0)(x_1-x_2)(x_1-x_3)(x_1-x_4)}y_1$$

$$+\frac{(x-x_0)(x-x_1)(x-x_3)(x-x_4)}{(x_2-x_0)(x_2-x_1)(x_2-x_3)(x_2-x_4)}y_2$$

$$+\frac{(x-x_0)(x-x_1)(x-x_2)(x-x_4)}{(x_3-x_0)(x_3-x_1)(x_3-x_2)(x_3-x_4)}y_3$$

$$+\frac{(x-x_0)(x-x_1)(x-x_2)(x-x_3)}{(x_4-x_0)(x_4-x_1)(x_4-x_2)(x_4-x_3)}y_4$$

$$\Rightarrow y = 276\,\frac{(1)(-1)(-2)(-4)}{(-3)(-5)(-6)(-8)}$$

$$+ 460\,\frac{(4)(-1)(-2)(-4)}{(3)(-2)(-3)(-5)}$$

$$+ 414\,\frac{(4)(1)(-2)(-4)}{(5)(2)(-1)(-3)}$$

$$+ 343\,\frac{(4)(1)(-1)(-4)}{(6)(3)(1)(-2)}$$

$$+ 110\,\frac{(4)(1)(-1)(-2)}{(8)(5)(3)(2)}$$

$\Rightarrow y = -3.066 + 163.555 + 441.6 - 152.44 + 3.666$

$\Rightarrow y = 453.311.$

***

Note :