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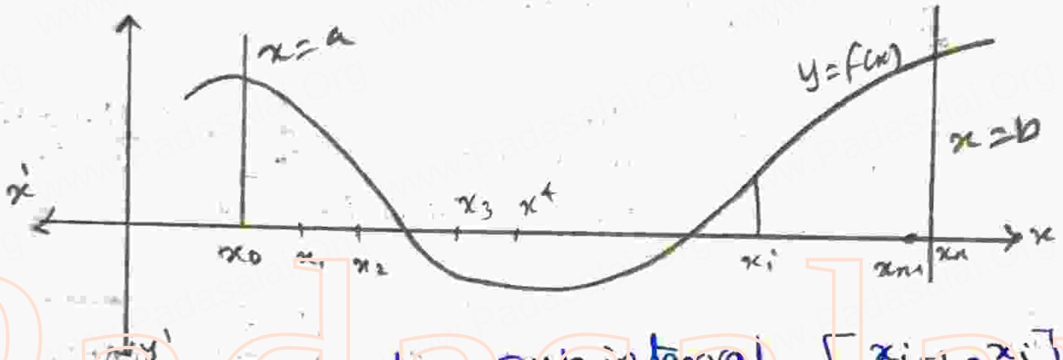
9. Applications Of Integration

Riemann Integral :-

Consider a real valued, bounded function $f(x)$ defined on the closed and bounded interval $[a, b]$.

The function $f(x)$ need not have the same sign on $[a, b]$; Partition the interval $[a, b]$ into n subintervals. $[x_0, x_1], [x_1, x_2], \ldots [x_{n-2}, x_{n-1}], [x_{n-1}, x_n]$ such that

$$a = x_0 < x_1 < x_2 < \ldots < x_{n-1} < x_n = b$$



In each sub interval $[x_{i-1}, x_i]$ $i = 1, 2, 3 \ldots n$, chose a real number ξ_i arbitrarily such that $x_{i-1} \leq \xi_i \leq x_i$.

Consider the sum

$$\sum_{i=1}^{n} f(\xi_i)(x_i - x_{i-1}) = f(\xi_1)(x_1 - x_0) + f(\xi_2)(x_2 - x_1) + \ldots + f(\xi_n)(x_n - x_{n-1})$$

is called a Riemann Sum of $f(x)$ Corresponding to the Partition $[x_0, x_1], [x_1, x_2) \ldots [x_{n-1}, x_n]$ of $[a, b]$

The limiting Process $n \to \infty$ and $\max(x_i - x_{i-1}) \to 0$ the sum tends to finite value. It is called definite integeral. and also it called the Riemann Integral.

Left-end rule :

$$\int_a^b f(x)\,dx = \lim_{\substack{n\to\infty \text{ and} \\ \max(x_i - x_{i-1})\to 0}} \sum_{i=1}^{n} f(x_{i-1})(x_i - x_{i-1})$$

Right-end rule :

$$\int_a^b f(x)\,dx = \lim_{\substack{n\to\infty \text{ and} \\ \max(x_i - x_{i-1})\to 0}} \sum_{i=1}^{n} f(x_i)(x_i - x_{i-1})$$

mid-Point rule :-

$$\int_a^b f(x)\,dx = \lim_{\substack{n\to\infty \text{ and} \\ \max(x_i - x_{i-1})\to 0}} \sum_{i=1}^{n} f\left(\frac{x_{i-1}+x_i}{2}\right)(x_i - x_{i-1})$$

9.1 / 9.4 Estimate the value of $\int_0^{0.5} x^2\,dx$ using the Riemann sums corresponding to 5 subintervals of equal width and applying
(i) left-end rule (ii) right-end rule
(iii) The mid point rule.

$a=0 \quad b=0.5 \quad n=5$

$$h = \Delta x = \frac{b-a}{n} = \frac{0.5-0}{5} = 0.1$$

$x_0 = 0 \quad x_1 = 0.1 \quad x_2 = 0.2 \quad x_3 = 0.3 \quad x_4 = 0.4$

$x_5 = 0.5$

(i) Left-end rule

$$S = [f(x_0) + f(x_1) + f(x_2) + \dots + f(x_{n-1})]\,\Delta x$$

$$S = [f(0) + f(0.1) + f(0.2) + f(0.3) + f(0.4)]\,0.1$$

$$= (0 + 0.01 + 0.04 + 0.09 + 0.16)\,0.1$$

$$= (0.3)(0.1) = 0.03.$$

$$\int_0^{0.5} x^2\,dx \cong 0.03.$$

(ii) Right-end Rule :-

$$S = [f(x_1)+f(x_2)+f(x_3)+\cdots+f(x_n)]\,\Delta x$$

$$= [f(0.1)+f(0.2)+f(0.3)+f(0.4)+f(0.5)]\,0.1$$

$$= (0.01+0.04+0.09+0.16+0.25)\,0.1$$

$$= (0.55)(0.1) = 0.055$$

$$\int_0^{0.5} f(x)\,dx = \int_0^{0.5} x^2\,dx \simeq 0.055$$

(iii) The mid Point rule :-

$$S = \left[f\left(\frac{x_0+x_1}{2}\right)+f\left(\frac{x_1+x_2}{2}\right)+f\left(\frac{x_2+x_3}{2}\right)+f\left(\frac{x_3+x_4}{2}\right)+f\left(\frac{x_4+x_5}{2}\right)\right]\Delta x$$

$$= [f(0.05)+f(0.15)+f(0.25)+f(0.35)+f(0.45)]\,0.1$$

$$= [0.0025+0.0225+0.0625+0.1225+0.2025]\,0.1$$

$$= (0.4125)(0.1)$$

$$= 0.04125$$

$$\int_0^{0.5} x^2\,dx \simeq 0.04125$$

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Ex : 9.1

$\frac{1}{96}$ Find an approximate value of $\int_1^{1.5} x\,dx$ by applying the left end rule with the partition $\{1.1, 1.2, 1.3, 1.4, 1.5\}$ (x_1, x_2, x_3, x_4, x_5)

$a = 1 \quad b = 1.5 \quad n = 5$

$$\Delta x = \frac{b-a}{n} = \frac{1.5-1}{5} = \frac{0.5}{5} = 0.1$$

Left-end rule :

$$S = [f(x_0)+f(x_1)+f(x_2)+\cdots+f(x_{n-1})]\,\Delta x$$

$$= [f(1)+f(1.1)+f(1.2)+f(1.3)+f(1.4)]\,0.1$$

$$= [1+1.1+1.2+1.3+1.4]\,0.1 = 6(0.1)$$

$$= 0.6$$

$$\int_1^{1.5} x\,dx \simeq 0.6$$

$\frac{2}{96}$. Find an approximate value of $\int_1^{1.5} x^2 dx$ by applying the right-end rule with partition $\{1.1, 1.2, 1.3, 1.4, 1.5\}$

$a = 1 \quad b = 1.5 \quad n = 5$

$$h = \Delta x = \frac{b-a}{n} = \frac{1.5-1}{5} = \frac{0.5}{5} = 0.1$$

$x_0 = 1 \quad x_1 = 1.1 \quad x_2 = 1.2 \quad x_3 = 1.3$

$x_4 = 1.4 \quad x_5 = 1.5$

Right end Rule :

$$S = [f(x_1) + f(x_2) + f(x_3) + f(x_4) + f(x_5)]\Delta x$$

$$= [f(1.1) + f(1.2) + f(1.3) + f(1.4) + f(1.5)]\, 0.1$$

$$= [1.21 + 1.44 + 1.69 + 1.96 + 2.25]\, 0.1$$

$$= (8.55)(0.1) = 0.855$$

$$\int_1^{1.5} x^2 dx \simeq 0.855.$$

$\frac{3}{96}$ Find an approximate value of $\int_1^{1.5} (2-x)dx$ by applying the mid point rule with the partition $\{1.1, 1.2, 1.3, 1.4, 1.5\}$

$a = 1 \quad b = 1.5 \quad n = 5$

$$h = \Delta x = \frac{b-a}{n} = \frac{1.5-1}{5} = \frac{0.5}{5} = 0.1$$

$x_0 = 1 \quad x_1 = 1.1 \quad x_2 = 1.2 \quad x_3 = 1.3 \quad x_4 = 1.4$

$x_5 = 1.5$

The mid Point rule :

$$S = \left[f\left(\frac{x_0+x_1}{2}\right) + f\left(\frac{x_1+x_2}{2}\right) + f\left(\frac{x_2+x_3}{2}\right) + f\left(\frac{x_3+x_4}{2}\right) + f\left(\frac{x_4+x_5}{2}\right)\right]\Delta x$$

$$S = \left[f\left(\frac{1+1.1}{2}\right) + f\left(\frac{1.1+1.2}{2}\right) + f\left(\frac{1.2+1.3}{2}\right) + f\left(\frac{1.3+1.4}{2}\right) + f\left(\frac{1.4+1.5}{2}\right)\right] 0.1$$

$$= f(1.05) + f(1.15) + f(1.25) + f(1.35) + f(1.45)]\, 0.1$$

$$= [0.95 + 0.85 + 0.75 + 0.65 + 0.55]\, 0.1$$

$$= (3.75)(0.1) = 0.375$$

$$\int_1^{1.5} (2-x)dx \simeq 0.375$$

Limit Formula to Evaluate $\int_a^b f(x)\,dx$.

$$R = \lim_{n\to\infty} \frac{b-a}{n} \sum_{r=1}^{n} f(a+r\Delta x) = \int_a^b f(x)\,dx.$$

$$\Delta x = \frac{b-a}{n}$$

9.2
97.

Evaluate $\int_0^1 x\,dx$ as the limit of a sum.

$a=0 \quad b=1 \qquad \Delta x = \frac{b-a}{n} = \frac{1-0}{n} = \frac{1}{n}$

$$f(a+r\Delta x) = f(0+r\cdot\tfrac{1}{n}) = f(\tfrac{r}{n})$$

$$\boxed{\int_a^b f(x)\,dx = \lim_{n\to\infty} \frac{b-a}{n} \cdot \sum_{r=1}^{n} f(a+r\Delta x)}$$

$$= \lim_{n\to\infty} \frac{1}{n} \sum_{r=1}^{n} f(\tfrac{r}{n})$$

$$= \lim_{n\to\infty} \frac{1}{n} \sum_{1}^{n} \frac{r}{n}$$

$$= \lim_{n\to\infty} \frac{1}{n^2} \sum_{1}^{n} r$$

$$= \lim_{n\to\infty} \frac{1}{n^2} \left[\frac{n(n+1)}{2}\right]$$

$$= \lim_{n\to\infty} \frac{1}{\cancel{n^2}} \left[\frac{\cancel{n^2}(1+\frac{1}{n})}{2}\right]$$

$$= \frac{1}{2}$$

$$\therefore \int_0^1 x\,dx = \frac{1}{2}.$$

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9.3
97

$\int_0^1 x^3\,dx$ as the limit of a sum.

$a=0 \quad b=1 \qquad \Delta x = \frac{b-a}{n} = \frac{1-0}{n} = \frac{1}{n}$

$$f(a+r\Delta x) = f(0+\tfrac{r}{n}) = f(\tfrac{r}{n})$$

$$\int_a^b f(x)\,dx = \lim_{n\to\infty} \frac{b-a}{n} \sum_{r=1}^{n} f(a+r\Delta x)$$

$$= \lim_{n\to\infty} \frac{1}{n} \sum_{r=1}^{n} f(\tfrac{r}{n})$$

$$= \lim_{n\to\infty} \frac{1}{n} \sum_{r=1}^{n} (\tfrac{r}{n})^3$$

$$= \lim_{n\to\infty} \frac{1}{n^4} \sum_{r=1}^{n} r^3$$

$$= \lim_{n\to\infty} \frac{1}{n^4} \left[\frac{n(n+1)}{2}\right]^2$$

$$= \lim_{n\to\infty} \frac{1}{\cancel{n^4}}\left[\frac{\cancel{n^2}\cdot \cancel{n^2}(1+\frac{1}{n})^2}{4}\right] = \frac{1}{4}$$

$$\therefore \int_0^1 x^3\,dx = \frac{1}{4}.$$

9.4 / 97 Evaluate $\int_1^4 (2x^2+3)\,dx$ as the limit of a sum.

$a=1 \quad b=4 \quad \Delta x = \frac{b-a}{n} = \frac{4-1}{n} = \frac{3}{n}$

$$f(a+r\Delta x) = f\left(1+r\left(\frac{3}{n}\right)\right) = f\left(1+\frac{3r}{n}\right)$$

$$f\left(1+\frac{3r}{n}\right) = 2\left(1+\frac{3r}{n}\right)^2+3$$

$$= 2\left\{1+\frac{9r^2}{n^2}+\frac{6r}{n}\right\}+3$$

$$= 2+\frac{18}{n^2}r^2+\frac{12}{n}r+3$$

$$= 5+\frac{18}{n^2}r^2+\frac{12}{n}r$$

$$\int_a^b f(x)\,dx = \lim_{n\to\infty}\frac{b-a}{n}\cdot\sum_{r=1}^{n} f(a+r\Delta x)$$

$$= \lim_{n\to\infty}\frac{3}{n}\sum_{r=1}^{n}\left(5+\frac{18}{n^2}r^2+\frac{12}{n}r\right)$$

$$= \lim_{n\to\infty}\frac{3}{n}\left[\sum_{r=1}^{n}5+\frac{18}{n^2}\sum_{r=1}^{n}r^2+\frac{12}{n}\sum_{r=1}^{n}r\right]$$

$$= \lim_{n\to\infty}\frac{3}{n}\left[5n+\frac{18}{\cancel{n^2}}\cdot\frac{\cancel{n}(n+1)(2n+1)}{6}+\frac{12}{\cancel{n}}\cdot\frac{\cancel{n}(n+1)}{2}\right]$$

$$= \lim_{n\to\infty}\frac{3}{n}\left[5n+3n\left(1+\frac{1}{n}\right)\left(2+\frac{1}{n}\right)+6n\left(1+\frac{1}{n}\right)\right]$$

$$= \lim_{n\to\infty} 3\left[5+3\left(1+\frac{1}{n}\right)\left(2+\frac{1}{n}\right)+6\left(1+\frac{1}{n}\right)\right]$$

$$= 3\left[5+3(2)+6(1)\right] = 3\,(5+6+6)$$

$$= 3(17) = 51$$

$$\int_1^4 (2x^2+3)\,dx = 51.$$

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Ex : 9.2

$\frac{1}{98}$ Evaluate the following integrals as the limits of sums.

(i) $\int_0^1 (5x+4)\,dx$ $\quad a=0 \quad b=1$

$\Delta x = \frac{b-a}{n} = \frac{1-0}{n} = \frac{1}{n}$

$f(a+r\Delta x) = f(0+r(\frac{1}{n})) = f(\frac{r}{n})$

$f(\frac{r}{n}) = 5(\frac{r}{n})+4$

$$\int_a^b f(x)\,dx = \lim_{n\to\infty} \frac{b-a}{n}\sum_{r=1}^{n} f(a+r\Delta x)$$

$$= \lim_{n\to\infty} \frac{1}{n}\sum_{r=1}^{n} \frac{5r}{n}+4$$

$$= \lim_{n\to\infty} \frac{1}{n}\left\{\frac{5}{n}\sum_{r=1}^{n} r + \sum_{r=1}^{n} 4\right\}$$

$$= \lim_{n\to\infty} \frac{1}{n}\left[\frac{5}{\cancel{n}}\left(\frac{\cancel{n}(n+1)}{2}\right) + 4n\right]$$

$$= \lim_{n\to\infty} \frac{1}{n}\left\{\frac{5n}{2}\left(1+\frac{1}{n}\right) + 4n\right\}$$

$$= \lim_{n\to\infty} \frac{n}{n}\left[\frac{5}{2}\left(1+\frac{1}{n}\right)+4\right]$$

$$= \frac{5}{2}+4 = \frac{13}{2}$$

$$\therefore \int_0^1 (5x+4)\,dx = \frac{13}{2}.$$

(ii) $\int_1^2 (4x^2-1)\,dx$

$a=1 \quad b=2 \quad \Delta x = \frac{b-a}{n} = \frac{2-1}{n} = \frac{1}{n}$

$f(a+r\Delta x) = f(1+\frac{r}{n}) = 4(1+\frac{r}{n})^2-1$

$= 4\left(1+\frac{r^2}{n^2}+\frac{2r}{n}\right)-1$

$= 4+\frac{4r^2}{n^2}+\frac{8r}{n}-1 = 3+\frac{4r^2}{n^2}+\frac{8r}{n}$

$$\int_a^b f(x)\,dx = \lim_{n\to\infty} \frac{b-a}{n}\sum_{r=1}^{n} f(a+r\Delta x)$$

$$= \lim_{n\to\infty} \frac{1}{n}\sum_{r=1}^{n}\left[\frac{4r^2}{n^2}+\frac{8r}{n}+3\right]$$

$$= \lim_{n\to\infty} \frac{1}{n}\left\{\frac{4}{n^2}\sum_{r=1}^{n} r^2 + \frac{8}{n}\sum_{r=1}^{n} r + \sum_{r=1}^{n} 3\right\}$$

$$= \lim_{n\to\infty} \frac{1}{n}\left\{\frac{4}{n^2}\cdot\frac{n(n+1)(2n+1)}{6} + \frac{8}{n}\cdot\frac{n(n+1)}{2} + 3n\right\}$$

$$= \lim_{n\to\infty} \frac{1}{n}\left\{\frac{4}{n^2}\cdot\frac{n^3(1+\frac{1}{n})(2+\frac{1}{n})}{6} + \frac{8}{n}\,\frac{n^2(1+\frac{1}{n})}{2} + 3n\right\}$$

$$\left[\frac{4}{6}(1)(2)+\frac{8}{2}+3\right] = \frac{25}{3}.$$

9.3 Fundametal Theorems of integral calculus and their applications :-

First Fundametal Theorem of Integral Calculus:

If $f(x)$ be a continuous function defined on a closed interval $[a,b]$ and

$$F(x) = \int_a^x f(u)\,du \quad a<x<b$$ then

$\frac{d}{dx}F(x) = f(x)$. In other words, $F(x)$ is an anti-derivative of $f(x)$.

Second Fundametal Theorem of Integral Calculus:

If $f(x)$ be a continuous function defined on a closed Interval $[a,b]$ and $F(x)$ is an anti-derivative of $f(x)$, then

$$\int_a^b f(x)\,dx = F(b) - F(a)$$

Some basic Properties:

* $\int_a^b f(x)\,dx = \int_a^b f(u)\,du$

* $\int_a^b f(x)\,dx = -\int_b^a f(x)\,dx$

* $\int_a^b f(x)\,dx = \int_a^c f(x)\,dx + \int_c^b f(x)\,dx \quad a<c<b$

* $\int_a^b [\alpha f(x) + \beta g(x)]\,dx = \alpha\int_a^b f(x)\,dx + \beta\int_a^b g(x)\,dx$

  [α, β are constants.

* If $x = g(u)$ then $\int_a^b f(x)\,dx$

  $$= \int_c^d f(g(u))\,\frac{dg(u)}{du}\,du$$

  where $g(c) = a \quad g(d) = b$

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9.5/99 Evaluate : $\int_0^3 (3x^2-4x+5)\,dx$

$$\int_0^3 (3x^2-4x+5)\,dx = \left[\frac{\cancel{3}x^3}{\cancel{3}} - \frac{\overset{2}{\cancel{4}}x^2}{\cancel{2}} + 5x\right]_0^3$$

$$= (27-18+15) - (0)$$

$$= 24.$$

$$\int_0^3 (3x^2-4x+5)\,dx = 24.$$

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9.6/99 Evaluate : $\int_0^1 \frac{2x+7}{5x^2+9}\,dx$

$$\int_0^1 \frac{2x+7}{5x^2+9}\,dx = \int_0^1 \frac{2x}{5x^2+9}\,dx + 7\int \frac{1}{5x^2+9}\,dx$$

$$= \frac{1}{5}\int_0^1 \frac{10x}{5x^2+9}\,dx + 7\int \frac{dx}{(\sqrt{5}x)^2+9}$$

$$\left[\int \frac{f'(x)}{f(x)}\,dx = \log|f(x)|+c \qquad \int \frac{dx}{x^2+a^2} = \frac{1}{a}\tan^{-1}\left(\frac{x}{a}\right)+c \right]$$

$$= \left[\frac{1}{5}\log|5x^2+9| + 7\cdot\frac{3}{\sqrt{5}}\cdot\tan^{-1}\left(\frac{\sqrt{5}x}{3}\right)\right]_0^1$$

$$= \left[\frac{1}{5}\log|5x+9|\right]_0^1 + \frac{21}{\sqrt{5}}\left[\tan^{-1}\left(\frac{\sqrt{5}x}{3}\right)\right]_0^1$$

$$= \frac{1}{5}\log 14 - \frac{1}{5}\log 9 + \frac{21}{\sqrt{5}}\left[\tan^{-1}\left(\frac{\sqrt{5}}{3}\right) - \tan^{-1}(0)\right]$$

$$= \frac{1}{5}\log\left(\frac{14}{9}\right) + \frac{21}{\sqrt{5}}\tan^{-1}\left(\frac{\sqrt{5}}{3}\right)$$

9.7/180 Evaluate : $\int_0^1 [2x]$ where $[\cdot]$ is the greatest integer function.

$$\int_0^1 [2x]\,dx = \int_0^{\frac{1}{2}} [2x]\,dx + \int_{1/2}^1 [2x]\,dx$$

$$= 0 + \int_{1/2}^1 1\,dx = [x]_{1/2}^1$$

$$= 1 - \frac{1}{2} = \frac{1}{2}$$

$$\int_0^1 [2x]\,dx = \frac{1}{2}.$$

9.8 / 100

Evaluate : $\int_0^{\pi/3} \frac{\sec x \tan x}{1+\sec^2 x}\, dx$

$u = \sec x$

$du = \sec x \tan x \, dx$

x	0	$\pi/3$
u	1	2

$$I = \int_1^2 \frac{du}{1+u^2}$$

$$= \left[\tan^{-1}(u)\right]_1^2$$

$$= \tan^{-1}(2) - \tan^{-1}(1)$$

$$= \tan^{-1} 2 - \pi/4$$

$$\therefore \int_0^{\pi/3} \frac{\sec x \tan x}{1+\sec^2 x}\, dx = \tan^{-1}(2) - \pi/4.$$

9.9 / 100

Evaluate : $\int_0^9 \frac{1}{x+\sqrt{x}}\, dx$

$u = \sqrt{x}$

$u^2 = x$

$2u\, du = dx$

x	0	9
u	0	3

$$I = \int_0^9 \frac{1}{x+\sqrt{x}}\, dx$$

$$= \int_0^3 \frac{2u\, du}{u^2+u}$$

$$= \int_0^3 \frac{2\cancel{u}\, du}{\cancel{u}(u+1)} = \left[2\log|u+1|\right]_0^3$$

$$= 2\log 4 - 2\log 1$$

$$= 2\log 4 - 0 = \log 4^2 = \log 16$$

$$\therefore \int_0^9 \frac{1}{x+\sqrt{x}}\, dx = \log 16.$$

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9.10 / 100

Evaluate : $\int_1^2 \frac{x}{(x+1)(x+2)}\, dx$

Consider $\frac{x}{(x+1)(x+2)} = \frac{A}{x+1} + \frac{B}{x+2}$

$$x = A(x+2) + B(x+1)$$

Put $x = -1 \Rightarrow -1 = A$ $\boxed{A = -1}$

Put $x = -2 \Rightarrow -2 = -B \Rightarrow \boxed{B = 2}$

$$\int_1^2 \frac{x}{(x+1)(x+2)}\, dx = \int_1^2 \frac{-1}{x+1}\, dx + 2\int \frac{dx}{x+2}$$

$$= \left[-\log|x+1| + 2\log(x+2|\right]_1^2$$

$$= \left[2\log|x+2| - \log|x+1|\right]_1^2$$

$$= \left[\log \frac{|x+2|^2}{x+1}\right]_1^2 = \log\left(\frac{16}{3}\right) - \log\left(\frac{9}{2}\right)$$

$$= \log\left(\frac{16}{3}\cdot\frac{2}{9}\right) = \log\left(\frac{32}{27}\right)$$

$$\therefore \int_1^2 \frac{x}{(x+1)(x+2)}\,dx = \log\left(\frac{32}{27}\right)$$

9.11 / 101. Evaluate : $\int_0^{\pi/2} \frac{\cos\theta}{(1+\sin\theta)(2+\sin\theta)}\,d\theta$

$u = \sin\theta$

$du = \cos\theta\, d\theta$

θ	0	$\pi/2$
u	0	1

$$I = \int_0^1 \frac{du}{(u+1)(u+2)}$$

$$= \int_0^1 \left[\frac{1}{u+1} - \frac{1}{u+2}\right] du$$

$$= \left[\log(u+1) - \log(u+2)\right]_0^1$$

$$= \left[\log\left(\frac{u+1}{u+2}\right)\right]_0^1 = \log\left(\frac{2}{3}\right) - \log\left(\frac{1}{2}\right)$$

$$= \log\left(\frac{2}{3}\cdot\frac{2}{1}\right) = \log\frac{4}{3}$$

$$\therefore \int_0^{\pi/2} \frac{\cos\theta\, d\theta}{(1+\sin\theta)(2+\sin\theta)} = \log\frac{4}{3}.$$

9.12 / 101. Evaluate : $\int_0^{1/\sqrt{2}} \frac{\sin^{-1}x}{(1-x^2)^{3/2}}\,dx$

$u = \sin^{-1}x$

$du = \frac{1}{\sqrt{1-x^2}}\,dx$

x	0	$\frac{1}{\sqrt{2}}$
u	0	$\frac{\pi}{4}$

$x = \sin u$

$$I = \int_0^{\pi/4} \frac{u\,du}{(1-x^2)^{3/\cancel{2}}}\cdot(1-\cancel{x^2})^{1/2}\,du$$

$$= \int_0^{\pi/4} \frac{u\,du}{1-x^2} = \int_0^{\pi/4} \frac{u\,du}{1-\sin^2 u}$$

$$= \int_0^{\pi/4} u\cdot\sec^2 u\,du$$

$$\boxed{\int u\,dv = uv - u'v_1 + u''v_2 \ldots}$$

$$= \left[u\cdot\tan u - 1\cdot\log\sec u\right]_0^{\pi/4}$$

$$= \left[u\tan u + \log\cos u\right]_0^{\pi/4}$$

$$= \left(\frac{\pi}{4}\cdot 1 + \log\frac{1}{\sqrt{2}}\right) - 0$$

$$= \frac{\pi}{4} + \log\left(\frac{1}{2}\right)^{1/2} = \frac{\pi}{4} + \frac{1}{2}\log\frac{1}{2} \quad (\text{or})$$

$$= \frac{\pi}{4} - \frac{1}{2}\log 2$$

9.13 / 102 Evaluate : $\int_0^{\pi/2} (\sqrt{\tan x} + \sqrt{\cot x})\, dx$

$$I = \int_0^{\pi/2} \left[\sqrt{\frac{\sin x}{\cos x}} + \sqrt{\frac{\cos x}{\sin x}}\right] dx$$

$$= \int_0^{\pi/2} \frac{\sin x + \cos x}{\sqrt{\sin x \cos x}}\, dx = \sqrt{2}\int_0^{\frac{\pi}{2}} \frac{\sin x + \cos x}{\sqrt{2 \sin x \cos x}}\, dx$$

$$= \sqrt{2}\int_0^{\pi/2} \frac{\sin x + \cos x}{\sqrt{1 - (\sin x - \cos x)^2}}\, dx \qquad \left[2ab = a^2 + b^2 - (a-b)^2\right]$$

$u = \sin x - \cos x$

$du = (\cos x + \sin x)\, dx$

x	0	$\pi/2$
u	-1	1

$$= \sqrt{2}\int_{-1}^{1} \frac{du}{\sqrt{1-u^2}}$$

$$= \sqrt{2}\left[\sin^{-1} u\right]_{-1}^{1}$$

$$= \sqrt{2}\left[\sin^{-1}(1) - \sin^{-1}(-1)\right]$$

$$= \sqrt{2}\left[\frac{\pi}{2} + \frac{\pi}{2}\right] = \sqrt{2}\,\pi.$$

$$\int_0^{\pi/2} (\sqrt{\tan x} + \sqrt{\cot x})\, dx = \sqrt{2}\,\pi$$

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9.14 / 102. Evaluate $\int_0^{1.5} [x^2]\, dx$ where $[\cdot]$ is the greatest integer function.

$$[x^2] = \begin{cases} 0 & 0 \le x < 1 \\ 1 & 1 \le x < \sqrt{2} \\ 2 & \sqrt{2} \le x < 1.5 \end{cases}$$

$$\int_0^{1.5} [x^2]\, dx = \int_0^1 [x^2]\, dx + \int_1^{\sqrt{2}} [x^2]\, dx + \int_{\sqrt{2}}^{1.5} [x^2]\, dx$$

$$= \int_0^1 0 \cdot dx + \int_1^{\sqrt{2}} 1 \cdot dx + \int_{\sqrt{2}}^{1.5} 2 \cdot dx$$

$$= [x]_1^{\sqrt{2}} + 2[x]_{\sqrt{2}}^{1.5}$$

$$= \sqrt{2} - 1 + 2(1.5 - \sqrt{2})$$

$$= \sqrt{2} - 1 + 3 - 2\sqrt{2}$$

$$= 2 - \sqrt{2}$$

Same model. Some more examples:

1. If [·] denotes the greatest integer function then find the value of $\int_1^2 [3x]\,dx$

$$[3x] = \begin{cases} 3 & 3 \le 3x < 4 \\ 4 & 4 \le 3x < 5 \\ 5 & 5 \le 3x < 6 \end{cases}$$

$$[3x] = \begin{cases} 3 & 1 \le x < 4/3 \\ 4 & 4/3 \le x < 5/3 \\ 5 & 5/3 \le x < 2 \end{cases}$$

$$\int_1^2 [3x]\,dx = \int_1^{4/3} [3x]\,dx + \int_{4/3}^{5/3} [3x]\,dx + \int_{5/3}^{2} [3x]\,dx$$

$$= \int_1^{4/3} 3\,dx + \int_{4/3}^{5/3} 4\,dx + \int_{5/3}^{2} 5\,dx$$

$$= (3x)_1^{4/3} + (4x)_{4/3}^{5/3} + (5x)_{5/3}^{2}$$

$$= 3\left(\frac{4}{3}-1\right) + 4\left(\frac{5}{3}-\frac{4}{3}\right) + 5\left(2-\frac{5}{3}\right)$$

$$= 1 + \frac{4}{3} + \frac{5}{3} = 4.$$

2. Let $f(x) = x - [x]$ for every real number x, where $[x]$ is the greatest integer less than or equal to x. Then find $\int_{-1}^{1} f(x)\,dx$

$$x - [x] = \begin{cases} x-(-1) = x+1 & -1 \le x < 0 \\ x - 0 = x & 0 \le x < 1 \end{cases}$$

$$\int_{-1}^{1} \{x - [x]\}\,dx = \int_{-1}^{0} (x+1)\,dx + \int_0^1 x\,dx$$

$$= \left(\frac{x^2}{2}+x\right)_{-1}^{0} + \left(\frac{x^2}{2}\right)_0^1$$

$$= 0 - \left(\frac{1}{2}-1\right) + \left(\frac{1}{2}-0\right)$$

$$= \frac{1}{2} + \frac{1}{2} = 1.$$

$$\therefore \int_{-1}^{1} \{x - [x]\}\,dx = 1.$$

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9.15 / 102. Evaluate : $\int_{-4}^{4} |x+3|\, dx$

$$|x+3| = \begin{cases} x+3 & x \geq -3 \\ -(x+3) & x < -3 \end{cases}$$

$$\int_{-4}^{4} |x+3|\, dx = \int_{-4}^{-3} |x+3|\, dx + \int_{-3}^{4} |x+3|\, dx$$

$$= \int_{-4}^{-3} -(x+3)\, dx + \int_{-3}^{4} (x+3)\, dx$$

$$= -\left[\frac{(x+3)^2}{2}\right]_{-4}^{-3} + \left[\frac{(x+3)^2}{2}\right]_{-3}^{4}$$

$$= -\left\{0 - \frac{1}{2}\right\} + \left\{\frac{49}{2} - 0\right\}$$

$$= \frac{1}{2} + \frac{49}{2} = \frac{50}{2} = 25.$$

$$\therefore \int_{-4}^{4} |x+3|\, dx = 25.$$

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Note:

To evaluate anti-derivatives of the type $\int \frac{dx}{a\cos x + b\sin x + c}$, we use the Substitution method by Putting

$u = \tan\frac{x}{2}$ $\quad \cos x = \frac{1-u^2}{1+u^2}$ $\quad \sin x = \frac{2u}{1+u^2}$

$dx = \frac{2du}{1+u^2}.$

9.16 / 103. Show that $\int_0^{\pi/2} \frac{dx}{4+\sin x} = \frac{1}{3}\log_e 2$

Put $u = \tan\frac{x}{2}$ $\quad \sin x = \frac{2\tan\frac{x}{2}}{1+\tan^2\frac{x}{2}} = \frac{2u}{1+u^2}$

$du = \frac{1}{2}\sec^2\frac{x}{2}\, dx$

$dx = \frac{2du}{1+u^2}$

x	0	$\pi/2$
u	0	1

$$\int_0^{\pi/2} \frac{dx}{4+5\sin x} = \int_0^1 \frac{2du/1+u^2}{4+5\left(\frac{2u}{1+u^2}\right)}$$

$$= \int_0^1 \frac{du}{2u^2+5u+2} = \frac{1}{2}\int_0^1 \frac{du}{u^2+\frac{5}{2}u+1}$$

$$= \frac{1}{2}\int_0^1 \frac{du}{\left(u+\frac{5}{4}\right)^2-\frac{25}{16}+1} = \frac{1}{2}\int_0^1 \frac{du}{\left(u+\frac{5}{4}\right)^2-\left(\frac{3}{4}\right)^2}$$

$$\boxed{\left[\int \frac{dx}{x^2-a^2} = \frac{1}{2a}\log\left|\frac{x-a}{x+a}\right|+c\right]}$$

$$= \frac{1}{2}\left[\frac{1}{2(3/4)}\log\left(\frac{u+\frac{5}{4}-\frac{3}{4}}{u+\frac{5}{4}+\frac{3}{4}}\right)\right]_0^1$$

$$= \frac{1}{3}\left[\log\left(\frac{u+\frac{1}{2}}{u+2}\right)\right]_0^1 = \frac{1}{3}\left\{\log\left(\frac{3/2}{3}\right)-\log\left(\frac{1/2}{2}\right)\right\}$$

$$= \frac{1}{3}\left[\log\left(\frac{1}{2}\right)-\log\left(\frac{1}{4}\right)\right] = \frac{1}{3}\log\left(\frac{1/2}{1/4}\right)$$

$$= \frac{1}{3}\log 2.$$

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9.17/103 Prove that $\int_0^{\pi/4} \frac{\sin 2x}{\sin^4 x+\cos^4 x}\,dx = \frac{\pi}{4}$

$$I = \int_0^{\pi/4} \frac{\sin 2x\, dx}{(\sin^2 x+\cos^2 x)^2 - 2\sin^2 x\cos^2 x}$$

$$= \int_0^{\pi/4} \frac{\sin 2x\, dx}{1-\frac{1}{2}(2\sin x\cos x)^2} = \int_0^{\pi/4} \frac{2\sin 2x\, dx}{2-\sin^2 2x}$$

$$= \int_0^{\pi/4} \frac{2\sin 2x\, dx}{1+\cos^2 2x}$$

$u = \cos 2x$

$du = -2\sin 2x . dx$

$-du = 2\sin 2x\, dx$

x	0	$\pi/4$
u	1	0

$$= \int_1^0 \frac{-du}{1+u^2} = \int_0^1 \frac{du}{1+u^2}$$

$$= \left[\tan^{-1} u\right]_0^1 = \tan^{-1}(1) - \tan^{-1}(0)$$

$$= \frac{\pi}{4} - 0$$

$$= \frac{\pi}{4}.$$

$$\therefore \int_0^{\pi/4} \frac{\sin 2x}{\sin^4 x+\cos^4 x}\,dx = \frac{\pi}{4}.$$

$\frac{9.18}{104}$ Prove that $\int_0^{\pi/4} \frac{dx}{a^2\sin^2 x + b^2\cos^2 x} = \frac{1}{ab}\tan^{-1}\left(\frac{a}{b}\right)$ where $a, b > 0$

$$I = \int_0^{\pi/4} \frac{dx}{a^2\sin^2 x + b^2\cos^2 x}$$

$\div \cos^2 x$

$$I = \int_0^{\pi/4} \frac{\sec^2 x\, dx}{a^2\tan^2 x + b^2}$$

$u = \tan x$

$du = \sec^2 x\, dx$

x	0	$\pi/4$
u	0	1

$$= \int_0^1 \frac{du}{a^2u^2 + b^2}$$

$$= \frac{1}{a^2}\int_0^1 \frac{du}{u^2 + \left(\frac{b}{a}\right)^2}$$

$$= \frac{1}{a^2}\left[\frac{1}{b/a}\tan^{-1}\left(\frac{u}{b/a}\right)\right]_0^1$$

$$= \frac{1}{ab}\left[\tan^{-1}\frac{a}{b} - \tan^{-1}(0)\right]$$

$$= \frac{1}{ab}\tan^{-1}\left(\frac{a}{b}\right)$$

$$\therefore \int_0^{\pi/4} \frac{dx}{a^2\sin^2 x + b^2\cos^2 x} = \frac{1}{ab}\tan^{-1}\left(\frac{a}{b}\right)$$

Ex : 9.3

$\frac{1}{113}$ Evaluate the following definite integrals :

(i) $\int_3^4 \frac{dx}{x^2-4}$

$$\boxed{\int \frac{dx}{x^2-a^2} = \frac{1}{2a}\log\left|\frac{x-a}{x+a}\right| + c}$$

$$= \left[\frac{1}{2(2)}\log\left|\frac{x-2}{x+2}\right|\right]_3^4$$

$$= \frac{1}{4}\left\{\log\left(\frac{2}{6}\right) - \log\left(\frac{1}{5}\right)\right\} = \frac{1}{4}\log\left(\frac{2}{6}\times\frac{5}{1}\right)$$

$$= \frac{1}{4}\log\left(\frac{5}{3}\right)$$

$$\int_3^4 \frac{dx}{x^2-4} = \frac{1}{4}\log\left(\frac{5}{3}\right)$$

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(ii) $\displaystyle\int_{-1}^{1} \frac{dx}{x^2+2x+5}$

$$\boxed{\int \frac{dx}{x^2+a^2} = \frac{1}{a}\tan^{-1}\left(\frac{x}{a}\right)+c}$$

$$= \int_{-1}^{1} \frac{dx}{(x+1)^2-1+5}$$

$$= \int_{-1}^{1} \frac{dx}{(x+1)^2+4}$$

$$= \left\{\frac{1}{2}\tan^{-1}\left[\frac{x+1}{2}\right]\right\}_{-1}^{1} = \frac{1}{2}\left[\tan^{-1}(1) - \tan^{-1}(0)\right]$$

$$= \frac{1}{2}\left(\frac{\pi}{4}\right) = \frac{\pi}{8}$$

$$\therefore \int_{-1}^{1} \frac{dx}{x^2+2x+5} = \frac{\pi}{8}$$

(iii) $\displaystyle\int_{0}^{1} \sqrt{\frac{1-x}{1+x}}\,dx$

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Method I :

$$\int_0^1 \sqrt{\frac{1-x}{1+x}\times\frac{1-x}{1-x}}\,dx = \int_0^1 \sqrt{\frac{(1-x)^2}{1-x^2}}$$

$$= \int_0^1 \frac{1-x}{\sqrt{1-x^2}}\,dx = \int_0^1 \frac{dx}{\sqrt{1-x^2}} - \int_0^1 \frac{x}{\sqrt{1-x^2}}\,dx$$

$$\boxed{\int \frac{f'(x)}{\sqrt{f(x)}}\,dx = 2\sqrt{f(x)}+c \qquad \int\frac{dx}{\sqrt{1-x^2}} = \sin^{-1}(x)+c}$$

$$= \left[\sin^{-1}x\right]_0^1 + \frac{1}{2}\left(2\sqrt{1-x^2}\right)_0^1$$

$$= \sin^{-1}(1) - \sin^{-1}(0) + \{0-1\}$$

$$= \frac{\pi}{2} - 1 \qquad \therefore \int_0^1 \sqrt{\frac{1-x}{1+x}}\,dx = \frac{\pi}{2} - 1.$$

Method II.

let $x = \sin\theta$

$dx = \cos\theta \cdot d\theta$

x	0	1
θ	0	$\frac{\pi}{2}$

$$\int_0^1 \sqrt{\frac{1-x}{1+x}}\,dx$$

$$= \int_0^{\pi/2} \sqrt{\frac{1-\sin\theta}{1+\sin\theta}}\,\cos\theta \cdot d\theta$$

$$= \int_0^{\pi/2} \sqrt{\frac{(1-\sin\theta)(1-\sin\theta)}{(1+\sin\theta)(1-\sin\theta)}}\,\cos\theta \cdot d\theta$$

$$= \int_0^{\pi/2} \frac{1-\sin\theta}{\cancel{\cos\theta}}\cdot\cancel{\cos\theta}\cdot d\theta$$

$$= \left[\theta + \cos\theta\right]_0^{\pi/2} = \left(\frac{\pi}{2}+0\right) - (0+1) = \frac{\pi}{2} - 1.$$

(iv) $\displaystyle\int_0^{\pi/2} e^x\left(\frac{1+\sin x}{1+\cos x}\right)dx$

$$\boxed{\int e^x[f(x)+f'(x)]\,dx = e^x f(x)+c}$$

$$I = \int_0^{\pi/2} e^x\left[\frac{1+2\sin\frac{x}{2}.\cos\frac{x}{2}}{2\cos^2\frac{x}{2}}\right]dx$$

$$= \int_0^{\pi/2} e^x\left[\frac{1}{2}\sec^2\frac{x}{2}+\tan\frac{x}{2}\right]dx$$

$$= \left[e^x\tan\frac{x}{2}\right]_0^{\pi/2}$$

$$\boxed{\begin{aligned} f(x) &= \tan\frac{x}{2}\\ f'(x) &= \frac{1}{2}\sec^2\frac{x}{2}\end{aligned}}$$

$$= e^{\pi/2}.\tan\frac{\pi}{4} - e^0.\tan 0$$

$$= e^{\pi/2}$$

$$\therefore \int_0^{\pi/2} e^x\left(\frac{1+\sin x}{1+\cos x}\right)dx = e^{\pi/2}.$$

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(v) $\displaystyle\int_0^{\pi/2}\sqrt{\cos\theta}\,\sin^3\theta\,d\theta$

$$= \int_0^{\pi/2}\sqrt{\cos\theta}\,.\sin^2\theta.\sin\theta.d\theta$$

$u = \cos\theta$

$du = -\sin\theta.d\theta$

$-du = \sin\theta.d\theta$

θ	0	$\pi/2$
u	1	0

$$= \int_1^0 \sqrt{u}\,(1-u^2)\,(-du)$$

$$= \int_0^1 (\sqrt{u} - u^{5/2})\,du$$

$$= \left[\frac{2}{3}u^{3/2} - \frac{2}{7}u^{7/2}\right]_0^1 = \frac{2}{3}-\frac{2}{7} = \frac{14-6}{21} = \frac{8}{21}$$

$$\therefore \int_0^{\pi/2}\sqrt{\cos\theta}\,.\sin^3\theta.d\theta = \frac{8}{21}.$$

(vi) $\displaystyle\int_0^1 \frac{1-x^2}{(1+x^2)^2}dx$

Dividing numerator and denominator by x^2.

$$= \int_0^1 \frac{\frac{1}{x^2}-1}{\left(x+\frac{1}{x}\right)^2}dx$$

$u = x+\frac{1}{x}$

$du = \left(1-\frac{1}{x^2}\right)dx$

$-du = \left(\frac{1}{x^2}-1\right)dx$

x	0	1
u	∞	2

$$= \int_\infty^2 -\frac{1}{u^2}du = \int_2^\infty \frac{1}{u^2}du$$

$$= \left[-\frac{1}{u}\right]_2^\infty = -\frac{1}{\infty}+\frac{1}{2} = \frac{1}{2}$$

$$\int_0^1 \frac{1-x^2}{(1+x^2)^2}dx = \frac{1}{2}$$

Properties of definite Integrals.

* $\int_a^b f(x)\,dx = \int_a^b f(a+b-x)\,dx$
* $\int_0^a f(x)\,dx = \int_0^a f(a-x)\,dx$
* $\int_{-a}^a f(x)\,dx = \begin{cases} 0 & f(x) \text{ is an odd} \\ 2\int_0^a f(x)\,dx & f(x) \text{ is an even.} \end{cases}$
* $\int_0^{2a} f(x)\,dx = \begin{cases} 0 & f(2a-x) = -f(x) \\ 2\int_0^a f(x)\,dx & f(2a-x) = f(x) \end{cases}$
* $\int_0^{2a} f(x)\,dx = \int_0^a f(x)\,dx + \int_0^a f(2a-x)\,dx$
* $\int_0^a x\,f(x)\,dx = \frac{a}{2}\int_0^a f(x)\,dx$ If $f(a-x) = f(x)$
* $\int_0^{2a} f(x)\,dx = 2\int_0^a f(x)\,dx$ If $f(a+x) = f(x)$

9.19
104

Evaluate : $\int_0^{\pi/4} \frac{dx}{\sin x + \cos x}$

$$I = \int_0^{\pi/4} \frac{dx}{\sin x + \cos x} = \int_0^{\pi/4} \frac{dx}{\sqrt{2}\left[\frac{1}{\sqrt{2}}\sin x + \frac{1}{\sqrt{2}}\cos x\right]}$$

$$= \frac{1}{\sqrt{2}}\int_0^{\pi/4} \frac{dx}{\cos\frac{\pi}{4}\cos x + \sin\frac{\pi}{4}\sin x} = \frac{1}{\sqrt{2}}\int_0^{\pi/4} \frac{dx}{\cos(\frac{\pi}{4} - x)}$$

$$\boxed{\int_0^a f(x)\,dx = \int_0^a f(a-x)\,dx.}$$

$$= \frac{1}{\sqrt{2}}\int_0^{\pi/4} \frac{1}{\cos x}\,dx$$

$$= \frac{1}{\sqrt{2}}\int_0^{\pi/4} \sec x\,dx$$

$$= \frac{1}{\sqrt{2}}\Big[\log(\sec x + \tan x)\Big]_0^{\pi/4}$$

$$= \frac{1}{\sqrt{2}}\left\{\log(\sqrt{2}+1) - \log 1\right\}$$

$$= \frac{1}{\sqrt{2}}\log(\sqrt{2}+1)$$

9.20 / 107.

Show that $\int_0^{\pi} g(\sin x)dx = 2\int_0^{\pi/2} g(\sin x)dx$ where $g(\sin x)$ is a function of $\sin x$.

$I = \int_0^{\pi} g(\sin x)\,dx$

$\boxed{\int_0^{2a} f(x)dx = 2\int_0^{a} f(x)dx \quad f(2a-x) = f(x)}$

$2a = \pi \qquad f(x) = g(\sin x)$

$f(2a-x) = g[\sin(\pi - x)]$

$= g(\sin x)$

$= f(x)$

$\therefore \int_0^{\pi} g(\sin x)\,dx = 2\int_0^{\pi/2} g(\sin x)\,dx$

9.22 / 108

Show that $\int_0^{2\pi} g(\cos x)dx = 2\int_0^{\pi} g(\cos x)dx$ where $g(\cos x)$ is a function of $\cos x$.

$2a = 2\pi \qquad f(x) = g(\cos x)$

$\boxed{\int_0^{2a} f(x)dx = 2\int_0^{a} f(x)dx \quad f(2a-x) = f(x)}$

$f(2a-x) = g(\cos(2\pi - x))$

$= g(\cos x)$

$= f(x)$

$\int_0^{2\pi} g(\cos x)\,dx = 2\int_0^{\pi} g(\cos x)\,dx$

9.21 / 108

Evaluate: $\int_0^{\pi} \frac{x}{1+\sin x}\,dx$

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$I = \int_0^{\pi} \frac{x}{1+\sin x}\,dx$

$= \int_0^{\pi} x \cdot \frac{1}{1+\sin x}\,dx$

$f(x) = \frac{1}{1+\sin x} \qquad f(\pi - x) = \frac{1}{1+\sin(\pi - x)} = \frac{1}{1+\sin x} = f(x)$

$f(\pi - x) = f(x)$

$\boxed{\int_0^{a} x f(x)dx = \frac{a}{2}\int_0^{a} f(x)dx \quad f(a-x) = f(x)}$

$\boxed{\int_0^{\pi} g(\sin x)dx = 2\int_0^{\pi/2} g(\sin x)dx \quad f(2a-x) = f(x)}$

$I = 2\int_0^{\pi/2} \frac{1}{1+\sin x}\,dx$

$= 2\int_0^{\pi/2} \frac{1}{1+\sin(\frac{\pi}{2} - x)}\,dx$

$= 2\int_0^{\pi/2} \frac{1}{1+\cos x}\,dx$

$$= \cancel{2}\int_0^{\pi/2} \frac{dx}{\cancel{2}\cos^2 x/2} = \int_0^{\pi/2} \frac{\sec^2 \frac{x}{2}}{1/2}\,dx$$

$$= 2\left[\tan\frac{x}{2}\right]_0^{\pi/2} = 2\left[\tan\frac{\pi}{4} - 0\right] = 2(1) = 2$$

$$\therefore \int_0^{\pi} \frac{x}{1+\sin x}\,dx = 2.$$

9.23/109. If $f(x) = f(a+x)$ then $\int_0^{2a} f(x)\,dx = 2\int_0^{a} f(x)\,dx$.

We know that $\int_0^{2a} f(x)\,dx = \int_0^{a} f(x)\,dx + \int_a^{2a} f(x)\,dx$

Consider $\int_a^{2a} f(x)\,dx$ Let $x = a+u$, $dx = du$

x	a	$2a$
u	0	a

$$= \int_0^{a} f(a+u)\,du$$

$$= \int_0^{a} f(a+x)\,dx = \int_0^{a} f(x)\,dx$$

$$\therefore \int_0^{2a} f(x)\,dx = \int_0^{a} f(x)\,dx + \int_0^{a} f(x)\,dx$$

$$= 2\int_0^{a} f(x)\,dx.$$

9.24/109 Evaluate: $\int_{-\pi/2}^{\pi/2} x\cos x\,dx$

$f(x) = x\cos x$

$$f(-x) = -x\cos(-x) = -x\cos x = -f(x)$$

$\therefore f(-x) = -f(x)$

$f(x)$ is an odd function.

$$\therefore \int_{-\pi/2}^{\pi/2} x\cos x\,dx = 0.$$

9.25/109 Evaluate: $\int_{-\log 2}^{\log 2} e^{-|x|}\,dx$

$f(x) = e^{-|x|}$ $f(-x) = e^{-|-x|} = e^{-|x|} = f(x)$

$f(-x) = f(x)$

$\therefore f(x)$ is an even function.

$$\therefore \int_{-\log 2}^{\log 2} e^{-|x|}\,dx = 2\int_{0}^{\log 2} e^{-|x|}\,dx. = 2\int_{0} e^{-x}\,dx$$

$$= 2\left[-e^{-x}\right]_{0}^{\log 2} = 2\left[-e^{-\log 2} + e^{0}\right]$$

$$= 2\left[-e^{\log \frac{1}{2}} + 1\right]$$

$$= 2\left[-\frac{1}{2} + 1\right] = 2\left(\frac{1}{2}\right) = 1.$$

9.26 / 110 Evaluate : $\int_{0}^{a} \frac{f(x)}{f(x)+f(a-x)}\,dx$

$$I = \int_{0}^{a} \frac{f(x)}{f(x)+f(a-x)}\,dx \quad \text{———} \quad (1)$$

$$\boxed{\int_{0}^{a} f(x)\,dx = \int_{0}^{a} f(a-x)\,dx}$$

$$I = \int_{0}^{a} \frac{f(a-x)}{f(a-x)+f(a-(a-x))}\,dx$$

$$= \int_{0}^{a} \frac{f(a-x)}{f(a-x)+f(x)}\,dx \quad \text{———} \quad (2)$$

(1) + (2)

$$I + I = \int_{0}^{a} \frac{f(x)+f(a-x)}{f(x)+f(a-x)}\,dx$$

$$2I = \int_{0}^{a} dx = [x]_{0}^{a}$$

$$2I = a$$

$$\boxed{I = a/2}$$

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9.27 / 110 Prove that $\int_{0}^{\pi/4} \log(1+\tan x)\,dx = \frac{\pi}{8}\log 2.$

$$I = \int_{0}^{\pi/4} \log(1+\tan x)\,dx \quad \text{———} \quad (1)$$

$$\boxed{\int_{0}^{a} f(x)\,dx = \int_{0}^{a} f(a-x)\,dx \qquad \tan\left(\frac{\pi}{4}-x\right) = \frac{1-\tan x}{1+\tan x}}$$

$$I = \int_{0}^{\pi/4} \log\left(1+\tan\left(\frac{\pi}{4}-x\right)\right)dx$$

$$= \int_{0}^{\pi/4} \log\left(1+\frac{1-\tan x}{1+\tan x}\right)dx$$

$$= \int_{0}^{\pi/4} \log\left(\frac{1+\cancel{\tan x}+1-\cancel{\tan x}}{1+\tan x}\right)dx$$

$$= \int_{0}^{\pi/4} \log\left(\frac{2}{1+\tan x}\right)dx \quad \text{———} \quad (2)$$

① + ②

$$I + I = \int_0^{\pi/4} \left[\log(1+\tan x) + \log \frac{2}{1+\tan x}\right] dx$$

$$= \int_0^{\pi/4} \log \left(\cancel{(1+\tan x)} \cdot \frac{2}{\cancel{1+\tan x}}\right) \cdot dx$$

$$2I = \log 2 \int_0^{\pi/4} dx = \log 2 \, [x]_0^{\pi/4}$$

$$2I = \log 2 \cdot \frac{\pi}{4}$$

$$\boxed{I = \frac{\pi}{8} \log 2.}$$

9.28 / 111. Show that $\int_0^1 [\tan^{-1}x + \tan^{-1}(1-x)] dx = \frac{\pi}{2} - \log_e 2$

$$I = \int_0^1 \tan^{-1}x \, dx + \int_0^1 \tan^{-1}(1-x) \, dx$$

$$= \int_0^1 \tan^{-1}x \, dx + \int_0^1 \tan^{-1}(1-(1-x)) \, dx$$

$$= \int_0^1 \tan^{-1}x \, dx + \int_0^1 \tan^{-1}x \, dx$$

$$= 2\int_0^1 \tan^{-1}x \cdot dx$$

$u = \tan^{-1}x$ $\quad \int dv = \int dx$

$du = \frac{1}{1+x^2} dx$ $\quad v = x$

$$\boxed{\int u \, dv = uv - \int v \, du}$$

$$\int \frac{f'(x)}{f(x)} dx = \log f(x) + c$$

$$= 2\left[x \tan^{-1}x\right]_0^1 - 2\int_0^1 \frac{x}{1+x^2} \cdot dx$$

$$= \left[2x \tan^{-1}x\right]_0^1 - \int_0^1 \frac{2x}{1+x^2} dx$$

$$= \left[2\tan^{-1}(1) - 0\right] - \left[\log(1+x^2)\right]_0^1$$

$$= 2\frac{\pi}{4} - \log 2$$

$$= \frac{\pi}{2} - \log 2$$

9.29 / 111

$$\int_2^3 \frac{\sqrt{x}}{\sqrt{5-x} + \sqrt{x}} dx$$

$$I = \int_2^3 \frac{\sqrt{x}}{\sqrt{5-x} + \sqrt{x}} dx \quad \text{——— ①}$$

$$\boxed{\int_a^b f(a+b-x) \, dx = \int_a^b f(x) \, dx}$$

$$I = \int_2^3 \frac{\sqrt{5-x}}{\sqrt{5-(5-x)} + \sqrt{5-x}} dx$$

$$= \int_2^3 \frac{\sqrt{5-x}}{\sqrt{x} + \sqrt{5-x}} dx \quad \text{——— ②}$$

① + ②

$$2I = \int_2^3 \frac{\sqrt{x} + \sqrt{5-x}}{\sqrt{x} + \sqrt{5-x}} dx = \int_2^3 dx = [x]_2^3 = 3 - 2 = 1$$

$$\boxed{I = \frac{1}{2}}$$

9.30 / 113 Evaluate: $\int_{-\pi}^{\pi} \frac{\cos^2 x}{1+a^x}\,dx$

$$I = \int_{-\pi}^{\pi} \frac{\cos^2 x}{1+a^x}\,dx \quad \text{—— (1)}$$

$$\boxed{\int_a^b f(x)\,dx = \int_a^b f(a+b-x)\,dx}$$

$$I = \int_{-\pi}^{\pi} \frac{\cos^2(\pi-\pi-x)}{1+a^{\pi-\pi-x}}\,dx$$

$$= \int_{-\pi}^{\pi} \frac{\cos^2 x}{1+a^{-x}}\,dx$$

$$= \int_{-\pi}^{\pi} \frac{\cos^2 x}{1+\frac{1}{a^x}}\,dx = \int_{-\pi}^{\pi} a^x \frac{\cos^2 x}{(1+a^x)}\,dx \quad \text{—— (2)}$$

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(1) + (2)

$$I + I = \int_{-\pi}^{\pi} \frac{\cos^2 x}{\cancel{a^x+1}} \cancel{(a^x+1)}\,dx$$

$$2I = \int_{-\pi}^{\pi} \cos^2 x\,dx = 2\int_0^{\pi} \cos^2 x\,.dx$$

$$I = \int_0^{\pi} \frac{1+\cos 2x}{2}\,dx$$

$$= \frac{1}{2}\left[x + \frac{\sin 2x}{2}\right]_0^{\pi} = \frac{1}{2}\{(\pi+0)-0\} = \frac{\pi}{2}$$

$$\therefore \int_{-\pi}^{\pi} \frac{\cos^2 x}{1+a^x}\,dx = \frac{\pi}{2}.$$

---

2 / 113 Evaluate the following integrals using properties of integration.

(i) $\int_{-5}^{5} x\cos\left(\frac{e^x-1}{e^x+1}\right)dx$

$$\boxed{\int_{-a}^{a} f(x)\,dx = 0 \quad f(-x) = -f(x)}$$

$$f(x) = x\cos\left(\frac{e^x-1}{e^x+1}\right)$$

$$f(-x) = -x\cos\left(\frac{e^{-x}-1}{e^{-x}+1}\right) = -x\cos\left[\frac{\frac{1}{e^x}-1}{\frac{1}{e^x}+1}\right]$$

$$= -x\cos\left[\frac{1-e^x}{1+e^x}\right] = -x\cos\left\{-\left(\frac{e^x-1}{e^x+1}\right)\right\}$$

$$= -x\cos\left(\frac{e^x-1}{e^x+1}\right) = -f(x)$$

$$\boxed{\cos(-\theta) = \cos\theta}$$

$\therefore f(-x) = -f(x)$

$f(x)$ is an odd function.

$$\int_{-5}^{5} x\cos\left(\frac{e^x-1}{e^x+1}\right)dx = 0$$

(ii) $\int_{-\pi/2}^{\pi/2} (x^5 + x\cos x + \tan^3 x + 1)\,dx$

$I = \int_{-\pi/2}^{\pi/2} (x^5 + x\cos x + \tan^3 x)\,dx + \int_{-\pi/2}^{\pi/2} dx$

$f(x) = x^5 + x\cos x + \tan^3 x$

$f(-x) = (-x)^5 - x\cos(-x) + [\tan(-x)]^3$

$= -[x^5 + x\cos x + \tan^3 x) = -f(x)$

$f(x)$ is an odd function.

$I = \int_{-\pi/2}^{\pi/2} (x^5 + x\cos x + \tan^3 x)\,dx + \int_{-\pi/2}^{\pi/2} dx$

$= 0 + [x]_{-\pi/2}^{\pi/2} = \frac{\pi}{2} + \frac{\pi}{2} = \pi$

(iii) $\int_{-\pi/4}^{\pi/4} \sin^2 x\,dx$

$f(x) = \sin^2 x$

$f(-x) = [\sin(-x)]^2$

$= \sin^2 x = f(x)$

$\boxed{\int_{-a}^{a} f(x)\,dx = 2\int_0^a f(x)\,dx \quad f(-x) = f(x)}$

$\therefore f(x)$ is an even function.

$\int_{-\pi/4}^{\pi/4} \sin^2 x\,dx = 2\int_0^{\pi/4} \sin^2 x\,dx$

$= \cancel{2}\int_0^{\pi/4} \frac{1-\cos 2x}{\cancel{2}}\,dx = \left[x - \frac{\sin 2x}{2}\right]_0^{\pi/4}$

$= \frac{\pi}{4} - \frac{1}{2}$

(iv) $\int_0^{2\pi} x \log\left(\frac{3+\cos x}{3-\cos x}\right) dx.$

$I = \int_0^{2\pi} x.\log\left(\frac{3+\cos x}{3-\cos x}\right) dx$

Let $f(x) = \log\left(\frac{3+\cos x}{3-\cos x}\right)$

$f(2\pi - x) = \log\left(\frac{3+\cos(2\pi-x)}{3-\cos(2\pi-x)}\right) = \log\left(\frac{3+\cos x}{3-\cos x}\right)$

$f(2\pi - x) = f(x)$

$\int_0^{2\pi} x\log\left(\frac{3+\cos x}{3-\cos x}\right) dx = \frac{2\pi}{2}\int_0^{\pi} \log\left(\frac{3+\cos x}{3-\cos x}\right) dx$

$= \pi\int_0^{\pi} \log\left(\frac{3+\cos x}{3-\cos x}\right) dx.$

$\boxed{\left[\int_0^a x f(x)\,dx = \frac{a}{2}\int_0^a f(x)\,dx\right]}$

$$I = \pi \int_0^{\pi} \log\left(\frac{3+\cos x}{3-\cos x}\right) dx$$

$$f(2a-x) = f(\pi - x) = \log\left(\frac{3+\cos(\pi-x)}{3-\cos(\pi-x)}\right)$$

$$= \log\left(\frac{3-\cos x}{3+\cos x}\right) = -\log\left(\frac{3+\cos x}{3-\cos x}\right)$$

$$f(2a-x) = -f(x)$$

$$\pi\int_0^{\pi} \log\left(\frac{3+\cos x}{3-\cos x}\right) dx = 0$$

$$\boxed{\int_0^{2a} f(x)\,dx = 0 \quad f(2a-x) = -f(x)}$$

(v) $\int_0^{2\pi} \sin^4 x \cos^3 x \, dx.$

$$f(x) = \sin^4 x \cos^3 x$$

$$f(2\pi - x) = [\sin(2\pi - x)]^4 \cdot [\cos(2\pi - x)]^3$$

$$= (-\sin x)^4 \cdot \cos^3 x$$

$$= \sin^4 x \cos^3 x$$

$$\boxed{\int_0^{2a} f(x)dx = \begin{cases} 0 & f(2a-x) = -f(x) \\ 2\int_0^a f(x)dx & f(2a-x) = f(x) \end{cases}}$$

$$f(2\pi - x) = f(x)$$

$$\int_0^{2\pi} \sin^4 x \cos^3 x \, dx = 2\int_0^{\pi} \sin^4 x \cos^3 x \, dx$$

$$g(x) = \sin^4 x \cos^3 x$$

$$g(2a-x) = g(\pi - x) = [\sin(\pi - x)]^4 \cdot [\cos(\pi - x)]^3$$

$$= \sin^4 x (-\cos^3 x)$$

$$= -\sin^4 x \cos^3 x$$

$$g(2a-x) = -g(x)$$

$$\therefore \int_0^{2\pi} \sin^4 x \cos^3 x \, dx = 2(0) = 0.$$

(vi) Evaluate :- $\int_0^1 |5x-3| \, dx$

$$|5x-3| = \begin{cases} 5x-3 & x \geq 3/5 \\ -(5x-3) & x < 3/5 \end{cases}$$

$$\int_0^1 |5x-3| dx = \int_0^{3/5} (3-5x)\,dx + \int_{3/5}^1 (5x-3)\,dx$$

$$= \left[3x - \frac{5x^2}{2}\right]_0^{3/5} + \left[\frac{5x^2}{2} - 3x\right]_{3/5}^1$$

$$= \left(\frac{9}{5} - \frac{45}{50}\right) + \left(\left(\frac{5}{2} - 3\right) - \left(\frac{45}{50} - \frac{9}{5}\right)\right]$$

$$= \frac{9}{5} - \frac{9}{10} + \frac{5}{2} - 3 - \frac{9}{10} + \frac{9}{5} = \frac{18}{5} - \frac{18}{10} + \frac{5}{2} - 3$$

$$= \frac{13}{10}$$

(vii) $\int_0^{\sin^2 x} \sin^{-1}\sqrt{t}\,dt + \int_0^{\cos^2 x} \cos^{-1}\sqrt{t}\,.dt$

$\int u\,dv = uv - \int v\,du.$

$u = \sin^{-1}\sqrt{t}$ $\quad \int dv = \int dt$

$du = \frac{1}{\sqrt{1-t}} \cdot \frac{1}{2\sqrt{t}}.dt$ $\quad v = t$

$u = \cos^{-1}(\sqrt{t})$ $\quad \int dv = \int dt$

$du = \frac{-1}{\sqrt{1-t}} \cdot \frac{1}{2\sqrt{t}}.dt$ $\quad v = t$

$$I = \left[t\sin^{-1}\sqrt{t}\right]_0^{\sin^2 x} - \frac{1}{2}\int_0^{\sin^2 x} \frac{\sqrt{t}}{\sqrt{1-t}}\,dt + \left[t\cos^{-1}\sqrt{t}\right]_0^{\cos^2 x} + \frac{1}{2}\int_0^{\cos^2 x} \frac{\sqrt{t}}{\sqrt{1-t}}\,dt$$

$$= x\sin^2 x - \frac{1}{2}\int_0^{\sin^2 x} \frac{\sqrt{t}}{\sqrt{1-t}}\,dt + x\cos^2 x + \frac{1}{2}\int_0^{\cos^2 x} \frac{\sqrt{t}}{\sqrt{1-t}}\,dt$$

$$= x(\sin^2 x + \cos^2 x) + \frac{1}{2}\int_{\sin^2 x}^{0} \frac{\sqrt{t}}{\sqrt{1-t}}\,dt + \frac{1}{2}\int_0^{\cos^2 x} \frac{\sqrt{t}}{\sqrt{1-t}}\,dt$$

$$= x + \frac{1}{2}\int_{\sin^2 x}^{\cos^2 x} \frac{\sqrt{t}}{\sqrt{1-t}}\,dt$$

$u = \sin^2\theta$

$du = 2\sin\theta\cos\theta . d\theta$

t	$\sin^2 x$	$\cos^2 x$
u	x	$\frac{\pi}{2} - x$

$$= x + \frac{1}{2}\int_x^{\pi/2 - x} \frac{\sin\theta}{\sqrt{\cos^2\theta}}(2\sin\theta\cos\theta)\,d\theta$$

$$= x + \frac{1}{\cancel{2}}\int_x^{\pi/2 - x} \frac{\cancel{2}\sin^2\theta . \cancel{\cos\theta}}{\cancel{\cos\theta}}\,d\theta \quad = x + \int_x^{\frac{\pi}{2} - x} \sin^2\theta . d\theta$$

$$= x + \int_x^{\pi/2 - x} \frac{1 - \cos 2\theta}{2}\,d\theta$$

$$= x + \frac{1}{2}\left[\theta - \frac{\sin 2\theta}{2}\right]_x^{\frac{\pi}{2} - x}$$

$$= x + \frac{1}{2}\left\{\left(\frac{\pi}{2} - x - \frac{\sin 2x}{2}\right) - \left(x - \frac{\sin 2x}{2}\right)\right\}$$

$$= x + \frac{1}{2}\left[\frac{\pi}{2} - x - \cancel{\frac{\sin 2x}{2}} - x + \cancel{\frac{\sin 2x}{2}}\right]$$

$$= x + \frac{1}{2}\left[\frac{\pi}{2} - 2x\right]$$

$$= x + \frac{\pi}{4} - x = \frac{\pi}{4}.$$

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(viii) $\int_0^1 \frac{\log(1+x)}{1+x^2}dx$

$x = \tan\theta$

$dx = \sec^2\theta \cdot d\theta$

x	0	1
θ	0	$\frac{\pi}{4}$

$I = \int_0^{\pi/4} \frac{\log(1+\tan\theta)}{1+\tan^2\theta}\sec^2\theta \cdot d\theta$

$I = \int_0^{\pi/4} \log(1+\tan\theta)\, d\theta$

$= \int_0^{\pi/4} \log\left(1+\tan\left(\frac{\pi}{4}-\theta\right)\right) d\theta$

$\int_0^a f(x)\,dx = \int_0^a f(a-x)\,dx.$

$= \int_0^{\pi/4} \log\left[1+\frac{1-\tan\theta}{1+\tan\theta}\right] d\theta$

$= \int_0^{\pi/4} \log\left(\frac{2}{1+\tan\theta}\right) d\theta$

$I = \int_0^{\frac{\pi}{4}} \log 2\, d\theta - \int_0^{\pi/4} \log(1+\tan\theta)\, d\theta$

$I = \log 2\,[\theta]_0^{\pi/4} - I$

$2I = \log 2 \cdot \frac{\pi}{4}$

$I = \frac{\pi}{8}\log 2$

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(ix) $\int_0^{\pi} \frac{x\sin x}{1+\sin x}dx.$

$\boxed{\int_0^a x f(x)\,dx = \frac{a}{2}\int_0^a f(x)\,dx}$

let $f(x) = \frac{\sin x}{1+\sin x}$

$f(\pi-x) = \frac{\sin(\pi-x)}{1+\sin(\pi-x)}$

$= \frac{\sin x}{1+\sin x} = f(x)$

$\therefore I = \int_0^{\pi} x\frac{\sin x}{1+\sin x}dx = \frac{\pi}{2}\int_0^{\pi} \frac{\sin x}{1+\sin x}dx$

$\boxed{\int_0^{\pi} g(\sin x)\,dx = 2\int_0^{\frac{\pi}{2}} g(\sin x)\,dx}$

$= \cancel{2}\frac{\pi}{\cancel{2}}\int_0^{\pi/2} \frac{\sin\left(\frac{\pi}{2}-x\right)}{1+\sin\left(\frac{\pi}{2}-x\right)}dx$

$= \cancel{2}\frac{\pi}{\cancel{2}}\int_0^{\pi/2} \frac{\cos x}{1+\cos x} \times \frac{1-\cos x}{1-\cos x}dx$

$= \pi\int_0^{\pi/2} \frac{\cos x(1-\cos x)}{\sin^2 x}dx = \pi\int_0^{\frac{\pi}{2}} \frac{\cos x - \cos^2 x}{\sin^2 x}dx$

$= \pi\int_0^{\pi/2} (\operatorname{cosec} x \cot x - \cot^2 x)\,dx$

$$= \pi \int_0^{\pi/2} \left[\operatorname{cosec}x \cot x - (\operatorname{cosec}^2 x - 1)\right] dx$$

$$= \pi \int_0^{\pi/2} (1 - \operatorname{cosec}^2 x + \operatorname{cosec}x \cot x)\, dx$$

$$= \pi \left[x + \cot x - \operatorname{cosec}x\right]_0^{\pi/2}$$

$$= \pi \left[\left(\frac{\pi}{2} + 0 - 1\right) - (0 + \infty - \infty)\right]$$

$$= \pi\left(\frac{\pi}{2} - 1\right) = \frac{\pi}{2}(\pi - 2)$$

(x) $\displaystyle\int_{\pi/8}^{3\pi/8} \frac{dx}{1+\sqrt{\tan x}}$ $\qquad \frac{\pi}{8} + \frac{3\pi}{8} = \frac{\pi}{2}$

$$I = \int_{\pi/8}^{3\pi/8} \frac{dx}{1+\sqrt{\tan x}} \quad \text{——— (1)}$$

$$\boxed{\int_a^b f(x)\,dx = \int_a^b f(a+b-x)\,dx.}$$

$$I = \int_{\pi/8}^{3\pi/8} \frac{dx}{1+\sqrt{\tan\left(\frac{3\pi}{8} + \frac{\pi}{8} - x\right)}}$$

$$I = \int_{\pi/8}^{3\pi/8} \frac{dx}{1+\sqrt{\cot x}} \quad \text{——— (2)}$$

(1) + (2)

$$I + I = \int_{\pi/8}^{3\pi/8} \left(\frac{1}{1+\sqrt{\tan x}} + \frac{1}{1+\sqrt{\cot x}}\right) dx$$

$$2I = \int_{\pi/8}^{3\pi/8} \left(\frac{1}{1+\frac{\sqrt{\sin x}}{\sqrt{\cos x}}} + \frac{1}{1+\frac{\sqrt{\cos x}}{\sqrt{\sin x}}}\right) dx$$

$$= \int_{\pi/8}^{3\pi/8} \left(\frac{\sqrt{\cos x}}{\sqrt{\sin x}+\sqrt{\cos x}} + \frac{\sqrt{\sin x}}{\sqrt{\cos x}+\sqrt{\sin x}}\right) dx$$

$$= \int_{\pi/8}^{3\pi/8} \left(\frac{\sqrt{\cos x}+\sqrt{\sin x}}{\sqrt{\sin x}+\sqrt{\cos x}}\right) dx = [x]_{\pi/8}^{3\pi/8}$$

$$2I = \frac{3\pi}{8} - \frac{\pi}{8} = \frac{2\pi}{8} = \frac{\pi}{4}$$

$$\boxed{I = \frac{\pi}{12}}$$

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(xi) $\int_0^{\pi} x\left[\sin^2(\sin x)+\cos^2(\cos x)\right]dx$.

$$I=\int_0^{\pi} x\left[\sin^2(\sin x)+\cos^2(\cos x)\right]dx \quad \text{——— (1)}$$

$$I=\int_0^{\pi}(\pi-x)\left[\sin^2[\sin(\pi-x)]+\cos^2(\cos(\pi-x))\right]dx$$

$$=\int_0^{\pi}(\pi-x)\left[\sin^2(\sin x)+\cos^2(\cos x)\right]dx \quad \text{——— (2)}$$

(1) + (2)

$$I+I=\pi\int_0^{\pi}\left[\sin^2(\sin x)+\cos^2(\cos x)\right]dx.$$

$$I=\frac{\pi}{2}\int_0^{\pi}\left[\sin^2(\sin x)+\cos^2(\cos x)\right]dx.$$

$$\boxed{\int_0^a f(x)dx=\int_0^a f(a-x)dx \qquad \int_0^{2a} f(2a-x)dx=2\int_0^a f(x)dx}$$

$$I=\frac{\pi}{\cancel{2}}\cdot\cancel{2}\int_0^{\pi/2}\left[\sin^2(\sin x)+\cos^2(\cos x)\right]dx \quad \text{——— (3)}$$

$$=\pi\int_0^{\pi/2}\left[\sin^2(\cos x)+\cos^2(\sin x)\right]dx \quad \text{——— (4)}$$

(3) + (4)

$$2I=\pi\int_0^{\pi/2}\left\{\left[\sin^2(\sin x)+\cos^2(\sin x)\right]+\left[\sin^2(\cos x)+\cos^2(\cos x)\right]\right\}dx$$

$$2I=\pi\int_0^{\pi/2}2\,dx=2\pi\left[x\right]_0^{\pi/2}$$

$$2I=\cancel{2}\pi\left[\frac{\pi}{\cancel{2}}\right]$$

$$\boxed{I=\frac{\pi^2}{2}.}$$

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Bernoulli's Formula :

$$\boxed{\int u\,dv=uv-u'v_1+u''v_2-\ldots}$$

9.31/114. Evaluate $\int_0^{\pi} x^2 \cos nx\, dx$ where is a Positive integer.

$u = x^2 \qquad \int dv = \int \cos nx\, dx$

$u' = 2x \qquad v = \frac{\sin nx}{n}$

$u'' = 2 \qquad v_1 = -\frac{\cos nx}{n^2} \qquad v_2 = \frac{-\sin nx}{n^3}$

$$\int u\,dv = uv - u'v_1 + u''v_2 - \cdots$$

$$\int_0^{\pi} x^2 \cos nx\, dx = \left[x^2 \frac{\sin nx}{n} - 2x\left(-\frac{\cos nx}{n^2}\right) + 2\left(-\frac{\sin nx}{n^3}\right)\right]_0^{\pi}$$

$$= \left(0 + \frac{2\pi}{n^2}(-1)^n - 0\right) - 0 \qquad \sin n\pi = 0,\ \cos n\pi = (-1)^n$$

$$= \frac{2\pi}{n^2}(-1)^n$$

9.32/114 Evaluate : $\int_0^1 e^{-2x}(1 + x - 2x^3)\, dx$

$u = 1 + x - 2x^3 \qquad \int dv = \int e^{-2x} dx$

$u' = 1 - 6x^2 \qquad v = -\frac{e^{-2x}}{2} \qquad v_1 = \frac{e^{-2x}}{4}$

$u'' = -12x \qquad v_2 = -\frac{e^{-2x}}{8} \qquad v_3 = \frac{e^{-2x}}{16}$

$u''' = -12$

$$\int u\,dv = uv - u'v_1 + u''v_2 - u'''v_3 + \cdots$$

$$\int_0^1 e^{-2x}(1 + x - 2x^3)\,dx = \left[(1 - x - 2x^3)\frac{e^{-2x}}{-2} - (1 - 6x^2)\left(\frac{e^{-2x}}{4}\right) + (-12x)\left(\frac{e^{-2x}}{-8}\right) - (-12)\frac{e^{-2x}}{16}\right]_0^1$$

$$= \left\{\frac{e^{-2x}}{16}\left[(1 + x - 2x^3)(-8) - (1 - 6x^2)(4) + 24x + 12\right]\right\}_0^1$$

$$= \left\{\frac{e^{-2x}}{16}\left(\cancel{-8} - 8x + 16x^3 \cancel{-4} + 24x^2 + 24x + \cancel{12}\right)\right\}_0^1$$

$$= \left[\frac{e^{-2x}}{16}\left(16x^3 + 24x^2 + 16x\right)\right]_0^1$$

$$= \frac{e^{-2}}{16}(16 + 24 + 16) - \frac{1}{16}(0) = \frac{e^{-2}}{16}(56)$$

$$= \frac{7}{2e^2}.$$

9.33/115 Evaluate $\int_0^{2\pi} x^2 \sin nx \, dx$ where n is Positive integer.

$$\int_0^{2\pi} x^2 \sin nx \, dx$$

$u = x^2$ $\qquad \int dv = \int \sin nx \, dx$

$u' = 2x$ $\qquad v = -\frac{\cos nx}{n}$

$u'' = 2$ $\qquad v_1 = -\frac{\sin nx}{n^2}$

$v_2 = \frac{\cos nx}{n^3}$

$$\int u\,dv = uv - u'v_1 + u''v_2 - u'''v_3 + \dots$$

$$\int_0^{2\pi} x^2 \sin nx \, dx = \left[x^2\left(-\frac{\cos nx}{n}\right) - 2x\left(-\frac{\sin nx}{n^2}\right) + 2\left(\frac{\cos nx}{n^3}\right)\right]_0^{2\pi}$$

$$= \left[(2\pi)^2\left(-\frac{1}{n}\right) - 0 + 2\left(\frac{1}{n^3}\right)\right] - \left[0 - 0 + \frac{2}{n^3}\right]$$

$$= -\frac{4\pi^2}{n} + \frac{2}{n^3} - \frac{2}{n^3} = -\frac{4\pi^2}{n}.$$

9.34/115 Evaluate : $\int_{-1}^{1} e^{-\lambda x}(1-x^2)\,dx$

$u = 1 - x^2$ $\qquad \int dv = \int e^{-\lambda x} dx$

$u' = -2x$ $\qquad v = -\frac{e^{-\lambda x}}{+\lambda}$

$u'' = -2$ $\qquad v_1 = \frac{e^{-\lambda x}}{\lambda^2}$ $\qquad v_2 = -\frac{e^{-\lambda x}}{\lambda^3}$

$$\int u\,dv = uv - u'v_1 + u''v_2 - \dots$$

$$\int_{-1}^{1} e^{-\lambda x}(1-x^2)\,dx = \left[(1-x^2)\left(-\frac{e^{-\lambda x}}{\lambda}\right) - (-2x)\left(\frac{e^{-\lambda x}}{\lambda^2}\right) + (-2)\left(-\frac{e^{-\lambda x}}{\lambda^3}\right)\right]_{-1}^{1}$$

$$= \left[0 + 2\left(\frac{e^{-\lambda}}{\lambda^2}\right) + 2\left(\frac{e^{-\lambda}}{\lambda^3}\right)\right] - \left[0 - 2\frac{e^{\lambda}}{\lambda^2} + 2\frac{e^{\lambda}}{\lambda^3}\right]$$

$$= \frac{2e^{-\lambda}}{\lambda^2} + \frac{2e^{-\lambda}}{\lambda^3} + \frac{2e^{\lambda}}{\lambda^2} - \frac{2e^{\lambda}}{\lambda^3}$$

$$= \frac{2}{\lambda^2}\left(e^{-\lambda} + e^{\lambda}\right) + \frac{2}{\lambda^3}\left(e^{\lambda} - e^{-\lambda}\right)$$

Ex: 9.4

Evaluate the following :-

1. $\int_0^1 x^3 e^{-2x} \cdot dx$

$u = x^3 \qquad \int dv = \int e^{-2x} dx$

$u' = 3x^2 \qquad v = -\frac{e^{-2x}}{2} \qquad v_1 = \frac{e^{-2x}}{4}$

$u'' = 6x$

$u''' = 6 \qquad v_2 = -\frac{e^{-2x}}{8} \qquad v_3 = \frac{e^{-2x}}{16}$

$$\int u\,dv = uv - u'v_1 + u''v_2 - u'''v_3 + \cdots$$

$$\int_0^1 x^3 e^{-2x} dx = \left[x^3\left(\frac{e^{-2x}}{-2}\right) - 3x^2\left(\frac{e^{-2x}}{4}\right) + 6x\left(\frac{e^{-2x}}{-8}\right) - 6\left(\frac{e^{-2x}}{16}\right)\right]_0^1$$

$$= \left[\frac{e^{-2}}{-2} - \frac{3e^{-2}}{4} + \frac{\cancel{6}^{3}e^{-2}}{-\cancel{8}_{4}} - \frac{\cancel{6}^{3}e^{-2}}{\cancel{16}_{8}}\right] - \left[0 - 0 + 0 - \frac{6}{16}\right]$$

$$= \frac{e^{-2}}{8}\left[-4 - 6 - 6 - 3\right] + \frac{3}{8}$$

$$= \frac{e^{-2}}{8}(-19) + \frac{3}{8} = \frac{1}{8}\left(3 - 19e^{-2}\right)$$

2. Evaluate : $\int_0^1 \frac{\sin(3\tan^{-1}x)}{1+x^2} \tan^{-1}x \, dx.$

Let $t = \tan^{-1}x$

$dt = \frac{1}{1+x^2} dx$

x	0	1
t	0	$\frac{\pi}{4}$

$$= \int_0^{\pi/4} t \cdot \sin 3t \; dt$$

$$= \left[t\left(\frac{-\cos 3t}{3}\right) - 1\left(\frac{-\sin 3t}{9}\right)\right]_0^{\pi/4}$$

$$= \left[\frac{-\pi}{12}\left(\frac{-1}{\sqrt{2}}\right) + \frac{1}{9}\left(\frac{1}{\sqrt{2}}\right)\right] - 0$$

$$= \frac{1}{\sqrt{2}}\left(\frac{\pi}{12} + \frac{1}{9}\right]$$

3. Evaluate : $\int_0^{1/\sqrt{2}} \frac{e^{\sin^{-1}x} \cdot \sin^{-1}x}{\sqrt{1-x^2}} dx$

$t = \sin^{-1}x$

$dt = \frac{1}{\sqrt{1-x^2}} dx$

x	0	$\frac{1}{\sqrt{2}}$
t	0	$\frac{\pi}{4}$

$$= \int_0^{\pi/4} t e^{t} \, dt$$

$$= \left[t e^t - e^t\right]_0^{\pi/4}$$

$$= \left(\frac{\pi}{4} e^{\pi/4} - e^{\pi/4}\right) - (0 - 1)$$

$$= 1 - e^{\pi/4}\left(\frac{\pi}{4} - 1\right)$$

4/115 $\int_0^{\pi/2} x^2 \cos 2x \, dx$

$$\int dv = \int \cos 2x \, dx$$

$u = x^2$ $\qquad v = \frac{\sin 2x}{2}$

$u' = 2x$ $\qquad v_1 = -\frac{\cos 2x}{4}$

$u'' = 2$ $\qquad v_2 = -\frac{\sin 2x}{8}$

$$\int u\,dv = uv - u'v_1 + u''v_2 - \cdots$$

$$\int_0^{\pi/2} x^2 \cos 2x \, dx = \left[x^2\left(\frac{\sin 2x}{2}\right) - 2x\left(\frac{-\cos 2x}{4}\right) + 2\left(\frac{-\sin 2x}{8}\right)\right]_0^{\pi/2}$$

$$= \left[0 + \frac{1}{2}\cdot\frac{\pi}{2}(-1) + 0\right] - [0 - 0 + 0]$$

$$= -\frac{\pi}{4}.$$

9.5 Improper Integrals :

9.35/116 Evaluate : $\int_b^{\infty} \frac{1}{a^2+x^2} dx \quad a>0 \quad b \in R$

$$\int_b^{\infty} \frac{1}{a^2+x^2} dx = \frac{1}{a}\left[\tan^{-1}\left(\frac{x}{a}\right)\right]_b^{\infty}$$

$$= \frac{1}{a}\left[\tan^{-1}(\infty) - \tan^{-1}\left(\frac{b}{a}\right)\right]$$

$$= \frac{1}{a}\left[\frac{\pi}{2} - \tan^{-1}\left(\frac{b}{a}\right)\right]$$

Results :-

$$\int_0^{\infty} \frac{dx}{a^2+x^2} = \frac{1}{a}\left[\frac{\pi}{2} - \tan^{-1}(0)\right] = \frac{\pi}{2a}.$$

$$\int_a^{\infty} \frac{dx}{a^2+x^2} = \frac{1}{a}\left[\frac{\pi}{2} + \tan^{-1}(1)\right] = \frac{\pi}{4a}$$

$$\int_{-\infty}^{\infty} \frac{dx}{a^2+x^2} = 2\int_0^{\infty} \frac{dx}{x^2+a^2} = 2\left(\frac{\pi}{2a}\right) = \frac{\pi}{a}$$

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9.36 / 116.

Evaluate : $\int_0^{\pi/2} \frac{dx}{4\sin^2 x + 5\cos^2 x}$

Dividing both Nr and Dr by $\cos^2 x$.

$$I = \int_0^{\pi/2} \frac{\sec^2 x}{4\tan^2 x + 5} du$$

$u = \tan x$

$du = \sec^2 x \, dx$

x	0	$\pi/2$
u	0	∞

$$= \int_0^{\infty} \frac{du}{4u^2+5}$$

$$= \frac{1}{4}\int_0^{\infty} \frac{du}{u^2+\frac{5}{4}} = \frac{1}{4}\left\{\frac{1}{\frac{\sqrt5}{2}} \tan^{-1}\left(\frac{u}{\sqrt5/2}\right)\right\}_0^{\infty}$$

$$= \left[\frac{1}{2\sqrt5} \tan^{-1} \frac{2u}{\sqrt5}\right]_0^{\infty}$$

$$= \frac{1}{2\sqrt5}\left[\tan^{-1}\infty - \tan^{-1} 0\right] = \frac{1}{2\sqrt5}\left(\frac{\pi}{2}\right)$$

$$= \frac{\pi}{4\sqrt5}.$$

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Ex : 9.5.

Evaluate the following :-

1. $\int_0^{\pi/2} \frac{dx}{1+5\cos^2 x}$

Dividing both Nr and Dr by $\cos^2 x$.

$$I = \int_0^{\pi/2} \frac{dx}{1+5\cos^2 x} = \int_0^{\pi/2} \frac{\sec^2 x \, dx}{\sec^2 x + 5}$$

$$= \int_0^{\pi/2} \frac{\sec^2 x \, dx}{1+\tan^2 x + 5} = \int_0^{\pi/2} \frac{\sec^2 x \, dx}{\tan^2 x + 6}$$

$u = \tan x.$

$du = \sec^2 x \, dx$

x	0	$\pi/2$
u	0	∞

$$= \int_0^{\infty} \frac{du}{u^2+6}$$

$$= \frac{1}{\sqrt6}\left(\tan^{-1}\left(\frac{u}{\sqrt6}\right)\right)_0^{\infty}$$

$$= \frac{1}{\sqrt6}\left[\tan^{-1}(\infty) - \tan^{-1}(0)\right]$$

$$= \frac{1}{\sqrt6}\left(\frac{\pi}{2}\right) = \frac{\pi}{2\sqrt6}.$$

2. Evaluate : $\int_0^{\pi/2} \frac{dx}{5+4\sin^2 x}$

Dividing both Nr and Dr by $\cos^2 x$

$$I = \int_0^{\pi/2} \frac{\sec^2 x}{5\sec^2 x + 4\tan^2 x}\,dx$$

$$= \int_0^{\pi/2} \frac{\sec^2 x}{5(1+\tan^2 x)+4\tan^2 x}\,dx$$

$$= \int_0^{\pi/2} \frac{\sec^2 x}{9\tan^2 x + 5}\,dx$$

$u = \tan x$

$du = \sec^2 x\, dx$

x	0	$\pi/2$
u	0	∞

$$= \int_0^{\infty} \frac{du}{9u^2+5}$$

$$= \frac{1}{9}\int_0^{\infty} \frac{du}{u^2+\frac{5}{9}}$$

$$= \frac{1}{9}\left\{\frac{1}{\frac{\sqrt{5}}{3}} \tan^{-1}\left(\frac{u}{\sqrt{5}/3}\right)\right\}_0^{\infty}$$

$$= \frac{1}{9}\left[\frac{3}{\sqrt{5}} \tan^{-1}\left(\frac{3u}{\sqrt{5}}\right)\right]_0^{\infty}$$

$$= \frac{1}{3\sqrt{5}} \cdot \frac{\pi}{2} = \frac{\pi}{6\sqrt{5}}.$$

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Reduction Formulae :-

$$I_n = \int_0^{\pi/2} \sin^n x\, dx = \frac{n-1}{n} I_{n-2}, \quad n \geq 2$$

$$I_n = \int_0^{\pi/2} \cos^n x\, dx = \frac{n-1}{n} I_{n-2}, \quad n \geq 2$$

$$I_{m,n} = \int_0^{\pi/2} \sin^m x \cos^n x\, dx = \frac{n-1}{m+n} \cdot I_{m,n-2}; \quad n \geq 2$$

$$I_{m,n} = \int_0^{\pi/2} x^m (1-x)^n\, dx = \frac{n}{m+n+1} I_{m,n-1} \quad n \geq 1$$

$$I_n = \left.\begin{array}{l} \int_0^{\pi/2} \sin^n x \, dx \\ \text{[OR]} \\ \int_0^{\pi/2} \cos^n x \, dx \end{array}\right\} = \left\{\begin{array}{ll} \frac{n-1}{n} \cdot \frac{n-3}{n-2} \cdot \frac{n-5}{n-4} \cdots \frac{2}{3} \cdot 1 & \text{[n is odd]} \\ \frac{n-1}{n} \cdot \frac{n-3}{n-2} \cdot \frac{n-5}{n-4} \cdots \frac{1}{2} \cdot \frac{\pi}{2} & \text{[n is even]} \end{array}\right.$$

$\frac{9.37}{119}$ Evaluate : $\int_0^{\pi/2} (\sin^2 x + \cos^4 x)\, dx$

$$I_n = \int_0^{\pi/2} \sin^2 x \, dx + \int_0^{\pi/2} \cos^4 x \cdot dx$$

[where n are even]

$$= \frac{1}{2} \cdot \frac{\pi}{2} + \frac{3}{4} \cdot \frac{1}{2} \cdot \frac{\pi}{2} = \frac{\pi}{4}\left(1 + \frac{3}{4}\right)$$

$$= \frac{\pi}{4}\left(\frac{7}{4}\right) = \frac{7\pi}{16}.$$

$\frac{9.38}{119}$ $\int_0^{\pi/2} \begin{vmatrix} \cos^4 x & 7 \\ \sin^5 x & 3 \end{vmatrix} dx$

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$$= \int_0^{\pi/2} (3\cos^4 x - 7\sin^5 x)\, dx$$

$$= 3\int_0^{\pi/2} \cos^4 x \, dx - 7\int_0^{\pi/2} \sin^5 x \, dx$$

(n is even) (n is odd)

$$= 3\left(\frac{3}{4} \cdot \frac{1}{2} \cdot \frac{\pi}{2}\right) - 7\left(\frac{4}{5} \cdot \frac{2}{3} \cdot 1\right)$$

$$= \frac{9\pi}{16} - \frac{56}{15}$$

Ex : 9.6

1. Evaluate the following :-

(i) $\int_0^{\pi/2} \sin^{10} x \, dx$ $n = 10$ n is even

$$= \frac{\cancel{9}^{3}}{\cancel{10}_{2}} \cdot \frac{7}{8} \cdot \frac{\cancel{5}}{\cancel{6}_{2}} \cdot \frac{3}{4} \cdot \frac{1}{2} \cdot \frac{\pi}{2} = \frac{63\pi}{512}$$

$$\int_0^{\pi/2} \sin^{10} x \, dx = \frac{63\pi}{512}$$

(ii) $\int_0^{\pi/2} \cos^7 x\, dx$ $n=7$, n is odd.

$$I = \frac{\cancel{6}^2}{7} \cdot \frac{4}{5} \cdot \frac{2}{\cancel{3}} \cdot 1 = \frac{16}{35}$$

(iii) $\int_0^{\pi/4} \sin^6 2x\, dx$

$u = 2x$

$du = 2dx$

$\frac{du}{2} = dx$

x	0	$\pi/4$
u	0	$\frac{\pi}{2}$

$$= \frac{1}{2}\int_0^{\pi/2} \sin^6 u\, du \quad [n \text{ is even}]$$

$$= \frac{1}{2}\left[\frac{5}{\cancel{6}_2} \cdot \frac{\cancel{3}}{4} \cdot \frac{1}{2} \cdot \frac{\pi}{2}\right]$$

$$= \frac{5\pi}{64}.$$

(iv) $\int_0^{\pi/6} \sin^5 3x\, dx$

$u = 3x$

$du = 3dx$

$\frac{du}{3} = dx$

x	0	$\pi/6$
u	0	$\pi/2$

$$I = \frac{1}{3}\int_0^{\pi/2} \sin^5 u\, du$$

$$= \frac{1}{3}\left[\frac{4}{5} \cdot \frac{2}{3} \cdot 1\right]$$

$$= \frac{8}{45}$$

(v) $\int_0^{\pi/2} \sin^2 x \cos^4 x\, dx$.

$$= \int_0^{\pi/2} (1-\cos^2 x)\cos^4 x\, dx$$

$$= \int_0^{\pi/2} \cos^4 x\, dx - \int_0^{\pi/2} \cos^6 x\, dx \quad [n \text{ is even}]$$

$$= \frac{3}{4}\cdot\frac{1}{2}\cdot\frac{\pi}{2} - \frac{5}{6}\cdot\frac{3}{4}\cdot\frac{1}{2}\cdot\frac{\pi}{2}$$

$$= \frac{3\pi}{16}\left(1-\frac{5}{6}\right) = \frac{\cancel{3}\pi}{16}\left(\frac{1}{\cancel{6}_2}\right)$$

$$= \frac{\pi}{32}.$$

(vii) $\int_0^{2\pi} \sin^7\left(\frac{x}{4}\right) dx$

$u = \frac{x}{4}$

$du = \frac{dx}{4}$

$4\,du = dx$

x	0	2π
u	0	$\frac{\pi}{2}$

$= 4 \int_0^{\pi/2} \sin^7 u \, du$

$= 4\left[\frac{\cancel{6}^{2}}{7} \cdot \frac{4}{5} \cdot \frac{2}{\cancel{3}} \cdot 1\right]$

$= \frac{64}{35}$

9.40 / 119. Evaluate: $\int_0^{2a} x^2 \sqrt{2ax - x^2}\, dx$

Put $x = 2a\cos^2\theta$

$dx = -4a\sin\theta\cos\theta \cdot d\theta$

x	0	$2a$
θ	$\frac{\pi}{2}$	0

$I = \int_0^{2a} x^2 \sqrt{2ax - x^2}\, dx$

$= \int_{\pi/2}^{0} 4a^2\cos^4\theta \sqrt{4a^2\cos^2\theta - 4a^2\cos^4\theta}\,(-4a\sin\theta\cos\theta)\, d\theta$

$= \int_{\pi/2}^{0} 4a^2\cos^4\theta \cdot 2a\cos\theta\sqrt{1-\cos^2\theta}\,(-4a\sin\theta\cos\theta)\, d\theta$

$= -32a^4 \int_{\pi/2}^{0} \cos^6\theta \sin^2\theta \, d\theta$

$= 32a^4 \int_0^{\pi/2} \cos^6\theta\,(1 - \cos^2\theta)\, d\theta$

$= 32a^4 \left\{ \int_0^{\pi/2} \cos^6\theta \, d\theta - \int_0^{\pi/2} \cos^8\theta \cdot d\theta \right\}$

$= 32a^4 \left[\frac{5}{\cancel{6}_2} \cdot \frac{3}{4} \cdot \frac{1}{2} \cdot \frac{\pi}{2} - \frac{7}{8} \cdot \frac{5}{\cancel{6}_2} \cdot \frac{\cancel{3}}{4} \cdot \frac{1}{2} \cdot \frac{\pi}{2} \right]$

$= 32a^4 \left[\frac{5\pi}{32}\left(1 - \frac{7}{8}\right)\right]$

$= \cancel{32}a^4 \left(\frac{5\pi}{\cancel{32}}\right)\left(\frac{1}{8}\right)$

$= \frac{5a^4\pi}{8}$

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If n is even and m is even.

$$\int_0^{\pi/2} \sin^m x \cos^n x \, dx = \frac{n-1}{m+n} \cdot \frac{n-3}{m+n-2} \cdot \frac{n-5}{m+n-4} \cdots \frac{1}{m+2} \cdot \frac{m-1}{m}$$

$$\left(\frac{m-3}{m-2}\right)\left(\frac{m-5}{m-4}\right) \cdots \frac{1}{2} \cdot \frac{\pi}{2}$$

If n is odd m is any Positive integer

$$\int_0^{\pi/2} \sin^m x \cos^n x \, dx = \frac{n-1}{m+n} \cdot \frac{n-3}{m+n-2} \cdot \frac{n-5}{m+n-4} \cdots \frac{2}{m+3} \cdot \frac{1}{m+1}$$

If one of m and n is odd than it is Convenient to get the Power of $\cos x$ as odd

$$\int_0^{\pi/2} \sin^m x \cos^n x \, dx = \int_0^{\pi/2} \sin^n x \cos^m x \, dx$$

$$= \frac{n-1}{n+m} \cdot \frac{m-3}{m+n-2} \cdot \frac{m-5}{m+n-4} \cdots \frac{2}{n+3} \cdot \frac{1}{n+1}.$$

9.39 / 118 Find the values of the following:-

(i) $\int_0^{\pi/2} \sin^4 x \cos^6 x \, dx$

$$= \frac{6-1}{6+4} \cdot \frac{6-3}{6+4-2} \cdot \frac{6-5}{6+4-4} \cdot \frac{4-1}{4} \cdot \frac{4-3}{4-2} \cdot \frac{\pi}{2}$$

$$= \frac{5}{10} \cdot \frac{3}{8} \cdot \frac{1}{6} \cdot \frac{3}{4} \cdot \frac{1}{2} \cdot \frac{\pi}{2} = \frac{3\pi}{12}.$$

$$\int_0^{\pi/2} \sin^4 x \cos^6 x \, . \, dx = \frac{3\pi}{12}$$

(ii) $\int_0^{\pi/2} \sin^5 x \cos^4 x \, dx = \frac{3}{9} \cdot \frac{1}{7} \cdot \frac{4}{5} \cdot \frac{2}{3} = \frac{8}{315}$

9.41 / 119

Evaluate : $\int_0^1 x^5(1-x^2)^5 \cdot dx$

$x = \sin\theta$, $dx = \cos\theta \cdot d\theta$

x	0	1
θ	0	$\frac{\pi}{2}$

$$= \int_0^{\pi/2} \sin^5\theta \, (1-\sin^2\theta)^5 \cdot \cos\theta \, d\theta$$

$$= \int_0^{\pi/2} \sin^5\theta \cdot \cos^{10}\theta \, (\cos\theta \cdot d\theta)$$

$$= \int_0^{\pi/2} \sin^5\theta \cdot \cos^{11}\theta \cdot d\theta$$

$$= \frac{\cancel{10}}{\cancel{16}_2} \cdot \frac{\cancel{8}}{14} \cdot \frac{\cancel{6}}{\cancel{12}_2} \cdot \frac{\cancel{4}}{\cancel{10}} \cdot \frac{\cancel{2}}{\cancel{8}_2} \cdot \frac{1}{6} = \frac{1}{336}.$$

(vii) / 120

$$\int_0^{\pi/2} \sin^3\theta \cdot \cos^5\theta \cdot d\theta.$$

$m = 3$ $n = 5$

$$\int_0^{\pi/2} \sin^3\theta \cdot \cos^5\theta \cdot d\theta = \frac{2}{8} \cdot \frac{4}{6} \cdot \frac{2}{4} \cdot \frac{1}{2} = \frac{1}{24}.$$

$$\int_0^1 x^m(1-x)^n \, dx = \frac{m! \times n!}{(m+n+1)!}.$$

9.42 / 120

Evaluate : $\int_0^1 x^3(1-x)^4 \, dx$

$m = 3$ $n = 4$

$$\int_0^1 x^3(1-x)^4 \, dx = \frac{3! \times 4!}{(3+4+1)!} = \frac{\cancel{6} \times \cancel{24}}{8 \times 7 \times \cancel{6} \times 5 \times \cancel{4!}}$$

$$= \frac{1}{280}$$

viii / 120

Evaluate : $\int_0^1 x^2(1-x)^3 \, dx$

$m = 2$ $n = 3$

$$\int_0^1 x^2(1-x)^3 \, dx = \frac{2! \, 3!}{(2+3+1)!} = \frac{\cancel{2} \times \cancel{6}}{6 \times 5 \times \cancel{4}_2 \times \cancel{3!}} = \frac{1}{60}$$

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Gamma Integral :

9.43 / 120 Prove that $\int_0^{\infty} e^{-x} x^n \, dx = n!$ where n is a Positive integer.

$$\int_0^{\infty} e^{-x} x^n . dx \qquad \left[\int u dv = uv - \int v du \right]$$

$$= \left[\left[x^n . (-e^{-x})\right]_0^{\infty} - \int_0^{\infty} - e^{-x} . (n x^{n-1}) dx \right]$$

$$I_n = n \int_0^{\infty} e^{-x} . x^{n-1} . dx$$

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$$I_n = n I_{n-1}$$

$$= n(n-1) I_{n-2}$$

$$\vdots$$

$$I_n = n(n-1)(n-2) \ldots 3 . 2 . 1 \; I_0$$

$$= n! \; I_0$$

$$= n! \int_0^{\infty} e^{-x} x^0 dx = n! \left[- e^{-x} \right]_0^{\infty}$$

$$= n! \left[0 - (-1) \right] = n!$$

$$\therefore \int_0^{\infty} x^n . e^{-x} . dx = n!$$

$\int_0^{\infty} e^{-x} x^{n-1} dx$ is called the Gamma Integral. It is denoted by $\Gamma(n)$

$$\boxed{\begin{aligned} \Gamma(n+1) &= n \Gamma(n) \\ \Gamma(1) &= 1 \\ \Gamma(n) &= \int_0^{\infty} x^{n-1} e^{-x} . dx = (n-1)! \end{aligned}}$$

9.44/121 Evaluate : $\int_0^\infty e^{-ax} x^n \, dx \quad a>0$

$\int_0^\infty e^{-ax} \cdot x^n \cdot dx$

$u = ax$

$du = a\,dx$

$\frac{du}{a} = dx$

x	0	∞
u	0	∞

$= \int_0^\infty e^{-u} \cdot \left(\frac{u}{a}\right)^n \cdot \frac{du}{a}$

$= \frac{1}{a^{n+1}} \int_0^\infty e^{-u} \cdot u^n \, du$

$= \frac{1}{a^{n+1}} \int_0^\infty e^{-x} \cdot x^n \cdot dx = \frac{1}{a^{n+1}} n!$

$\int_0^\infty x^n \cdot e^{-ax} \, dx = \frac{n!}{a^{n+1}}$

9.45/121 Show that $\Gamma(n) = 2\int_0^\infty e^{-x^2} x^{2n-1} \, dx$

$2\int_0^\infty e^{-x^2} x^{2n-1} \cdot dx$

$x = \sqrt{u}$

$dx = \frac{1}{2\sqrt{u}} \cdot du$

x	0	∞
u	0	∞

$= 2\int_0^\infty e^{-u} \cdot (\sqrt{u})^{2n-1} \cdot \frac{1}{2\sqrt{u}} \cdot du$

$= 2 \times \frac{1}{2} \int_0^\infty e^{-u} \cdot (\sqrt{u})^{2n-1-1} \cdot du$

$= 2 \times \frac{1}{2} \int_0^\infty e^{-u} \cdot u^{n-1} \, du$

$= \int_0^\infty e^{-u} \cdot u^{n-1} du = \int_0^\infty e^{-x} x^{n-1} \cdot dx = (n-1)!$

$= \Gamma n$

$\therefore \Gamma(n) = 2\int_0^\infty e^{-x^2} \cdot x^{2n-1} \cdot dx$

9.46/122 Evaluate $\int_0^\infty \frac{x^n}{n^x} dx$ where n is +ve integer $n \geq 2$.

$n = e^{\log_e n}$

$I = \int_0^\infty x^n \cdot n^{-x} \cdot dx$

$= \int_0^\infty (e^{\log n})^{-x} x^n \, dx = \int_0^\infty e^{-x \log n} \cdot x^n \, dx$

$$I = \int_0^\infty e^{-u}\left(\frac{u}{\log n}\right)^n \cdot \frac{du}{\log n}$$

$u = x\log n$

$du = \log n \cdot dx$

$\frac{du}{\log n} = dx$

x	0	∞
u	0	∞

$$= \frac{1}{(\log n)^{n+1}} \int_0^\infty e^{-u} \cdot u^n \, du$$

$$= \frac{1}{(\log n)^{n+1}} \int_0^\infty e^{-u} u^{(n+1)-1} \cdot du$$

$$= \frac{\Gamma(n+1)}{(\log n)^{n+1}} = \frac{n!}{(\log n)^{n+1}}$$

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Ex : 9.7.

Evaluate the following :-

1. (i) $\int_0^\infty x^5 e^{-3x} dx$

$n = 5$

$a = 3$

$$\int_0^\infty x^n e^{-ax} dx = \frac{n!}{a^{n+1}}$$

$$\int_0^\infty x^5 e^{-3x} dx = \frac{5!}{3^{5+1}} = \frac{120}{3^6}$$

(ii) $\int_0^{\pi/2} \frac{e^{-\tan x}}{\cos^6 x} \cdot dx$

$$= \int_0^{\pi/2} e^{-\tan x} \cdot \sec^6 x \cdot dx$$

$u = \tan x$

$du = \sec^2 x \cdot dx$

x	0	$\pi/2$
u	0	∞

$$= \int_0^{\pi/2} e^{-\tan x} \cdot \sec^4 x \cdot (\sec^2 x \, dx)$$

$$= \int_0^\infty e^{-u} (\sec^2 x)^2 \cdot du$$

$$= \int_0^\infty e^{-u} (1+u^2)^2 du$$

$$= \int_0^\infty e^{-u} (1 + u^4 + 2u^2) du$$

$$= \int_0^\infty e^{-u}\,du + \int_0^\infty e^{-u} u^4\,du + 2\int_0^\infty e^{-u} \cdot u^2$$

$$= \left(-e^{-u}\right)_0^\infty + \frac{4!}{1^5} + 2\left(\frac{2!}{1^3}\right)$$

$$= 1 + 24 + 4 = 29.$$

$\frac{2}{122.}$ If $\int_0^\infty e^{-\alpha x^2} \cdot x^3\,dx = 32$ $\alpha > 0$, find α.

$$\int_0^\infty e^{-\alpha x^2} \cdot x^2 (x\,dx) = 32$$

$u = x^2$

$du = 2x\,dx$

$\frac{du}{2} = x \cdot dx$

x	0	∞
u	0	∞

$$\int_0^\infty e^{-\alpha u} \cdot u \cdot \frac{du}{2} = 32$$

$$\int_0^\infty e^{-\alpha u} u \cdot du = 64$$

$$\frac{1!}{\alpha^2} = 64$$

$$\alpha^2 = \frac{1}{64}$$

$$\alpha = \pm \frac{1}{8}$$

$\alpha \neq -\frac{1}{8}$

$\boxed{\alpha = \frac{1}{8}}$ $\boxed{\alpha > 0}$

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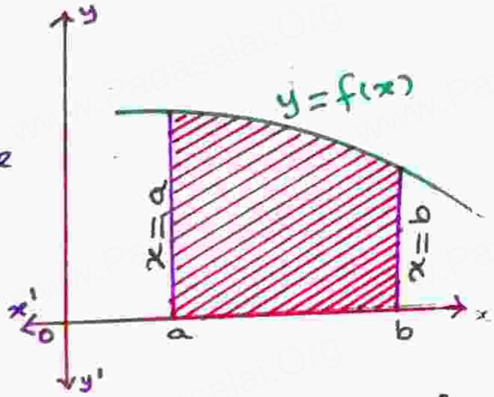
9.8 Evaluation of a Bounded Plane Area by Integration.

Area of the region bounded by a curve, x-axis and the lines $x=a$ and $x=b$.



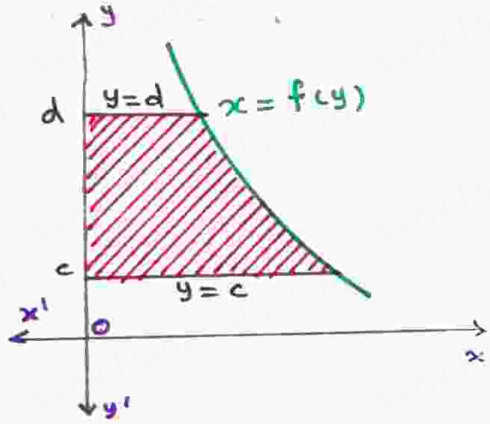
Let $y=f(x)$ be a continuous function defined on $[a,b]$ which is positive [f(x) lies on or above x-axis] on the interval $[a,b]$. Then the area bounded by the curve $y=f(x)$ the x-axis and the ordinates $x=a$ and $x=b$ is given by

$$A=\int_a^b y\,dx \quad (\text{or}) \quad \int_a^b f(x)\,dx.$$

If $f(x)\le 0$ [f(x) lies on or below x-axis] for all x in $a\le x\le b$ then area is given by

$$\text{Area}=\left|\int_a^b y\,dx\right| \quad (\text{or}) \quad \left|\int_a^b f(x)\,dx\right|$$

Area of the region bounded by a curve, y-axis and the lines $y=c$ and $y=d$.



Let $x=f(y)$ be a continuous function of y on $[c,d]$. The area bounded by the curve $x=f(y)$ and the abscissae $y=c$ and $y=d$ to the right of y-axis is

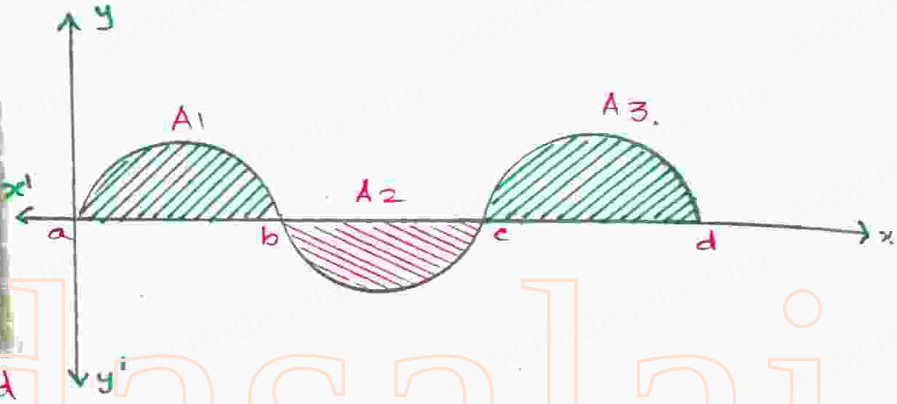
$$\text{Area}=\int_c^d x\,dy \quad (\text{or}) \quad \int_c^d f(y)\,dy.$$

If the curve lies to the left of y-axis between the lines $y=c$ and $y=d$

$$\text{Area} = \left| \int_c^d x\, dy \right| \text{ (or) } \left| \int_c^d f(y)\, dy \right|$$

Remark:

If the continuous curve f crosses the x-axis, then the integral $\int_a^b f(x)\,dx$ gives the algebraic sum of the areas between the curve and the axis, counting area above as positive and below as negative.



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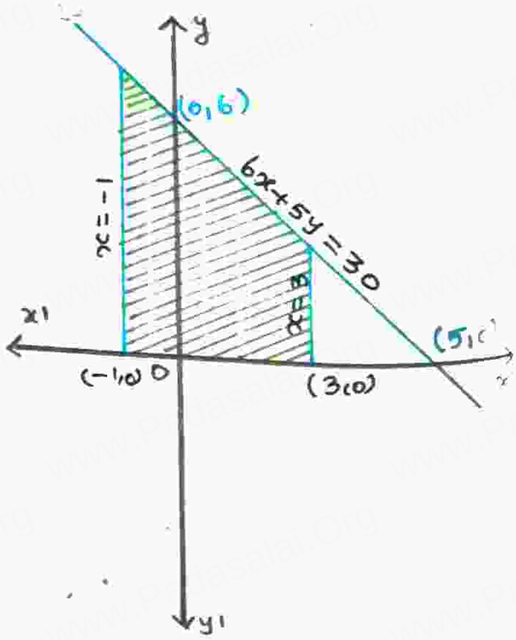
$$\text{Area} = \int_a^d f(x)\,dx = A_1 + A_2 + A_3$$

$$= \int_a^b f(x)\,dx + \left| \int_b^c f(x)\,dx \right| + \int_c^d f(x)\,dx.$$

9.47
125. Find the area of the region bounded by the line $6x+5y=30$, x-axis and the lines $x=-1$ and $x=3$.



$$6x + 5y = 30$$

$$5y = -6x + 30$$

$$y = -\frac{1}{5}(6x-30)$$

Limits: $x=-1$ $x=3$.

$$\text{Required area} = \int_a^b y\,dx$$

$$= \int_{-1}^{3} \frac{-1}{5}(6x-30)\,dx$$

$$= -\frac{6}{5}\int_{-1}^{3}(x-5)\,dx$$

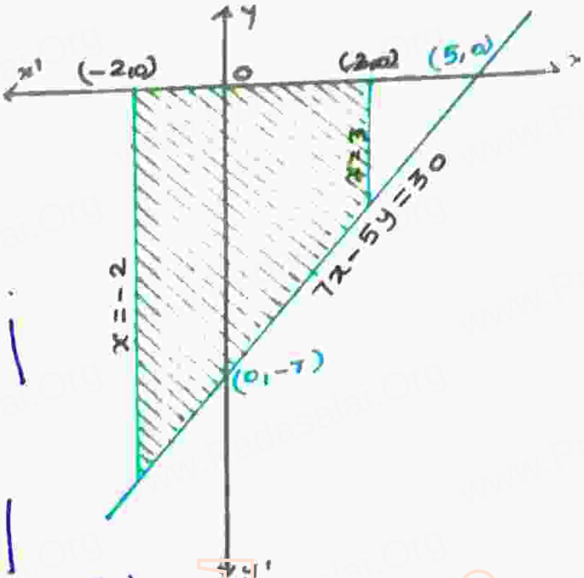
$$= -\frac{6}{5}\left[\frac{x^2}{2} - 5x\right]_{-1}^{3}$$

$$= \frac{6}{5}\left[5x - \frac{x^2}{2}\right]_{-1}^{3}$$

$$= \frac{6}{5}\left\{\left(15 - \frac{9}{2}\right) - \left(-5 - \frac{1}{2}\right)\right\}$$

$$= \frac{6}{5}\{20 - 4\} = \frac{6}{5}(16) = \frac{96}{5} \text{ sq. units.}$$

9.48 / 125. Find the area of the region bounded by the line $7x - 5y = 35$, x-axis and the lines $x = -2$ and $x = 3$.



$$7x - 5y = 35$$
$$5y = 7x - 35$$
$$y = \frac{7}{5}(x - 5)$$

Limits: $x = -2$ and $x = 3$.

$$\text{Required area} = \left|\int_a^b y\,dx\right|$$

$$= \left|\int_{-2}^{3} \frac{7}{5}(x-5)\,dx\right|$$

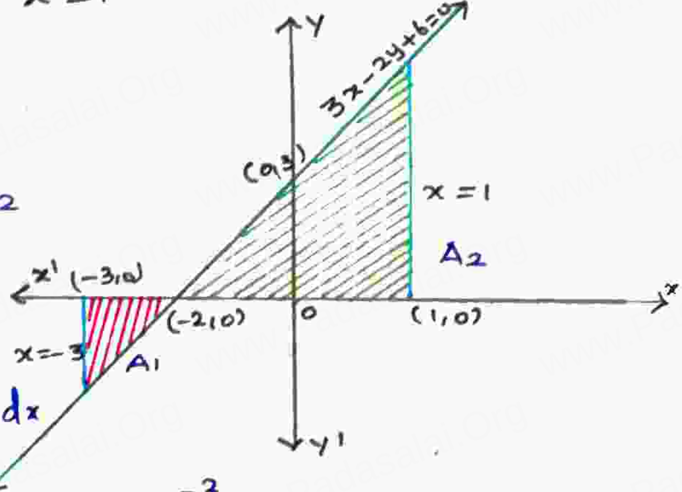
$$= \left|\frac{7}{5}\left[\frac{x^2}{2} - 5x\right]_{-2}^{3}\right|$$

$$= \left|\frac{7}{5}\left\{\left(\frac{9}{2} - 15\right) - \left(\frac{4}{2} + 10\right)\right\}\right|$$

$$= \left|\frac{7}{5}\left(\frac{5}{2} - 25\right)\right| = \frac{7}{5}\left(\frac{45}{2}\right) = \frac{63}{2} \text{ sq. units.}$$

Ex : 9.8

1 / 134 Find the area of the region bounded by $3x - 2y + 6 = 0$; $x = -3$, $x = 1$ and x-axis



$$3x - 2y + 6 = 0$$
$$2y = 3x + 6$$
$$y = \frac{3}{2}(x + 2)$$

$$\text{Required area} = A_1 + A_2$$

$$= \int_{-3}^{-2} -y\,dx + \int_{-2}^{1} y\,dx$$

$$= \frac{3}{2}\int_{-2}^{1}(x+2)\,dx - \frac{3}{2}\int_{-3}^{-2}(x+2)\,dx$$

$$= \frac{3}{2}\left[\frac{x^2}{2} + 2x\right]_{-2}^{1} - \frac{3}{2}\left[\frac{x^2}{2} + 2x\right]_{-3}^{-2}$$

$$= \frac{3}{2}\left\{\left(\frac{1}{2} + 2\right) - \left(\frac{4}{2} - 4\right)\right\} - \frac{3}{2}\left\{\left(\frac{4}{2} - 4\right) - \left(\frac{9}{2} - 6\right)\right\}$$

$$= \frac{3}{2}\left[-\frac{3}{2} + 6\right] - \frac{3}{2}\left[\frac{-5}{2} + 2\right] = \frac{3}{2}\left[-\frac{3}{2} + 6 + \frac{5}{2} - 2\right]$$

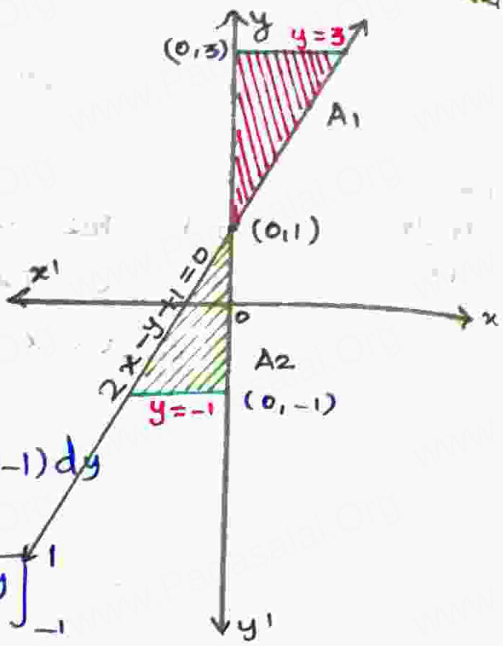
$$= \frac{3}{2}(5) = \frac{15}{2} \text{ sq. units.}$$

$\frac{2}{134}$. Find the area of the region bounded by $2x-y+1=0$, $y=-1$ and $y=3$, and y-axis.

$2x-y+1=0$

$2x=y-1$

$x=\frac{1}{2}(y-1)$



Required area $= A_1 + A_2$

$$= \int_1^3 x\,dy - \int_{-1}^1 x\,dy$$

$$= \int_1^3 \frac{1}{2}(y-1)\,dy - \int_{-1}^1 \frac{1}{2}(y-1)\,dy$$

$$= \frac{1}{2}\left[\frac{y^2}{2}-y\right]_1^3 - \frac{1}{2}\left[\frac{y^2}{2}-y\right]_{-1}^1$$

$$= \frac{1}{2}\left\{\left(\frac{9}{2}-3\right)-\left(\frac{1}{2}-1\right)\right\} - \frac{1}{2}\left\{\left(\frac{1}{2}-1\right)-\left(\frac{1}{2}+1\right)\right\}$$

$$= \frac{1}{2}\left\{\frac{8}{2}-2\right\} - \frac{1}{2}\left\{0-2\right\}$$

$$= \frac{1}{2}[2+2] = 2 \text{ sq. units.}$$

$\frac{9.52}{127}$. Find the area of the region bounded by x-axis, the curve $y=\sin x$ the lines $x=0$ and $x=2\pi$

$y=\sin x$

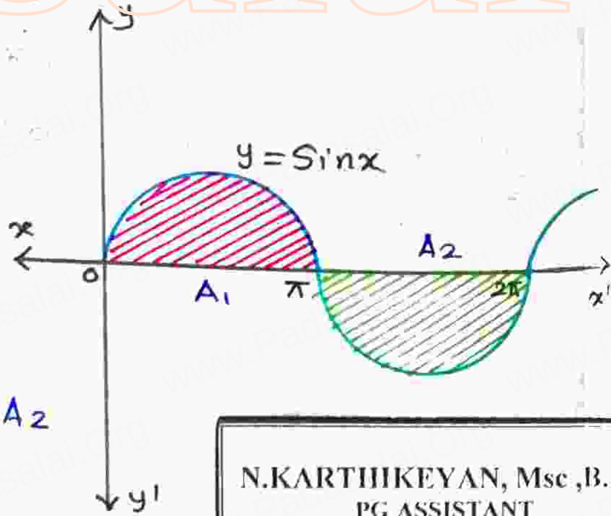
Put $y=0$

$\sin x = 0$

$x = \sin^{-1}(0)$

$x = 0, \pi, 2\pi \ldots$

$x = n\pi$



Required area $= A_1 + A_2$

$$A = \int_0^\pi y\,dx - \int_\pi^{2\pi} y\,dx$$

$$= \int_0^\pi \sin x\,dx - \int_\pi^{2\pi} \sin x\,dx$$

$$= [-\cos x]_0^\pi + [\cos x]_\pi^{2\pi}$$

$$= (1+1) + (1+1) = 2+2 = 4 \text{ sq. units.}$$

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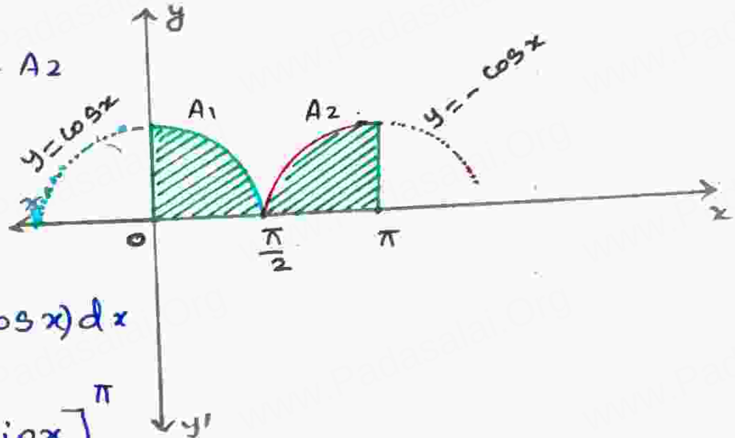
9.53
128

Find the area of the region bounded by x-axis, the curve $y = |\cos x|$, the lines $x = 0$ and $x = \pi$

$$|\cos x| = \begin{cases} \cos x & 0 \le x \le \pi/2 \\ -\cos x & \frac{\pi}{2} < x \le \pi \end{cases}$$

Required area $= A_1 + A_2$



$$= \int_0^{\pi/2} y\,dx + \int_{\pi/2}^{\pi} y\,dx$$

$$= \int_0^{\pi/2} \cos x\,dx + \int_{\pi/2}^{\pi} (-\cos x)\,dx$$

$$= [\sin x]_0^{\pi/2} - [\sin x]_{\pi/2}^{\pi}$$

$$= [\sin \tfrac{\pi}{2} - \sin 0] - [\sin \pi - \sin \tfrac{\pi}{2}]$$

$$= 1 - 0 - 0 + 1 = 2 \text{ sq. units.}$$

9.51
127.

Find the area of the region bounded by the y-axis and the Parabola $x = 5 - 4y - y^2$

$x = 5 - 4y - y^2$

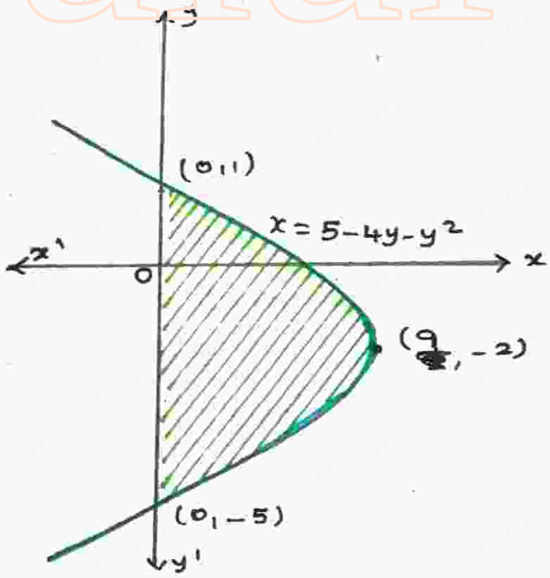
$y^2 + 4y = -x + 5$

$(y+2)^2 - 4 = -x + 5$

$(y+2)^2 = -x + 9$

$(y+2)^2 = -(x - 9)$

∴ The Parabola is open leftward.



Put $x = 0$

$5 - 4y - y^2 = 0$

$y^2 + 4y - 5 = 0$

$(y-1)(y+5) = 0$

$y = 1 \quad y = -5$

$$\text{R. A} = \int_c^d x\,dy = \int_{-5}^{1} (5 - 4y - y^2)\,dy = \left[5y - \frac{\cancel{4}^{2}y^2}{\cancel{2}} - \frac{y^3}{3}\right]_{-5}^{1}$$

$$= \left(5 - 2 - \tfrac{1}{3}\right) - \left(-25 - 50 + \tfrac{125}{3}\right)$$

$$= 3 - \tfrac{1}{3} + 75 - \tfrac{125}{3} = 78 - \tfrac{126}{3} = 78 - 42$$

$$= 36 \text{ sq. units.}$$

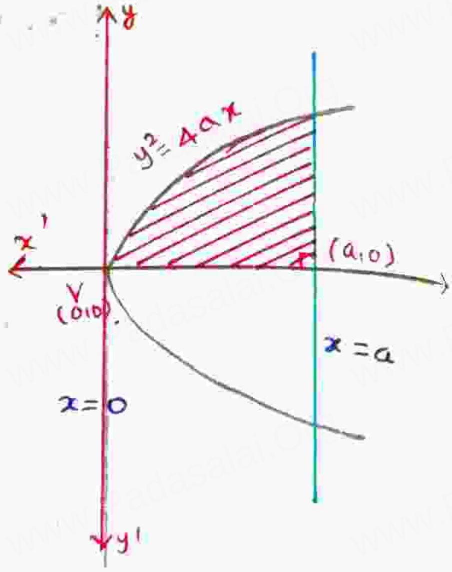
9.50 / 126

Find the area of the region bounded between the parabola $y^2 = 4ax$ and its latus rectum.

$y^2 = 4ax$

$y = 2\sqrt{a}\sqrt{x}$

Limits: $x = 0$ and $x = a$



Since the curve is symmetrical about x-axis.

$$\text{Required area} = 2\int_a^b y\,dx$$

$$= 2\int_0^a 2\sqrt{a}\sqrt{x}\,dx$$

$$= 4\sqrt{a}\left[\frac{2}{3}x^{3/2}\right]_0^a$$

$$= \frac{8}{3}\sqrt{a}\left(a^{3/2} - 0\right) = \frac{8}{3}a^2 \text{ sq. units.}$$

9.49 / 126

Find the area of the region bounded by the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$.

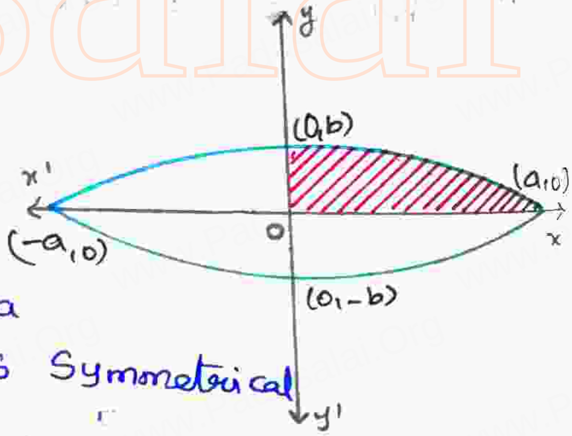
Equation of the ellipse

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$$

Put $y = 0$

$$\frac{x^2}{a^2} = 1$$

$x^2 = a^2 \Rightarrow x = \pm a$



Limits: $x = 0$ and $x = a$

Since the curve is symmetrical about both the axes.

$$\text{Required area} = 4\int_a^b y\,dx$$

$$\frac{y^2}{b^2} = 1 - \frac{x^2}{a^2}$$

$$y^2 = \frac{b^2}{a^2}(a^2 - x^2)$$

$$y = \frac{b}{a}\sqrt{a^2 - x^2}$$

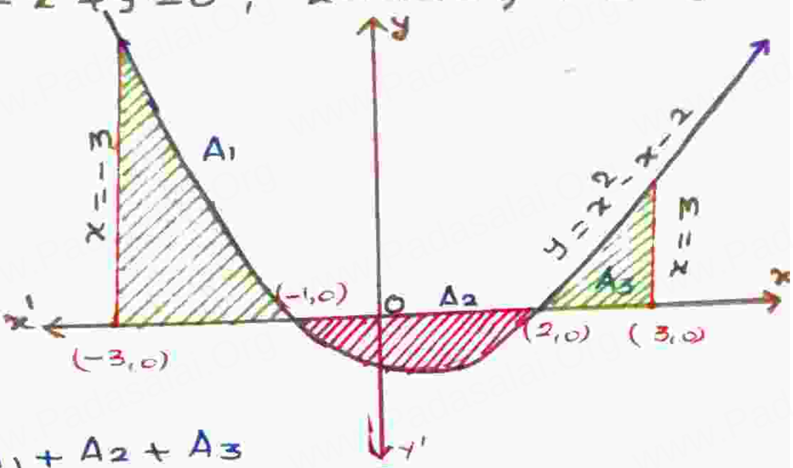
$$= 4\int_0^a \frac{b}{a}\sqrt{a^2 - x^2}\,dx$$

$$= \frac{4b}{a}\left[\frac{x}{2}\sqrt{a^2 - x^2} + \frac{a^2}{2}\sin^{-1}\left(\frac{x}{a}\right)\right]_0^a$$

$$= \frac{4b}{a}\left[0 + \frac{a^2}{2}\sin^{-1}(1)\right] = \frac{4b}{a} \cdot \frac{a^2}{2} \cdot \frac{\pi}{2}$$

$$= \pi ab \text{ sq. units}$$

3/134 Find the area of the region bounded by the curve $2+x-x^2+y=0$, x-axis, $x=-3$ and $x=3$.



$2+x-x^2+y=0$

$y=x^2-x-2$

put $y=0$

$x^2-x-2=0$

$(x-2)(x+1)=0$

$x=-1, 2$

Required area $= A_1+A_2+A_3$

$$=\int_{-3}^{-1} y\,dx - \int_{-1}^{2} y\,dx + \int_{2}^{3} y\,dx$$

$$=\int_{-3}^{-1}(x^2-x-2)dx - \int_{-1}^{2}(x^2-x-2)dx + \int_{2}^{3}(x^2-x-2)dx$$

$$=\left[\frac{x^3}{3}-\frac{x^2}{2}-2x\right]_{-3}^{-1} - \left[\frac{x^3}{3}-\frac{x^2}{2}-2x\right]_{-1}^{2} + \left[\frac{x^3}{3}-\frac{x^2}{2}-2x\right]_{2}^{3}$$

$$=\left\{\left(\frac{-1}{3}-\frac{1}{2}+2\right)-\left(\frac{-27}{3}-\frac{9}{2}+6\right)\right\}-\left\{\left(\frac{8}{3}-\frac{4}{2}-4\right)-\left(\frac{-1}{3}-\frac{1}{2}+2\right)\right\}$$
$$+\left\{\left(\frac{27}{3}-\frac{9}{2}-6\right)-\left(\frac{8}{3}-\frac{4}{2}-4\right)\right\}$$

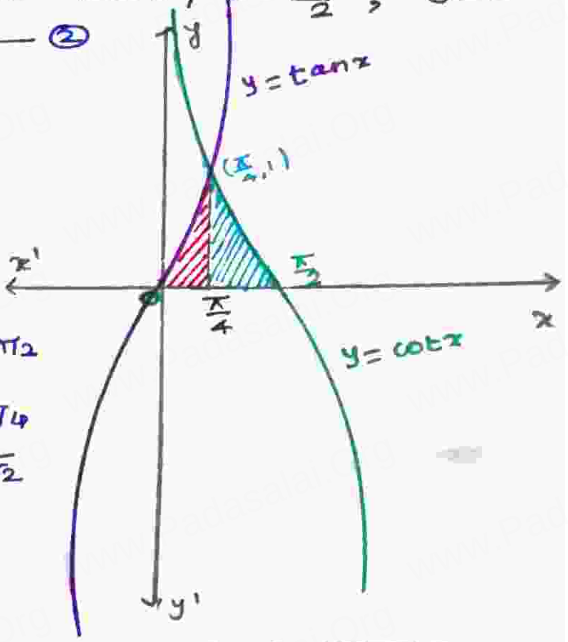
$$=\left\{\frac{26}{3}+\frac{8}{2}-4\right\}-\left\{\frac{9}{3}-\frac{3}{2}-6\right\}+\left\{\frac{19}{3}-\frac{5}{2}-2\right\}$$

$$=\left\{\frac{26}{3}-\frac{9}{3}+\frac{19}{3}\right\}+\left\{+\frac{8}{2}+\frac{3}{2}-\frac{5}{2}\right\}+\left\{-4+6-2\right\}$$

$$=\left\{\frac{36}{3}\right\}+\left\{\frac{6}{2}\right\}+0 = 12+3 = 15 \text{ sq. units.}$$

6/134 Find the area of the region bounded by $y=\tan x$ and $y=\cot x$ and the lines $x=0$, $x=\frac{\pi}{2}$, $y=0$

$y=\tan x$ — ① $y=\cot x$ — ②



$\tan x = \cot x$

$\boxed{x=\pi/4}$

Required area $= \int_a^b y\,dx$

$$=\int_0^{\pi/4}\tan x\,dx + \int_{\pi/4}^{\pi/2}\cot x\,dx$$

$$=\left[\log\sec x\right]_0^{\pi/4} + \left[\log\sin x\right]_{\pi/4}^{\pi/2}$$

$$=\log\sqrt{2}-\cancel{\log 1}+\cancel{\log 1}-\log\frac{1}{\sqrt{2}}$$

$$=\log\sqrt{2}+\log\sqrt{2} = 2\log\sqrt{2}$$

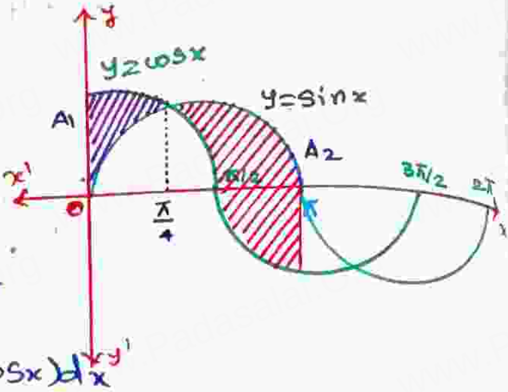
$$=\log 2 \text{ sq. units}$$

$\frac{5}{134}$ Find the area of the region bounded between the curves $y = \sin x$ and $y = \cos x$ and the lines $x = 0$ and $x = \pi$

$y = \sin x \qquad y = \cos x$

$\sin x = \cos x$

$\boxed{x = \pi/4}$



Required area $= A_1 + A_2$

$$= \int_0^{\pi/4} (\cos x - \sin x)\, dx + \int_{\pi/4}^{\pi} (\sin x - \cos x)\, dx$$

$$= \left[\sin x + \cos x\right]_0^{\pi/4} + \left[-\cos x - \sin x\right]_{\pi/4}^{\pi}$$

$$= \left(\frac{1}{\sqrt{2}} + \frac{1}{\sqrt{2}}\right) - (0+1) + (1-0) - \left(-\frac{1}{\sqrt{2}} - \frac{1}{\sqrt{2}}\right)$$

$$= \frac{2}{\sqrt{2}} - 1 + 1 + \frac{2}{\sqrt{2}} = \frac{4}{\sqrt{2}} = 2\sqrt{2} \text{ sq. units.}$$

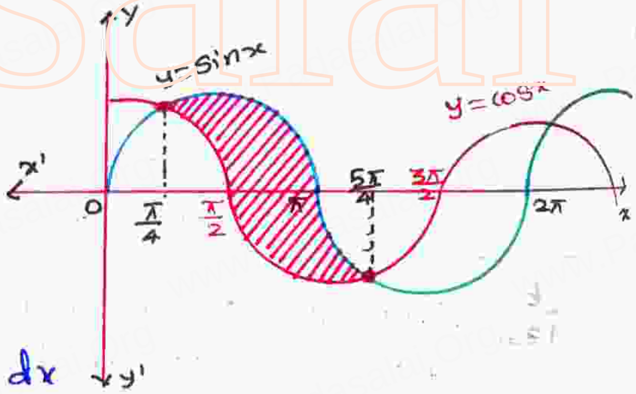
$\frac{9.56}{130}$ Find the area of the region bounded by $y = \cos x$, $y = \sin x$, the lines $x = \frac{\pi}{4}$ and $x = \frac{5\pi}{4}$

$y = \cos x$

$y = \sin x$

Limits:

$x = \frac{\pi}{4}$ and $x = \frac{5\pi}{4}$.



$$R.A = \int_a^b (y_U - y_L)\, dx$$

$$= \int_{\pi/4}^{5\pi/4} (\sin x - \cos x)\, dx$$

$$= \left[-\cos x - \sin x\right]_{\pi/4}^{5\pi/4}$$

$$= \left(-\cos \frac{5\pi}{4} - \sin \frac{5\pi}{4}\right) - \left(-\cos \frac{\pi}{4} - \sin \frac{\pi}{4}\right)$$

$$= \frac{1}{\sqrt{2}} + \frac{1}{\sqrt{2}} + \frac{1}{\sqrt{2}} + \frac{1}{\sqrt{2}} = \frac{4}{\sqrt{2}} = 2\sqrt{2} \text{ sq. units.}$$

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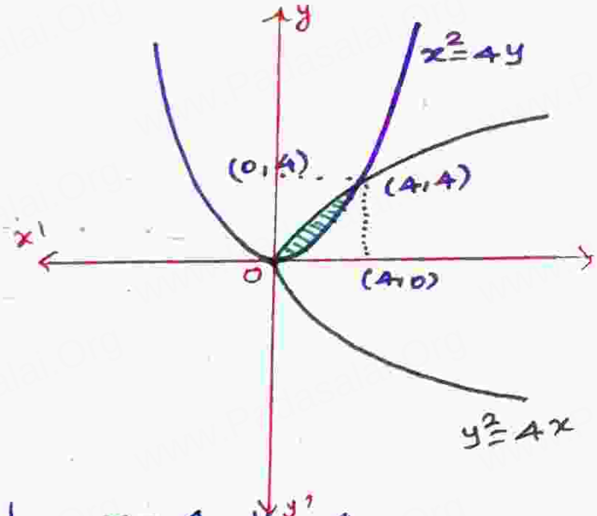
9.54 / 128 Find the area of the region bounded between the Parabolas $y^2 = 4x$ and $x^2 = 4y$.

$y^2 = 4x$ —— ①

$x^2 = 4y$ —— ②



Solving ① & ②

$$\left(\frac{x^2}{4}\right)^2 = 4x$$

$$x^4 = 64x$$

$$x^4 - 64x = 0$$

$$x(x^3 - 64) = 0$$

$$x = 0 \quad x = 4$$

when $x = 0$, $y = 0$ when $x = 4$, $y = 4$

∴ The Point of intersections are $(0,0)$ and $(4,4)$

Limits: $x = 0$ and $x = 4$.

Required area $= \int_a^b (y_U - y_L)\,dx$

$$= \int_0^4 \left(2\sqrt{x} - \frac{x^2}{4}\right)dx = \left[2\cdot\frac{2}{3}x^{3/2} - \frac{x^3}{12}\right]_0^4$$

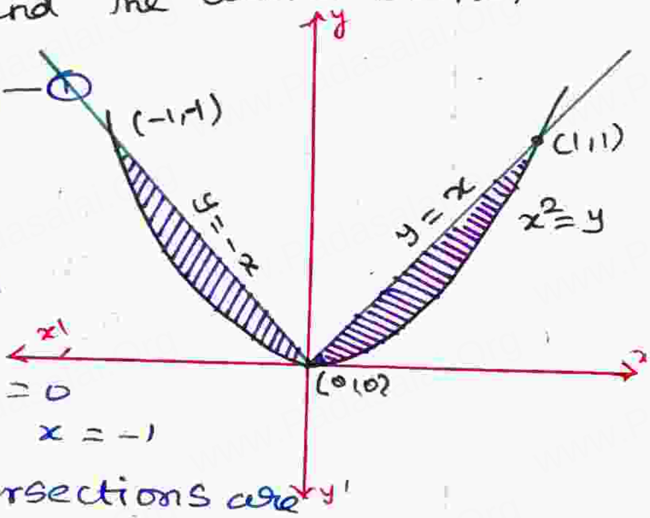
$$= \frac{4}{3}(4)^{3/2} - \frac{64}{12} = \frac{4}{3}(8) - \frac{16}{3} = \frac{32}{3} - \frac{16}{3}$$

$$= \frac{16}{3} \text{ sq. units.}$$

9.55 / 130 Find the area of the region bounded between the Parabola $x^2 = y$ and the curve $y = |x|$

$$y = |x| = \begin{cases} x & x \geq 0 \\ -x & x < 0 \end{cases} \text{ —— ①}$$

$x^2 = y$ —— ②



Solving ① & ②

$x^2 = x$	$x^2 = -x$
$x^2 - x = 0$	$x^2 + x = 0$
$x(x-1) = 0$	$x(x+1) = 0$
$x = 0 \quad x = 1$	$x = 0 \quad x = -1$

∴ The Point of intersections are $(0,0)$ $(1,1)$ & $(-1,-1)$

Since both curves are symmetrical about y-axis

Required area $= 2\int_a^b [y_U - y_L]\,dx$

$$= 2\int_0^1 (x - x^2)\,dx$$

$$= 2\left[\frac{x^2}{2} - \frac{x^3}{3}\right]_0^1 = 2\left[\frac{1}{2} - \frac{1}{3}\right]$$

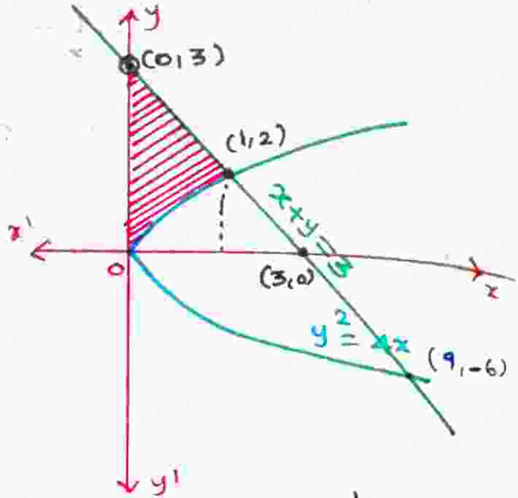
$$= 2\left[\frac{3-2}{6}\right] = \frac{1}{3} \text{ sq. units.}$$

9.58 / 131

Find the area of the region in the first quadrant bounded by the Parabola $y^2 = 4x$ the line $x + y = 3$ and y-axis.

$y^2 = 4x$ ——— ①

$x + y = 3$ ——— ②



Solving ① & ②

$y^2 = 4(3-y)$

$y^2 = 12 - 4y$

$y^2 + 4y - 12 = 0$

$(y+6)(y-2) = 0$

$y = 2 \quad y = -6$

when $y = 2, \; x = 1$

when $y = -6 \quad x = 9$

∴ The Point of intersections are $(1,2)$ and $(9,-6)$.

Take the Limits: $x = 0$ and $x = 1$

Required area $= \int_a^b [y_U - y_L]\, dx$

$$= \int_0^1 [(3-x) - 2\sqrt{x}]\, dx$$

$$= \left[3x - \frac{x^2}{2} - 2 \cdot \frac{2}{3} x^{3/2}\right]_0^1$$

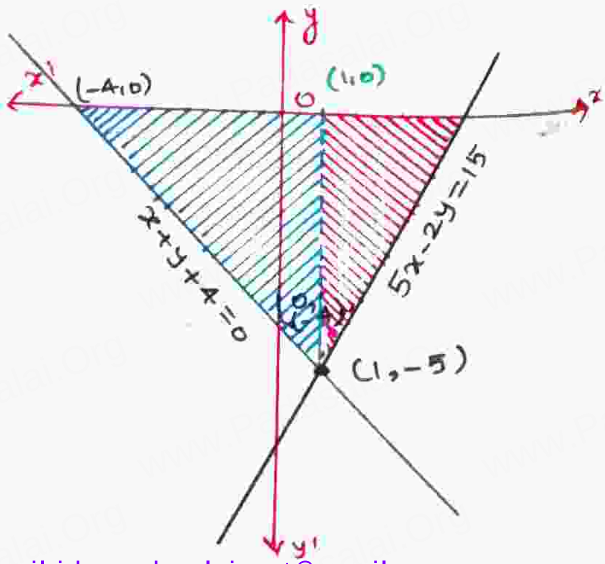
$$= 3 - \frac{1}{2} - \frac{4}{3} = \frac{18-3-8}{6} = \frac{7}{6} \text{ Sq. units.}$$

9.59 / 131.

Find by integration, the area of the region bounded by the lines $5x - 2y = 15$; $x + y + 4 = 0$ and x-axis.

$5x - 2y = 15$ ——— ①

$x + y + 4 = 0$ ——— ②



Solving ① & ②

x	y	1	
-2	-15	5	-2
1	4	1	1

$$\frac{x}{-8+15} = \frac{y}{-15-20} = \frac{1}{5+2}$$

$$\frac{x}{7} = \frac{y}{-35} = \frac{1}{7}$$

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$\therefore (x, y) = (1, -5)$

Required area $= A_1 + A_2$

$$= \left|\int_{-4}^{1} y\,dx\right| + \left|\int_{1}^{3} y\,dx\right|$$

$$= \left|\int_{-4}^{1}(-4-x)dx\right| + \left|\int_{1}^{3} \frac{5}{2}(x-3)dx\right|$$

$$= \left|\left[-4x-\frac{x^2}{2}\right]_{-4}^{1}\right| + \left|\frac{5}{2}\left(\frac{x^2}{2}-3x\right)_{1}^{3}\right|$$

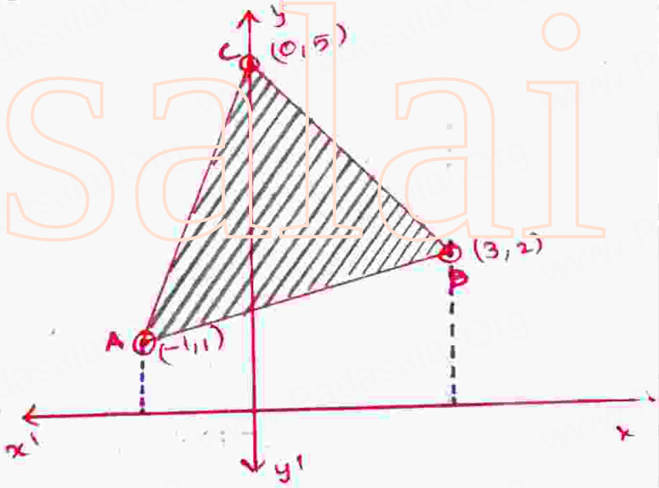
$$= \left|\left(-4-\frac{1}{2}\right)-\left(16-\frac{16}{2}\right)\right| + \frac{5}{2}\left|\left(\frac{9}{2}-9\right)-\left(\frac{1}{2}-3\right)\right|$$

$$= \left|-20+\frac{15}{2}\right| + \frac{5}{2}\left|\frac{8}{2}-6\right|$$

$$= \left|-\frac{25}{2}\right| + \frac{5}{2}\left|-2\right| = \frac{25}{2}+5 = \frac{35}{2} \text{ Sq. units.}$$

9.60 / 132. Using integration, find the area of the region bounded by triangle ABC, whose vertices A, B and C are $(-1,1)$, $(3,2)$ and $(0,5)$ respectively.

$A(-1,1)$ $B(3,2)$ and $C(0,5)$ be the vertices of ΔABC.



Eqn of AB : $\frac{y-1}{2-1} = \frac{x+1}{3+1}$

$\Rightarrow y = \frac{1}{4}(x+5)$

Eqn of BC : $\frac{y-5}{2-5} = \frac{x-0}{3-0}$

$\Rightarrow y = -x+5$

Eqn of AC : $\frac{y-1}{5-1} = \frac{x+1}{0+1}$

$\Rightarrow y = 4x+5.$

$\therefore$ Area of ΔABC = Ar(DACO) + Ar(OCBE) − ar(DABE)

$$A = \int_{-1}^{0}(4x+5)dx + \int_{0}^{3}(-x+5)dx - \frac{1}{4}\int_{-1}^{3}(x+5)dx$$

$$= \left[\frac{4x^2}{2}+5x\right]_{-1}^{0} + \left[-\frac{x^2}{2}+5x\right]_{0}^{3} - \frac{1}{4}\left[\frac{x^2}{2}+5x\right]_{-1}^{3}$$

$$= \left[0-(2-5)\right] + \left[-\frac{9}{2}+15\right] - \frac{1}{4}\left[\left(\frac{9}{2}+15\right)-\left(\frac{1}{2}-5\right)\right]$$

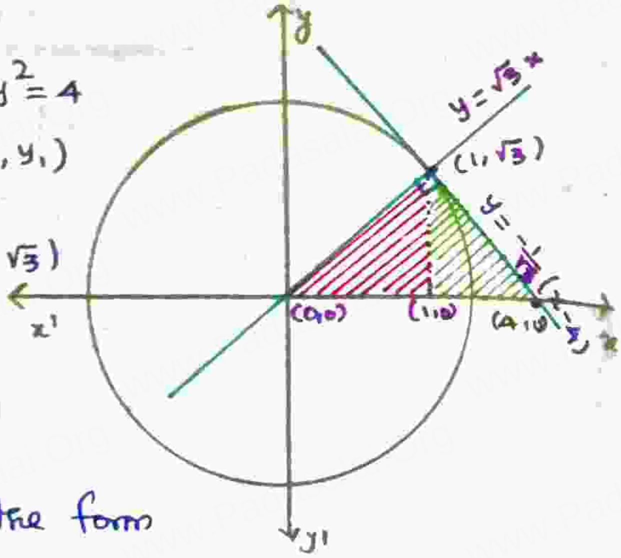
$$= 3+\frac{21}{2} - \frac{1}{4}\left(\frac{39}{2}+\frac{9}{2}\right)$$

$$= 3+\frac{21}{2} - \frac{1}{4}\left(\frac{48}{2}\right) = 3+\frac{21}{2}-6 = \frac{21}{2}-3 = \frac{15}{2}$$

$\therefore$ The Required area $= \frac{15}{2}$ Sq. units.

9.61
133.

Using integration, find the area of the region which is bounded by x-axis, the tangent and normal to the circle $x^2+y^2=4$ drawn at $(1,\sqrt{3})$



Eqn of the Circle : $x^2+y^2=4$

Eqn of the tangent at (x_1,y_1)

$$xx_1+yy_1=4$$

Eqn of the tangent at $(1,\sqrt{3})$

$$\boxed{x+\sqrt{3}y=4}$$

$$y=\frac{-x+4}{\sqrt{3}}$$

$$y=\frac{1}{\sqrt{3}}(-x+4)$$

Eqn of the normal is of the form

$$\sqrt{3}x-y+k=0$$

It Passes through $(1,\sqrt{3})$

$$\sqrt{3}-\sqrt{3}+k=0$$

$$\boxed{k=0}$$

$\therefore$ Eqn of the normal

$$\boxed{y=\sqrt{3}x}$$

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$\therefore$ The lines are $y=\sqrt{3}x$ $y=-\frac{1}{\sqrt{3}}(x-4)$

Point of intersection is $(1,\sqrt{3})$

Required Area $=A_1+A_2$

$$=\int_0^1 y_1dx+\int_1^4 y_2dx$$

$$=\int_0^1\sqrt{3}x\,dx+\int_1^4-\frac{1}{\sqrt{3}}(x-4)dx$$

$$=\left[\sqrt{3}\frac{x^2}{2}\right]_0^1-\frac{1}{\sqrt{3}}\left[\frac{x^2}{2}-4x\right]_1^4$$

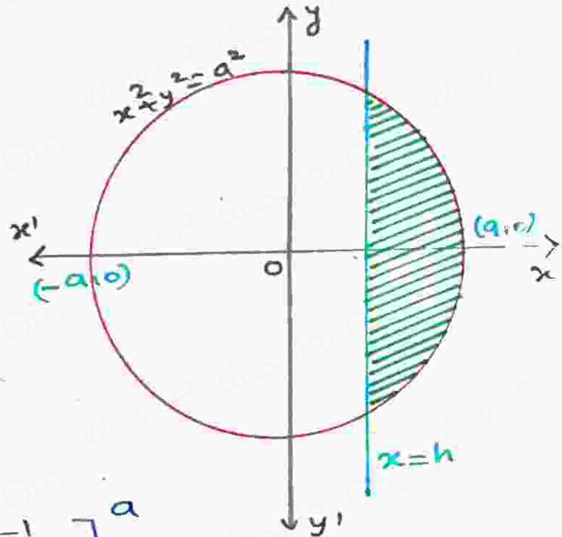
$$=\frac{\sqrt{3}}{2}-\frac{1}{\sqrt{3}}\left((8-16)-\left(\frac{1}{2}-4\right)\right)$$

$$=\frac{\sqrt{3}}{2}-\frac{1}{\sqrt{3}}\left[-8+\frac{7}{2}\right]$$

$$=\frac{\sqrt{3}}{2}-\frac{1}{\sqrt{3}}\left(\frac{-9}{2}\right)=\frac{\sqrt{3}}{2}+\frac{9}{2\sqrt{3}}=\frac{\sqrt{3}}{2}+\frac{3\sqrt{3}}{2}$$

$$=\frac{4\sqrt{3}}{2}=2\sqrt{3}\text{ Sq. units.}$$

9.57 / 131. The region enclosed by the circle $x^2+y^2=a^2$ is divided into two segments by the lines $x=h$. Find the area of the smaller segment.



Equation of the circle

$$x^2+y^2=a^2$$

Limits: $x=h$ and $x=a$

$$y^2=a^2-x^2$$
$$y=\sqrt{a^2-x^2}$$

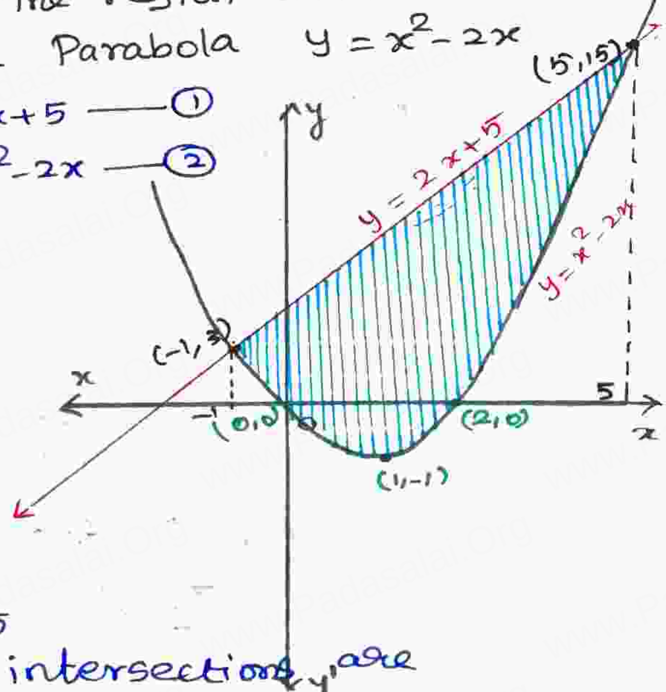
Required area $= \int_a^b y\,dx$

$$= \int_h^a \sqrt{a^2-x^2}\,dx$$
$$= \left[\frac{x}{2}\sqrt{a^2-x^2}+\frac{a^2}{2}\sin^{-1}\frac{x}{a}\right]_h^a$$
$$= 0+\frac{a^2}{2}\sin^{-1}(1)-\left[\frac{h}{2}\sqrt{a^2-h^2}+\frac{a^2}{2}\sin^{-1}\left(\frac{h}{a}\right)\right]$$
$$= \frac{a^2}{2}\cdot\frac{\pi}{2}-\frac{h}{2}\sqrt{a^2-h^2}-\frac{a^2}{2}\sin^{-1}\left(\frac{h}{a}\right)$$
$$= \frac{a^2}{2}\left[\frac{\pi}{2}-\sin^{-1}\left(\frac{h}{a}\right)\right]-\frac{h}{2}\sqrt{a^2-h^2}$$
$$= \frac{a^2}{2}\cos^{-1}\left(\frac{h}{a}\right)-\frac{h}{2}\sqrt{a^2-h^2}$$
$$= \frac{1}{2}\left[a^2\cos^{-1}\left(\frac{h}{a}\right)-h\sqrt{a^2-h^2}\right]$$

Since the curve symmetrical about x-axis

$$R.A = 2\left\{\frac{1}{2}\left(a^2\cos^{-1}\left(\frac{h}{a}\right)-h\sqrt{a^2-h^2}\right)\right\}$$
$$= \left[a^2\cos^{-1}\left(\frac{h}{a}\right)-h\sqrt{a^2-h^2}\right] \text{ sq. units.}$$

4 / 134 Find the area of the region bounded by the line $y=2x+5$ and the Parabola $y=x^2-2x$



$y=2x+5$ ——(1)

$y=x^2-2x$ ——(2)

$$x^2-2x=0$$
$$x(x-2)=0$$
$$x=0 \quad x=2$$

Solving (1) & (2)

$$x^2-2x=2x+5$$
$$x^2-4x-5=0$$
$$(x-5)(x+1)=0$$
$$x=-1,\ 5$$

when $x=-1$, $y=3$

when $x=5$, $y=15$

∴ The Point of intersections are $(-1,3)$ and $(5,15)$

Required area $= \int_a^b [y_U - y_L]\, dx$

$= \int_{-1}^{5} [(2x+5) - (x^2 - 2x)]\, dx$

$= \int_{-1}^{5} (2x + 5 - x^2 + 2x)\, dx$

$= \int_{-1}^{5} (4x + 5 - x^2)\, dx$

$= \left[\frac{4x^2}{2} + 5x - \frac{x^3}{3}\right]_{-1}^{5}$

$= \left(50 + 25 - \frac{125}{3}\right) - \left(2 - 5 + \frac{1}{3}\right)$

$= 75 - \frac{125}{3} + 3 - \frac{1}{3}$

$= 78 - \frac{126}{3} = 78 - 42 = 36$ sq. units.

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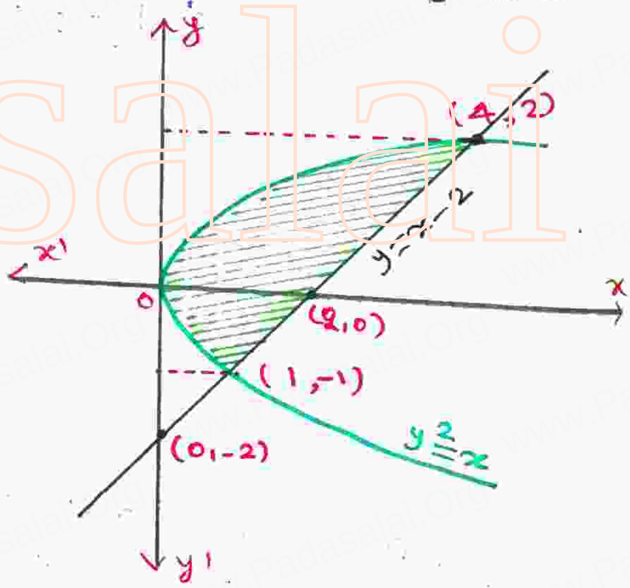
$\frac{7}{134}$ Find the area of the region bounded by the Parabola $y^2 = x$ and the line $y = x - 2$

$y^2 = x$ —— (1)

$y = x - 2$

$x = y + 2$ —— (2)



Solving (1) & (2)

$y^2 = y + 2$

$y^2 - y - 2 = 0$

$(y - 2)(y + 1) = 0$

$y = 2 \quad y = -1$

when $y = 2, \quad x = 4$

when $y = -1 \quad x = 1$

$\therefore$ The Point of intersections are (4,2) and (1,−1)

Required area $= \int_c^d [x_U - x_L]\, dy$

$= \int_{-1}^{4} \{(y+2) - y^2\}\, dy$

$= \left[\frac{y^2}{2} + 2y - \frac{y^3}{3}\right]_{-1}^{2}$

$= \left(\frac{4}{2} + 4 - \frac{8}{3}\right) - \left(\frac{1}{2} - 2 + \frac{1}{3}\right)$

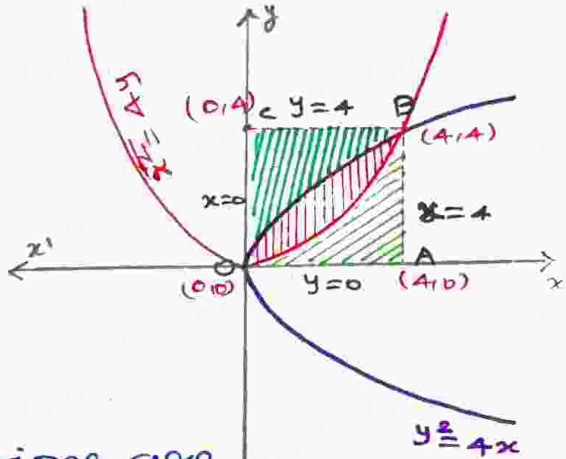
$= \frac{3}{2} + 6 - \frac{9}{3} = \frac{3}{2} + 3$

$= \frac{9}{2}$ sq. units.

8/135. Father of a Family wishes to divide his square field bounded by $x=0$, $x=4$, $y=4$ and $y=0$ along the curve $y^2=4x$ and $x^2=4y$ into three equal parts for his wife, daughter, and son. Is it possible to divide? If so find the area to be divided among them.

$y^2=4x$ ——— ①

$x^2=4y$ ——— ②



Point of intersection :-

$\left(\frac{y^2}{4}\right)^2 = 4y$

$y^4 = 64y$

$y(y^3-64)=0$

$y=0 \quad y=4$

when $y=0$, $x=0$

when $y=4$ $x=4$

∴ The point of intersections are $(0,0)$ and $(4,4)$

(i) Area of the field $OAB = \int_a^b y\,dx$

$= \int_0^4 \frac{x^2}{4}dx = \frac{1}{4}\left[\frac{x^3}{3}\right]_0^4$

$= \frac{1}{4}\left(\frac{64}{3}\right) = \frac{16}{3}$ sq. units.

(ii) Area of the field $OCD = \int_c^d x\,dy$

$= \int_0^4 \frac{y^2}{4}\cdot dy = \frac{1}{4}\left[\frac{y^3}{3}\right]_0^4$

$= \frac{1}{4}\left(\frac{64}{3}\right) = \frac{16}{3}$ square units.

(iii) area of the Remaining Portion

= area of square $OABC - [ar(OAB)+ar(OCB)]$

$= 4^2 - \left(\frac{16}{3}+\frac{16}{3}\right) = 16 - \frac{32}{3} = \frac{48-32}{3} = \frac{16}{3}$

$= \frac{16}{3}$ sq. units

Yes it is possible to divide into three equal parts.

∴ The area $= \frac{16}{3}$ sq. units

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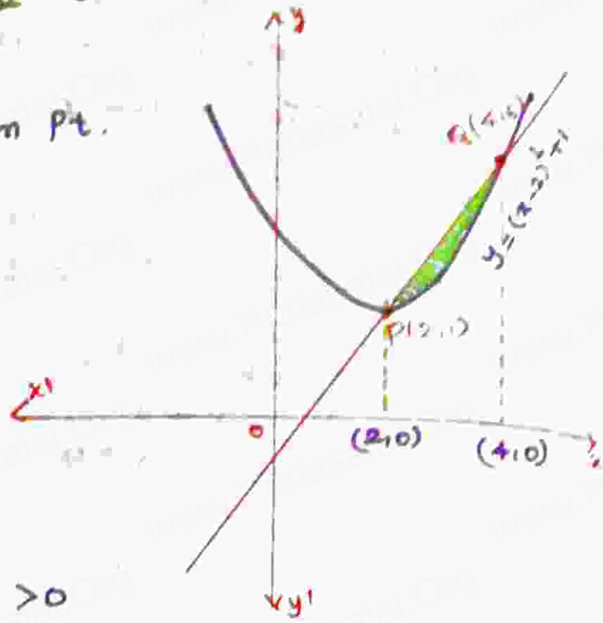
$\frac{9}{135}$ The curve $y=(x-2)^2+1$ has a minimum point at P. A point Q on the curve is such that the slope of PQ is 2. Find the area bounded by the curve and the chord PQ.



(i) To find the minimum Pt.

$$y=(x-2)^2+1$$

$$\frac{dy}{dx}=2(x-2)$$

say $\frac{dy}{dx}=0$

$$2(x-2)=0$$

$$x-2=0$$

$$\boxed{x=2}$$

$$\frac{d^2y}{dx^2}=2(1)=2$$

when $x=2$, $\frac{d^2y}{dx^2}=2>0$

$\therefore$ y is minimum at $x=2$

when $x=2$, $y=1$.

$\therefore$ The minimum Point P is $(2,1)$

Slope of PQ $=m=2$

Eqn of PQ : $y-y_1=m(x-x_1)$

$$y-1=2(x-2)$$

$$y-1=2x-4$$

$$y=2x-3$$

$\therefore$ Eqn of PQ : $\boxed{y=2x-3}$

(ii) Point of intersection of the curve and the chord.

$$2x-3=(x-2)^2+1$$

$$2x-3=x^2-4x+4+1$$

$$x^2-4x+5-2x+3=0$$

$$x^2-6x+8=0$$

$$(x-2)(x-4)=0$$

$$x=2,4$$

when $x=2$, $y=1$

when $x=4$, $y=5$

$\therefore$ The Point of intersections are $(2,1)$ and $(4,5)$

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Required area $= \int_a^b (y_U - y_L)\,dx$

$$= \int_2^4 \{(2x-3) - [(x-2)^2+1]\}\,dx$$

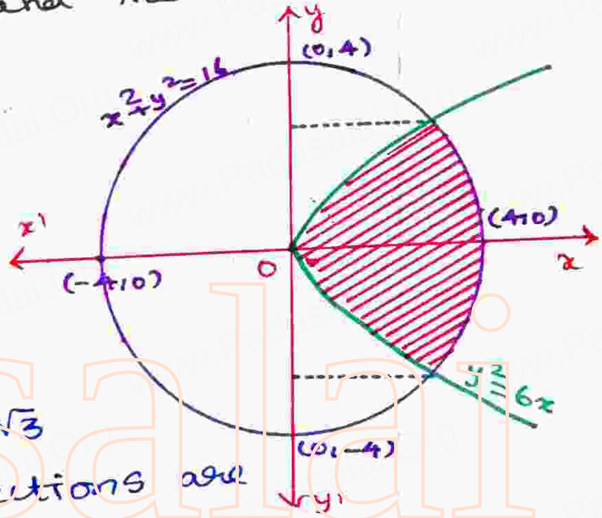
$$= \int_2^4 [2x-3-x^2-4+4x-1]\,dx$$

$$= \int_2^4 (6x-x^2-8)\,dx = \left[\frac{\cancel{6}^3 x^2}{\cancel{2}} - \frac{x^3}{3} - 8x\right]_2^4$$

$$= \left(48 - \frac{64}{3} - 32\right) - \left(12 - \frac{8}{3} - 16\right)$$

$$= 16 - \frac{64}{3} + 4 + \frac{8}{3} = 20 - \frac{56}{3} = \frac{4}{3} \text{ sq. units}$$

$\frac{10}{135}$ Find the area of the region common to the circle $x^2+y^2=16$ and the Parabola $y^2=6x$.



$x^2+y^2=16$ ——— ①

$y^2=6x$ ——— ②

Solving ① & ②

$x^2+6x-16=0$

$(x+8)(x-2)=0$

$x=2$ $\quad \boxed{x \neq -8}$

when $x=2$ $\quad y^2=12$

$y = \pm 2\sqrt{3}$

∴ The Point of intersections are $(2, 2\sqrt{3})$ and $(2, -2\sqrt{3})$

Required area $= \int_c^d (x_1 - x_2)\,dy$

$$= \int_{-2\sqrt{3}}^{2\sqrt{3}} \left[\sqrt{16-y^2} - \frac{y^2}{6}\right] dy$$

$$= 2\int_0^{2\sqrt{3}} \left(\sqrt{16-y^2} - \frac{y^2}{6}\right) dy$$

$$= 2\left[\frac{y}{2}\sqrt{16-y^2} + \frac{16}{2}\sin^{-1}\left(\frac{y}{4}\right) - \frac{y^3}{3(6)}\right]_0^{2\sqrt{3}}$$

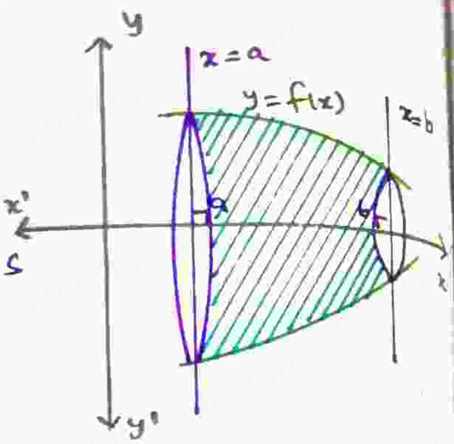
$$= 2\left[\frac{\cancel{2}\sqrt{3}}{\cancel{2}}\sqrt{16-12} + 8\sin^{-1}\left(\frac{\sqrt{3}}{2}\right) - \frac{\cancel{12}^2(2\sqrt{3})}{3(\cancel{6})}\right]$$

$$= 2\left[2\sqrt{3} + 8\cdot\frac{\pi}{3} - \frac{4\sqrt{3}}{3}\right]$$

$$= 2\left[\frac{2\sqrt{3}}{3} + \frac{8\pi}{3}\right] = \frac{4}{3}(\sqrt{3}+4\pi)$$

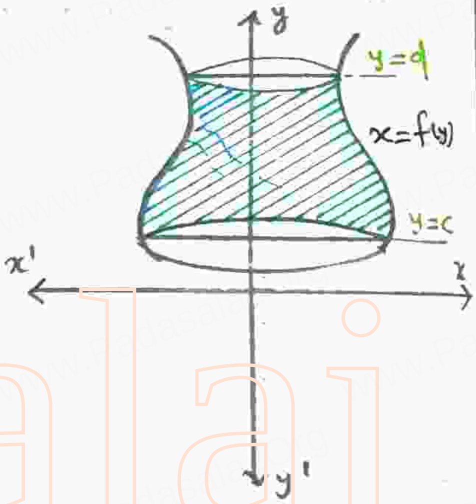
$$= \frac{4}{3}(4\pi+\sqrt{3}) \text{ sq. units.}$$

Volume of a Solid obtained by revolving area about an axis :



* The volume of the Solid of revolution of the region bounded by the curve $y = f(x)$, x-axis and the lines $x = a$ and $x = b$ about x-axis

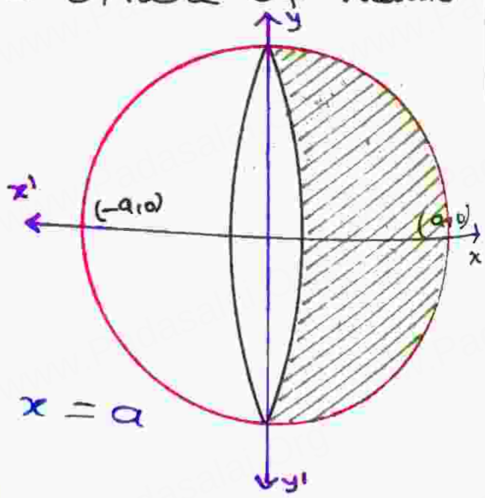
$$\text{Volume} = \int_a^b \pi y^2 \, dx$$



* The volume of the Solid of revolution of the region bounded by the curve $x = f(y)$, y-axis and the lines $y = c$ and $y = d$ about y-axis.

$$\text{Volume} = \int_c^d \pi x^2 \, dy$$

9.62 / 136 Find the volume of a Sphere of radius 'a'



Eqn of the circle

$$x^2 + y^2 = a^2$$

$$y^2 = a^2 - x^2$$

Put $y = 0$ $\quad x^2 = a^2$

$$x = \pm a$$

Limits : $x = -a$ and $x = a$

$$\text{Volume} = \int_a^b \pi y^2 \cdot dx$$

$$= \pi \int_{-a}^{a} (a^2 - x^2) \, dx$$

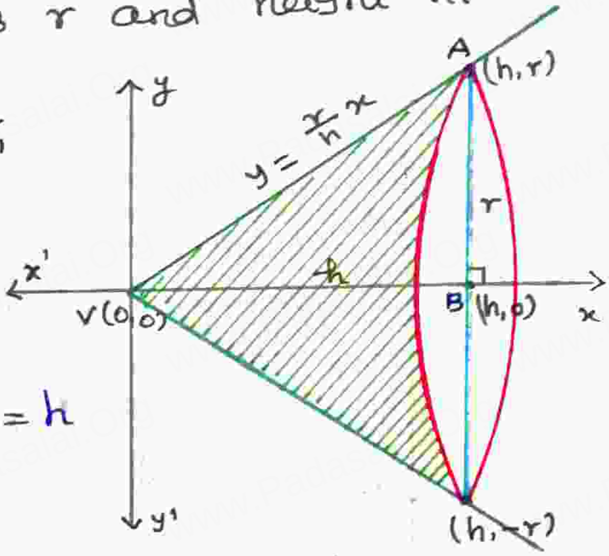
$$= 2\pi \int_0^a (a^2 - x^2) \, dx$$

$$= 2\pi \left[a^2 x - \frac{x^3}{3} \right]_0^a$$

$$= 2\pi \left[a^3 - \frac{a^3}{3} \right] = 2\pi \left(\frac{2a^3}{3} \right) = \frac{4}{3}\pi a^3.$$

$\therefore$ The volume of the sphere is $\frac{4}{3}\pi a^3$ cu.units.

9.63 / 136 Find the volume of a right circular cone of base radius r and height h.



Slope of VA $= \frac{y_2 - y_1}{x_2 - x_1}$

$$= \frac{r-0}{h-0} = \frac{r}{h}$$

Eqn of VA : $y = mx$

$$y = \frac{r}{h} x$$

Limits : $x = 0$ and $x = h$

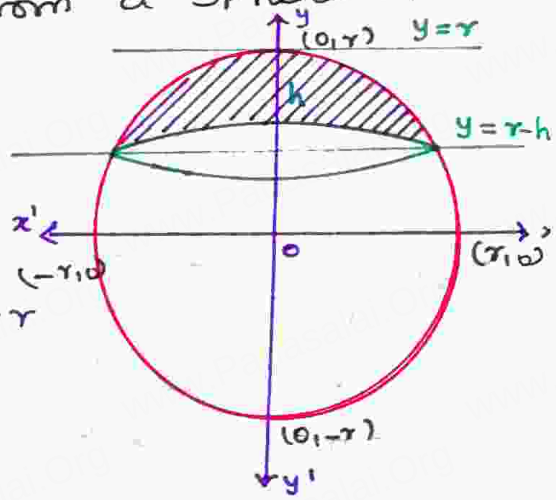
$$\text{volume} = \int_a^b \pi y^2 \, dx$$

$$= \pi \int_0^h \frac{r^2}{h^2} x^2 dx = \frac{\pi r^2}{h^2} \left[\frac{x^3}{3} \right]_0^h$$

$$= \frac{\pi r^2}{h^2} \left(\frac{h^3}{3} \right) = \frac{1}{3}\pi r^2 h \text{ cu. units.}$$

$\therefore$ vol. of the cone $= \frac{1}{3}\pi r^2 h$ cu. units.

9.64 / 137. Find the volume of the spherical cap of height 'h' cut of from a sphere of radius 'r'



Eqn of the circle

$$x^2 + y^2 = r^2$$

$$x^2 = r^2 - y^2$$

Limits : $y = r - h$ and $y = r$

$$\text{volume} = \pi \int_c^d x^2 \, dy$$

$$= \pi \int_{r-h}^{r} (r^2 - y^2) dy$$

$$= \pi \left[r^2 y - \frac{y^3}{3} \right]_{r-h}^{r}$$

$$= \pi \left[\left(r^3 - \frac{r^3}{3} \right) - \left(r^2 (r-h) - \frac{(r-h)^3}{3} \right) \right]$$

$$= \pi \left[\frac{2r^3}{3} - \left[r^3 - hr^2 - \left(\frac{r^3 - 3r^2 h + 3rh^2 - h^3}{3} \right) \right] \right]$$

$$= \pi\left[\frac{2r^3}{3} - \frac{(3r^3 - 3hr^2 - r^3 + 3r^2h - 3rh^2 + h^3)}{3}\right]$$

$$= \frac{\pi}{3}\left[2r^3 - 3r^3 + \cancel{3r^2h} + r^3 - \cancel{3r^2h} + 3rh^2 - h^3\right]$$

$$= \frac{\pi}{3}\left[3r^3 - 3r^3 + 3rh^2 - h^3\right]$$

$$= \frac{\pi}{3}\left[3rh^2 - h^3\right]$$

$$= \frac{1}{3}\pi h^2(3r - h)$$

$\therefore$ The volume of the cap is $\frac{1}{3}\pi h^2(3r-h)$ cu. units.

$\frac{9.65}{137}$ Find the volume of the solid formed by revolving the region bounded by the Parabola $y = x^2$, x-axis, ordinates $x = 0$ and $x = 1$ about x-axis.

$$y = x^2$$
$$y^2 = x^4$$

limits : $x = 0$ and $x = 1$

$$\text{Volume} = \pi\int_a^b y^2\,dx$$

$$= \pi\int_0^1 x^4\,dx$$

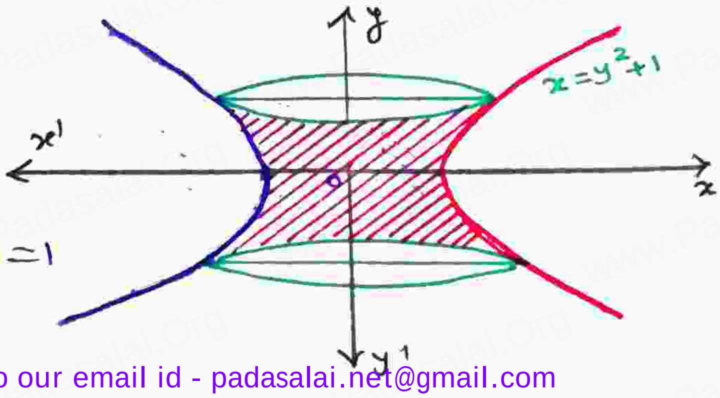
$$= \pi\left[\frac{x^5}{5}\right]_0^1$$

$$= \frac{\pi}{5} \text{ cu. units.}$$

$\frac{9.67}{138}$ Find the volume of the solid generated by revolving about y-axis the origin bounded between the Parabola $x = y^2 + 1$ the y-axis and the lines $y = 1$ and $y = -1$

$$x = y^2 + 1$$
$$x^2 = (y^2+1)^2$$

limits : $y = -1$ and $y = 1$



$$\text{Volume} = \pi\int_{c}^{d} x^2\,dy = \pi\int_{-1}^{1}(y^2+1)^2\,dy$$

$$= 2\pi\int_{0}^{1}(y^2+1)^2\,dy = 2\pi\int_{0}^{1}(y^4+2y^2+1)\,dy$$

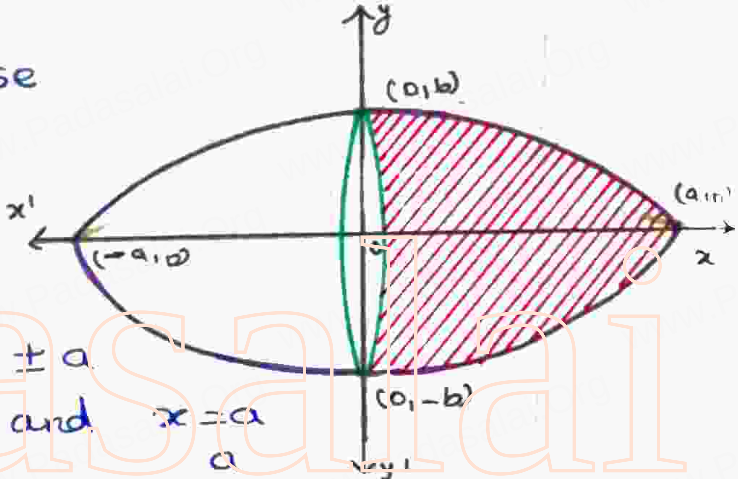
$$= 2\pi\left[\frac{y^5}{5}+\frac{2y^3}{3}+y\right]_0^1$$

$$= 2\pi\left[\frac{1}{5}+\frac{2}{3}+1\right] = 2\pi\left[\frac{3+10+15}{15}\right]$$

$$= \frac{56\pi}{15} \text{ cu. units.}$$

9.66
137

Find the volume of the solid formed by revolving the region bounded by the ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$; $a>b$ about the major axis.



Eqn of the ellipse

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$$

Put $y=0$

$$\frac{x^2}{a^2}=1$$

$$x^2=a^2 \Rightarrow x=\pm a$$

Limits: $x=-a$ and $x=a$

$$\text{Volume} = \pi\int_{a}^{b} y^2\,dx = \pi\int_{-a}^{a}\frac{b^2}{a^2}(a^2-x^2)\,dx$$

$$= 2\frac{b^2}{a^2}\pi\int_{0}^{a}(a^2-x^2)\,dx$$

$$= 2\frac{\pi b^2}{a^2}\left[a^2x-\frac{x^3}{3}\right]_0^a$$

$$= \frac{2\pi b^2}{a^2}\left[a^3-\frac{a^3}{3}\right]$$

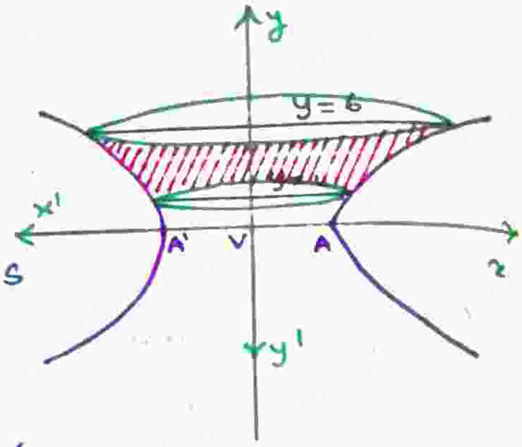
$$= \frac{2\pi b^2}{a^2}\left[\frac{2a^3}{3}\right] = \frac{4}{3}\pi ab^2 \text{ cu. units}$$

$\therefore$ The volume of the ellipse $= \frac{4}{3}\pi ab^2$ cu. units

[Revolving about major axis]

9.68
138

Find the volume of the solid generated by revolving about y-axis the region bounded between the curve $y = \frac{3}{4}\sqrt{x^2-16}$, $x \geq 4$, the y-axis, and the lines $y=1$ and $y=6$.



$$y = \frac{3}{4}\sqrt{x^2-16}$$

$$y^2 = \frac{9}{16}(x^2-16)$$

$\Rightarrow \frac{x^2}{16} - \frac{y^2}{9} = 1$ which is a hyperbola.

Limits: $y=1$ and $y=6$

$$\text{volume} = \pi\int_c^d x^2\,dy$$

$$\frac{x^2}{16} = 1+\frac{y^2}{9}$$

$$x^2 = \frac{16}{9}(9+y^2)$$

$$x = \frac{4}{3}\sqrt{9+y^2}$$

$$= \pi\int_1^6 \frac{16}{9}(9+y^2)\,dy$$

$$= \frac{16\pi}{9}\left[9y+\frac{y^3}{3}\right]_1^6$$

$$= \frac{16\pi}{9}\left[\left(54+\frac{216}{3}\right)-\left(9+\frac{1}{3}\right)\right]$$

$$= \frac{16\pi}{9}\left[54+72-9-\frac{1}{3}\right] = \frac{16\pi}{9}\left[\frac{350}{3}\right]$$

$$= \frac{5600\pi}{27} \text{ cu. units}$$

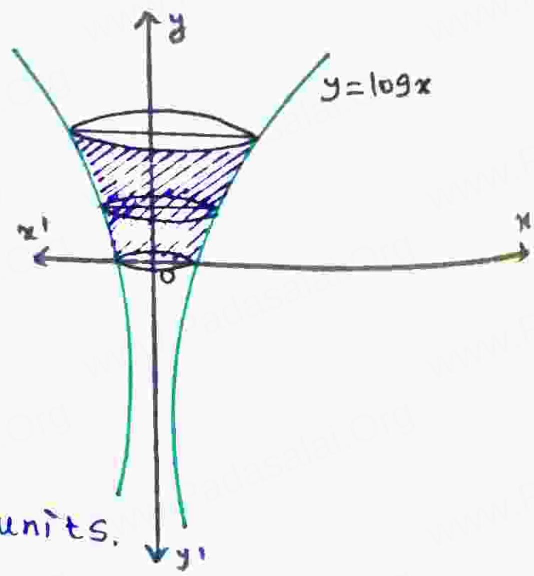
9.69
139.

Find the volume of the solid generated by revolving about y-axis the region bounded by the curves $y=\log x$, $y=0$, $x=0$ and $y=2$

$$y = \log x$$

$$e^y = x$$

$$x^2 = e^{2y}$$



Limits: $y=0$ and $y=2$

$$\text{volume} = \pi\int_c^d x^2\,dy$$

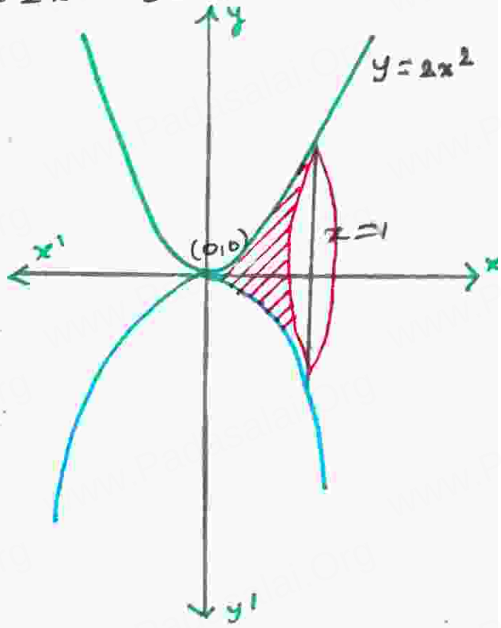
$$= \pi\int_0^2 e^{2y}\,dy$$

$$= \pi\left[\frac{e^{2y}}{2}\right]_0^2$$

$$= \frac{\pi}{2}(e^4-1) \text{ cu. units.}$$

Ex : 9.9

$\frac{1}{139}$ Find, by integration, the volume of the solid generated by revolving about the x-axis, the region enclosed by $y = 2x^2$, $y = 0$ and $x = 1$.



$y = 2x^2$

$y^2 = 4x^4$

Limits : $x = 0$ and $x = 1$

$$\text{volume} = \pi \int_a^b y^2\, dx$$

$$= \pi \int_0^1 4x^4 . dx$$

$$= \pi \left[\frac{4x^5}{5}\right]_0^1$$

$$= \frac{4\pi}{5} \text{ cu. units.}$$

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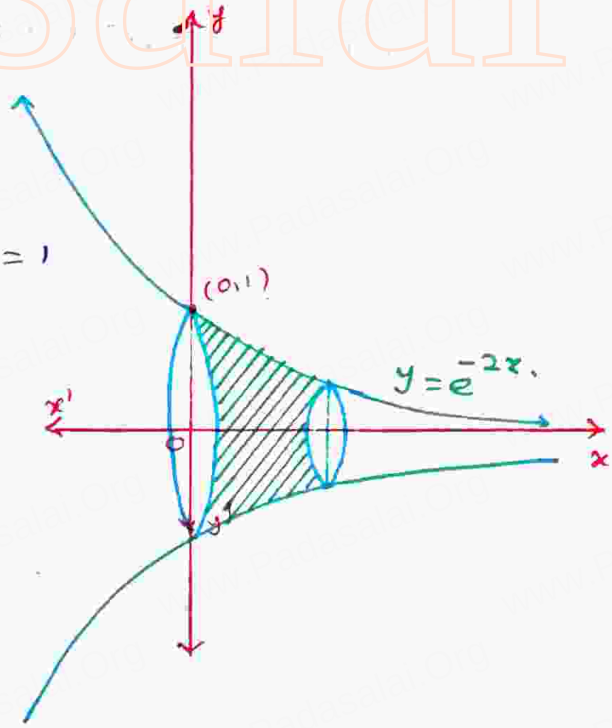
$\frac{2}{139}$ Find, by integration, the volume of the solid generated by revolving about the x-axis, the region enclosed by $y = e^{-2x}$ $y = 0$, $x = 0$ and $x = 1$



$y = e^{-2x}$

$y^2 = (e^{-2x})^2$

$= e^{-4x}.$

Limits: $x = 0$ and $x = 1$

$$\text{volume} = \pi \int_a^b y^2\, dx$$

$$= \pi \int_0^1 e^{-4x} . dx$$

$$= \pi \left[-\frac{e^{-4x}}{4}\right]_0^1$$

$$= \frac{\pi}{4}\left[-e^{-4} + 1\right]$$

$$= \frac{\pi}{4}(1 - e^{-4}) \text{ cu. units.}$$

$\frac{3}{139}$ Find, by integration, the volume of the solid generated by revolving about the y-axis the region enclosed by $x^2 = 1+y$ and $y = 3$.

$x^2 = y+1$

Limits: $y = -1$ and $y = 3$

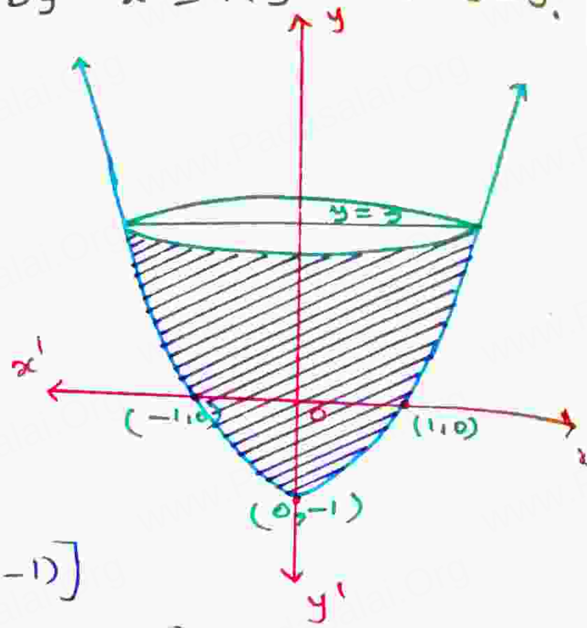
$$\text{volume} = \pi \int_c^d x^2 \, dy$$

$$= \pi \int_{-1}^{3} (y+1) \, dy$$

$$= \pi \left[\frac{y^2}{2} + y \right]_{-1}^{3}$$

$$= \pi \left[\left(\frac{9}{2} + 3 \right) - \left(\frac{1}{2} - 1 \right) \right]$$

$$= \pi \left[\frac{8}{2} + 4 \right] = 8\pi \text{ cu. units.}$$



$\frac{4}{139}$ The region enclosed between the graphs of $y = x$ and $y = x^2$ is denoted by R, find the volume generated when R is rotated through $360°$ about x-axis.

$y = x$ —— ① $y = x^2$ —— ②

Solving ① & ②

$x^2 - x = 0$

$x(x-1) = 0$

$x = 1 \quad x = 0$

Limits: $x = 0 \quad x = 1$

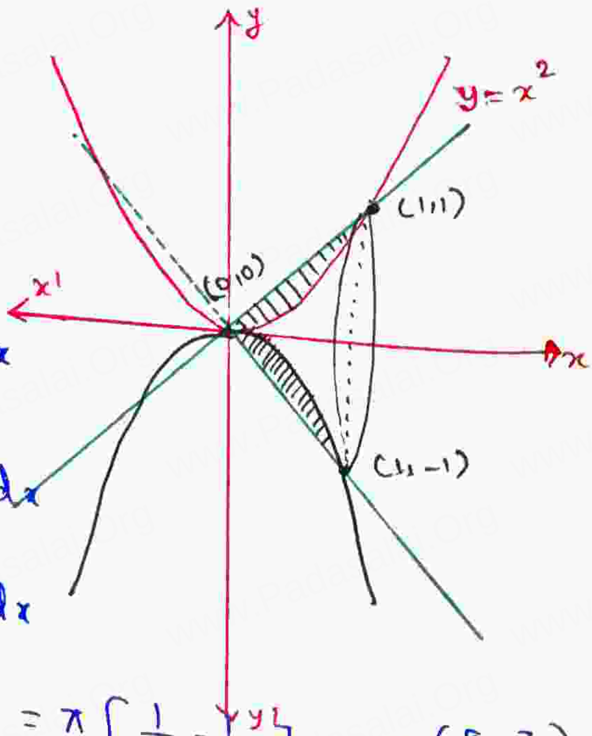
$$\text{Volume} = \pi \int_a^b (y_1^2 - y_2^2) \, dx$$

$$= \pi \int_0^1 [(x)^2 - (x^2)^2] \, dx$$

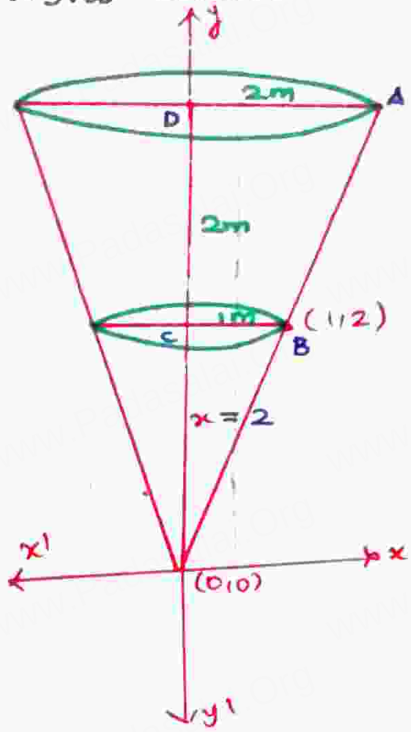
$$= \pi \int_0^1 (x^2 - x^4) \, dx$$

$$= \pi \left[\frac{x^3}{3} - \frac{x^5}{5} \right]_0^1 = \pi \left[\frac{1}{3} - \frac{1}{5} \right] = \pi \left(\frac{5-3}{15} \right)$$

$$= \frac{2\pi}{15} \text{ cu. units.}$$



$\frac{5}{139}$ Find, by integration, the volume of the container which is in the shape of a right circular conical frustum.



Consider the ΔADO and ΔBCO

$$\frac{x}{1} = \frac{x+2}{2}$$

$$2x = x+2$$

$$\boxed{x=2}$$

$\therefore$ B is (1, 2) and A(2, 4)

Equation of OA

$$\frac{y-y_1}{y_2-y_1} = \frac{x-x_1}{x_2-x_1}$$

$$\frac{y-2}{4-2} = \frac{x-1}{2-1}$$

$$\frac{y-2}{2} = x-1$$

$$y - \cancel{2} = 2x - \cancel{2}$$

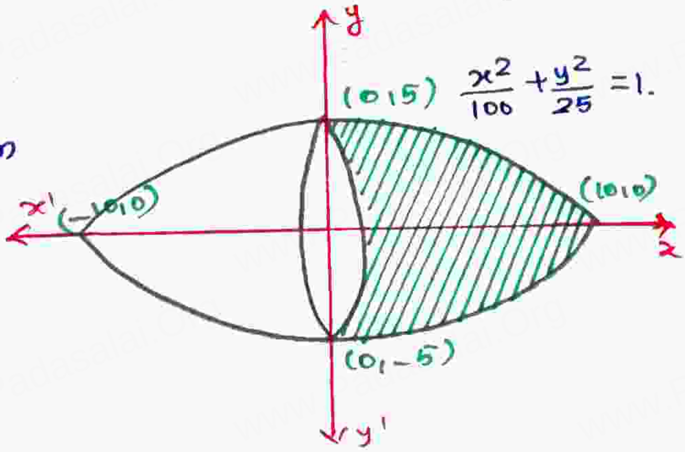
$$\boxed{y = 2x} \qquad y^2 = 4x^2$$

Volume of the frustum $= \pi \int_c^d x^2\, dy$

$$= \pi \int_2^4 \frac{y^2}{4}\, dy = \frac{\pi}{4}\left[\frac{y^3}{3}\right]_2^4$$

$$= \frac{\pi}{4}\left(\frac{64}{3} - \frac{8}{3}\right) = \frac{\pi}{4}\left(\frac{56}{3}\right) = \frac{14\pi}{3} \text{ cu. units.}$$

$\frac{6}{139}$ A watermelon has an ellipsoid shape which can be obtained by revolving an ellipse with major-axis 20cm and minor axis 10cm about its major axis. Find its volume using integration.



$2a = 20 \qquad 2b = 10$

$a = 10\text{cm} \qquad b = 5\text{cm}$

Eqn of the ellipse

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$$

$$\frac{x^2}{100} + \frac{y^2}{25} = 1$$

Put $y = 0 \Rightarrow \frac{x^2}{100} = 1$

$$x^2 = 100$$

$$\boxed{x = \pm 10}$$

$$\frac{y^2}{25} = 1 - \frac{x^2}{100}$$

$$y^2 = \frac{25}{100}(100 - x^2)$$

$$= \frac{1}{4}(100 - x^2)$$

Limits: $x = -10$ and $x = 10$

$$\text{Required volume} = \pi \int_a^b y^2\,dx$$

$$= \frac{\pi}{4} \int_{-10}^{10} (100 - x^2)\,dx$$

$$= \frac{\pi}{4} \times 2 \int_0^{10} (100 - x^2)\,dx$$

$$= \frac{\pi}{2} \left[100x - \frac{x^3}{3}\right]_0^{10}$$

$$= \frac{\pi}{2} \left(1000 - \frac{1000}{3}\right)$$

$$= \frac{\pi}{2} \left(\frac{2(1000)}{3}\right)$$

$$= \frac{1000}{3} \pi \text{ cu. units.}$$

$$\therefore \text{Volume} = \frac{1000\pi}{3} \text{ cu. units.}$$

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