Padasalai

Higher Secondary - First Year – Mathematics

Chapter : 1 - Sets, Relations and Functions- Key Points

• Set
– Subset, super set, trivial subset, proper subset, improper subset
– Empty set, power set, universal set, singleton set, finite set, infinite set
– Cardinality of a set
– Union, Intersection, Complement, Set Difference, Symmetric Difference
– Properties and De Morgan Laws
– Cartesian Product
Intervals
– Constants, dependent and independent variables
– Open, Closed, finite and infinite intervals and neighbourhoods;

• Relations
– Domain and range of relation
– Extreme relations (empty and universal)
– Inverse of a relation
– Reflexive, Symmetric, Transitive, Equivalence Relations

• Functions
– Definition, domain, co-domain, range, image, pre-image,
– Tabular, graphical, analytical and piecewise representations,
– Identity function, constant function, zero function, modulus function, signum function, greatest integer function, smallest integer function,
– Injective, surjective and bijective functions,
– Vertical test and Horizontal test,
– Composition of functions, inverse of a function,
– Addition and multiplication of real valued functions,
– Polynomial function, linear function, exponential function, logarithmic function, rationalfunction, reciprocal function,
– Odd and Even functions.

Graphing functions
– Reflection, translation, dilation
– Drawing graph of some seems to be complicated functions.

Prepared by, S.Saravana Kumar, Maths Teacher, Jazz Public Matric Higher Secondary School, Melur, Madurai District, Phone : 7092681321, 9597410308

Sets: a *set* is a collection of well-defined objects.

Exp :

(i) The collection of all beautiful flowers in Ooty Rose Garden.
(ii) The collection of all old men in Tamilnadu.

the ***union*** of two sets A and B is denoted by $A \cup B$ and is defined as
$A \cup B = \{x : x \in A \text{ or } x \in B\}$

the ***intersection*** as
$A \cap B = \{x : x \in A \text{ and } x \in B\}$.

Two sets A and B are ***disjoint*** if they do not have any common element. That is, A and B are disjoint if $A \cap B = \emptyset$.

If A is a set, then the set of all subsets of A is called the ***power set*** of A and is usually denoted as $P(A)$. That is, $P(A) = \{B : B \subseteq A\}$. The number of elements in $P(A)$ is $2n$, where n is the number of elements in A.

all sets under consideration in a mathematical process are assumed to be subsets of some fixed set. This basic set is called the ***universal set.*** For example, depending on the situation, for the set of prime numbers, the universal set can be any one of the sets containing the set of prime numbers.

The ***complement*** of A with respect to U is denoted as $A\_$ or Ac and defined as
$A\_ = \{x : x \in U \text{ and } x /\in A\}$.
The ***set difference*** of the set A to the set B is denoted by either $A - B$ or $A \_ B$ and is defined as
$A - B = \{a : a \in A \text{ and } a /\in B\}$.
Note that,
(i) $U - A = A\_$ (ii) $A - A = \emptyset$ (iii) $\emptyset - A = \emptyset$ (iv) $A - \emptyset = A$ (v) $A - U = \emptyset$.

The ***symmetric difference*** between two sets A and B is denoted by $A\Delta B$ and is defined as $A\Delta B = (A - B) \cup (B - A)$. Actually the elements of $A\Delta B$ are the elements of $A \cup B$ which are not in $A \cap B$. Thus $A\Delta B = (A \cup B) - (A \cap B)$.

A set X is said to be a ***finite*** *set* if it has k elements for some $k \in W$. In this case, we say the finite set X is of ***cardinality*** k and is denoted by $n(X)$. A set is an ***infinite set*** if it is not finite. For an infinite set A, the cardinality is infinity. If $n(A) = 1$, then it is called a ***singleton set***. Note that

$n(\emptyset) = 0$ and $n(\{\emptyset\}) = 1$

Prepared by, S.Saravana Kumar, Maths Teacher, Jazz Public Matric Higher Secondary School, Melur, Madurai District, Phone : 7092681321, 9597410308

Properties of Set Operations

We now list out some of the properties.

Commutative

(i) $A \cup B = B \cup A$ (ii) $A \cap B = B \cap A$.

Associative

(i) $(A \cup B) \cup C = A \cup (B \cup C)$ (ii) $(A \cap B) \cap C = A \cap (B \cap C)$.

Distributive

(i) $A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$ (ii) $A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$.

Identity

(i) $A \cup \emptyset = A$ (ii) $A \cap U = A$.

Idempotent

(i) $A \cup A = A$ (ii) $A \cap A = A$.

Absorption

(i) $A \cup (A \cap B) = A$ (ii) $A \cap (A \cup B) = A$.

De Morgan Laws

(i) $(A \cup B)\_ = A\_ \cap B\_$ (ii) $(A \cap B)\_ = A\_ \cup B\_$

(iii) $A - (B \cup C) = (A - B) \cap (A - C)$ (iv) $A - (B \cap C) = (A - B) \cup (A - C)$.

On Symmetric Difference

(i) $A \Delta B = B \Delta A$ (ii) $(A \Delta B)\Delta C = A \Delta (B \Delta C)$

(iii) $A \cap (B \Delta C) = (A \cap B)\Delta(A \cap C)$.

On Empty Set and Universal Set

(i) $\emptyset\_ = U$ (ii) $U\_ = \emptyset$

(iii) $A \cup A\_ = U$ (iv) $A \cap A\_ = \emptyset$

(v) $A \cup U = U$ (vi) $A \cap U = A$.

On Cardinality

(i) For any two finite sets A and B, $n(A \cup B) = n(A) + n(B) - n(A \cap B)$.

(ii) If A and B are disjoint finite sets, then $n(A \cup B) = n(A) + n(B)$.

(iii) For any three finite sets A,B and C,

$n(A \cup B \cup C) = n(A) + n(B) + n(C) - n(A \cap B) - n(A \cap C) - n(B \cap C) + n(A \cap B \cap C)$.

Constants and Variables, Intervals and Neighbourhoods

Constants and VariablesA quantity that remains unaltered throughout a mathematical process is called a ***constant.*** A quantitythat varies in a mathematical process is called a ***variable.*** A variable is **an *independent variabl***e whenit takes any arbitrary (independent) value not depending on any other variables, whereas if its value depends on other variables, then it is called a ***dependent variable.***

examples:

(i) area of a rectangle $A = lb$. (ii) area of a circle $A = \pi r^2$. (iii) volume of a cuboid $V = lbh$

Intervals and Neighbourhoods

The system R of real numbers can be represented by the points on a line and a point on the line can be related to a unique real number as in Figure 1.2. By this, we mean that any real number can be identified as a point on the line. With this identification we call the line as the ***real line.***

Prepared by, S.Saravana Kumar, Maths Teacher, Jazz Public Matric Higher Secondary School, Melur, Madurai District, Phone : 7092681321, 9597410308

Definition 1.1
A subset I of R is said to be an *interval* if
(i) I contains at least two elements and
(ii) $a, b \in I$ and $a < c < b$ then $c \in I$.
Geometrically, intervals correspond to rays and line segments on the real line.

Type of Intervals

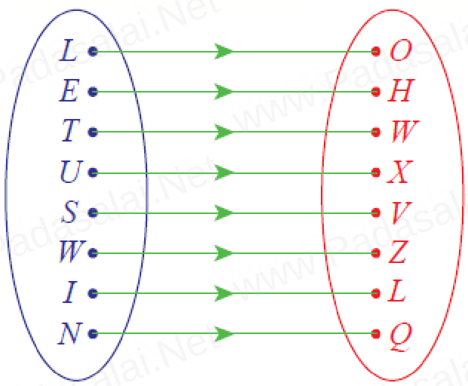
Interval	Notation	Set	Diagramatic Representation
finite	(a, b)	$\{x : a < x < b\}$	a b
	$[a, b]$	$\{x : a \leq x \leq b\}$	a b
	$(a, b]$	$\{x : a < x \leq b\}$	a b
	$[a, b)$	$\{x : a \leq x < b\}$	a b
infinite	(a, ∞)	$\{x : a < x < \infty\}$	a
	$[a, \infty)$	$\{x : a \leq x < \infty\}$	a
	$(-\infty, b)$	$\{x : -\infty < x < b\}$	b
	$(-\infty, b]$	$\{x : -\infty < x \leq b\}$	b
	$(-\infty, \infty)$ or $\mathbb{R}$	$\{x : -\infty < x < \infty\}$ or the set of real numbers	

Neighbourhood

Neighbourhood of a point 'a' is any open interval containing 'a'. In particular, if _ is a positive number, usually very small, then the _-neighbourhood of 'a' is the open interval $(a - E, a + \text{E})$. The set $(a - E, a + E) - \{a\}$ is called ***deleted neighbourhood*** of 'a' and it is denoted as $0 < |x - a| < E$

Relations

Cryptography: The study of the techniques used in creating coding and decoding these ciphers is called cryptography. One of the earliest methods of coding a message was a simple substitution. For example, eachletter in a message might be replaced by the letter that appears three places later in the alphabet

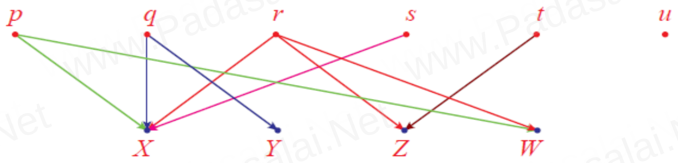


Definition of Relation

(i)

	p	p	q	q	r	r	r	s	t
	X	W	X	Y	X	Z	W	X	Z

(ii)



(iii) $\{ (p, X), (p, W), (q, X), (q, Y), (r, X), (r, Z), (r, W), (s, X), (t, Z) \}$

(iv) $pRX, pRW, qRX, qRY, rRX, rRZ, rRW, sRX, tRZ.$

Definition 1.2

Let A and B be any two non-empty sets. A ***relation*** R from A to B is defined as a subset of the Cartesian product of A and B. Symbolically $R \subseteq A \times B$.

The domain of a relation is a subset of the first set in the Cartesian product and the range is a subset of second set. Usually we call the second set as **co-domain** of the relation.

For any set A, $\emptyset$ and $A \times A$ are subsets of $A \times A$. These two relations are called ***extreme relations.*** The former relation is an ***empty relation*** and the later is an ***universal relation***

Type of Relations

Consider the following examples:

(i) Let $S = \{1, 2, 3, 4\}$ and $R = \{(1, 1), (1, 3), (2, 3)\}$ on S.

(ii) Let $S = \{1, 2, 3, . . . 10\}$ and define "m is related to n, if m divides n".

(iii) Let C be the set of all circles in a plane and define "a circle C is related to a circle C_, if theradius of C is equal to the radius of C_".

(iv) In the set S of all people define "a is related to b, if a is a brother of b".

(v) Let S be the set of all people. Define the relation on S by the rule "mother of".

Definition 1.3

Let S be any non-empty set. Let R be a relation on S. Then

- R is said to be ***reflexive*** if a is related to a for all $a \in S$.
- R is said to be ***symmetric*** if a is related to b implies that b is related to a.
- R is said to be ***transitive*** if "a is related to b and b is related to c" implies that a is related to c.

These three relations are called ***basic relations***.

Let S be any non-empty set. Let R be a relation on S. Then R is

- **reflexive if "$(a, a) \in R$ for all $a \in S$".**
- **symmetric if "$(a, b) \in R \Rightarrow (b, a) \in R$".**
- **transitive if "$(a, b), (b, c) \in R \Rightarrow (a, c) \in R$".**

Prepared by, S.Saravana Kumar, Maths Teacher, Jazz Public Matric Higher Secondary School, Melur, Madurai District, Phone : 7092681321, 9597410308

Definition 1.4
Let S be any set. A relation on S is said to be an ***equivalence relation*** if it is reflexive, symmetric and transitive.

Let us consider the following two relations.
(1) In the set $S1$ of all people, define a relation $R1$ by the rule: "a is related to b, if a is a brother of b".
(2) In the set $S2$ of all males, define a relation $R2$ by the rule: "a is related to b, if a is a brother of b".

Definition 1.5
If R is a relation from A to B, then the relation $R-1$ defined from B to A by
$R^{-1} = \{(b, a) : (a, b) \in R\}$ is called the ***inverse*** of the relation R.

Functions
Definition 1.6
Let A and B be two sets. A relation f from A to B, a subset of $A\times B$, is called a function from A to B if it satisfies the following:
(i) for all $a \in A$, there is an element $b \in B$ such that $(a, b) \in f$.
(ii) if $(a, b) \in f$ and $(a, c) \in f$ then $b = c$.

A is called the ***domain*** of f and B is called the ***co-domain*** of f. If (a, b) is in f, then we write $f(a) = b$; the element b is called the ***image*** of a and the element a is called a ***pre-image*** of b and $f(a)$is known as the value of f at a. The set $\{b : (a, b) \in f$ for some $a \in A\}$ is called the ***range*** of the
function. If B is a subset of R, then we say that the function is a ***real-valued function***.

Two functions f and g are said to be ***equal functions*** if their domains are same and $f(a) = g(a)$ for all a in the domain.

Ways of Representing Functions

(a) Tabular Representation of a Function
When the elements of the domain are listed like $x1, x2, x3 \ldots xn$, we can use this tabular form.Here, the values of the arguments $x1, x2, x3 \ldots xn$ and the corresponding values of the function $y1, y2, y3 \ldots yn$ are written out in a definite order.

x	x_1	x_2	$\ldots$	x_n
y	y_1	y_2	$\ldots$	y_n

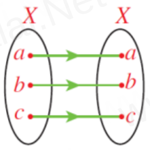
(b) Graphical Representation of a Function
When the domain and the co-domain are subsets of R, many functions can be represented using a graph with x-axis representing the domain and y-axis representing the co-domain in the (x, y)-plane. the variable x is treated as independent variable and y as a dependentvariable. The variable x is called the *argument* and $f(x)$ is called the value

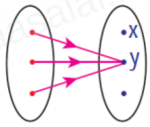
(c) Analytical Representation of a Function
If the functional relation $y = f(x)$ is such that f denotes an analytical expression

Vertical Line Test: Testing whether a given curve represents a function or not by drawing vertical lines is called ***vertical line test*** or simply vertical test.

Some Elementary Functions

(i) Let X be any non-empty set. The function $f: X \to X$ defined by $f(x) = x$ for all $x \in X$ is called the ***identity function*** on X

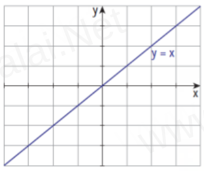


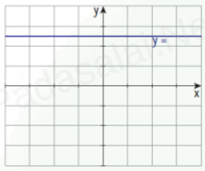


Let X and Y be two sets. Let c be a fixed element of Y. The function $f: X \to Y$ defined by $f(x) = c$ for all $x \in X$ is called a ***constant function***

The value of a constant function is same for all values of x throughout the domain.

If X is any set, then the constant function defined by $f(x) = 0$ for all x is called the ***zero function***. So zero function is a particular case of constant function.





The function $f: R \to R$ defined by $f(x) = |x|$, where $|x|$ is the modulus or absolute value of x, is called the ***modulus function* or *absolute value function***.

The function $f : \mathbb{R} \to \mathbb{R}$ defined by $f(x) = \begin{cases} \frac{x}{|x|} & \text{if} \quad x \neq 0 \\ 0 & \text{if} \quad x = 0 \end{cases}$ is called the ***signum function***.

This function is denoted by sgn. (See Figure 1.24)

The function $f : \mathbb{R} \to \mathbb{R}$ defined by $f(x)$ is the greatest integer less than or equal to x is called the integral part function or the ***greatest integer function*** or the ***floor function***. This function is denoted by $\lfloor x \rfloor$. (See Figure 1.25.)

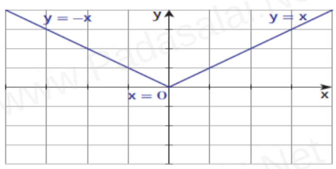
The function $f : \mathbb{R} \to \mathbb{R}$defined by $f(x)$ is the smallest integer greater than or equal to x is called the ***smallest integer function*** or the ***ceil function*** (See Figure 1.26.). This function is denoted by $\lceil \cdot \rceil$; that is $f(x)$ is denoted by $\lceil x \rceil$.

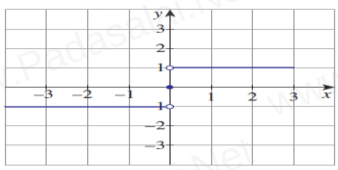
The functions (v) and (vi) are also called ***step functions***.

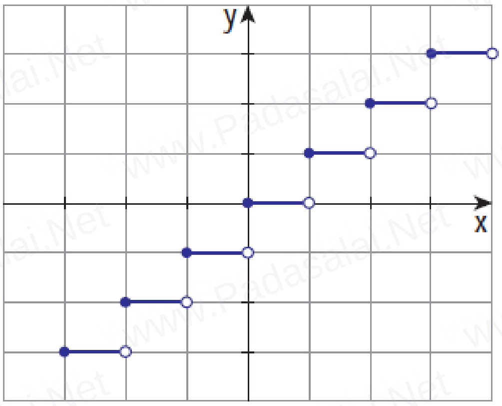
Let us note that $\lfloor 1\frac{1}{5} \rfloor = 1$, $\lfloor 7.23 \rfloor = 7$, $\lfloor -2\frac{1}{2} \rfloor = -3$ (not -2), $\lfloor 6 \rfloor = 6$ and $\lfloor -4 \rfloor = -4$.

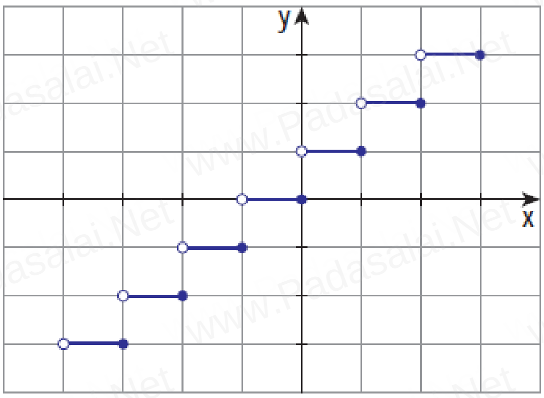
Let us note that $\lceil 1\frac{1}{5} \rceil = 2$, $\lceil 7.23 \rceil = 8$, $\lceil -2\frac{1}{2} \rceil = -2$ (not -3), $\lceil 6 \rceil = 6$ and $\lceil -4 \rceil = -4$.

$$|x| = \begin{cases} -x & \text{if} & x < 0 \\ 0 & \text{if} & x = 0 \\ x & \text{if} & x > 0 \end{cases} \quad \text{or } |x| = \begin{cases} -x & \text{if} & x \leq 0 \\ x & \text{if} & x > 0 \end{cases} \quad \text{or } |x| = \begin{cases} -x & \text{if} & x < 0 \\ x & \text{if} & x \geq 0 \end{cases}$$









Types of Functions

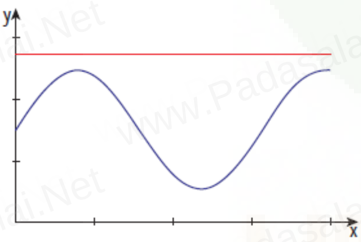
Definition 1.7

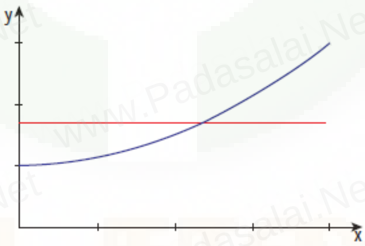
A function $f : A \to B$ is said to be ***one-to-one*** if $x, y \in A$, $x \neq y \Rightarrow f(x) \neq f(y)$ [or equivalently $f(x) = f(y) \Rightarrow x = y$]. A function $f : A \to B$ is said to be *onto*, if for each $b \in B$ there exists at least one element $a \in A$ such that $f(a) = b$. That is, the range of f is B.

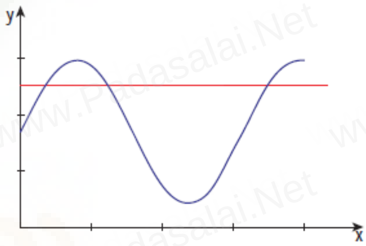
Another name for one-to-one function is ***injective function***; onto function is ***surjective function***. A function $f : A \to B$ is said to be ***bijective*** if it is both one-to-one and onto.
A constant
function is not onto unless the co-domain contains only one element

Horizontal Test

Similar to the vertical line test we have a test called horizontal test to check whether a function is one-to-one, onto or not. Let a function be given as a curve in the plane. If the horizontal line through a point y in the co-domain meets the curve at some points, then the x-coordinate of all the points give pre-images for y.







(i) If the horizontal line through a point y in the co-domain does not meet the curve, then there willbe no pre-image for y and hence the function is not onto.
(ii) If the horizontal line through atleast one point meets the curve at more than one point, then thefunction is not one-to-one.
(iii) If for all y in the range the horizontal line through y meets the curve at only one point, then thefunction is one-to-one.

Testing whether a given curve represents a one-to-one function, onto function or not by drawing horizontal lines is called *horizontal line test* or simply horizontal test

Prepared by, S.Saravana Kumar, Maths Teacher, Jazz Public Matric Higher Secondary School, Melur, Madurai District, Phone : 7092681321, 9597410308

Operations on Functions

Composition

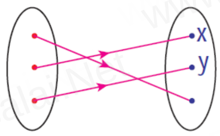
Let there be two functions f and g as given in the Figure 1.36 and Figure 1.37. Let us note that the co-domain of f and the domain of g are the same. Let us cut off Figure 1.37 of g and paste it on the of f so that the domain Y of g is pasted on co-domain Y of f.

Definition 1.8

Let $f : X \to Y$ and $g : Y \to Z$ be two functions. Then the function $h : X \to Z$ defined as $h(x) = g(f(x))$ for every $x \in X$ is called the *composition* of f with g. It is denoted by $g \circ f$ (Read this as f composite with g).

Inverse of a Function

Let there be a bijection $f : X \to Y$



Definition 1.9

Let $f : X \to Y$ be a bijection. The function $g : Y \to X$ defined by $g(y) = x$ if $f(x) = y$, is called the ***inverse*** of f and is denoted by $f–1$.

If a function f has an inverse, then we say that f is ***invertible.*** There is a nice relationship between composition of functions and inverse.

Definition 1.10A function $f : X \to Y$ is said to be ***invertible*** if there exists a function $g : Y \to X$ such that$g \circ f = IX$ and $f \circ g = IY$ where IX and IY are identity functions on X and Y respectively. Inthis case, g is called the inverse of f and g is denoted by $f–1$.Working Rule to Find the Inverse of Functions from R to R:

Let f : R → R be the given function.

i. write $y = f(x)$;

ii. write x in terms of y;

iii. write $f–1(y)$ = the expression in y.

iv. replace y as x.

Definition 1.11

Let X be any set. Let f and g be real valued functions defined on X. Define, for all $x \in X$

- $(f + g)(x) = f(x) + g(x)$.
- $(f - g)(x) = f(x) - g(x)$.
- $(fg)(x) = f(x)g(x)$.
- $fg(x) = f(x)\ g(x)$, where $g(x) =/ 0$.
- $(cf)(x) = cf(x)$, where c is a real constant.
- $(-f)(x) = -f(x)$.

Theorem 1.3: If f and g are real-valued functions, then $f(g + h) = fg + fh$.

Proof. Let X be any set and f and g be real-valued functions defined on X. Let $x \in X$.

$(f(g + h))(x) = f(x)(g + h)(x)$ (by the definition of product)

$= f(x)[g(x) + h(x)]$ (by the definition of addition)

$= f(x)g(x) + f(x)h(x)$ (by the distributivity of reals)

$= (fg)(x) + (fh)(x)$ (by the definition of product)

$= (fg + fh)(x)$ (by the definition of addition)

Thus $(f(g + h))(x) = (fg + fh)(x)$ for all $x \in X$; hence $f(g + h) = fg + fh$.

Some Special Functions
(i) The function $f : R \to R$ defined by
$f(x) = a0xn + a1xn-1 + a2xn-2 + \ldots + an-1x + an,$
where ai are constants, is called a ***polynomial function***. Since the right hand side of the equalitydefining the function is a polynomial, this function is called a polynomial function.
(ii) The function $f : R \to R$ defined by $f(x) = ax + b$ where a _= 0 and b are constants, is called ***alinear function***. A function which is not linear is called a ***non-linear function***.
Clearly a linear function is a polynomial function. The graph of this function is a straight line; a straight line is called a linear curve; so this function is called a linear function. (one maycome across different definitions for linear functions in higher study of mathematics.)
(iii) Let a be a non-negative constant. Consider the function $f : R \to R$ defined by $f(x) = ax$.
If $a = 0$, x _= 0 then the function becomes the zero function and if $a = 1$, then function $f : R \to R$ defined by $f(x) = ax$ is the constant function $f(x) = 1$.
When $a > 1$, the function $f(x) = ax$ is called an ***exponential function***. Moreover, any function having x in the "power" is called as an exponential function.

(iv) Let $a > 1$ be a constant. The function $f : (0,\infty) \to R$ defined by $f(x) = \log a\ x$ is called a ***logarithmic function***. In fact, the inverse of an exponential function $f(x) = ax$ on a suitable domain is called a logarithmic function. [See, Figure 1.44].
(v) The real valued function f defined by $f(x) = p(x)$
$q(x)$ on a suitable domain, where $p(x)$ and $q(x)$
are polynomials, $q(x)$ _= 0, is called ***a rational function.*** In fact, the domain of these function
are the sets obtained from R by removing the real numbers at which $q(x) = 0$.

If f is a real valued function such that $f(x)$ _= 0, then the real valued function g defined by $g(x) = 1\ f(x)$ on a suitable domain is called the *reciprocal function* of f. The domain of g is the set obtained from R by removing the real numbers at which $f(x) = 0$. For example, the largest possible domain of $f(x) = 1\ x-1$ is $R - \{1\}$.

Definition 1.12
A function $f : R \to R$ is said to be an *odd function* if $f(-x) = -f(x)$ for all $\in R$. It is said to be an ***even function*** if $f(-x) = f(x)$ for all $x \in R$.

Theorem 1.4: The product of an odd function and an even function is an odd function.
Proof. Let f be an odd function and g be an even function. Let $h = fg$. Now
$h(-x) = (fg)(-x) = f(-x)g(-x) = -f(x)g(x)$ (as f is odd and g is even) $= -h(x)$
Thus h is an odd function. This shows that fg is an odd function.

If one function is not odd then don't think that the function is an even function. There are plenty of functions which are neither even nor odd.

Graphing Functions using Transformations
(i) Reflection (ii) Translation (iii) Dilations.

Reflection

The ***reflection*** of the graph of a function with respect to a line _ is the graph that is symmetric toit with respect to _. A reflection is the mirror image of the graph where line _ is the mirror of thereflection.

f_ is the mirror image of f with respect to _. Every point of f has a corresponding image in f_. Some useful reflections of $y = f(x)$ are

(i) The graph $y = -f(x)$ is the reflection of the graph of f about the x-axis.

(ii) The graph $y = f(-x)$ is the reflection of the graph of f about the y-axis.

(iii) The graph of $y = f-1(x)$ is the reflection of the graph of f in $y = x$.

Translation

A ***translation*** of a graph is a vertical or horizontal shift of the graph that produces congruent graphs.

The graph of

$y = f(x + c)$, $c > 0$ causes the shift to the left.

$y = f(x - c)$, $c > 0$ causes the shift to the right.

$y = f(x) + d$, $d > 0$ causes the shift to the upward.

$y = f(x) - d$, $d > 0$ causes the shift to the downward.

Dilation

Dilation is also a transformation which causes the curve stretches (expands) or compresses (contracts).

Multiplying a function by a positive constant vertically stretches or compresses its graph; that is, the graph moves away from x-axis or towards x-axis.

If the positive constant is greater than one, the graph moves away from the x-axis. If the positive constant is less than one, the graph moves towards the x-axis.