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UNIT - 7

APPLICATIONS OF DIFFERENTIAL CALCULUS

7. Applications of Differential Calculus

Example 7.1 For the function $f(x) = x^2$, $x \in [0, 2]$ compute the average rate of changes in the subintervals $[0, 0.5]$, $[0.5, 1]$, $[1, 1.5]$, $[1.5, 2]$ and the instantaneous rate of changes at the points $x = 0.5, 1, 1.5, 2$.

Solution: Given $f(x) = x^2$

The average rate of changes between the

$$\begin{aligned} \text{intervals } x = a \text{ and } x = b \text{ is } & \frac{f(b) - f(a)}{b - a} \\ &= \frac{b^2 - a^2}{b - a} \\ &= \frac{(b - a)(b + a)}{b - a} \\ &= b + a \end{aligned}$$

Hence in the interval $[0, 0.5] = 0.5 + 0 = 0.5$

$$[0.5, 1] = 1 + 0.5 = 1.5$$

$$[1, 1.5] = 1.5 + 1 = 2.5$$

$$[1.5, 2] = 2 + 1.5 = 3.5$$

The instantaneous rate of change $f'(x) = 2x$

Hence the instantaneous rate of changes at the

point $x = 0.5$ is $f'(0.5) = 2(0.5) = 1$

$x = 1$ is $f'(1) = 2(1) = 2$

$x = 1.5$ is $f'(1.5) = 2(1.5) = 3$

$x = 2$ is $f'(2) = 2(2) = 4$

Example 7.2 The temperature in Celsius in a long rod of length 10 m, insulated at both ends, is a function of length x given by $T = x(10 - x)$. Prove that the rate of change of temperature at the midpoint of the rod is zero.

Solution: Length of the rod = 10 m

To prove the rate of change of temperature at the mid point of the rod $x = 5$ is zero.

Given Temperature $T = x(10 - x)$

$$= 10x - x^2$$

$$T' = 10 - 2x$$

$$\text{At } x = 5, \quad T' = 10 - 2(5)$$

$$= 10 - 10$$

$$= 0 \text{ Hence proved.}$$

Example 7.3 A person learnt 100 words for an English test. The number of words the person remembers in t days after learning is given by $W(t) = 100 \times (1 - 0.1t)^2, 0 \leq t \leq 10$.

What is the rate at which the person forgets the words 2 days after learning?

Solution: The number of words the person remembers given by

$$W(t) = 100 \times (1 - 0.1t)^2, 0 \leq t \leq 10.$$

$$\begin{aligned} W'(t) &= 100 \times 2(1 - 0.1t)(-0.1) \\ &= -20 \times (1 - 0.1t) \end{aligned}$$

$$\begin{aligned} \text{At } t = 2, \quad W'(2) &= -20 \times (1 - 0.2) \\ &= -20 \times (0.8) \\ &= -16 \end{aligned}$$

The rate at which the person forgets the words 2 days after learning is 16 words.

Example 7.4 A particle moves so that the distance moved is according to the law $s(t) = \frac{t^3}{3} - t^2 + 3$. At what time the velocity and acceleration are zero respectively?

Solution: Given $s(t) = \frac{t^3}{3} - t^2 + 3$

To find the time at velocity $s'(t)$ and acceleration $s''(t)$ becomes zero.

$$s(t) = \frac{t^3}{3} - t^2 + 3$$

$$\begin{aligned} s'(t) &= \frac{3t^2}{3} - 2t \\ &= t^2 - 2t \end{aligned}$$

$$s''(t) = 2t - 2$$

(i) when velocity $s'(t) = 0$

$$t^2 - 2t = 0$$

$$t(t - 2) = 0$$

Gives, $t = 0$, and $t = 2$

(ii) when acceleration $s''(t) = 0$

$$2t - 2 = 0$$

$$2t = 2$$

$$t = 1$$

Example 7.5 A particle is fired straight up from the ground to reach a height of s feet in t seconds, where $s(t) = 128t - 16t^2$.

(1) Compute the maximum height of the particle reached.

(2) What is the velocity when the particle hits the ground?

Solution: Given $s(t) = 128t - 16t^2$

$$\text{Velocity } s'(t) = 128 - 32t$$

The particle reaches the maximum height when velocity becomes zero.

$$\therefore 128 - 32t = 0$$

$$32t = 128$$

$$t = \frac{128}{32} = 4$$

(1) The time taken by the particle to reach the maximum height $t = 4$ seconds.

Substituting in $t = 4$ in $S(t)$, we get the maximum height reached

$$\begin{aligned} s(4) &= 128(4) - 16(4)^2 \\ &= 512 - 16(16) \\ &= 512 - 256 \\ &= 256 \text{ feet.} \end{aligned}$$

The time taken to reach the maximum height is $t = 4$ seconds, hence the time taken for the downward direction is also $t = 4$ seconds.

(2) The total time to reach the ground is $t = 8$ s

Substituting $t = 8$, in $s'(t)$, we the velocity it strikes the ground

$$\begin{aligned} \therefore s'(8) &= 128 - 32(8) \\ &= 128 - 256 \\ &= -128 \text{ ft/s} \end{aligned}$$

The velocity when the particle hits the ground is 128 ft/s

Example 7.6

A particle moves along a horizontal line such that its position at any time $t \geq 0$ is given by $s(t) = t^3 - 6t^2 + 9t + 1$, where s is measured in metres and t in seconds?

- (1) At what time the particle is at rest?
- (2) At what time the particle changes direction?
- (3) Find the total distance travelled by the particle in the first 2 seconds.

Solution: $s(t) = t^3 - 6t^2 + 9t + 1$

$$V(t) = s'(t) = 3t^2 - 12t + 9$$

- (1) The time when the particle comes to rest when velocity becomes zero

$$3t^2 - 12t + 9 = 0$$

$$\div 3, \quad t^2 - 4t + 3 = 0$$

$$(t - 1)(t - 3) = 0$$

$$(t - 1) = 0, \text{ gives } t = 1 \text{ and}$$

$$(t - 3) = 0, \text{ gives } t = 3$$

Hence at $t = 1, t = 3$ the particle is at rest.

- (2) The particle changes direction when $V(t)$ changes its sign.

When t lies between 0 and 1 $V(t) > 0$

When t lies between 1 and 3 $V(t) < 0$

When t lies above 3 $V(t) > 0$

The particle changes direction when $t = 1$ and $t = 3$

- (3) The total distance travelled by the particle in the first 2 seconds.

$$= |s(0) - s(1)| + |s(1) - s(2)|$$

$$\text{Now, } s(0) = (0)^3 - 6(0)^2 + 9(0) + 1 = 1$$

$$s(1) = (1)^3 - 6(1)^2 + 9(1) + 1$$

$$= 1 - 6 + 9 + 1$$

$$= 11 - 6$$

$$= 5$$

$$s(2) = (2)^3 - 6(2)^2 + 9(2) + 1$$

$$= 8 - 24 + 18 + 1$$

$$= 27 - 24$$

$$= 3$$

$$\therefore |s(0) - s(1)| + |s(1) - s(2)|$$

$$= |1 - 5| + |5 - 3|$$

$$= |-4| + |2|$$

$$= 4 + 2$$

$$= 6 \text{ metres.}$$

The total distance travelled by the particle

in the first 2 seconds = 6 metres.

Example 7.7 If we blow air into a balloon of spherical shape at a rate of 1000 cm^3 per second. At what rate the radius of the balloon changes when the radius is 7cm? Also compute the rate at which the surface area changes.

Solution:

Volume of the spherical balloon $V = \frac{4}{3}\pi r^3$

Given $\frac{dV}{dt} = 1000 \text{ cm}^3 \text{ per second}$

To find $\frac{dr}{dt}$ and $\frac{dS}{dt}$ when radius $r = 7 \text{ cm}$

$$V = \frac{4}{3}\pi r^3$$

$$\frac{dV}{dt} = \frac{4}{3}\pi 3r^2 \times \frac{dr}{dt}$$

$$1000 = 4\pi \times 7 \times 7 \times \frac{dr}{dt}$$

$$\frac{1000}{4 \times \pi \times 7 \times 7} = \frac{dr}{dt}$$

$$\frac{dr}{dt} = \frac{250}{49\pi} \text{ cm/s}$$

(i) The radius changes at the rate of $\frac{250}{49\pi} \text{ cm/s}$

Surface area of the sphere $S = 4\pi r^2$

$$\frac{dS}{dt} = 4\pi \times 2r \times \frac{dr}{dt}$$

$$= 4\pi \times 2 \times 7 \times \frac{250}{49\pi}$$

$$= 4 \times 2 \times \frac{250}{7}$$

$$\frac{dS}{dt} = \frac{2000}{7} \text{ cm}^2/\text{s}$$

(ii) The Surface area changes at the

$$\text{rate of } \frac{2000}{7} \text{ cm}^2/\text{s}$$

Example 7.8 The price of a product is related to the number of units available (supply) by the equation $Px + 3P - 16x = 234$, where P is the price of the product per unit in Rupees and x is the number of units. Find the rate at which the price is changing with respect to time when 90

units are available and the supply is increasing at a rate of 15 units/week.

Solution: The supply and price of the product is x and P . It is related by

$$Px + 3P - 16x = 234$$

To find $\frac{dP}{dt}$, when $x = 90$ and $\frac{dx}{dt} = 15$

Given $Px + 3P - 16x = 234$

$$Px + 3P = 234 + 16x$$

$$P(x + 3) = 234 + 16x$$

$$P = \frac{(234+16x)}{(x+3)}$$

$$\frac{dP}{dt} = \frac{(x+3)(234+16x)' - (234+16x)(x+3)'}{(x+3)^2}$$

$$= \frac{(x+3)\left(16\frac{dx}{dt}\right) - (234+16x)\left(\frac{dx}{dt}\right)}{(x+3)^2}$$

$$= \frac{[(x+3)(16) - (234+16x)]\left(\frac{dx}{dt}\right)}{(x+3)^2}$$

$$= \frac{[16x+48-234-16x]\left(\frac{dx}{dt}\right)}{(x+3)^2}$$

$$= \frac{[-186]\left(\frac{dx}{dt}\right)}{(x+3)^2}$$

$$= \frac{[-186](15)}{(90+3)^2}$$

$$= \frac{[-186](15)}{(93)^2}$$

$$= \frac{[-186](15)}{(93)(93)}$$

$$= \frac{(-2)(15)}{(93)}$$

$$= \frac{(-2)(5)}{(31)}$$

$$= \frac{-10}{31}$$

$$\cong -0.32$$

Hence the price decreasing at the rate

of Rs. 0.32 per unit.

Example 7.9 Salt is poured from a conveyer belt at a rate of 30 cubic meters per minute forming a conical pile with a circular base

whose height and diameter of base are always equal. How fast is the height of the pile increasing when the pile is 10 metre high?

Solution:



Given: Diameter = Height

$$2r = h$$

$$r = \frac{h}{2}$$

Rate of change of volume $\frac{dV}{dt} = 30$

To find rate of change of height $\frac{dh}{dt}$ at $h = 10$

We know the volume of the cone

$$V = \frac{1}{3}\pi r^2 h$$

$$= \frac{1}{3}\pi \left(\frac{h}{2}\right) \left(\frac{h}{2}\right) h$$

$$= \frac{\pi}{12} h^3$$

$$\frac{dV}{dt} = \frac{\pi}{12} 3h^2 \left(\frac{dh}{dt}\right)$$

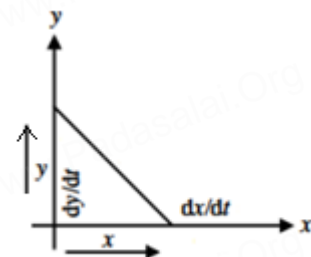
$$30 = \frac{\pi}{4} \times 10 \times 10 \times \left(\frac{dh}{dt}\right)$$

$$\frac{30 \times 4}{10 \times 10 \times \pi} = \frac{dh}{dt}$$

$$\frac{dh}{dt} = \frac{6}{5\pi} \text{ m/min}$$

Example 7.10 (Two variable related rate problem) A road running north to south crosses a road going east to west at the point P . Car A is driving north along the first road, and car B is driving east along the second road. At a particular time car A 10 kilometers to the north of P and traveling at 80 km/hr, while car B is 15 kilometers to the east of P and traveling at 100 km/hr. How fast is the distance between the two cars changing?

Solution:



Given $x = 10$ and $y = 15$

To find the change of the distance between

the cars $\frac{dz}{dt}$, when $\frac{dx}{dt} = 80$, and $\frac{dy}{dt} = 100$

By using Pythagoras Theorem,

$$x^2 + y^2 = z^2$$

$$(10)^2 + (15)^2 = z^2$$

$$100 + 225 = z^2$$

$$z^2 = 325$$

$$= 25 \times 13$$

$$z = 5\sqrt{13}$$

$$x^2 + y^2 = z^2$$

Differentiating with respect to t

$$2x \frac{dx}{dt} + 2y \frac{dy}{dt} = 2z \frac{dz}{dt}$$

$$\div 2, \quad x \frac{dx}{dt} + y \frac{dy}{dt} = z \frac{dz}{dt}$$

$$(10)(80) + (15)(100) = (5\sqrt{13}) \frac{dz}{dt}$$

Dividing by 5,

$$(2)(80) + (3)(100) = (\sqrt{13}) \frac{dz}{dt}$$

$$160 + 300 = (\sqrt{13}) \frac{dz}{dt}$$

$$460 = (\sqrt{13}) \frac{dz}{dt}$$

$$\frac{dz}{dt} = \frac{460}{\sqrt{13}}$$

$$= \frac{460}{3.65}$$

$$\cong 127.6 \text{ Km/hr}$$

The distance between changes at the rate of 127.6 Km/hr

EXERCISE 7.1

1. A point moves along a straight line in such a way that after t seconds its distance from the origin is $s = 2t^2 + 3t$ metres.

(i) Find the average velocity of the points between $t = 3$ and $t = 6$ seconds.

(ii) Find the instantaneous velocities at $t = 3$ and $t = 6$ seconds.

Solution:

Given $s(t) = 2t^2 + 3t$

The average velocity of the points between

$t = a$ and $t = b$ is $\frac{S(b)-S(a)}{b-a}$

$$s(a) = s(3) = 2(3)^2 + 3(3)$$

$$= 2(9) + 9$$

$$= 18 + 9$$

$$= 27$$

$$s(b) = s(6) = 2(6)^2 + 3(6)$$

$$= 2(36) + 18$$

$$= 72 + 18$$

$$= 90$$

The average velocity of the points $= \frac{S(6)-S(3)}{6-3}$

$$= \frac{90-27}{3}$$

$$= \frac{63}{3}$$

$$= 21 \text{ m/s}$$

(ii) Given $s(t) = 2t^2 + 3t$

Hence instantaneous velocity $= S'(t)$

$$S'(t) = 4t + 3$$

Velocity at $t = 3$ seconds $S'(3) = 4(3) + 3$

$$= 12 + 3$$

$$= 15 \text{ m/s}$$

Velocity at $t = 6$ seconds $S'(6) = 4(6) + 3$

$$= 24 + 3$$

$$= 27 \text{ m/s}$$

2. A camera is accidentally knocked off an edge of a cliff 400 ft high. The camera falls a distance of $s = 16t^2$ in t seconds.

(i) How long does the camera fall before it hits the ground?

(ii) What is the average velocity with which the camera falls during the last 2 seconds?

(iii) What is the instantaneous velocity of the camera when it hits the ground?

Solution:

(i) Given $s = 16t^2$. The camera falls down from the height of 400 ft.

∴ Time taken to hit the ground is to find t when $s=400$ feet.

$$400 = 16t^2$$

$$\frac{400}{16} = t^2$$

$$\frac{4 \times 100}{16} = t^2$$

$$\pm \frac{2 \times 10}{4} = t$$

$$\pm 5 = t$$

Since $t = -5$, is not possible

$$t = 5$$

∴ Time taken for the camera to hit the ground is $t = 5$ seconds.

(ii) To find the average velocity with which the camera falls during the last 2 seconds is to find the average velocity at $t = 3$ and $t = 5$ seconds.

We know that the average velocity of the

points between $t = a$ and $t = b$ is $\frac{s(b)-s(a)}{b-a}$

$$\text{Given } s = 16t^2$$

$$s(a) = s(3) = 16(3)^2$$

$$= 16(9)$$

$$= 144$$

$$s(b) = s(5) = 16(5)^2$$

$$= 16(25)$$

$$= 400$$

$$\begin{aligned} \text{The average velocity of the points} &= \frac{s(5)-s(3)}{5-3} \\ &= \frac{400-144}{2} \\ &= \frac{256}{2} \\ &= 128 \text{ ft/s} \end{aligned}$$

(iii) To find the instantaneous velocity of the camera when it hits the ground is to find the instantaneous velocity at $t = 5$ seconds.

$$\text{Given } s(t) = 16t^2$$

Hence instantaneous velocity $= S'(t)$

$$S'(t) = 32t$$

Velocity at $t = 5$ seconds $S'(5) = 32(5)$

$$= 160$$

$$= 160 \text{ ft/s}$$

3. A particle moves along a line according to the law $s(t) = 2t^3 - 9t^2 + 12t - 4$, where $t \geq 0$.

(i) At what times the particle changes direction?

(ii) Find the total distance travelled by the particle in the first 4 seconds.

(iii) Find the particle's acceleration each time the velocity is zero.

Solution:

(i) To find the times the particle changes

direction is to find t when $S'(t) = 0$

$$\text{Given } s(t) = 2t^3 - 9t^2 + 12t - 4$$

$$s'(t) = 6t^2 - 18t + 12$$

when $s'(t) = 0$ gives,

$$6t^2 - 18t + 12 = 0$$

$$\div 6, \quad t^2 - 3t + 2 = 0$$

$$(t-1)(t-2) = 0$$

$$t-1 = 0, \text{ gives } t = 1 \text{ and}$$

$$t-2 = 0, \text{ gives } t = 2$$

∴ The particle changes direction at $t = 1$ and $t = 2$ seconds.

(ii) To find the total distance travelled by the particle in the first 4 seconds

S at $t = 0, 1, 2, 3$ and 4 seconds.

$$\text{Given } s(t) = 2t^3 - 9t^2 + 12t - 4$$

$$s(0) = -4$$

$$s(1) = 2 - 9 + 12 - 4$$

$$= 14 - 13$$

$$= 1$$

$$s(2) = 2(2)^3 - 9(2)^2 + 12(2) - 4$$

$$\begin{aligned}
 &= 2(8) - 9(4) + 12(2) - 4 \\
 &= 16 - 36 + 24 - 4 \\
 &= 40 - 40 \\
 &= 0
 \end{aligned}$$

$$\begin{aligned}
 s(3) &= 2(3)^3 - 9(3)^2 + 12(3) - 4 \\
 &= 2(27) - 9(9) + 12(3) - 4 \\
 &= 54 - 81 + 36 - 4 \\
 &= 90 - 85 \\
 &= 5
 \end{aligned}$$

$$\begin{aligned}
 s(4) &= 2(4)^3 - 9(4)^2 + 12(4) - 4 \\
 &= 2(64) - 9(16) + 12(4) - 4 \\
 &= 128 - 144 + 48 - 4 \\
 &= 176 - 148 \\
 &= 28
 \end{aligned}$$

t	0	1	2	3	4
S	-4	1	0	5	28

Total distance travelled in first 4 seconds

$$\begin{aligned}
 &= |s(0) - s(1)| + |s(1) - s(2)| \\
 &\quad + |s(2) - s(3)| + |s(3) - s(4)| \\
 &= |-4 - 1| + |1 - 0| + |0 - 5| + |5 - 28| \\
 &= |-5| + |1| + |-5| + |-23| \\
 &= 5 + 1 + 5 + 23 \\
 &= 34 \text{ meters.}
 \end{aligned}$$

(iii) To find the particle's acceleration each time the velocity is zero.

$$\text{Given } s(t) = 2t^3 - 9t^2 + 12t - 4$$

$$\text{Velocity } s'(t) = 6t^2 - 18t + 12$$

when $s'(t) = 0$ gives,

$$6t^2 - 18t + 12 = 0$$

$$\div 6, \quad t^2 - 3t + 2 = 0$$

$$(t - 1)(t - 2) = 0$$

$$t - 1 = 0, \text{ gives } t = 1 \text{ and}$$

$$t - 2 = 0, \text{ gives } t = 2$$

$$\text{Acceleration } s''(t) = 12t - 18$$

$$\begin{aligned}
 \text{When } t = 1, \text{ Acceleration } s''(1) &= 12 - 18 \\
 &= -6 \text{ m/s}^2
 \end{aligned}$$

$$\begin{aligned}
 \text{When } t = 2, \text{ Acceleration } s''(2) &= 12(2) - 18 \\
 &= 24 - 18 \\
 &= 6 \text{ m/s}^2
 \end{aligned}$$

4. If the volume of a cube of side length x is $v = x^3$. Find the rate of change of the volume with respect to x when $x = 5$ units.

Solution: Given volume of the cube $v = x^3$

$$\text{The rate of change of the volume} = \frac{dv}{dt}$$

$$\therefore \frac{dv}{dt} = 3x^2$$

$$\frac{dv}{dt} \text{ at } x = 5 = 3(5)^2$$

$$= 3(25)$$

$$= 75 \text{ units.}$$

5. If the mass $m(x)$ (in kilograms) of a thin rod of length x (in metres) is given by, $m(x) = \sqrt{3x}$ then what is the rate of change of mass with respect to the length when it is $x = 3$ and $x = 27$ metres.

$$\begin{aligned}
 \text{Solution: Given } m(x) &= \sqrt{3x} \\
 &= \sqrt{3}\sqrt{x}
 \end{aligned}$$

$$\text{Rate of change of mass} = m'(x)$$

$$m'(x) = \sqrt{3} \frac{1}{2\sqrt{x}}$$

(i) Rate of change of mass at $x = 3$

$$m'(3) = \sqrt{3} \frac{1}{2\sqrt{3}}$$

$$= \frac{1}{2} \text{ Kg/m}$$

(ii) Rate of change of mass at $x = 27$

$$m'(27) = \sqrt{3} \frac{1}{2\sqrt{27}}$$

$$= \sqrt{3} \frac{1}{2\sqrt{9 \times 3}}$$

$$= \sqrt{3} \frac{1}{2 \times 3\sqrt{3}}$$

$$= \frac{1}{6} \text{ Kg/m}$$

6. A stone is dropped into a pond causing ripples in the form of concentric circles. The radius r of the outer ripple is increasing at a constant rate at 2 cm per second. When the radius is 5 cm find the rate of changing of the total area of the disturbed water?

Solution:

Given The rate of increasing radius $\frac{dr}{dt} = 2 \text{ cm/s}$

To find the rate of change of area of the

circle $\frac{dA}{dt}$ when $r = 5$

Area of the circle $A = \pi r^2$

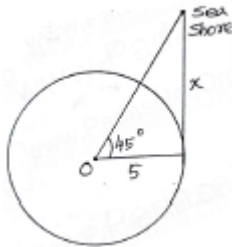
$$\frac{dA}{dt} = 2\pi r \times \frac{dr}{dt}$$

$$= 2\pi(5) \times (2)$$

$$= 20\pi \text{ sq.cm/s}$$

7. A beacon makes one revolution every 10 seconds. It is located on a ship which is anchored 5 km from a straight shore line. How fast is the beam moving along the shore line when it makes an angle of 45° with the shore?

Solution:



Given : one revolution every 10 seconds.

It covers in 10 sec $= 2\pi$

$$\therefore 1 \text{ sec} = \frac{2\pi}{10} = \frac{\pi}{5}$$

$$\text{Revolution in 1 second } \frac{d\theta}{dt} = \frac{\pi}{5}$$

$$\text{From the given data } \tan \theta = \frac{x}{5}$$

$$\text{Gives, } x = 5 \tan \theta$$

$$\therefore \frac{dx}{dt} = 5 \sec^2 \theta \times \frac{d\theta}{dt}$$

$$\text{when } \theta = 45^\circ \text{ and } \frac{d\theta}{dt} = \frac{\pi}{5}$$

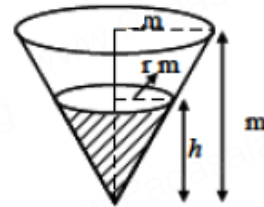
$$\frac{dx}{dt} = 5 \sec^2 45^\circ \times \frac{\pi}{5}$$

$$= 5 \sqrt{2}^2 \times \frac{\pi}{5} = 2 \times \pi$$

The beam moving along the shore line at a rate of $2\pi \text{ km/s}$

8. A conical water tank with vertex down of 12 metres height has a radius of 5 metres at the top. If water flows into the tank at a rate 10 cubic m/min, how fast is the depth of the water increases when the water is 8 metres deep?

Solution:



Given: Height of the tank $h = 12 \text{ m}$

Radius at the top $r = 5 \text{ m}$

$$\text{Hence } \frac{r}{h} = \frac{5}{12} \Rightarrow r = \frac{5}{12} h$$

Rate of change of volume $\frac{dV}{dt} = 10$

To find rate of change of height $\frac{dh}{dt}$ at $h = 8$

We know the volume of the cone

$$V = \frac{1}{3} \pi r^2 h$$

$$= \frac{1}{3} \pi \left(\frac{5}{12} h \right) \left(\frac{5}{12} h \right) h$$

$$= \frac{5 \times 5}{3 \times 12 \times 12} \pi h^3$$

$$\frac{dV}{dt} = \frac{5 \times 5}{3 \times 12 \times 12} \pi 3h^2 \times \frac{dh}{dt}$$

$$\frac{dV}{dt} = \frac{25}{144} \pi \times h^2 \times \frac{dh}{dt}$$

Substituting $\frac{dV}{dt} = 10$ and $h = 8$, we get

$$10 = \frac{25}{144} \pi \times 8 \times 8 \times \frac{dh}{dt}$$

$$\frac{10 \times 144}{25 \times 8 \times 8 \times \pi} = \frac{dh}{dt}$$

$$\frac{2 \times 5 \times 16 \times 9}{25 \times 8 \times 8 \times \pi} = \frac{dh}{dt}$$

$$\frac{9}{10\pi} = \frac{dh}{dt}$$

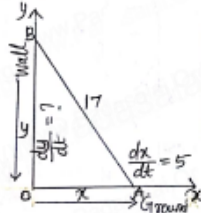
So, the rate of change of height $\frac{dh}{dt} = \frac{9}{10\pi} \text{ m/min}$

9. A ladder 17 metre long is leaning against the wall. The base of the ladder is pulled away from the wall at a rate of 5 m/s. When the base of the ladder is 8 metres from the wall.

(i) How fast is the top of the ladder moving down the wall?

(ii) At what rate, the area of the triangle formed by the ladder, wall, and the floor, is changing?

Solution:



Given length of the ladder $AB = 17$ m.

Let the foot of the ladder is x m away from the wall. That is let $OA = x$

Similarly let the top of the ladder touches the wall y m away from the floor.

That is let $OB = y$

By using Pythagoras Theorem,

$$x^2 + y^2 = 17^2$$

When $x = 8$, $8^2 + y^2 = 17^2$

$$y^2 = 17^2 - 8^2$$

$$= (17 + 8)(17 - 8)$$

$$= (25)(9)$$

$$y = (5)(3)$$

$$y = 15$$

(i) To find the rate at which the ladder slips

down the wall $= \frac{dy}{dt}$ when $\frac{dx}{dt} = 5$

From $x^2 + y^2 = 17^2$

Differentiating with respect to t

$$2x \frac{dx}{dt} + 2y \frac{dy}{dt} = 0$$

$$\div 2, \quad x \frac{dx}{dt} + y \frac{dy}{dt} = 0$$

$$(8)(5) + (15) \frac{dy}{dt} = 0$$

$$\frac{dy}{dt} = -\frac{40}{15}$$

$$= -\frac{8}{3}$$

Ladder slips down the wall at the rate of $\frac{8}{3}$ m/s

(ii) To find the rate at which the area of the

triangle OAB changes $= \frac{dA}{dt}$

Area of triangle OAB $A = \frac{1}{2}xy$

Differentiating with respect to t

$$\frac{dA}{dt} = \frac{1}{2} \left(x \frac{dy}{dt} + y \frac{dx}{dt} \right)$$

$$= \frac{1}{2} \left[(8) \left(-\frac{8}{3} \right) + (15)(5) \right]$$

$$= \frac{1}{2} \left(-\frac{64}{3} + 75 \right)$$

$$= \frac{1}{2} \left(\frac{-64 + 225}{3} \right)$$

$$= \frac{1}{2} \left(\frac{121}{3} \right)$$

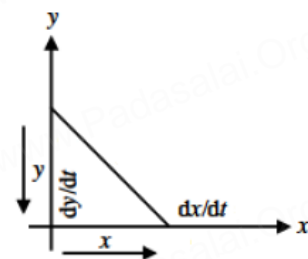
$$= \frac{121}{6}$$

$$= 26.83$$

The area of the triangle OAB changes at the rate $\frac{dA}{dt} = 26.83$ sq. m/s

10, A police jeep, approaching an orthogonal intersection from the northern direction, is chasing a speeding car that has turned and moving straight east. When the jeep is 0.6 km north of the intersection and the car is 0.8 km to the east. The police determine with a radar that the distance between them and the car is increasing at 20 km/hr. If the jeep is moving at 60 km/hr at the instant of measurement, what is the speed of the car?

Solution:



Given $x = 0.8$ and $y = 0.6$

The distance between the car and the jeep

changes at the rate $\frac{dz}{dt} = 20$

To find $\frac{dx}{dt}$, when $\frac{dy}{dt} = -60$

By using Pythagoras Theorem,

$$\begin{aligned}
 x^2 + y^2 &= z^2 \\
 (0.8)^2 + (0.6)^2 &= z^2 \\
 0.64 + 0.36 &= z^2 \\
 1 &= z^2 \\
 z &= 1 \\
 x^2 + y^2 &= z^2
 \end{aligned}$$

Differentiating with respect to t

$$\begin{aligned}
 2x \frac{dx}{dt} + 2y \frac{dy}{dt} &= 2z \frac{dz}{dt} \\
 \div 2, \quad x \frac{dx}{dt} + y \frac{dy}{dt} &= z \frac{dz}{dt} \\
 (0.8) \left(\frac{dx}{dt} \right) + (0.6)(-60) &= (1)(20) \\
 (0.8) \left(\frac{dx}{dt} \right) - 36 &= 20 \\
 \frac{8}{10} \left(\frac{dx}{dt} \right) &= 20 + 36 \\
 \frac{dx}{dt} &= 56 \times \frac{10}{8} \\
 \frac{dx}{dt} &= 70
 \end{aligned}$$

Hence the speed of the car changes at the rate

$$\frac{dx}{dt} = 70 \text{ km/hr.}$$

Example 7.11 Find the equations of tangent and normal to the curve $y = x^2 + 3x - 2$ at the point (1, 2)

Solution: $y = x^2 + 3x - 2$

$$\begin{aligned}
 \frac{dy}{dx} &= 2x + 3 \\
 \frac{dy}{dx} \text{ at the point } (1, 2) &= 2(1) + 3 \\
 &= 2 + 3
 \end{aligned}$$

$$\text{Slope } m = 5$$

Equation of the tangent with slope m through

the point (x_1, y_1) is $y - y_1 = m(x - x_1)$

Substituting $m = 5$, and the point (1, 2)

Tangent equation is $y - 2 = 5(x - 1)$

$$\begin{aligned}
 y - 2 &= 5x - 5 \\
 5x - y - 5 + 2 &= 0 \\
 5x - y - 3 &= 0
 \end{aligned}$$

Normal is perpendicular to tangent.

Hence normal is of the form $x + 5y + k = 0$

It passes through the point (1, 2)

$$\text{Hence, } 1 + 5(2) + k = 0$$

$$1 + 10 + k = 0$$

$$11 + k = 0$$

$$k = -11$$

Equation of the normal is $x + 5y - 11 = 0$

Example 7.12 For what value of x the tangent of the curve $y = x^3 - 3x^2 + x - 2$ is parallel to the line $y = x$.

Solution: Given $y = x^3 - 3x^2 + x - 2$

$$\frac{dy}{dx} = 3x^2 - 6x + 1$$

Slope of the first curve $m_1 = 3x^2 - 6x + 1$

and $y = x$

$$\frac{dy}{dx} = 1$$

Slope of the second curve $m_2 = 1$

Since the tangents are parallel, $m_1 = m_2$

$$\therefore 3x^2 - 6x + 1 = 1$$

$$3x^2 - 6x + 1 - 1 = 0$$

$$3x^2 - 6x = 0$$

$$3x(x - 2) = 0$$

So, $3x = 0 \Rightarrow x = 0$

and $x - 2 = 0 \Rightarrow x = 2$

Substituting $x = 0$ in $y = x^3 - 3x^2 + x - 2$,

we get $y = -2$ and

$x = 2$ in $y = x^3 - 3x^2 + x - 2$,

we get $y = 2^3 - 3(2^2) + 2 - 2$

$$= 8 - 12 + 2 - 2$$

$$= 10 - 14$$

$$y = -4$$

The tangent is parallel to the line $y = x$ at the points (0, -2) and (2, -4)

Example 7.13 Find the equation of the tangent and normal to the Lissajous curve given by $x = 2 \cos 3t$ and $y = 3 \sin 2t$, $t \in \mathbb{R}$

Solution: $x = 2 \cos 3t$

$$\frac{dx}{dt} = 2 \times 3(-\sin 3t)$$

$$\frac{dx}{dt} = -6 \sin 3t$$

$$y = 3 \sin 2t$$

$$\frac{dy}{dt} = 3 \times 2(\cos 2t)$$

$$\frac{dy}{dt} = 6 \cos 2t$$

$$\text{Slope of the tangent } m = \frac{dy}{dx} = \frac{6 \cos 2t}{-6 \sin 3t}$$

$$m = -\frac{\cos 2t}{\sin 3t}$$

Equation of the tangent with slope m through the point (x_1, y_1) is $y - y_1 = m(x - x_1)$

$$\text{Substituting } m = -\frac{\cos 2t}{\sin 3t},$$

and the point $(2 \cos 3t, 3 \sin 2t)$

Equation of tangent is

$$(y - 3 \sin 2t) = -\frac{\cos 2t}{\sin 3t}(x - 2 \cos 3t)$$

$$(y - 3 \sin 2t) \sin 3t = -\cos 2t(x - 2 \cos 3t)$$

$$y \sin 3t - 3 \sin 2t \sin 3t = -x \cos 2t + 2 \cos 2t \cos 3t$$

$$x \cos 2t + y \sin 3t = 3 \sin 2t \sin 3t + 2 \cos 2t \cos 3t$$

Normal is perpendicular to tangent.

Hence normal is of the form

$$x \sin 3t - y \cos 2t + k = 0$$

It passes through $(2 \cos 3t, 3 \sin 2t)$

$$(2 \cos 3t) \sin 3t - (3 \sin 2t) \cos 2t + k = 0$$

$$2 \cos 3t \sin 3t - 3 \sin 2t \cos 2t + k = 0$$

$$2 \cos 3t \sin 3t - \frac{3}{2}(2 \sin 2t \cos 2t) + k = 0$$

$$\sin 6t - \frac{3}{2} \sin 4t + k = 0$$

$$k = \frac{3}{2} \sin 4t - \sin 6t$$

So equation of the normal is

$$x \sin 3t - y \cos 2t + \frac{3}{2} \sin 4t - \sin 6t = 0$$

Example 7.14 Find the acute angle between $y = x^2$ and $y = (x - 3)^2$.

Solution: Given $y = x^2$ and $y = (x - 3)^2$.

$$\text{So, } x^2 = (x - 3)^2$$

$$x^2 = x^2 - 6x + 9$$

$$x^2 - 6x + 9 - x^2 = 0$$

$$-6x + 9 = 0$$

$$6x = 9$$

$$x = \frac{9}{6} = \frac{3}{2}$$

When $x = \frac{3}{2}$, $y = x^2$ gives, $y = \frac{9}{4}$

The point of intersection is $\left(\frac{3}{2}, \frac{9}{4}\right)$

$$\text{From } y = x^2$$

$$\frac{dy}{dx} = 2x$$

At the point $\left(\frac{3}{2}, \frac{9}{4}\right)$

$$\frac{dy}{dx} = 2\left(\frac{3}{2}\right) = 3$$

Slope of the first curve $m_1 = 3$

$$\text{From } y = (x - 3)^2$$

$$y = x^2 - 6x + 9$$

$$\frac{dy}{dx} = 2x - 6$$

At the point $\left(-\frac{3}{2}, \frac{9}{4}\right)$

$$\frac{dy}{dx} = 2\left(-\frac{3}{2}\right) - 6 = 3 - 6 = -3$$

Slope of the second curve $m_2 = -3$

If θ is the angle between the curves, then

$$\tan \theta = \left| \frac{m_1 - m_2}{1 + m_1 m_2} \right|$$

$$= \left| \frac{3 - (-3)}{1 + (3)(-3)} \right|$$

$$= \left| \frac{3+3}{1-9} \right|$$

$$= \left| \frac{6}{-8} \right|$$

$$= \left| \frac{3}{-4} \right|$$

$$= \frac{3}{4}$$

$$\theta = \tan^{-1} \left(\frac{3}{4} \right)$$

The angle between the curves $\theta = \tan^{-1} \left(\frac{3}{4} \right)$

Example 7.15 Find the acute angle between the curves $y = x^2$ and $x = y^2$ at their points of intersection $(0, 0)$, $(1, 1)$.

Solution: From $y = x^2$

$$\frac{dy}{dx} = m_1 = 2x \text{ and}$$

$$\text{From } x = y^2$$

$$1 = 2y \frac{dy}{dx}$$

$$\frac{dy}{dx} = m_2 = \frac{1}{2y}$$

(i) At $(0, 0)$

$$m_1 = 2x = 0 \text{ and}$$

$$m_2 = \frac{1}{2y} = \frac{1}{0} = \infty$$

If θ is the angle between the curves, then

$$\tan \theta = \left| \frac{m_1 - m_2}{1 + m_1 m_2} \right|$$

$$= \left| \frac{0 - \infty}{1 + \infty} \right| = \infty$$

$$\theta = \tan^{-1}(\infty) = \frac{\pi}{2}$$

The angle between the curves $\theta = \frac{\pi}{2}$

(i) At $(1, 1)$

$$m_1 = 2x = 2 \text{ and}$$

$$m_2 = \frac{1}{2y} = \frac{1}{2}$$

If θ is the angle between the curves, then

$$\tan \theta = \left| \frac{m_1 - m_2}{1 + m_1 m_2} \right|$$

$$= \left| \frac{2 - \left(\frac{1}{2}\right)}{1 + (2)\left(\frac{1}{2}\right)} \right|$$

$$= \left| \frac{\frac{4-1}{2}}{1+1} \right|$$

$$= \left(\frac{\frac{3}{2}}{\frac{2}{1}} \right)$$

$$= \left(\frac{3}{2} \times \frac{1}{2} \right)$$

$$= \frac{3}{4}$$

$$\theta = \tan^{-1} \left(\frac{3}{4} \right)$$

The angle between the curves $\theta = \tan^{-1} \left(\frac{3}{4} \right)$

Example 7.16

Find the angle of intersection of the curve $y = \sin x$ with the positive x -axis.

Solution: Given $y = \sin x$

$$\frac{dy}{dx} = \cos x$$

The given curve intersects with positive x -axis

On x -axis $y = 0$, gives $\sin x = 0$

$$\text{Hence } x = n\pi$$

Slope of the tangent $\frac{dy}{dx} = m = \cos x$ gives

$$m = \cos n\pi = (-1)^n$$

$$\text{Then } \tan \theta = (-1)^n$$

$$\theta = \tan^{-1}(-1)^n$$

$$\text{If } n \text{ is even, } \theta = \tan^{-1}(1) = \frac{\pi}{4}$$

$$\text{If } n \text{ is odd, } \theta = \tan^{-1}(-1) = \frac{3\pi}{4}$$

Example 7.17 If the curves $ax^2 + by^2 = 1$ and $cx^2 + dy^2 = 1$ intersect each other

orthogonally then, $\frac{1}{a} - \frac{1}{b} = \frac{1}{c} - \frac{1}{d}$.

Solution: $ax^2 + by^2 = 1$

$$cx^2 + dy^2 = 1$$

$$\text{Solving, } \Delta = \begin{vmatrix} a & b \\ c & d \end{vmatrix}$$

$$= ad - bc$$

$$\Delta_{x^2} = \begin{vmatrix} 1 & b \\ 1 & d \end{vmatrix}$$

$$= d - b$$

$$\Delta_{y^2} = \begin{vmatrix} a & 1 \\ c & 1 \end{vmatrix}$$

$$= a - c$$

$$\therefore x^2 = \frac{d-b}{ad-bc}$$

$$y^2 = \frac{a-c}{ad-bc}$$

$$\text{Given } ax^2 + by^2 = 1$$

$$2ax + 2by \frac{dy}{dx} = 0$$

$$2by \frac{dy}{dx} = -2ax$$

$$\frac{dy}{dx} = -\frac{2ax}{2by}$$

$$m_1 = -\frac{ax}{by} \text{ and}$$

$$cx^2 + dy^2 = 1$$

$$2cx + 2dy \frac{dy}{dx} = 0$$

$$2dy \frac{dy}{dx} = -2cx$$

$$\frac{dy}{dx} = -\frac{2cx}{2dy}$$

$$m_2 = -\frac{cx}{dy}$$

Given the curves cut orthogonally.

$$\therefore m_1 m_2 = -1$$

$$\left(-\frac{ax}{by}\right)\left(-\frac{cx}{dy}\right) = -1$$

$$\frac{acx^2}{bdy^2} = -1$$

$$acx^2 = -bdy^2$$

$$acx^2 + bdy^2 = 0$$

Substituting the values of x^2 and y^2

$$ac\left(\frac{d-b}{ad-bc}\right) + bd\left(\frac{a-c}{ad-bc}\right) = 0$$

$$ac(d-b) + bd(a-c) = 0$$

$$\div \text{ by } abcd,$$

$$\frac{ac}{abcd}(d-b) + \frac{bd}{abcd}(a-c) = 0$$

$$\frac{(d-b)}{bd} + \frac{(a-c)}{ac} = 0$$

$$\frac{d}{bd} - \frac{b}{bd} + \frac{a}{ac} - \frac{c}{ac} = 0$$

$$\frac{1}{b} - \frac{1}{d} + \frac{1}{c} - \frac{1}{a} = 0$$

$$\therefore \frac{1}{a} - \frac{1}{b} = \frac{1}{c} - \frac{1}{d}$$

Example 7.18 Prove that the ellipse $x^2 + 4y^2 = 8$ and the hyperbola $x^2 - 2y^2 = 4$ intersect orthogonally.

Solution: $x^2 + 4y^2 = 8$ (1)

$$x^2 - 2y^2 = 4$$
 (2)

$$(1) - (2) \text{ gives, } 6y^2 = 4$$

$$y^2 = \frac{4}{6} = \frac{2}{3}$$

Multiplying (2) by 3,

$$3x^2 - 6y^2 = 12$$

$$\text{Substituting } 6y^2 = 4$$

$$3x^2 - 4 = 12$$

$$3x^2 = 12 + 4$$

$$3x^2 = 16$$

$$x^2 = \frac{16}{3}$$

$$\text{Given } x^2 + 4y^2 = 8$$

$$2x + 8y \frac{dy}{dx} = 0$$

$$8y \frac{dy}{dx} = -2x$$

$$\frac{dy}{dx} = -\frac{2x}{8y}$$

$$m_1 = -\frac{x}{4y}$$

$$x^2 - 2y^2 = 4$$

$$2x - 4y \frac{dy}{dx} = 0$$

$$4y \frac{dy}{dx} = 2x$$

$$\frac{dy}{dx} = \frac{2x}{4y}$$

$$m_2 = \frac{x}{2y}$$

$$\text{Now } m_1 \times m_2 = \left(-\frac{x}{4y}\right) \times \left(\frac{x}{2y}\right) = -\frac{x^2}{8y^2}$$

Substituting the values of x^2 and y^2

$$m_1 \times m_2 = -\frac{16/3}{8(2/3)}$$

$$= -\frac{16/3}{16/3} = -1$$

So, the given curves cut orthogonally.

EXERCISE 7.2

1. Find the slope of the tangent to the curves at the respective given points.

(i) $y = x^4 + 2x^2 - x$ at $x = 1$

(ii) $x = a \cos^3 t, y = b \sin^3 t$ at $t = \frac{\pi}{2}$.

Solution:

(i) $y = x^4 + 2x^2 - x$

$$\frac{dy}{dx} = 4x^3 + 4x - 1$$

at $x = 1$, Slope $\frac{dy}{dx} = 4(1)^3 + 4(1) - 1$

$$= 4 + 4 - 1$$

$$m = 8 - 1 = 7$$

(ii) $x = a \cos^3 t$

$$\frac{dx}{dt} = 3a \cos^2 t (-\sin t)$$

$$y = b \sin^3 t$$

$$\frac{dy}{dt} = 3b \sin^2 t (\cos t)$$

$$\text{Slope } \frac{dy}{dx} = \frac{3b \sin^2 t (\cos t)}{3a \cos^2 t (-\sin t)}$$

$$= -\frac{b \sin t}{a \cos t}$$

$$\text{at } t = \frac{\pi}{2}, \frac{dy}{dx} = -\frac{b \sin(\frac{\pi}{2})}{a \cos(\frac{\pi}{2})}$$

$$= -\frac{b(1)}{a(0)}$$

$$= \infty$$

2. Find the point on the curve $y = x^2 - 5x + 4$ at which the tangent is parallel to the line $3x + y = 7$.

Solution: $y = x^2 - 5x + 4$

$$\frac{dy}{dx} = 2x - 5$$

Slope $m_1 = 2x - 5$ and from,

$$3x + y = 7$$

$$y = 7 - 3x$$

$$\frac{dy}{dx} = -3$$

$$m_2 = -3$$

Given the curve is parallel to the line

$$\therefore m_1 = m_2$$

$$2x - 5 = -3$$

$$2x = -3 + 5$$

$$2x = 2 \text{ gives}$$

$$x = 1$$

Since to find the point on the curve,

substituting $x = 1$, in

$$y = x^2 - 5x + 4$$

$$= 1 - 5 + 4$$

$$= 5 - 5$$

$$= 0$$

So, at the point $(1, 0)$ the curve is parallel to the given line.

3. Find the points on the curve

$y = x^3 - 6x^2 + x + 3$ where the normal is parallel to the line $x + y = 1729$.

Solution: Given $y = x^3 - 6x^2 + x + 3$

$$\frac{dy}{dx} = 3x^2 - 12x + 1$$

Slope of the tangent $m = 3x^2 - 12x + 1$

Hence slope of the normal $= -\frac{1}{m}$ and

$$x + y = 1729$$

$$1 + \frac{dy}{dx} = 0$$

$$\frac{dy}{dx} = -1$$

Slope of the line $m = -1$

Since the normal parallel to the line,

$$-\frac{1}{m} = -1$$

$$\frac{1}{m} = 1$$

$$\frac{1}{(3x^2 - 12x + 1)} = 1$$

$$3x^2 - 12x + 1 = 1$$

$$3x^2 - 12x + 1 - 1 = 0$$

$$3x^2 - 12x = 0$$

$$3x(x - 4) = 0$$

$$\text{So, } 3x = 0 \Rightarrow x = 0$$

$$\text{and } x - 4 = 0 \Rightarrow x = 4$$

Substituting $x = 0$ in $y = x^3 - 6x^2 + x + 3$,

we get $y = 3$ and

$$x = 4 \text{ in } y = x^3 - 6x^2 + x + 3$$

$$\text{we get } y = 4^3 - 6(4^2) + 4 + 3$$

$$= 64 - 96 + 4 + 3$$

$$= 71 - 96$$

$$y = -25$$

The normal is parallel to the line at the points $(0, 4)$ and $(4, -25)$

4. Find the points on the curve

$y^2 - 4xy = x^2 + 5$ for which the tangent is horizontal.

Solution: $y^2 - 4xy = x^2 + 5$

$$2y \frac{dy}{dx} - 4 \left(x \frac{dy}{dx} + y \right) = 2x$$

$$2y \frac{dy}{dx} - 4x \frac{dy}{dx} - 4y = 2x$$

$$2y \frac{dy}{dx} - 4x \frac{dy}{dx} = 2x + 4y$$

Dividing by 2,

$$\left(y \frac{dy}{dx} - 2x \frac{dy}{dx} \right) = x + 2y$$

$$(y - 2x) \frac{dy}{dx} = x + 2y$$

$$\frac{dy}{dx} = \frac{x+2y}{(y-2x)}$$

When the tangent is horizontal, it is parallel

to x axis. So, $\frac{dy}{dx} = 0$

$$\text{Gives, } \frac{x+2y}{(y-2x)} = 0$$

$$x + 2y = 0$$

$$x = -2y$$

Substituting $x = -2y$ in

$$y^2 - 4xy = x^2 + 5 \text{ we get}$$

$$y^2 - 4(-2y)y = (-2y)^2 + 5$$

$$y^2 + 8y^2 = 4y^2 + 5$$

$$9y^2 - 4y^2 = 5$$

$$5y^2 = 5$$

$$y^2 = 1$$

$$y = \pm 1$$

When $y = 1$, $x = -2y$ gives $x = -2$ and

$y = -1$, $x = -2y$ gives $x = 2$

At the points $(-2, 1)$ and $(2, -1)$ the tangent is horizontal.

5. Find the tangent and normal to the following curves at the given points on the curve.

(i) $y = x^2 - x^4$ at $(1, 0)$

(ii) $y = x^4 + 2e^x$ at $(0, 2)$

(iii) $y = x \sin x$ at $\left(\frac{\pi}{2}, \frac{\pi}{2}\right)$

(iv) $x = \cos t$, $y = 2\sin^2 t$, at $t = \frac{\pi}{3}$

Solution:

(i) $y = x^2 - x^4$

$$\frac{dy}{dx} = 2x - 4x^3$$

At $(1, 0)$

$$\frac{dy}{dx} = 2(1) - 4(1)^3$$

$$= 2 - 4$$

$$m = -2$$

Slope $m = -2$

Equation of the tangent with slope m through

the point (x_1, y_1) is $y - y_1 = m(x - x_1)$

Substituting $m = -2$, and the point $(1, 0)$

Tangent equation is $y - 0 = -2(x - 1)$

$$y = -2x + 2$$

$$2x + y - 2 = 0$$

Normal is perpendicular to tangent.

Hence normal is of the form $x - 2y + k = 0$

It passes through the point $(1, 0)$

$$\text{Hence, } 1 - 2(0) + k = 0$$

$$1 - 0 + k = 0$$

$$1 + k = 0$$

$$k = -1$$

Equation of the normal is $x - 2y - 1 = 0$

(ii) $y = x^4 + 2e^x$

$$\frac{dy}{dx} = 4x^3 + 2e^x$$

at $(0, 2)$

$$\frac{dy}{dx} = 4(0)^3 + 2e^{(0)}$$

$$= 0 + 2(1)$$

$$m = 2$$

Slope $m = 2$

Equation of the tangent with slope m through

the point (x_1, y_1) is $y - y_1 = m(x - x_1)$

Substituting $m = 2$, and the point $(0, 2)$

Tangent equation is $y - 2 = 2(x - 0)$

$$y - 2 = 2x + 0$$

$$2x - y + 2 = 0$$

Normal is perpendicular to tangent.

Hence normal is of the form $x + 2y + k = 0$

It passes through the point $(0, 2)$

$$\text{Hence, } 0 + 2(2) + k = 0$$

$$0 + 4 + k = 0$$

$$4 + k = 0$$

$$k = -4$$

Equation of the normal is $x + 2y - 4 = 0$

(iii) $y = x \sin x$

$$\frac{dy}{dx} = x(\cos x) + \sin x$$

$$\text{at } \left(\frac{\pi}{2}, \frac{\pi}{2}\right)$$

$$\frac{dy}{dx} = \frac{\pi}{2} \left(\cos \frac{\pi}{2}\right) + \sin \frac{\pi}{2}$$

$$= \frac{\pi}{2}(0) + 1$$

$$= 0 + 1$$

$$m = 1$$

$$\text{Slope } m = 1$$

Equation of the tangent with slope m through

the point (x_1, y_1) is $y - y_1 = m(x - x_1)$

Substituting $m = 1$, and the point $\left(\frac{\pi}{2}, \frac{\pi}{2}\right)$

Tangent equation is $y - \frac{\pi}{2} = 1 \left(x - \frac{\pi}{2}\right)$

$$y - \frac{\pi}{2} = x - \frac{\pi}{2}$$

$$x - y = 0$$

Normal is perpendicular to tangent.

Hence normal is of the form $x + y + k = 0$

It passes through the point $\left(\frac{\pi}{2}, \frac{\pi}{2}\right)$

$$\text{Hence, } \frac{\pi}{2} + \frac{\pi}{2} + k = 0$$

$$\pi + k = 0 \text{ gives } k = -\pi$$

Equation of the normal is $x + y - \pi = 0$

(iv) $x = \cos t$, and $y = 2\sin^2 t$

$$\frac{dx}{dt} = -\sin t \quad \frac{dy}{dt} = 4 \sin t \cos t$$

$$\text{So, } \frac{dy}{dx} = \frac{4 \sin t \cos t}{-\sin t}$$

$$= -4 \cos t$$

$$\text{at } t = \frac{\pi}{3}$$

$$\frac{dy}{dx} = -4 \cos t$$

$$= -4 \cos \left(\frac{\pi}{3}\right)$$

$$= -4 \left(\frac{1}{2}\right)$$

$$m = -2$$

Slope $m = -2$

$$(x_1, y_1) = \left(\cos \frac{\pi}{3}, 2\sin^2 \frac{\pi}{3}\right)$$

$$= \left(\frac{1}{2}, 2\left(\frac{\sqrt{3}}{2}\right)^2\right)$$

$$= \left(\frac{1}{2}, 2\left(\frac{3}{4}\right)\right)$$

$$= \left(\frac{1}{2}, \frac{3}{2}\right)$$

Equation of the tangent with slope m through the point (x_1, y_1) is $y - y_1 = m(x - x_1)$

Substituting $m = -2$, and the point $\left(\frac{1}{2}, \frac{3}{2}\right)$

Tangent equation is $y - \frac{3}{2} = -2 \left(x - \frac{1}{2}\right)$

Multiplying by 2,

$$2y - 3 = -4 \left(x - \frac{1}{2}\right)$$

$$2y - 3 = -4x + 2$$

$$4x + 2y - 3 - 2 = 0$$

$$4x + 2y - 5 = 0$$

Normal is perpendicular to tangent.

Hence normal is of the form $2x - 4y + k = 0$

It passes through the point $\left(\frac{1}{2}, \frac{3}{2}\right)$

$$\text{Hence, } 2 \left(\frac{1}{2}\right) - 4 \left(\frac{3}{2}\right) + k = 0$$

$$1 - 6 + k = 0$$

$$-5 + k = 0$$

$$k = 5$$

Equation of the normal is $2x - 4y + 5 = 0$

6. Find the equations of the tangents to the curve $y = 1 + x^3$ for which the tangent is orthogonal with the line $x + 12y = 12$.

Solution: Given $y = 1 + x^3$

$$\frac{dy}{dx} = 3x^2$$

Slope of the tangent $m_1 = 3x^2$

Equation of the given line $x + 12y = 12$

$$1 + 12 \frac{dy}{dx} = 0$$

$$12 \frac{dy}{dx} = -1$$

$$\frac{dy}{dx} = -\frac{1}{12}$$

Slope of the line $m_2 = -\frac{1}{12}$

Since the tangent is orthogonal to the line,

$$m_1 \times m_2 = -1$$

$$3x^2 \times -\frac{1}{12} = -1$$

$$\frac{x^2}{4} = 1$$

$$x^2 = 4$$

$$x = \pm 2$$

The points lie on the curve $y = 1 + x^3$

When $x = 2$, $y = 1 + 2^3$

$$= 1 + 8$$

$$y = 9$$

When $x = -2$, $y = 1 + (-2)^3$

$$= 1 - 8$$

$$y = -7$$

Hence (2, 9) and (-2, -7) are the points.

At $x = \pm 2$

Slope of the tangent $m = 3x^2$

$$= 3(\pm 2)^2$$

$$m = 12$$

Equation of the tangent with slope m through

the point (x_1, y_1) is $y - y_1 = m(x - x_1)$

(i) Substituting $m = 12$, and the point (2, 9)

Tangent equation is $y - 9 = 12(x - 2)$

$$y - 9 = 12x - 24$$

$$12x - y - 24 + 9 = 0$$

$$12x - y - 15 = 0$$

(ii) Substituting $m = 12$, and the point (-2, -7)

Tangent equation is $y + 7 = 12(x + 2)$

$$y + 7 = 12x + 24$$

$$12x - y + 24 - 7 = 0$$

$$12x - y + 17 = 0$$

7. Find the equations of the tangents to the curve $y = \frac{x+1}{x-1}$ which are parallel to the line $x + 2y = 6$.

Solution: Given $y = \frac{x+1}{x-1}$

$$\frac{dy}{dx} = \frac{(x-1)(1) - (x+1)(1)}{(x-1)^2}$$

$$= \frac{x-1-x-1}{(x-1)^2}$$

$$= \frac{-2}{(x-1)^2}$$

Slope of the tangent $m_1 = \frac{-2}{(x-1)^2}$

Equation of the given line $x + 2y = 6$

$$1 + 2 \frac{dy}{dx} = 0$$

$$2 \frac{dy}{dx} = -1$$

$$\frac{dy}{dx} = -\frac{1}{2}$$

Slope of the line $m_2 = -\frac{1}{2}$

Since the tangent is parallel to the line,

$$m_1 = m_2$$

$$\frac{-2}{(x-1)^2} = -\frac{1}{2}$$

$$\frac{2}{(x-1)^2} = \frac{1}{2}$$

$$4 = (x-1)^2$$

$$(x-1)^2 = 4$$

$$x^2 - 2x + 1 = 4$$

$$x^2 - 2x + 1 - 4 = 0$$

$$x^2 - 2x - 3 = 0$$

$$(x-3)(x+1) = 0$$

$x - 3 = 0$, gives $x = 3$ and

$$x + 1 = 0, \text{ gives } x = -1$$

The points lie on the curve $y = \frac{x+1}{x-1}$

$$\text{When } x = 3, y = \frac{3+1}{3-1}$$

$$= \frac{4}{2}$$

$$y = 2$$

$$\text{When } x = -1, y = \frac{-1+1}{-1-1}$$

$$= 0$$

$$y = 0$$

Hence (3, 2) and (-1, 0) are the points.

(i) At $x = 3$,

$$\text{Slope of the tangent } m = \frac{-2}{(x-1)^2}$$

$$= \frac{-2}{(3-1)^2}$$

$$= \frac{-2}{(2)^2}$$

$$= \frac{-2}{4}$$

$$m = -\frac{1}{2}$$

Equation of the tangent with slope m through

the point (x_1, y_1) is $y - y_1 = m(x - x_1)$

Substituting $m = -\frac{1}{2}$, and the point (3, 2)

Tangent equation is $y - 2 = -\frac{1}{2}(x - 3)$

$$2(y - 2) = -(x - 3)$$

$$2y - 4 = -x + 3$$

$$x + 2y - 4 - 3 = 0$$

$$x + 2y - 7 = 0$$

(ii) At $x = -1$,

$$\text{Slope of the tangent } m = \frac{-2}{(x-1)^2}$$

$$= \frac{-2}{(-1-1)^2}$$

$$= \frac{-2}{(-2)^2}$$

$$= \frac{-2}{4}$$

$$m = -\frac{1}{2}$$

Equation of the tangent with slope m through

the point (x_1, y_1) is $y - y_1 = m(x - x_1)$

Substituting $m = -\frac{1}{2}$, and the point (-1, 0)

Tangent equation is $y - 0 = -\frac{1}{2}(x + 1)$

$$2y = -(x + 1)$$

$$2y = -x - 1$$

$$x + 2y + 1 = 0$$

$$x + 2y + 1 = 0$$

8. Find the equation of tangent and normal to the curve given by $x = 7 \cos t$ and $y = 2 \sin t$, $t \in \mathbb{R}$ at any point on the curve.

Solution: $x = 7 \cos t$

$$\frac{dx}{dt} = -7 \sin t$$

$$y = 2 \sin t$$

$$\frac{dy}{dt} = 2 \cos t$$

$$\text{Slope of the tangent } m = \frac{dy}{dx} = \frac{-2 \cos t}{7 \sin t}$$

$$m = -\frac{2 \cos t}{7 \sin t}$$

Equation of the tangent with slope m through

the point (x_1, y_1) is $y - y_1 = m(x - x_1)$

Substituting $m = -\frac{2 \cos t}{7 \sin t}$,

and the point $(7 \cos t, 2 \sin t)$

Equation of tangent is

$$(y - 2 \sin t) = -\frac{2 \cos t}{7 \sin t}(x - 7 \cos t)$$

$$(y - 2 \sin t)7 \sin t = -2 \cos t(x - 7 \cos t)$$

$$(7 \sin t)y - 14 \sin^2 t = -(2 \cos t)x + 14 \cos^2 t$$

$$(2 \cos t)x + (7 \sin t)y - 14 \sin^2 t - 14 \cos^2 t = 0$$

$$(2 \cos t)x + (7 \sin t)y - 14(\sin^2 t + \cos^2 t) = 0$$

$$(2 \cos t)x + (7 \sin t)y - 14(1) = 0$$

$$(2 \cos t)x + (7 \sin t)y - 14 = 0$$

Normal is perpendicular to tangent.

Hence normal is of the form

$$(7 \sin t)x - (2 \cos t)y + k = 0$$

It passes through $(7 \cos t, 2 \sin t)$

$$(7 \sin t)(7 \cos t) - (2 \cos t)(2 \sin t) + k = 0$$

$$49 \sin t \cos t - 4 \sin t \cos t + k = 0$$

$$45 \sin t \cos t + k = 0$$

$$k = -45 \sin t \cos t$$

So equation of the normal is

$$(7 \sin t)x - (2 \cos t)y - 45 \sin t \cos t = 0$$

9. Find the angle between the rectangular hyperbola $xy = 2$ and the parabola $x^2 + 4y = 0$.

Solution:

Given $xy = 2$

Gives $y = \frac{2}{x}$

Substituting $y = \frac{2}{x}$ in $x^2 + 4y = 0$, we get

$$x^2 + 4\left(\frac{2}{x}\right) = 0$$

$$x^2 + \frac{8}{x} = 0$$

Multiplying by x , $x^3 + 8 = 0$

$$x^3 = -8 = (-2)^3$$

Gives $x = -2$

Substituting $x = -2$, in $xy = 2$ we get

$$(-2)y = 2$$

$$-2y = 2$$

gives, $y = -1$

So, the point of intersection is $(-2, -1)$

Again from $xy = 2$

$$x \frac{dy}{dx} + y(1) = 0$$

$$x \frac{dy}{dx} + y = 0$$

$$x \frac{dy}{dx} = -y$$

$$\frac{dy}{dx} = -\frac{y}{x}$$

Slope of the first curve at $(-2, -1)$

$$m_1 = -\frac{1}{2} \text{ and from}$$

$$x^2 + 4y = 0$$

$$2x + 4 \frac{dy}{dx} = 0$$

$$4 \frac{dy}{dx} = -2x$$

$$\frac{dy}{dx} = -\frac{2x}{4}$$

Slope of the second curve at $(-2, -1)$

$$m_2 = -\frac{x}{2} = \frac{2}{2} = 1$$

If θ is the angle between the curves, then

$$\tan \theta = \left| \frac{m_1 - m_2}{1 + m_1 m_2} \right|$$

$$= \left| \frac{\left(-\frac{1}{2}\right) - 1}{1 + \left(-\frac{1}{2}\right)(1)} \right|$$

$$= \left| \frac{\frac{-1-2}{2}}{1 - \frac{1}{2}} \right|$$

$$= \left| \frac{\frac{-3}{2}}{\frac{1}{2}} \right|$$

$$= \left| \frac{-3}{2} \times \frac{2}{1} \right|$$

$$= |-3|$$

$$\tan \theta = 3$$

$$\theta = \tan^{-1}(3)$$

10. Show that the two curves $x^2 - y^2 = r^2$ and $xy = c^2$ where c, r are constants, cut orthogonally.

Solution: Given $x^2 - y^2 = r^2$

$$2x - 2y \frac{dy}{dx} = 0$$

$$-2y \frac{dy}{dx} = -2x$$

$$2y \frac{dy}{dx} = 2x$$

$$\frac{dy}{dx} = \frac{2x}{2y}$$

$$\frac{dy}{dx} = \frac{x}{y}$$

Let (x_1, y_1) be point of intersection.

Then slope of the first curve at (x_1, y_1) $m_1 = \frac{x_1}{y_1}$

Again from $xy = c^2$

$$x \frac{dy}{dx} + y(1) = 0$$

$$x \frac{dy}{dx} + y = 0$$

$$x \frac{dy}{dx} = -y$$

$$\frac{dy}{dx} = -\frac{y}{x}$$

Slope of the second curve at (x_1, y_1) $m_2 = -\frac{y_1}{x_1}$

$$\begin{aligned}\text{Product of the slopes } m_1 \times m_2 &= \left(\frac{x_1}{y_1}\right) \times \left(-\frac{y_1}{x_1}\right) \\ &= -1\end{aligned}$$

Hence the two given curves cut orthogonally.

Example 7.19 Compute the value of 'c' satisfied by the Rolle's theorem for the function

$$f(x) = x^2(1-x)^2, x \in [0,1].$$

Solution: $f(x)$ is continuous in $[0,1]$ and

$f(x)$ is differentiable in $(0,1)$

$$\text{Given } f(x) = x^2(1-x)^2, x \in [0,1]$$

$$f(a) = f(0) = 0^2(1-0)^2 = 0$$

$$f(b) = f(1) = 1^2(1-1)^2 = 0$$

$$\text{Hence } f(a) = f(b)$$

$$\begin{aligned}f(x) &= x^2(1-x)^2 \\ &= x^2(x^2 - 2x + 1) \\ &= x^4 - 2x^3 + x^2\end{aligned}$$

$$f'(x) = 4x^3 - 6x^2 + 2x$$

$$f'(c) = 4c^3 - 6c^2 + 2c$$

Substituting $f'(c) = 0$, we get

$$4c^3 - 6c^2 + 2c = 0$$

$$c(4c^2 - 6c + 2) = 0$$

$$\text{Gives, } c = 0 \text{ and } (4c^2 - 6c + 2) = 0$$

$$\text{When } 4c^2 - 6c + 2 = 0$$

$$\text{Dividing by 2, } 2c^2 - 3c + 1 = 0$$

$$(2c - 1)(c - 1) = 0$$

$$2c - 1 = 0 \Rightarrow 2c = 1 \Rightarrow c = \frac{1}{2} \text{ and}$$

$$c - 1 = 0 \Rightarrow c = 1$$

Hence $c = 0, c = \frac{1}{2}, c = 1$ are the values.

By Rolle's Theorem, when $f'(c) = 0, c \in [a, b]$

$$\therefore c = \frac{1}{2} \in [0,1]$$

Example 7.20 Find the values in the interval $\left(\frac{1}{2}, 2\right)$ satisfied by the Rolle's theorem for the

$$\text{function } f(x) = x + \frac{1}{x}, x \in \left[\frac{1}{2}, 2\right].$$

Solution: $f(x)$ is continuous in $\left[\frac{1}{2}, 2\right]$ and

$f(x)$ is differentiable in $\left(\frac{1}{2}, 2\right)$

$$\text{Given } f(x) = x + \frac{1}{x}, x \in \left[\frac{1}{2}, 2\right]$$

$$f(a) = f\left(\frac{1}{2}\right) = \frac{1}{2} + 2 = \frac{5}{2}$$

$$f(b) = f(2) = 2 + \frac{1}{2} = \frac{5}{2}$$

$$\text{Hence } f(a) = f(b)$$

$$f(x) = x + \frac{1}{x}$$

$$f'(x) = 1 - \frac{1}{x^2}$$

$$f'(c) = 1 - \frac{1}{c^2}$$

Substituting $f'(c) = 0$, we get

$$1 - \frac{1}{c^2} = 0$$

$$c^2 - 1 = 0$$

$$c^2 = 1$$

$$c = \pm 1 \text{ gives}$$

$$c = 1 \text{ and } c = -1$$

By Rolle's Theorem, when $f'(c) = 0, c \in [a, b]$

$$\therefore c = 1 \in \left[\frac{1}{2}, 2\right]$$

Example 7.21 Compute the value of 'c' satisfied by Rolle's theorem for the function

$$f(x) = \log\left(\frac{x^2+6}{5x}\right) \text{ in the interval } [2,3].$$

Solution: $f(x)$ is continuous in $[2,3]$ and

$f(x)$ is differentiable in $(2,3)$

$$f(x) = \log\left(\frac{x^2+6}{5x}\right)$$

$$f(x) = \log(x^2 + 6) - \log(5x)$$

$$f(a) = f(2) = \log(4 + 6) - \log(10)$$

$$= \log(10) - \log(10)$$

$$f(a) = 0$$

$$f(b) = f(3) = \log(9 + 6) - \log(15)$$

$$= \log(15) - \log(15)$$

$$f(b) = 0$$

$$\text{Hence, } f(a) = f(b) = 0$$

From $f(x) = \log\left(\frac{x^2+6}{5x}\right)$

$$f'(x) = \frac{1}{x^2+6}(2x) - \frac{1}{5x}(5)$$

$$= \frac{2x}{x^2+6} - \frac{1}{x}$$

$$f'(c) = \frac{2c}{c^2+6} - \frac{1}{c}$$

When $f'(c) = 0$, gives

$$\frac{2c}{c^2+6} - \frac{1}{c} = 0$$

$$\frac{2c}{c^2+6} = \frac{1}{c}$$

$$(2c)c = c^2 + 6$$

$$2c^2 = c^2 + 6$$

$$2c^2 - c^2 = 6$$

$$c^2 = 6$$

$$c = \pm \sqrt{6}$$

Since $c = \sqrt{6} \in [2, 3]$, the satisfied

value is $c = \sqrt{6}$

Example 7.22 Without actually solving show that the equation $x^4 + 2x^3 - 2 = 0$ has only one real root in the interval $(0, 1)$.

Solution: Let $f(x) = x^4 + 2x^3 - 2$

$f(x)$ is continuous in $[0, 1]$ and

$f(x)$ is differentiable in $(0, 1)$

$$\text{Now, } f'(x) = 4x^3 + 6x^2$$

Substituting $f'(x) = 0$

$$4x^3 + 6x^2 = 0$$

$$2x^2(2x + 3) = 0$$

$$2x^2 = 0 \text{ and } 2x + 3 = 0$$

$$2x^2 = 0 \Rightarrow x = 0 \text{ (twice)}$$

$$2x + 3 = 0 \Rightarrow x = -\frac{3}{2}$$

$$\text{But } x = 0, -\frac{3}{2} \notin (0, 1)$$

Hence by the Rolle's theorem there do not exist

$a, b \in (0, 1)$ such that, $f(a) = 0 = f(b)$.

Therefore the equation $f(x) = 0$ cannot have

two roots in the interval $(0, 1)$.

But, $f(0) = -2 < 0$ and $f(1) = 1 > 0$

tells us the curve $y = f(x)$ crosses the x -axis between 0 and 1 only once by the Intermediate value theorem. Therefore the equation $x^4 + 2x^3 - 2 = 0$ has only one real root in the interval $(0, 1)$

Example 7.23 Prove using the Rolle's theorem that between any two distinct real zeros of the polynomial $a_n x^n + a_{n-1} x^{n-1} + \dots + a_1 x + a_0$ there is a zero of the polynomial $n a_n x^{n-1} + (n-1) a_{n-1} x^{n-2} + \dots + a_1$.

Solution:

$$P(x) = a_n x^n + a_{n-1} x^{n-1} + \dots + a_1 x + a_0$$

Let α, β be the two zeros of the polynomial.

$\therefore P(x)$ is continuous on $[\alpha, \beta]$

$P(x)$ is differentiable on (α, β)

Hence $P(\alpha) = P(\beta)$

By Rolle's theorem there exists a value $\gamma \in$

(α, β) such that $P'(\gamma) = 0$

So,

$$P(\gamma) = n a_n \gamma^{n-1} + (n-1) a_{n-1} \gamma^{n-2} + \dots + a_1$$

Example 7.24 Prove that there is a zero of the polynomial, $2x^3 - 9x^2 - 11x + 12$ in the interval $(2, 7)$ given that 2 and 7 are the zeros of the polynomial $x^4 - 6x^3 - 11x^2 + 24x + 28$

Solution: $f(x) = x^4 - 6x^3 - 11x^2 + 24x + 28$

$f(x)$ is continuous on $[2, 7]$

$f(x)$ is differentiable on $(2, 7)$

$$f(a) = f(2) = (2)^4 - 6(2)^3 - 11(2)^2 + 24(2) + 28$$

$$= 16 - 6(8) - 11(4) + 48 + 28$$

$$= 16 - 48 - 44 + 48 + 28$$

$$= 44 - 48 - 44 + 48$$

$$= 0$$

$$f(b) = f(7) = (7)^4 - 6(7)^3 - 11(7)^2 + 24(7) + 28$$

$$= 2401 - 6(343) - 11(49) + 168 + 28$$

$$= 2401 - 2058 - 539 + 196$$

$$= 2597 - 2597 = 0$$

$$\text{Hence } f(a) = f(b) = 0$$

By Rolle's theorem there exists a value $c \in (a, b)$ such that $f'(c) = 0$

$$f(x) = x^4 - 6x^3 - 11x^2 + 24x + 28$$

$$f'(x) = 4x^3 - 12x^2 - 22x + 24$$

$$f'(c) = 4c^3 - 12c^2 - 22c + 24$$

When $f'(c) = 0$, gives

$$4c^3 - 12c^2 - 22c + 24 = 0$$

Dividing by 2,

$$2c^3 - 6c^2 - 11c + 12 = 0$$

Hence there is a zero of the above polynomial lies in the given interval $(2, 7)$.

Example 7.25 Find the values in the interval $(1, 2)$ of the mean value theorem satisfied by the function $f(x) = x - x^2$ for $1 \leq x \leq 2$.

Solution: Mean value theorem states that

$f(x)$ is continuous on $[a, b]$

$f(x)$ is differentiable on (a, b)

$$f(a) \neq f(b)$$

There exists a least possible value $c \in (a, b)$

$$\text{such that } f'(c) = \frac{f(b) - f(a)}{b - a}$$

$$\text{Given } f(x) = x - x^2 \text{ for } 1 \leq x \leq 2$$

$f(x)$ is continuous on $[1, 2]$

$f(x)$ is differentiable on $(1, 2)$

$$f(a) = f(1) = 1 - 1^2 = 0$$

$$f(b) = f(2) = 2 - (2)^2$$

$$= 2 - 4$$

$$= -2$$

Hence, $f(a) \neq f(b)$

$$\text{Now } f(x) = x - x^2$$

$$f'(x) = 1 - 2x$$

$$f'(c) = 1 - 2c$$

$$f'(c) = \frac{f(b) - f(a)}{b - a} \text{ gives,}$$

$$1 - 2c = \frac{-2 - 0}{2 - 1}$$

$$1 - 2c = \frac{-2}{1}$$

$$1 - 2c = -2$$

$$-2c = -3$$

$$2c = 3$$

$$c = \frac{3}{2}$$

$\therefore c = \frac{3}{2} \in (1, 2)$ is the required value.

Example 7.26 A truck travels on a toll road with a speed limit of 80 km/hr. The truck completes a 164 km journey in 2 hours. At the end of the toll road the trucker is issued with a speed violation ticket. Justify this using the Mean Value Theorem.

Solution:

Let $f(t)$ be the distance travelled by the trucker in ' t ' hours.

This is a continuous function in $[0, 2]$ and differentiable in $(0, 2)$.

$$\text{Now, } f(0) = 0 \text{ and } f(2) = 164.$$

By an application of the Mean Value Theorem, there exists a time c such that,

$$f'(c) = \frac{f(b) - f(a)}{b - a}$$

$$= \frac{164 - 0}{2 - 0}$$

$$= \frac{164}{2}$$

$$= 82 > 80$$

Therefore at some point of time, during the travel in 2 hours the trucker must have travelled with a speed more than 80 km which justifies the issuance of a speed violation ticket.

Example 7.27

Suppose $f(x)$ is a differentiable function for all x with $f'(x) \leq 29$ and $f(2) = 17$. What is the maximum value of $f(7)$?

Solution: By an application of the Mean Value

Theorem, there exists a time $c \in (2, 7)$ such

$$\text{that, } f'(c) = \frac{f(b) - f(a)}{b - a}$$

$$\frac{f(7) - f(2)}{7 - 2} = f'(c) \leq 29$$

$$\frac{f(7)-17}{5} \leq 29$$

$$\begin{aligned} f(7) &\leq 29(5) + 17 \\ &\leq 145 + 17 \\ &\leq 162 \end{aligned}$$

Therefore maximum value of $f(7)$ is 162.

Example 7.28 Prove, using mean value theorem, that $|\sin \alpha - \sin \beta| \leq |\alpha - \beta|$, $\alpha, \beta \in \mathbb{R}$.

Solution:

Let $f(x) = \sin x$ which is a differentiable function in any open interval. Consider an interval $[\alpha, \beta]$. Applying the mean value theorem there exists $c \in (\alpha, \beta)$ such that,

$$\frac{\sin \beta - \sin \alpha}{\beta - \alpha} = f'(c) = \cos c$$

$$\text{Therefore } \left| \frac{\sin \beta - \sin \alpha}{\beta - \alpha} \right| = |\cos c| \leq 1$$

$$\text{Hence, } |\sin \alpha - \sin \beta| \leq |\alpha - \beta|$$

Example 7.29

A thermometer was taken from a freezer and placed in boiling water. It took 22 seconds for the thermometer to raise from -10°C to 100°C . Show that the rate of change of temperature at some time t is 5°C per second.

Solution:

Let $f(t)$ be the temperature at time t . By the mean value theorem,

$$\begin{aligned} f'(c) &= \frac{f(b)-f(a)}{b-a} \\ &= \frac{100-(-10)}{22-0} \\ &= \frac{100+10}{22} \\ &= \frac{110}{22} \\ &= 5^\circ\text{C per second.} \end{aligned}$$

Hence the instantaneous rate of change of temperature at some time t should be 5°C per second.

EXERCISE 7.3

1. Explain why Rolle's theorem is not applicable to the following functions in the respective intervals.

$$(i) f(x) = \left| \frac{1}{x} \right|, x \in [-1, 1]$$

$$(ii) f(x) = \tan x, x \in [0, \pi]$$

$$(iii) f(x) = x - 2 \log x, x \in [2, 7]$$

Solution:

$$(i) f(x) = \left| \frac{1}{x} \right|, x \in [-1, 1]$$

Given $x \in [-1, 1]$.

At $x = 0$, $f(x)$ is not continuous, hence

Rolle's Theorem is not applicable.

$$(ii) f(x) = \tan x, x \in [0, \pi]$$

Given $x \in [0, \pi]$.

At $x = \frac{\pi}{2}$, $f(x)$ is not continuous, hence

Rolle's Theorem is not applicable.

$$(iii) f(x) = x - 2 \log x, x \in [2, 7]$$

$f(x)$ is continuous on $[2, 7]$

$f(x)$ is differentiable on $(2, 7)$

$$f(a) = f(2) = 2 - 2 \log 2 \quad \text{and}$$

$$f(b) = f(7) = 7 - 2 \log 7$$

Hence, $f(a) \neq f(b)$ hence

Rolle's Theorem is not applicable.

2. Using the Rolle's theorem, determine the values of x at which the tangent is parallel to the x -axis for the following functions :

$$(i) f(x) = x^2 - x, x \in [0, 1]$$

$$(ii) f(x) = \frac{x^2 - 2x}{x + 2}, x \in [-1, 6]$$

$$(iii) f(x) = \sqrt{x} - \frac{x}{3}, x \in [0, 9]$$

Solution:

$$(i) f(x) = x^2 - x, x \in [0, 1]$$

$f(x)$ is continuous on $[0, 1]$

$f(x)$ is differentiable on $(0, 1)$

$$f(x) = x^2 - x$$

$$f(a) = f(0) = 0 \quad \text{and}$$

$$f(b) = f(1) = 1 - 1 = 0$$

Hence, $f(a) = f(b)$ hence

There exists a least possible value 'c' such

$$\text{that } f'(c) = 0$$

$$\text{From } f(x) = x^2 - x$$

$$f'(x) = 2x - 1$$

$$f'(c) = 2c - 1$$

When $f'(c) = 0$, we get

$$2c - 1 = 0$$

$$2c = 1$$

$$c = \frac{1}{2} \in (0,1)$$

$$\text{At } x = \frac{1}{2}, y = f(x) = x^2 - x$$

$$y = f\left(\frac{1}{2}\right) = \left(\frac{1}{2}\right)^2 - \frac{1}{2}$$

$$y = \frac{1}{4} - \frac{1}{2}$$

$$= \frac{1-2}{4}$$

$$= -\frac{1}{4}$$

Hence at the point $\left(\frac{1}{2}, -\frac{1}{4}\right)$ the tangent is parallel to x -axis.

$$(ii) f(x) = \frac{x^2 - 2x}{x + 2}, x \in [-1, 6]$$

$f(x)$ is continuous on $[-1, 6]$

$f(x)$ is differentiable on $(-1, 6)$

$$f(x) = \frac{x^2 - 2x}{x + 2}$$

$$f(a) = f(-1)$$

$$= \frac{(-1)^2 - 2(-1)}{-1 + 2}$$

$$= \frac{1+2}{1} = 3$$

$$f(b) = f(6)$$

$$= \frac{(6)^2 - 2(6)}{6 + 2}$$

$$= \frac{36-12}{8}$$

$$= \frac{24}{8} = 3$$

Hence, $f(a) = f(b)$ hence

There exists a least possible value 'c' such

that $f'(c) = 0$

$$\text{From } f(x) = \frac{x^2 - 2x}{x + 2}$$

$$f'(x) = \frac{(x+2)(x^2-2x)' - (x^2-2x)(x+2)'}{(x+2)^2}$$

$$= \frac{(x+2)(2x-2) - (x^2-2x)(1)}{(x+2)^2}$$

$$= \frac{(2x^2-2x+4x-4) - (x^2-2x)}{(x+2)^2}$$

$$= \frac{(2x^2+2x-4) - (x^2-2x)}{(x+2)^2}$$

$$= \frac{2x^2+2x-4-x^2+2x}{(x+2)^2}$$

$$f'(x) = \frac{x^2+4x-4}{(x+2)^2}$$

$$f'(c) = \frac{c^2+4c-4}{(c+2)^2}$$

When $f'(c) = 0$

$$\text{We get } \frac{c^2+4c-4}{(c+2)^2} = 0$$

$$c^2 + 4c - 4 = 0$$

$$c = \frac{-4 \pm \sqrt{16+16}}{2}$$

$$= \frac{-4 \pm \sqrt{16 \times 2}}{2}$$

$$= \frac{-4 \pm 4\sqrt{2}}{2}$$

$$= \frac{2(-2 \pm 2\sqrt{2})}{2}$$

$$= -2 \pm 2\sqrt{2}$$

$$c = -2 + 2\sqrt{2} \in (-1, 6)$$

$$\text{At } x = -2 + 2\sqrt{2},$$

$$y = f(x) = \frac{x^2 - 2x}{x + 2}$$

$$y = f(-2 + 2\sqrt{2})$$

$$= \frac{(-2+2\sqrt{2})^2 - 2(-2+2\sqrt{2})}{(-2+2\sqrt{2}) + 2}$$

$$= \frac{4+8-8\sqrt{2}+4-4\sqrt{2}}{2\sqrt{2}}$$

$$= \frac{16-12\sqrt{2}}{2\sqrt{2}} = \frac{2(8-6\sqrt{2})}{2\sqrt{2}}$$

$$= \frac{8-6\sqrt{2}}{\sqrt{2}} \times \frac{\sqrt{2}}{\sqrt{2}}$$

$$= \frac{8\sqrt{2}-6(2)}{2}$$

$$y = \frac{2(4\sqrt{2}-6)}{2} = 4\sqrt{2} - 6$$

Hence at the point $(-2 + 2\sqrt{2}, 4\sqrt{2} - 6)$

the tangent is parallel to x -axis.

(iii) $f(x) = \sqrt{x} - \frac{x}{3}, x \in [0,9]$

$f(x)$ is continuous on $[0,9]$

$f(x)$ is differentiable on $(0,9)$

$$f(a) = f(0) = \sqrt{0} - \frac{0}{3} = 0$$

$$f(b) = f(9) = \sqrt{9} - \frac{9}{3} = 3 - 3 = 0$$

Hence, $f(a) = f(b)$ hence

There exists a least possible value 'c' such that $f'(c) = 0$

$$\text{From } f(x) = \sqrt{x} - \frac{x}{3}$$

$$f'(x) = \frac{1}{2\sqrt{x}} - \frac{1}{3}$$

$$f'(c) = \frac{1}{2\sqrt{c}} - \frac{1}{3}$$

When $f'(c) = 0$

$$\frac{1}{2\sqrt{c}} - \frac{1}{3} = 0$$

$$\frac{1}{2\sqrt{c}} = \frac{1}{3}$$

$$3 = 2\sqrt{c}$$

Squaring, $4c = 9$

$$c = \frac{9}{4} \in (0,9)$$

At $x = \frac{9}{4}$,

$$y = f(x) = \sqrt{\frac{9}{4}} - \frac{\frac{9}{4}}{3}$$

$$= \frac{3}{2} - \frac{9}{4} \times \frac{1}{3}$$

$$= \frac{3}{2} - \frac{3}{4}$$

$$= \frac{6-3}{4}$$

$$= \frac{3}{4}$$

Hence at the point $\left(\frac{9}{4}, \frac{3}{4}\right)$ the tangent is parallel to x-axis.

3. Explain why Lagrange's mean value theorem is not applicable to the following functions in the respective intervals :

(i) $f(x) = \frac{x+1}{x}, x \in [-1, 2]$

(ii) $f(x) = |3x + 1|, x \in [-1, 3]$

Solution:

(i) $f(x) = \frac{x+1}{x}, x \in [-1, 2]$

When $x = 0$,

$$f(0) = \frac{0+1}{0}$$

$$= \frac{1}{0}$$

$$= \infty$$

Hence $f(x)$ is discontinuous, So Lagrange's mean value theorem is not applicable.

(ii) $f(x) = |3x + 1|, x \in [-1, 3]$

$f(x)$ is continuous on $[-1, 3]$

$f(x)$ is not differentiable on $(-1, 3)$

So Lagrange's mean value theorem is not applicable.

4. Using the Lagrange's mean value theorem determine the values of x at which the tangent is parallel to the secant line at the end points of the given interval:

(i) $f(x) = x^3 - 3x + 2, x \in [-2, 2]$

(ii) $f(x) = (x-2)(x-7), x \in [3, 11]$

Solution:

(i) $f(x) = x^3 - 3x + 2, x \in [-2, 2]$

$f(x)$ is continuous on $[-2, 2]$

$f(x)$ is differentiable on $(-2, 2)$

Now, $f(a) = f(-2) = (-2)^3 - 3(-2) + 2$

$$= -8 + 6 + 2$$

$$= -8 + 8$$

$$= 0 \text{ and}$$

$$f(b) = f(2) = (2)^3 - 3(2) + 2$$

$$= 8 - 6 + 2$$

$$= 10 - 6$$

$$= 4$$

Hence, $f(a) \neq f(b)$ hence

By an application of the Mean Value

Theorem, there exists a time c such that,

$$f'(c) = \frac{f(b) - f(a)}{b - a}$$

$$f(x) = x^3 - 3x + 2$$

$$f'(x) = 3x^2 - 3$$

$$f'(c) = 3c^2 - 3$$

$$\therefore f'(c) = \frac{f(b) - f(a)}{b - a}$$

$$3c^2 - 3 = \frac{4 - 0}{2 - (-2)}$$

$$= \frac{4}{4}$$

$$3c^2 - 3 = 1$$

$$3c^2 = 4$$

$$c^2 = \frac{4}{3}$$

$$c = \pm \frac{2}{\sqrt{3}}$$

$$c = \pm \frac{2}{\sqrt{3}} \in [-2, 2]$$

At $x = \pm \frac{2}{\sqrt{3}}$ the tangent is parallel to the secant.

$$(ii) f(x) = (x - 2)(x - 7), x \in [3, 11]$$

$$f(x) = x^2 - 9x + 14$$

$f(x)$ is continuous on $[3, 11]$

$f(x)$ is differentiable on $(3, 11)$

$$\text{Now, } f(a) = f(3) = (3)^2 - 9(3) + 14$$

$$= 9 - 27 + 14$$

$$= 23 - 27$$

$$= -4 \text{ and}$$

$$f(b) = f(11) = (11)^2 - 9(11) + 14$$

$$= 121 - 99 + 14$$

$$= 135 - 99$$

$$= 36$$

Hence, $f(a) \neq f(b)$ hence

By an application of the Mean Value

Theorem, there exists a time c such that,

$$f'(c) = \frac{f(b) - f(a)}{b - a}$$

$$f(x) = x^2 - 9x + 14$$

$$f'(x) = 2x - 9$$

$$f'(c) = 2c - 9$$

$$2c - 9 = \frac{36 - 4}{11 - 3}$$

$$2c - 9 = \frac{40}{8} = 5$$

$$2c = 5 + 9$$

$$2c = 14$$

$$c = \frac{14}{2} = 7 \in (3, 11)$$

At $x = 7$ the tangent is parallel to the secant.

5. Show that the value in the conclusion of the mean value theorem for

(i) $f(x) = \frac{1}{x}$ on a closed interval of positive numbers $[a, b]$ is \sqrt{ab}

(ii) $f(x) = Ax^2 + Bx + C$ on any interval $[a, b]$ is $\frac{a+b}{2}$

Solution:

$$(i) f(x) = \frac{1}{x}, x \in [a, b]$$

$f(x)$ is continuous on $[a, b]$

$f(x)$ is differentiable on (a, b)

$$\text{Now, } f(a) = \frac{1}{a} \text{ and}$$

$$f(b) = \frac{1}{b}$$

Hence, $f(a) \neq f(b)$ hence

By an application of the Mean Value

Theorem, there exists a time c such that,

$$f'(c) = \frac{f(b) - f(a)}{b - a}$$

$$f(x) = \frac{1}{x}$$

$$f'(x) = -\frac{1}{x^2}$$

$$f'(c) = -\frac{1}{c^2}$$

$$-\frac{1}{c^2} = \frac{\frac{1}{b} - \frac{1}{a}}{b - a}$$

$$= \left(\frac{1}{b} - \frac{1}{a}\right) \times \frac{1}{b - a}$$

$$= \left(\frac{a - b}{ab}\right) \times \frac{1}{b - a}$$

$$= -\frac{1}{ab}$$

$$\therefore c^2 = ab$$

$$c = \pm \sqrt{ab}$$

$$c = \sqrt{ab} \in (a, b)$$

Hence proved.

(ii) $f(x) = Ax^2 + Bx + C, x \in [a, b]$

$f(x)$ is continuous on $[a, b]$

$f(x)$ is differentiable on (a, b)

Now, $f(a) = Aa^2 + Ba + C$ and

$$f(b) = Ab^2 + Bb + C$$

Hence, $f(a) \neq f(b)$ hence

By an application of the Mean Value

Theorem, there exists a time c such that,

$$f'(c) = \frac{f(b) - f(a)}{b - a}$$

$$f(x) = Ax^2 + Bx + C$$

$$f'(x) = 2Ax + B$$

$$f'(c) = 2Ac + B$$

$$2Ac + B = \frac{(Ab^2 + Bb + C) - (Aa^2 + Ba + C)}{b - a}$$

$$= \frac{Ab^2 + Bb + C - Aa^2 - Ba - C}{b - a}$$

$$= \frac{Ab^2 + Bb - Aa^2 - Ba}{b - a}$$

$$= \frac{Ab^2 - Aa^2 + Bb - Ba}{b - a}$$

$$= \frac{A(b^2 - a^2) + B(b - a)}{b - a}$$

$$= \frac{A(b - a)(b + a) + B(b - a)}{b - a}$$

$$= \frac{(b - a)[A(b + a) + B]}{b - a}$$

$$2Ac + B = A(b + a) + B$$

$$2Ac = A(b + a)$$

$$2c = (b + a)$$

$$c = \frac{(b + a)}{2}, \in (a, b)$$

Hence proved.

6. A race car driver is racing at 20th km. If his speed never exceeds 150 km/hr, what is the maximum distance he can cover in the next two hours.

Solution:

Let $f(x)$ be the distance covered.

Given $x \in [0, 2]$

When $x = 0, f(a) = f(0) = 20$ and

$$x = 2, f(b) = f(2) = ?$$

$f(x)$ is continuous on $[a, b]$

$f(x)$ is differentiable on (a, b)

Now, $f(a) = 20$ and

$$f(b) = ?$$

Hence, $f(a) \neq f(b)$ hence

By an application of the Mean Value

Theorem, there exists a time c such that,

$$f'(c) = \frac{f(b) - f(a)}{b - a}$$

$$\frac{f(b) - 20}{2 - 0} \leq 150$$

$$\frac{f(b) - 20}{2} \leq 150$$

$$f(b) - 20 \leq 300$$

$$f(b) \leq 300 + 20$$

$$f(b) \leq 320$$

Hence the covers a maximum distance of

320 Km, in next two hours.

7. Suppose that for a function $f(x), f'(x) \leq 1$ for all $1 \leq x \leq 4$. Show that $f(4) - f(1) \leq 3$.

Solution:

Given $f'(x) \leq 1$

$f(x)$ is continuous on $[1, 4]$

$f(x)$ is differentiable on $(1, 4)$

By an application of the Mean Value

Theorem, there exists a time c such that,

$$f'(c) = \frac{f(b) - f(a)}{b - a}$$

$$\therefore \frac{f(b) - f(a)}{b - a} = f'(x)$$

$$\frac{f(4) - f(1)}{4 - 1} = f'(x) \leq 1$$

$$\frac{f(4) - f(1)}{3} \leq 1$$

$$f(4) - f(1) \leq 3. \text{ Hence proved}$$

8. Does there exist a differentiable function

$f(x)$ such that $f(0) = -1, f(2) = 4$ and

$f'(x) \leq 2$ for all x . Justify your answer.

Solution:

Given $f'(x) \leq 2$

$f(x)$ is continuous on $[0,2]$

$f(x)$ is differentiable on $(0,2)$

By an application of the Mean Value

Theorem, there exists a time c such that,

$$\begin{aligned} f'(c) &= \frac{f(b)-f(a)}{b-a} \\ &= \frac{f(2)-f(0)}{2-0} \\ &= \frac{4+1}{2} \end{aligned}$$

$$c = \frac{5}{2} = 2.5 \notin (0, 2)$$

9. Show that there lies a point on the curve

$f(x) = x(x+3)e^{-\frac{\pi}{2}}, -3 \leq x \leq 0$ where tangent drawn is parallel to the x - axis.

Solution:

$$f(x) = x(x+3)e^{-\frac{\pi}{2}}$$

$f(x)$ is continuous on $[-3,0]$

$f(x)$ is differentiable on $(-3,0)$

$$f(x) = (x^2 + 3x)e^{-\frac{\pi}{2}}$$

$$f(a) = f(-3) = 0 \text{ and}$$

$$f(b) = f(0) = 0$$

Hence, $f(a) = f(b)$ hence

There exists a least possible value 'c' such

that $f'(c) = 0$

$$f(x) = (x^2 + 3x)e^{-\frac{\pi}{2}}$$

$$f'(x) = (2x + 3)e^{-\frac{\pi}{2}}$$

$$f'(c) = (2c + 3)e^{-\frac{\pi}{2}}$$

When $f'(c) = 0$,

$$(2c + 3)e^{-\frac{\pi}{2}} = 0$$

$$2c + 3 = 0$$

$$2c = -3$$

$$c = -\frac{3}{2} \in (-3, 0)$$

$$\text{At } x = -\frac{3}{2}, y = f(x) = (x^2 + 3x)e^{-\frac{\pi}{2}}$$

$$\begin{aligned} &= \left(\frac{9}{4} + 3\left(-\frac{3}{2}\right)\right)e^{-\frac{\pi}{2}} \\ &= \left(\frac{9}{4} - \frac{9}{2}\right)e^{-\frac{\pi}{2}} \\ &= \left(\frac{9-18}{4}\right)e^{-\frac{\pi}{2}} \\ &= \left(-\frac{9}{4}\right)e^{-\frac{\pi}{2}} \end{aligned}$$

Hence at the point $\left(-\frac{3}{2}, \left(-\frac{9}{4}\right)e^{-\frac{\pi}{2}}\right)$ the tangent is parallel to the x - axis.

10. Using mean value theorem prove that for, $a > 0, b > 0, |e^{-a} - e^{-b}| < |a - b|$.

Solution:

Given $f(x) = e^{-x}$

$f(x)$ is continuous on $[a, b]$

$f(x)$ is differentiable on (a, b)

By an application of the Mean Value Theorem,

there exists a time $c \in (a, b)$ such that,

$$f'(c) = \frac{f(b)-f(a)}{b-a}$$

$$f(x) = e^{-x}$$

$$f'(x) = -e^{-x}$$

$$f'(c) = -e^{-c}$$

$$-e^{-c} = \frac{e^{-b} - e^{-a}}{b-a}$$

$$= \frac{-(e^{-a} - e^{-b})}{-(a-b)}$$

$$-e^{-c} = \frac{(e^{-a} - e^{-b})}{(a-b)}$$

Taking modulus on either sides,

$$|-e^{-c}| = \frac{|e^{-a} - e^{-b}|}{|a-b|}$$

$$\frac{|e^{-a} - e^{-b}|}{|a-b|} < 1$$

$$\text{So, } |e^{-a} - e^{-b}| < |a - b|.$$

Example 7.30

Expand $\log(1+x)$ as a Maclaurin's series up to 4 non-zero terms for $-1 < x \leq 1$.

Solution:

$$\begin{aligned} f(x) &= f(0) + \frac{x}{1!}f'(0) + \frac{x^2}{2!}f''(0) \\ &\quad + \frac{x^3}{3!}f'''(0) + \frac{x^4}{4!}f^{(4)}(0) + \dots \end{aligned}$$

Function and its derivatives	$\log(1+x)$ and its derivatives	Value at $x=0$
$f(x)$	$\log(1+x)$	0
$f'(x)$	$\frac{1}{(1+x)}$	1
$f''(x)$	$-\frac{1}{(1+x)^2}$	-1
$f'''(x)$	$\frac{2}{(1+x)^3}$	2
$f^{(v)}(x)$	$-\frac{6}{(1+x)^4}$	-6

Substituting the values and on simplification we get the required expansion of the function given by

$$\log(1+x) = 0 + \frac{x}{1}(1) + \frac{x^2}{2}(-1) + \frac{x^3}{6}(2) + \frac{x^4}{24}(-6) + \dots$$

$$\log(1+x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \dots$$

Example 7.31 Expand $\tan x$ in ascending powers of x up to 5th power for $-\frac{\pi}{2} < x < \frac{\pi}{2}$

Function and its derivatives	$\tan x$ and its derivatives	Value at $y=0$
$y = f(x)$	$\tan x$	0
$y_1 = f'(x)$	$\sec^2 x = 1+y^2$	1
$y_2 = f''(x)$	$2yy_1 = 2y(1+y^2) = 2y+2y^3$	0
$y_3 = f'''(x)$	$= 2y_1 + 6y^2y_1$ $= 2(1+y^2) + 6y^2(1+y^2)$ $= 2+2y^2+6y^2+6y^4$ $= 2+8y^2+6y^4$	2
$y_4 = f^{(v)}(x)$	$= 16yy_1 + 24y^3y_1$ $= 16y(1+y^2) + 24y^3(1+y^2)$ $= 16y+16y^3+24y^3+24y^5$ $= 16y+40y^3+24y^5$	0
$y_5 = f^{(v)}(x)$	$= 16y_1 + 120y^2y_1 + 120y^4y_1$	16

Substituting the values and on simplification we get the required expansion of the function given by

$$f(x) = f(0) + \frac{x}{1!}f'(0) + \frac{x^2}{2!}f''(0) + \frac{x^3}{3!}f'''(0) + \frac{x^4}{4!}f^{(v)}(0) + \dots$$

$$\tan x = 0 + \frac{x}{1}(1) + \frac{x^2}{2}(0) + \frac{x^3}{6}(2) + \frac{x^4}{24}(0) + \frac{x^5}{120}(16) \dots$$

$$\tan x = x + \frac{x^3}{3} + \frac{x^5}{15}(2) + \dots$$

$$\tan x = x + \frac{x^3}{3} + \frac{2x^5}{15} + \dots$$

Example 7.32

Write the Taylor series expansion of $\frac{1}{x}$ about $x=2$ by finding the first three non-zero terms.

Solution:

Function and its derivatives	$\frac{1}{x}$ and its derivatives	Value at $x=2$
$f(x)$	$\frac{1}{x}$	$\frac{1}{2}$
$f'(x)$	$-\frac{1}{x^2}$	$-\frac{1}{4}$
$f''(x)$	$\frac{2}{x^3}$	$\frac{1}{4}$
$f'''(x)$	$-\frac{6}{x^4}$	$-\frac{3}{8}$
$f^{(v)}(x)$	$\frac{24}{x^5}$	$\frac{3}{4}$

$$f(x) = f(a) + \frac{(x-a)}{1!}f'(a) + \frac{(x-a)^2}{2!}f''(a) + \frac{(x-a)^3}{3!}f'''(a) + \frac{(x-a)^4}{4!}f^{(v)}(a) + \dots$$

$$\frac{1}{x} = f(2) + \frac{(x-2)}{1}f'(2) + \frac{(x-2)^2}{2}f''(2) + \frac{(x-2)^3}{6}f'''(2) + \frac{(x-2)^4}{24}f^{(v)}(2) + \dots$$

$$\frac{1}{x} = \frac{1}{2} + \frac{(x-2)}{1}\left(-\frac{1}{4}\right) + \frac{(x-2)^2}{2}\left(\frac{1}{4}\right) + \frac{(x-2)^3}{6}\left(-\frac{3}{8}\right) + \frac{(x-2)^4}{24}\left(\frac{3}{4}\right) + \dots$$

$$\frac{1}{x} = \frac{1}{2} - \frac{(x-2)}{4} + \frac{(x-2)^2}{8} - \frac{(x-2)^3}{16} + \frac{(x-2)^4}{32} + \dots$$

EXERCISE 7.4

1. Write the Maclaurin series expansion of the following functions:

- (i) e^x (ii) $\sin x$ (iii) $\cos x$
 (iv) $\log(1-x)$; $-1 \leq x < 1$
 (v) $\tan^{-1}(x)$; $-1 \leq x \leq 1$
 (vi) $\cos 2x$
 Solution:
 (i) e^x

Function and its derivatives	e^x and its derivatives	Value at $x = 0$
$f(x)$	e^x	1
$f'(x)$	e^x	1
$f''(x)$	e^x	1
$f'''(x)$	e^x	1
$f^{(v)}(x)$	e^x	1

Substituting the values and on simplification we get the required expansion of the function given by

$$f(x) = f(0) + \frac{x}{1!} f'(0) + \frac{x^2}{2!} f''(0) + \frac{x^3}{3!} f'''(0) + \frac{x^4}{4!} f^{(v)}(0) + \dots$$

$$e^x = 1 + \frac{x}{1}(1) + \frac{x^2}{2}(1) + \frac{x^3}{6}(1) + \dots$$

(ii) $\sin x$

Function and its derivatives	$\sin x$ and its derivatives	Value at $x = 0$
$f(x)$	$\sin x$	0
$f'(x)$	$\cos x$	1
$f''(x)$	$-\sin x$	0
$f'''(x)$	$-\cos x$	-1
$f^{(v)}(x)$	$\sin x$	0

Substituting the values and on simplification we get the required expansion of the function given by

$$f(x) = f(0) + \frac{x}{1!} f'(0) + \frac{x^2}{2!} f''(0) + \frac{x^3}{3!} f'''(0) + \frac{x^4}{4!} f^{(v)}(0) + \dots$$

$$\sin x = 0 + \frac{x}{1!}(1) + \frac{x^2}{2!}(0) + \frac{x^3}{3!}(-1) + \dots$$

$$\sin x = \frac{x}{1!} - \frac{x^3}{3!} + \dots$$

(iii) $\cos x$

Function and its derivatives	$\cos x$ and its derivatives	Value at $x = 0$
$f(x)$	$\cos x$	1
$f'(x)$	$-\sin x$	0
$f''(x)$	$-\cos x$	-1
$f'''(x)$	$\sin x$	0
$f^{(v)}(x)$	$\cos x$	1

Substituting the values and on simplification we get the required expansion of the function given by

$$f(x) = f(0) + \frac{x}{1!} f'(0) + \frac{x^2}{2!} f''(0) + \frac{x^3}{3!} f'''(0) + \frac{x^4}{4!} f^{(v)}(0) + \dots$$

$$\cos x = 1 + \frac{x}{1!}(0) + \frac{x^2}{2!}(-1) + \frac{x^3}{3!}(0) + \dots$$

$$\cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \dots$$

(iv) $\log(1-x)$

Function and its derivatives	$\log(1-x)$ and its derivatives	Value at $x = 0$
$f(x)$	$\log(1-x)$	0
$f'(x)$	$\frac{-1}{(1-x)}$	-1
$f''(x)$	$-\frac{1}{(1-x)^2}$	-1
$f'''(x)$	$-\frac{2}{(1-x)^3}$	-2
$f^{(v)}(x)$	$-\frac{6}{(1-x)^4}$	-6

Substituting the values and on simplification we get the required expansion of the function given by

$$f(x) = f(0) + \frac{x}{1!} f'(0) + \frac{x^2}{2!} f''(0) + \frac{x^3}{3!} f'''(0) + \frac{x^4}{4!} f^{(v)}(0) + \dots$$

$$\log(1-x) = 0 + \frac{x}{1}(-1) + \frac{x^2}{2}(-1) + \dots$$

$$\log(1-x) = -x - \frac{x^2}{2} - \frac{x^3}{3} - \frac{x^4}{4} + \dots$$

$$+ \frac{x^3}{6}(-2) + \frac{x^4}{24}(-6) + \dots$$

(v) $\tan^{-1}(x)$

Function and its derivatives	$\tan^{-1}(x)$ and its derivatives	Value at $x = 0$
$f(x)$	$\tan^{-1}(x)$	0
$f'(x)$	$\frac{1}{1+x^2}$ $= 1 - x^2 + x^4 - x^6$	1
$f''(x)$	$-2x + 4x^3 - 6x^5$	0
$f'''(x)$	$-2 + 12x^2 - 30x^4$	-2
$f^{(v)}(x)$	$24x - 120x^3$	0

Substituting the values and on simplification we get the required expansion of the function given by

$$f(x) = f(0) + \frac{x}{1!}f'(0) + \frac{x^2}{2!}f''(0) + \frac{x^3}{3!}f'''(0) + \frac{x^4}{4!}f^{(v)}(0) + \dots$$

$$\tan^{-1}(x) = 0 + \frac{x}{1!}(1) + \frac{x^2}{2!}(0) + \frac{x^3}{3!}(-2) + \dots$$

$$\tan^{-1}(x) = x - \frac{2x^3}{3!} + \frac{24x^5}{5!} \dots$$

$$\tan^{-1}(x) = x - \frac{x^3}{3} + \frac{x^5}{5} \dots$$

(vi) $\cos^2 x$

Function and its derivatives	$\cos^2 x$ and its derivatives	Value at $x = 0$
$f(x)$	$\cos^2 x$	1
$f'(x)$	$-2\cos x \sin x$ $= -\sin 2x$	0
$f''(x)$	$-2\cos 2x$	-2
$f'''(x)$	$4\sin 2x$	0
$f^{(v)}(x)$	$8\cos 2x$	8

Substituting the values and on simplification we get the required expansion of the function given by

$$f(x) = f(0) + \frac{x}{1!}f'(0) + \frac{x^2}{2!}f''(0) + \dots$$

$$+ \frac{x^3}{3!}f'''(0) + \frac{x^4}{4!}f^{(v)}(0) + \dots$$

$$\cos^2 x = 1 + \frac{x}{1!}(0) + \frac{x^2}{2!}(-2) + \frac{x^3}{3!}(0) + \frac{x^4}{4!}(8) + \dots$$

$$\cos^2 x = 1 + \frac{x^2}{2!}(-2) + \frac{x^4}{4!}(8) + \dots$$

2. Write down the Taylor series expansion, of the function $\log x$ about $x = 1$ up to three non-zero terms for $x > 0$.

Solution:

Function and its derivatives	$\log x$ and its derivatives	Value at $x = 1$
$f(x)$	$\log x$	0
$f'(x)$	$\frac{1}{x}$	1
$f''(x)$	$-\frac{1}{x^2}$	-1
$f'''(x)$	$\frac{2}{x^3}$	2
$f^{(v)}(x)$	$-\frac{6}{x^4}$	-6

$$f(x) = f(a) + \frac{(x-a)}{1!}f'(a) + \frac{(x-a)^2}{2!}f''(a) + \frac{(x-a)^3}{3!}f'''(a) + \frac{(x-a)^4}{4!}f^{(v)}(a) + \dots$$

$$\log x = 0 + \frac{(x-2)}{1}(1) + \frac{(x-2)^2}{2}(-1) + \frac{(x-2)^3}{6}(2) + \frac{(x-2)^4}{24}(-6) + \dots$$

$$\log x = \frac{(x-2)}{1} - \frac{(x-2)^2}{2} + \frac{(x-2)^3}{3} - \frac{(x-2)^4}{4} + \dots$$

3. Expand $\sin x$ in ascending powers $x - \frac{\pi}{4}$ up to three non-zero terms.

Solution:

Function and its derivatives	$\sin x$ and its derivatives	Value at $x = \frac{\pi}{4}$
$f(x)$	$\sin x$	$\frac{1}{\sqrt{2}}$
$f'(x)$	$\cos x$	$\frac{1}{\sqrt{2}}$
$f''(x)$	$-\sin x$	$-\frac{1}{\sqrt{2}}$
$f'''(x)$	$-\cos x$	$-\frac{1}{\sqrt{2}}$
$f^{(v)}(x)$	$\sin x$	$\frac{1}{\sqrt{2}}$

$$f(x) = f(a) + \frac{(x-a)}{1!} f'(a) + \frac{(x-a)^2}{2!} f''(a) + \frac{(x-a)^3}{3!} f'''(a) + \frac{(x-a)^4}{4!} f^{(4)}(a) + \dots$$

$$\sin x = \frac{1}{\sqrt{2}} + \frac{(x-\frac{\pi}{4})}{1} \left(\frac{1}{\sqrt{2}}\right) + \frac{(x-\frac{\pi}{4})^2}{2} \left(-\frac{1}{\sqrt{2}}\right) + \frac{(x-\frac{\pi}{4})^3}{6} \left(\frac{1}{\sqrt{2}}\right) + \frac{(x-\frac{\pi}{4})^4}{24} \left(\frac{1}{\sqrt{2}}\right) + \dots$$

4. Expand the polynomial $f(x) = x^2 - 3x + 2$ in powers of $x - 1$.

Solution:

Function and its derivatives	$f(x)$ and its derivatives	Value at $x = 1$
$f(x)$	$x^2 - 3x + 2$	0
$f'(x)$	$2x - 3$	-1
$f''(x)$	2	2
$f'''(x)$	0	0

$$f(x) = f(a) + \frac{(x-a)}{1!} f'(a) + \frac{(x-a)^2}{2!} f''(a) + \frac{(x-a)^3}{3!} f'''(a) + \frac{(x-a)^4}{4!} f^{(4)}(a) + \dots$$

$$x^2 - 3x + 2 = 0 + \frac{(x-1)}{1} (-1) + \frac{(x-1)^2}{2} (2) + \frac{(x-1)^3}{6} (0)$$

$$x^2 - 3x + 2 = -(x - 1) + (x - 1)^2$$

Example 7.33 Evaluate: $\lim_{x \rightarrow 1} \left(\frac{x^2 - 3x + 2}{x^2 - 4x + 3} \right)$.

Solution: $\lim_{x \rightarrow 1} \left(\frac{x^2 - 3x + 2}{x^2 - 4x + 3} \right)$

$$= \frac{(1)^2 - 3(1) + 2}{(1)^2 - 4(1) + 3}$$

$$= \frac{1 - 3 + 2}{1 - 4 + 3} = \frac{3 - 3}{4 - 4} = \frac{0}{0} \text{ Indeterminate form.}$$

Applying l'hospital's Rule,

$$\lim_{x \rightarrow 1} \left(\frac{x^2 - 3x + 2}{x^2 - 4x + 3} \right) = \lim_{x \rightarrow 1} \left(\frac{2x - 3}{2x - 4} \right)$$

$$= \frac{2(1) - 3}{2(1) - 4} = \frac{2 - 3}{2 - 4} = \frac{-1}{-2} = \frac{1}{2}$$

$$\therefore \lim_{x \rightarrow 1} \left(\frac{x^2 - 3x + 2}{x^2 - 4x + 3} \right) = \frac{1}{2}$$

Example 7.34

Compute the limit: $\lim_{x \rightarrow a} \left(\frac{x^n - a^n}{x - a} \right)$.

Solution: $\lim_{x \rightarrow a} \left(\frac{x^n - a^n}{x - a} \right)$

$$= \frac{a^n - a^n}{a - a} = \frac{0}{0} \text{ Indeterminate form.}$$

Applying l'hospital's Rule,

$$\lim_{x \rightarrow a} \left(\frac{x^n - a^n}{x - a} \right) = \lim_{x \rightarrow a} \left(\frac{nx^{n-1} - 0}{1 - 0} \right)$$

$$= \frac{na^{n-1}}{1}$$

$$\therefore \lim_{x \rightarrow a} \left(\frac{x^n - a^n}{x - a} \right) = na^{n-1}$$

Example 7.35

Evaluate the limit: $\lim_{x \rightarrow 0} \left(\frac{\sin mx}{x} \right)$.

Solution: $\lim_{x \rightarrow 0} \left(\frac{\sin mx}{x} \right)$

$$= \frac{\sin m(0)}{(0)}$$

$$= \frac{\sin 0}{(0)} = \frac{0}{0} \text{ Indeterminate form.}$$

Applying l'hospital's Rule,

$$\lim_{x \rightarrow 0} \left(\frac{\sin mx}{x} \right) = \lim_{x \rightarrow 0} \left(\frac{m \cos mx}{1} \right)$$

$$= \frac{m \cos m(0)}{1}$$

$$= \frac{m \cos 0}{1} = m$$

$$\therefore \lim_{x \rightarrow 0} \left(\frac{\sin mx}{x} \right) = m$$

Example 7.36 Evaluate the limit: $\lim_{x \rightarrow 0} \left(\frac{\sin x}{x^2} \right)$.

Solution: $\lim_{x \rightarrow 0} \left(\frac{\sin x}{x^2} \right)$

$$= \frac{\sin(0)}{(0)^2}$$

$$= \frac{\sin 0}{(0)} = \frac{0}{0} \text{ Indeterminate form.}$$

Applying l'hospital's Rule,

$$\lim_{x \rightarrow 0} \left(\frac{\sin x}{x^2} \right) = \lim_{x \rightarrow 0^+} \left(\frac{\cos x}{2x} \right)$$

$$= \frac{\cos(0)}{2(0)} = \frac{1}{0} = \infty \text{ and also,}$$

$$\lim_{x \rightarrow 0} \left(\frac{\sin x}{x^2} \right) = \lim_{x \rightarrow 0^-} \left(\frac{\cos x}{2x} \right)$$

$$= \frac{\cos(-0)}{2(-0)} = \frac{1}{-0} = -\infty$$

As the left limit and the right limit are not the same we conclude that the limit does not exist.

Example 7.37 If $\lim_{\theta \rightarrow 0} \left(\frac{1 - \cos m\theta}{1 - \cos n\theta} \right) = 1$,
then prove that $m = \pm n$

Solution: $\lim_{\theta \rightarrow 0} \left(\frac{1 - \cos m\theta}{1 - \cos n\theta} \right)$

$$= \frac{1 - \cos m(0)}{1 - \cos n(0)}$$

$$= \frac{1 - \cos 0}{1 - \cos 0}$$

$$= \frac{1 - 1}{1 - 1} = \frac{0}{0} \text{ Indeterminate form.}$$

Applying l'hospital's Rule,

$$\begin{aligned} \lim_{\theta \rightarrow 0} \left(\frac{1 - \cos m\theta}{1 - \cos n\theta} \right) &= \lim_{\theta \rightarrow 0} \left[\frac{0 - (-m \sin m\theta)}{0 - (-n \sin n\theta)} \right] \\ &= \lim_{\theta \rightarrow 0} \left[\frac{m \sin m\theta}{n \sin n\theta} \right] \\ &= \lim_{\theta \rightarrow 0} \frac{m}{n} \left(\frac{\sin m\theta}{\sin n\theta} \right) \\ &= \frac{m}{n} \left(\frac{\sin m(0)}{\sin n(0)} \right) \\ &= \frac{m}{n} \left(\frac{\sin 0}{\sin 0} \right) = \frac{0}{0} \text{ Indeterminate form.} \end{aligned}$$

Applying l'hospital's Rule,

$$\begin{aligned} \lim_{\theta \rightarrow 0} \frac{m}{n} \left(\frac{\sin m\theta}{\sin n\theta} \right) &= \lim_{\theta \rightarrow 0} \frac{m}{n} \left(\frac{m \cos m\theta}{n \cos n\theta} \right) \\ &= \lim_{\theta \rightarrow 0} \frac{m^2}{n^2} \left(\frac{\cos m\theta}{\cos n\theta} \right) \\ &= \frac{m^2}{n^2} \left(\frac{\cos m(0)}{\cos n(0)} \right) \\ &= \frac{m^2}{n^2} \left(\frac{\cos 0}{\cos 0} \right) \\ &= \frac{m^2}{n^2} \left(\frac{1}{1} \right) \\ &= \frac{m^2}{n^2} \end{aligned}$$

$$\therefore \lim_{\theta \rightarrow 0} \left(\frac{1 - \cos m\theta}{1 - \cos n\theta} \right) = \frac{m^2}{n^2}$$

Given $\lim_{\theta \rightarrow 0} \left(\frac{1 - \cos m\theta}{1 - \cos n\theta} \right) = 1$

Hence, $\frac{m^2}{n^2} = 1$, gives

$$m^2 = n^2$$

So, $m = \pm n$ Hence proved.

Example 7.38 Evaluate: $\lim_{x \rightarrow 1^-} \left[\frac{\log(1-x)}{\cot(\pi x)} \right]$.

Solution: $\lim_{x \rightarrow 1^-} \left[\frac{\log(1-x)}{\cot(\pi x)} \right]$

$$= \left[\frac{\log(1-1)}{\cot(\pi)} \right]$$

$$= \left[\frac{\log(0)}{\cot(\pi)} \right]$$

$$= \frac{-(-\infty)}{\infty} = \frac{\infty}{\infty} \text{ Indeterminate form}$$

Applying l'hospital's Rule,

$$\begin{aligned} \lim_{x \rightarrow 1^-} \left[\frac{\log(1-x)}{\cot(\pi x)} \right] &= \lim_{x \rightarrow 1^-} \left[\frac{\frac{1}{(1-x)}(-1)}{-\pi \operatorname{cosec}^2(\pi x)} \right] \\ &= \lim_{x \rightarrow 1^-} \left[\frac{\frac{1}{(1-x)}}{\pi \operatorname{cosec}^2(\pi x)} \right] \\ &= \frac{1}{\pi \operatorname{cosec}^2(\pi)} \\ &= \frac{1}{\pi \left(\frac{1}{0} \right)} \\ &= \frac{\infty}{\infty} \text{ Indeterminate form} \end{aligned}$$

Applying l'hospital's Rule,

$$\begin{aligned} \lim_{x \rightarrow 1^-} \left[\frac{\frac{1}{(1-x)}}{\pi \operatorname{cosec}^2(\pi x)} \right] &= \lim_{x \rightarrow 1^-} \left[\frac{\frac{1}{(1-x)}}{\pi \frac{1}{\sin^2(\pi x)}} \right] \\ &= \lim_{x \rightarrow 1^-} \left[\frac{1}{(1-x)} \times \frac{\sin^2(\pi x)}{\pi} \right] \\ &= \lim_{x \rightarrow 1^-} \left[\frac{\sin^2(\pi x)}{\pi - \pi x} \right] \\ &= \frac{\sin^2(\pi)}{\pi - \pi} = \frac{0}{0} \text{ Indeterminate form.} \end{aligned}$$

Applying l'hospital's Rule,

$$\begin{aligned} \lim_{x \rightarrow 1^-} \left[\frac{\sin^2(\pi x)}{\pi - \pi x} \right] &= \lim_{x \rightarrow 1^-} \left[\frac{2\sin(\pi x)\cos(\pi x)\pi}{-\pi} \right] \\ &= \lim_{x \rightarrow 1^-} [-2\sin(\pi x)\cos(\pi x)] \\ &= -2\sin(-\pi)\cos(-\pi) \\ &= 2\sin(\pi)\cos(\pi) \\ &= 2(0)(-1) \\ &= 0 \end{aligned}$$

$$\therefore \lim_{x \rightarrow 1^-} \left[\frac{\log(1-x)}{\cot(\pi x)} \right] = 0$$

Example 7.39 Evaluate: $\lim_{x \rightarrow 0^+} \left(\frac{1}{x} - \frac{1}{e^x - 1} \right)$

Solution: $\lim_{x \rightarrow 0^+} \left(\frac{1}{x} - \frac{1}{e^x - 1} \right)$

$$= \left(\frac{1}{0} - \frac{1}{e^0 - 1} \right)$$

$$= \left(\frac{1}{0} - \frac{1}{1 - 1} \right)$$

$$= \left(\frac{1}{0} - \frac{1}{0} \right)$$

$= (\infty - \infty)$ Indeterminate form

Applying l'hospital's Rule,

$$\lim_{x \rightarrow 0^+} \left(\frac{1}{x} - \frac{1}{e^x - 1} \right) = \lim_{x \rightarrow 0^+} \left[\frac{e^x - 1 - x}{x(e^x - 1)} \right]$$

$$= \left[\frac{e^0 - 1 - 0}{0(e^0 - 1)} \right]$$

$$= \frac{1 - 1}{0(1 - 1)}$$

$= \frac{0}{0}$ Indeterminate form

Applying l'hospital's Rule,

$$\lim_{x \rightarrow 0^+} \left[\frac{e^x - 1 - x}{x(e^x - 1)} \right] = \lim_{x \rightarrow 0^+} \left[\frac{e^x - 1}{x(e^x) + (e^x - 1)} \right]$$

$$= \left[\frac{e^0 - 1}{0(e^0) + (e^0 - 1)} \right]$$

$$= \left[\frac{1 - 1}{0(1) + (1 - 1)} \right]$$

$= \frac{0}{0}$ Indeterminate form

Applying l'hospital's Rule, $\lim_{x \rightarrow 0^+} \left[\frac{e^x - 1}{x(e^x) + (e^x - 1)} \right] =$

$$\lim_{x \rightarrow 0^+} \left[\frac{e^x}{x(e^x) + e^x + (e^x)} \right]$$

$$= \left[\frac{e^0}{0(e^0) + e^0 + (e^0)} \right]$$

$$= \frac{1}{0(1) + 1 + 1}$$

$$= \frac{1}{2}$$

$$\therefore \lim_{x \rightarrow 0^+} \left(\frac{1}{x} - \frac{1}{e^x - 1} \right) = \frac{1}{2}$$

Example 7.40 Evaluate: $\lim_{x \rightarrow 0^+} x \log x$

Solution: $\lim_{x \rightarrow 0^+} x \log x$

$= 0(-\infty)$ Indeterminate form.

Applying l'hospital's Rule,

$$\lim_{x \rightarrow 0^+} x \log x = \lim_{x \rightarrow 0^+} \frac{\log x}{\frac{1}{x}}$$

$$= \left[\frac{\log 0}{\frac{1}{0}} \right]$$

$= \frac{\infty}{\infty}$ Indeterminate form

Applying l'hospital's Rule,

$$\lim_{x \rightarrow 0^+} \frac{\log x}{\frac{1}{x}} = \lim_{x \rightarrow 0^+} \frac{\frac{1}{x}}{-\left(\frac{1}{x^2}\right)}$$

$$= \lim_{x \rightarrow 0^+} \left(\frac{1}{x} \right) \times \left(\frac{x^2}{-1} \right)$$

$$= \lim_{x \rightarrow 0^+} \left(\frac{x}{-1} \right)$$

$$= \left(\frac{0}{-1} \right)$$

$$= 0$$

$$\therefore \lim_{x \rightarrow 0^+} x \log x = 0$$

Example 7.41 Evaluate: $\lim_{x \rightarrow \infty} \left(\frac{x^2 + 17x + 29}{x^4} \right)$

Solution: $\lim_{x \rightarrow \infty} \left(\frac{x^2 + 17x + 29}{x^4} \right)$

$= \frac{\infty}{\infty}$ Indeterminate form

Applying l'hospital's Rule,

$$\lim_{x \rightarrow \infty} \left(\frac{x^2 + 17x + 29}{x^4} \right) = \lim_{x \rightarrow \infty} \left(\frac{2x + 17}{4x^3} \right)$$

$= \frac{\infty}{\infty}$ Indeterminate form

Applying l'hospital's Rule,

$$\lim_{x \rightarrow \infty} \left(\frac{2x + 17}{4x^3} \right) = \lim_{x \rightarrow \infty} \left(\frac{2}{12x^2} \right)$$

$$= \frac{2}{\infty}$$

$$= 0$$

$$\therefore \lim_{x \rightarrow \infty} \left(\frac{x^2 + 17x + 29}{x^4} \right) = 0$$

Example 7.42 Evaluate: $\lim_{x \rightarrow \infty} \left(\frac{e^x}{x^m} \right), m \in \mathbb{N}$

Solution: $\lim_{x \rightarrow \infty} \left(\frac{e^x}{x^m} \right)$

$$= \frac{e^{\infty}}{\infty^m}$$

$= \frac{\infty}{\infty}$ Indeterminate form

Applying l'hospital's Rule,

$$\lim_{x \rightarrow \infty} \left(\frac{e^x}{x^m} \right) = \lim_{x \rightarrow \infty} \left(\frac{e^x}{m!} \right)$$

$$= \frac{e^\infty}{m!}$$

$$= \infty$$

$$\therefore \lim_{x \rightarrow \infty} \left(\frac{e^x}{x^m} \right) = \infty$$

Example 7.43 Using the l' hospital Rule prove

that: $\lim_{x \rightarrow 0^+} (1+x)^{\frac{1}{x}} = e$

Solution: $\lim_{x \rightarrow 0^+} (1+x)^{\frac{1}{x}}$

$$= (1+0)^{\frac{1}{0}}$$

$$= (1)^\infty \text{ Indeterminate form}$$

Let $A = (1+x)^{\frac{1}{x}}$

Taking log on either side,

$$\log A = \log(1+x)^{\frac{1}{x}}$$

$$\lim_{x \rightarrow 0^+} \log A = \lim_{x \rightarrow 0^+} \frac{1}{x} \log(1+x)$$

$$= \lim_{x \rightarrow 0^+} \frac{\log(1+x)}{x}$$

$$= \frac{\log(1+0)}{0}$$

$$= \frac{\log(1)}{0}$$

$$= \frac{0}{0} \text{ Indeterminate form}$$

Applying l'hospital's Rule,

$$\lim_{x \rightarrow 0^+} \log A = \lim_{x \rightarrow 0^+} \frac{\log(1+x)}{x}$$

$$= \lim_{x \rightarrow 0^+} \frac{\frac{1}{(1+x)}}{1}$$

$$= \lim_{x \rightarrow 0^+} \frac{1}{(1+x)}$$

$$= \frac{1}{(1+0)}$$

$$= \frac{1}{1}$$

$$\lim_{x \rightarrow 0^+} \log A = 1$$

By Composite function theorem,

$$\log \lim_{x \rightarrow 0^+} A = 1$$

Taking exponential on both side,

$$\lim_{x \rightarrow 0^+} A = e$$

$$\lim_{x \rightarrow 0^+} (1+x)^{\frac{1}{x}} = e$$

Example 7.44 Evaluate: $\lim_{x \rightarrow \infty} (1+2x)^{\frac{1}{2 \log x}}$

Solution: $\lim_{x \rightarrow \infty} (1+2x)^{\frac{1}{2 \log x}}$

$$= (1+\infty)^{\frac{1}{2 \log \infty}}$$

$$= (\infty)^\infty$$

$$= (\infty)^0 \text{ Indeterminate form}$$

Let $A = (1+2x)^{\frac{1}{2 \log x}}$

Taking log on either side,

$$\log A = \log (1+2x)^{\frac{1}{2 \log x}}$$

$$\lim_{x \rightarrow \infty} \log A = \lim_{x \rightarrow \infty} \frac{1}{2 \log x} \log(1+2x)$$

$$= \lim_{x \rightarrow \infty} \frac{\log(1+2x)}{2 \log x}$$

$$= \frac{\log(1+\infty)}{2 \log \infty}$$

$$= \frac{\log(\infty)}{\infty}$$

$$= \frac{\infty}{\infty} \text{ Indeterminate form}$$

Applying l'hospital's Rule,

$$\lim_{x \rightarrow \infty} \log A = \lim_{x \rightarrow \infty} \frac{\log(1+2x)}{2 \log x}$$

$$= \lim_{x \rightarrow \infty} \frac{\frac{1}{(1+2x)} \times 2}{\frac{2}{x}}$$

$$= \lim_{x \rightarrow \infty} \frac{2}{(1+2x)} \times \frac{x}{2}$$

$$= \lim_{x \rightarrow \infty} \frac{x}{(1+2x)}$$

$$= \frac{\infty}{\infty} \text{ Indeterminate form}$$

Applying l'hospital's Rule

$$\lim_{x \rightarrow \infty} \log A = \lim_{x \rightarrow \infty} \frac{x}{(1+2x)}$$

$$= \lim_{x \rightarrow \infty} \frac{1}{(2)}$$

$$= \frac{1}{2}$$

By Composite function theorem,

$$\log \lim_{x \rightarrow \infty} A = \frac{1}{2}$$

Taking exponential on both side,

$$\lim_{x \rightarrow \infty} A = e^{\frac{1}{2}}$$

$$\lim_{x \rightarrow \infty} (1 + 2x)^{\frac{1}{2 \log x}} = e^{\frac{1}{2}} = \sqrt{e}$$

Example 7.45 Evaluate: $\lim_{x \rightarrow 1} x^{\frac{1}{1-x}}$

Solution: $\lim_{x \rightarrow 1} x^{\frac{1}{1-x}}$

$$= x^{\frac{1}{1-x}}$$

$$= 1^{\frac{1}{1-1}}$$

$$= 1^{\frac{1}{0}}$$

$$= 1^{\infty} \text{ Indeterminate form}$$

$$\text{Let } A = x^{\frac{1}{1-x}}$$

Taking log on either side,

$$\log A = \log x^{\frac{1}{1-x}}$$

$$\lim_{x \rightarrow 1} \log A = \lim_{x \rightarrow 1} \frac{1}{1-x} \log(x)$$

$$= \lim_{x \rightarrow 1} \frac{\log x}{1-x}$$

$$= \frac{\log(1)}{1-1}$$

$$= \frac{\log(1)}{0}$$

$$= \frac{0}{0} \text{ Indeterminate form}$$

Applying l'hôpital's Rule,

$$\lim_{x \rightarrow 1} \log A = \lim_{x \rightarrow 1} \frac{\log x}{1-x}$$

$$= \lim_{x \rightarrow 1} \frac{\frac{1}{x}}{-1}$$

$$= \lim_{x \rightarrow 1} \frac{-1}{x}$$

$$= \frac{-1}{1}$$

$$= -1$$

By Composite function theorem,

$$\log \lim_{x \rightarrow 1} A = -1$$

Taking exponential on both side,

$$\lim_{x \rightarrow 1} A = e^{-1} = \frac{1}{e}$$

$$\lim_{x \rightarrow 1} x^{\frac{1}{1-x}} = \frac{1}{e}$$

EXERCISE 7.5

Evaluate the following limits, if necessary

use l'Hôpital Rule :

$$1. \lim_{x \rightarrow 0} \left(\frac{1 - \cos x}{x^2} \right)$$

$$\text{Solution: } \lim_{x \rightarrow 0} \left(\frac{1 - \cos x}{x^2} \right)$$

$$= \frac{1 - \cos(0)}{(0)^2}$$

$$= \frac{1-1}{0} = \frac{0}{0} \text{ Indeterminate form}$$

Applying l'hôpital's Rule,

$$\lim_{x \rightarrow 0} \left(\frac{1 - \cos x}{x^2} \right) = \lim_{x \rightarrow 0} \left(\frac{0 - (-\sin x)}{2x} \right)$$

$$= \lim_{x \rightarrow 0} \left(\frac{0 + \sin x}{2x} \right)$$

$$= \frac{0 + \sin 0}{0}$$

$$= \frac{0}{0} \text{ Indeterminate form}$$

Applying l'hôpital's Rule,

$$\lim_{x \rightarrow 0} \left(\frac{\sin x}{2x} \right) = \lim_{x \rightarrow 0} \left(\frac{\cos x}{2} \right)$$

$$= \frac{\cos(0)}{2}$$

$$= \frac{1}{2}$$

$$\therefore \lim_{x \rightarrow 0} \left(\frac{1 - \cos x}{x^2} \right) = \frac{1}{2}$$

$$2. \lim_{x \rightarrow \infty} \frac{2x^2 - 3}{x^2 - 5x + 3}$$

$$\text{Solution: } \lim_{x \rightarrow \infty} \frac{2x^2 - 3}{x^2 - 5x + 3}$$

$$= \frac{\infty}{\infty} \text{ Indeterminate form}$$

Applying l'hôpital's Rule

$$\lim_{x \rightarrow \infty} \frac{2x^2 - 3}{x^2 - 5x + 3} = \lim_{x \rightarrow \infty} \frac{4x}{2x - 5}$$

$$= \frac{\infty}{\infty} \text{ Indeterminate form}$$

Applying l'hôpital's Rule

$$\lim_{x \rightarrow \infty} \frac{4x}{2x - 5} = \lim_{x \rightarrow \infty} \frac{4}{2}$$

$$= \frac{4}{2} = 2$$

$$\therefore \lim_{x \rightarrow \infty} \frac{2x^2 - 3}{x^2 - 5x + 3} = 2$$

$$3. \lim_{x \rightarrow \infty} \frac{x}{\log x}$$

Solution: $\lim_{x \rightarrow \infty} \frac{x}{\log x}$

$$= \frac{\infty}{\log \infty}$$

$$= \frac{\infty}{\infty} \text{ Indeterminate form}$$

Applying l'hospital's Rule

$$\lim_{x \rightarrow \infty} \frac{x}{\log x} = \lim_{x \rightarrow \infty} \frac{1}{\frac{1}{x}}$$

$$= \lim_{x \rightarrow \infty} x$$

$$= \infty$$

$$\therefore \lim_{x \rightarrow \infty} \frac{x}{\log x} = \infty$$

$$4. \lim_{x \rightarrow \frac{\pi}{2}} \frac{\sec x}{\tan x}$$

Solution: $\lim_{x \rightarrow \frac{\pi}{2}} \frac{\sec x}{\tan x}$

$$= \frac{\sec \frac{\pi}{2}}{\tan \frac{\pi}{2}}$$

$$= \frac{\infty}{\infty} \text{ Indeterminate form}$$

Applying l'hospital's Rule

$$\lim_{x \rightarrow \frac{\pi}{2}} \frac{\sec x}{\tan x} = \lim_{x \rightarrow \frac{\pi}{2}} \frac{1/\cos x}{\sin x/\cos x}$$

$$= \lim_{x \rightarrow \frac{\pi}{2}} \left(\frac{1}{\cos x} \times \frac{\cos x}{\sin x} \right)$$

$$= \lim_{x \rightarrow \frac{\pi}{2}} \frac{1}{\sin x}$$

$$= \frac{1}{\sin \frac{\pi}{2}}$$

$$= \frac{1}{1} = 1$$

$$\therefore \lim_{x \rightarrow \frac{\pi}{2}} \frac{\sec x}{\tan x} = 1$$

$$5. \lim_{x \rightarrow \infty} e^{-x} \sqrt{x}$$

Solution:

$$\lim_{x \rightarrow \infty} e^{-x} \sqrt{x} = \lim_{x \rightarrow \infty} \frac{\sqrt{x}}{e^x}$$

$$= \frac{\sqrt{\infty}}{e^{\infty}}$$

$$= \frac{\infty}{\infty} \text{ Indeterminate form}$$

Applying l'hospital's Rule,

$$\lim_{x \rightarrow \infty} \frac{\sqrt{x}}{e^x} = \lim_{x \rightarrow \infty} \frac{\frac{1}{2\sqrt{x}}}{e^x}$$

$$= \lim_{x \rightarrow \infty} \frac{1}{2\sqrt{x}e^x}$$

$$= \frac{1}{2\sqrt{\infty}e^{\infty}}$$

$$= \frac{1}{\infty}$$

$$= 0$$

$$\therefore \lim_{x \rightarrow \infty} e^{-x} \sqrt{x} = 0$$

$$6. \lim_{x \rightarrow 0} \left(\frac{1}{\sin x} - \frac{1}{x} \right)$$

Solution: $\lim_{x \rightarrow 0} \left(\frac{1}{\sin x} - \frac{1}{x} \right)$

$$= \frac{1}{\sin 0} - \frac{1}{0}$$

$$= \frac{1}{0} - \frac{1}{0}$$

$$= \infty - \infty \text{ Indeterminate form.}$$

Applying l'hospital's Rule,

$$\lim_{x \rightarrow 0} \left(\frac{1}{\sin x} - \frac{1}{x} \right) = \lim_{x \rightarrow 0} \left[\frac{x - \sin x}{(\sin x)x} \right]$$

$$= \left[\frac{0 - \sin 0}{(\sin 0)0} \right]$$

$$= \left[\frac{0-0}{(0)0} \right]$$

$$= \frac{0}{0} \text{ Indeterminate form.}$$

Applying l'hospital's Rule,

$$\lim_{x \rightarrow 0} \left[\frac{x - \sin x}{(\sin x)x} \right] = \lim_{x \rightarrow 0} \left[\frac{1 - \cos x}{(\cos x)x + (\sin x)1} \right]$$

$$= \left[\frac{1 - \cos 0}{(\cos 0)0 + (\sin 0)1} \right]$$

$$= \left[\frac{1-1}{(1)0 + (0)1} \right]$$

$$= \frac{0}{0} \text{ Indeterminate form.}$$

Applying l'hospital's Rule, $\lim_{x \rightarrow 0} \left[\frac{1 - \cos x}{(\cos x)x + (\sin x)1} \right] =$

$$\lim_{x \rightarrow 0} \left[\frac{-(-\sin x)}{(-\sin x)x + (\cos x) + \cos x} \right]$$

$$= \lim_{x \rightarrow 0} \left[\frac{\sin x}{(-\sin x)x + 2\cos x} \right]$$

$$= \left[\frac{\sin 0}{(-\sin 0)0 + 2\cos 0} \right]$$

$$= \left[\frac{0}{(0)0 + 2(1)} \right]$$

$$= \frac{0}{2}$$

$$= 0$$

$$\therefore \lim_{x \rightarrow 0} \left(\frac{1}{\sin x} - \frac{1}{x} \right) = 0$$

$$7. \lim_{x \rightarrow 1^+} \left(\frac{2}{x^2 - 1} - \frac{x}{x - 1} \right)$$

$$\text{Solution: } \lim_{x \rightarrow 1^+} \left(\frac{2}{x^2 - 1} - \frac{x}{x - 1} \right)$$

$$= \frac{2}{1^2 - 1} - \frac{1}{1 - 1}$$

$$= \frac{2}{0} - \frac{1}{0}$$

$= \infty - \infty$ Indeterminate form.

Applying l'hospital's Rule,

$$\lim_{x \rightarrow 1^+} \left(\frac{2}{x^2 - 1} - \frac{x}{x - 1} \right) = \lim_{x \rightarrow 1^+} \left(\frac{2}{x^2 - 1} - \frac{x}{x - 1} \right)$$

$$= \lim_{x \rightarrow 1^+} \left(\frac{2}{(x - 1)(x + 1)} - \frac{x}{x - 1} \right)$$

$$= \lim_{x \rightarrow 1^+} \left(\frac{2 - x(x + 1)}{x^2 - 1} \right)$$

$$= \lim_{x \rightarrow 1^+} \left(\frac{2 - x^2 - x}{x^2 - 1} \right)$$

$$= \frac{2 - 1^2 - 1}{1^2 - 1}$$

$$= \frac{2 - 2}{1 - 1}$$

$$= \frac{0}{0} \text{ Indeterminate form.}$$

Applying l'hospital's Rule,

$$\lim_{x \rightarrow 1^+} \left(\frac{2 - x^2 - x}{x^2 - 1} \right) = \lim_{x \rightarrow 1^+} \left(\frac{-2x - 1}{2x} \right)$$

$$= \frac{-2(1) - 1}{2(1)}$$

$$= \frac{-3}{2}$$

$$\lim_{x \rightarrow 1^+} \left(\frac{2}{x^2 - 1} - \frac{x}{x - 1} \right) = \frac{-3}{2}$$

$$8. \lim_{x \rightarrow 0^+} x^x$$

$$\text{Solution: } \lim_{x \rightarrow 0^+} x^x$$

$= 0^0$ Indeterminate form.

$$\text{Let } A = x^x$$

Taking log on either side,

$$\log A = \log x^x$$

$$\lim_{x \rightarrow 0^+} \log A = \lim_{x \rightarrow 0^+} x \log(x)$$

$$= \lim_{x \rightarrow 0^+} \frac{\log x}{\frac{1}{x}}$$

$$= \frac{\log(0)}{\frac{1}{0}}$$

$$= \frac{-\infty}{\infty} \text{ Indeterminate form}$$

Applying l'hospital's Rule,

$$\lim_{x \rightarrow 0^+} \log A = \lim_{x \rightarrow 0^+} \frac{\log x}{\frac{1}{x}}$$

$$= \lim_{x \rightarrow 0^+} \frac{\frac{1}{x}}{\left(-\frac{1}{x^2}\right)}$$

$$= \lim_{x \rightarrow 0^+} \frac{1}{x} \times \frac{x^2}{-1}$$

$$= \lim_{x \rightarrow 0^+} \frac{x}{-1}$$

$$= \frac{0}{-1}$$

$$= 0$$

By Composite function theorem,

$$\log \lim_{x \rightarrow 0^+} A = 0$$

Taking exponential on both side,

$$\lim_{x \rightarrow 0^+} A = e^0 = 1$$

$$\therefore \lim_{x \rightarrow 0^+} x^x = 1$$

$$9. \lim_{x \rightarrow \infty} \left(1 + \frac{1}{x} \right)^x$$

$$\text{Solution: } \lim_{x \rightarrow \infty} \left(1 + \frac{1}{x} \right)^x$$

$$= \left(1 + \frac{1}{\infty} \right)^{\infty}$$

$$= (1 + 0)^{\infty}$$

$$= 1^{\infty} \text{ Indeterminate form.}$$

$$\text{Let } A = \left(1 + \frac{1}{x} \right)^x$$

Taking log on either side,

$$\log A = \log \left(1 + \frac{1}{x} \right)^x$$

$$\lim_{x \rightarrow \infty} \log A = \lim_{x \rightarrow \infty} x \log \left(1 + \frac{1}{x} \right)$$

$$= \lim_{x \rightarrow \infty} \frac{\log \left(1 + \frac{1}{x} \right)}{\frac{1}{x}}$$

$$= \frac{\log \left(1 + \frac{1}{\infty} \right)}{\frac{1}{\infty}}$$

$$\begin{aligned}
 &= \frac{\log(1+0)}{0} \\
 &= \frac{\log(1)}{0} \\
 &= \frac{0}{0} \text{ Indeterminate form}
 \end{aligned}$$

Applying l'hôpital's Rule,

$$\begin{aligned}
 \lim_{x \rightarrow \infty} \log A &= \lim_{x \rightarrow \infty} \frac{\log\left(1 + \frac{1}{x}\right)}{\frac{1}{x}} \\
 &= \lim_{x \rightarrow \infty} \frac{\left(1 + \frac{1}{x}\right)\left(-\frac{1}{x^2}\right)}{\left(-\frac{1}{x^2}\right)} \\
 &= \lim_{x \rightarrow \infty} \left(1 + \frac{1}{x}\right) \\
 &= \left(1 + \frac{1}{\infty}\right) \\
 &= (1 + 0) \\
 &= 1
 \end{aligned}$$

By Composite function theorem,

$$\log \lim_{x \rightarrow \infty} A = 1$$

Taking exponential on both side,

$$\begin{aligned}
 \lim_{x \rightarrow \infty} A &= e^1 = e \\
 \therefore \lim_{x \rightarrow \infty} \left(1 + \frac{1}{x}\right)^x &= e
 \end{aligned}$$

$$10. \lim_{x \rightarrow \frac{\pi}{2}} (\sin x)^{\tan x}$$

$$\text{Solution: } \lim_{x \rightarrow \frac{\pi}{2}} (\sin x)^{\tan x}$$

$$\begin{aligned}
 &= \left(\sin \frac{\pi}{2}\right)^{\tan \frac{\pi}{2}} \\
 &= (1)^\infty \\
 &= 1^\infty \text{ Indeterminate form.}
 \end{aligned}$$

$$\text{Let } A = (\sin x)^{\tan x}$$

Taking log on either side,

$$\begin{aligned}
 \log A &= \log (\sin x)^{\tan x} \\
 &= \tan x \log (\sin x) \\
 &= \frac{\log(\sin x)}{\frac{1}{\tan x}} \\
 &= \frac{\log(\sin x)}{\cot x}
 \end{aligned}$$

$$\lim_{x \rightarrow \frac{\pi}{2}} \log A = \lim_{x \rightarrow \frac{\pi}{2}} \frac{\log(\sin x)}{\cot x}$$

$$\begin{aligned}
 &= \frac{\log\left(\sin \frac{\pi}{2}\right)}{\cot \frac{\pi}{2}} \\
 &= \frac{\log(1)}{0} \\
 &= \frac{0}{0} \text{ Indeterminate form}
 \end{aligned}$$

Applying l'hôpital's Rule,

$$\begin{aligned}
 \lim_{x \rightarrow \frac{\pi}{2}} \log A &= \lim_{x \rightarrow \frac{\pi}{2}} \frac{\frac{1}{\sin x}(\cos x)}{-\operatorname{cosec}^2 x} \\
 &= \lim_{x \rightarrow \frac{\pi}{2}} \frac{\cos x}{\sin x} \times \frac{\sin^2 x}{-1} \\
 &= \lim_{x \rightarrow \frac{\pi}{2}} \frac{\cos x \sin x}{-1} \\
 &= \frac{\cos\left(\frac{\pi}{2}\right) \sin\left(\frac{\pi}{2}\right)}{-1} \\
 &= \frac{(0)(1)}{-1} \\
 &= 0
 \end{aligned}$$

Taking exponential on both side,

$$\lim_{x \rightarrow \frac{\pi}{2}} A = e^0 = 1$$

$$\therefore \lim_{x \rightarrow \frac{\pi}{2}} (\sin x)^{\tan x} = 1$$

$$11. \lim_{x \rightarrow 0^+} (\cos x)^{\frac{1}{x^2}}$$

$$\text{Solution: } \lim_{x \rightarrow 0^+} (\cos x)^{\frac{1}{x^2}}$$

$$\begin{aligned}
 &= (\cos 0)^{\frac{1}{0}} \\
 &= (1)^\infty \\
 &= 1^\infty \text{ Indeterminate form.}
 \end{aligned}$$

$$\text{Let } A = (\cos x)^{\frac{1}{x^2}}$$

Taking log on either side,

$$\begin{aligned}
 \log A &= \log (\cos x)^{\frac{1}{x^2}} \\
 &= \frac{1}{x^2} \log (\cos x) \\
 &= \frac{\log (\cos x)}{x^2}
 \end{aligned}$$

$$\begin{aligned}
 \lim_{x \rightarrow 0^+} \log A &= \lim_{x \rightarrow 0^+} \frac{\log (\cos x)}{x^2} \\
 &= \frac{\log (\cos 0)}{0}
 \end{aligned}$$

$$= \frac{\log(1)}{0}$$

$$= \frac{0}{0} \text{ Indeterminate form}$$

Applying l'hôpital's Rule,

$$\lim_{x \rightarrow 0^+} \log A = \lim_{x \rightarrow 0^+} \frac{\frac{1}{\cos x}(-\sin x)}{2x}$$

$$= \lim_{x \rightarrow 0^+} \frac{-\sin x}{(\cos x)2x}$$

$$= \frac{-\sin 0}{(\cos 0)(0)}$$

$$= \frac{0}{0} \text{ Indeterminate form}$$

Applying l'hôpital's Rule,

$$\lim_{x \rightarrow 0^+} \frac{-\sin x}{(\cos x)2x} = \lim_{x \rightarrow 0^+} \frac{-(\cos x)}{(\cos x)(2) + 2x(-\sin x)}$$

$$= \frac{-(\cos 0)}{(\cos 0)(2) + 0(-\sin 0)}$$

$$= \frac{-(1)}{(1)(2) + 0(0)}$$

$$= \frac{-1}{2}$$

By Composite function theorem,

$$\log \lim_{x \rightarrow 0^+} A = -\frac{1}{2}$$

Taking exponential on both side,

$$\lim_{x \rightarrow 0^+} A = e^{-\frac{1}{2}} = \frac{1}{\sqrt{e}}$$

12. If an initial amount A_0 of money is invested at an interest rate r compounded n times a year, the value of the investment after t years is $A = A_0 \left(1 + \frac{r}{n}\right)^{nt}$. If the interest is compounded continuously, (that is as $n \rightarrow \infty$), show that the amount after t years is $A = A_0 e^{rt}$

Solution: $\log_{n \rightarrow \infty} A_0 \left(1 + \frac{r}{n}\right)^{nt}$

$$= A_0 \left(1 + \frac{r}{\infty}\right)^{\infty t}$$

$$= A_0 (1 + 0)^{\infty}$$

$$= A_0 (1)^{\infty} \text{ Indeterminate form}$$

$$\text{Let } Y = \left(1 + \frac{r}{n}\right)^{nt}$$

Taking log on either side,

$$\log Y = \log \left(1 + \frac{r}{n}\right)^{nt}$$

$$= (nt) \log \left(1 + \frac{r}{n}\right)$$

$$= (t) \frac{\log \left(1 + \frac{r}{n}\right)}{\frac{1}{n}}$$

$$\log_{n \rightarrow \infty} Y = \log_{n \rightarrow \infty} (t) \frac{\log \left(1 + \frac{r}{n}\right)}{\frac{1}{n}}$$

$$= (t) \log_{n \rightarrow \infty} \frac{\log \left(1 + \frac{r}{n}\right)}{\frac{1}{n}}$$

$$= (t) \log_{n \rightarrow \infty} \left[\frac{\frac{1}{\left(1 + \frac{r}{n}\right)} \left(-\frac{r}{n^2}\right)}{\left(-\frac{1}{n^2}\right)} \right]$$

$$= (t) \log_{n \rightarrow \infty} \left[\frac{1}{\left(1 + \frac{r}{n}\right)} \left(-\frac{r}{n^2}\right) \times \left(-\frac{n^2}{1}\right) \right]$$

$$= (t) \log_{n \rightarrow \infty} \frac{1}{\left(1 + \frac{r}{n}\right)} (r)$$

$$= (rt) \log_{n \rightarrow \infty} \frac{1}{\left(1 + \frac{r}{n}\right)}$$

$$= (rt) \frac{1}{\left(1 + \frac{r}{\infty}\right)}$$

$$= (rt) \frac{1}{(1+0)}$$

$$= (rt) (1)$$

$$= (rt)$$

By Composite function theorem,

$$\log \lim_{x \rightarrow \infty} Y = rt$$

Taking exponential on both side,

$$\lim_{x \rightarrow \infty} Y = e^{rt}$$

$$\text{Hence, } \log_{n \rightarrow \infty} A_0 \left(1 + \frac{r}{n}\right)^{nt} = A_0 e^{rt}$$

Example 7.46

Prove that the function $f(x) = x^2 + 2$ is strictly increasing in the interval $(2, 7)$ and strictly decreasing in the interval $(-2, 0)$.

Solution: Given $f(x) = x^2 + 2$

$$f'(x) = 2x$$

In the interval $(2, 7)$ let $x = 3$

$$f'(x) = 2(3) = 6 > 0$$

So, $f(x)$ is strictly increasing function.

In the interval $(-2, 0)$ let $x = -1$

$$f'(x) = 2(-1) = -2 < 0$$

So, $f(x)$ is strictly decreasing function.

Example 7.47

Prove that the function $f(x) = x^2 - 2x - 3$ is strictly increasing in $(2, \infty)$.

Solution: $f(x) = x^2 - 2x - 3$

$$f'(x) = 2x - 2$$

In the interval $(2, \infty)$ let $x = 3$

$$f'(x) = 2(3) - 2$$

$$= 6 - 2 = 4 > 0$$

So, $f(x)$ is strictly increasing function.

Example 7.48

Find the absolute maximum and absolute minimum values of the function

$$f(x) = 2x^3 + 3x^2 - 12x \text{ on } [-3, 2]$$

Solution: $f(x) = 2x^3 + 3x^2 - 12x$

$$f'(x) = 6x^2 + 6x - 12$$

$$\text{When, } f'(x) = 0$$

$$6x^2 + 6x - 12 = 0$$

Dividing by 6,

$$x^2 + x - 2 = 0$$

$$(x + 2)(x - 1) = 0$$

$$x = -2 \text{ and } x = 1$$

since $x = -2, 1 \in [-3, 2]$

The critical numbers are $x = -2, 1$ and

given the end points are $x = -3, 2$

$$\text{At } x = -2$$

$$f(-2) = 2(-2)^3 + 3(-2)^2 - 12(-2)$$

$$= 2(-8) + 3(4) + 24$$

$$= -16 + 12 + 24$$

$$= -16 + 36$$

$$f(-2) = 20$$

$$\text{At } x = 1$$

$$f(1) = 2(1)^3 + 3(1)^2 - 12(1)$$

$$= 2 + 3 - 12$$

$$= 5 - 12$$

$$f(1) = -7$$

$$\text{At } x = -3$$

$$f(-3) = 2(-3)^3 + 3(-3)^2 - 12(-3)$$

$$= 2(-27) + 3(9) + 36$$

$$= -54 + 27 + 36$$

$$= -54 + 63$$

$$f(-3) = 9$$

$$\text{At } x = 2$$

$$f(2) = 2(2)^3 + 3(2)^2 - 12(2)$$

$$= 2(8) + 3(4) - 24$$

$$= 16 + 12 - 24$$

$$= 28 - 24$$

$$f(2) = 4$$

Hence, absolute maximum value of $f(x) = 20$

and absolute minimum value of $f(x) = -7$

Example 7.49

Find the absolute extrema of the function

$$f(x) = 3 \cos x \text{ on the closed interval } [0, 2\pi].$$

Solution: $f(x) = 3 \cos x$

$$f'(x) = 3(-\sin x)$$

$$\text{When, } f'(x) = 0$$

$$-3 \sin x = 0$$

$$\sin x = 0$$

$$\text{Since } \sin \pi = 0, x = \pi \in [0, 2\pi]$$

The critical number is $x = \pi$ and

given the end points are $x = 0, 2\pi$

$$\text{At } x = \pi$$

$$f(x) = 3 \cos(\pi)$$

$$= 3(-1)$$

$$= -3$$

$$\text{At } x = 0$$

$$f(x) = 3 \cos(0)$$

$$= 3(1)$$

$$= 3$$

$$\text{At } x = 2\pi$$

$$f(x) = 3 \cos(2\pi)$$

$$= 3(1)$$

$$= 3$$

Hence, absolute maximum value of $f(x) = 3$

and absolute minimum value of $f(x) = -3$

Example 7.50

Find the intervals of monotonicity and hence find the local extrema for the function $f(x) = x^2 - 4x + 4$.

Solution: $f(x) = x^2 - 4x + 4$

$$f'(x) = 2x - 4$$

When, $f'(x) = 0$

$$2x - 4 = 0$$

$$2x = 4$$

$x = 2$ is the critical number.

\therefore The interval are $(-\infty, 2)$ and $(2, \infty)$

In the interval $(-\infty, 2)$ let $x = 0$

$$f'(x) = 2x - 4$$

$$= 2(0) - 4$$

$$= -4 < 0$$

So, $f(x)$ is strictly decreasing function.

In the interval $(2, \infty)$ let $x = 3$

$$f'(x) = 2x - 4$$

$$f'(x) = 2(3) - 4 = 6 - 4 = 2 > 0$$

So, $f(x)$ is strictly increasing function.

Because $f'(x)$ changes its sign from negative to positive when passing through $x = 2$ for the function $f(x)$, it has a local minimum at $x = 2$.

The local minimum value is

$$f(2) = (2)^2 - 4(2) + 4$$

$$= 4 - 8 + 4$$

$$= 8 - 8$$

$$f(2) = 0$$

Example 7.51 Find the intervals of monotonicity and hence find the local extrema for the

function $f(x) = x^{\frac{2}{3}}$

Solution: $f(x) = x^{\frac{2}{3}}$

$$f'(x) = \frac{2}{3} x^{\frac{2}{3}-1}$$

$$= \frac{2}{3} x^{\frac{2-3}{3}}$$

$$= \frac{2}{3} x^{-\frac{1}{3}}$$

$$= \frac{2}{3x^{\frac{1}{3}}}$$

$f'(x)$ can't be equal to zero for $x \in \mathbb{R}$ hence

$f'(x)$ does not exist at $x = 0$.

\therefore The interval are $(-\infty, 0)$ and $(0, \infty)$

In the interval $(-\infty, 0)$ $f'(x) < 0$

So, $f(x)$ is strictly decreasing function.

In the interval $(0, \infty)$ $f'(x) > 0$

So, $f(x)$ is strictly increasing function.

Because $f'(x)$ changes its sign from negative to positive when passing through $x = 0$ for the

function $f(x)$, it has a local minimum at $x = 0$

The local minimum value is $f(0) = 0$. Local

minimum occurs at $x = 0$ which is not a

stationary point.

Example 7.52

Prove that the function $f(x) = x - \sin x$ is increasing on the real line. Also discuss for the existence of local extrema.

Solution: $f(x) = x - \sin x$

$$f'(x) = 1 - \cos x$$

When $f(x) = 0$, we get

$$1 - \cos x = 0$$

$$\cos x = 1$$

$$\text{So, } x = 2n\pi, n \in \mathbb{Z}$$

$\therefore f(x)$ is increasing on the real line. Since there is no change $f'(x)$ passing through $x = 2n\pi$, by the first derivative test there is no local extrema.

Example 7.53

Discuss the monotonicity and local extrema of the function $f(x) = \log(1+x) - \frac{x}{1+x}$, $x > -1$ and hence find the domain where, $\log(1+x) > \frac{x}{1+x}$.

Solution: $f(x) = \log(1+x) - \frac{x}{1+x}$

$$f'(x) = \frac{1}{(1+x)} - \left[\frac{(1+x)(1-x(1))}{(1+x)^2} \right]$$

$$\begin{aligned}
 &= \frac{1}{(1+x)} - \left[\frac{1+x-x}{(1+x)^2} \right] \\
 &= \frac{1}{(1+x)} - \left[\frac{1}{(1+x)^2} \right] \\
 &= \frac{(1+x)-1}{(1+x)^2} \\
 &= \frac{1+x-1}{(1+x)^2} \\
 f'(x) &= \frac{x}{(1+x)^2}
 \end{aligned}$$

When $f'(x) = 0$,

$$\frac{x}{(1+x)^2} = 0 \text{ gives}$$

$$x = 0$$

\therefore The interval are $(-\infty, 0)$ and $(0, \infty)$

In the interval $(-\infty, 0)$ $f'(x) < 0$

So, $f(x)$ is strictly decreasing function.

In the interval $(0, \infty)$ $f'(x) > 0$

So, $f(x)$ is strictly increasing function.

Because $f'(x)$ changes its sign from negative to

positive when passing through $x = 0$ for the

function $f(x)$, it has a local minimum at $x = 0$

The local minimum value is

$$\begin{aligned}
 f(0) &= \log(1+0) - \frac{0}{1+0} \\
 &= 0
 \end{aligned}$$

Further when $x > 0$,

$$f(x) > f(0)$$

$$\log(1+x) - \frac{x}{1+x} > 0$$

$$\log(1+x) > \frac{x}{1+x} \quad \text{Proved.}$$

Example 7.54

Find the intervals of monotonicity and local extrema of the function $f(x) = x \log x + 3x$.

Solution: $f(x) = x \log x + 3x$

$$\begin{aligned}
 f'(x) &= x \left(\frac{1}{x} \right) + \log x (1) + 3 \\
 &= 1 + \log x + 3 \\
 &= 4 + \log x
 \end{aligned}$$

When $f'(x) = 0$,

$$4 + \log x = 0$$

$$\log x = -4$$

$$x = e^{-4}$$

\therefore The interval are $(-\infty, e^{-4})$ and (e^{-4}, ∞)

In the interval $(-\infty, e^{-4})$ $f'(x) < 0$

So, $f(x)$ is strictly decreasing function.

In the interval (e^{-4}, ∞) $f'(x) > 0$

So, $f(x)$ is strictly increasing function.

Because $f'(x)$ changes its sign from negative to

positive when passing through $x = e^{-4}$ for the

function $f(x)$, it has a local minimum at $x =$

$$e^{-4}, f(x) = x \log x + 3x$$

$$f(e^{-4}) = e^{-4} \log e^{-4} + 3e^{-4}$$

$$= e^{-4}(-4) \log e + 3e^{-4}$$

$$= e^{-4}(-4)(1) + 3e^{-4}$$

$$= -4e^{-4} + 3e^{-4}$$

$$= -e^{-4}$$

Local minimum is $-e^{-4}$

Example 7.55

Find the intervals of monotonicity and local

extrema of the function $f(x) = \frac{1}{1+x^2}$

Solution: $f(x) = \frac{1}{1+x^2}$

$$f'(x) = \frac{-2x}{(1+x^2)^2}$$

When $f'(x) = 0$,

$$\frac{-2x}{(1+x^2)^2} = 0 \text{ gives}$$

$$x = 0$$

\therefore The interval are $(-\infty, 0)$ and $(0, \infty)$

In the interval $(-\infty, 0)$ $f'(x) > 0$

So, $f(x)$ is strictly increasing function.

In the interval $(0, \infty)$ $f'(x) < 0$

So, $f(x)$ is strictly decreasing function.

Because $f'(x)$ changes its sign from positive to

negative when passing through $x = 0$ for the

function $f(x)$, it has a local maximum at $x = 0$,

$$f(0) = \frac{1}{1+0} = 1$$

Example 7.56

Find the intervals of monotonicity and local extrema of the function $f(x) = \frac{x}{1+x^2}$

Solution: $f(x) = \frac{x}{1+x^2}$

$$f'(x) = \frac{(1+x^2)(1) - x(2x)}{(1+x^2)^2}$$

$$= \frac{1+x^2-2x^2}{(1+x^2)^2}$$

$$= \frac{1-x^2}{(1+x^2)^2}$$

When $f'(x) = 0$,

$$\frac{1-x^2}{(1+x^2)^2} = 0 \text{ gives}$$

$$1 - x^2 = 0$$

$$x^2 = 1$$

$$x = \pm 1$$

Hence stationary points are $x = 1, -1$

\therefore The interval are $(-\infty, -1)$, $(-1, 1)$ and $(1, \infty)$

In the interval $(-\infty, -1)$ $f'(x) < 0$

So, $f(x)$ is strictly decreasing function.

In the interval $(-1, 1)$ $f'(x) > 0$

So, $f(x)$ is strictly increasing function and

In the interval $(1, \infty)$ $f'(x) < 0$

So, $f(x)$ is strictly decreasing function

Because $f'(x)$ changes its sign from negative to positive when passing through $x = -1$ for the function $f(x)$, it has a local minimum at $x =$

$$-1, f(x) = \frac{(-1)}{1+(-1)^2}$$

$$= \frac{-1}{1+1}$$

$$= -\frac{1}{2}$$

Local minimum is $-\frac{1}{2}$

Because $f'(x)$ changes its sign from positive to negative when passing through $x = 1$ for the

function $f(x)$, it has a local maximum at $x = 1$,

$$f(x) = \frac{(1)}{1+(1)^2}$$

$$= \frac{1}{1+1}$$

$$= \frac{1}{2}$$

Local maximum is $-\frac{1}{2}$

EXERCISE 7.6

1. Find the absolute extrema of the following functions on the given closed interval.

(i) $f(x) = x^2 - 12x + 10$; $[1, 2]$

(ii) $f(x) = 3x^4 - 4x^3$; $[-1, 2]$

(iii) $f(x) = 6x^{\frac{4}{3}} - 3x^{\frac{1}{3}}$; $[-1, 1]$

(iv) $f(x) = 2 \cos x + \sin 2x$; $\left[0, \frac{\pi}{2}\right]$

Solution: (i) $f(x) = x^2 - 12x + 10$; $[1, 2]$

$$f(x) = x^2 - 12x + 10$$

$$f'(x) = 2x - 12$$

When, $f'(x) = 0$

$$2x - 12 = 0$$

$$2x = 12$$

$$x = 6$$

The critical number $x = 6$ and

given the end points are $x = 1, 2$

At $x = 6$

$$f(6) = (6)^2 - 12(6) + 10$$

$$= 36 - 72 + 10$$

$$= 46 - 72$$

$$= -26$$

$$f(6) = -26$$

At $x = 1$

$$f(1) = (1)^2 - 12(1) + 10$$

$$= 1 - 12 + 10$$

$$= 11 - 12$$

$$f(1) = -1$$

At $x = 2$

$$f(2) = (2)^2 - 12(2) + 10$$

$$= 4 - 24 + 10$$

$$= 14 - 24$$

$$= -10$$

Hence, absolute maximum value of $f(x) = -1$

and absolute minimum value of $f(x) = -26$

$$(ii) \quad f(x) = 3x^4 - 4x^3; [-1, 2]$$

$$f(x) = 3x^4 - 4x^3$$

$$f'(x) = 12x^3 - 12x^2$$

$$\text{When, } f'(x) = 0$$

$$12x^3 - 12x^2 = 0$$

$$12x^2(1 - x) = 0$$

$$12x^2 = 0, \text{ gives } x = 0$$

$$1 - x = 0, \text{ gives } x = 1$$

The critical numbers $x = 0, 1$ and

given the end points are $x = -1, 2$

$$\text{At } x = 0$$

$$f(0) = 3(0)^4 - 4(0)^3$$

$$= 0$$

$$f(0) = 0$$

$$\text{At } x = 1$$

$$f(1) = 3(1)^4 - 4(1)^3$$

$$= 3 - 4$$

$$f(1) = -1$$

$$\text{At } x = -1$$

$$f(-1) = 3(-1)^4 - 4(-1)^3$$

$$= 3(1) - 4(-1)$$

$$= 3 + 4$$

$$f(-1) = 7$$

$$\text{At } x = 2$$

$$f(2) = 3(2)^4 - 4(2)^3$$

$$= 3(16) - 4(8)$$

$$= 48 - 32$$

$$= 16$$

Hence, absolute maximum value of $f(x) = 16$

and absolute minimum value of $f(x) = -1$

$$(iii) \quad f(x) = 6x^{\frac{4}{3}} - 3x^{\frac{1}{3}}; [-1, 1]$$

$$f(x) = 6x^{\frac{4}{3}} - 3x^{\frac{1}{3}}$$

$$f'(x) = 6\left(\frac{4}{3}\right)x^{\frac{4}{3}-1} - 3\left(\frac{1}{3}\right)x^{\frac{1}{3}-1}$$

$$= 8x^{\frac{1}{3}} - x^{\frac{1}{3}-1}$$

$$= x^{\frac{1}{3}}(8 - x^{-1})$$

$$= x^{\frac{1}{3}}\left(8 - \frac{1}{x}\right)$$

When $f'(x) = 0$, we get

$$x^{\frac{1}{3}}\left(8 - \frac{1}{x}\right) = 0$$

$$\text{Hence } x = 0 \text{ and } 8 - \frac{1}{x} = 0 \text{ gives } x = \frac{1}{8}$$

So, critical numbers are $x = \frac{1}{8}, 0$ and the end

points are $x = -1, 1$

At $x = 0$,

$$f(0) = 6(0)^{\frac{4}{3}} - 3(0)^{\frac{1}{3}} = 0$$

At $x = \frac{1}{8}$,

$$f\left(\frac{1}{8}\right) = 6\left(\frac{1}{8}\right)^{\frac{4}{3}} - 3\left(\frac{1}{8}\right)^{\frac{1}{3}}$$

$$= 6\left[\left(\frac{1}{2}\right)^3\right]^{\frac{4}{3}} - 3\left[\left(\frac{1}{2}\right)^3\right]^{\frac{1}{3}}$$

$$= 6\left(\frac{1}{2}\right)^4 - 3\left(\frac{1}{2}\right)^1$$

$$= 6\left(\frac{1}{16}\right) - 3\left(\frac{1}{2}\right)$$

$$= \frac{3}{8} - \frac{3}{2}$$

$$= \frac{3-12}{8}$$

$$= -\frac{9}{8}$$

At $x = -1$,

$$f(-1) = 6(-1)^{\frac{4}{3}} - 3(-1)^{\frac{1}{3}}$$

$$= 6[(-1)^3]^{\frac{4}{3}} - 3[(-1)^3]^{\frac{1}{3}}$$

$$= 6(-1)^4 - 3(-1)^1$$

$$= 6(1) - 3(-1)$$

$$= 6 + 3$$

$$= 9$$

At $x = 1$,

$$f(1) = 6(1)^{\frac{4}{3}} - 3(1)^{\frac{1}{3}}$$

$$= 6(1) - 3(1)$$

$$= 6 - 3$$

$$= 3$$

Hence, absolute maximum value of $f(x) = 9$

and absolute minimum value of $f(x) = -\frac{9}{8}$

$$(iv) \quad f(x) = 2 \cos x + \sin 2x; \left[0, \frac{\pi}{2}\right]$$

$$f(x) = 2 \cos x + \sin 2x$$

$$f'(x) = 2(-\sin x) + 2 \cos 2x$$

When $f'(x) = 0$,

$$-2 \sin x + 2 \cos 2x = 0$$

$$2 \cos 2x = 2 \sin x$$

$$\cos 2x = \sin x$$

$$1 - 2\sin^2 x = \sin x$$

$$2\sin^2 x + \sin x - 1 = 0$$

$$(2 \sin x - 1)(\sin x + 1) = 0$$

$2 \sin x - 1 = 0$, gives

$$2 \sin x = 1$$

$$\sin x = \frac{1}{2}$$

$$x = \frac{\pi}{6} \text{ and }$$

$\sin x + 1 = 0$, gives

$$\sin x = -1$$

$x = -\frac{\pi}{2}, \frac{3\pi}{2}$ they does not belong to the given

interval. So the critical number is $x = \frac{\pi}{6}$ and

the end points are $x = 0, \frac{\pi}{2}$

At $x = \frac{\pi}{6}$

$$f\left(\frac{\pi}{6}\right) = 2 \cos \frac{\pi}{6} + \sin 2\left(\frac{\pi}{6}\right)$$

$$= 2 \cos \frac{\pi}{6} + \sin \frac{\pi}{3}$$

$$= 2\left(\frac{\sqrt{3}}{2}\right) + \frac{\sqrt{3}}{2}$$

$$= \frac{3\sqrt{3}}{2}$$

At $x = 0$

$$f(0) = 2 \cos 0 + \sin 2(0)$$

$$= 2 \cos 0 + \sin 0$$

$$= 2(1) + 0$$

$$= 2$$

At $x = \frac{\pi}{2}$

$$f\left(\frac{\pi}{2}\right) = 2 \cos \frac{\pi}{2} + \sin 2\left(\frac{\pi}{2}\right)$$

$$= 2 \cos \frac{\pi}{2} + \sin \pi$$

$$= 2(0) + 0$$

$$= 0$$

Hence, absolute maximum value of $f(x) = \frac{3\sqrt{3}}{2}$

and absolute minimum value of $f(x) = 0$

2. Find the intervals of monotonicities and hence find the local extremum for the following functions:

$$(i) f(x) = 2x^3 + 3x^2 - 12x$$

$$(ii) f(x) = \frac{x}{x-5} \quad (iii) f(x) = \frac{e^x}{1-e^x}$$

$$(iv) f(x) = \frac{x^3}{3} - \log x$$

$$(v) f(x) = \sin x \cos x + 5, x \in (0, 2\pi)$$

Solution:

$$(i) f(x) = 2x^3 + 3x^2 - 12x$$

$$f'(x) = 6x^2 + 6x - 12$$

When, $f'(x) = 0$

$$6x^2 + 6x - 12 = 0$$

Dividing by 6,

$$x^2 + x - 2 = 0$$

$$(x+2)(x-1) = 0$$

$$x+2 = 0, \text{ gives } x = -2 \text{ and}$$

$$x-1 = 0, \text{ gives } x = 1$$

The critical numbers $x = -2, 1$

\therefore The interval are $(-\infty, -2)$, $(-2, 1)$ and $(1, \infty)$

In the interval $(-\infty, -2)$ let $x = -3$

$$f'(-3) = 6(-3)^2 + 6(-3) - 12$$

$$= 6(9) - 18 - 12$$

$$= 54 - 30$$

$$= 24 > 0$$

So, $f(x)$ is strictly increasing function.

In the interval $(-2, 1)$ let $x = 0$

$$f'(0) = 6(0)^2 + 6(0) - 12$$

$$= 0 + 0 - 12$$

$$= -12 < 0$$

So, $f(x)$ is strictly decreasing function.

Because $f'(x)$ changes its sign from positive to negative when passing through $x = -2$ for the function $f(x)$, has a local maximum at $x = -2$

$$\begin{aligned} f(-2) &= 2(-2)^3 + 3(-2)^2 - 12(-2) \\ &= 2(-8) + 3(4) - 12(-2) \\ &= -16 + 12 + 24 \\ &= -16 + 36 \\ &= 20 \end{aligned}$$

Local maximum is 20

In the interval $(1, \infty)$ let $x = 2$

$$\begin{aligned} f'(2) &= 6(2)^2 + 6(2) - 12 \\ &= 6(4) + 12 - 12 \\ &= 24 \\ &= 24 > 0 \end{aligned}$$

So, $f(x)$ is strictly increasing function. Because

$f'(x)$ changes its sign from negative to positive when passing through $x = 1$ for the function

$f(x)$, has a local minimum at $x = 1$

$$\begin{aligned} f(1) &= 2(1)^3 + 3(1)^2 - 12(1) \\ &= 2(1) + 3(1) - 12(1) \\ &= 2 + 3 - 12 \\ &= 5 - 12 \\ &= -7 \end{aligned}$$

Local minimum is -7

$$\begin{aligned} \text{(ii) } f(x) &= \frac{x}{x-5} \\ f'(x) &= \frac{(x-5)(1) - x(1)}{(x-5)^2} \\ &= \frac{x-5-x}{(x-5)^2} \\ &= \frac{-5}{(x-5)^2} \end{aligned}$$

$f'(x)$ cannot be equal to zero. And $f'(x)$

does not exist at $x = 5$.

Hence the critical number is

$x = 5$. \therefore The interval are $(-\infty, 5)$ and $(5, \infty)$

In the interval $(-\infty, 5)$ $f'(x) < 0$

So, $f(x)$ is strictly decreasing function.

In the interval $(5, \infty)$ $f'(x) < 0$

So, $f(x)$ is strictly decreasing function.

Since there is no change $f'(x)$ passing through $x = 5$, by the first derivative test there is no local extrema.

$$\begin{aligned} \text{(iii) } f(x) &= \frac{e^x}{1-e^x} \\ f'(x) &= \frac{(1-e^x)(e^x) - e^x(e^x)}{(1-e^x)^2} \\ &= \frac{e^x - e^{2x} + e^{2x}}{(1-e^x)^2} \\ &= \frac{e^x}{(1-e^x)^2} \end{aligned}$$

$f'(x)$ cannot be equal to zero. And $f'(x)$ does not exist at $x = 0$.

Hence the critical number is $x = 0$.

\therefore The interval are $(-\infty, 0)$ and $(0, \infty)$

In the interval $(-\infty, 0)$ $f'(x) > 0$

So, $f(x)$ is strictly increasing function.

In the interval $(0, \infty)$ $f'(x) > 0$

So, $f(x)$ is strictly increasing function.

Since there is no change $f'(x)$ passing through $x = 0$, by the first derivative test there is no local extrema.

$$\begin{aligned} \text{(iv) } f(x) &= \frac{x^3}{3} - \log x \\ f'(x) &= \frac{3x^2}{3} - \frac{1}{x} \\ &= x^2 - \frac{1}{x} \\ &= \frac{x^3-1}{x} \end{aligned}$$

When $f'(x) = 0$,

$$x^3 - 1 = 0$$

$$(x-1)(x^2+x+1) = 0$$

$x-1 = 0$, Gives $x = 1$.

Hence there is a critical point at $x = 1$.

\therefore The interval are $(-\infty, 1)$ and $(1, \infty)$

In the interval $(-\infty, 1)$ $f'(x) < 0$

So, $f(x)$ is strictly decreasing function.

In the interval $(1, \infty)$ $f'(x) > 0$

So, $f(x)$ is strictly increasing function.

Because $f'(x)$ changes its sign from negative to positive when passing through $x = 1$ for the function $f(x)$, it has a local minimum at $x = 1$,

$$\begin{aligned} f(1) &= \frac{1}{3} - \log 1 \\ &= \frac{1}{3} - 0 \\ &= \frac{1}{3}, \text{ So local minimum is } \frac{1}{3} \end{aligned}$$

$$(v) f(x) = \sin x \cos x + 5, x \in (0, 2\pi)$$

$$f(x) = \sin x \cos x + 5$$

$$\begin{aligned} f'(x) &= \sin x (-\sin x) + \cos x (\cos x) \\ &= -\sin^2 x + \cos^2 x \\ &= \cos^2 x - \sin^2 x \\ &= \cos 2x \end{aligned}$$

When $f'(x) = 0$,

$$\cos 2x = 0$$

We know that $\cos \frac{\pi}{2} = 0$, hence

$$2x = \frac{\pi}{2} \text{ gives, } x = \frac{\pi}{4}$$

The values $\frac{\pi}{4}, \frac{3\pi}{4}, \frac{5\pi}{4}$ and $\frac{7\pi}{4} \in (0, 2\pi)$

\therefore The interval are $(0, \frac{\pi}{4}), (\frac{\pi}{4}, \frac{3\pi}{4}), (\frac{3\pi}{4}, \frac{5\pi}{4}),$
 $(\frac{5\pi}{4}, \frac{7\pi}{4})$ and $(\frac{7\pi}{4}, 2\pi)$

In the interval $(0, \frac{\pi}{4})$ $f'(x) > 0$

So, $f(x)$ is strictly increasing function.

In the interval $(\frac{\pi}{4}, \frac{3\pi}{4})$ $f'(x) < 0$

So, $f(x)$ is strictly decreasing function.

In the interval $(\frac{3\pi}{4}, \frac{5\pi}{4})$ $f'(x) > 0$

So, $f(x)$ is strictly increasing function.

In the interval $(\frac{5\pi}{4}, \frac{7\pi}{4})$ $f'(x) < 0$

So, $f(x)$ is strictly decreasing function.

In the interval $(\frac{7\pi}{4}, 2\pi)$ $f'(x) > 0$

So, $f(x)$ is strictly increasing function.

Because $f'(x)$ changes its sign from positive to negative when passing through $x = \frac{\pi}{4}, \frac{5\pi}{4}$ for

the function $f(x)$, it has a local maximum at

$$x = \frac{\pi}{4}, \frac{5\pi}{4}$$

$$f(x) = \sin x \cos x + 5$$

$$f\left(\frac{\pi}{4}\right) = \sin \frac{\pi}{4} \cos \frac{\pi}{4} + 5$$

$$= \left(\frac{1}{\sqrt{2}}\right)\left(\frac{1}{\sqrt{2}}\right) + 5$$

$$= \frac{1}{2} + 5$$

$$= \frac{1+10}{2} = \frac{11}{2}$$

$$f\left(\frac{5\pi}{4}\right) = \sin\left(\frac{5\pi}{4}\right) \cos\left(\frac{5\pi}{4}\right) + 5$$

$$= \sin\left(\pi + \frac{\pi}{4}\right) \cos\left(\pi + \frac{\pi}{4}\right) + 5$$

$$= \left[-\sin\left(\frac{\pi}{4}\right)\right]\left[-\cos\left(\frac{\pi}{4}\right)\right] + 5$$

$$= \left(\frac{1}{\sqrt{2}}\right)\left(\frac{1}{\sqrt{2}}\right) + 5$$

$$= \frac{1}{2} + 5$$

$$= \frac{1+10}{2} = \frac{11}{2}$$

Because $f'(x)$ changes its sign from negative to positive when passing through $x = \frac{3\pi}{4}, \frac{7\pi}{4}$ for

the function $f(x)$, it has a local maximum at

$$x = \frac{3\pi}{4}, \frac{7\pi}{4}$$

$$f(x) = \sin x \cos x + 5$$

$$f\left(\frac{3\pi}{4}\right) = \sin\left(\frac{3\pi}{4}\right) \cos\left(\frac{3\pi}{4}\right) + 5$$

$$= \sin\left(\frac{\pi}{2} + \frac{\pi}{4}\right) \cos\left(\frac{\pi}{2} + \frac{\pi}{4}\right) + 5$$

$$= \left[\cos\left(\frac{\pi}{4}\right)\right]\left[-\sin\left(\frac{\pi}{4}\right)\right] + 5$$

$$= \left(\frac{1}{\sqrt{2}}\right)\left(-\frac{1}{\sqrt{2}}\right) + 5$$

$$= -\frac{1}{2} + 5$$

$$= \frac{-1+10}{2} = \frac{9}{2}$$

$$\begin{aligned}
 f\left(\frac{7\pi}{4}\right) &= \sin\left(\frac{7\pi}{4}\right) \cos\left(\frac{7\pi}{4}\right) + 5 \\
 &= \sin\left(\frac{3\pi}{2} + \frac{\pi}{4}\right) \cos\left(\frac{3\pi}{2} + \frac{\pi}{4}\right) + 5 \\
 &= \left[-\cos\left(\frac{\pi}{4}\right)\right] \left[\sin\left(\frac{\pi}{4}\right)\right] + 5 \\
 &= \left(-\frac{1}{\sqrt{2}}\right) \left(\frac{1}{\sqrt{2}}\right) + 5 \\
 &= -\frac{1}{2} + 5 \\
 &= \frac{-1+10}{2} = \frac{9}{2}
 \end{aligned}$$

So local minimum is $\frac{9}{2}$ and local maximum is $\frac{11}{2}$

Example 7.57

Determine the intervals of concavity of the curve $f(x) = (x-1)^3 \cdot (x-5)$, $x \in \mathbb{R}$ and, points of inflection if any.

Solution: $f(x) = (x-1)^3 \cdot (x-5)$

$$\begin{aligned}
 f(x) &= (x^3 - 3x^2 + 3x - 1) \cdot (x-5) \\
 &= x^4 - 3x^3 + 3x^2 - x - 5x^3 + 15x^2 - 15x + 5 \\
 f(x) &= x^4 - 8x^3 + 18x^2 - 16x + 5 \\
 f'(x) &= 4x^3 - 24x^2 + 36x - 16 \\
 f''(x) &= 12x^2 - 48x + 36 \\
 \text{When } f''(x) &= 0, \\
 12x^2 - 48x + 36 &= 0
 \end{aligned}$$

Dividing by 12,

$$\begin{aligned}
 x^2 - 4x + 3 &= 0 \\
 (x-3)(x-1) &= 0 \\
 x-3 &= 0, \text{ gives } x = 3 \text{ and} \\
 x-1 &= 0, \text{ gives } x = 1 \\
 \therefore \text{The interval are } &(-\infty, 1), (1, 3) \text{ and } (3, \infty)
 \end{aligned}$$

In the interval $(-\infty, 1)$ let $x = 0$

$$\begin{aligned}
 f''(0) &= 12(0)^2 - 48(0) + 36 \\
 &= 0 - 0 + 36 \\
 &= 36 > 0
 \end{aligned}$$

Hence $f(x)$ is concave upward in the $(-\infty, 1)$

In the interval $(1, 3)$ let $x = 2$

$$f''(2) = 12(2)^2 - 48(2) + 36$$

$$\begin{aligned}
 &= 48 - 96 + 36 \\
 &= 84 - 96 \\
 &= -12 < 0
 \end{aligned}$$

Hence $f(x)$ is concave downward in the $(3, \infty)$

In the interval $(3, \infty)$ let $x = 4$

$$\begin{aligned}
 f''(4) &= 12(4)^2 - 48(4) + 36 \\
 &= 192 - 192 + 36 \\
 &= 36 > 0
 \end{aligned}$$

Hence $f(x)$ is concave upward in the $(3, \infty)$

$f''(x)$ changes the direction at $x = 1, 3$

We get the points of inflection at $x = 1, 3$

(i) at $x = 1$

$$\begin{aligned}
 f(x) &= (x-1)^3 \cdot (x-5) \\
 f(1) &= (1-1)^3 \cdot (1-5) \\
 &= 0
 \end{aligned}$$

(ii) at $x = 3$

$$\begin{aligned}
 f(x) &= (x-1)^3 \cdot (x-5) \\
 f(3) &= (3-1)^3 \cdot (3-5) \\
 &= (2)^3 \cdot (-2) \\
 &= 8 \cdot (-2) \\
 &= -16
 \end{aligned}$$

The points of inflections are $(1, 0), (3, -16)$

Example 7.58 Determine the intervals of concavity of the curve $y = 3 + \sin x$.

Solution:

$$y = 3 + \sin x$$

$$y' = \frac{dy}{dx} = \cos x$$

$$y'' = -\sin x$$

Substituting $y'' = 0$, we get

$$\sin x = 0, \text{ gives } x = 0, n\pi$$

As the limit $(-\pi, \pi)$ we get $(-\pi, 0)$ and $(0, \pi)$ as intervals.

In the interval $(-\pi, 0)$, let $x = -\frac{\pi}{2}$

$$y'' = -\sin x$$

$$= -\sin\left(-\frac{\pi}{2}\right)$$

$$= \sin\left(\frac{\pi}{2}\right) = 1 > 0$$

Hence $f(x)$ is concave upward in the $(-\pi, 0)$

In the interval $(0, \pi)$, let $x = \frac{\pi}{2}$

$$y'' = -\sin x$$

$$= -\sin\left(\frac{\pi}{2}\right)$$

$$= -1 < 0$$

Hence $f(x)$ is concave downward in the $(0, \pi)$

As y'' changes the sign as it passes through

$x = 0$, we get a point of inflection at $x = 0$

Substituting $x = 0$ in the given function,

$$y = 3 + \sin x \text{ we get}$$

$$y = 3 + \sin(0)$$

$$y = 3 + 0$$

$$y = 3$$

Hence the point of inflection is $(0, 3)$

In general the intervals are $(n\pi, (n+1)\pi)$, $n \in \mathbb{Z}$

Hence $(n\pi, 3)$ is also the point of inflection.

Example 7.59 Find the local extremum of the function $f(x) = x^4 + 32x$.

Solution:

$$f(x) = x^4 + 32x$$

$$f'(x) = 4x^3 + 32$$

$$f''(x) = 12x^2$$

When $f'(x) = 0$, we get

$$4x^3 + 32 = 0$$

$$4x^3 = -32$$

$$x^3 = -8$$

$$x = -2$$

$$\text{At } x = -2, f''(x) = 12x^2$$

$$f''(-2) = 12(-2)^2$$

$$= 12(4)$$

$$= 48 > 0$$

So, the function has local minima at $x = -2$

at $x = -2$, $f(x) = x^4 + 32x$ has

$$f(-2) = (-2)^4 + 32(-2)$$

$$= 16 - 64$$

$$= -48$$

So the local minimum is -48 and the extreme point is $(-2, -48)$

Example 7.60

Find the local extrema of the function $f(x) = 4x^6 - 6x^4$.

Solution: $f(x) = 4x^6 - 6x^4$

$$f'(x) = 24x^5 - 24x^3$$

$$f''(x) = 120x^4 - 72x^2$$

When $f'(x) = 0$, we get

$$24x^5 - 24x^3 = 0$$

$$24x^3(x^2 - 1) = 0$$

$$24x^3 = 0 \text{ gives } x = 0$$

$$x^2 - 1 = 0 \text{ gives } x = \pm 1$$

The critical points are $x = -1, 0, 1$

Substituting $x = -1$ in

$$f''(x) = 120x^4 - 72x^2 \text{ we get,}$$

$$f''(-1) = 120(-1)^4 - 72(-1)^2$$

$$= 120(1) - 72(1)$$

$$= 120 - 72$$

$$= 48 > 0$$

Hence $f(x)$ has local minimum at $x = -1$

Substituting $x = 1$ in

$$f''(x) = 120x^4 - 72x^2 \text{ we get,}$$

$$f''(1) = 120(1)^4 - 72(1)^2$$

$$= 120(1) - 72(1)$$

$$= 120 - 72$$

$$= 48 > 0$$

Hence $f(x)$ has local minimum at $x = 1$

Substituting $x = 0$ in

$$f''(x) = 120x^4 - 72x^2 \text{ we get,}$$

$$f''(-1) = 120(0) - 72(0) \\ = 0$$

Hence the second derivative test does not give any information about local extrema at $x = 0$

So, to try by first derivative test,

Intervals are $(-\infty, -1)$, $(-1, 0)$, $(0, 1)$, $(1, \infty)$

In $(-\infty, -1)$ $f'(x) < 0$

So, $f(x)$ is strictly decreasing function.

In $(-1, 0)$ $f'(x) > 0$

So, $f(x)$ is strictly increasing function.

In $(0, 1)$ $f'(x) < 0$

So, $f(x)$ is strictly decreasing function.

In $(1, \infty)$ $f'(x) > 0$

So, $f(x)$ is strictly increasing function.

Because $f'(x)$ changes its sign from negative to positive when passing through $x = -1$ for the function $f(x)$, it has a local minimum at $x = -1$ and the local minimum value of the function is

$$f(-1) = 4(-1)^6 - 6(-1)^4$$

$$= 4(1) - 6(1)$$

$$= 4 - 6$$

$$= -2$$

Also $f'(x)$ changes its sign from positive to negative when passing through $x = 0$ for the

function $f(x)$, it has a local maximum is

$$f(0) = 4(0)^6 - 6(0)^4$$

$$= 0$$

And $f'(x)$ changes its sign from negative to positive when passing through $x = 1$ for the function $f(x)$, it has a local minimum at $x = 1$ and the local minimum value of the function is

$$f(1) = 4(1)^6 - 6(1)^4$$

$$= 4(1) - 6(1)$$

$$= 4 - 6$$

$$= -2$$

Hence the local minimum is -2 and

the local maximum is 0 .

Example 7.61

Find the local maximum and minimum of the function x^2y^2 on the line $x + y = 10$

Solution: Given $x + y = 10$ gives

$$y = 10 - x$$

Substituting the value of y in x^2y^2

$$f(x) = x^2(10 - x)^2$$

$$= x^2(100 + x^2 - 20x)$$

$$= 100x^2 + x^4 - 20x^3$$

$$f'(x) = 200x + 4x^3 - 60x^2$$

$$f''(x) = 200 + 12x^2 - 120x$$

When $f'(x) = 0$, we get

$$200x + 4x^3 - 60x^2 = 0$$

$$4x(50 + x^2 - 15x) = 0$$

$$4x = 0 \text{ gives } x = 0$$

$$x^2 - 15x + 50 = 0 \text{ gives}$$

$$(x - 5)(x - 10) = 0$$

$$\text{Hence } x = 5, 10$$

So, the critical points are $x = 0, 5, 10$

Substituting $x = 0$ in $f''(x)$

$$f''(0) = 200 + 12(0)^2 - 120(0)$$

$$= 200 > 0$$

So, has local minimum at $x = 0$ and the local minimum value of the function is $f(0)$ and

$$f(0) = (0)^2(10 - 0)^2$$

$$= 0$$

Substituting $x = 5$ in $f''(x)$

$$f''(5) = 200 + 12(5)^2 - 120(5)$$

$$= 200 + 12(25) - 120(5)$$

$$= 200 + 300 - 600$$

$$= 500 - 600$$

$$= -100 < 0$$

So, has local maximum at $x = 5$ and the local maximum value of the function is $f(5)$ and

$$f(5) = (5)^2(10 - 5)^2$$

$$= (5)^2(5)^2$$

$$= 25 \times 25$$

$$= 625$$

Substituting $x = 10$ in $f''(x)$

$$f''(10) = 200 + 12(10)^2 - 120(10)$$

$$= 200 + 12(100) - 120(10)$$

$$= 200 + 1200 - 1200$$

$$= 200 > 0$$

So, has local minimum at $x = 10$ and the local minimum value of the function is $f(10)$ and

$$f(10) = (10)^2(10 - 10)^2$$

$$= 0$$

Hence the local minimum is 0 and

the local maximum is 625

EXERCISE 7.7

1. Find intervals of concavity and points of inflexion for the following functions:

(i) $f(x) = x(x - 4)^3$

(ii) $f(x) = \sin x + \cos x$, $0 < x < 2\pi$

(iii) $f(x) = \frac{1}{2}(e^x - e^{-x})$

Solution:

(i) $f(x) = x(x - 4)^3$

$$= x(x^3 - 12x^2 + 48x - 64)$$

$$= x^4 - 12x^3 + 48x^2 - 64x$$

$$f'(x) = 4x^3 - 36x^2 + 96x - 64$$

$$f''(x) = 12x^2 - 72x + 96$$

When $f''(x) = 0$, we get

$$12x^2 - 72x + 96 = 0$$

Dividing by 12,

$$x^2 - 6x + 8 = 0$$

$$(x - 2)(x - 4) = 0$$

$$\text{Hence } x = 2, 4$$

\therefore The interval are $(-\infty, 2)$, $(2, 4)$ and $(4, \infty)$

In the interval $(-\infty, 2)$ let $x = 0$

$$f''(0) = 12(0)^2 - 72(0) + 96$$

$$= 96 > 0$$

Hence $f(x)$ is concave upward in the $(-\infty, 2)$

In the interval $(2, 4)$ let $x = 3$

$$f''(3) = 12(3)^2 - 72(3) + 96$$

$$= 12(9) - 216 + 96$$

$$= 108 - 216 + 96$$

$$= 204 - 216$$

$$= -12 < 0$$

Hence $f(x)$ is concave downward in the $(2, 4)$

In the interval $(4, \infty)$ let $x = 5$

$$f''(5) = 12(5)^2 - 72(5) + 96$$

$$= 12(25) - 360 + 96$$

$$= 300 - 360 + 96$$

$$= 396 - 360$$

$$= 36 > 0$$

Hence $f(x)$ is concave upward in the $(2, 4)$

$f''(x)$ changes the direction at $x = 2, 4$

When $x = 2$, $f(x) = x(x - 4)^3$ gives

$$f(2) = 2(2 - 4)^3$$

$$= 2(-2)^3$$

$$= 2(-8)$$

$$= -16$$

When $x = 4$, $f(x) = x(x - 4)^3$ gives

$$f(4) = 2(4 - 4)^3$$

$$= 2(0)^3$$

$$= 0$$

The points of inflections are $(2, -16)$, $(4, 0)$

(ii) $f(x) = \sin x + \cos x$, $0 < x < 2\pi$

$$f'(x) = \cos x - \sin x$$

$$f''(x) = -\sin x - \cos x$$

When $f''(x) = 0$,

$$-\sin x - \cos x = 0$$

$$\sin x = -\cos x$$

$$\frac{\sin x}{\cos x} = -1$$

$$\tan x = -1 \text{ gives}$$

$$x = \frac{3\pi}{4}, \frac{7\pi}{4} \in (0, 2\pi)$$

Hence the intervals are $(0, \frac{3\pi}{4}), (\frac{3\pi}{4}, \frac{7\pi}{4}), (\frac{7\pi}{4}, 2\pi)$

$$\text{In } (0, \frac{3\pi}{4}) f''(x) < 0$$

Hence $f(x)$ is concave downward in the $(0, \frac{3\pi}{4})$

$$\text{In } (\frac{3\pi}{4}, \frac{7\pi}{4}) f''(x) > 0$$

Hence $f(x)$ is concave upward in the $(\frac{3\pi}{4}, \frac{7\pi}{4})$

$$\text{In } (\frac{7\pi}{4}, 2\pi) f''(x) < 0$$

Hence $f(x)$ is concave downward in $(\frac{7\pi}{4}, 2\pi)$

$f''(x)$ changes the direction at $x = \frac{3\pi}{4}, \frac{7\pi}{4}$

$$\text{At } x = \frac{3\pi}{4}, f(x) = \sin x + \cos x,$$

$$\begin{aligned} f\left(\frac{3\pi}{4}\right) &= \sin\left(\frac{3\pi}{4}\right) + \cos\left(\frac{3\pi}{4}\right) \\ &= \sin\left(\frac{\pi}{2} + \frac{\pi}{4}\right) + \cos\left(\frac{\pi}{2} + \frac{\pi}{4}\right) \\ &= \cos\left(\frac{\pi}{4}\right) - \sin\left(\frac{\pi}{4}\right) \\ &= \frac{1}{\sqrt{2}} - \frac{1}{\sqrt{2}} \\ &= 0 \end{aligned}$$

$$\text{At } x = \frac{7\pi}{4}, f(x) = \sin x + \cos x,$$

$$\begin{aligned} f\left(\frac{7\pi}{4}\right) &= \sin\left(\frac{7\pi}{4}\right) + \cos\left(\frac{7\pi}{4}\right) \\ &= \sin\left(2\pi - \frac{\pi}{4}\right) + \cos\left(2\pi - \frac{\pi}{4}\right) \\ &= \sin\left(-\frac{\pi}{4}\right) + \cos\left(-\frac{\pi}{4}\right) \\ &= -\sin\left(\frac{\pi}{4}\right) + \cos\left(\frac{\pi}{4}\right) \\ &= -\frac{1}{\sqrt{2}} + \frac{1}{\sqrt{2}} \\ &= 0 \end{aligned}$$

The points of inflections are $(\frac{3\pi}{4}, 0), (\frac{7\pi}{4}, 0)$

$$(iii) f(x) = \frac{1}{2}(e^x - e^{-x})$$

$$f'(x) = \frac{1}{2}[e^x - (-e^{-x})]$$

$$= \frac{1}{2}(e^x + e^{-x})$$

$$f''(x) = \frac{1}{2}(e^x - e^{-x})$$

When $f''(x) = 0,$

$$\frac{1}{2}(e^x - e^{-x}) = 0 \text{ gives}$$

$$e^x - e^{-x} = 0$$

$$e^x = e^{-x}$$

$$e^x = \frac{1}{e^x}$$

$$e^x e^x = 1$$

$$e^{2x} = e^0 \text{ gives}$$

$$2x = 0 \Rightarrow x = 0$$

\therefore The interval are $(-\infty, 0)$ and $(0, \infty)$

$$\text{In } (-\infty, 0) f''(x) < 0$$

Hence $f(x)$ is concave downward in $(-\infty, 0)$

$$\text{In } (0, \infty) f''(x) > 0$$

Hence $f(x)$ is concave upward in the $(0, \infty)$

$f''(x)$ changes the direction at $x = 0$

$$\text{At } x = 0, f(x) = \frac{1}{2}(e^x - e^{-x})$$

$$f(0) = \frac{1}{2}(e^0 - e^{-0})$$

$$= \frac{1}{2}(1 - 1)$$

$$= 0$$

So, the point of inflection is $(0, 0)$

2. Find the local extrema for the following functions using second derivative test:

$$(i) f(x) = -3x^5 + 5x^3 \quad (ii) f(x) = x \log x$$

$$(iii) f(x) = x^2 e^{-2x}$$

$$\text{Solution: } f(x) = -3x^5 + 5x^3$$

$$f'(x) = -15x^4 + 15x^2$$

$$f''(x) = -60x^3 + 30x$$

When $f'(x) = 0$

$$-15x^4 + 15x^2 = 0$$

$$-15x^2(x^2 - 1) = 0$$

$$-15x^2 = 0, \text{ gives } x = 0 \text{ and}$$

$$x^2 - 1 = 0, \text{ gives } x = \pm 1$$

Hence the critical points are $x = 0, -1, 1$

$$\text{At } x = 0, f''(x) = -60x^3 + 30x$$

$$f''(0) = -60(0)^3 + 30(0) = 0$$

$$\text{At } x = -1, f''(x) = -60x^3 + 30x$$

$$f''(-1) = -60(-1)^3 + 30(-1)$$

$$= -60(-1) + 30(-1)$$

$$= 60 - 30$$

$$= 30 > 0$$

So, $f(x)$ has local minimum at $x = -1$

Local minimum value is $f(-1)$

$$f(-1) = -3(-1)^5 + 5(-1)^3$$

$$= -3(-1) + 5(-1)$$

$$= 3 - 5$$

$$= -2$$

At $x = 1$, $f''(x) = -60x^3 + 30x$

$$f''(1) = -60(1)^3 + 30(1)$$

$$= -60 + 30$$

$$= -30 < 0$$

So, $f(x)$ has local maximum at $x = 1$

Local maximum value is $f(1)$

$$f(1) = -3(1)^5 + 5(1)^3$$

$$= -3 + 5 = 2$$

Hence the local minimum is -2 and

the local maximum is 2

(ii) $f(x) = x \log x$

$$f'(x) = x \left(\frac{1}{x} \right) + \log x \quad (1)$$

$$= 1 + \log x$$

$$f''(x) = \frac{1}{x}$$

When $f'(x) = 0$

$$1 + \log x = 0$$

$$\log_e x = -1$$

$$x = e^{-1} = \frac{1}{e} \text{ is the critical point}$$

Substituting $x = \frac{1}{e}$ in $f''(x) = \frac{1}{x}$

$$f''\left(\frac{1}{e}\right) = \frac{1}{\frac{1}{e}} = e > 0$$

Hence, $f(x)$ has local minimum at $x = \frac{1}{e}$. The

local minimum value of the function is $f\left(\frac{1}{e}\right)$

$$f\left(\frac{1}{e}\right) = \left(\frac{1}{e}\right) \log\left(\frac{1}{e}\right)$$

$$= \left(\frac{1}{e}\right) [\log(1) - \log(e)]$$

$$= \left(\frac{1}{e}\right) [0 - 1]$$

$$= -\frac{1}{e}$$

Hence the local minimum is $-\frac{1}{e}$

(iii) $f(x) = x^2 e^{-2x}$

$$f'(x) = x^2(-2e^{-2x}) + e^{-2x}(2x)$$

$$= -2x^2(e^{-2x}) + e^{-2x}(2x)$$

$$f''(x) = -2[x^2(-2e^{-2x}) + e^{-2x}(2x)]$$

$$+ e^{-2x}(2) + (2x)(-2e^{-2x})$$

$$= 4x^2 e^{-2x} - 4xe^{-2x} + 2e^{-2x} - 4xe^{-2x}$$

$$= 4x^2 e^{-2x} - 8xe^{-2x} + 2e^{-2x}$$

$$= 2e^{-2x}(2x^2 - 4x + 1)$$

When $f'(x) = 0$

$$-2x^2(e^{-2x}) + e^{-2x}(2x) = 0$$

$$2x(e^{-2x})(-x + 1) = 0 \text{ we get}$$

$2x = 0$, gives $x = 0$ and

$-x + 1 = 0$, gives $x = 1$ and $e^{-2x} \neq 0$

The critical points are $x = 0, 1$

At $x = 0$, $f''(x) = 2e^{-2x}(2x^2 - 4x + 1)$

$$f''(0) = 2e^{-2(0)}(2(0)^2 - 4(0) + 1)$$

$$= 2e^0(0 - 0 + 1)$$

$$= 2(1)(1)$$

$$= 2 > 0$$

So, $f(x)$ has local minimum at $x = 0$

Local maximum value is $f(0) = 0$

At $x = 1$, $f''(x) = 2e^{-2x}(2x^2 - 4x + 1)$

$$f''(1) = 2e^{-2(1)}(2(1)^2 - 4(1) + 1)$$

$$= 2e^{-2}(2 - 4 + 1)$$

$$= 2e^{-2}(3 - 4)$$

$$= 2e^{-2}(-1) < 0$$

So, $f(x)$ has local maximum at $x = 1$

Local maximum value is $f(1)$

$$f(1) = (1)^2 e^{-2(1)}$$

$$= e^{-2}$$

Hence the local minimum is 0 and

the local maximum is e^{-2}

3. For the function $f(x) = 4x^3 + 3x^2 - 6x + 1$ find the intervals of monotonicity, local extrema, intervals of concavity and points of inflection.

Solution: $f(x) = 4x^3 + 3x^2 - 6x + 1$

$$f'(x) = 12x^2 + 6x - 6$$

Substituting $f'(x) = 0$, we get

$$12x^2 + 6x - 6 = 0$$

Dividing by 6,

$$2x^2 + x - 1 = 0$$

$$(2x - 1)(x + 1) = 0$$

$2x - 1 = 0$ gives $2x = 1$ which gives $x = \frac{1}{2}$

$x + 1 = 0$ gives $x = -1$

The critical points are $x = -1, \frac{1}{2}$

Intervals are $(-\infty, -1), (-1, \frac{1}{2}), (\frac{1}{2}, \infty)$

In $(-\infty, -1)$ let $x = -2$

Then $f'(x) = 12x^2 + 6x - 6$ gives

$$f'(-2) = 12(-2)^2 + 6(-2) - 6$$

$$= 12(4) - 12 - 6$$

$$= 48 - 18$$

$$= 30 > 0$$

$f(x)$ is strictly increasing function in $(-\infty, -1)$

In $(-1, \frac{1}{2})$ let $x = 0$

Then $f'(x) = 12x^2 + 6x - 6$ gives

$$f'(0) = 12(0)^2 + 6(0) - 6$$

$$= -6 < 0$$

$f(x)$ is strictly decreasing function in $(-1, \frac{1}{2})$

In $(\frac{1}{2}, \infty)$ let $x = 1$

Then $f'(x) = 12x^2 + 6x - 6$ gives

$$f'(1) = 12(1)^2 + 6(1) - 6$$

$$= 12 + 6 - 6$$

$$= 18 - 6$$

$$= 12 > 0$$

$f(x)$ is strictly increasing function in $(\frac{1}{2}, \infty)$

Also $f'(x)$ changes its sign from positive to negative when passing through $x = -1$ for the function $f(x)$, it has a local maximum is

$$f(-1) = 4(-1)^3 + 3(-1)^2 - 6(-1) + 1$$

$$= 4(-1) + 3(1) - 6(-1) + 1$$

$$= -4 + 3 + 6 + 1$$

$$= -4 + 10$$

$$= 6$$

And $f'(x)$ changes its sign from negative to positive when passing through $x = \frac{1}{2}$ for the

function $f(x)$, it has a local minimum at $x = \frac{1}{2}$

and the local minimum value of the function is

$$f\left(\frac{1}{2}\right) = 4\left(\frac{1}{2}\right)^3 + 3\left(\frac{1}{2}\right)^2 - 6\left(\frac{1}{2}\right) + 1$$

$$= 4\left(\frac{1}{8}\right) + 3\left(\frac{1}{4}\right) - 3 + 1$$

$$= \frac{1}{2} + \frac{3}{4} - 2$$

$$= \frac{2+3-8}{4}$$

$$= \frac{5-8}{4}$$

$$= \frac{-3}{4}$$

Hence the local minimum is $\frac{-3}{4}$ and

the local maximum is 6

And from $f'(x) = 12x^2 + 6x - 6$

$$f''(x) = 24x + 6$$

Substituting $f''(x) = 0$, we get

$$24x = -6$$

$$x = -\frac{6}{24} = -\frac{1}{4}$$

Intervals are $(-\infty, -\frac{1}{4}), (-\frac{1}{4}, \infty)$

In $(-\infty, -\frac{1}{4})$ let $x = -1$

Then $f''(x) = 24x + 6$ gives

$$\begin{aligned}
 f''(-1) &= 24(-1) + 6 \\
 &= -24 + 6 \\
 &= -18 < 0
 \end{aligned}$$

Hence $f(x)$ is concave downward in $(-\frac{1}{4}, \infty)$

In $(-\frac{1}{4}, \infty)$ let $x = 0$

$$\begin{aligned}
 \text{Then } f''(x) &= 24x + 6 \text{ gives} \\
 f''(0) &= 24(0) + 6 \\
 &= 6 > 0
 \end{aligned}$$

Hence $f(x)$ is concave upward in $(-\frac{1}{4}, \infty)$

$f''(x)$ changes the direction at $x = -\frac{1}{4}$

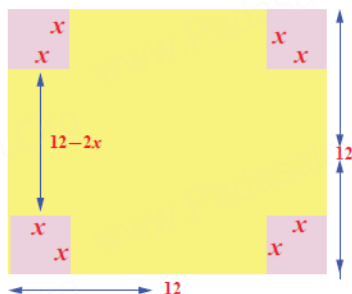
$$\begin{aligned}
 f\left(-\frac{1}{4}\right) &= 4\left(-\frac{1}{4}\right)^3 + 3\left(-\frac{1}{4}\right)^2 - 6\left(-\frac{1}{4}\right) + 1 \\
 &= 4\left(-\frac{1}{64}\right) + 3\left(\frac{1}{16}\right) + \frac{6}{4} + 1 \\
 &= -\frac{1}{16} + \frac{3}{16} + \frac{6}{4} + 1 \\
 &= \frac{-1+3+24+16}{16} \\
 &= \frac{-1+43}{16} \\
 &= \frac{42}{16} \\
 &= \frac{21}{8}
 \end{aligned}$$

So, the point of inflection is $(-\frac{1}{4}, \frac{21}{8})$

Example 7.62

We have a 12 square unit piece of thin material and want to make an open box by cutting small squares from the corners of our material and folding the sides up. The question is, which cut produces the box of maximum volume?

Solution:



Let x = length of the cut on each side of the little squares.

V = the volume of the folded box.

The length of the base after two cuts along each edge of size x is $(12 - x)$. The depth of the box after folding is x , so the volume is

$$V = (12 - 2x) \times (12 - 2x) \times x$$

At $x = 0$, volume becomes zero.

Hence $x \in (0, 6)$

$$\begin{aligned}
 V &= (144 + 4x^2 - 48x)x \\
 &= 144x + 4x^3 - 48x^2
 \end{aligned}$$

$$V' = 144 + 12x^2 - 96x$$

$$V'' = 24x - 96$$

Substituting $V' = 0$, we get

$$12x^2 - 96x + 144 = 0$$

Dividing by 12,

$$x^2 - 8x + 12 = 0$$

$$(x - 2)(x - 6) = 0$$

$$x - 2 = 0, \text{ gives } x = 2$$

$$x - 6 = 0, \text{ gives } x = 6 \text{ but } x \neq 6$$

So, $x = 2$ is the only stationary point.

When $x = 2$, $V'' = 24x - 96$ becomes

$$\begin{aligned}
 V'' &= 24(2) - 96 \\
 &= 48 - 96 \\
 &= -48 < 0
 \end{aligned}$$

Hence volume of the box is maximum at $x = 2$

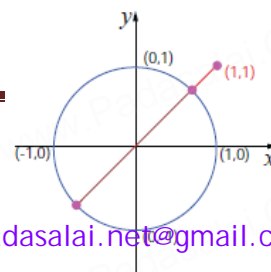
Hence the maximum volume of the box is $V(2)$

$$\begin{aligned}
 V(2) &= [(12 - 2(2)) \times (12 - 2(2))] \times (2) \\
 &= [(12 - 4) \times (12 - 4)] \times (2) \\
 &= (8)(8)(2) \\
 &= (64)(2) \\
 &= 128 \text{ units.}
 \end{aligned}$$

So the maximum cut can be 2 units.

Example 7.63 Find the points on the unit circle $x^2 + y^2 = 1$ nearest and farthest from $(1, 1)$.

Solution:



So, the required points are $\left(\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}\right), \left(-\frac{1}{\sqrt{2}}, -\frac{1}{\sqrt{2}}\right)$

Example 7.64 A steel plant is capable of producing x tones per day of a low-grade steel and y tones per day of a high-grade steel, where $y = \frac{40-5x}{10-x}$. If the fixed market price of low-grade steel is half that of high-grade steel, then what should be optimal productions in low-grade steel and high-grade steel in order to have maximum receipts.

Solution:

Per day the low grade steel produced = x tones

Let the market price per ton be Rupees p

Then the cost = px

Per day the high grade steel produced = y tones

Let the market price per ton be Rupees $2p$

Then the cost = $2py$

Then total receipt $R = px + 2py$

Substituting the given value $y = \frac{40-5x}{10-x}$, we get

$$R = px + 2p \left(\frac{40-5x}{10-x} \right)$$

$$= \frac{px(10-x) + 2p(40-5x)}{10-x}$$

$$= \frac{10px - px^2 + 80p - 10px}{10-x}$$

$$R = \frac{-px^2 + 80p}{10-x}$$

$$R' = \frac{(10-x)(-2px) - (-px^2 + 80p)(-1)}{(10-x)^2}$$

$$= \frac{-20px + 2px^2 - px^2 + 80p}{(10-x)^2}$$

$$= \frac{-20px + px^2 + 80p}{(10-x)^2}$$

$$= \frac{p(x^2 - 20x + 80)}{(10-x)^2}$$

Substituting $R' = 0$, we get

Given point is $(1, 1)$. Let other point be (x, y) .

Then distance $d = \sqrt{(x-1)^2 + (y-1)^2}$

$$\begin{aligned} \text{Let } D &= (x-1)^2 + (y-1)^2 \\ &= x^2 + 1 - 2x + y^2 + 1 - 2y \\ &= x^2 + y^2 - 2x - 2y + 2 \end{aligned}$$

$$D' = 2x + 2yy' - 2 - 2y'$$

From $x^2 + y^2 = 1$

$$2x + 2yy' = 0$$

$$2yy' = -2x$$

$$yy' = -x$$

$$y' = -\frac{x}{y}$$

Substituting the value in D' we get

$$\begin{aligned} D' &= 2x + 2yy' - 2 - 2y' \\ &= 2x + 2y \left(-\frac{x}{y} \right) - 2 - 2 \left(-\frac{x}{y} \right) \\ &= 2x - 2x - 2 + \frac{2x}{y} \\ &= -2 + \frac{2x}{y} \end{aligned}$$

Substituting $D' = 0$, we get

$$-2 + \frac{2x}{y} = 0$$

$$\frac{2x}{y} = 2$$

$$2x = 2y \text{ gives } x = y$$

Since the points lie on the circle $x^2 + y^2 = 1$

we get $x^2 + x^2 = 1$

$$2x^2 = 1$$

$$x^2 = \frac{1}{2}$$

$$x = \pm \frac{1}{\sqrt{2}}$$

$$\frac{p(x^2 - 20x + 80)}{(10 - x)^2} = 0$$

$$x^2 - 20x + 80 = 0$$

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

$$= \frac{20 \pm \sqrt{400 - 320}}{2}$$

$$= \frac{20 \pm \sqrt{80}}{2}$$

$$= \frac{20 \pm \sqrt{16 \times 5}}{2}$$

$$= \frac{20 \pm 4\sqrt{5}}{2}$$

$$= \frac{2(10 \pm 2\sqrt{5})}{2}$$

$$x = 10 \pm 2\sqrt{5}$$

$$R'' = p \left[\frac{(10 - x)^2(2x - 20) - (x^2 - 20x + 80)2(10 - x)(-1)}{(10 - x)^4} \right]$$

$$= p \frac{[(10 - x)^2(2x - 20) + (x^2 - 20x + 80)2(10 - x)]}{(10 - x)^4}$$

$$= p \frac{(10 - x)[(10 - x)(2x - 20) + (x^2 - 20x + 80)2]}{(10 - x)^4}$$

$$= p \frac{[(10 - x)(2x - 20) + (x^2 - 20x + 80)2]}{(10 - x)^3}$$

$$= p \frac{[20x - 2x^2 - 200 + 20x + 2x^2 - 40x + 160]}{(10 - x)^3}$$

$$= \frac{-40p}{(10 - x)^3}$$

$$\text{At } x = 10 - 2\sqrt{5}, R'' < 0$$

$$\text{Hence Receipt } R \text{ is maximum at } x = 10 - 2\sqrt{5}$$

$$\text{When } x = 10 - 2\sqrt{5}, \text{ then } y = \frac{40 - 5x}{10 - x}$$

$$y = \frac{40 - 5(10 - 2\sqrt{5})}{10 - (10 - 2\sqrt{5})}$$

$$= \frac{40 - 50 + 10\sqrt{5}}{10 - 10 + 2\sqrt{5}}$$

$$= \frac{-10 + 10\sqrt{5}}{2\sqrt{5}}$$

$$= \frac{10(\sqrt{5} - 1)}{2\sqrt{5}} = \frac{2 \times \sqrt{5} \times \sqrt{5}(\sqrt{5} - 1)}{2\sqrt{5}}$$

$$= \sqrt{5}(\sqrt{5} - 1)$$

$$= 5 - \sqrt{5}$$

So, the plant should produce low grade steels

$10 - 2\sqrt{5}$ tones and produce high grade

$5 - \sqrt{5}$ tones.

Example 7.65 Prove that among all the rectangles of the given area square has the least perimeter.

Solution:

Let the dimensions of the rectangle be x, y

Then its area $A = xy$

Perimeter of the rectangle $P = 2x + 2y$

Substituting the value $y = \frac{A}{x}$ in P

$$P = 2x + 2\left(\frac{A}{x}\right)$$

$$P' = 2 - \frac{2A}{x^2}$$

$$P'' = \frac{4A}{x^3}$$

Substituting $P' = 0$, we get

$$2 - \frac{2A}{x^2} = 0$$

$$2 = \frac{2A}{x^2}$$

$$2x^2 = 2A \text{ gives}$$

$$x^2 = A \Rightarrow x = \sqrt{A}$$

Substituting $x = \sqrt{A}$ in P'' ,

$P'' > 0$. Hence perimeter P is minimum at $x =$

\sqrt{A} . Then $y = \frac{A}{x}$ becomes

$$y = \frac{A}{\sqrt{A}} = \frac{\sqrt{A}\sqrt{A}}{\sqrt{A}} = \sqrt{A}$$

Since $x = \sqrt{A}$, $y = \sqrt{A}$ the given rectangle is a square when the perimeter is minimum.

EXERCISE 7.8

1. Find two positive numbers whose sum is 12 and their product is maximum.

Solution:

Let x, y be the two numbers then the sum

$$x + y = 12 \text{ gives}$$

$$y = 12 - x$$

Product of the numbers $P = xy$

$$= x(12 - x)$$

$$P = 12x - x^2$$

$$P' = 12 - 2x$$

$$P'' = -2$$

Substituting $P' = 0$, we get

$$12 - 2x = 0$$

$$2x = 12 \text{ gives}$$

$$x = 6$$

Since $P'' = -2 < 0$, Product P is maximum at $x = 6$. Then $y = 12 - x$, gives $y = 12 - 6 = 6$.

The required two numbers are 6, 6.

2. Find two positive numbers whose product is 20 and their sum is minimum.

Solution: Let x, y be the two numbers then

the product $xy = 20$ gives

$$y = \frac{20}{x}$$

Sum of the numbers $S = x + y$

$$= x + \left(\frac{20}{x}\right)$$

$$S' = 1 - \frac{20}{x^2}$$

$$S'' = \frac{40}{x^3}$$

Substituting $S' = 0$, we get

$$1 - \frac{20}{x^2} = 0$$

$$1 = \frac{20}{x^2}$$

$$x^2 = 20 \text{ gives}$$

$$x = \pm\sqrt{20}$$

$$x = \pm\sqrt{4 \times 5}$$

Since the number is positive,

$$x = 2\sqrt{5}$$

Since $S'' = \frac{40}{x^3} > 0$, when $x = 2\sqrt{5}$ sum S is

minimum at $x = 2\sqrt{5}$. Then $y = \frac{20}{x}$, gives

$$y = \frac{20}{2\sqrt{5}} = \frac{10}{\sqrt{5}} = \frac{10 \times \sqrt{5}}{\sqrt{5} \times \sqrt{5}} = \frac{10 \times \sqrt{5}}{5} = 2\sqrt{5}$$

The required two numbers are $2\sqrt{5}, 2\sqrt{5}$

3. Find the smallest possible value of $x^2 + y^2$ given that $x + y = 10$.

Solution:

Given $x + y = 10$ gives

$$y = 10 - x$$

Let $f(x) = x^2 + y^2$ gives

$$= x^2 + (10 - x)^2$$

$$= x^2 + 100 + x^2 - 20x$$

$$f(x) = 2x^2 - 20x + 100$$

$$f'(x) = 4x - 20$$

$$f''(x) = 4$$

Substituting $f'(x)$, we get

$$4x - 20 = 0$$

$$4x = 20 \text{ gives}$$

$$x = 5$$

Since $f''(x) = 4 > 0$, $f(x) = x^2 + y^2$ is minimum at $x = 5$.

When $x = 5$, $y = 10 - x$ gives

$$y = 10 - 5 = 5$$

The required two values are $x = 5, y = 5$ and the smallest possible value of $x^2 + y^2$ is $(5)^2 + (5)^2 = 25 + 25 = 50$.

4. A garden is to be laid out in a rectangular area and protected by wire fence. What is the largest possible area of the fenced garden with 40 metres of wire.

Solution:

Let the dimensions of the rectangle be x, y

Then its area $A = xy$

Perimeter of the rectangle $40 = 2x + 2y$

Then, $x + y = 20$ gives

$$y = 20 - x$$

Then its area $A = x(20 - x)$

$$= 20x - x^2$$

$$A' = 20 - 2x$$

$$A'' = -2$$

Substituting $A' = 0$, we get

$$20 - 2x = 0 \text{ gives}$$

$$2x = 20$$

$$x = 10$$

Since $A'' = -2 < 0$, area is maximum

At $x = 10$, then $y = 20 - x$ gives $y = 10$

Hence the maximum area is

$$A = (10)(10) = 100 \text{ sq. meters.}$$

5. A rectangular page is to contain 24 cm^2 of print. The margins at the top and bottom of the page are 1.5 cm and the margins at other sides of the page are 1 cm. What should be the dimensions of the page so that the area of the paper used is minimum?

Solution:

Let x and y be dimension of the printed area.

Given $xy = 24$ gives

$$y = \frac{24}{x}$$

From by the given data, dimensions of the paper are $x + 3, y + 2$

Hence area of the paper

$$A = (x + 3)(y + 2)$$

$$= xy + 2x + 3y + 6$$

$$= 24 + 2x + 3\left(\frac{24}{x}\right) + 6$$

$$A = 2x + \frac{72}{x} + 30$$

$$A' = 2 - \frac{72}{x^2}$$

$$A'' = \frac{144}{x^3}$$

Substituting $A' = 0$, we get

$$2 - \frac{72}{x^2} = 0$$

$$2x^2 = 72$$

$$x^2 = 36 \text{ gives}$$

$$x = 6$$

$$\text{At } x = 6, A'' = \frac{144}{x^3} = \frac{144}{6^3} > 0$$

Hence printed area of the paper is minimum at

$$x = 6, \text{ then } y = \frac{24}{x} = \frac{24}{6} = 4$$

The dimensions of the paper are

$$x + 3 = 6 + 3 = 9 \text{ cm, and}$$

$$y + 2 = 4 + 2 = 6 \text{ cm}$$

6. A farmer plans to fence a rectangular pasture adjacent to a river. The pasture must contain 1,80,000 sq.mtrs in order to provide enough grass for herds. No fencing is needed along the river. What is the length of the minimum needed fencing material?

Solution:

Let x and y be dimension of the pasture area.

Given $xy = 180000$ gives

$$y = \frac{180000}{x}$$

Fencing required $= x + 2y$

$$F = x + 2\left(\frac{180000}{x}\right)$$

$$F = x + \frac{360000}{x}$$

$$F' = 1 - \frac{360000}{x^2}$$

$$F'' = \frac{720000}{x^3}$$

Substituting $F' = 0$, we get

$$1 - \frac{360000}{x^2} = 0$$

$$x^2 = 360000 \text{ gives}$$

$$x = 600$$

$$\text{At } x = 600, F'' = \frac{720000}{x^3} = \frac{720000}{600^3} > 0$$

Hence pasture area is minimum at $x = 600$,

$$\text{then } y = \frac{180000}{600} = \frac{1800}{6} = 300$$

Hence minimum length of the fence

needed $= x + 2y$

$$= 600 + 2(300)$$

$$= 600 + 600 = 1200 \text{ meters.}$$

7. Find the dimensions of the rectangle with maximum area that can be inscribed in a circle of radius 10 cm.

Solution:

Let us take the circle to be a circle with centre (0, 0) and radius r 10 units and PQRS be the rectangle inscribed in it. Let P(x, y) be the vertex of the rectangle that lies on the first quadrant. Let θ be the angle made by OP with the x- axis.

then $x = 10 \cos \theta$ and $y = 10 \sin \theta$

Now the dimensions of the rectangle are

$$2x = 20 \cos \theta ; 2y = 20 \sin \theta , 0 \leq \theta \leq \frac{\pi}{2}$$

Area of the rectangle $A = (2x)(2y)$

$$= (20 \cos \theta)(20 \sin \theta)$$

$$= 200 (2 \sin \theta \cos \theta)$$

$$= 200 (\sin 2\theta)$$

Area is maximum when $\sin 2\theta$ is maximum.

We know the maximum value of $\sin 2\theta = 1$

It gives $2\theta = \frac{\pi}{2}$ that is $\theta = \frac{\pi}{4}$, the area of the rectangle is maximum.

$$\text{When } \theta = \frac{\pi}{4}, 2x = 20 \cos \theta$$

$$= 20 \cos \frac{\pi}{4}$$

$$= 20 \times \frac{1}{\sqrt{2}}$$

$$= \frac{20}{\sqrt{2}} \times \frac{\sqrt{2}}{\sqrt{2}}$$

$$= \frac{20\sqrt{2}}{2} = 10\sqrt{2} \text{ and}$$

$$2y = 20 \sin \theta$$

$$= 20 \sin \frac{\pi}{4}$$

$$= 20 \times \frac{1}{\sqrt{2}}$$

$$= \frac{20}{\sqrt{2}} \times \frac{\sqrt{2}}{\sqrt{2}}$$

$$= \frac{20\sqrt{2}}{2} = 10\sqrt{2}$$

So, dimensions of the rectangle are $10\sqrt{2}, 10\sqrt{2}$

8. Prove that among all the rectangles of the given perimeter, the square has the maximum area.

Solution:

Let the dimensions of the rectangle be x, y

Perimeter of the rectangle $P = 2x + 2y$

$$\frac{P}{2} = x + y$$

$$y = \frac{P}{2} - x$$

Then its area $A = xy$

$$= x \left(\frac{P}{2} - x \right)$$

$$A = \frac{P}{2}x - x^2$$

$$A' = \frac{P}{2} - 2x$$

$$A'' = -2$$

Substituting $A' = 0$, we get

$$\frac{P}{2} - 2x = 0$$

$$\frac{P}{2} = 2x$$

$$x = \frac{P}{4}$$

Since $A'' = -2 < 0$, area is maximum,

at $x = \frac{P}{4}$, then $y = \frac{P}{2} - x$ gives

$$y = \frac{P}{2} - \frac{P}{4}$$

$$= \frac{2P-P}{4}$$

$$y = \frac{P}{4}$$

$$\text{Since, } x = y = \frac{P}{4},$$

the given rectangle is a square when the area is maximum.

9. Find the dimensions of the largest rectangle that can be inscribed in a semi circle of radius r cm.

Solution:

Let us take the circle to be a circle with centre

(0, 0) and radius r cm and PQRS be the

rectangle inscribed in the semi circle.

Let $P(x, y)$ be the vertex of the rectangle that lies on the first quadrant. Let θ be the angle made by OP with the x -axis.

then $x = r \cos \theta$ and $y = r \sin \theta$

Now the dimensions of the rectangle are

$$x = r \cos \theta ; 2y = 2r \sin \theta, 0 \leq \theta \leq \frac{\pi}{2}$$

Area of the rectangle $A = (x)(2y)$

$$= (r \cos \theta)(2r \sin \theta)$$

$$= r^2 (2 \sin \theta \cos \theta)$$

$$= r^2 (\sin 2\theta)$$

Area is maximum when $\sin 2\theta$ is maximum.

We know the maximum value of $\sin 2\theta = 1$

It gives $2\theta = \frac{\pi}{2}$ that is $\theta = \frac{\pi}{4}$, the area of the

rectangle is maximum. Hence maximum area of the rectangle is $A = r^2 (1) = r^2$ sq. cm

10. A manufacturer wants to design an open box having a square base and a surface area of 108 sq.cm. Determine the dimensions of the box for the maximum volume.

Solution: Given the box base is square.

So, let length, breadth and height of the box be

$$l = x, b = x \text{ and } h = y.$$

Then box base area = x^2

Area of the 4 sides of the box = $4xy$

Surface area of the open box $S = x^2 + 4xy$

$$\text{Given } x^2 + 4xy = 108$$

$$4xy = 108 - x^2$$

$$y = \frac{108 - x^2}{4x}$$

Volume of the box $V = x \times x \times y$

$$= x^2 \times \frac{108 - x^2}{4x}$$

$$V = x \times \frac{108 - x^2}{4}$$

$$V = \frac{1}{4} (108x - x^3)$$

$$V' = \frac{1}{4} (108 - 3x^2)$$

$$V'' = \frac{1}{4} (-6x)$$

Substituting $V' = 0$, we get

$$\frac{1}{4} (108 - 3x^2) = 0$$

$$108 - 3x^2 = 0$$

$$3x^2 = 108$$

$$x^2 = \frac{108}{3}$$

$$x^2 = 36 \text{ gives}$$

$$x = \pm 6$$

Since, $x \neq -6$, we get $x = 6$.

$$\text{When } x = 6, V'' = \frac{1}{4} (-6x)$$

$$V'' = \frac{1}{4} (-36) < 0$$

So, volume is maximum at $x = 6$

$$\text{When } x = 6, \text{ then } y = \frac{108 - x^2}{4x} = \frac{108 - 36}{4(6)} = \frac{72}{24} = 3$$

Hence dimensions of the box are 6cm, 6cm, 3cm.

11. The volume of a cylinder is given by the formula $V = \pi r^2 h$. Find the greatest and least values of V if $r + h = 6$.

Solution: Given $r + h = 6$ which gives

$$h = 6 - r$$

Hence, volume of the cylinder $V = \pi r^2 h$

$$V = \pi r^2 (6 - r)$$

$$= \pi (6r^2 - r^3)$$

$$V' = \pi (12r - 3r^2)$$

$$V'' = \pi (12 - 6r)$$

Substituting $V' = 0$, we get

$$\pi (12r - 3r^2) = 0$$

$$12r - 3r^2 = 0$$

$$3r(4 - r) = 0$$

$$3r = 0, \text{ gives } r = 0 \text{ and}$$

$$4 - r = 0, \text{ gives } r = 4$$

$$\text{When } r = 0, V'' = \pi (12 - 6r) = 12\pi > 0$$

Hence volume is minimum at $r = 0$.

Hence the least value of V is 0

When $r = 4$, $V'' = \pi(12 - 24) = -12\pi < 0$

Hence volume is maximum at $r = 4$.

Then $h = 6 - r$, gives $h = 6 - 4 = 2$

Hence the greatest value of $V = \pi r^2 h$

$$V = \pi(4^2)(2)$$

$$= \pi(16)(2)$$

$$V = 32\pi$$

Hence the least value of V is 0 and

Hence the greatest value of V is 32π

12. A hollow cone with base radius a cm and height b cm is placed on a table. Show that the volume of the largest cylinder that can hide underneath is $\frac{4}{9}$ times volume of the cone.

Solution: Given radius of cone $= a$

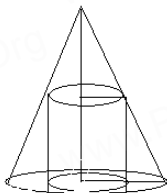
Height $= b$

Hence volume of cone $= \frac{1}{3}\pi a^2 b$

Let the radius of cylinder $= x$ and height of the cylinder be $= y$

Then the volume of cylinder $V = \pi x^2 y$

To prove the maximum volume $= \frac{4}{9}\left(\frac{1}{3}\pi a^2 b\right)$



Let the height of cone $OC = b$ and

radius $CE = a$

Radius of cylinder $AB = CE = x$

Height of cylinder $AC = y$

Triangle OAB and OCD are similar, hence the corresponding sides are proportional.

$$\text{Hence, } \frac{OA}{OC} = \frac{AB}{CD}$$

$$\frac{b-y}{b} = \frac{x}{a}$$

$$b - y = \frac{b}{a}x$$

$$y = b - \frac{b}{a}x$$

Substituting the value in Volume of cylinder,

$V = \pi x^2 y$ becomes

$$V = \pi x^2 \left(b - \frac{b}{a}x\right)$$

$$= \pi b \left(x^2 - \frac{x^3}{a}\right)$$

$$V' = \pi b \left(2x - \frac{3x^2}{a}\right)$$

$$V'' = \pi b \left(2 - \frac{6x}{a}\right)$$

Substituting $V' = 0$, we get

$$\pi b \left(2x - \frac{3x^2}{a}\right) = 0 \text{ gives}$$

$$2x - \frac{3x^2}{a} = 0$$

$$x \left(2 - \frac{3x}{a}\right) = 0$$

$$x = 0, \text{ and } 2 - \frac{3x}{a} = 0 \text{ gives}$$

$$2 = \frac{3x}{a}$$

$$2a = 3x$$

$$x = \frac{2a}{3}$$

When $x = \frac{2a}{3}$, $V'' = \pi b \left(2 - \frac{6x}{a}\right)$ gives

$$V'' = \pi b \left[2 - \left(\frac{6}{a}\right)\left(\frac{2a}{3}\right)\right]$$

$$= \pi b(2 - 4)$$

$$V'' = \pi b(-2) < 0$$

So, volume of cylinder is maximum when $x =$

$\frac{2a}{3}$ then $y = b - \frac{b}{a}x$ gives

$$y = b - \frac{b}{a}\left(\frac{2a}{3}\right)$$

$$= b - \frac{2b}{3}$$

$$= \frac{3b-2b}{3}$$

$$y = \frac{b}{3}$$

Maximum volume of cylinder $V = \pi x^2 y$

$$V = \pi \left(\frac{2a}{3}\right)^2 \left(\frac{b}{3}\right)$$

$$= \pi \left(\frac{4a^2}{9}\right) \left(\frac{b}{3}\right)$$

$$V = \frac{4}{9} \left(\frac{1}{3} \pi a^2 b \right) \text{ Hence proved.}$$

Example 7.66

Find the asymptotes of the function $f(x) = \frac{1}{x}$

Example 7.67 Find the slant (oblique)

asymptote for the function $f(x) = \frac{x^2 - 6x + 7}{x + 5}$

Example 7.68

Find the asymptotes of the curve $f(x) = \frac{2x^2 - 8}{x^2 - 16}$

Example 7.69

Sketch the curve $y = f(x) = x^2 - x - 6$

Example 7.70

Sketch the curve $y = f(x) = x^3 - 6x + 9$

Example 7.71

Sketch the curve $y = \frac{x^2 - 3x}{(x - 1)}$

Example 7.72

Sketch the graph of the function $y = \frac{3x}{x^2 - 1}$

EXERCISE 7.9

1. Find the asymptotes of the following curves:

$$\begin{aligned} \text{(i)} f(x) &= \frac{x^2}{x^2 - 1} & \text{(ii)} f(x) &= \frac{x^2}{x + 1} \\ \text{(iii)} f(x) &= \frac{3x}{\sqrt{x^2 + 1}} & \text{(iv)} f(x) &= \frac{x^2 - 6x - 1}{x + 3} \\ \text{(v)} f(x) &= \frac{x^2 + 6x - 4}{3x - 6} \end{aligned}$$

2. Sketch the graphs of the following functions:

$$\begin{aligned} \text{(i)} y &= -\frac{1}{3}(x^3 - 3x + 2) & \text{(ii)} y &= x\sqrt{4 - x} \\ \text{(iii)} y &= \frac{x^2 + 1}{x^2 - 4} & \text{(iv)} y &= \frac{1}{1 + e^{-x}} \\ \text{(v)} y &= \frac{x^3}{24} - \log x \end{aligned}$$

EXERCISE 7.10

Choose the correct or the most suitable answer from the given four alternatives:

1. The volume of a sphere is increasing in volume at the rate of $3\pi cm^3/sec$. The rate of change of its radius when radius is $\frac{1}{2} cm$
- (1) $3 cm/s$ (2) $2 cm/s$

$$(3) 1 cm/s \quad (4) \frac{1}{2} cm/s$$

2. A balloon rises straight up at 10 m/s. An observer is 40 m away from the spot where the balloon left the ground. Find the rate of change of the balloon's angle of elevation in radian per second when the balloon is 30 metres above the ground.

$$\begin{aligned} (1) \frac{3}{25} \text{ radians/sec} & \quad (2) \frac{4}{25} \text{ radians/sec} \\ (3) \frac{1}{5} \text{ radians/sec} & \quad (4) \frac{1}{3} \text{ radians/sec} \end{aligned}$$

3. The position of a particle moving along a horizontal line of any time t is given by $s(t) = 3t^2 - 2t - 8$. The time at which the particle is at rest is

$$\begin{aligned} (1) t = 0 & \quad (2) t = \frac{1}{3} \\ (3) t = 1 & \quad (4) t = 3 \end{aligned}$$

4. A stone is thrown up vertically. The height it reaches at time t seconds is given by $x = 80t - 16t^2$. The stone reaches the maximum height in time t seconds is given by

$$(1) 2 \quad (2) 2.5 \quad (3) 3 \quad (4) 3.5$$

5. Find the point on the curve $6y = x^3 + 2$ at which y - coordinate changes 8 times as fast as x - coordinate is

$$\begin{aligned} (1) (4, 11) & \quad (2) (4, -11) \\ (3) (-4, 11) & \quad (4) (-4, -11) \end{aligned}$$

6. The abscissa of the point on the curve $f(x) = \sqrt{8 - 2x}$ at which the slope of the tangent is -0.25 ?

$$(1) -8 \quad (2) -4 \quad (3) -2 \quad (4) 0$$

7. The slope of the line normal to the curve $f(x) = 2 \cos 4x$ at $x = \frac{\pi}{2}$ is

$$(1) -4\sqrt{3} \quad (2) -4 \quad (3) \frac{\sqrt{3}}{12} \quad (4) 4\sqrt{3}$$

8. The tangent to the curve $y^2 - xy + 9 = 0$ is vertical when

$$\begin{aligned} (1) y = 0 & \quad (2) y = \pm \sqrt{3} \\ (3) y = \frac{1}{2} & \quad (4) y = \pm 3 \end{aligned}$$

9. Angle between $y^2 = x$ and $x^2 = y$ at the origin is

- (1) $\tan^{-1}\left(\frac{3}{4}\right)$ (2) $\tan^{-1}\left(\frac{4}{3}\right)$
 (3) $\frac{\pi}{2}$ (4) $\frac{\pi}{4}$

10. What is the value of the

$$\lim_{x \rightarrow 0} \left(\cot x - \frac{1}{x} \right) ?$$

- (1) 0 (2) 1 (3) 2 (4) \leq

11. The function $\sin^4 x + \cos^4 x$ is increasing in the interval

- (1) $\left[\frac{5\pi}{8}, \frac{3\pi}{4}\right]$ (2) $\left[\frac{\pi}{2}, \frac{5\pi}{8}\right]$
 (3) $\left[\frac{\pi}{4}, \frac{\pi}{2}\right]$ (4) $\left[0, \frac{\pi}{4}\right]$

12. The number given by the Rolle's theorem for the function $x^3 - 3x^2, x \in [0, 3]$ is

- (1) 1 (2) $\sqrt{2}$ (3) $\frac{3}{2}$ (4) 2

13. The number given by the Mean value theorem for the function $\frac{1}{x}, x \in [1, 9]$ is

- (1) 2 (2) 2.5 (3) 3 (4) 3.5

14. The minimum value of the function $|3 - x| + 9$ is

- (1) 0 (2) 3 (3) 6 (4) 9

15. The maximum slope of the tangent to the curve $y = e^x \sin x, x \in [0, 2\pi]$ is at

- (1) $x = \frac{\pi}{4}$ (2) $x = \frac{\pi}{2}$
 (3) $x = \pi$ (4) $x = \frac{3\pi}{2}$

16. The maximum value of the function $x^2 e^{-2x}, x > 0$ is

- (1) $\frac{1}{e}$ (2) $\frac{1}{2e}$ (3) $\frac{1}{e^2}$ (4) $\frac{4}{e^4}$

17. One of the closest points on the curve $x^2 - y^2 = 4$ to the point (6, 0) is

- (1) (2, 0) (2) $(\sqrt{5}, 1)$
 (3) $(3, \sqrt{5})$ (4) $(\sqrt{13}, -\sqrt{3})$

18. The maximum product of two positive numbers, when their sum of the squares is 200, is

- (1) 100 (2) $25\sqrt{7}$
 (3) 28 (4) $24\sqrt{14}$

19. The curve $y = ax^4 + bx^2$ with $ab > 0$

- (1) has no horizontal tangent
 (2) is concave up
 (3) is concave down
 (4) has no points of inflection

20. The point of inflection of the curve $y = (x - 1)^3$ is

- (1) (0, 0) (2) (0, 1)
 (3) (1, 0) (4) (1, 1)

DEPARTMENT OF MATHEMATICS

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VELLORE DT -632503

8. Differentials and Partial Derivatives

Example 8.1

Find the linear approximation for

$f(x) = \sqrt{1+x}$, $x \geq -1$, at $x_0 = 3$. Use the linear approximation to estimate $f(3.2)$.

Solution:

Linear approximation

$$L(x) = f(x_0) + f'(x_0)(x - x_0)$$

Given $x_0 = 3$, $x = 3.2$

$$f(x_0) = f(3) = \sqrt{1+3}$$

$$= \sqrt{4}$$

$$= 2$$

$$f(x) = \sqrt{1+x}$$

$$f'(x) = \frac{1}{2\sqrt{1+x}}$$

$$f'(x_0) = f'(3) = \frac{1}{2\sqrt{1+3}}$$

$$= \frac{1}{2\sqrt{4}}$$

$$= \frac{1}{4}$$

Thus, Linear approximation

$$L(x) = f(x_0) + f'(x_0)(x - x_0)$$

$$= 2 + \frac{1}{4}(x - 3)$$

$$= 2 + \frac{x}{4} - \frac{3}{4}$$

$$= \frac{8-3}{4} + \frac{x}{4}$$

$$= \frac{5}{4} + \frac{x}{4}$$

$$\text{And, } f(x) = \sqrt{1+x}$$

$$f(3.2) = \sqrt{1+3.2}$$

$$= \sqrt{4.2}$$

$$\text{So, } L(3.2) = \frac{5}{4} + \frac{3.2}{4}$$

$$= 1.25 + 0.8$$

$$= 2.05$$

Example 8.2

Use linear approximation to find an approximate value of $\sqrt{9.2}$ without using a calculator.

Solution:

Let $y = \sqrt{x}$, let $x = 9$ and $dx = 0.2$

$$y = \sqrt{x}$$

$$= \sqrt{9}$$

$$y = 3$$

From $y = \sqrt{x}$

$$dy = \frac{1}{2\sqrt{x}} dx$$

$$= \frac{1}{2\sqrt{9}} (0.2)$$

$$= \frac{1}{3} (0.1)$$

$$= 0.033$$

$$f(x) = y + dy$$

$$f(\sqrt{9.2}) = 3 + 0.033 = 3.033$$

Example 8.3

Let us assume that the shape of a soap bubble is a sphere. Use linear approximation to approximate the increase in the surface area of a soap bubble as its radius increases from 5 cm to 5.2 cm. Also, calculate the percentage error.

Solution:

Radius of the sphere $r = 5$

Increase radius $dr = 0.2$

Surface area of the sphere $S = 4\pi r^2$

Approximate change in SA $dS = 4\pi(2r)dr$

$$= 4\pi(2 \times 5)(0.2)$$

$$= 4\pi(10)(0.2)$$

$$= 4\pi(2)$$

Approximate change in SA $dS = 8\pi$

Actual change in surface area $= S(5.2) - S(5)$

$$S(5.2) - S(5) = 4\pi(5.2)^2 - 4\pi(5)^2$$

$$= 4\pi[(5.2)^2 - (5)^2]$$

$$= 4\pi[(5.2 + 5) \times (5.2 - 5)]$$

$$= 4\pi[(10.2) \times (0.2)]$$

$$= 4\pi(2.04)$$

$$= 8.16\pi$$

$$\text{Relative error} = \frac{\text{Actual error} - \text{approximate error}}{\text{Actual error}}$$

$$= \frac{8.16\pi - 8\pi}{8.16\pi}$$

$$= \frac{0.16\pi}{8.16\pi}$$

$$= \frac{16}{816}$$

$$= 0.01960$$

$$\text{Percentage error} = \text{Relative error} \times 100$$

$$= 0.01960 \times 100$$

$$= 01.960 \%$$

Example 8.4

A right circular cylinder has radius $r = 10$ cm. and height $h = 20$ cm. Suppose that the radius of the cylinder is increased from 10 cm to 10.1 cm and the height does not change. Estimate the change in the volume of the cylinder. Also, calculate the relative error and percentage error.

Solution:

$$\text{Height of the cylinder } h = 20$$

$$\text{Radius of the cylinder } r = 10$$

$$\text{Increase radius } dr = 0.1$$

$$\text{Volume of the cylinder } V = \pi r^2 h$$

$$V = \pi r^2 (20)$$

$$V = 20\pi r^2$$

$$\text{Approximate change in } V \quad dV = 20\pi(2r)dr$$

$$= 20\pi(2 \times 10)(0.1)$$

$$= 20\pi(20)(0.1)$$

$$= 20\pi(2)$$

$$\text{Approximate change in } V \quad dV = 40\pi$$

$$\text{Actual change in Volume} = V(10.1) - V(10)$$

$$V(10.1) - V(10) = 20\pi(10.1)^2 - 20\pi(10)^2$$

$$= 20\pi[(10.1)^2 - (10)^2]$$

$$= 20\pi[(10.1 + 10) \times (10.1 - 10)]$$

$$= 20\pi[(20.1) \times (0.1)]$$

$$= 20\pi(2.01)$$

$$= 40.2\pi$$

$$\text{Relative error} = \frac{\text{Actual error} - \text{approximate error}}{\text{Actual error}}$$

$$= \frac{40.2\pi - 40\pi}{40.2\pi}$$

$$= \frac{0.2\pi}{40.2\pi}$$

$$= \frac{2}{402}$$

$$= \frac{1}{201}$$

$$= 0.00497$$

$$\text{Percentage error} = \text{Relative error} \times 100$$

$$= 0.00497 \times 100$$

$$= 0.497 \%$$

EXERCISE 8.1

1. Let $f(x) = \sqrt[3]{x}$. Find the linear approxi. at $x = 27$. Use the linear approximation to approximate $\sqrt[3]{27.2}$

Solution:

Linear approximation

$$L(x) = f(x_0) + f'(x_0)(x - x_0)$$

$$\text{Given } x_0 = 27, x = 27.2$$

$$f(x) = \sqrt[3]{x}$$

$$f(x_0) = f(27) = \sqrt[3]{27} = 3$$

$$f(x) = (x)^{\frac{1}{3}}$$

$$f'(x) = \frac{1}{3}(x)^{\frac{1}{3}-1}$$

$$= \frac{1}{3}(x)^{-\frac{2}{3}}$$

$$f'(x) = \frac{1}{3} \times \frac{1}{(x)^{\frac{2}{3}}}$$

$$f'(x_0) = \frac{1}{3} \times \frac{1}{(27)^{\frac{2}{3}}}$$

$$= \frac{1}{3} \times \frac{1}{(3)^{3 \times \frac{2}{3}}}$$

$$= \frac{1}{3} \times \frac{1}{(3)^2}$$

$$= \frac{1}{3} \times \frac{1}{9}$$

$$f'(x_0) = \frac{1}{27}$$

Linear approximation

$$L(x) = f(x_0) + f'(x_0)(x - x_0)$$

$$= 3 + \frac{1}{27}(x - 27)$$

To find $\sqrt[3]{27.2}$

$$\begin{aligned} L(27.2) &= 3 + \frac{1}{27}(27.2 - 27) \\ &= 3 + \frac{1}{27}(0.2) \\ &= 3 + \frac{0.2}{27} \\ &= 3 + 0.007 \end{aligned}$$

$$\sqrt[3]{27.2} = 3.007$$

2. Use the linear approximation to find approximate values of

(i) $(123)^{\frac{2}{3}}$ (ii) $\sqrt[4]{15}$ (iii) $\sqrt[3]{26}$

Solution: (i) $(123)^{\frac{2}{3}}$

Linear approximation

$$L(x) = f(x_0) + f'(x_0)(x - x_0)$$

Given $x_0 = 125, x = 123$

$$f(x) = (123)^{\frac{2}{3}}$$

$$f(x_0) = f(125) = (125)^{\frac{2}{3}}$$

$$= (5)^{3 \times \frac{2}{3}}$$

$$= (5)^2$$

$$= 25$$

$$f(x) = (x)^{\frac{2}{3}}$$

$$f'(x) = \frac{2}{3}(x)^{\frac{2}{3}-1}$$

$$= \frac{2}{3}(x)^{-\frac{1}{3}}$$

$$f'(x) = \frac{2}{3} \times \frac{1}{(x)^{\frac{1}{3}}}$$

$$f'(x_0) = \frac{2}{3} \times \frac{1}{(125)^{\frac{1}{3}}}$$

$$= \frac{2}{3} \times \frac{1}{(5)^{3 \times \frac{1}{3}}}$$

$$= \frac{2}{3} \times \frac{1}{(5)^1}$$

$$= \frac{2}{3} \times \frac{1}{5}$$

$$f'(x_0) = \frac{2}{15}$$

Linear approximation

$$L(x) = f(x_0) + f'(x_0)(x - x_0)$$

$$= 25 + \frac{2}{15}(x - 125)$$

To find $(123)^{\frac{2}{3}}$

$$L(123) = 25 + \frac{2}{15}(123 - 125)$$

$$= 25 + \frac{2}{15}(-2)$$

$$= 25 - \frac{4}{15}$$

$$= 25 - 0.266$$

$$(123)^{\frac{2}{3}} = 24.734$$

(ii) $\sqrt[4]{15}$

Linear approximation

$$L(x) = f(x_0) + f'(x_0)(x - x_0)$$

Given $x_0 = 16, x = 15$

$$f(x) = \sqrt[4]{15} = (15)^{\frac{1}{4}}$$

$$f(x_0) = f(16) = (16)^{\frac{1}{4}}$$

$$= (2)^{4 \times \frac{1}{4}}$$

$$= (2)^1$$

$$= 2$$

$$f(x) = (x)^{\frac{1}{4}}$$

$$f'(x) = \frac{1}{4}(x)^{\frac{1}{4}-1}$$

$$= \frac{1}{4}(x)^{-\frac{3}{4}}$$

$$f'(x) = \frac{1}{4} \times \frac{1}{(x)^{\frac{3}{4}}}$$

$$f'(x_0) = \frac{1}{4} \times \frac{1}{(16)^{\frac{3}{4}}}$$

$$= \frac{1}{4} \times \frac{1}{(2)^{4 \times \frac{3}{4}}}$$

$$= \frac{1}{4} \times \frac{1}{(2)^3}$$

$$= \frac{1}{4} \times \frac{1}{8}$$

$$f'(x_0) = \frac{1}{32}$$

Linear approximation

$$L(x) = f(x_0) + f'(x_0)(x - x_0)$$

$$= 2 + \frac{1}{32}(x - 16)$$

To find $\sqrt[4]{15}$

$$\begin{aligned} L(15) &= 2 + \frac{1}{32}(15 - 16) \\ &= 2 + \frac{1}{32}(-1) \\ &= 2 - \frac{1}{32} \\ &= 2 - 0.03125 \end{aligned}$$

$$\sqrt[4]{15} = 1.96875$$

(iii) $\sqrt[3]{26}$

Linear approximation

$$L(x) = f(x_0) + f'(x_0)(x - x_0)$$

Given $x_0 = 27, x = 26$

$$f(x) = \sqrt[3]{26} = (26)^{\frac{1}{3}}$$

$$f(x_0) = f(27) = (27)^{\frac{1}{3}}$$

$$= (3)^{3 \times \frac{1}{3}}$$

$$= (3)^1$$

$$= 3$$

$$f(x) = (x)^{\frac{1}{3}}$$

$$f'(x) = \frac{1}{3}(x)^{\frac{1}{3}-1}$$

$$= \frac{1}{3}(x)^{-\frac{2}{3}}$$

$$f'(x) = \frac{1}{3} \times \frac{1}{(x)^{\frac{2}{3}}}$$

$$f'(x_0) = \frac{1}{3} \times \frac{1}{(27)^{\frac{2}{3}}}$$

$$= \frac{1}{3} \times \frac{1}{(3)^{3 \times \frac{2}{3}}}$$

$$= \frac{1}{3} \times \frac{1}{(3)^2}$$

$$= \frac{1}{3} \times \frac{1}{9}$$

$$f'(x_0) = \frac{1}{27}$$

Linear approximation

$$L(x) = f(x_0) + f'(x_0)(x - x_0)$$

$$= 3 + \frac{1}{27}(x - 27)$$

To find $\sqrt[3]{26}$

$$L(26) = 3 + \frac{1}{27}(26 - 27)$$

$$= 3 + \frac{1}{27}(-1)$$

$$= 3 - \frac{1}{27}$$

$$= 3 - 0.0370$$

$$\sqrt[3]{26} = 2.963$$

3. Find a linear approximation for the following functions at the indicated points.

(i) $f(x) = x^3 - 5x + 12, x_0 = 2$

(ii) $g(x) = \sqrt{x^2 + 9}, x_0 = -4$

(iii) $h(x) = \frac{x}{x+1}, x_0 = 1$

Solution:

(i) $f(x) = x^3 - 5x + 12, x_0 = 2$

$$f(x) = x^3 - 5x + 12$$

$$f(x_0) = f(2) = (2)^3 - 5(2) + 12$$

$$= 8 - 10 + 12$$

$$= 20 - 10$$

$$f(2) = 10$$

$$f'(x) = 3x^2 - 5$$

$$f'(x_0) = f'(2) = 3(2)^2 - 5$$

$$= 3(4) - 5$$

$$= 12 - 5$$

$$= 7$$

Linear approximation

$$L(x) = f(x_0) + f'(x_0)(x - x_0)$$

$$= f(2) + f'(2)(x - 2)$$

$$= 10 + 7(x - 2)$$

$$= 10 + 7x - 14$$

$$= 7x - 4$$

(ii) $g(x) = \sqrt{x^2 + 9}, x_0 = -4$

$$g(x) = \sqrt{x^2 + 9}$$

$$g(x_0) = g(-4) = \sqrt{(-4)^2 + 9}$$

$$= \sqrt{16 + 9}$$

$$= \sqrt{25}$$

$$g(-4) = 5$$

$$g'(x) = \frac{1}{2\sqrt{x^2+9}} \times 2x$$

$$= \frac{x}{\sqrt{x^2+9}}$$

$$g'(x_0) = g'(-4) = \frac{(-4)}{\sqrt{(-4)^2+9}}$$

$$= \frac{-4}{\sqrt{16+9}}$$

$$= \frac{-4}{\sqrt{25}}$$

$$= -\frac{4}{5}$$

Linear approximation

$$L(x) = g(x_0) + g'(x_0)(x - x_0)$$

$$= g(-4) + g'(-4)(x + 4)$$

$$= 5 - \frac{4}{5}(x + 4)$$

$$= 5 - \frac{4x}{5} - \frac{16}{5}$$

$$= 5 - \frac{16}{5} - \frac{4x}{5}$$

$$= \frac{25-16}{5} - \frac{4x}{5}$$

$$= \frac{9}{5} - \frac{4x}{5}$$

$$= \frac{9-4x}{5} = \frac{1}{5}(9-4x)$$

$$(iii) h(x) = \frac{x}{x+1}, x_0 = 1$$

$$h(x) = \frac{x}{x+1}$$

$$h(x_0) = h(1) = \frac{1}{1+1}$$

$$= \frac{1}{2}$$

$$h'(x) = \frac{(x+1)(x)' - x(x+1)'}{(x+1)^2}$$

$$= \frac{(x+1)(1) - x(1)}{(x+1)^2}$$

$$= \frac{x+1-x}{(x+1)^2}$$

$$= \frac{1}{(x+1)^2}$$

$$h'(x_0) = h'(1) = \frac{1}{(1+1)^2}$$

$$= \frac{1}{(2)^2}$$

$$= \frac{1}{4}$$

Linear approximation

$$L(x) = h(x_0) + h'(x_0)(x - x_0)$$

$$= h(1) + h'(1)(x - 1)$$

$$= \frac{1}{2} + \frac{1}{4}(x - 1)$$

$$= \frac{1}{2} + \frac{x}{4} - \frac{1}{4}$$

$$= \frac{1}{2} - \frac{1}{4} + \frac{x}{4}$$

$$= \frac{2-1}{4} + \frac{x}{4}$$

$$= \frac{1}{4} + \frac{x}{4}$$

$$= \frac{x+1}{4} = \frac{1}{4}(x+1)$$

4. The radius of a circular plate is measured as 12.65 cm instead of the actual length 12.5 cm. find the following in calculating the area of the circular plate:

(i) Absolute error (ii) Relative error

(iii) Percentage error

Solution:

Change in radius $dr = 12.65 - 12.5 = 0.15$

Area of the circular plate $A = \pi r^2$

$$A' = 2\pi r dr$$

$$= 2\pi(12.5)(0.15)$$

$$= \pi(12.5)(0.3)$$

$$= 3.75\pi$$

Actual change in Area = $A(12.65) - A(12.5)$

$$A(12.65) - A(12.5) = \pi(12.65)^2 - \pi(12.5)^2$$

$$= \pi[(12.65)^2 - (12.5)^2]$$

$$= \pi[(12.65) + (12.5)(12.65) - (12.5)]$$

$$= \pi[(25.15)(0.15)]$$

$$= \pi(3.7725)$$

(i) Absolute error

$$= \text{Actual change in Area} - \text{Approximate value}$$

$$= \pi(3.7725) - 3.75\pi$$

$$= 0.0225\pi$$

$$\begin{aligned}
 \text{(ii) Relative error} &= \frac{\text{Actual error} - \text{approximate error}}{\text{Actual error}} \\
 &= \frac{\pi(3.7725) - 3.75\pi}{\pi(3.7725)} \\
 &= \frac{0.0225\pi}{\pi(3.7725)} \\
 &= \frac{0.0225}{3.7725} \\
 &= \frac{225}{37725} \\
 &= 0.00596 \\
 &= 0.006
 \end{aligned}$$

$$\begin{aligned}
 \text{(iii) Percentage error} &= \text{Relative error} \times 100 \% \\
 &= 0.006 \times 100 \% \\
 &= 0.6 \times 100 \%
 \end{aligned}$$

5. A sphere is made of ice having radius 10 cm.

Its radius decreases from 10 cm to 9.8 cm.

Find approximations for the following:

(i) change in the volume

(ii) change in the surface area

Solution:

Change in radius $dr = -0.2$

Volume of the sphere $V = \frac{4}{3}\pi r^3$

$$\begin{aligned}
 V' &= \frac{4}{3} \times 3\pi r^2 dr \\
 &= 4\pi r^2 dr \\
 &= 4\pi(10)^2(-0.2) \\
 &= 4\pi(100)(-0.2) \\
 &= 400\pi(-0.2)
 \end{aligned}$$

$$V' = -80\pi$$

Volume decreases by $80\pi \text{ cm}^3$

Surface Area of the sphere $S = 4\pi r^2$

$$\begin{aligned}
 S' &= 4 \times 2\pi r dr \\
 &= 8\pi r dr \\
 &= 8\pi(10)(-0.2) \\
 &= 80\pi(-0.2)
 \end{aligned}$$

$$S' = -16\pi$$

Surface Area decreases by $16\pi \text{ cm}^2$

6. The time T , taken for a complete oscillation of a single pendulum with length l , is given by the equation $T = 2\pi\sqrt{\frac{l}{g}}$, where g is a constant. Find the approximate percentage error in the calculated value of T corresponding to an error of 2 percent in the value of l .

$$\begin{aligned}
 \text{Solution: } T &= 2\pi\sqrt{\frac{l}{g}} \\
 T &= 2\pi\left(\frac{l}{g}\right)^{\frac{1}{2}}
 \end{aligned}$$

Taking log on both sides,

$$\begin{aligned}
 \log T &= \log\left(2\pi\left(\frac{l}{g}\right)^{\frac{1}{2}}\right) \\
 &= \log(2\pi) + \log\left(\frac{l}{g}\right)^{\frac{1}{2}} \\
 &= \log(2\pi) + \frac{1}{2} \log\left(\frac{l}{g}\right) \\
 &= \log(2\pi) + \frac{1}{2} [\log(l) - \log(g)]
 \end{aligned}$$

D.w.r.t.l,

$$\begin{aligned}
 \frac{1}{T} \frac{dT}{dl} &= 0 + \frac{1}{2} \left(\frac{1}{l} - 0\right) \\
 \frac{dT}{T} &= \frac{1}{2} \left(\frac{1}{l}\right) dl \\
 \frac{dT}{T} &= \frac{1}{2} \left(\frac{dl}{l}\right)
 \end{aligned}$$

Percentage error

$$\begin{aligned}
 \frac{dT}{T} \times 100 &= \frac{1}{2} \left(\frac{dl}{l}\right) \times 100 \\
 &= \frac{1}{2} \left(\frac{0.02l}{l}\right) \times 100 \\
 &= \frac{1}{2} (0.02) \times 100 \\
 &= (0.01) \times 100 \\
 &= 1\%
 \end{aligned}$$

7. Show that the percentage error in the n^{th} root of a number is approximately $\frac{1}{n}$ times the percentage error in the number.

Solution: Let x be the number.

$$\therefore y = (x)^{\frac{1}{n}}$$

Taking log on both sides,

$$\begin{aligned}\log y &= \log(x)^{\frac{1}{n}} \\ &= \frac{1}{n} \log(x)\end{aligned}$$

D.w.r.t.x,

$$\begin{aligned}\frac{1}{y} \frac{dy}{dx} &= \frac{1}{n} \times \frac{1}{x} \\ \frac{dy}{y} &= \frac{1}{n} \times \frac{dx}{x} \\ \frac{dy}{y} \times 100 &= \frac{1}{n} \times \frac{dx}{x} \times 100 \\ &= \frac{1}{n} \times (\text{times the percentage of the number})\end{aligned}$$

Example 8.5

Let $f, g : (a, b) \rightarrow \mathbb{R}$ be differentiable functions.

Show that $d(fg) = fdg + gdf$

Solution:

Let $f, g : (a, b) \rightarrow \mathbb{R}$ be differentiable functions.

Then (fg) is also a differentiable function on \mathbb{R}

$$\therefore d(fg) = f(dg) + g(df)$$

Example 8.6 Let $g(x) = x^2 + \sin x$.

Calculate the differential dg

Solution:

$$\begin{aligned}g(x) &= x^2 + \sin x \\ \therefore \frac{dg}{dx} &= 2x + \cos x \\ dg &= (2x + \cos x) dx\end{aligned}$$

Example 8.7

If the radius of a sphere, with radius 10 cm, has to decrease by 0.1 cm, approximately how much will its volume decrease?

Solution:

Change in radius $dr = -0.1$

Volume of the sphere $V = \frac{4}{3}\pi r^3$

$$\frac{dV}{dr} = \frac{4}{3} \times 3\pi r^2$$

$$\begin{aligned}dV &= \frac{4}{3} \times 3\pi r^2 dr \\ &= 4\pi r^2 dr \\ &= 4\pi(10)^2(-0.1) \\ &= 4\pi(100)(-0.1) \\ &= 400\pi(-0.1) \\ dV &= -40\pi\end{aligned}$$

Volume of the sphere decreases by $40\pi \text{ cm}^3$.

EXERCISE 8.2

1. Find differential dy for each of the following functions:

(i) $y = \frac{(1-2x)^3}{3-4x}$

Solution: $y = \frac{(1-2x)^3}{3-4x}$

$$\begin{aligned}\frac{dy}{dx} &= \frac{(3-4x)[(1-2x)^3]' - (1-2x)^3(3-4x)'}{(3-4x)^2} \\ &= \frac{(3-4x)[3(1-2x)^2(-2)] - (1-2x)^3(-4)}{(3-4x)^2} \\ &= \frac{(1-2x)^2[(3-4x)(-6) + 4(1-2x)]}{(3-4x)^2} \\ &= \frac{(1-2x)^2[-18+24x+4-8x]}{(3-4x)^2} \\ dy &= \frac{(1-2x)^2(16x-14)}{(3-4x)^2} dx\end{aligned}$$

(ii) $y = [3 + \sin(2x)]^{2/3}$

Solution: $y = [3 + \sin(2x)]^{2/3}$

$$\begin{aligned}\frac{dy}{dx} &= \frac{2}{3} [3 + \sin(2x)]^{\frac{2}{3}-1} (2 \cos 2x) \\ &= \frac{4}{3} [3 + \sin(2x)]^{-\frac{1}{3}} (\cos 2x) \\ dy &= \frac{4}{3} [3 + \sin(2x)]^{-\frac{1}{3}} (\cos 2x) dx\end{aligned}$$

(iii) $y = e^{x^2-5x+7} \cos(x^2 - 1)$

Solution:

$$\begin{aligned}y &= e^{x^2-5x+7} \cos(x^2 - 1) \\ \frac{dy}{dx} &= e^{x^2-5x+7} \sin(x^2 - 1) (2x) \\ &\quad + \cos(x^2 - 1) (e^{x^2-5x+7})(2x - 5) \\ \frac{dy}{dx} &= e^{x^2-5x+7} [2x \sin(x^2 - 1) + \cos(x^2 - 1)(2x - 5)] \\ dy &= e^{x^2-5x+7} [2x \sin(x^2 - 1) + \cos(x^2 - 1)(2x - 5)] dx\end{aligned}$$

2. Find df for $f(x) = x^2 + 3x$ and evaluate

it for (i) $x = 2$ and $dx = 0.1$

(ii) $x = 3$ and $dx = 0.02$

Solution: (i) $x = 2$ and $dx = 0.1$

$$f(x) = x^2 + 3x$$

$$\frac{df}{dx} = 2x + 3$$

$$df = (2x + 3)dx$$

$$= [2(2) + 3](0.1)$$

$$= (4 + 3)(0.1)$$

$$= (7)(0.1)$$

$$df = 0.7$$

(ii) $x = 3$ and $dx = 0.02$

$$f(x) = x^2 + 3x$$

$$\frac{df}{dx} = 2x + 3$$

$$df = (2x + 3)dx$$

$$= [2(3) + 3](0.02)$$

$$= (6 + 3)(0.02)$$

$$= (9)(0.02)$$

$$df = 0.18$$

3. Find Δf and df for the function f for the

indicated values of $x, \Delta x$ and compare

(i) $(x) = x^3 - 2x^2$; $x = 2, \Delta x = dx = 0.5$

(ii) $(x) = x^2 + 2x + 3$; $x = -0.5, \Delta x = dx = 0.1$

Solution:

(i) $(x) = x^3 - 2x^2$; $x = 2, \Delta x = dx = 0.5$

$$f(x) = x^3 - 2x^2$$

$$\frac{df}{dx} = 3x^2 - 4x$$

$$df = (3x^2 - 4x)dx$$

$$= [3(2)^2 - 4(2)](0.5)$$

$$= (12 - 8)(0.5)$$

$$= (4)(0.5)$$

$$df = 2$$

We know, $\Delta f = f(x + \Delta x) - f(x)$

Given $x = 2, \Delta x = 0.5$

$$f(x) = x^3 - 2x^2$$

$$f(x + \Delta x) = f(2 + 0.5) = f(2.5)$$

$$f(2.5) = (2.5)^3 - 2(2.5)^2$$

$$= 15.625 - 2(6.25)$$

$$= 15.625 - 12.50$$

$$= 3.125$$

$$f(x) = x^3 - 2x^2$$

$$f(2) = (2)^3 - 2(2)^2$$

$$= 8 - 2(4)$$

$$= 8 - 8$$

$$= 0$$

$$\therefore \Delta f = f(x + \Delta x) - f(x)$$

$$= 3.125 - 0$$

$$= 3.125$$

(ii) $(x) = x^2 + 2x + 3$; $x = -0.5, \Delta x = dx = 0.1$

$$f(x) = x^2 + 2x + 3$$

$$\frac{df}{dx} = 2x + 2$$

$$df = (2x + 2)dx$$

$$= [2(-0.5) + 2](0.1)$$

$$= (-1 + 2)(0.1)$$

$$= (1)(0.1)$$

$$df = 0.1$$

We know, $\Delta f = f(x + \Delta x) - f(x)$

Given $x = -0.5, \Delta x = 0.1$

$$f(x) = x^2 + 2x + 3$$

$$f(x + \Delta x) = f(-0.5 + 0.1) = f(-0.4)$$

$$f(-0.4) = (-0.4)^2 + 2(-0.4) + 3$$

$$= 0.16 - 0.8 + 3$$

$$= 3.16 - 0.8$$

$$= 2.36$$

$$f(x) = x^2 + 2x + 3$$

$$f(-0.5) = (-0.5)^2 + 2(-0.5) + 3$$

$$= 0.25 - 1 + 3$$

$$= 0.25 + 2$$

$$= 2.25$$

$$\therefore \Delta f = f(x + \Delta x) - f(x)$$

$$= 2.36 - 2.25$$

$$= 0.11$$

4. Assuming $\log_{10} e = 0.4343$, find an approximate value of $\log_{10} 1003$.

Solution:

Let $y = \log x$, $x = 1000$, $dx = 3 \log_{10} e$

$$\frac{dy}{dx} = \frac{1}{x}$$

$$dy = \frac{1}{x} dx$$

$$= \frac{1}{1000} (3 \log_{10} e)$$

$$= \frac{1}{1000} (3 \times 0.4343)$$

$$= \frac{1}{1000} (1.3029)$$

$$= 0.0013029$$

$$y = \log x$$

$$= \log 1000$$

$$= \log(10)^3$$

$$= 3 \log(10)$$

$$= 3 \times 1 = 3$$

$$\log_{10} 1003 = y + dy$$

$$= 3 + 0.0013029$$

$$= 3.0013029$$

5. The trunk of a tree has diameter 30 cm.

During the following year, the circumference grew 6 cm. (i) Approximately, how much did the tree's diameter grow?

(ii) What is the percentage increase in area of the tree's cross-section?

Solution:

(i) Given diameter $d = 30$ cm

$$\text{Circumference } P = 2\pi r$$

$$\frac{dP}{dr} = 2\pi$$

$$dP = 2\pi dr$$

Given increase in perimeter $dP = 6$

$$\therefore 6 = 2\pi dr$$

$$dr = \frac{6}{2\pi}$$

$$dr = \frac{3}{\pi}$$

Change in radius $dr = \frac{3}{\pi}$

\therefore Change in diameter $2dr = \frac{6}{\pi}$

(ii) Area $A = \pi r^2$

$$\frac{dA}{dr} = \pi(2r)$$

$$dA = 2\pi r dr$$

$$\frac{dA}{A} = \frac{2\pi r dr}{\pi r^2}$$

$$= \frac{2dr}{r}$$

$$= \frac{2(\frac{3}{\pi})}{(15)}$$

$$= \frac{6}{\pi(15)}$$

$$\frac{dA}{A} = \frac{2}{5\pi}$$

$$\text{So, \% error} = \frac{dA}{A} \times 100$$

$$= \frac{2}{5\pi} \times 100$$

$$= \frac{40}{\pi}$$

6. An egg of a particular bird is very nearly spherical. If the radius to the inside of the shell is 5 mm and radius to the outside of the shell is 5.3 mm, find the volume of the shell approximately.

Solution:

$$\text{Volume of the sphere } V = \frac{4}{3}\pi r^3$$

$$\frac{dV}{dr} = \frac{4}{3} \times 3\pi r^2$$

$$dV = \frac{4}{3} \times 3\pi r^2 dr$$

Here, $r = 5$ cm, and $dr = 0.3$

$$dV = \frac{4}{3} \times 3\pi(5)^2(0.3)$$

$$= 4 \times \pi(25)(0.3)$$

$$= 100 \pi(0.3)$$

$$dV = 30 \pi$$

7. Assume that the cross section of the artery of human is circular. A drug is given to a patient to dilate his arteries. If the radius of an artery is increased from 2 mm to 2.1 mm, how much is cross-sectional area increased approximately?

Solution:

Area of the circle $A = \pi r^2$

$$\frac{dA}{dr} = \pi(2r)$$

$$dA = 2\pi r dr$$

Here, $r = 2\text{mm}$, and $dr = 0.1$

$$dA = 2\pi(2)(0.1)$$

$$= 4\pi(0.1)$$

$$\text{So, } dA = 0.4 \pi$$

8. In a newly developed city, it is estimated that the voting population (in thousands) will increase according to $V(t) = 30 + 12t^2 - t^3$, $0 \leq t \leq 8$ where t is the time in years. Find the approximate change in voters for the time change from 4 to $4\frac{1}{6}$ year.

Solution: $V(t) = 30 + 12t^2 - t^3$

$$\frac{dV}{dt} = 12(2t) - 3t^2$$

$$dV = (24t - 3t^2)dt$$

Here, $t = 4$, and $dt = \frac{1}{6}$

$$\begin{aligned} \therefore dV &= [24(t) - 3(t)^2]dt \\ &= [24(4) - 3(4)^2] \left(\frac{1}{6}\right) \\ &= [96 - 3(16)] \left(\frac{1}{6}\right) \\ &= (96 - 48) \left(\frac{1}{6}\right) \\ &= (48) \left(\frac{1}{6}\right) \\ &= 8 \text{ (in thousands)} \end{aligned}$$

9. The relation between the number of words y a person learns in x hours is given by $y = 52\sqrt{x}$, $0 \leq x \leq 9$. What is the approximate number of words learned when x changes from

(i) 1 to 1.1 hour? (ii) 4 to 4.1 hour?

Solution:

$$y = 52\sqrt{x}$$

$$\frac{dy}{dx} = 52 \frac{1}{2\sqrt{x}}$$

$$= \frac{26}{\sqrt{x}}$$

$$dy = \frac{26}{\sqrt{x}} dx$$

(i) 1 to 1.1 hour

When $x = 1$, and $dx = 0.1$

$$dy = \frac{26}{\sqrt{1}} (0.1)$$

$$= 26 (0.1)$$

$$dy = 2.6$$

Approximately 3 words.

(ii) 4 to 4.1 hour

When $x = 4$, and $dx = 0.1$

$$dy = \frac{26}{\sqrt{4}} (0.1)$$

$$= \frac{26}{2} (0.1)$$

$$= 13 (0.1)$$

$$dy = 1.3$$

Approximately 1 word.

10. A circular plate expands uniformly under the influence of heat. If its radius increases from 10.5 cm to 10.75 cm, then find an approximate change in the area and the approximate percentage change in the area.

Solution: Area $A = \pi r^2$

$$\frac{dA}{dr} = \pi(2r)$$

$$dA = 2\pi r dr$$

(i) $r = 10.5\text{cm}$, and $dr = 0.25$

$$dA = 2\pi(10.5)(0.25)$$

$$= 0.5\pi(10.5)$$

$$dA = 5.25\pi$$

$$\frac{dA}{A} = \frac{2\pi r dr}{\pi r^2}$$

$$= \frac{2dr}{r}$$

$$= \frac{2(0.25)}{10.5}$$

$$= \frac{0.5}{10.5}$$

$$= 0.0476$$

$$\text{So, \% error} = \frac{dA}{A} \times 100$$

$$= 0.0476 \times 100$$

$$= 4.76 \%$$

11. A coat of paint of thickness 0.2 cm is

applied to the faces of a cube whose edge is

10 cm. Use the differentials to find

approximately how many cubic centimeters

of paint is used to paint this cube. Also

calculate the exact amount of paint used to

paint this cube.

Solution:

Volume of the cube $V = a^3$

$$\frac{dV}{da} = 3a^2$$

$$dV = 3a^2 da$$

$a = 10\text{cm}$, and $da = 0.2$

$$dV = 3(10)^2 (0.2)$$

$$= 0.6 (100)$$

$$dV = 60$$

$$\text{Exact amount of paint used} = (10.2)^3 - (10)^3$$

$$= 1061.208 - 1000$$

$$= 61.208$$

EXERCISE 8.3

1. Evaluate $\lim_{(x,y) \rightarrow (1,2)} g(x, y)$, if the limit

exists, where $g(x, y) = \frac{3x^2 - xy}{x^2 + y^2 + 3}$

Solution:

$$\begin{aligned} \lim_{(x,y) \rightarrow (1,2)} g(x, y) &= \lim_{(x,y) \rightarrow (1,2)} \frac{3x^2 - xy}{x^2 + y^2 + 3} \\ &= \frac{3(1)^2 - (1)(2)}{(1)^2 + (2)^2 + 3} \\ &= \frac{3 - 2}{1 + 4 + 3} \\ &= \frac{1}{8} \end{aligned}$$

2. Evaluate $\lim_{(x,y) \rightarrow (0,0)} \cos\left(\frac{x^3 + y^2}{x + y + 2}\right)$.

If the limit exists.

Solution:

$$\begin{aligned} \lim_{(x,y) \rightarrow (0,0)} \cos\left(\frac{x^3 + y^2}{x + y + 2}\right) \\ &= \cos\left(\frac{0 + 0}{0 + 0 + 2}\right) \\ &= \cos\left(\frac{0}{2}\right) \\ &= \cos(0) \\ &= 1 \end{aligned}$$

3. Let $f(x, y) = \frac{y^2 - xy}{\sqrt{x} - \sqrt{y}}$ for $(x, y) \neq (0, 0)$.

Show that $\lim_{(x,y) \rightarrow (0,0)} f(x, y) = 0$.

Solution:

$$\begin{aligned} \lim_{(x,y) \rightarrow (0,0)} f(x, y) \\ &= \lim_{(x,y) \rightarrow (0,0)} \frac{y^2 - xy}{\sqrt{x} - \sqrt{y}} \times \frac{\sqrt{x} + \sqrt{y}}{\sqrt{x} + \sqrt{y}} \\ &= \lim_{(x,y) \rightarrow (0,0)} \frac{y(y - x)(\sqrt{x} + \sqrt{y})}{(x - y)} \\ &= \lim_{(x,y) \rightarrow (0,0)} \frac{-y(x - y)(\sqrt{x} + \sqrt{y})}{(x - y)} \\ &= \lim_{(x,y) \rightarrow (0,0)} -y(\sqrt{x} + \sqrt{y}) \\ &= (0 + 0) \\ &= 0 \text{ Hence proved.} \end{aligned}$$

4. Evaluate $\lim_{(x,y) \rightarrow (0,0)} \cos\left(\frac{e^x \sin y}{y}\right)$, if the limit exists.

Solution:

$$\begin{aligned} \lim_{(x,y) \rightarrow (0,0)} \cos\left(\frac{e^x \sin y}{y}\right) &= \cos\left(\lim_{(x,y) \rightarrow (0,0)} \frac{e^x \sin y}{y}\right) \\ &= \cos\left(\lim_{(x,y) \rightarrow (0,0)} e^x \lim_{(x,y) \rightarrow (0,0)} \frac{\sin y}{y}\right) \\ &= \cos(e^0 \times 1) \\ &= \cos(1 \times 1) \\ &= \cos 1 \end{aligned}$$

5. Let $g(x, y) = \frac{x^2 y}{x^4 + y^2}$ for $(x, y) \neq (0, 0)$ and $f(0, 0) = 0$.

- (i) Show that $\lim_{(x,y) \rightarrow (0,0)} g(x, y) = 0$ along every line $y = mx, m \in \mathbb{R}$.

Solution:

$$\begin{aligned} \lim_{(x,y) \rightarrow (0,0)} g(x, y) &= \lim_{(x,y) \rightarrow (0,0)} \frac{x^2 y}{x^4 + y^2} \\ \text{When } y &= mx \\ &= \lim_{(x,y) \rightarrow (0,0)} \frac{x^2 mx}{x^4 + m^2 x^2} \\ &= \lim_{(x,y) \rightarrow (0,0)} \frac{x^2(mx)}{x^2(x^2 + m^2)} \\ &= \lim_{(x,y) \rightarrow (0,0)} \frac{(mx)}{(x^2 + m^2)} \\ &= \frac{0}{0 + m^2} \\ &= 0 \end{aligned}$$

- (ii) Show that $\lim_{(x,y) \rightarrow (0,0)} g(x, y) = \frac{k}{1 + k^2}$, along every parabola $y = kx^2, k \in \mathbb{R} \setminus \{0\}$

Solution:

$$\begin{aligned} \lim_{(x,y) \rightarrow (0,0)} g(x, y) &= \lim_{(x,y) \rightarrow (0,0)} \frac{x^2 y}{x^4 + y^2} \\ \text{When } y &= kx^2 \\ &= \lim_{(x,y) \rightarrow (0,0)} \frac{x^2 kx^2}{x^4 + k^2 x^4} \\ &= \lim_{(x,y) \rightarrow (0,0)} \frac{x^4(k)}{x^4(1 + k^2)} \end{aligned}$$

6. Show that $f(x, y) = \frac{x^2 - y^2}{y^2 + 1}$ is continuous at every $(x, y) \in \mathbb{R}^2$.

Solution:

- (i) Let $(a, b) \in \mathbb{R}^2$ be an arbitrary point.

$$f(a, b) = \frac{a^2 - b^2}{b^2 + 1} \text{ is defined for } \forall (a, b) \in \mathbb{R}^2$$

- (ii) $\lim_{(x,y) \rightarrow (a,b)} f(x, y)$

$$\begin{aligned} &= \lim_{(x,y) \rightarrow (a,b)} \frac{x^2 - y^2}{y^2 + 1} \\ &= \frac{a^2 - b^2}{b^2 + 1} \end{aligned}$$

Limit exists at $(a, b) \in \mathbb{R}^2$

- (iii) $\lim_{(x,y) \rightarrow (a,b)} f(x, y) = L$

$$\begin{aligned} &= f(a, b) \\ &= \frac{a^2 - b^2}{b^2 + 1} \end{aligned}$$

Hence f is continuous at the point on \mathbb{R}^2

7. Let $g(x, y) = \frac{e^y \sin x}{x}$, for $x \neq 0$ and $g(0, 0) = 1$. Show that g is continuous at $(0, 0)$

Solution:

$$\begin{aligned} |g(x, y) - g(0, 0)| &= \left| \frac{e^y \sin x}{x} - 1 \right| \\ &= \left| \frac{e^y \sin x - x}{x} \right| \end{aligned}$$

- (i) $g(0, 0) = 1$

Hence g is defined.

- (ii) $\lim_{(x,y) \rightarrow (0,0)} g(x, y)$

$$\begin{aligned} &= \lim_{(x,y) \rightarrow (0,0)} \frac{e^y \sin x}{x} \\ &= \lim_{(x,y) \rightarrow (0,0)} e^y \lim_{(x,y) \rightarrow (0,0)} \frac{\sin x}{x} \\ &= (e^0 \times 1) \\ &= (1 \times 1) \\ &= 1 \end{aligned}$$

Hence limit exists at $(0, 0)$

- (iii) $\lim_{(x,y) \rightarrow (0,0)} g(x, y) = 1 = g(0, 0)$

Hence g is continuous at $(0,0)$

Example 8.11

Let $f(x, y) = 0$ if $xy \neq 0$ and $f(x, y) = 1$ if $xy = 0$.

(i) Calculate: $\frac{\partial f}{\partial x}(0,0), \frac{\partial f}{\partial y}(0,0)$

(ii) Show that f is not continuous at $(0,0)$.

Solution :

Note that the function f takes value 1 on the x , y - axes and 0 everywhere else on \mathbb{R}^2 .

So let us calculate

$$\frac{\partial f}{\partial x}(0,0) = \lim_{h \rightarrow 0} \frac{f(0+h,0) - f(0,0)}{h} = \lim_{h \rightarrow 0} \frac{1-1}{h} = 0$$

$$\frac{\partial f}{\partial y}(0,0) = \lim_{k \rightarrow 0} \frac{f(0,0+k) - f(0,0)}{k} = \lim_{k \rightarrow 0} \frac{1-1}{k} = 0$$

This completes (i)

Now for (ii) let us calculate the limit of f as

$(x, y) \rightarrow (0,0)$ along the line $y = x$. Then

$$\lim_{(x,y) \rightarrow (0,0)} f(x, y) = 0, \text{ because the line } y = x$$

when $x \neq 0, f(x, y) = 0$, but $f(0,0) = 1 \neq 0$,

hence f is not continuous at $(0,0)$.

Example 8.12

Let $F(x, y) = x^3y + y^2x + 7$ for all $(x, y) \in \mathbb{R}^2$. Calculate $\frac{\partial F}{\partial x}(-1,3)$ and $\frac{\partial F}{\partial y}(-2,1)$

Solution: $F(x, y) = x^3y + y^2x + 7$

$$\frac{\partial F}{\partial x} = 3x^2y + y^2$$

$$\text{So, } \frac{\partial F}{\partial x}(-1,3) = 3(-1)^2(3) + (3)^2$$

$$= 3(1)(3) + 9$$

$$= 9 + 9 = 18$$

$$\frac{\partial F}{\partial y} = x^3y + 2yx$$

$$\text{So, } \frac{\partial F}{\partial y}(-2,1) = (-2)^3(1) + 2(1)(-2)$$

$$= -8(1) - 4$$

$$= -8 - 4 = -12$$

Example 8.13

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Let $f(x, y) = \sin(xy^2) + e^{x^3+5y}$ for all $(x, y) \in \mathbb{R}^2$. Calculate $\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}, \frac{\partial^2 f}{\partial y \partial x}, \frac{\partial^2 f}{\partial x \partial y}$

Solution: $f(x, y) = \sin(xy^2) + e^{x^3+5y}$

$$\frac{\partial f}{\partial x} = \cos(xy^2)(y^2) + e^{x^3+5y}(3x^2)$$

$$\frac{\partial f}{\partial y} = \cos(xy^2)(x2y) + e^{x^3+5y}(5)$$

$$\frac{\partial^2 f}{\partial y \partial x} = \cos(xy^2)(2y) + (y^2)[- \sin(xy^2)](2xy)$$

$$+ (e^{x^3+5y})5(3x^2) + 0$$

$$= 2y \cos(xy^2) - 2xy^3[\sin(xy^2)]$$

$$+ 15x^2(e^{x^3+5y})$$

$$\frac{\partial^2 f}{\partial x \partial y} = \cos(xy^2)(2y) + (x2y)[- \sin(xy^2)](y^2)$$

$$+ 5e^{x^3+5y}(3x^2)$$

$$= (2y) \cos(xy^2) - 2xy^3 \sin(xy^2)$$

$$+ 15x^2e^{x^3+5y}$$

Example 8.14

Let $W(x, y) = xy + \frac{e^y}{y^2+1}$ for all $(x, y) \in \mathbb{R}^2$.

Calculate $\frac{\partial^2 W}{\partial y \partial x}, \frac{\partial^2 W}{\partial x \partial y}$

Solution: $W(x, y) = xy + \frac{e^y}{y^2+1}$

$$\frac{\partial W}{\partial x} = y$$

$$\frac{\partial^2 W}{\partial y \partial x} = 1 \dots\dots\dots (i)$$

$$W(x, y) = xy + \frac{e^y}{y^2+1}$$

$$\frac{\partial W}{\partial y} = x + \frac{(y^2+1)e^y - e^y(2y)}{(y^2+1)^2}$$

$$\frac{\partial^2 W}{\partial x \partial y} = 1 \dots\dots\dots (ii)$$

Example 8.15

Let $u(x, y) = e^{-2y} \cos(2x)$ for all $(x, y) \in \mathbb{R}^2$.

Prove that u is a harmonic function in \mathbb{R}^2 .

Solution:

$$u(x, y) = e^{-2y} \cos(2x)$$

$$u_x = e^{-2y}[- \sin(2x)2]$$

$$\begin{aligned}
 &= e^{-2y}[-2 \sin(2x)] \\
 u_{xx} &= e^{-2y}[-2 \cos(2x) 2] \\
 &= e^{-2y}[-4 \cos(2x)] \\
 u_{xx} &= -4 \cos(2x) e^{-2y} \\
 u_y &= \cos(2x) e^{-2y}(-2) \\
 &= -2 \cos(2x) e^{-2y} \\
 u_{yy} &= -2 \cos(2x) e^{-2y}(-2) \\
 u_{yy} &= 4 \cos(2x) e^{-2y}
 \end{aligned}$$

Thus,

$$\begin{aligned}
 u_{xx} + u_{yy} &= -4 \cos(2x) e^{-2y} + 4 \cos(2x) e^{-2y} \\
 &= 0
 \end{aligned}$$

Since, u satisfies the Laplace's equation, it is a harmonic function.

EXERCISE 8.4

1. Find the partial derivatives of the following functions at the indicated points.

- (i) $f(x, y) = 3x^2 - 2xy + y^2 + 5x + 2$, $(2, -5)$
(ii) $g(x, y) = 3x^2 + y^2 + 5x + 2$, $(1, -2)$
(iii) $h(x, y, z) = x \sin(xy) + z^2x$, $(2, \frac{\pi}{4}, 1)$
(iv) $G(x, y) = e^{x+3y} \log(x^2 + y^2)$, $(-1, 1)$

Solution:

- (i) $f(x, y) = 3x^2 - 2xy + y^2 + 5x + 2$, $(2, -5)$

$$f(x, y) = 3x^2 - 2xy + y^2 + 5x + 2$$

$$\frac{\partial f}{\partial x} = 6x - 2y + 5$$

At $(2, -5)$

$$\frac{\partial f}{\partial x} = 6(2) - 2(-5) + 5$$

$$= 12 + 10 + 5$$

$$= 27$$

$$f(x, y) = 3x^2 - 2xy + y^2 + 5x + 2$$

$$\frac{\partial f}{\partial y} = -2x + 2y$$

At $(2, -5)$

$$\frac{\partial f}{\partial y} = -2(2) + 2(-5)$$

$$= -4 - 10 = -14$$

- (ii) $g(x, y) = 3x^2 + y^2 + 5x + 2$, $(1, -2)$

$$g(x, y) = 3x^2 + y^2 + 5x + 2$$

$$\frac{\partial g}{\partial x} = 6x + 5$$

At $(1, -2)$

$$\frac{\partial g}{\partial x} = 6(1) + 5$$

$$= 6 + 5$$

$$= 11$$

$$g(x, y) = 3x^2 + y^2 + 5x + 2$$

$$\frac{\partial g}{\partial y} = 2y$$

At $(1, -2)$

$$\frac{\partial g}{\partial y} = 2(-2)$$

$$= -4$$

- (iii) $h(x, y, z) = x \sin(xy) + z^2x$, $(2, \frac{\pi}{4}, 1)$

$$h(x, y, z) = x \sin(xy) + z^2x$$

$$\frac{\partial h}{\partial x} = x[\cos(xy) y] + \sin(xy) + z^2$$

At $(2, \frac{\pi}{4}, 1)$

$$\frac{\partial h}{\partial x} = (2) \left[\cos\left(\frac{\pi}{2}\right) \frac{\pi}{4} \right] + \sin\left(\frac{\pi}{2}\right) + (1)^2$$

$$= (2) \left[(0) \frac{\pi}{4} \right] + 1 + 1$$

$$= 2$$

$$h(x, y, z) = x \sin(xy) + z^2x$$

$$\frac{\partial h}{\partial y} = x[\cos(xy) x] + \sin(xy)(0) + 0$$

At $(2, \frac{\pi}{4}, 1)$

$$\frac{\partial h}{\partial y} = (2) \left[\cos\left(\frac{\pi}{2}\right) 2 \right] + \sin\left(\frac{\pi}{2}\right) (0) + 0$$

$$= (2) [(0) 2] + 1(0) + 0$$

$$= 4(0) + 0$$

$$= 0$$

$$h(x, y, z) = x \sin(xy) + z^2x$$

$$\frac{\partial h}{\partial z} = 2zx$$

At $(2, \frac{\pi}{4}, 1)$

$$\frac{\partial h}{\partial z} = 2(1)(2) = 4$$

- (iv) $G(x, y) = e^{x+3y} \log(x^2 + y^2)$, $(-1, 1)$

$$G(x, y) = e^{x+3y} \log(x^2 + y^2)$$

$$\begin{aligned} \frac{\partial G}{\partial x} &= e^{x+3y} \frac{1}{(x^2+y^2)} (2x) + \log(x^2 + y^2) e^{x+3y} (1) \\ &= e^{x+3y} \frac{2x}{(x^2+y^2)} + \log(x^2 + y^2) e^{x+3y} \\ &= e^{x+3y} \left[\frac{2x}{(x^2+y^2)} + \log(x^2 + y^2) \right] \end{aligned}$$

At $(-1, 1)$

$$\begin{aligned} \frac{\partial G}{\partial x} &= e^{-1+3} \left[\frac{-2}{(1+1)} + \log(1+1) \right] \\ &= e^2 \left[\frac{-2}{(2)} + \log(2) \right] \\ &= e^2 [\log(2) - 1] \end{aligned}$$

$$G(x, y) = e^{x+3y} \log(x^2 + y^2)$$

$$\begin{aligned} \frac{\partial G}{\partial y} &= e^{x+3y} \frac{1}{(x^2+y^2)} (2y) + \log(x^2 + y^2) e^{x+3y} (3) \\ &= e^{x+3y} \frac{2y}{(x^2+y^2)} + 3 \log(x^2 + y^2) e^{x+3y} \\ &= e^{x+3y} \left[\frac{2y}{(x^2+y^2)} + 3 \log(x^2 + y^2) \right] \end{aligned}$$

At $(-1, 1)$

$$\begin{aligned} \frac{\partial G}{\partial y} &= e^{-1+3} \left[\frac{2}{(1+1)} + 3 \log(1+1) \right] \\ &= e^2 \left[\frac{2}{(2)} + 3 \log(2) \right] \\ &= e^2 [1 + \log 2^3] \\ &= e^2 [1 + \log 8] \end{aligned}$$

2. For each of the following functions find the f_x, f_y , and show that $f_{xy} = f_{yx}$

(i) $f(x, y) = \frac{3x}{y + \sin x}$

(ii) $f(x, y) = \tan^{-1} \left(\frac{x}{y} \right)$

(iii) $f(x, y) = \cos(x^2 - 3xy)$

Solution:

(i) $f(x, y) = \frac{3x}{y + \sin x}$

$$\begin{aligned} \frac{\partial f}{\partial x} &= \frac{(y + \sin x)(3) - 3x(\cos x)}{(y + \sin x)^2} \\ &= \frac{3y + 3 \sin x - 3x \cos x}{(y + \sin x)^2} \end{aligned}$$

$$\begin{aligned} \frac{\partial^2 f}{\partial y \partial x} &= \frac{(y + \sin x)^2 (3) - (3y + 3 \sin x - 3x \cos x) 2(y + \sin x)}{(y + \sin x)^4} \\ &= \frac{(y + \sin x) [(y + \sin x)(3) - (3y + 3 \sin x - 3x \cos x) 2]}{(y + \sin x)^4} \\ &= \frac{(y + \sin x) [(3y + 3 \sin x) - (6y + 6 \sin x - 6x \cos x)]}{(y + \sin x)^4} \end{aligned}$$

$$\begin{aligned} &= \frac{(y + \sin x) [3y + 3 \sin x - 6y - 6 \sin x + 6x \cos x]}{(y + \sin x)^4} \\ &= \frac{(-3 \sin x - 3y + 6x \cos x)}{(y + \sin x)^3} \\ &= \frac{-3(\sin x + y - 2x \cos x)}{(y + \sin x)^3} \dots \dots \dots (1) \end{aligned}$$

$$\begin{aligned} \frac{\partial f}{\partial y} &= \frac{(y + \sin x)(0) - 3x(1+0)}{(y + \sin x)^2} \\ &= \frac{-3x}{(y + \sin x)^2} \end{aligned}$$

$$\begin{aligned} \frac{\partial^2 f}{\partial x \partial y} &= \frac{(y + \sin x)^2 (-3) - (-3x) 2(y + \sin x)(\cos x)}{(y + \sin x)^4} \\ &= \frac{(y + \sin x) [(y + \sin x)(-3) + 6x(\cos x)]}{(y + \sin x)^4} \\ &= \frac{[(y + \sin x)(-3) + 6x(\cos x)]}{(y + \sin x)^3} \\ &= \frac{(-3y - 3 \sin x + 6x \cos x)}{(y + \sin x)^3} \\ &= \frac{-3(y + \sin x - 2x \cos x)}{(y + \sin x)^3} \dots \dots \dots (2) \end{aligned}$$

From (1) and (2)

$f_{xy} = f_{yx}$ is proved.

(ii) $f(x, y) = \tan^{-1} \left(\frac{x}{y} \right)$

$$\begin{aligned} \frac{\partial f}{\partial x} &= \frac{1}{1 + \left(\frac{x}{y} \right)^2} \left(\frac{1}{y} \right) \\ &= \frac{1}{1 + \left(\frac{x^2}{y^2} \right)} \left(\frac{1}{y} \right) \end{aligned}$$

$$= \frac{1}{\frac{x^2 + y^2}{y^2}} \left(\frac{1}{y} \right)$$

$$= \frac{y^2}{x^2 + y^2} \left(\frac{1}{y} \right)$$

$$= \frac{y}{x^2 + y^2}$$

$$\begin{aligned} \frac{\partial^2 f}{\partial y \partial x} &= \frac{(x^2 + y^2)(1) - y(2y)}{(x^2 + y^2)^2} \\ &= \frac{x^2 + y^2 - 2y^2}{(x^2 + y^2)^2} \end{aligned}$$

$$\frac{\partial^2 f}{\partial y \partial x} = \frac{x^2 - y^2}{(x^2 + y^2)^2} \dots \dots \dots (1)$$

$$\frac{\partial f}{\partial y} = \frac{1}{1 + \left(\frac{x}{y} \right)^2} \left(\frac{-x}{y^2} \right)$$

$$= \frac{1}{1 + \left(\frac{x^2}{y^2} \right)} \left(\frac{-x}{y^2} \right)$$

$$\begin{aligned}
 &= \frac{1}{x^2 + y^2} \left(\frac{-x}{y^2} \right) \\
 &= \frac{y^2}{x^2 + y^2} \left(\frac{-x}{y^2} \right) \\
 &= \frac{-x}{x^2 + y^2} \\
 \frac{\partial^2 f}{\partial x \partial y} &= \frac{(x^2 + y^2)(-1) - (-x)(2x)}{(x^2 + y^2)^2} \\
 &= \frac{-x^2 - y^2 + 2x^2}{(x^2 + y^2)^2} \\
 \frac{\partial^2 f}{\partial x \partial y} &= \frac{x^2 - y^2}{(x^2 + y^2)^2} \dots\dots\dots (2)
 \end{aligned}$$

From (1) and (2)

$f_{xy} = f_{yx}$ is proved.

(iii) $f(x, y) = \cos(x^2 - 3xy)$

$$\begin{aligned}
 \frac{\partial f}{\partial x} &= -\sin(x^2 - 3xy)(2x - 3y) \\
 &= -(2x - 3y)\sin(x^2 - 3xy)
 \end{aligned}$$

$$\begin{aligned}
 \frac{\partial^2 f}{\partial y \partial x} &= -(2x - 3y)\cos(x^2 - 3xy)(-3x) \\
 &\quad -\sin(x^2 - 3xy)(-3) \\
 &= 3x(2x - 3y)\cos(x^2 - 3xy) + 3\sin(x^2 - 3xy) \dots\dots\dots(1)
 \end{aligned}$$

$$\begin{aligned}
 \frac{\partial f}{\partial y} &= -\sin(x^2 - 3xy)(-3x) \\
 &= 3x\sin(x^2 - 3xy)
 \end{aligned}$$

$$\begin{aligned}
 \frac{\partial^2 f}{\partial x \partial y} &= 3x\cos(x^2 - 3xy)(2x - 3y) \\
 &\quad + \sin(x^2 - 3xy)(3) \\
 &= 3x(2x - 3y)\cos(x^2 - 3xy) + 3\sin(x^2 - 3xy) \dots\dots\dots(2)
 \end{aligned}$$

From (1) and (2)

$f_{xy} = f_{yx}$ is proved.

3. If $U(x, y, z) = \frac{x^2 + y^2}{xy} + 3z^2y$,

find $\frac{\partial U}{\partial x}$, $\frac{\partial U}{\partial y}$ and $\frac{\partial U}{\partial z}$

Solution:

$$\begin{aligned}
 U(x, y, z) &= \frac{x^2 + y^2}{xy} + 3z^2y \\
 \frac{\partial U}{\partial x} &= \frac{(xy)(2x + 0) - (x^2 + y^2)(y)}{(xy)^2} + 0 \\
 &= \frac{2x^2y - x^2y - y^3}{(xy)^2} \\
 &= \frac{x^2y - y^3}{(xy)^2}
 \end{aligned}$$

$$\begin{aligned}
 &= \frac{y(x^2 - y^2)}{x^2y^2} \\
 &= \frac{(x^2 - y^2)}{x^2y} \\
 U(x, y, z) &= \frac{x^2 + y^2}{xy} + 3z^2y \\
 \frac{\partial U}{\partial y} &= \frac{(xy)(0 + 2y) - (x^2 + y^2)(x)}{(xy)^2} + 3z^2 \\
 &= \frac{2xy^2 - x^3 - xy^2}{(xy)^2} + 3z^2 \\
 &= \frac{xy^2 - x^3}{(xy)^2} + 3z^2 \\
 &= \frac{x(y^2 - x^2)}{x^2y^2} + 3z^2 \\
 &= \frac{(y^2 - x^2)}{xy^2} + 3z^2
 \end{aligned}$$

$$U(x, y, z) = \frac{x^2 + y^2}{xy} + 3z^2y$$

$$\begin{aligned}
 \frac{\partial U}{\partial z} &= 0 + 3y(2z) \\
 &= 6yz
 \end{aligned}$$

4. If $U(x, y, z) = \log(x^3 + y^3 + z^3)$,

find $\frac{\partial U}{\partial x} + \frac{\partial U}{\partial y} + \frac{\partial U}{\partial z}$

Solution:

$$U(x, y, z) = \log(x^3 + y^3 + z^3)$$

$$\begin{aligned}
 \frac{\partial U}{\partial x} &= \frac{1}{(x^3 + y^3 + z^3)} 3x^2 \\
 &= \frac{3x^2}{(x^3 + y^3 + z^3)}
 \end{aligned}$$

$$\begin{aligned}
 \frac{\partial U}{\partial y} &= \frac{1}{(x^3 + y^3 + z^3)} 3y^2 \\
 &= \frac{3y^2}{(x^3 + y^3 + z^3)}
 \end{aligned}$$

$$\begin{aligned}
 \frac{\partial U}{\partial z} &= \frac{1}{(x^3 + y^3 + z^3)} 3z^2 \\
 &= \frac{3z^2}{(x^3 + y^3 + z^3)}
 \end{aligned}$$

$$\begin{aligned}
 \frac{\partial U}{\partial x} + \frac{\partial U}{\partial y} + \frac{\partial U}{\partial z} &= \frac{3x^2 + 3y^2 + 3z^2}{(x^3 + y^3 + z^3)} \\
 &= \frac{3(x^2 + y^2 + z^2)}{(x^3 + y^3 + z^3)}
 \end{aligned}$$

5. For each of the following functions find

the g_{xy} , g_{xx} , g_{yy} and g_{yx} .

(i) $g(x, y) = xe^y + 3x^2y$

(ii) $g(x, y) = \log(5x + 3y)$

(iii) $g(x, y) = x^2 + 3xy - 7y + \cos(5x)$

Solution:

(i) $g(x, y) = xe^y + 3x^2y$

$$g_x = e^y + 6xy$$

$$g_{xx} = 6y$$

$$g_{yx} = e^y + 6x$$

$$g_y = xe^y + 3x^2$$

$$g_{yy} = xe^y$$

$$g_{xy} = e^y + 6x$$

6. Let $w(x, y, z) = \frac{1}{\sqrt{x^2+y^2+z^2}}$.

Show that $\frac{\partial^2 w}{\partial x^2} + \frac{\partial^2 w}{\partial y^2} + \frac{\partial^2 w}{\partial z^2} = 0$

Solution:

$$w(x, y, z) = \frac{1}{\sqrt{x^2+y^2+z^2}}$$

$$= \frac{1}{(x^2+y^2+z^2)^{\frac{1}{2}}}$$

$$= (x^2 + y^2 + z^2)^{-\frac{1}{2}}$$

$$\frac{\partial w}{\partial x} = -\frac{1}{2}(x^2 + y^2 + z^2)^{-\frac{3}{2}}(2x)$$

$$= -x(x^2 + y^2 + z^2)^{-\frac{3}{2}}$$

$$\frac{\partial^2 w}{\partial x^2} = -x \left(-\frac{3}{2} \right) (x^2 + y^2 + z^2)^{-\frac{5}{2}}(2x)$$

$$+ (x^2 + y^2 + z^2)^{-\frac{3}{2}}(-1)$$

$$= (3x^2)(x^2 + y^2 + z^2)^{-\frac{3}{2}-1} - (x^2 + y^2 + z^2)^{-\frac{3}{2}}$$

$$= (x^2 + y^2 + z^2)^{-\frac{3}{2}}[(3x^2)(x^2 + y^2 + z^2)^{-1} - 1]$$

$$= (x^2 + y^2 + z^2)^{-\frac{3}{2}} \left[\frac{3x^2}{(x^2+y^2+z^2)} - 1 \right]$$

$$= (x^2 + y^2 + z^2)^{-\frac{3}{2}} \left[\frac{3x^2 - x^2 - y^2 - z^2}{(x^2+y^2+z^2)} \right]$$

$$\frac{\partial^2 w}{\partial x^2} = (x^2 + y^2 + z^2)^{-\frac{3}{2}} \left[\frac{2x^2 - y^2 - z^2}{(x^2+y^2+z^2)} \right]$$

Similarly,

$$\frac{\partial^2 w}{\partial y^2} = (x^2 + y^2 + z^2)^{-\frac{3}{2}} \left[\frac{2y^2 - x^2 - z^2}{(x^2+y^2+z^2)} \right]$$

$$\frac{\partial^2 w}{\partial z^2} = (x^2 + y^2 + z^2)^{-\frac{3}{2}} \left[\frac{2z^2 - x^2 - y^2}{(x^2+y^2+z^2)} \right]$$

$$\frac{\partial^2 w}{\partial x^2} + \frac{\partial^2 w}{\partial y^2} + \frac{\partial^2 w}{\partial z^2}$$

$$= (x^2 + y^2 + z^2)^{-\frac{3}{2}} \left[\frac{2x^2 - y^2 - z^2}{(x^2+y^2+z^2)} + \frac{2y^2 - x^2 - z^2}{(x^2+y^2+z^2)} + \frac{2z^2 - x^2 - y^2}{(x^2+y^2+z^2)} \right]$$

$$= (x^2 + y^2 + z^2)^{-\frac{3}{2}} \left[\frac{2x^2 - 2x^2 + 2y^2 - 2y^2 + 2z^2 - 2z^2}{(x^2+y^2+z^2)} \right]$$

$$= (x^2 + y^2 + z^2)^{-\frac{3}{2}}[0]$$

$$\therefore \frac{\partial^2 w}{\partial x^2} + \frac{\partial^2 w}{\partial y^2} + \frac{\partial^2 w}{\partial z^2} = 0$$

7. If $V(x, y) = e^x(x \cos y - y \sin y)$ then

prove that $\frac{\partial^2 V}{\partial x^2} + \frac{\partial^2 V}{\partial y^2} = 0$

Solution:

$$V(x, y) = e^x(x \cos y - y \sin y)$$

$$V = xe^x(\cos y) - e^x y \sin y$$

$$\frac{\partial V}{\partial x} = (\cos y)[xe^x + e^x] - y \sin y e^x$$

$$\frac{\partial^2 V}{\partial x^2} = (\cos y)[xe^x + e^x + e^x] - y \sin y e^x$$

$$= (\cos y)[xe^x + 2e^x] - y \sin y e^x$$

$$= xe^x \cos y + 2e^x \cos y - y \sin y e^x$$

$$V = xe^x(\cos y) - e^x y \sin y$$

$$\frac{\partial V}{\partial y} = xe^x(-\sin y) - e^x(y \cos y + \sin y)$$

$$= -xe^x \sin y - e^x y \cos y - e^x \sin y$$

$$\frac{\partial^2 V}{\partial y^2} = -xe^x \cos y - e^x(-y \sin y + \cos y)$$

$$-e^x \cos y$$

$$= -xe^x \cos y - 2e^x \cos y + y \sin y e^x$$

$$\therefore \frac{\partial^2 V}{\partial x^2} + \frac{\partial^2 V}{\partial y^2} = 0$$

8. If $w(x, y) = xy + \sin(xy)$, then

prove that $\frac{\partial^2 w}{\partial y \partial x} = \frac{\partial^2 w}{\partial x \partial y}$

Solution:

$$w(x, y) = xy + \sin(xy)$$

$$\frac{\partial w}{\partial x} = y + \cos(xy)y$$

$$= y + y \cos(xy)$$

$$\frac{\partial^2 w}{\partial y \partial x} = 1 + y[-\sin(xy)x] + \cos(xy)(1)$$

$$= 1 - xy \sin(xy) + \cos(xy) \dots (i)$$

$$w(x, y) = xy + \sin(xy)$$

$$\frac{\partial w}{\partial y} = x + \cos(xy)x$$

$$= x + x \cos(xy)$$

$$\frac{\partial^2 w}{\partial x \partial y} = 1 + x[-\sin(xy)y] + \cos(xy)(1)$$

$$= 1 - xy \sin(xy) + \cos(xy) \dots (ii)$$

From (i) and (ii) $\frac{\partial^2 w}{\partial y \partial x} = \frac{\partial^2 w}{\partial x \partial y}$ is proved.

9. If $V(x, y, z) = x^3 + y^3 + z^3 + 3xyz$,

show that $\frac{\partial^2 v}{\partial y \partial z} = \frac{\partial^2 v}{\partial z \partial y}$

Solution:

$$V(x, y, z) = x^3 + y^3 + z^3 + 3xyz$$

$$\frac{\partial V}{\partial z} = 3z^2 + 3xy$$

$$\frac{\partial^2 v}{\partial y \partial z} = 3x \dots (i)$$

$$\frac{\partial V}{\partial y} = 3y^2 + 3xz$$

$$\frac{\partial^2 v}{\partial z \partial y} = 3x \dots (ii)$$

From (i) and (ii) $\frac{\partial^2 v}{\partial y \partial z} = \frac{\partial^2 v}{\partial z \partial y}$ is proved.

10. A firm produces two types of calculators each week, x number of type A and y number of type B. The weekly revenue and cost functions (in rupees) are $R(x, y) = 80x + 90y + 0.04xy - 0.05x^2 - 0.05y^2$ and $C(x, y) = 8x + 6y + 2000$ respectively.

(i) Find the profit function $P(x, y)$,

(ii) Find $\frac{\partial P}{\partial x}(1200, 1800)$ and $\frac{\partial P}{\partial y}$ at

$(1200, 1800)$ and interpret these results.

Solution:

Given Revenue = $R(x, y)$ and

Cost = $C(x, y)$

So, Profit $P(x, y) = R(x, y) - C(x, y)$

$$= (80x + 90y + 0.04xy - 0.05x^2 - 0.05y^2) - (8x + 6y + 2000)$$

(i) $P(x, y) = 72x + 84y + 0.04xy - 0.05x^2 - 0.05y^2 - 2000$

$$\frac{\partial P}{\partial x} = 72 + 0.04y - 0.1x$$

At $(1200, 1800)$

$$\frac{\partial P}{\partial x} = 72 + 0.04(1800) - 0.1(1200)$$

$$= 72 + 72.00 - 120.0$$

$$= 144 - 120$$

$$\frac{\partial P}{\partial x} = 24$$

$$\frac{\partial P}{\partial y} = 84 + 0.04x - 0.1y$$

At $(1200, 1800)$

$$\frac{\partial P}{\partial y} = 84 + 0.04(1200) - 0.1(1800)$$

$$= 84 + 48.00 - 180.0$$

$$= 132 - 180$$

$$\frac{\partial P}{\partial y} = -48$$

(ii) $\frac{\partial P}{\partial x} = 24$ and $\frac{\partial P}{\partial y} = -48$ At $(1200, 1800)$,

shows Profit increases when keeping y as constant.

Example 8.16

If $w(x, y, z) = x^2y + y^2z + z^2x$, $x, y, z \in \mathbb{R}$, find the differential dw .

Solution:

$$dw = w_x dx + w_y dy + w_z dz$$

Given $w(x, y, z) = x^2y + y^2z + z^2x$

$$w_x = 2xy + z^2$$

$$w_y = x^2 + 2yz$$

$$w_z = y^2 + 2zx$$

$$\therefore dw = (2xy + z^2)dx + (x^2 + 2yz)dy$$

$$+ (y^2 + 2zx)dz$$

Example 8.17

Let $U(x, y, z) = x^2 - xy + 3 \sin z$, $x, y, z \in \mathbb{R}$.

Find the linear approximation for U at

$(2, -1, 0)$.

Solution:

Linear approximation $L(x, y, z)$

$$= U(x_0, y_0, z_0) + U_x(x_0, y_0, z_0)(x - x_0)$$

$$+ U_y(x_0, y_0, z_0)(y - y_0) + U_z(x_0, y_0, z_0)(z - z_0)$$

$$L(x, y, z) = U(x_0, y_0, z_0) + \sum U_x(x_0, y_0, z_0)(x - x_0)$$

$$U(x, y, z) = x^2 - xy + 3 \sin z$$

$$\begin{aligned} U(x_0, y_0, z_0) &= U(2, -1, 0) \\ &= (2)^2 - (2)(-1) + 3 \sin(0) \\ &= 4 + 2 + 3(0) \end{aligned}$$

$$U(x_0, y_0, z_0) = 6$$

$$U_x = 2x - y$$

$$U_y = -x$$

$$U_z = 3 \cos z$$

$$\text{At } (2, -1, 0), U_x = 2(2) - (-1)$$

$$= 4 + 1$$

$$= 5$$

$$U_y = -(2)$$

$$= -2$$

$$U_z = 3 \cos(0)$$

$$= 3(1)$$

$$= 3$$

$$\therefore L(x, y, z) = 6 + 5(x - 2) - 2(y + 1) + 3(z - 0)$$

$$\begin{aligned} L(x, y, z) &= 6 + 5x - 10 - 2y - 2 + 3z \\ &= 5x - 2y + 3z - 6 \end{aligned}$$

The required linear approximation

for U at $(x_0, y_0, z_0) = 5x - 2y + 3z - 6$

EXERCISE 8.5

1. If $w(x, y) = x^3 - 3xy + 2y^2$, $x, y \in \mathbb{R}$, find the linear approximation for w at $(1, -1)$.

Solution: Linear approximation

$$L(x, y) = U(x_0, y_0) + \sum U_x(x_0, y_0)(x - x_0)$$

$$w(x, y) = x^3 - 3xy + 2y^2$$

$$w(x_0, y_0) = w(1, -1)$$

$$= (1)^3 - 3(1)(-1) + 2(-1)^2$$

$$= 1 + 3 + 2(1)$$

$$= 4 + 2$$

$$w(x_0, y_0) = 6$$

$$w_x = 3x^2 - 3y$$

$$w_y = -3x + 4y$$

At $(1, -1)$

$$w_x = 3(1)^2 - 3(-1)$$

$$= 3 + 3 = 6$$

$$w_y = -3(1) + 4(-1)$$

$$= -3 - 4$$

$$= -7$$

$$\therefore L(x, y, z) = 6 + 6(x - 1) - 7(y + 1)$$

$$= 6 + 6x - 6 - 7y - 7$$

$$= 6x - 7y - 7$$

The required linear approximation

for w at $(x_0, y_0) = 6x - 7y - 7$

2. Let $z(x, y) = x^2y + 3xy^4$, $x, y \in \mathbb{R}$. Find the linear approximation for z at $(2, -1)$.

Solution: Linear approximation

$$L(x, y) = U(x_0, y_0) + \sum U_x(x_0, y_0)(x - x_0)$$

$$z(x, y) = x^2y + 3xy^4$$

$$z(x_0, y_0) = z(2, -1)$$

$$= (2)^2(-1) + 3(2)(-1)^4$$

$$= 4(-1) + 6(1)$$

$$= -4 + 6$$

$$z(x_0, y_0) = 2$$

$$z_x = 2xy + 3y^4$$

$$z_y = x^2 + 12xy^3$$

At $(2, -1)$

$$z_x = 2(2)(-1) + 3(-1)^4$$

$$= -4 + 3$$

$$= -1$$

$$z_y = (2)^2 + 12(2)(-1)^3$$

$$= 4 - 24$$

$$= -20$$

$$\therefore L(x, y, z) = 2 - 1(x - 2) - 20(y + 1)$$

$$= 2 - x + 2 - 20y - 20$$

$$= -x - 20y - 16$$

The required linear approximation

for z at $(x_0, y_0) = -(x + 20y + 16)$

3. If $v(x, y) = x^2 - xy + \frac{1}{4}y^2 + 7, x, y \in \mathbb{R}$,
find the differential dv .

Solution: $dv = v_x dx + v_y dy$

$$\text{Given } v(x, y) = x^2 - xy + \frac{1}{4}y^2 + 7$$

$$v_x = 2x - y$$

$$v_y = -x + \frac{1}{4}(2y)$$

$$= -x + \frac{y}{2}$$

$$\therefore dv = (2x - y)dx + \left(-x + \frac{y}{2}\right)dy$$

4. Let $W(x, y, z) = x^2 - xy + 3 \sin z, x, y, z \in \mathbb{R}$.
Find the linear approximation at $(2, -1, 0)$

Solution: Example:- 8.17

5. Let $V(x, y, z) = xy + yz + zx, x, y, z \in \mathbb{R}$.
Find the differential dV

Solution:

$$dV = V_x dx + V_y dy + V_z dz$$

$$\text{Given } V(x, y, z) = xy + yz + zx$$

$$V_x = y + z$$

$$V_y = x + z$$

$$V_z = y + x$$

$$\therefore dV = (y + z)dx + (x + z)dy + (y + x)dz$$

Example 8.18 Verify $\frac{dF}{dt} = \frac{\partial F}{\partial x} \frac{dx}{dt} + \frac{\partial F}{\partial y} \frac{dy}{dt}$ for

$$F(x, y) = x^2 - 2y^2 + 2xy$$

$$\text{at } x(t) = \cos t, y(t) = \sin t, t \in [0, 2\pi].$$

Solution: $F(x, y) = x^2 - 2y^2 + 2xy$

$$\text{At } x(t) = \cos t, y(t) = \sin t$$

$$\begin{aligned} F(t) &= (\cos t)^2 - 2(\sin t)^2 + 2(\cos t)(\sin t) \\ &= \cos^2 t - 2\sin^2 t + 2 \sin t \cos t \end{aligned}$$

$$\begin{aligned} \frac{dF}{dt} &= 2 \cos t (-\sin t) - 2(2 \sin t \cos t) \\ &\quad + 2[\sin t (-\sin t) + \cos t (\cos t)] \\ &= -2 \cos t \sin t - 4 \cos t \sin t - 2\sin^2 t + 2\cos^2 t \end{aligned}$$

$$\frac{dF}{dt} = -6 \cos t \sin t - 2\sin^2 t + 2\cos^2 t. \quad (1)$$

$$F(x, y) = x^2 - 2y^2 + 2xy$$

$$\frac{\partial F}{\partial x} = 2x + 2y$$

$$\frac{\partial F}{\partial y} = -4y + 2x$$

$$x(t) = \cos t$$

$$\frac{dx}{dt} = -\sin t$$

$$y(t) = \sin t$$

$$\frac{dy}{dt} = \cos t$$

$$\begin{aligned} \frac{\partial F}{\partial x} \frac{dx}{dt} + \frac{\partial F}{\partial y} \frac{dy}{dt} &= (2x + 2y)(-\sin t) \\ &\quad + (-4y + 2x)(\cos t) \\ &= (2 \cos t + 2 \sin t)(-\sin t) \\ &\quad + (-4 \sin t + 2 \cos t)(\cos t) \\ &= -2 \cos t \sin t - 2\sin^2 t - 4 \cos t \sin t + 2\cos^2 t \\ &= -6 \cos t \sin t - 2\sin^2 t + 2\cos^2 t. \dots (2) \end{aligned}$$

From (1) and (2)

$$\frac{dF}{dt} = \frac{\partial F}{\partial x} \frac{dx}{dt} + \frac{\partial F}{\partial y} \frac{dy}{dt} \text{ is verified.}$$

Example 8.19

Let $g(x, y) = x^2 - yx + \sin(x + y)$,
 $x(t) = e^{3t}, y(t) = t^2, t \in \mathbb{R}$ Find $\frac{dg}{dt}$

Solution: $\frac{dg}{dt} = \frac{\partial g}{\partial x} \frac{dx}{dt} + \frac{\partial g}{\partial y} \frac{dy}{dt}$

$$g(x, y) = x^2 - yx + \sin(x + y)$$

$$\frac{\partial g}{\partial x} = 2x - y + \cos(x + y)$$

$$\frac{\partial g}{\partial y} = -x + \cos(x + y)$$

$$x(t) = e^{3t}$$

$$\frac{dx}{dt} = 3e^{3t}$$

$$y(t) = t^2$$

$$\frac{dy}{dt} = 2t$$

$$\begin{aligned} \therefore \frac{dg}{dt} &= \frac{\partial g}{\partial x} \frac{dx}{dt} + \frac{\partial g}{\partial y} \frac{dy}{dt} \\ &= (2x - y + \cos(x + y))(3e^{3t}) \\ &\quad + (-x + \cos(x + y))(2t) \\ &= (2e^{3t} - t^2 + \cos(e^{3t} + t^2))(3e^{3t}) \end{aligned}$$

$$\begin{aligned}
 &+(-e^{3t} + \cos(e^{3t} + t^2))(2t) \\
 &= 6e^{6t} - 3t^2e^{3t} + 3e^{3t}\cos(e^{3t} + t^2) \\
 &\quad - 2te^{3t} + 2t\cos(e^{3t} + t^2)
 \end{aligned}$$

Example 8.20

Let $g(x, y) = 2y + x^2$,

$x = 2r - s, y = r^2 + 2s, r, s \in \mathbb{R}$. Find $\frac{\partial g}{\partial r}, \frac{\partial g}{\partial s}$

Solution: (i) $\frac{\partial g}{\partial r} = \frac{\partial g}{\partial x} \frac{\partial x}{\partial r} + \frac{\partial g}{\partial y} \frac{\partial y}{\partial r}$

$$g(x, y) = 2y + x^2$$

$$\frac{\partial g}{\partial x} = 2x$$

$$\frac{\partial g}{\partial y} = 2$$

$$x = 2r - s$$

$$\frac{\partial x}{\partial r} = 2$$

$$y = r^2 + 2s$$

$$\frac{\partial y}{\partial r} = 2r$$

$$\therefore \frac{\partial g}{\partial r} = \frac{\partial g}{\partial x} \frac{\partial x}{\partial r} + \frac{\partial g}{\partial y} \frac{\partial y}{\partial r}$$

$$= (2x)(2) + (2)(2r)$$

$$= 4x + 4r$$

$$= 4(2r - s) + 4r$$

$$= 8r - 4s + 4r$$

$$\frac{\partial g}{\partial r} = 12r - 4s$$

$$(ii) \frac{\partial g}{\partial s} = \frac{\partial g}{\partial x} \frac{\partial x}{\partial s} + \frac{\partial g}{\partial y} \frac{\partial y}{\partial s}$$

$$g(x, y) = 2y + x^2$$

$$\frac{\partial g}{\partial x} = 2x$$

$$\frac{\partial g}{\partial y} = 2$$

$$x = 2r - s$$

$$\frac{\partial x}{\partial s} = -1$$

$$y = r^2 + 2s$$

$$\frac{\partial y}{\partial s} = 2$$

$$\therefore \frac{\partial g}{\partial s} = \frac{\partial g}{\partial x} \frac{\partial x}{\partial s} + \frac{\partial g}{\partial y} \frac{\partial y}{\partial s}$$

$$= (2x)(-1) + (2)(2)$$

$$= -2x + 4$$

$$= -2(2r - s) + 4$$

$$= -4r + 2s + 4r$$

$$\frac{\partial g}{\partial s} = -4r + 2s + 4$$

EXERCISE 8.6

1. If $u(x, y) = x^2y + 3xy^4, x = e^t$ and $y = \sin t$, find $\frac{du}{dt}$ and evaluate it at $t = 0$

Solution: $\frac{du}{dt} = \frac{\partial u}{\partial x} \frac{dx}{dt} + \frac{\partial u}{\partial y} \frac{dy}{dt}$

$$u(x, y) = x^2y + 3xy^4$$

$$\frac{\partial u}{\partial x} = 2xy + 3y^4$$

$$\frac{\partial u}{\partial y} = x^2 + 12xy^3$$

$$x(t) = e^t$$

$$\frac{dx}{dt} = e^t$$

$$y(t) = \sin t$$

$$\frac{dy}{dt} = \cos t$$

$$\frac{du}{dt} = \left(\frac{\partial u}{\partial x}\right)\left(\frac{dx}{dt}\right) + \left(\frac{\partial u}{\partial y}\right)\left(\frac{dy}{dt}\right)$$

$$= (2xy + 3y^4)(e^t) + (x^2 + 12xy^3)(\cos t)$$

$$= [2(e^t)(\sin t) + 3(\sin t)^4](e^t)$$

$$+ [(e^t)^2 + 12(e^t)(\sin t)^3](\cos t)$$

$$= 2e^{2t} \sin t + 3e^t \sin^4 t + e^{2t} \cos t + 12e^t \cos t \sin^3 t$$

$$\frac{du}{dt} = e^t(2e^t \sin t + 3\sin^4 t + e^t \cos t + 12 \cos t \sin^3 t)$$

at $t = 0$,

$$\frac{du}{dt} = e^0(2e^0 \sin 0 + 3\sin^4 0 + e^0 \cos 0 + 12 \cos 0 \sin^3 0)$$

$$= 1(0 + 0 + 1 + 0)$$

$$= 1(1)$$

$$\frac{du}{dt} = 1$$

2. If $u(x, y, z) = xy^2z^3$,

$x = \sin t, y = \cos t, z = 1 + e^{2t}$ find $\frac{du}{dt}$

Solution: $\frac{du}{dt} = \frac{\partial u}{\partial x} \frac{dx}{dt} + \frac{\partial u}{\partial y} \frac{dy}{dt} + \frac{\partial u}{\partial z} \frac{dz}{dt}$

$$u(x, y, z) = xy^2z^3$$

$$\frac{\partial u}{\partial x} = y^2z^3$$

$$\frac{\partial u}{\partial y} = x(2y)z^3 = 2xyz^3$$

$$\frac{\partial u}{\partial z} = xy^2(3z^2) = 3xy^2z^2$$

$$x = \sin t$$

$$\frac{dx}{dt} = \cos t$$

$$x = \cos t$$

$$\frac{dy}{dt} = -\sin t$$

$$z = 1 + 2e^{2t}$$

$$\frac{dz}{dt} = 2(e^{2t}) = 2e^{2t}$$

$$\begin{aligned} \frac{du}{dt} &= (y^2z^3)(\cos t) + (2xyz^3)(-\sin t) \\ &\quad + (3xy^2z^2)(2e^{2t}) \\ &= y^2z^3 \cos t - 2xyz^3 \sin t + 6xy^2z^2e^{2t} \\ &= yz^2(yz \cos t - 2xz \sin t + 6xye^{2t}) \\ &= \cos t (1 + e^{2t})^2 \end{aligned}$$

$$[\cos^2 t (1 + e^{2t}) - 2\sin^2 t (1 + e^{2t}) + 6 \sin t \cos t e^{2t}]$$

3. If $w(x, y, z) = x^2 + y^2 + z^2$,

$x = e^t, y = e^t \sin t, z = e^t \cos t$, find $\frac{dw}{dt}$

Solution: $\frac{dw}{dt} = \frac{\partial w}{\partial x} \frac{dx}{dt} + \frac{\partial w}{\partial y} \frac{dy}{dt} + \frac{\partial w}{\partial z} \frac{dz}{dt}$

$$w(x, y, z) = x^2 + y^2 + z^2$$

$$\frac{\partial w}{\partial x} = 2x$$

$$\frac{\partial w}{\partial y} = 2y$$

$$\frac{\partial w}{\partial z} = 2z$$

$$x = e^t$$

$$\frac{dx}{dt} = e^t$$

$$y = e^t \sin t$$

$$\frac{dy}{dt} = e^t \cos t + \sin t e^t$$

$$z = e^t \cos t$$

$$\frac{dz}{dt} = e^t(-\sin t) + \cos t e^t$$

$$\frac{\partial w}{\partial x} \frac{dx}{dt} = (2x)(e^t)$$

$$= (2e^t)e^t$$

$$= 2e^{2t}$$

$$\frac{\partial w}{\partial y} \frac{dy}{dt} = (2y)(e^t \cos t + \sin t e^t)$$

$$= (2e^t \sin t)(e^t \cos t + \sin t e^t)$$

$$= (2e^{2t} \sin t)(\cos t + \sin t)$$

$$\frac{\partial w}{\partial z} \frac{dz}{dt} = (2e^t \cos t)(-e^t \sin t + \cos t e^t)$$

$$= (2e^{2t} \cos t)(-\sin t + \cos t)$$

$$\frac{dw}{dt} = \frac{\partial w}{\partial x} \frac{dx}{dt} + \frac{\partial w}{\partial y} \frac{dy}{dt} + \frac{\partial w}{\partial z} \frac{dz}{dt}$$

$$= 2e^{2t}(1 + \sin t \cos t + \sin^2 t - \sin t \cos t + \cos^2 t)$$

$$= 2e^{2t}(1 + \sin^2 t + \cos^2 t)$$

$$= 2e^{2t}(1 + 1)$$

$$= 2e^{2t}(2)$$

$$\frac{dw}{dt} = 4e^{2t}$$

4. Let $U(x, y, z) = xyz$,

$x = e^{-t}, y = e^{-t} \cos t, z = \sin t, t \in \mathbb{R}$, find $\frac{dU}{dt}$

Solution: $\frac{dU}{dt} = \frac{\partial U}{\partial x} \frac{dx}{dt} + \frac{\partial U}{\partial y} \frac{dy}{dt} + \frac{\partial U}{\partial z} \frac{dz}{dt}$

$$U(x, y, z) = xyz$$

$$\frac{\partial U}{\partial x} = yz$$

$$\frac{\partial U}{\partial y} = xz$$

$$\frac{\partial U}{\partial z} = xy$$

$$x = e^{-t}$$

$$\frac{dx}{dt} = -e^{-t}$$

$$y = e^{-t} \cos t$$

$$\frac{dy}{dt} = e^{-t}(-\sin t) + \cos t(-e^{-t})$$

$$= e^{-t}(-\sin t - \cos t)$$

$$= -e^{-t}(\sin t + \cos t)$$

$$z = \sin t$$

$$\frac{dz}{dt} = \cos t$$

$$\frac{\partial U}{\partial x} \frac{dx}{dt} = (yz)(-e^{-t})$$

$$= (e^{-t} \cos t)(\sin t)(-e^{-t})$$

$$= -e^{-2t} \cos t \sin t$$

$$\begin{aligned}\frac{\partial U}{\partial y} \frac{dy}{dt} &= (xz)[-e^{-t}(\sin t + \cos t)] \\ &= (e^{-t})(\sin t)[-e^{-t}(\sin t + \cos t)] \\ &= (-e^{-2t})(\sin^2 t + \sin t \cos t)\end{aligned}$$

$$\begin{aligned}\frac{\partial w}{\partial z} \frac{dz}{dt} &= (e^{-t})(e^{-t} \cos t)(\cos t) \\ &= (e^{-2t})(\cos^2 t)\end{aligned}$$

$$\begin{aligned}\frac{dw}{dt} &= \frac{\partial w}{\partial x} \frac{dx}{dt} + \frac{\partial w}{\partial y} \frac{dy}{dt} + \frac{\partial w}{\partial z} \frac{dz}{dt} \\ &= -e^{-2t}(\sin t \cos t + \sin^2 t + \sin t \cos t - \cos^2 t) \\ &= -e^{-2t}[2\sin t \cos t - (\cos^2 t - \sin^2 t)] \\ \frac{dw}{dt} &= -e^{-2t}(\sin 2t - \cos 2t)\end{aligned}$$

5. If $w(x, y) = 6x^3 - 3xy + 2y^2$,
 $x = e^s, y = \cos s, s \in \mathbb{R}$,
 find $\frac{dw}{ds}$, and evaluate at $s = 0$

Solution: $\frac{dw}{ds} = \frac{\partial w}{\partial x} \frac{dx}{ds} + \frac{\partial w}{\partial y} \frac{dy}{ds}$

$$w(x, y) = 6x^3 - 3xy + 2y^2$$

$$\frac{\partial w}{\partial x} = 18x^2 - 3y$$

$$\frac{\partial w}{\partial y} = -3x + 4y$$

$$x = e^s$$

$$\frac{dx}{ds} = e^s$$

$$y = \cos s$$

$$\frac{dy}{ds} = -\sin s$$

$$\frac{\partial w}{\partial x} \frac{dx}{ds} = (18x^2 - 3y)(e^s)$$

$$= [18(e^s)^2 - 3(\cos s)]e^s$$

$$= e^s[18(e^{2s}) - 3(\cos s)]$$

$$\frac{\partial w}{\partial y} \frac{dy}{ds} = (-3x + 4y)(-\sin s)$$

$$= [-3(e^s) + 4(\cos s)](-\sin s)$$

$$= (2e^{2t} \sin t)(\cos t + \sin t)$$

$$\frac{dw}{ds} = \frac{\partial w}{\partial x} \frac{dx}{ds} + \frac{\partial w}{\partial y} \frac{dy}{ds}$$

$$= 18e^{3s} - 3\cos s e^s + 3e^s \sin s - 4\sin s \cos s$$

$$\text{at } s = 0$$

$$\begin{aligned}\frac{dw}{dt} &= 18e^0 - 3\cos 0 e^0 + 3e^0 \sin 0 - 4\sin 0 \cos 0 \\ &= 18 - 3 + 0 - 0 \\ \frac{dw}{dt} &= 15\end{aligned}$$

6. If $z(x, y) = x \tan^{-1}(xy)$,

$x = t^2, y = se^t, s, t \in \mathbb{R}$, find $\frac{\partial z}{\partial s}$ and $\frac{\partial z}{\partial t}$
 at $s = t = 1$.

Solution: (i) $\frac{\partial z}{\partial s} = \frac{\partial z}{\partial x} \frac{\partial x}{\partial s} + \frac{\partial z}{\partial y} \frac{\partial y}{\partial s}$

$$z(x, y) = x \tan^{-1}(xy)$$

$$\frac{\partial z}{\partial x} = x \left[\frac{1}{1+(xy)^2} (y) \right] + \tan^{-1}(xy)(1)$$

$$= x \left(\frac{y}{1+x^2y^2} \right) + \tan^{-1}(xy)$$

$$\frac{\partial z}{\partial x} = \frac{xy}{1+x^2y^2} + \tan^{-1}(xy)$$

$$x = t^2$$

$$\frac{\partial x}{\partial s} = 0$$

$$\frac{\partial z}{\partial x} \frac{\partial x}{\partial s} = \left[\frac{xy}{1+x^2y^2} + \tan^{-1}(xy) \right] (0)$$

$$\frac{\partial z}{\partial x} \frac{\partial x}{\partial s} = 0$$

$$z(x, y) = x \tan^{-1}(xy)$$

$$\frac{\partial z}{\partial y} = x \left[\frac{1}{1+(xy)^2} (x) \right] + \tan^{-1}(xy)(0)$$

$$= \left(\frac{x^2}{1+(xy)^2} \right)$$

$$y = se^t$$

$$\frac{\partial y}{\partial s} = e^t$$

$$\frac{\partial z}{\partial x} \frac{\partial y}{\partial s} = \left(\frac{x^2}{1+(xy)^2} \right) (e^t)$$

at $x = t^2, y = se^t$.

$$\frac{\partial z}{\partial s} = \left(\frac{t^4}{1+(t^2se^t)^2} \right) (e^t)$$

$$\frac{\partial z}{\partial s} = \frac{t^4 e^t}{1+(t^2se^t)^2}$$

at $s = t = 1$.

$$\frac{\partial z}{\partial s} = \frac{e}{1+e^2}$$

(ii) $\frac{\partial z}{\partial t} = \frac{\partial z}{\partial x} \frac{\partial x}{\partial t} + \frac{\partial z}{\partial y} \frac{\partial y}{\partial t}$

$$z(x, y) = x \tan^{-1}(xy)$$

$$\frac{\partial z}{\partial x} = x \left[\frac{1}{1+(xy)^2} (y) \right] + \tan^{-1}(xy)(1)$$

$$= x \left(\frac{y}{1+x^2y^2} \right) + \tan^{-1}(xy)$$

$$\frac{\partial z}{\partial x} = \frac{xy}{1+x^2y^2} + \tan^{-1}(xy)$$

$$x = t^2$$

$$\frac{\partial x}{\partial t} = 2t$$

$$\frac{\partial z}{\partial x} \frac{\partial x}{\partial t} = \left[\frac{xy}{1+x^2y^2} + \tan^{-1}(xy) \right] (2t)$$

at $s = t = 1$ then $x = 1, y = e$.

$$\frac{\partial z}{\partial x} \frac{\partial x}{\partial t} = \left[\frac{e}{1+e^2} + \tan^{-1}(e) \right] (2)$$

$$= \left[\frac{2e}{1+e^2} + 2\tan^{-1}(e) \right]$$

$$z(x, y) = x \tan^{-1}(xy)$$

$$\frac{\partial z}{\partial y} = x \left[\frac{1}{1+(xy)^2} (x) \right] + \tan^{-1}(xy)(0)$$

$$= \left(\frac{xy}{1+(xy)^2} \right)$$

$$y = se^t$$

$$\frac{\partial y}{\partial t} = se^t$$

$$\frac{\partial z}{\partial y} \frac{\partial y}{\partial t} = \left(\frac{xy}{1+(xy)^2} \right) (se^t)$$

at $s = t = 1$ then $x = 1, y = e$

$$\frac{\partial z}{\partial y} \frac{\partial y}{\partial t} = \left(\frac{1}{1+e^2} \right) (e)$$

$$= \frac{e}{1+e^2}$$

$$\frac{\partial z}{\partial t} = \frac{\partial z}{\partial x} \frac{\partial x}{\partial t} + \frac{\partial z}{\partial y} \frac{\partial y}{\partial t}$$

$$\frac{\partial z}{\partial t} = \frac{2e}{1+e^2} + 2\tan^{-1}(e) + \frac{e}{1+e^2}$$

$$\frac{\partial z}{\partial t} = \frac{3e}{1+e^2} + 2\tan^{-1}(e)$$

7. Let $U(x, y) = e^x \sin y$ where

$$x = st^2, y = s^2t, s, t \in \mathbb{R},$$

find $\frac{\partial U}{\partial s}, \frac{\partial U}{\partial t}$ and evaluate them at $s = t = 1$.

Solution:

$$\frac{\partial U}{\partial s} = \frac{\partial U}{\partial x} \frac{\partial x}{\partial s} + \frac{\partial U}{\partial y} \frac{\partial y}{\partial s}$$

$$U(x, y) = e^x \sin y$$

$$\frac{\partial U}{\partial x} = \sin y e^x$$

$$x = st^2$$

$$\frac{\partial x}{\partial s} = t^2$$

$$\frac{\partial U}{\partial x} \frac{\partial x}{\partial s} = (\sin y e^x)(t^2)$$

$$s = t = 1 \text{ then } x = 1, y = 1$$

$$\frac{\partial U}{\partial x} \frac{\partial x}{\partial s} = [\sin(1) e^1](1)$$

$$= e \sin(1)$$

$$\frac{\partial U}{\partial y} = e^x \cos y$$

$$y = s^2t$$

$$\frac{\partial y}{\partial s} = 2st$$

$$\frac{\partial U}{\partial y} \frac{\partial y}{\partial s} = (e^x \cos y)(2st)$$

$$s = t = 1 \text{ then } x = 1, y = 1$$

$$\frac{\partial U}{\partial y} \frac{\partial y}{\partial s} = [e \cos(1)](2)$$

$$= 2e \cos(1)$$

$$\frac{\partial U}{\partial s} = e \sin(1) + 2e \cos(1)$$

8. Let $z(x, y) = x^3 - 3x^2y^3$ where $x = se^t$,

$y = se^{-t}, s, t \in \mathbb{R}$, find $\frac{\partial z}{\partial s}$ and $\frac{\partial z}{\partial t}$

Solution:

$$\frac{\partial z}{\partial s} = \frac{\partial z}{\partial x} \frac{\partial x}{\partial s} + \frac{\partial z}{\partial y} \frac{\partial y}{\partial s}$$

$$z(x, y) = x^3 - 3x^2y^3$$

$$\frac{\partial z}{\partial x} = 3x^2 - 6xy^3$$

$$x = se^t$$

$$\frac{\partial x}{\partial s} = e^t$$

$$\frac{\partial z}{\partial x} \frac{\partial x}{\partial s} = (3x^2 - 6xy^3)(e^t)$$

$$= [3(se^t)^2 - 6(se^t)(se^{-t})^3](e^t)$$

$$\begin{aligned}
 &= (3s^2 e^{2t} - 6se^t s^3 e^{-3t})(e^t) \\
 &= (3s^2 e^{2t} - 6s^4 e^{-2t})(e^t) \\
 \frac{\partial z}{\partial x} \frac{\partial x}{\partial s} &= (3s^2 e^t)(e^{2t} - 2s^2 e^{-2t})
 \end{aligned}$$

$$z(x, y) = x^3 - 3x^2 y^3$$

$$\frac{\partial z}{\partial y} = -9x^2 y^2$$

$$y = se^{-t}$$

$$\frac{\partial y}{\partial s} = e^{-t}$$

$$\frac{\partial z}{\partial y} \frac{\partial y}{\partial s} = (-9x^2 y^2)(e^{-t})$$

$$= [-9(x)^2 (y)^2](e^{-t})$$

$$= [-9(se^t)^2 (se^{-t})^2](e^{-t})$$

$$= -9s^2 e^{2t} s^2 e^{-2t} e^{-t}$$

$$= -9s^4 e^{-t}$$

$$\frac{\partial z}{\partial s} = (3s^2 e^t)(e^{2t} - 2s^2 e^{-2t}) + (-9s^4 e^{-t})$$

$$= (3s^2 e^t)(e^{2t} - 2s^2 e^{-2t} - 3s^2 e^{-2t})$$

$$9. W(x, y, z) = xy + yz + zx,$$

$$x = u - v, y = uv, z = u + v, u, v \in \mathbb{R}.$$

$$\text{Find } \frac{dw}{du}, \frac{dw}{dv} \text{ and } \frac{dw}{dz} \text{ evaluate them at } \left(\frac{1}{2}, 1\right)$$

$$\text{Solution: (i) } \frac{dw}{du} = \frac{\partial w}{\partial x} \frac{\partial x}{\partial u} + \frac{\partial w}{\partial y} \frac{\partial y}{\partial u} + \frac{\partial w}{\partial z} \frac{\partial z}{\partial u}$$

$$W(x, y, z) = xy + yz + zx$$

$$\frac{\partial w}{\partial x} = y + z$$

$$\frac{\partial w}{\partial y} = x + z$$

$$\frac{\partial w}{\partial z} = y + x$$

$$x = u - v$$

$$\frac{\partial x}{\partial u} = 1$$

$$y = uv$$

$$\frac{\partial y}{\partial u} = v$$

$$z = u + v$$

$$\frac{\partial z}{\partial u} = 1$$

$$\begin{aligned}
 \frac{dw}{du} &= \left(\frac{\partial w}{\partial x}\right)\left(\frac{\partial x}{\partial u}\right) + \left(\frac{\partial w}{\partial y}\right)\left(\frac{\partial y}{\partial u}\right) + \left(\frac{\partial w}{\partial z}\right)\left(\frac{\partial z}{\partial u}\right) \\
 &= (y + z)(1) + (x + z)(v) + (y + x)(1) \\
 &= (y + z) + (x + z)(v) + (y + x)
 \end{aligned}$$

$$\text{At } x = u - v, y = uv, z = u + v$$

$$\begin{aligned}
 \frac{dw}{du} &= (uv + u + v) + (u - v + u + v)(v) \\
 &\quad + (uv + u - v) \\
 &= (uv + u + v) + (2u)(v) + (uv + u - v) \\
 &= uv + u + v + 2uv + uv + u - v
 \end{aligned}$$

$$\frac{dw}{du} = 4uv + 2u$$

$$\text{at } \left(\frac{1}{2}, 1\right)$$

$$\begin{aligned}
 \frac{dw}{du} &= 4\left(\frac{1}{2}\right)(1) + 2\left(\frac{1}{2}\right) \\
 &= 2 + 1
 \end{aligned}$$

$$\frac{dw}{du} = 3$$

$$(ii) \frac{dw}{dv} = \frac{\partial w}{\partial x} \frac{\partial x}{\partial v} + \frac{\partial w}{\partial y} \frac{\partial y}{\partial v} + \frac{\partial w}{\partial z} \frac{\partial z}{\partial v}$$

$$W(x, y, z) = xy + yz + zx$$

$$\frac{\partial w}{\partial x} = y + z$$

$$\frac{\partial w}{\partial y} = x + z$$

$$\frac{\partial w}{\partial z} = y + x$$

$$x = u - v$$

$$\frac{\partial x}{\partial v} = -1$$

$$y = uv$$

$$\frac{\partial y}{\partial v} = u$$

$$z = u + v$$

$$\frac{\partial z}{\partial v} = 1$$

$$\begin{aligned}
 \frac{dw}{dv} &= \left(\frac{\partial w}{\partial x}\right)\left(\frac{\partial x}{\partial v}\right) + \left(\frac{\partial w}{\partial y}\right)\left(\frac{\partial y}{\partial v}\right) + \left(\frac{\partial w}{\partial z}\right)\left(\frac{\partial z}{\partial v}\right) \\
 &= (y + z)(-1) + (x + z)(u) + (y + x)(1) \\
 &= -(y + z) + (x + z)(u) + (y + x)
 \end{aligned}$$

$$\text{At } x = u - v, y = uv, z = u + v$$

$$\begin{aligned}\frac{dw}{dv} &= -(uv + u + v) + (u - v + u + v)(u) \\ &\quad + (uv + u - v) \\ &= -(uv + u + v) + (2u)(u) + (uv + u - v) \\ &= -uv - u - v + 2u^2 + uv + u - v\end{aligned}$$

$$\frac{dw}{dv} = 2u^2 - 2v$$

$$\text{at } \left(\frac{1}{2}, 1\right)$$

$$\frac{dw}{dv} = 2\left(\frac{1}{2}\right)^2 - 2(1)$$

$$= 2\left(\frac{1}{4}\right) - 2$$

$$= \frac{1}{2} - 2$$

$$= \frac{1-4}{2}$$

$$\frac{dw}{dv} = -\frac{3}{2}$$

Example 8.21

Show that $F(x, y) = \frac{x^2 + 5xy - 10y^2}{3x + 7y}$ is a homogeneous function of degree 1.

Solution:

$$\text{Given } F(x, y) = \frac{x^2 + 5xy - 10y^2}{3x + 7y}$$

$$\therefore F(tx, ty) = \frac{(tx)^2 + 5(tx)(ty) - 10(ty)^2}{3(tx) + 7(ty)}$$

$$= \frac{t^2x^2 + 5t^2xy - 10t^2y^2}{3tx + 7ty}$$

$$= \frac{t^2(x^2 + 5xy - 10y^2)}{t(3x + 7y)}$$

$$= \frac{t(x^2 + 5xy - 10y^2)}{(3x + 7y)}$$

$$= t F(x, y) \text{ for all } t \in \mathbb{R}.$$

So, F is a homogeneous function of degree 1.

Example 8.22 If $u = \sin^{-1}\left(\frac{x+y}{\sqrt{x}+\sqrt{y}}\right)$,

Show that $x \frac{\partial u}{\partial x} + y \frac{\partial u}{\partial y} = \frac{1}{2} \tan u$

Solution:

$$\text{Given } u = \sin^{-1}\left(\frac{x+y}{\sqrt{x}+\sqrt{y}}\right)$$

$$\sin u(x, y) = \left(\frac{x+y}{\sqrt{x}+\sqrt{y}}\right)$$

$$\sin u(tx, ty) = \left(\frac{tx+ty}{\sqrt{tx}+\sqrt{ty}}\right)$$

$$= \frac{t(x+y)}{\sqrt{t}(\sqrt{x}+\sqrt{y})}$$

$$= \frac{\sqrt{t}\sqrt{t}(x+y)}{\sqrt{t}(\sqrt{x}+\sqrt{y})}$$

$$= \frac{\sqrt{t}(x+y)}{(\sqrt{x}+\sqrt{y})}$$

$$= \frac{t^{\frac{1}{2}}(x+y)}{(\sqrt{x}+\sqrt{y})}$$

$$= t^{\frac{1}{2}} \sin u(x, y) \text{ for all } t \in \mathbb{R}.$$

So, $\sin u(x, y)$ is a homogeneous function of degree $\frac{1}{2}$

By Euler's theorem,

$$x \frac{\partial f}{\partial x} + y \frac{\partial f}{\partial y} = \frac{1}{2} f$$

Now, substituting $f = \sin u$ in the above equation, we get

$$x \frac{\partial(\sin u)}{\partial x} + y \frac{\partial(\sin u)}{\partial y} = \frac{1}{2} \sin u$$

$$x \cos u \frac{\partial u}{\partial x} + y \cos u \frac{\partial u}{\partial y} = \frac{1}{2} \sin u$$

$$\cos u \left(x \frac{\partial u}{\partial x} + y \frac{\partial u}{\partial y} \right) = \frac{1}{2} \sin u$$

$$x \frac{\partial u}{\partial x} + y \frac{\partial u}{\partial y} = \frac{1}{2} \frac{\sin u}{\cos u}$$

$$x \frac{\partial u}{\partial x} + y \frac{\partial u}{\partial y} = \frac{1}{2} \tan u \quad \text{Proved.}$$

EXERCISE 8.7

1. In each of the following cases, determine whether the following function is homogeneous or not.

If it is so, find the degree.

(i) $f(x, y) = x^2y + 6x^3 + 7$

Solution:

$$f(x, y) = x^2y + 6x^3 + 7$$

$$f(tx, ty) = (tx)^2(ty) + 6(tx)^3 + 7$$

$$= t^2x^2ty + 6t^3x^3 + 7$$

$$= t^3x^2y + 6t^3x^3 + 7$$

It is not homogeneous function.

(ii) $h(x, y) = \frac{6x^2y^3 - \pi y^5 + 9x^4y}{2020x^2 + 2019y^2}$

$$h(tx, ty) = \frac{6(tx)^2(ty)^3 - \pi(ty)^5 + 9(tx)^4(ty)}{2020(tx)^2 + 2019(ty)^2}$$

$$\begin{aligned}
&= \frac{6t^2x^2y^3 - \pi t^5y^5 + 9t^4x^4ty}{2020t^2x^2 + 2019t^2y^2} \\
&= \frac{6t^5x^2y^3 - \pi t^5y^5 + 9t^5x^4y}{2020t^2x^2 + 2019t^2y^2} \\
&= \frac{t^5(6x^2y^3 - \pi y^5 + 9x^4y)}{t^2(2020x^2 + 2019y^2)} \\
&= \frac{t^3(6x^2y^3 - \pi y^5 + 9x^4y)}{(2020x^2 + 2019y^2)} \\
&= t^3h(x, y)
\end{aligned}$$

It is a homogeneous function of degree 3.

$$\begin{aligned}
\text{(iii)} \quad g(x, y, z) &= \frac{\sqrt{3x^2 + 5y^2 + z^2}}{4x + 7y} \\
g(tx, ty, tz) &= \frac{\sqrt{3(tx)^2 + 5(ty)^2 + (tz)^2}}{4(tx) + 7(ty)} \\
&= \frac{\sqrt{3t^2x^2 + 5t^2y^2 + t^2z^2}}{4tx + 7ty} \\
&= \frac{\sqrt{t^2(3x^2 + 5y^2 + z^2)}}{t(4x + 7y)} \\
&= \frac{t\sqrt{(3x^2 + 5y^2 + z^2)}}{t(4x + 7y)} \\
&= \frac{t^0\sqrt{(3x^2 + 5y^2 + z^2)}}{(4x + 7y)}
\end{aligned}$$

It is a homogeneous function of degree 0.

$$\begin{aligned}
\text{(iv)} \quad u(x, y, z) &= xy + \sin\left(\frac{y^2 - 2z^2}{xy}\right) \\
u(tx, ty, tz) &= txt + \sin\left(\frac{(ty)^2 - 2(tz)^2}{(tx)(ty)}\right) \\
&= t^2xy + \sin\left(\frac{t^2y^2 - 2t^2z^2}{t^2xy}\right) \\
&= t^2xy + \sin\left[\frac{t^2(y^2 - 2z^2)}{t^2(xy)}\right] \\
&= t^2xy + \sin\left(\frac{y^2 - 2z^2}{xy}\right)
\end{aligned}$$

It is not homogeneous function.

2. Prove that $f(x, y) = x^3 - 2x^2y + 3xy^2 + y^3$ is homogeneous; what is the degree? Verify Euler's Theorem for f .

Solution:

$$\begin{aligned}
f(x, y) &= x^3 - 2x^2y + 3xy^2 + y^3 \\
f(tx, ty) &= (tx)^3 - 2(tx)^2(ty) + 3(tx)(ty)^2 + (ty)^3 \\
&= t^3x^3 - 2t^2x^2ty + 3txt^2y^2 + t^3y^3 \\
&= t^3x^3 - 2t^3x^2y + 3t^3xy^2 + t^3y^3 \\
&= t^3(x^3 - 2x^2y + 3xy^2 + y^3)
\end{aligned}$$

It is a homogeneous function of degree 3.

By Euler's theorem, $x \frac{\partial f}{\partial x} + y \frac{\partial f}{\partial y} = 3f$

To verify,

$$f(x, y) = x^3 - 2x^2y + 3xy^2 + y^3$$

$$\frac{\partial f}{\partial x} = 3x^2 - 4xy + 3y^2$$

$$\frac{\partial f}{\partial y} = -2x^2 + 6xy + 3y^2$$

$$x \frac{\partial f}{\partial x} + y \frac{\partial f}{\partial y} = x(3x^2 - 4xy + 3y^2)$$

$$\begin{aligned}
&+ y(-2x^2 + 6xy + 3y^2) \\
&= 3x^3 - 4x^2y + 3xy^2 + y^3 - 2x^2y + 6xy^2 + 3y^3 \\
&= 3x^3 - 6x^2y + 9xy^2 + 3y^3 \\
&= 3(x^3 - 2x^2y + 3xy^2 + y^3) \\
&= 3f
\end{aligned}$$

Hence Euler's theorem, $x \frac{\partial f}{\partial x} + y \frac{\partial f}{\partial y} = 3f$

is verified.

3. Prove that $g(x, y) = x \log\left(\frac{y}{x}\right)$ is

homogeneous; what is the degree?

Verify Euler's Theorem for g .

Solution:

$$g(x, y) = x \log\left(\frac{y}{x}\right)$$

$$g(tx, ty) = tx \log\left(\frac{ty}{tx}\right)$$

$$= tx \log\left(\frac{y}{x}\right)$$

$$= tg(x, y)$$

It is a homogeneous function of degree 1.

By Euler's theorem, $x \frac{\partial g}{\partial x} + y \frac{\partial g}{\partial y} = g$

$$g(x, y) = x \log\left(\frac{y}{x}\right)$$

$$\frac{\partial g}{\partial x} = x \left[\frac{1}{\left(\frac{y}{x}\right)} y \left(\frac{-1}{x^2}\right) \right] + \log\left(\frac{y}{x}\right)$$

$$= x \left[\left(\frac{x}{y}\right) \left(\frac{-y}{x^2}\right) \right] + \log\left(\frac{y}{x}\right)$$

$$= \left[\left(\frac{x^2}{y}\right) \left(\frac{-y}{x^2}\right) \right] + \log\left(\frac{y}{x}\right)$$

$$\frac{\partial g}{\partial x} = -1 + \log\left(\frac{y}{x}\right)$$

$$g(x, y) = x \log\left(\frac{y}{x}\right)$$

$$\frac{\partial g}{\partial y} = x \left[\frac{1}{\left(\frac{y}{x}\right)} \left(\frac{1}{x}\right) (1) \right] + \log\left(\frac{y}{x}\right) (0)$$

$$= x \left[\left(\frac{x}{y}\right) \left(\frac{1}{x}\right) \right] + 0$$

$$= \frac{x}{y}$$

$$\therefore x \frac{\partial g}{\partial x} + y \frac{\partial g}{\partial y} = x \left[-1 + \log\left(\frac{y}{x}\right) \right] + y \left(\frac{x}{y}\right)$$

$$= -x + x \log\left(\frac{y}{x}\right) + x$$

$$x \frac{\partial g}{\partial x} + y \frac{\partial g}{\partial y} = x \log\left(\frac{y}{x}\right)$$

Hence, $x \frac{\partial g}{\partial x} + y \frac{\partial g}{\partial y} = g$ is verified.

4. If $u(x, y) = \frac{x^2 + y^2}{\sqrt{x+y}}$, prove that

$$x \frac{\partial u}{\partial x} + y \frac{\partial u}{\partial y} = \frac{3}{2} u$$

Solution:

$$u(x, y) = \frac{x^2 + y^2}{\sqrt{x+y}}$$

$$u(tx, ty) = \frac{(tx)^2 + (ty)^2}{\sqrt{(tx) + (ty)}}$$

$$= \frac{t^2 x^2 + t^2 y^2}{\sqrt{t(x+y)}}$$

$$= \frac{t^2 (x^2 + y^2)}{\sqrt{t(x+y)}}$$

$$= \frac{t^2 (x^2 + y^2)}{\sqrt{t} \sqrt{(x+y)}}$$

$$= \frac{t^{\frac{3}{2}} (x^2 + y^2)}{\sqrt{(x+y)}}$$

$$= t^{\frac{3}{2}} u(x, y)$$

It is a homogeneous function of degree $\frac{3}{2}$.

Hence by Euler's theorem $x \frac{\partial u}{\partial x} + y \frac{\partial u}{\partial y} = \frac{3}{2} u$

5. If $v(x, y) = \log\left(\frac{x^2 + y^2}{x+y}\right)$, prove that

$$x \frac{\partial v}{\partial x} + y \frac{\partial v}{\partial y} = 1$$

Solution:

$$v(x, y) = \log\left(\frac{x^2 + y^2}{x+y}\right)$$

$$e^v = \frac{x^2 + y^2}{x+y}$$

$$\text{Let } f = e^v(tx, ty) = \frac{(tx)^2 + (ty)^2}{(tx) + (ty)}$$

$$= \frac{t^2(x^2 + y^2)}{t(x+y)}$$

$$= \frac{t(x^2 + y^2)}{(x+y)}$$

It is a homogeneous function of degree 1.

By Euler's theorem, $x \frac{\partial f}{\partial x} + y \frac{\partial f}{\partial y} = f$

But $f = e^v$

$$\text{So, } x \frac{\partial e^v}{\partial x} + y \frac{\partial e^v}{\partial y} = e^v$$

$$x e^v \frac{\partial v}{\partial x} + y e^v \frac{\partial v}{\partial y} = e^v$$

$$e^v \left(x \frac{\partial v}{\partial x} + y \frac{\partial v}{\partial y} \right) = e^v$$

$$x \frac{\partial v}{\partial x} + y \frac{\partial v}{\partial y} = \frac{e^v}{e^v} = 1 \text{ is proved.}$$

6. If $w(x, y, z) = \log\left(\frac{5x^3y^4 + 7y^2xz^4 - 75y^3z^4}{x^2 + y^2}\right)$,

$$\text{find } x \frac{\partial w}{\partial x} + y \frac{\partial w}{\partial y} + z \frac{\partial w}{\partial z}$$

Solution:

$$w(x, y, z) = \log\left(\frac{5x^3y^4 + 7y^2xz^4 - 75y^3z^4}{x^2 + y^2}\right)$$

$$e^w = \frac{5x^3y^4 + 7y^2xz^4 - 75y^3z^4}{x^2 + y^2}$$

$$\text{Let } f = e^w = \frac{5x^3y^4 + 7y^2xz^4 - 75y^3z^4}{x^2 + y^2}$$

It is a homogeneous function of degree 5.

By Euler's theorem, $x \frac{\partial f}{\partial x} + y \frac{\partial f}{\partial y} + z \frac{\partial f}{\partial z} = 5f$

But $f = e^w$

$$\text{So, } x \frac{\partial e^w}{\partial x} + y \frac{\partial e^w}{\partial y} + z \frac{\partial e^w}{\partial z} = 5e^w$$

$$x e^w \frac{\partial w}{\partial x} + y e^w \frac{\partial w}{\partial y} + z e^w \frac{\partial w}{\partial z} = 5e^w$$

$$e^w \left(x \frac{\partial w}{\partial x} + y \frac{\partial w}{\partial y} + z \frac{\partial w}{\partial z} \right) = 5e^w$$

$$\therefore x \frac{\partial w}{\partial x} + y \frac{\partial w}{\partial y} + z \frac{\partial w}{\partial z} = 5$$

with an error of 0.1 cm, then the error in our calculation of the volume is

- (1) 0.4 cu.cm (2) 0.45 cu.cm
(3) 2 cu.cm (4) **4.8 cu.cm**

EXERCISE 8.8

1. A circular template has a radius of 10 cm.

The measurement of radius has an approximate error of 0.02 cm. Then the percentage error in calculating area of this template is

- (1) 0.2% (2) **0.4%**
(3) 0.04% (4) 0.08%

2. The percentage error of fifth root of 31 is approximately how many times the percentage error in 31?

- (1) $\frac{1}{31}$ (2) $\frac{1}{5}$ (3) 5 (4) 31

3. If $u(x, y) = e^{x^2+y^2}$, then $\frac{\partial u}{\partial x}$ is equal to

- (1) $e^{x^2+y^2}$ (2) **$2xu$** (3) x^2u (4) y^2u

4. If $v(x, y) = \log(e^x + e^y)$,

then $\frac{\partial v}{\partial x} + \frac{\partial v}{\partial y}$ is equal to

- (1) $e^x + e^y$ (2) $\frac{1}{e^x+e^y}$ (3) 2 (4) **1**

5. If $w(x, y) = x^y$, $x > 0$, then $\frac{\partial w}{\partial x}$ is equal to

- (1) $x^y \log x$ (2) $y \log x$
(3) **yx^{y-1}** (4) $x \log y$

6. If $f(x, y) = e^{xy}$, then $\frac{\partial^2 f}{\partial x \partial y}$ is equal to

- (1) xye^{xy} (2) **$(1 + xy)e^{xy}$**
(3) $(1 + y)e^{xy}$ (4) $(1 + x)e^{xy}$

7. If we measure the side of a cube to be 4 cm

8. The change in the surface area $S = 6x^2$ of a cube when the edge length varies from x_0 to $x_0 + dx$ is

- (1) $12x_0 + dx$ (2) **$12x_0 dx$**
(3) $6x_0 dx$ (4) $6x_0 + dx$

9. The approximate change in the volume V of a cube of side x metres caused by increasing the side by 1% is

- (1) $0.3x dx m^3$ (2) $0.03x m^3$
(3) **$0.03x^2 m^3$** (4) $0.03x^3 m^3$

10. If $g(x, y) = 3x^2 - 5y + 2y^2$,

$x(t) = e^t$ and $y(t) = \cos t$, then $\frac{dg}{dt}$ is

- (1) **$6e^{2t} + 5 \sin t - 4 \cos t \sin t$**
(2) $6e^{2t} - 5 \sin t + 4 \cos t \sin t$
(3) $3e^{2t} + 5 \sin t + 4 \cos t \sin t$
(4) $3e^{2t} - 5 \sin t + 4 \cos t \sin t$

11. If $f(x) = \frac{x}{x+1}$, then its differential is

- (1) $\frac{-1}{(x+1)^2} dx$ (2) **$\frac{1}{(x+1)^2} dx$**
(3) $\frac{1}{x+1} dx$ (4) $\frac{-1}{x+1} dx$

12. If $u(x, y) = x^2 + 3xy + y - 2019$,

then $\frac{\partial u}{\partial x(4, -5)}$ is equal to

- (1) -4 (2) -3 (3) **-7** (4) 13

13. Linear approximation for $g(x) = \cos x$

at $x = \frac{\pi}{2}$ is

(1) $x + \frac{\pi}{2}$ (2) $-x + \frac{\pi}{2}$

(3) $x - \frac{\pi}{2}$ (4) $-x - \frac{\pi}{2}$

14. If $(x, y, z) = x^2(y - z) + y^2(z - x) + z^2(x - y)$,

then $\frac{\partial w}{\partial x} + \frac{\partial w}{\partial y} + \frac{\partial w}{\partial z}$ is

(1) $xy + yz + zx$ (2) $x(y + z)$

(3) $y(z + x)$ (4) 0

15. If $f(x, y, z) = xy + yz + zx$,

then $f_x - f_z$ is equal to

(1) $z - x$ (2) $y - z$

(3) $x - z$ (4) $y - x$



DEPARTMENT OF MATHEMATICS
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Chapter No: 9 Applications of Integration

EXERCISE 9.1

1. Find an approximate value of $\int_1^{1.5} x dx$ by applying the left-end rule with the partition $\{1.1, 1.2, 1.3, 1.4, 1.5\}$.

Solution: Given $n = 5, x_0 = 1, x_1 = 1.1,$
 $x_2 = 1.2, x_3 = 1.3, x_4 = 1.4$ and $x_5 = 1.5$
 $\Delta x = 1.1 - 1 = 0.1$

By left end rule,

$$\int_a^b f(x) dx \cong f(x_0)\Delta x + f(x_1)\Delta x + f(x_2)\Delta x \\ + f(x_3)\Delta x + f(x_4)\Delta x$$

$$\int_1^{1.5} x dx \cong f(1)\Delta x + f(1.1)\Delta x + f(1.2)\Delta x \\ + f(1.3)\Delta x + f(1.4)\Delta x \\ \cong 1\Delta x + 1.1\Delta x + 1.2\Delta x + 1.3\Delta x + 1.4\Delta x \\ \cong (1 + 1.1 + 1.2 + 1.3 + 1.4)\Delta x \\ \cong (6)0.1$$

$$\int_1^{1.5} x dx \cong 0.6$$

2. Find an approximate value of $\int_1^{1.5} x^2 dx$ by applying the right-end rule with the partition $\{1.1, 1.2, 1.3, 1.4, 1.5\}$.

Solution: Given $n = 5, x_0 = 1, x_1 = 1.1,$
 $x_2 = 1.2, x_3 = 1.3, x_4 = 1.4$ and $x_5 = 1.5$
 $\Delta x = 1.1 - 1 = 0.1$

By right end rule,

$$\int_a^b f(x) dx \cong f(x_1)\Delta x + f(x_2)\Delta x + f(x_3)\Delta x \\ + f(x_4)\Delta x + f(x_5)\Delta x$$

$$\int_1^{1.5} x^2 dx \cong f(1.1)\Delta x + f(1.2)\Delta x + f(1.3)\Delta x \\ + f(1.4)\Delta x + f(1.5)\Delta x \\ \cong (1.1)^2\Delta x + (1.2)^2\Delta x + (1.3)^2\Delta x \\ + (1.4)^2\Delta x + (1.5)^2\Delta x \\ \cong (1.21 + 1.44 + 1.69 + 1.96 + 2.25)\Delta x \\ \cong (8.55)0.1$$

$$\int_1^{1.5} x^2 dx \cong 0.855$$

3. Find an approximate value of $\int_1^{1.5} (2 - x) dx$ by applying the mid-point rule with the partition $\{1.1, 1.2, 1.3, 1.4, 1.5\}$

Solution: Given $n = 5, x_0 = 1, x_1 = 1.1,$
 $x_2 = 1.2, x_3 = 1.3, x_4 = 1.4$ and $x_5 = 1.5$
 $\Delta x = 1.1 - 1 = 0.1$

By midpoint rule,

$$\int_a^b f(x) dx \cong f\left(\frac{x_0+x_1}{2}\right)\Delta x + f\left(\frac{x_1+x_2}{2}\right)\Delta x \\ + f\left(\frac{x_2+x_3}{2}\right)\Delta x + f\left(\frac{x_3+x_4}{2}\right)\Delta x + f\left(\frac{x_4+x_5}{2}\right)\Delta x \\ \cong f\left(\frac{1+1.1}{2}\right)\Delta x + f\left(\frac{1.1+1.2}{2}\right)\Delta x + f\left(\frac{1.2+1.3}{2}\right)\Delta x \\ + f\left(\frac{1.3+1.4}{2}\right)\Delta x + f\left(\frac{1.4+1.5}{2}\right)\Delta x \\ \cong f\left(\frac{2.1}{2}\right)\Delta x + f\left(\frac{2.3}{2}\right)\Delta x + f\left(\frac{2.5}{2}\right)\Delta x \\ + f\left(\frac{2.7}{2}\right)\Delta x + f\left(\frac{2.9}{2}\right)\Delta x \\ \cong f(1.05)\Delta x + f(1.15)\Delta x + f(1.25)\Delta x \\ + f(1.35)\Delta x + f(1.45)\Delta x$$

$$\int_1^{1.5} (2 - x) dx \cong (2 - 1.05)\Delta x + (2 - 1.15)\Delta x \\ + (2 - 1.25)\Delta x + (2 - 1.35)\Delta x + (2 - 1.45)\Delta x \\ \cong (0.95 + 0.85 + 0.75 + 0.65 + 0.55)\Delta x \\ \cong (3.75)0.1$$

$$\int_1^{1.5} (2 - x) dx \cong 0.375$$

EXERCISE 9.2

1. Evaluate the following integrals as the limits of sums:

(i) $\int_0^1 (5x + 4) dx$

Solution:

$$\int_0^1 f(x) dx = \lim_{n \rightarrow \infty} \frac{1}{n} \sum_{r=1}^n f\left(\frac{r}{n}\right)$$

$$\text{Given } f(x) = (5x + 4)$$

$$f\left(\frac{r}{n}\right) = \left(\frac{5r}{n} + 4\right)$$

$$\sum_{r=1}^n f\left(\frac{r}{n}\right) = \sum_{r=1}^n f\left(\frac{5r}{n} + 4\right)$$

$$= \sum_{r=1}^n f\left(\frac{5r}{n}\right) + \sum_{r=1}^n f(4)$$

$$= \frac{5}{n} \sum_{r=1}^n f(r) + 4 \sum_{r=1}^n f(1)$$

$$= \frac{5}{n}(1 + 2 + 3 + \dots n) + 4(1 + 1 + \dots n)$$

$$= \frac{5}{n} \times \frac{n(n+1)}{2} + 4(n)$$

$$= \frac{5(n+1)}{2} + 4(n)$$

$$\sum_{r=1}^n f\left(\frac{r}{n}\right) = \frac{5(n+1)}{2} + 4(n)$$

$$\lim_{n \rightarrow \infty} \frac{1}{n} \sum_{r=1}^n f\left(\frac{r}{n}\right)$$

$$= \lim_{n \rightarrow \infty} \frac{1}{n} \left[\frac{5}{2}(n+1) + 4(n) \right]$$

$$= \lim_{n \rightarrow \infty} \frac{1}{n} \times n \left[\frac{5}{2} \left(1 + \frac{1}{n}\right) + 4 \right]$$

$$= \lim_{n \rightarrow \infty} \left[\frac{5}{2} \left(1 + \frac{1}{n}\right) + 4 \right]$$

$$= \frac{5}{2} \left(1 + \frac{1}{\infty}\right) + 4$$

$$= \frac{5}{2}(1 + 0) + 4$$

$$= \frac{5}{2}(1) + 4$$

$$= \frac{5}{2} + 4$$

$$= \frac{5+8}{2}$$

$$= \frac{13}{2}$$

$$\therefore \int_0^1 (5x + 4) dx = \frac{13}{2}$$

$$(ii) \int_1^2 (4x^2 - 1) dx$$

Solution:

$$\int_a^b f(x) dx = \lim_{n \rightarrow \infty} \frac{b-a}{n} \sum_{r=1}^n f\left[a + (b-a)\frac{r}{n}\right]$$

Here $a = 1, b = 2$ and $f(x) = 4x^2 - 1$

$$= \lim_{n \rightarrow \infty} \frac{2-1}{n} \sum_{r=1}^n f\left[1 + (2-1)\frac{r}{n}\right]$$

$$= \lim_{n \rightarrow \infty} \frac{1}{n} \sum_{r=1}^n f\left[1 + (1)\frac{r}{n}\right]$$

$$= \lim_{n \rightarrow \infty} \frac{1}{n} \sum_{r=1}^n f\left[1 + \frac{r}{n}\right]$$

$$f(x) = 4x^2 - 1$$

$$f\left[1 + \frac{r}{n}\right] = 4\left(1 + \frac{r}{n}\right)^2 - 1$$

$$= 4\left(1 + \frac{2r}{n} + \frac{r^2}{n^2}\right) - 1$$

$$= 4 + 8\frac{r}{n} + 4\frac{r^2}{n^2} - 1$$

$$= 3 + 8\frac{r}{n} + 4\frac{r^2}{n^2}$$

$$\sum_{r=1}^n f\left[1 + \frac{r}{n}\right] = \sum_{r=1}^n \left(3 + 8\frac{r}{n} + 4\frac{r^2}{n^2}\right)$$

$$= \sum_{r=1}^n (3) + \sum_{r=1}^n \left(8\frac{r}{n}\right) + \sum_{r=1}^n \left(4\frac{r^2}{n^2}\right)$$

$$= 3 \sum_{r=1}^n (1) + \frac{8}{n} \sum_{r=1}^n (r) + \frac{4}{n^2} \sum_{r=1}^n (r^2)$$

$$= 3(n) + \frac{8}{n} \times \frac{n(n+1)}{2} + \frac{4}{n^2} \times \frac{n(n+1)(2n+1)}{6}$$

$$= 3n + 4(n+1) + \frac{2}{3n}(n+1)(2n+1)$$

$$\lim_{n \rightarrow \infty} \frac{1}{n} \sum_{r=1}^n f\left[1 + \frac{r}{n}\right]$$

$$= \lim_{n \rightarrow \infty} \frac{1}{n} \left[3n + 4n \left(1 + \frac{1}{n}\right) + \frac{2}{3n} n \left(1 + \frac{1}{n}\right) n \left(2 + \frac{1}{n}\right) \right]$$

$$= \lim_{n \rightarrow \infty} \frac{1}{n} \left[3n + 4n \left(1 + \frac{1}{n}\right) + \frac{2}{3} n \left(1 + \frac{1}{n}\right) \left(2 + \frac{1}{n}\right) \right]$$

$$= \lim_{n \rightarrow \infty} \left[3 + 4 \left(1 + \frac{1}{n}\right) + \frac{2}{3} \left(1 + \frac{1}{n}\right) \left(2 + \frac{1}{n}\right) \right]$$

$$= \left[3 + 4 \left(1 + \frac{1}{\infty}\right) + \frac{2}{3} \left(1 + \frac{1}{\infty}\right) \left(2 + \frac{1}{\infty}\right) \right]$$

$$= \left[3 + 4(1 + 0) + \frac{2}{3}(1 + 0)(2 + 0) \right]$$

$$= \left[3 + 4(1) + \frac{2}{3}(1)(2) \right]$$

$$= 3 + 4 + \frac{4}{3}$$

$$= 7 + \frac{4}{3}$$

$$= \frac{21+4}{3}$$

$$= \frac{25}{3}$$

$$\int_1^2 (4x^2 - 1) dx = \frac{25}{3}$$

EXERCISE 9.3

1. Evaluate the following definite integrals:

$$(i) \int_3^4 \frac{dx}{x^2 - 4}$$

Solution: $I = \int_3^4 \frac{dx}{x^2 - 4}$

$$\int \frac{dx}{x^2 - a^2} = \frac{1}{2a} \log \left| \frac{x-a}{x+a} \right| + c$$

$$\therefore I = \frac{1}{2(2)} \left[\log \left| \frac{x-2}{x+2} \right| \right]_3^4$$

$$= \frac{1}{4} \left[\log \left| \frac{4-2}{4+2} \right| - \log \left| \frac{3-2}{3+2} \right| \right]$$

$$= \frac{1}{4} \left[\log \left| \frac{2}{6} \right| - \log \left| \frac{1}{5} \right| \right]$$

$$= \frac{1}{4} \log \left(\frac{1}{3} \times \frac{5}{1} \right)$$

$$I = \frac{1}{4} \log \left(\frac{5}{3} \right)$$

$$(ii) \int_{-1}^1 \frac{dx}{x^2 + 2x + 5}$$

Solution: $I = \int_{-1}^1 \frac{dx}{x^2 + 2x + 5}$

$$\int \frac{dx}{a^2 + x^2} = \frac{1}{a} \tan^{-1} \left(\frac{x}{a} \right)$$

$$x^2 + 2x + 5 = x^2 + 2x + 1 + 4$$

$$= (x^2 + 2x + 1) + (4)$$

$$= (x + 1)^2 + (2)^2$$

$$\therefore I = \int_{-1}^1 \frac{dx}{x^2 + 2x + 5} = \int_{-1}^1 \frac{dx}{(x+1)^2 + 2^2}$$

$$= \frac{1}{2} \left[\tan^{-1} \left(\frac{x+1}{2} \right) \right]_{-1}^1$$

$$= \frac{1}{2} \left[\tan^{-1} \left(\frac{1+1}{2} \right) - \tan^{-1} \left(\frac{-1+1}{2} \right) \right]$$

$$= \frac{1}{2} \left[\tan^{-1} \left(\frac{2}{2} \right) - \tan^{-1} \left(\frac{0}{2} \right) \right]$$

$$= \frac{1}{2} [\tan^{-1}(1) - \tan^{-1}(0)]$$

$$= \frac{1}{2} \left[\frac{\pi}{4} - 0 \right]$$

$$= \frac{\pi}{8}$$

Hence $\int_{-1}^1 \frac{dx}{x^2 + 2x + 5} = \frac{\pi}{8}$

$$(iii) \int_0^1 \sqrt{\frac{1-x}{1+x}} dx$$

Solution: Let $I = \int_0^1 \sqrt{\frac{1-x}{1+x}} dx$

$$I = \int_0^1 \sqrt{\frac{1-x}{1+x}} \times \frac{1-x}{1-x} dx$$

$$= \int_0^1 \sqrt{\frac{(1-x)^2}{1-x^2}} dx$$

Substitute $x = \sin t$

Then $\frac{dx}{dt} = \cos t$

$$dx = \cos t dt$$

When $x = 0, t = 0$ and when $x = 1, t = \frac{\pi}{2}$

$$I = \int_0^{\frac{\pi}{2}} \sqrt{\frac{(1-\sin t)^2}{1-\sin^2 t}} \cos t dt$$

$$= \int_0^{\frac{\pi}{2}} \sqrt{\frac{(1-\sin t)^2}{\cos^2 t}} \cos t dt$$

$$= \int_0^{\frac{\pi}{2}} \frac{(1-\sin t)}{\cos t} \cos t dt$$

$$= \int_0^{\frac{\pi}{2}} (1 - \sin t) dt$$

$$= (t + \cos t) \Big|_0^{\frac{\pi}{2}}$$

$$= \left(\frac{\pi}{2} + \cos \frac{\pi}{2} \right) - (0 + \cos 0)$$

$$= \left(\frac{\pi}{2} + 0 \right) - (0 + 1)$$

$$= \frac{\pi}{2} - 1$$

$$\int_0^1 \sqrt{\frac{1-x}{1+x}} dx = \frac{\pi}{2} - 1$$

$$(iv) \int_0^{\frac{\pi}{2}} \left(\frac{1 + \sin x}{1 + \cos x} \right) dx$$

Solution:

$$\sin 2x = 2 \sin x \cos x$$

So, $\sin x = 2 \sin \frac{x}{2} \cos \frac{x}{2}$

$$\cos 2x = 2 \cos^2 x - 1$$

So, $1 + \cos 2x = 2 \cos^2 x$

$$1 + \cos x = 2 \cos^2 \frac{x}{2}$$

$$I = \int_0^{\frac{\pi}{2}} \left(\frac{1 + \sin x}{1 + \cos x} \right) dx$$

$$= \int_0^{\frac{\pi}{2}} \left(\frac{1 + 2 \sin \frac{x}{2} \cos \frac{x}{2}}{2 \cos^2 \frac{x}{2}} \right) dx$$

$$= \int_0^{\frac{\pi}{2}} \left(\frac{1}{2 \cos^2 \frac{x}{2}} + \frac{2 \sin \frac{x}{2} \cos \frac{x}{2}}{2 \cos^2 \frac{x}{2}} \right) dx$$

$$= \int_0^{\frac{\pi}{2}} \left(\frac{1}{2 \cos^2 \frac{x}{2}} + \frac{\sin \frac{x}{2}}{\cos \frac{x}{2}} \right) dx$$

$$= \int_0^{\frac{\pi}{2}} \left[\frac{1}{2} \sec^2 \frac{x}{2} + 2 \left(\frac{1}{2} \tan \frac{x}{2} \right) \right] dx$$

$$= \left[\tan \frac{x}{2} + 2 \log \sec \frac{x}{2} \right]_0^{\frac{\pi}{2}}$$

$$\begin{aligned}
 &= \left(\tan \frac{\pi}{4} + 2 \log \sec \frac{\pi}{4} \right) - (\tan 0 + 2 \log \sec 0) \\
 &= (1 + 2 \log \sqrt{2}) - (0 + 2 \log 1) \\
 &= (1 + \log \sqrt{2}^2) - [0 + 2(0)] \\
 &= (1 + \log 2) - (0 + 0) \\
 &= 1 + \log 2
 \end{aligned}$$

$$(v) \int_0^{\frac{\pi}{2}} \sqrt{\cos \theta} \sin^3 \theta \, d\theta$$

Solution: Put $\cos \theta = x$

$$\text{Then, } -\sin \theta \frac{d\theta}{dx} = 1$$

$$\sin \theta \, d\theta = -dx$$

$$\sin^3 \theta = \sin^2 \theta \sin \theta$$

$$= (1 - \cos^2 \theta) \sin \theta$$

$$= (1 - x^2) \sin \theta$$

$$\text{When } \theta = 0, x = \cos 0 = 1$$

$$\theta = \frac{\pi}{2}, x = \cos \frac{\pi}{2} = 0$$

$$\int_0^{\frac{\pi}{2}} \sqrt{\cos \theta} \sin^3 \theta \, d\theta = \int_1^0 \sqrt{x} \sin^2 \theta \sin \theta \, d\theta$$

$$= \int_1^0 \sqrt{x} (1 - x^2) (-dx)$$

$$= \int_0^1 \sqrt{x} (1 - x^2) dx$$

$$= \int_0^1 x^{\frac{1}{2}} (1 - x^2) dx$$

$$= \int_0^1 \left(x^{\frac{1}{2}} - x^{\frac{5}{2}} \right) dx$$

$$= \left[\frac{x^{\frac{1}{2}+1}}{\frac{1}{2}+1} - \frac{x^{\frac{5}{2}+1}}{\frac{5}{2}+1} \right]_0^1$$

$$= \left[\frac{x^{\frac{3}{2}}}{\frac{3}{2}} - \frac{x^{\frac{7}{2}}}{\frac{7}{2}} \right]_0^1$$

$$= \left[\frac{2}{3} x^{\frac{3}{2}} - \frac{2}{7} x^{\frac{7}{2}} \right]_0^1$$

$$= \left(\frac{2}{3} 1^{\frac{3}{2}} - \frac{2}{7} 1^{\frac{7}{2}} \right) - 0$$

$$= \frac{2}{3} - \frac{2}{7}$$

$$= \frac{14-6}{21}$$

$$\int_0^{\frac{\pi}{2}} \sqrt{\cos \theta} \sin^3 \theta \, d\theta = \frac{8}{21}$$

$$(vi) \int_0^1 \frac{1-x^2}{(1+x^2)^2} dx$$

$$\int_0^1 \frac{1-x^2}{(1+x^2)^2} dx = \int_0^1 \frac{2-1-x^2}{(1+x^2)^2} dx$$

$$= \int_0^1 \frac{2-(1+x^2)}{(1+x^2)^2} dx$$

$$= \int_0^1 \frac{2}{(1+x^2)^2} dx - \int_0^1 \frac{(1+x^2)}{(1+x^2)^2} dx$$

$$= \int_0^1 \frac{2}{(1+x^2)^2} dx - \int_0^1 \frac{1}{(1+x^2)} dx$$

$$\text{Let } I_1 = \int_0^1 \frac{2}{(1+x^2)^2} dx$$

Substitute $x = \tan \theta$

$$\text{Then, } dx = \sec^2 \theta \, d\theta$$

$$(1+x^2) = (1+\tan^2 \theta) = \sec^2 \theta$$

$$\text{When } x = 0, \theta = \tan^{-1} 0 = 0$$

$$x = 1, \theta = \frac{\pi}{4}$$

$$I_1 = 2 \int_0^{\frac{\pi}{4}} \frac{1}{(\sec^2 \theta)^2} \sec^2 \theta \, d\theta$$

$$= 2 \int_0^{\frac{\pi}{4}} \frac{1}{\sec^2 \theta} d\theta$$

$$= 2 \int_0^{\frac{\pi}{4}} \cos^2 \theta \, d\theta$$

$$= 2 \int_0^{\frac{\pi}{4}} \left(\frac{1+\cos 2\theta}{2} \right) d\theta$$

$$= \int_0^{\frac{\pi}{4}} (1 + \cos 2\theta) d\theta$$

$$= \left(\theta + \frac{\sin 2\theta}{2} \right)_0^{\frac{\pi}{4}}$$

$$= \left(\frac{\pi}{4} + \frac{1}{2} \sin \frac{\pi}{2} \right) - (0 + 0)$$

$$= \frac{\pi}{4} + \frac{1}{2} (1)$$

$$= \frac{\pi}{4} + \frac{1}{2}$$

$$\text{Let } I_2 = \int_0^1 \frac{1}{(1+x^2)} dx$$

$$= (\tan^{-1} x)_0^1$$

$$= (\tan^{-1} 1) - (\tan^{-1} 0)$$

$$= \frac{\pi}{4}$$

$$\int_0^1 \frac{1-x^2}{(1+x^2)^2} dx = I_1 + I_2$$

$$= \frac{\pi}{4} + \frac{1}{2} - \frac{\pi}{4}$$

$$= \frac{1}{2}$$

2. Evaluate the following integrals using properties of integration:

(i) $\int_{-5}^5 x \cos\left(\frac{e^x - 1}{e^x + 1}\right) dx$

Solution:

$$I = \int_{-5}^5 x \cos\left(\frac{e^x - 1}{e^x + 1}\right) dx$$

$$f(x) = x \cos\left(\frac{e^x - 1}{e^x + 1}\right)$$

$$f(-x) = -x \cos\left(\frac{e^{-x} - 1}{e^{-x} + 1}\right)$$

$$= -x \cos\left(\frac{1 - e^x}{1 + e^x}\right)$$

$$= -x \left[\cos\left(\frac{1 - e^x}{1 + e^x}\right) \right]$$

$$= -x \left[\cos - \left(\frac{e^x - 1}{e^x + 1}\right) \right]$$

$\cos \theta$ is an even function, because

$$\cos -(\theta) = \cos \theta$$

$$f(-x) = -x \left[\cos\left(\frac{e^x - 1}{e^x + 1}\right) \right]$$

$$= -f(x)$$

$\therefore f(x)$ is an odd function.

$$I = \int_{-5}^5 x \cos\left(\frac{e^x - 1}{e^x + 1}\right) dx = 0$$

(ii) $\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} (x^5 + x \cos x + \tan^3 x + 1) dx$

Solution:

$$I = \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} (x^5 + x \cos x + \tan^3 x + 1) dx$$

$$= \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} (x^5) dx + \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} (x \cos x) dx$$

$$+ \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} (\tan^3 x) dx + \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} 1 dx$$

$x^5, x \cos x, \tan^3 x$ are odd functions.

$$I = 0 + 0 + 0 + \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} 1 dx$$

$$= \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} 1 dx$$

$$= (x)_{-\frac{\pi}{2}}^{\frac{\pi}{2}}$$

$$= \left[\frac{\pi}{2} - \left(-\frac{\pi}{2}\right) \right]$$

$$= \frac{\pi}{2} + \frac{\pi}{2} = \pi$$

(iii) $\int_{-\frac{\pi}{4}}^{\frac{\pi}{4}} \sin^2 x dx$

Solution: $f(x) = \sin^2 x$

$$= \sin x \times \sin x$$

$$f(-x) = \sin(-x) \times \sin(-x)$$

$$= \sin x \times \sin x$$

$$= \sin^2 x$$

$$= f(x)$$

Hence, $f(x)$ is an even function.

$$\therefore I = 2 \int_0^{\frac{\pi}{4}} \sin^2 x dx$$

$$= 2 \int_0^{\frac{\pi}{4}} \left(\frac{1 - \cos 2x}{2} \right) dx$$

$$= \frac{2}{2} \int_0^{\frac{\pi}{4}} (1 - \cos 2x) dx$$

$$= \left(x - \frac{\sin 2x}{2} \right)_0^{\frac{\pi}{4}}$$

$$= \left(\frac{\pi}{4} - \frac{1}{2} \sin 2 \left(\frac{\pi}{4} \right) \right) - 0$$

$$= \left(\frac{\pi}{4} - \frac{1}{2} \sin \left(\frac{\pi}{2} \right) \right)$$

$$= \frac{\pi}{4} - \frac{1}{2}$$

(iv) $\int_0^{2\pi} x \log \left(\frac{3 + \cos x}{3 - \cos x} \right) dx$

Solution:

$$I = \int_0^{2\pi} x \log \left(\frac{3 + \cos x}{3 - \cos x} \right) dx$$

$$f(x) = x \log \left(\frac{3 + \cos x}{3 - \cos x} \right)$$

$$f(2\pi - x) = (2\pi - x) \log \left[\frac{3 + \cos(2\pi - x)}{3 - \cos(2\pi - x)} \right]$$

$$= (2\pi - x) \log \left(\frac{3 + \cos x}{3 - \cos x} \right)$$

$$= 2\pi \log \left(\frac{3 + \cos x}{3 - \cos x} \right) - x \log \left(\frac{3 + \cos x}{3 - \cos x} \right)$$

$$= 2\pi \log \left(\frac{3 + \cos x}{3 - \cos x} \right) - f(x)$$

$$f(2\pi - x) + f(x) = 2\pi \log \left(\frac{3 + \cos x}{3 - \cos x} \right)$$

$$= 0$$

Since, $f(2a - x) + f(x) = 0$

$$\therefore \int_0^{2\pi} x \log \left(\frac{3 + \cos x}{3 - \cos x} \right) dx = 0$$

$$(v) \int_0^{2\pi} \sin^4 x \cos^3 x \, dx$$

Solution:

If $f(2a - x) = f(x)$, then

$$\int_0^{2a} f(x) \, dx = 2 \int_0^a f(x) \, dx$$

and If $f(2a - x) = -f(x)$, then

$$\int_0^{2a} f(x) \, dx = 0$$

Let $f(x) = \sin^4 x \cos^3 x$

$$f(2\pi - x) = \sin^4(2\pi - x) \cos^3(2\pi - x)$$

$$= \sin^4(-x) \cos^3(-x)$$

$$= \sin^4 x \cos^3 x$$

$$= f(x)$$

$$\therefore I = 2 \int_0^{\pi} \sin^4 x \cos^3 x \, dx$$

Again, $f(x) = \sin^4 x \cos^3 x$

$$f(\pi - x) = \sin^4(\pi - x) \cos^3(\pi - x)$$

$$= -\sin^4(\pi - x) \cos^3(\pi - x)$$

$$= -f(\pi - x)$$

$$\therefore I = 0$$

$$(vi) \int_0^1 |5x - 3| \, dx$$

Solution:

$$|5x - 3| = \begin{cases} 5x - 3; & x \geq 3/5 \\ 3 - 5x; & x < 3/5 \end{cases}$$

$$\text{Let } I = \int_0^1 |5x - 3| \, dx$$

$$= \int_0^{3/5} (3 - 5x) \, dx + \int_{3/5}^1 (5x - 3) \, dx$$

$$= \left(3x - 5 \frac{x^2}{2} \right)_0^{3/5} + \left(5 \frac{x^2}{2} - 3x \right)_{3/5}^1$$

$$= \frac{9}{5} - \frac{5}{2} \left(\frac{9}{25} \right) + \left(\frac{5}{2} - 3 \right) - \left[\frac{5}{2} \left(\frac{9}{25} \right) - \frac{9}{5} \right]$$

$$= \frac{9}{5} - \frac{9}{10} + \frac{5}{2} - 3 - \frac{9}{10} + \frac{9}{5}$$

$$= \frac{18 - 9 + 25 - 30 - 9 + 18}{10}$$

$$= \frac{61 - 48}{10}$$

$$= \frac{13}{10}$$

$$(vii) \int_0^{\sin^2 x} \sin^{-1} \sqrt{t} \, dt + \int_0^{\cos^2 x} \cos^{-1} \sqrt{t} \, dt$$

Solution:

$$I = \int_0^{\sin^2 x} \sin^{-1} \sqrt{t} \, dt + \int_0^{\cos^2 x} \cos^{-1} \sqrt{t} \, dt$$

$$\text{Let } I_1 = \int_0^{\sin^2 x} \sin^{-1} \sqrt{t} \, dt$$

$$\text{Put } \sin^2 x = t \quad \text{when } x = 0, t = 0$$

$$\text{Then, } 2 \sin x \cos x \, dx = dt$$

$$\sin^{-1} \sqrt{t} \, dt = \sin^{-1} \sqrt{\sin^2 x} \, 2 \sin x \cos x \, dx$$

$$= \sin^{-1}(\sin x) \, 2 \sin x \cos x \, dx$$

$$= x \, 2 \sin x \cos x \, dx$$

$$= x \sin 2x \, dx$$

$$\therefore I_1 = \int_0^t x \sin 2x \, dx$$

$$\text{Let } I_2 = \int_0^{\cos^2 x} \cos^{-1} \sqrt{t} \, dt$$

$$\text{Put } \cos^2 x = t \quad \text{when } x = 0, t = \frac{\pi}{2}$$

$$\text{Then, } 2 \cos x (-\sin x) \, dx = dt$$

$$-2 \sin x \cos x \, dx = dt$$

$$\cos^{-1} \sqrt{t} \, dt = \cos^{-1} \sqrt{\cos^2 x} \, (-2 \sin x \cos x) \, dx$$

$$= \cos^{-1}(\cos x) \, (-2 \sin x \cos x) \, dx$$

$$= x \, (-2 \sin x \cos x) \, dx$$

$$= -x \sin 2x \, dx$$

$$\therefore I_2 = \int_{\frac{\pi}{2}}^t (-x \sin 2x) \, dx$$

$$I_2 = \int_t^{\frac{\pi}{2}} x \sin 2x \, dx$$

$$I = I_1 + I_2 \text{ gives}$$

$$= \int_0^t x \sin 2x \, dx + \int_t^{\frac{\pi}{2}} x \sin 2x \, dx$$

$$= \int_0^{\frac{\pi}{2}} x \sin 2x \, dx$$

We know that $\int u \, dv = uv - \int v \, du$

Taking $u = x$, and $dv = \sin 2x$

$$du = dx \text{ and}$$

$$\int dv = \int \sin 2x \, dx$$

$$v = -\frac{\cos 2x}{2}$$

$$\int x \sin 2x \, dx = x \left(-\frac{\cos 2x}{2} \right) - \int -\frac{\cos 2x}{2} \, dx$$

$$\begin{aligned}
 &= x \left(-\frac{\cos 2x}{2} \right) + \frac{1}{2} \int \cos 2x \, dx \\
 &= \left(-x \frac{\cos 2x}{2} \right) + \frac{1}{2} \left(\frac{\sin 2x}{2} \right) \\
 &= \left(-\frac{x}{2} \cos 2x \right) + \frac{1}{4} (\sin 2x)
 \end{aligned}$$

$$\begin{aligned}
 \therefore \int_0^{\frac{\pi}{2}} x \sin 2x \, dx &= \left(-\frac{x}{2} \cos 2x \right) \Big|_0^{\frac{\pi}{2}} + \frac{1}{4} (\sin 2x) \Big|_0^{\frac{\pi}{2}} \\
 &= \left(-\frac{\pi}{4} \cos \left(2 \cdot \frac{\pi}{2} \right) \right) - 0 + \frac{1}{4} \left(\sin 2 \left(\frac{\pi}{2} \right) - 0 \right) \\
 &= \left(-\frac{\pi}{4} \cos \pi \right) - 0 + \frac{1}{4} (\sin \pi - 0) \\
 &= \left(-\frac{\pi}{4} (-1) \right) - 0 + \frac{1}{4} (0 - 0) \\
 &= \frac{\pi}{4} + 0 \\
 \therefore I &= \frac{\pi}{4}
 \end{aligned}$$

$$(viii) \int_0^1 \frac{\log(1+x)}{1+x^2} dx$$

Solution:

$$\text{Let } I = \int_0^1 \frac{\log(1+x)}{1+x^2} dx$$

Substitute $x = \tan \theta$, then $dx = \sec^2 \theta \, d\theta$

When $x = 0$, $\theta = 0$ and $x = 1$, $\theta = \frac{\pi}{4}$

$$\begin{aligned}
 \therefore I &= \int_0^{\frac{\pi}{4}} \frac{\log(1+\tan \theta)}{1+\tan^2 \theta} \sec^2 \theta \, d\theta \\
 &= \int_0^{\frac{\pi}{4}} \frac{\log(1+\tan \theta)}{\sec^2 \theta} \sec^2 \theta \, d\theta
 \end{aligned}$$

$$I = \int_0^{\frac{\pi}{4}} \log(1 + \tan \theta) \, d\theta$$

We know that $\int_0^a f(x) \, dx = \int_0^a f(a-x) \, dx$

$$\text{Hence, } I = \int_0^{\frac{\pi}{4}} \log \left[1 + \tan \left(\frac{\pi}{4} - \theta \right) \right] d\theta$$

$$\text{Since, } \tan(A-B) = \frac{1 - \tan A \tan B}{1 + \tan A \tan B}$$

$$\tan \left(\frac{\pi}{4} - \theta \right) = \frac{1 - \tan \frac{\pi}{4} \tan \theta}{1 + \tan \frac{\pi}{4} \tan \theta}$$

$$\text{Since, } \tan \frac{\pi}{4} = 1$$

$$\tan \left(\frac{\pi}{4} - \theta \right) = \frac{1 - (1) \tan \theta}{1 + (1) \tan \theta} = \frac{1 - \tan \theta}{1 + \tan \theta}$$

$$I = \int_0^{\frac{\pi}{4}} \log \left[1 + \tan \left(\frac{\pi}{4} - \theta \right) \right] d\theta \text{ gives}$$

$$I = \int_0^{\frac{\pi}{4}} \log \left[1 + \left(\frac{1 - \tan \theta}{1 + \tan \theta} \right) \right] d\theta$$

$$= \int_0^{\frac{\pi}{4}} \log \left(\frac{1 + \tan \theta + 1 - \tan \theta}{1 + \tan \theta} \right) d\theta$$

$$I = \int_0^{\frac{\pi}{4}} \log \left(\frac{2}{1 + \tan \theta} \right) d\theta$$

$$\text{Now, } \log \left(\frac{m}{n} \right) = \log m - \log n$$

$$\therefore I = \int_0^{\frac{\pi}{4}} [\log(2) - \log(1 + \tan \theta)] d\theta$$

$$I = \int_0^{\frac{\pi}{4}} \log(2) \, d\theta - \int_0^{\frac{\pi}{4}} \log(1 + \tan \theta) \, d\theta$$

$$= \int_0^{\frac{\pi}{4}} \log(2) \, d\theta - I$$

$$I + I = \int_0^{\frac{\pi}{4}} \log(2) \, d\theta$$

$$2I = \log(2) \int_0^{\frac{\pi}{4}} d\theta$$

$$= \log 2 \left(\theta \right) \Big|_0^{\frac{\pi}{4}}$$

$$= \log 2 \left(\frac{\pi}{4} - 0 \right)$$

$$= \log 2 \left(\frac{\pi}{4} \right)$$

$$2I = \left(\frac{\pi}{4} \right) \log 2$$

$$\text{Hence, } I = \left(\frac{\pi}{8} \right) \log 2$$

$$\int_0^1 \frac{\log(1+x)}{1+x^2} dx = \left(\frac{\pi}{8} \right) \log 2$$

$$(ix) \int_0^{\pi} \frac{x \sin x}{1 + \sin x} dx$$

Solution:

$$\text{Let } I = \int_0^{\pi} \frac{x \sin x}{1 + \sin x} dx \quad \dots \dots (1)$$

We know that $\int_0^a f(x) \, dx = \int_0^a f(a-x) \, dx$

$$\text{Hence, } I = \int_0^{\pi} \frac{(\pi-x) \sin(\pi-x)}{1 + \sin(\pi-x)} dx$$

$$I = \int_0^{\pi} \frac{(\pi-x) \sin x}{1 + \sin x} dx \quad \dots \dots (2)$$

$$(1) + (2)$$

$$2I = \int_0^{\pi} \frac{x \sin x}{1 + \sin x} dx + \int_0^{\pi} \frac{(\pi-x) \sin x}{1 + \sin x} dx$$

$$= \int_0^{\pi} \frac{\sin x}{1 + \sin x} (x + \pi - x) dx$$

$$= \int_0^{\pi} \frac{\sin x}{1 + \sin x} (\pi) dx$$

$$= (\pi) \int_0^{\pi} \frac{\sin x}{1 + \sin x} dx$$

$$\begin{aligned}
&= \pi \int_0^{\pi} \frac{\sin x}{1+\sin x} \times \frac{1-\sin x}{1-\sin x} dx \\
&= \pi \int_0^{\pi} \frac{\sin x(1-\sin x)}{1-\sin^2 x} dx \\
&= \pi \int_0^{\pi} \frac{\sin x - \sin^2 x}{\cos^2 x} dx \\
&= \pi \int_0^{\pi} \left(\frac{\sin x}{\cos^2 x} - \frac{\sin^2 x}{\cos^2 x} \right) dx \\
&= \pi \int_0^{\pi} \left(\frac{1}{\cos x} \times \frac{\sin x}{\cos x} - \tan^2 x \right) dx \\
&= \pi \int_0^{\pi} [\sec x \tan x - (\sec^2 x - 1)] dx \\
&= \pi \int_0^{\pi} (\sec x \tan x - \sec^2 x + 1) dx \\
&= \pi \left[\int_0^{\pi} \sec x \tan x dx - \int_0^{\pi} \sec^2 x dx + \int_0^{\pi} dx \right]
\end{aligned}$$

$$\begin{aligned}
&= \pi[(\sec x)_0^{\pi} - (\tan x)_0^{\pi} + (x)_0^{\pi}] \\
&= \pi[(\sec \pi - \sec 0) - (\tan \pi - \tan 0) + (\pi - 0)]
\end{aligned}$$

$$2I = \pi[-2 + \pi]$$

$$= -2\pi + \pi^2$$

$$\therefore I = -\pi + \frac{\pi^2}{2}$$

$$(x) \int_{\frac{\pi}{8}}^{\frac{3\pi}{8}} \frac{1}{1+\sqrt{\tan x}} dx$$

$$\text{Let } I = \int_{\frac{\pi}{8}}^{\frac{3\pi}{8}} \frac{1}{1+\sqrt{\tan x}} dx \quad \dots\dots (1)$$

$$\text{We know } \int_a^b f(x) dx = \int_a^b f(a+b-x) dx$$

$$a = \frac{\pi}{8}, b = \frac{3\pi}{8} \text{ So, } a+b = \frac{\pi+3\pi}{8} = \frac{4\pi}{8} = \frac{\pi}{2}$$

$$\text{Hence } I = \int_{\frac{\pi}{8}}^{\frac{3\pi}{8}} \frac{1}{1+\sqrt{\tan(\frac{\pi}{2}-x)}} dx$$

$$= \int_{\frac{\pi}{8}}^{\frac{3\pi}{8}} \frac{1}{1+\sqrt{\cot x}} dx$$

$$= \int_{\frac{\pi}{8}}^{\frac{3\pi}{8}} \frac{1}{1+\frac{1}{\sqrt{\tan x}}} dx$$

$$= \int_{\frac{\pi}{8}}^{\frac{3\pi}{8}} \frac{1}{\frac{\sqrt{\tan x}+1}{\sqrt{\tan x}}} dx$$

$$= \int_{\frac{\pi}{8}}^{\frac{3\pi}{8}} \frac{\sqrt{\tan x}}{1+\sqrt{\tan x}} dx \quad \dots\dots (2)$$

(1) + (2) gives,

$$2I = \int_{\frac{\pi}{8}}^{\frac{3\pi}{8}} \frac{1+\sqrt{\tan x}}{1+\sqrt{\tan x}} dx$$

$$= \int_{\frac{\pi}{8}}^{\frac{3\pi}{8}} dx$$

$$= (x)_{\frac{\pi}{8}}^{\frac{3\pi}{8}}$$

$$= \left(\frac{3\pi}{8} - \frac{\pi}{8} \right)$$

$$= \left(\frac{3\pi-\pi}{8} \right)$$

$$= \left(\frac{2\pi}{8} \right)$$

$$2I = \frac{2\pi}{8}$$

$$\therefore I = \frac{\pi}{8}$$

$$(xi) \int_0^{\pi} x[\sin^2(\sin x) + \cos^2(\cos x)] dx$$

Solution:

$$\text{Let } I = \int_0^{\pi} x[\sin^2(\sin x) + \cos^2(\cos x)] dx \dots (1)$$

$$\text{We know that } \int_0^a f(x) dx = \int_0^a f(a-x) dx$$

$$I = \int_0^{\pi} (\pi-x)[\sin^2(\sin(\pi-x)) + \cos^2(\cos(\pi-x))] dx$$

$$= \int_0^{\pi} (\pi-x)[\sin^2(\sin x) + \cos^2(\cos x)] dx \dots (2)$$

(1) + (2) gives,

$$2I = \int_0^{\pi} (x+\pi-x)[\sin^2(\sin x) + \cos^2(\cos x)] dx$$

$$2I = \pi \int_0^{\pi} [\sin^2(\sin x) + \cos^2(\cos x)] dx$$

$$\therefore I = \frac{\pi}{2} \int_0^{\pi} [\sin^2(\sin x) + \cos^2(\cos x)] dx$$

$$\text{But } \int_0^{2a} f(x) dx = 2 \int_0^a f(x) dx$$

$$\therefore I = \frac{\pi}{2} (2) \int_0^{\frac{\pi}{2}} [\sin^2(\sin x) + \cos^2(\cos x)] dx$$

$$I = \pi \int_0^{\frac{\pi}{2}} [\sin^2(\sin x) + \cos^2(\cos x)] dx \dots (3)$$

$$\text{Since, } \int_0^a f(x) dx = \int_0^a f(a-x) dx$$

$$I = \pi \int_0^{\frac{\pi}{2}} \left[\sin^2 \left(\sin \left(\frac{\pi}{2} - x \right) \right) + \cos^2 \left(\cos \left(\frac{\pi}{2} - x \right) \right) \right] dx$$

$$= \pi \int_0^{\frac{\pi}{2}} [\sin^2(\cos x) + \cos^2(\sin x)] dx \dots (4)$$

(3) + (4) gives,

$$\begin{aligned}
 2I &= 2\pi \int_0^{\frac{\pi}{2}} [1] dx \\
 &= 2\pi \int_0^{\frac{\pi}{2}} dx \\
 &= 2\pi (x)_0^{\frac{\pi}{2}} \\
 &= 2\pi \left(\frac{\pi}{2} - 0\right) \\
 2I &= 2\pi \left(\frac{\pi}{2}\right) \\
 I &= \pi \left(\frac{\pi}{2}\right) \\
 I &= \frac{\pi^2}{2}
 \end{aligned}$$

$$\therefore \int_0^{\pi} x [\sin^2(\sin x) + \cos^2(\cos x)] dx = \frac{\pi^2}{2}$$

EXERCISE 9.4

Evaluate the following:

1. $\int_0^1 x^3 e^{-2x} dx$.

Solution:

$$\begin{aligned}
 dv &= e^{-2x} \\
 u &= x^3 & v &= \frac{e^{-2x}}{-2} \\
 u' &= 3x^2 & v_1 &= \frac{e^{-2x}}{4} \\
 u'' &= 6x & v_2 &= \frac{e^{-2x}}{-8} \\
 u''' &= 6 & v_3 &= \frac{e^{-2x}}{16}
 \end{aligned}$$

Applying Bernoulli's formula we get,

$$\begin{aligned}
 \int_0^1 x^3 e^{-2x} dx &= [uv - u'v_1 + u''v_2 - u'''v_3]_0^1 \\
 &= \left[x^3 \left(\frac{e^{-2x}}{-2}\right) - 3x^2 \left(\frac{e^{-2x}}{4}\right) + 6x \left(\frac{e^{-2x}}{-8}\right) - 6 \left(\frac{e^{-2x}}{16}\right) \right]_0^1 \\
 &= \frac{e^{-2}}{-2} - \frac{3e^{-2}}{2} - \frac{3e^{-2}}{4} - \frac{3e^{-2}}{8} - \left(-\frac{3}{8}\right) \\
 &= \frac{e^{-2}}{-2} - \frac{3e^{-2}}{2} - \frac{3e^{-2}}{4} - \frac{3e^{-2}}{8} + \frac{3}{8}
 \end{aligned}$$

2. $\int_0^1 \frac{\sin(3 \tan^{-1} x)(\tan^{-1} x)}{1+x^2} dx$.

Solution: Let $t = \tan^{-1} x$

$$dt = \frac{1}{1+x^2} dx = \frac{dx}{1+x^2}$$

$$\begin{aligned}
 \int \frac{\sin(3 \tan^{-1} x)(\tan^{-1} x)}{1+x^2} dx &= \int \sin(3t)(t) dt \\
 &= \int t \sin 3t dt
 \end{aligned}$$

$$dv = \sin 3t$$

$$u = t \quad v = -\frac{\cos 3t}{3}$$

$$u' = 1 \quad v_1 = -\frac{\sin 3t}{9}$$

When $x = 0, t = 0$ and $x = 1, t = \frac{\pi}{4}$

Applying Bernoulli's formula we get,

$$\begin{aligned}
 \int_0^{\frac{\pi}{4}} t \sin 3t dt &= [uv - u'v_1]_0^{\frac{\pi}{4}} \\
 &= \left[t \left(-\frac{\cos 3t}{3}\right) - 1 \left(-\frac{\sin 3t}{9}\right) \right]_0^{\frac{\pi}{4}} \\
 &= \left[-t \left(\frac{\cos 3t}{3}\right) + \left(\frac{\sin 3t}{9}\right) \right]_0^{\frac{\pi}{4}} \\
 &= \frac{1}{9} [-3t \cos 3t + \sin 3t]_0^{\frac{\pi}{4}} \\
 &= \frac{1}{9} \left[\left(-\frac{3\pi}{4} \cos \frac{3\pi}{4} + \sin \frac{3\pi}{4}\right) - 0 \right]
 \end{aligned}$$

Since, $\cos \frac{3\pi}{4} = -\frac{1}{\sqrt{2}}$ and $\sin \frac{3\pi}{4} = \frac{1}{\sqrt{2}}$

$$\begin{aligned}
 I &= \frac{1}{9} \left[-\frac{3\pi}{4} \left(-\frac{1}{\sqrt{2}}\right) + \left(\frac{1}{\sqrt{2}}\right) \right] \\
 &= \frac{1}{9} \left[\frac{3\pi}{4\sqrt{2}} + \frac{1}{\sqrt{2}} \right] \\
 &= \frac{1}{9\sqrt{2}} \left[\frac{3\pi}{4} + 1 \right] \\
 &= \frac{1}{9\sqrt{2}} \left[\frac{3\pi+4}{4} \right] \\
 I &= \frac{1}{36\sqrt{2}} (3\pi + 4)
 \end{aligned}$$

3. $\int_0^{\frac{1}{\sqrt{2}}} \frac{e^{\sin^{-1} x} \sin^{-1} x}{\sqrt{1-x^2}} dx$

Solution: Let $t = \sin^{-1} x$

$$dt = \frac{1}{\sqrt{1-x^2}} dx = \frac{dx}{\sqrt{1-x^2}}$$

$$\begin{aligned}
 \int \frac{e^{\sin^{-1} x} \sin^{-1} x}{\sqrt{1-x^2}} dx &= \int e^t t dt \\
 &= \int t e^t dt
 \end{aligned}$$

$$dv = e^t$$

$$u = t \quad v = e^t$$

$$u' = 1 \quad v_1 = e^t$$

When $x = 0, t = 0$ and $x = \frac{1}{\sqrt{2}}, t = \frac{\pi}{4}$

Applying Bernoulli's formula we get,

$$\begin{aligned}
 \int_0^{\frac{\pi}{4}} t e^t dt &= [t e^t - 1(e^t)]_0^{\frac{\pi}{4}} \\
 &= [t e^t - e^t]_0^{\frac{\pi}{4}} \\
 &= \left[\left(\frac{\pi}{4} e^{\frac{\pi}{4}} - e^{\frac{\pi}{4}} \right) - (0 - 1) \right] \\
 &= \frac{\pi}{4} e^{\frac{\pi}{4}} - e^{\frac{\pi}{4}} + 1 \\
 I &= e^{\frac{\pi}{4}} \left(\frac{\pi}{4} - 1 \right) + 1
 \end{aligned}$$

4. $\int_0^{\frac{\pi}{2}} x^2 \cos 2x dx$

Solution:

$$\begin{aligned}
 dv &= \cos 2x \\
 u &= x^2 & v &= \frac{\sin 2x}{2} \\
 u' &= 2x & v_1 &= \frac{\cos 2x}{-4} \\
 u'' &= 2 & v_2 &= \frac{\sin 2x}{-8}
 \end{aligned}$$

Applying Bernoulli's formula we get,

$$\begin{aligned}
 \int_0^{\frac{\pi}{2}} x^2 \cos 2x dx &= [uv - u'v_1 + u''v_2]_0^{\frac{\pi}{2}} \\
 &= \left[x^2 \left(\frac{\sin 2x}{2} \right) - 2x \left(\frac{\cos 2x}{-4} \right) + 2 \left(\frac{\sin 2x}{-8} \right) \right]_0^{\frac{\pi}{2}} \\
 &= \left[\frac{x^2}{2} (\sin 2x) + \frac{1}{2} x (\cos 2x) - \frac{1}{4} (\sin 2x) \right]_0^{\frac{\pi}{2}} \\
 &= \left[\left(\frac{\pi}{2} \right)^2 (\sin \pi) + \frac{1}{2} \left(\frac{\pi}{2} \right) (\cos \pi) - \frac{1}{4} (\sin \pi) \right] - 0
 \end{aligned}$$

Since $\sin \pi = 0$ and $\cos \pi = -1$

$$I = \frac{\left(\frac{\pi}{2}\right)^2}{2} (0) + \frac{1}{2} \left(\frac{\pi}{2}\right) (-1) - \frac{1}{4} (0)$$

$$\therefore \int_0^{\frac{\pi}{2}} x^2 \cos 2x dx = -\frac{\pi}{4}$$

EXERCISE 9.5

1. Evaluate the following:

(i) $\int_0^{\frac{\pi}{2}} \frac{dx}{1+5\cos^2 x}$

Solution:

$$\begin{aligned}
 \text{Let } I &= \int_0^{\frac{\pi}{2}} \frac{dx}{1+5\cos^2 x} \\
 &= \int_0^{\frac{\pi}{2}} \frac{dx}{\cos^2 x \left(\frac{1}{\cos^2 x} + 5 \right)}
 \end{aligned}$$

$$= \int_0^{\frac{\pi}{2}} \frac{\sec^2 x}{(\sec^2 x + 5)} dx$$

Since $\sec^2 x = 1 + \tan^2 x$

$$\begin{aligned}
 I &= \int_0^{\frac{\pi}{2}} \frac{\sec^2 x}{(1+\tan^2 x+5)} dx \\
 &= \int_0^{\frac{\pi}{2}} \frac{\sec^2 x}{(6+\tan^2 x)} dx
 \end{aligned}$$

Put $\tan x = t$, then $\sec^2 x dx = dt$

When $x = 0, t = 0$ and $x = \frac{\pi}{2}, t = \infty$

$$\begin{aligned}
 I &= \int_0^{\infty} \frac{1}{(6+t^2)} dt \\
 &= \int_0^{\infty} \frac{1}{(\sqrt{6}^2+t^2)} dt
 \end{aligned}$$

We know that $\int_0^{\infty} \frac{1}{(a^2+x^2)} dx = \frac{\pi}{2a}$

$$\text{So, } I = \frac{\pi}{2\sqrt{6}}$$

(ii) $\int_0^{\frac{\pi}{2}} \frac{dx}{5+4\sin^2 x}$

Solution:

$$\begin{aligned}
 \text{Let } I &= \int_0^{\frac{\pi}{2}} \frac{dx}{5+4\sin^2 x} \\
 &= \int_0^{\frac{\pi}{2}} \frac{dx}{5+4(1-\cos^2 x)} \\
 &= \int_0^{\frac{\pi}{2}} \frac{dx}{5+4-4\cos^2 x} \\
 &= \int_0^{\frac{\pi}{2}} \frac{dx}{9-4\cos^2 x} \\
 &= \int_0^{\frac{\pi}{2}} \frac{dx}{\cos^2 x \left(\frac{9}{\cos^2 x} - 4 \right)} \\
 &= \int_0^{\frac{\pi}{2}} \frac{\sec^2 x}{(9\sec^2 x - 4)} dx
 \end{aligned}$$

Since $\sec^2 x = 1 + \tan^2 x$

$$\begin{aligned}
 I &= \int_0^{\frac{\pi}{2}} \frac{\sec^2 x}{9(1+\tan^2 x) - 4} dx \\
 &= \int_0^{\frac{\pi}{2}} \frac{\sec^2 x}{9+9\tan^2 x - 4} dx \\
 &= \int_0^{\frac{\pi}{2}} \frac{\sec^2 x}{5+9\tan^2 x} dx
 \end{aligned}$$

Put $\tan x = t$, then $\sec^2 x dx = dt$

When $x = 0, t = 0$ and $x = \frac{\pi}{2}, t = \infty$

$$I = \int_0^{\infty} \frac{1}{(5+9t^2)} dt$$

$$\begin{aligned}
 &= \int_0^{\infty} \frac{1}{9 \left(\frac{5}{9} + t^2 \right)} dt \\
 &= \frac{1}{9} \int_0^{\infty} \frac{1}{\left(\frac{5}{9} + t^2 \right)} dt \\
 &= \frac{1}{9} \int_0^{\infty} \frac{1}{\left(\frac{\sqrt{5}}{3} \right)^2 + t^2} dt
 \end{aligned}$$

We know that $\int_0^{\infty} \frac{1}{(a^2 + x^2)} dx = \frac{\pi}{2a}$

$$\begin{aligned}
 \text{So, } I &= \frac{1}{9} \times \frac{\pi}{2 \left(\frac{\sqrt{5}}{3} \right)} \\
 &= \frac{1}{9} \times \frac{3\pi}{2\sqrt{5}} \\
 &= \frac{1}{3} \times \frac{\pi}{2\sqrt{5}} \\
 I &= \frac{\pi}{6\sqrt{5}}
 \end{aligned}$$

EXERCISE 9.6

1. Evaluate the following:

(i) $\int_0^{\frac{\pi}{2}} \sin^{10} x \, dx$

$$\begin{aligned}
 I &= \frac{9}{10} \times \frac{7}{8} \times \frac{5}{6} \times \frac{3}{4} \times \frac{1}{2} \times \frac{\pi}{2} \\
 I &= \frac{63\pi}{512}
 \end{aligned}$$

(ii) $\int_0^{\frac{\pi}{2}} \cos^7 x \, dx$

$$\begin{aligned}
 I &= \frac{6}{7} \times \frac{4}{5} \times \frac{2}{3} \\
 I &= \frac{16}{35}
 \end{aligned}$$

(iii) $\int_0^{\frac{\pi}{4}} \sin^6 2x \, dx$

Substitute, $2x = t$

$$2dx = dt$$

$$dx = \frac{dt}{2}$$

When $x = 0, t = 0$ and $x = \frac{\pi}{4}, t = \frac{\pi}{2}$

$$\begin{aligned}
 \int_0^{\frac{\pi}{4}} \sin^6 2x \, dx &= \int_0^{\frac{\pi}{2}} \sin^6 t \frac{dt}{2} \\
 &= \frac{1}{2} \int_0^{\frac{\pi}{2}} \sin^6 t \, dt \\
 &= \frac{1}{2} \times \frac{5}{6} \times \frac{3}{4} \times \frac{1}{2} \times \frac{\pi}{2} \\
 I &= \frac{5\pi}{64}
 \end{aligned}$$

(iv) $\int_0^{\frac{\pi}{6}} \sin^5 3x \, dx$

Substitute, $3x = t$

$$3dx = dt$$

$$dx = \frac{dt}{3}$$

When $x = 0, t = 0$ and $x = \frac{\pi}{6}, t = \frac{\pi}{2}$

$$\begin{aligned}
 \int_0^{\frac{\pi}{6}} \sin^5 3x \, dx &= \int_0^{\frac{\pi}{2}} \sin^5 t \frac{dt}{3} \\
 &= \frac{1}{3} \int_0^{\frac{\pi}{2}} \sin^5 t \, dt \\
 I &= \frac{1}{3} \times \frac{4}{5} \times \frac{2}{3} \\
 I &= \frac{8}{45}
 \end{aligned}$$

(v) $\int_0^{\frac{\pi}{2}} \sin^2 x \cos^4 x \, dx$

$$\begin{aligned}
 \int_0^{\frac{\pi}{2}} \sin^2 x \cos^4 x \, dx &= \int_0^{\frac{\pi}{2}} (1 - \cos^2 x) \cos^4 x \, dx \\
 &= \int_0^{\frac{\pi}{2}} (\cos^4 x - \cos^6 x) \, dx \\
 &= \int_0^{\frac{\pi}{2}} \cos^4 x \, dx - \int_0^{\frac{\pi}{2}} \cos^6 x \, dx \\
 &= \frac{3}{4} \times \frac{1}{2} \times \frac{\pi}{2} - \frac{5}{6} \times \frac{3}{4} \times \frac{1}{2} \times \frac{\pi}{2} \\
 &= \frac{3\pi}{16} - \frac{5\pi}{32} \\
 &= \frac{6\pi - 5\pi}{32} \\
 &= \frac{\pi}{32}
 \end{aligned}$$

(vi) $\int_0^{2\pi} \sin^7 \frac{x}{4} \, dx$

Substitute, $\frac{x}{4} = t$

Then, $x = 4t$

$$dx = 4dt$$

When $x = 0, t = 0$ and $x = 2\pi, t = \frac{\pi}{2}$

$$\begin{aligned}
 \int_0^{2\pi} \sin^7 \frac{x}{4} \, dx &= \int_0^{\frac{\pi}{2}} \sin^7 t (4dt) \\
 &= 4 \int_0^{\frac{\pi}{2}} \sin^7 t \, dt \\
 I &= 4 \times \frac{6}{7} \times \frac{4}{5} \times \frac{2}{3} \\
 I &= \frac{64}{35}
 \end{aligned}$$

$$(vii) \int_0^{\frac{\pi}{2}} \sin^3 \theta \cos^5 \theta d\theta$$

We know that, $\int_0^{\frac{\pi}{2}} \sin^m \theta \cos^n \theta d\theta$

$$= \frac{[(m-1)(m-3)\dots 2.1][(n-1)(n-3)\dots 2.1]}{(m+n)(m+n-2)(m+n-4)\dots}$$

Here $m = 3$ and $n = 5$

$$\begin{aligned} \int_0^{\frac{\pi}{2}} \sin^3 \theta \cos^5 \theta d\theta &= \frac{[(3-1)][(5-1)(5-3)]}{(3+5)(8-2)(8-4)(8-6)} \\ &= \frac{(2)(4)(2)}{(8)(6)(4)(2)} \\ &= \frac{(2)}{(8)(6)} \\ &= \frac{1}{24} \end{aligned}$$

$$(viii) \int_0^1 x^2 (1-x)^3 dx$$

We know $\int_0^1 x^m (1-x)^n dx = \frac{m! n!}{(m+n+1)!}$

Here $m = 2$ and $n = 3$

$$\begin{aligned} \int_0^1 x^m (1-x)^n dx &= \frac{2! 3!}{(6)!} \\ &= \frac{1 \times 2 \times 1 \times 2 \times 3}{1 \times 2 \times 3 \times 4 \times 5 \times 6} \\ &= \frac{1}{60} \end{aligned}$$

EXERCISE 9.7

Evaluate the following

$$1. (i) \int_0^{\infty} x^5 e^{-3x} dx$$

We know $\int_0^{\infty} x^n e^{-ax} dx = \frac{n!}{a^{n+1}}$

Here $n = 5$ and $a = 3$

$$\int_0^{\infty} x^5 e^{-3x} dx = \frac{5!}{3^{5+1}} = \frac{5!}{3^6}$$

$$(ii) \int_0^{\frac{\pi}{2}} \frac{e^{-\tan x}}{\cos^6 x} dx$$

$$= \int_0^{\frac{\pi}{2}} e^{-\tan x} \sec^6 x dx$$

$$= \int_0^{\frac{\pi}{2}} e^{-\tan x} \sec^4 x \sec^2 x dx$$

Put $\tan x = t$, then $\sec^2 x dx = dt$

When $x = 0, t = 0$ and $x = \frac{\pi}{2}, t = \infty$

$$I = \int_0^{\infty} e^{-t} (\sec^2 x)^2 dt$$

$$= \int_0^{\infty} e^{-t} (1+t^2)^2 dt$$

$$= \int_0^{\infty} e^{-t} (1+t^4+2t^2) dt$$

$$= \int_0^{\infty} e^{-t} dt + \int_0^{\infty} e^{-t} t^4 dt + 2 \int_0^{\infty} e^{-t} t^2 dt$$

$$= (-e^{-t})_0^{\infty} + \frac{4!}{1^4+1} + 2 \frac{2!}{1^2+1}$$

$$= -e^{-\infty} + e^0 + 4! + 2(2!)$$

$$= 0 + 1 + 24 + 4$$

$$= 29$$

$$2. \text{ If } \int_0^{\infty} e^{-\alpha x^2} x^3 dx = 32, \alpha > 0, \text{ find } \alpha$$

Solution:

$$\int_0^{\infty} e^{-\alpha x^2} x^3 dx = 32$$

$$\int_0^{\infty} e^{-\alpha x^2} x^2 x dx = 32$$

Substitute $x^2 = t$, then $2x dx = dt$

$$\text{Hence } x dx = \frac{dt}{2}$$

When $x = 0, t = 0$ and $x = \infty, t = \infty$

$$\int_0^{\infty} e^{-\alpha t} t \frac{dt}{2} = 32$$

$$\int_0^{\infty} e^{-\alpha t} t dt = 64$$

Here $n = 1$ and $a = \alpha$

$$\frac{1!}{\alpha^{1+1}} = 64$$

$$\frac{1}{\alpha^2} = 64$$

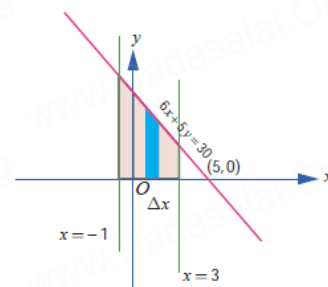
$$\alpha^2 = \frac{1}{64} \text{ gives,}$$

$$\alpha = \frac{1}{8}$$

Example 9.47

Find the area of the region bounded by the line $6x + 5y = 30$, x -axis and the lines $x = -1$ and $x = 3$.

Solution:



Equation of the line $6x + 5y = 30$

On x axis $y = 0$, $6x = 30$ gives $x = 5$

So, the line passes through $(5,0)$

On y axis $x = 0$, $5y = 30$ gives $y = 6$

So, the line passes through $(0,6)$

The area between the line and $x = -1$, $x = 3$ about x axis, lies above the x axis.

So the required Area $A = \int_a^b y \, dx$

From $6x + 5y = 30$

$$5y = 30 - 6x$$

$$y = \frac{1}{5}(30 - 6x)$$

$$\therefore A = \int_{-1}^3 \frac{1}{5}(30 - 6x) \, dx$$

$$= \frac{1}{5} \int_{-1}^3 (30 - 6x) \, dx$$

$$= \frac{1}{5} \left[30x - 6 \frac{x^2}{2} \right]_{-1}^3$$

$$= \frac{1}{5} [30x - 3x^2]_{-1}^3$$

$$= \frac{1}{5} [(30(3) - 3(3)^2) - (30(-1) - 3(-1)^2)]$$

$$= \frac{1}{5} [(30(3) - 3(9)) - (30(-1) - 3(1))]$$

$$= \frac{1}{5} [(90 - 27) - (-30 - 3)]$$

$$= \frac{1}{5} [(63) - (-33)]$$

$$= \frac{1}{5} (63 + 33)$$

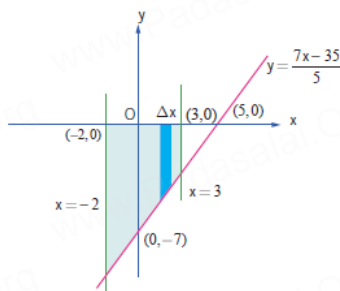
$$= \frac{1}{5} (96)$$

$$A = \frac{96}{5} \text{ sq. units.}$$

Example 9.48

Find the area of the region bounded by the line $7x - 5y = 35$, x - axis and the lines $x = -2$ and $x = 3$.

Solution:



Equation of the line $7x - 5y = 35$

On x axis $y = 0$, $7x = 35$ gives $x = 5$

So, the line passes through $(5,0)$

On y axis $x = 0$, $-5y = 35$ gives $y = -7$

So, the line passes through $(0, -7)$

The area between the line and $x = -2$, $x = 3$ about x axis, lies below the x axis.

So the required Area $A = \int_a^b (-y) \, dx$

From $7x - 5y = 35$

$$-5y = 35 - 7x$$

$$-y = \frac{1}{5}(35 - 7x)$$

$$\therefore A = \int_{-2}^3 \frac{1}{5}(35 - 7x) \, dx$$

$$= \frac{1}{5} \int_{-2}^3 (35 - 7x) \, dx$$

$$= \frac{1}{5} \left[35x - 7 \frac{x^2}{2} \right]_{-2}^3$$

$$= \frac{1}{10} [70x - 7x^2]_{-2}^3$$

$$= \frac{1}{10} [(70(3) - 7(3)^2) - (70(-2) - 7(-2)^2)]$$

$$= \frac{1}{10} [(70(3) - 7(9)) - (70(-2) - 7(4))]$$

$$= \frac{1}{10} [(210 - 63) - (-140 - 28)]$$

$$= \frac{1}{10} [(147) - (-168)]$$

$$= \frac{1}{10} (147 + 168)$$

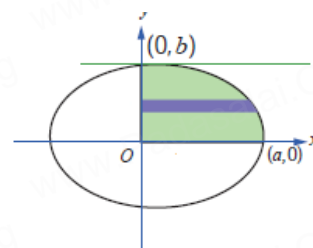
$$= \frac{1}{10} (315)$$

$$A = \frac{63}{2} \text{ sq. units.}$$

Example 9.49

Find the area of the region bounded by the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$

Solution:



Parametric equation of the ellipse is

$$x = a \cos \theta \text{ and } y = b \sin \theta$$

$$\text{So, } dy = b \cos \theta d\theta$$

The ellipse is symmetrical about both the axis.

On I quadrant $\theta = 0$ to $\frac{\pi}{2}$

$$\text{We know Area } A = \int_a^b y dx$$

$$\text{Hence area } A = 4 \int_0^{\frac{\pi}{2}} (a \cos \theta) b \cos \theta d\theta$$

$$= 4ab \int_0^{\frac{\pi}{2}} \cos^2 \theta d\theta$$

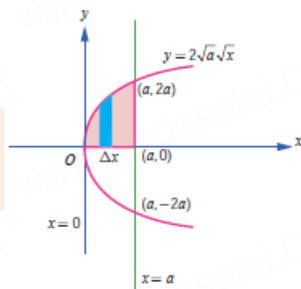
$$= 4ab \times \frac{1}{2} \times \frac{\pi}{2}$$

$$= \pi ab \text{ sq. units.}$$

Example 9.50

Find the area of the region bounded between the parabola $y^2 = 4ax$ and its latus rectum.

Solution:



The parabola $y^2 = 4ax$ is open right, passes through the origin. Its focus is $F(a, 0)$. The latus rectum is $x = a$. So, the limit is $x = 0$ to $x = a$. The parabola is symmetrical about x axis.

$$\text{Area } A = 2 \int_a^b y dx$$

$$\text{From } y^2 = 4ax, y = 2\sqrt{a}\sqrt{x}$$

$$\therefore A = 2 \int_0^a 2\sqrt{a}\sqrt{x} dx$$

$$= 4\sqrt{a} \int_0^a \sqrt{x} dx$$

$$= 4\sqrt{a} \int_0^a x^{\frac{1}{2}} dx$$

$$= 4\sqrt{a} \left(\frac{x^{\frac{3}{2}}}{\frac{3}{2}} \right)_0^a$$

$$= 4\sqrt{a} \times \frac{2}{3} \left(x^{\frac{3}{2}} \right)_0^a$$

$$= 4\sqrt{a} \times \frac{2}{3} \left(a^{\frac{3}{2}} - 0 \right)$$

$$= 4\sqrt{a} \times \frac{2}{3} (a\sqrt{a})$$

$$= \frac{8}{3} a^2$$

Example 9.51

Find the area of the region bounded by the y -axis and the parabola $x = 5 - 4y - y^2$.

Solution:

$$\text{Given } x = 5 - 4y - y^2$$

$$\text{On } y \text{ axis } x = 0$$

$$0 = 5 - 4y - y^2$$

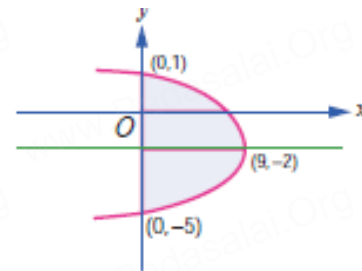
$$y^2 + 4y - 5 = 0$$

$$(y + 5)(y - 1) = 0$$

$$y + 5 = 0, \text{ gives } y = -5 \text{ and}$$

$$y - 1 = 0, \text{ gives } y = 1$$

The curve passes through $(0, -5)$, $(0, 1)$



$$\text{Area } A = \int_a^b x dy$$

$$= \int_{-5}^1 (5 - 4y - y^2) dy$$

$$= \left(5y - 4\frac{y^2}{2} - \frac{y^3}{3} \right)_{-5}^1$$

$$= \left(5y - 2y^2 - \frac{y^3}{3} \right)_{-5}^1$$

$$= \left[\left(5(1) - 2(1)^2 - \frac{(1)^3}{3} \right) - \left(5(-5) - 2(-5)^2 - \frac{(-5)^3}{3} \right) \right]$$

$$= \left[\left(5 - 2 - \frac{1}{3} \right) - \left(-25 - 2(25) - \frac{(-125)}{3} \right) \right]$$

$$= \left[\left(3 - \frac{1}{3} \right) - \left(-75 + \frac{125}{3} \right) \right]$$

$$= 3 - \frac{1}{3} + 75 - \frac{125}{3}$$

$$= 78 - \frac{126}{3}$$

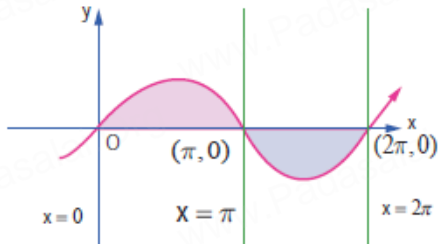
$$= \frac{234 - 126}{3}$$

$$= \frac{108}{3} \quad \text{Area } A = 36 \text{ square units.}$$

Example 9.52

Find the area of the region bounded by x - axis, the sine curve $y = \sin x$, the lines $x = 0$ and $x = 2\pi$

Solution:



The portion of the curve $x = 0$ to $x = \pi$, lies above x - axis and the portion of the curve $x = \pi$ to $x = 2\pi$, lies below x - axis.

$$\begin{aligned}\text{So, Area } A &= \int_a^b y \, dx + \int_c^d (-y) \, dx \\ &= \int_0^\pi \sin x \, dx + \int_\pi^{2\pi} (-\sin x) \, dx \\ &= -(\cos x)_0^\pi + (\cos x)_\pi^{2\pi} \\ &= -\cos \pi + \cos 0 + \cos 2\pi - \cos \pi \\ &= -(-1) + 1 + 1 - (-1) \\ &= 1 + 1 + 1 + 1 \\ &= 4 \text{ square units.}\end{aligned}$$

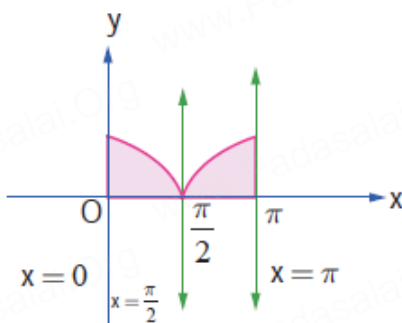
Example 9.53

Find the area of the region bounded by x - axis, the curve $y = |\cos x|$, the lines $x = 0$ and $x = \pi$

Solution:

$$y = |\cos x|, \text{ then}$$

$$\text{The given curve is } y = \begin{cases} \cos x, & 0 \leq x \leq \frac{\pi}{2} \\ -\cos x, & \frac{\pi}{2} \leq x \leq \pi \end{cases}$$



It lies above the x - axis,

$$A = \int_a^b y \, dx$$

$$= \int_0^{\frac{\pi}{2}} y \, dx + \int_{\frac{\pi}{2}}^\pi y \, dx$$

$$= \int_0^{\frac{\pi}{2}} \cos x \, dx + \int_{\frac{\pi}{2}}^\pi (-\cos x) \, dx$$

$$= \int_0^{\frac{\pi}{2}} \cos x \, dx - \int_{\frac{\pi}{2}}^\pi \cos x \, dx$$

$$= (\sin x)_0^{\frac{\pi}{2}} - (\sin x)_{\frac{\pi}{2}}^\pi$$

$$= \left(\sin \frac{\pi}{2} - \sin 0 \right) - \left(\sin \pi - \sin \frac{\pi}{2} \right)$$

$$= 1 - 0 - (0 - 1)$$

$$= 1 - (-1)$$

$$= 1 + 1$$

$$= 2 \text{ square units.}$$

Example 9.54

Find the area of the region bounded between the parabolas $y^2 = 4x$ and $x^2 = 4y$

Solution:

$y^2 = 4x$, is open up and $x^2 = 4y$ open left.

Solving the given equation for x

From, $x^2 = 4y$ gives $x^4 = 16y^2$

Substituting $y^2 = 4x$ in the above

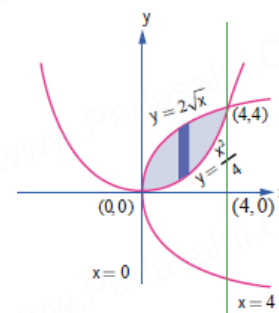
$$x^4 = 16(4x)$$

$$x^4 = 64x$$

$$x^4 - 64x = 0$$

$$x(x^3 - 64) = 0 \text{ gives,}$$

$$x = 0, \text{ and } x^3 - 64 = 0 \Rightarrow x^3 = 64 \Rightarrow x = 4$$



$$\text{So, Area } A = \int_a^b (y_U - y_L) \, dx$$

$$\text{Upper curve } y^2 = 4x \text{ gives } y = 2\sqrt{x}$$

$$\text{Lower curve } x^2 = 4y \text{ gives } y = \frac{x^2}{4}$$

$$\text{Area } A = \int_0^4 (y_U - y_L) \, dx$$

$$\begin{aligned}
 A &= \int_0^{\frac{\pi}{2}} \left(2\sqrt{x} - \frac{x^2}{4} \right) dx \\
 &= \left(2 \left(\frac{x^{\frac{3}{2}}}{\frac{3}{2}} \right) - \frac{1}{4} \left(\frac{x^3}{3} \right) \right) \bigg|_0^4 \\
 &= \left[\frac{4}{3} (4)^{\frac{3}{2}} - \frac{1}{12} \times (4)^3 \right] - 0 \\
 &= \left[\frac{4}{3} (4)\sqrt{4} - \frac{64}{12} \right] \\
 &= \left[\frac{16}{3} (2) - \frac{16}{3} \right] \\
 &= \frac{32}{3} - \frac{16}{3} \\
 &= \frac{32-16}{3} \\
 A &= \frac{16}{3} \text{ square units.}
 \end{aligned}$$

Example 9.55

Find the area of the region bounded between the parabola $x^2 = y$ and the curve $y = |x|$.

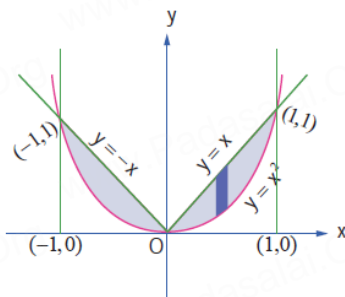
Solution:

$y = |x|$, then

The given curve is $y = \begin{cases} x, & \text{if } x \geq 0 \\ -x, & \text{if } x \leq 0 \end{cases}$

The parabola $x^2 = y$ intersects at $(1,1)$, $(-1,1)$

It is symmetrical,



$$\text{So, Area } A = 2 \int_a^b (y_U - y_L) dx$$

Upper curve $y = |x|$

Lower curve $y = x^2$

$$\text{Area } A = 2 \int_0^1 (y_U - y_L) dx$$

$$A = 2 \int_0^1 (x - x^2) dx$$

$$= 2 \left(\frac{x^2}{2} - \frac{x^3}{3} \right) \bigg|_0^1$$

$$= 2 \left(\frac{1}{2} - \frac{1}{3} \right)$$

$$\begin{aligned}
 &= 2 \left(\frac{3-2}{6} \right) \\
 &= 2 \left(\frac{1}{6} \right) \\
 &= \frac{1}{3} \text{ square units.}
 \end{aligned}$$

Example 9.56

Find the area of the region bounded by $y = \cos x$, $y = \sin x$, the lines $x = \frac{\pi}{4}$ and $x = \frac{5\pi}{4}$

Solution:

Upper curve $y = \sin x$, $x = \frac{\pi}{4}$ and $x = \frac{5\pi}{4}$

Lower curve $y = \cos x$, $x = \frac{\pi}{4}$ and $x = \frac{5\pi}{4}$

$$\text{So, Area } A = \int_{\frac{\pi}{4}}^{\frac{5\pi}{4}} (y_U - y_L) dx$$

$$= \int_{\frac{\pi}{4}}^{\frac{5\pi}{4}} (\sin x - \cos x) dx$$

$$= (-\cos x - \sin x) \bigg|_{\frac{\pi}{4}}^{\frac{5\pi}{4}}$$

$$= -\left(\cos \frac{5\pi}{4} + \sin \frac{5\pi}{4} \right) + \left(\cos \frac{\pi}{4} + \sin \frac{\pi}{4} \right)$$

$$= -\left(\left(-\frac{1}{\sqrt{2}} \right) + \left(-\frac{1}{\sqrt{2}} \right) \right) + \left(\left(\frac{1}{\sqrt{2}} \right) + \left(\frac{1}{\sqrt{2}} \right) \right)$$

$$= -\left(-\frac{1}{\sqrt{2}} - \frac{1}{\sqrt{2}} \right) + \left(\frac{1}{\sqrt{2}} + \frac{1}{\sqrt{2}} \right)$$

$$= \left(\frac{1}{\sqrt{2}} + \frac{1}{\sqrt{2}} \right) + \left(\frac{1}{\sqrt{2}} + \frac{1}{\sqrt{2}} \right)$$

$$= 4 \left(\frac{1}{\sqrt{2}} \right)$$

$$= 2\sqrt{2}\sqrt{2} \left(\frac{1}{\sqrt{2}} \right)$$

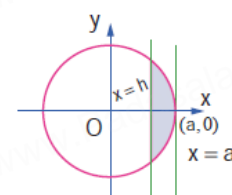
$$= 2\sqrt{2} \text{ Square units.}$$

Example 9.57

The region enclosed by the circle $x^2 + y^2 = a^2$ is divided into two segments by the line $x = h$. Find the area of the smaller segment.

Solution:

$$A = 2 \int_a^h \sqrt{a^2 - x^2} dx$$



$$\begin{aligned}
 &= 2 \left[\frac{x\sqrt{a^2-x^2}}{2} + \frac{a^2}{2} \sin^{-1} \left(\frac{x}{a} \right) \right]_a^h \\
 &= 2 \left[\left(0 + \frac{a^2}{2} \sin^{-1}(1) \right) - \left(\frac{h\sqrt{a^2-h^2}}{2} + \frac{a^2}{2} \sin^{-1} \left(\frac{h}{a} \right) \right) \right] \\
 &= a^2 \left(\frac{\pi}{2} \right) - h\sqrt{a^2-h^2} - a^2 \sin^{-1} \left(\frac{h}{a} \right) \\
 &= a^2 \left(\frac{\pi}{2} - \sin^{-1} \left(\frac{h}{a} \right) \right) - h\sqrt{a^2-h^2} \\
 &= a^2 \cos^{-1} \left(\frac{h}{a} \right) - h\sqrt{a^2-h^2}
 \end{aligned}$$

Example 9.58

Find the area of the region in the first quadrant bounded by the parabola $y^2 = 4x$, the line $x + y = 3$ and y -axis.

Solution: Solving, $y^2 = 4x$ and $x + y = 3$,

Substituting, $y = 3 - x$

$$(3 - x)^2 = 4x$$

$$9 + x^2 - 6x = 4x$$

$$x^2 - 10x + 9 = 0$$

$$(x - 9)(x - 1) = 0$$

$$x = 9, 1$$

When $x = 9$ in $x + y = 3$,

$$9 + y = 3$$

$$y = 3 - 9$$

$$y = -6$$

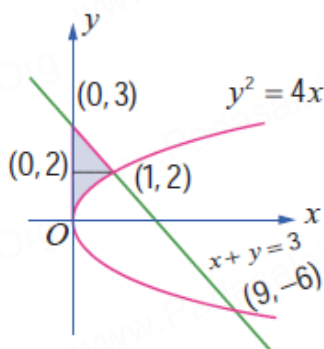
When $x = 1$ in $x + y = 3$,

$$1 + y = 3$$

$$y = 3 - 1$$

$$y = 2$$

$\therefore (1, 2)$ and $(9, -6)$ is the point of intersection.



$$x = \begin{cases} \frac{y^2}{4}, & 0 \leq y \leq 2 \\ 3 - y, & 2 \leq y \leq 3 \end{cases}$$

$$A = \int_0^2 x \, dy + \int_2^3 x \, dy$$

$$= \int_0^2 \frac{y^2}{4} \, dy + \int_2^3 (3 - y) \, dy$$

$$= \left(\frac{y^3}{12} \right)_0^2 + \left(3y - \frac{y^2}{2} \right)_2^3$$

$$= \left(\frac{8}{12} \right) - 0 + \left(9 - \frac{9}{2} \right) - \left(6 - \frac{4}{2} \right)$$

$$= \frac{2}{3} + 9 - \frac{9}{2} - 6 + 2$$

$$= \frac{2}{3} - \frac{9}{2} + 11 - 6$$

$$= \frac{4-27}{6} + 5$$

$$= \frac{-23}{6} + 5$$

$$= \frac{-23+30}{6}$$

$$= \frac{7}{6}$$

Example 9.59

Find, by integration, the area of the region bounded by the lines $5x - 2y = 15$, $x + y + 4 = 0$ and the x -axis.

Solution:

The lines $5x - 2y = 15$, $x + y + 4 = 0$

intersect at $(1, -5)$. The line $5x - 2y = 15$

meets x -axis at $(3, 0)$. The line $x + y + 4 = 0$

meets the x -axis at $(-4, 0)$. The required area

is shaded in diagram. It lies below the x -axis. It

can be computed either by considering vertical

strips or horizontal strips.

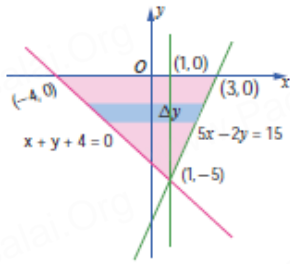
When we do by horizontal strips, there is no

need to subdivide the region. In this case, the

area is bounded on the right by the line

$5x - 2y = 15$ and on the left by $x + y + 4 = 0$.

So, we get



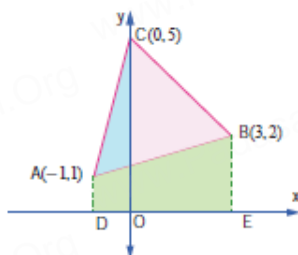
$$\text{Area } A = \int_{-5}^0 (x_R - x_L) dy$$

$$\begin{aligned} &= \int_{-5}^0 \left[7 + \frac{7y}{5} \right] dy \\ &= \left[7y + \frac{7y^2}{10} \right]_{-5}^0 \\ &= 0 - \left[-35 + \frac{7 \times 25}{10} \right] \\ &= - \left[-35 + \frac{7 \times 5}{2} \right] \\ &= - \left[-35 + \frac{35}{2} \right] \\ &= - \left[\frac{-70 + 35}{2} \right] \\ &= - \left[\frac{-35}{2} \right] \\ &= \frac{35}{2} \end{aligned}$$

Example 9.60

Using integration find the area of the region bounded by triangle ABC , whose vertices A , B , and C are $(-1, 1)$, $(3, 2)$, and $(0, 5)$ respectively.

Solution:



$$\text{Equation of line is } \frac{y - y_1}{y_2 - y_1} = \frac{x - x_1}{x_2 - x_1}$$

$$\begin{aligned} \text{Equation of AB is } \frac{y - 1}{2 - 1} &= \frac{x + 1}{3 + 1} \\ \frac{y - 1}{1} &= \frac{x + 1}{4} \end{aligned}$$

$$4(y - 1) = 1(x + 1)$$

$$4y - 4 = x + 1$$

$$4y = x + 1 + 4$$

$$4y = x + 5$$

$$y = \frac{x+5}{4}$$

$$\text{Equation of BC is } \frac{y - 5}{2 - 5} = \frac{x - 0}{3 - 0}$$

$$\frac{y - 5}{-3} = \frac{x}{3}$$

$$3(y - 5) = -3x$$

$$3y - 15 = -3x$$

$$3y = -3x + 15$$

$$\text{Dividing by 3, } y = -x + 5$$

$$\text{Equation of AC is } \frac{y - 1}{5 - 1} = \frac{x + 1}{0 + 1}$$

$$\frac{y - 1}{4} = \frac{x + 1}{1}$$

$$1(y - 1) = 4(x + 1)$$

$$y - 1 = 4x + 4$$

$$y = 4x + 4 + 1$$

$$y = 4x + 5$$

$$\begin{aligned} \text{Area of } \Delta ABC &= \text{Area of DACO} + \text{Area of OCBE} \\ &\quad - \text{Area of DABE} \end{aligned}$$

$$= \int_{-1}^0 (4x + 5) dx + \int_0^3 (-x + 5) dx - \frac{1}{4} \int_{-1}^3 (x + 5) dx$$

$$= \left[\frac{4x^2}{2} + 5x \right]_{-1}^0 + \left[-\frac{x^2}{2} + 5x \right]_0^3 - \frac{1}{4} \left[\frac{x^2}{2} + 5x \right]_{-1}^3$$

$$= 0 - (2 - 5) + \left(-\frac{9}{2} + 15 \right) - 0 - \frac{1}{4} \left(\frac{9}{2} + 15 \right)$$

$$+ \frac{1}{4} \left(\frac{1}{2} - 5 \right)$$

$$= 0 - (-3) + \left(\frac{-9+30}{2} \right) - \frac{1}{4} \left(\frac{9+30}{2} \right) + \frac{1}{4} \left(\frac{1-10}{2} \right)$$

$$= 3 + \left(\frac{21}{2} \right) - \frac{1}{4} \left(\frac{39}{2} \right) + \frac{1}{4} \left(\frac{-9}{2} \right)$$

$$= 3 + \left(\frac{21}{2} \right) - \frac{39}{8} - \frac{9}{8}$$

$$= 3 + \left(\frac{21}{2} \right) - \frac{39+9}{8}$$

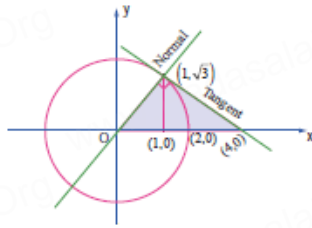
$$= 3 + \left(\frac{21}{2} \right) - \frac{48}{8}$$

$$= 3 + \left(\frac{21}{2} \right) - 6$$

$$= \frac{21}{2} - 3 = \frac{21-6}{2} = \frac{15}{2}$$

Example 9.61 Using integration, find the area of the region which is bounded by x -axis, the tangent and normal to the circle $x^2 + y^2 = 4$ drawn at $(1, \sqrt{3})$.

Solution:



Equation of the tangent to $x^2 + y^2 = 4$ is

$xx_1 + yy_1 = 4$. Given $(1, \sqrt{3})$ is the point.

Equation of the tangent is $x + \sqrt{3}y = 4$

So, Equation of normal is of the form

$\sqrt{3}x - y + k = 0$. It passes $(1, \sqrt{3})$

So, $\sqrt{3}(1) - \sqrt{3} + k = 0$, gives $k = 0$

So, Equation of normal is $\sqrt{3}x - y = 0$

The required area is the area of the region

bounded between $y = \sqrt{3}x$ and $x + \sqrt{3}y = 4$,

$y = 0$ and $y = \sqrt{3}$. It can be obtained by,

$$\begin{aligned} \text{Area } A &= \int_0^{\sqrt{3}} (x_R - x_L) dy \\ &= \int_0^{\sqrt{3}} \left[(4 - y\sqrt{3}) - \frac{y}{\sqrt{3}} \right] dy \\ &= \left[4y - \frac{y^2}{2}\sqrt{3} - \frac{y^2}{2\sqrt{3}} \right]_0^{\sqrt{3}} \\ &= 4\sqrt{3} - \frac{3}{2}\sqrt{3} - \frac{3}{2\sqrt{3}} \\ &= \frac{4\sqrt{3}(2\sqrt{3}) - 3\sqrt{3}(\sqrt{3}) - 3}{2\sqrt{3}} \\ &= \frac{24 - 9 - 3}{2\sqrt{3}} \\ &= \frac{12}{2\sqrt{3}} \\ &= \frac{12}{2\sqrt{3}} = \frac{6}{\sqrt{3}} \times \frac{\sqrt{3}}{\sqrt{3}} \\ &= \frac{6\sqrt{3}}{3} = 2\sqrt{3} \end{aligned}$$

EXERCISE 9.8

1. Find the area of the region bounded by $3x - 2y + 6 = 0$, $x = -3$, $x = 1$ and x -axis.

Solution:

$$3x - 2y + 6 = 0$$

on x -axis, $y = 0$

$$\therefore 3x + 6 = 0$$

$$3x = -6$$

$$x = -2$$

The line passes through $(-2, 0)$ and

$x = -3, x = 1$

From $3x - 2y + 6 = 0$

$$3x + 6 = 2y$$

$$y = \frac{3x+6}{2}$$

$$\text{Area } A = \int_{-3}^{-2} (-y) dx + \int_{-2}^1 y dx$$

$$= -\frac{1}{2} \int_{-3}^{-2} (3x + 6) dx + \frac{1}{2} \int_{-2}^1 (3x + 6) dx$$

$$= -\frac{1}{2} \left[3 \frac{x^2}{2} + 6x \right]_{-3}^{-2} + \frac{1}{2} \left[3 \frac{x^2}{2} + 6x \right]_{-2}^1$$

$$= -\frac{1}{2} \left[\left(3 \frac{4}{2} - 12 \right) - \left(3 \frac{9}{2} - 18 \right) \right]$$

$$+ \frac{1}{2} \left[\left(3 \frac{1}{2} + 6 \right) - \left(3 \frac{4}{2} - 12 \right) \right]$$

$$= -\frac{1}{2} \left[(6 - 12) - \left(\frac{27}{2} - 18 \right) \right]$$

$$+ \frac{1}{2} \left[\left(\frac{3}{2} + 6 \right) - (6 - 12) \right]$$

$$= -\frac{1}{2} \left[(-6) - \left(\frac{27-36}{2} \right) \right] + \frac{1}{2} \left[\left(\frac{3+12}{2} \right) - (-6) \right]$$

$$= -\frac{1}{2} \left[(-6) - \left(-\frac{9}{2} \right) \right] + \frac{1}{2} \left[\left(\frac{15}{2} \right) + 6 \right]$$

$$= -\frac{1}{2} \left[(-6) + \frac{9}{2} \right] + \frac{1}{2} \left[\frac{15}{2} + 6 \right]$$

$$= -\frac{1}{2} \left[\frac{-12+9}{2} \right] + \frac{1}{2} \left[\frac{15+12}{2} \right]$$

$$= -\frac{1}{2} \left[\frac{-3}{2} \right] + \frac{1}{2} \left[\frac{27}{2} \right]$$

$$= \frac{3}{4} + \frac{27}{4}$$

$$= \frac{3+27}{4}$$

$$= \frac{30}{4} = \frac{15}{2}$$

2. Find the area of the region bounded by $2x - y + 1 = 0$, $y = -1$, $y = 3$ and y -axis.

Solution:

$$2x - y + 1 = 0$$

on y -axis, $x = 0$

$$\therefore -y + 1 = 0$$

$$y = 1$$

The line passes through $(0, 1)$ and

$$y = -1, y = 3$$

From $2x - y + 1 = 0$

$$2x = y - 1$$

$$x = \frac{y-1}{2}$$

$$\begin{aligned} \text{Area } A &= \int_{-1}^1 (-x) dy + \int_{-1}^3 x dy \\ &= -\frac{1}{2} \int_{-1}^1 (y-1) dy + \frac{1}{2} \int_{-1}^3 (y-1) dy \\ &= -\frac{1}{2} \left[\frac{y^2}{2} - y \right]_{-1}^1 + \frac{1}{2} \left[\frac{y^2}{2} - y \right]_{-1}^3 \\ &= -\frac{1}{2} \left[\left(\frac{1}{2} - 1 \right) - \left(\frac{1}{2} + 1 \right) \right] \\ &\quad + \frac{1}{2} \left[\left(\frac{9}{2} - 3 \right) - \left(\frac{1}{2} - 1 \right) \right] \\ &= -\frac{1}{2} \left[\left(\frac{1-2}{2} \right) - \left(\frac{1+2}{2} \right) \right] + \frac{1}{2} \left[\left(\frac{9-6}{2} \right) - \left(\frac{1-2}{2} \right) \right] \\ &= -\frac{1}{2} \left[\left(-\frac{1}{2} \right) - \left(\frac{3}{2} \right) \right] + \frac{1}{2} \left[\left(\frac{3}{2} \right) - \left(-\frac{1}{2} \right) \right] \\ &= -\frac{1}{2} \left[-\frac{1}{2} - \frac{3}{2} \right] + \frac{1}{2} \left[\frac{3}{2} + \frac{1}{2} \right] \\ &= -\frac{1}{2} \left[-\frac{1+3}{2} \right] + \frac{1}{2} \left[\frac{3+1}{2} \right] \\ &= -\frac{1}{2} \left[-\frac{4}{2} \right] + \frac{1}{2} \left[\frac{4}{2} \right] \\ &= \frac{4}{4} + \frac{4}{4} \\ &= 1 + 1 = 2 \end{aligned}$$

3. Find the area of the region bounded by the curve $2 + x - x^2 + y = 0$, x -axis, $x = -3$ and $x = 3$.

Solution: To find the limit put $y = 0$, in

$$2 + x - x^2 + y = 0 \text{ gives}$$

$$2 + x - x^2 = 0$$

$$x^2 - x - 2 = 0$$

$$(x-2)(x+1) = 0$$

$$x = 2 \text{ and } x = -1$$

$$\text{Area } A = \int_{-3}^{-1} y dx + \int_{-1}^2 (-y) dx + \int_2^3 y dx$$

$$\text{Let } A_1 = \int_{-3}^{-1} (x^2 - x - 2) dx$$

$$= \left[\frac{x^3}{3} - \frac{x^2}{2} - 2x \right]_{-3}^{-1}$$

$$= \left[\left(-\frac{1}{3} - \frac{1}{2} + 2 \right) - \left(-\frac{27}{3} - \frac{9}{2} + 6 \right) \right]$$

$$= \left[\left(\frac{-2-3}{6} + 2 \right) - \left(-9 - \frac{9}{2} + 6 \right) \right]$$

$$= \left[\left(-\frac{5}{6} + 2 \right) - \left(-3 - \frac{9}{2} \right) \right]$$

$$= \left[\left(\frac{-5+12}{6} \right) - \left(\frac{-6-9}{2} \right) \right]$$

$$= \left(\frac{7}{6} \right) - \left(\frac{-15}{2} \right)$$

$$= \frac{7}{6} + \frac{15}{2}$$

$$= \frac{7+45}{6}$$

$$= \frac{52}{6}$$

$$A_2 = \int_{-1}^2 -(x^2 - x - 2) dx$$

$$= \int_{-1}^2 (x^2 - x - 2) dx$$

$$= \left[\frac{x^3}{3} - \frac{x^2}{2} - 2x \right]_{-1}^2$$

$$= \left[\left(\frac{8}{3} - \frac{4}{2} - 4 \right) - \left(-\frac{1}{3} - \frac{1}{2} + 2 \right) \right]$$

$$= \left[\left(\frac{-2-3}{6} + 2 \right) - \left(\frac{8}{3} - 2 - 4 \right) \right]$$

$$= \left[\left(-\frac{5}{6} + 2 \right) - \left(\frac{8}{3} - 6 \right) \right]$$

$$= \left[\left(\frac{-5+12}{6} \right) - \left(\frac{8-18}{3} \right) \right]$$

$$= \left(\frac{7}{6} \right) - \left(\frac{-10}{3} \right)$$

$$= \frac{7}{6} + \frac{10}{3}$$

$$= \frac{7+20}{6}$$

$$= \frac{27}{6}$$

$$A_3 = \int_2^3 (x^2 - x - 2) dx$$

$$= \left[\frac{x^3}{3} - \frac{x^2}{2} - 2x \right]_2^3$$

$$= \left[\left(\frac{27}{3} - \frac{9}{2} - 6 \right) - \left(\frac{8}{3} - \frac{4}{2} - 4 \right) \right]$$

$$\begin{aligned}
 &= \left[\left(9 - \frac{9}{2} - 6 \right) - \left(\frac{8}{3} - 2 - 4 \right) \right] \\
 &= \left[\left(3 - \frac{9}{2} \right) - \left(\frac{8}{3} - 6 \right) \right] \\
 &= \left[\left(\frac{6-9}{2} \right) - \left(\frac{8-18}{3} \right) \right] \\
 &= \left(\frac{-3}{2} \right) - \left(\frac{-10}{3} \right) \\
 &= \frac{-3}{2} + \frac{10}{3} \\
 &= \frac{-9+20}{6} \\
 &= \frac{11}{6}
 \end{aligned}$$

$$\text{Area } A = A_1 + A_2 + A_3$$

$$\begin{aligned}
 &= \frac{52}{6} + \frac{27}{6} + \frac{11}{6} \\
 &= \frac{52+27+11}{6} \\
 &= \frac{90}{6} \\
 &= 15
 \end{aligned}$$

4. Find the area of the region bounded by the line $y = 2x + 5$ and parabola $y = x^2 - 2x$

Solution:

To solve substitute $y = 2x + 5$ in

$y = x^2 - 2x$, we get,

$$2x + 5 = x^2 - 2x$$

$$x^2 - 2x - 2x - 5 = 0$$

$$x^2 - 4x - 5 = 0$$

$$(x - 5)(x + 1) = 0 \text{ which gives}$$

$$x = 5 \text{ and } x = -1$$

$$\begin{aligned}
 \text{Area } A &= \int_{-1}^5 (y_U - y_L) dx \\
 &= \int_{-1}^5 (2x + 5 - x^2 + 2x) dx \\
 &= \int_{-1}^5 (4x + 5 - x^2) dx \\
 &= \left[4 \frac{x^2}{2} + 5x - \frac{x^3}{3} \right]_{-1}^5 \\
 &= \left[2x^2 + 5x - \frac{x^3}{3} \right]_{-1}^5 \\
 &= \left[\left(50 + 25 - \frac{125}{3} \right) - \left(2 - 5 + \frac{1}{3} \right) \right] \\
 &= \left[\left(75 - \frac{125}{3} \right) - \left(-3 + \frac{1}{3} \right) \right]
 \end{aligned}$$

$$\begin{aligned}
 &= \left[\left(\frac{225-125}{3} \right) - \left(\frac{-9+1}{3} \right) \right] \\
 &= \left[\left(\frac{100}{3} \right) - \left(\frac{-8}{3} \right) \right] \\
 &= \frac{100}{3} + \frac{8}{3} \\
 &= \frac{108}{3} \\
 &= 36
 \end{aligned}$$

5. Find the area of the region bounded between the curves $y = \sin x$ and $y = \cos x$ and the lines $x = 0$ and $x = \pi$

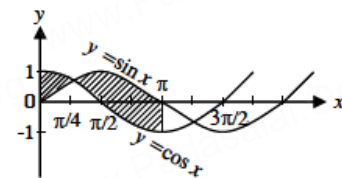
Solution:

To solve substitute $y = \sin x$ in

$y = \cos x$, we get,

$\sin x = \cos x$ gives,

$$x = \frac{\pi}{4}$$



$$\begin{aligned}
 \text{Area } A &= \int_0^{\pi/4} (\cos x - \sin x) dx + \int_{\pi/4}^{\pi} (\sin x - \cos x) dx \\
 &= [\sin x + \cos x]_0^{\pi/4} + [-\cos x - \sin x]_{\pi/4}^{\pi} \\
 &= \left[\sin \frac{\pi}{4} + \cos \frac{\pi}{4} \right] - [\sin 0 + \cos 0] \\
 &\quad + [-\cos \pi - \sin \pi] - \left[-\cos \frac{\pi}{4} - \sin \frac{\pi}{4} \right] \\
 &= \left[\frac{1}{\sqrt{2}} + \frac{1}{\sqrt{2}} \right] - [0 + 1] \\
 &\quad + [-(-1) - 0] - \left[-\frac{1}{\sqrt{2}} - \frac{1}{\sqrt{2}} \right] \\
 &= \left[\frac{2}{\sqrt{2}} \right] - 1 + 1 - \left[\frac{-2}{\sqrt{2}} \right] \\
 &= \frac{2}{\sqrt{2}} + \frac{2}{\sqrt{2}} \\
 &= \frac{2+2}{\sqrt{2}} \\
 &= \frac{4}{\sqrt{2}} \\
 &= \frac{2 \times \sqrt{2} \times \sqrt{2}}{\sqrt{2}} \\
 &= 2\sqrt{2}
 \end{aligned}$$

6. Find the area of the region bounded by

$y = \tan x$, $y = \cot x$ and the lines

$x = 0$ and $x = \frac{\pi}{2}$, $y = 0$.

Solution:

To solve substitute $y = \tan x$ in

$y = \cot x$, we get,

$\tan x = \cot x$ gives,

$$x = \frac{\pi}{4}$$

$$\text{Since, } \tan \frac{\pi}{4} = \cot \frac{\pi}{4} = 1$$

$$\text{Area } A = \int_0^{\frac{\pi}{4}} \tan x \, dx + \int_{\frac{\pi}{4}}^{\frac{\pi}{2}} \cot x \, dx$$

$$= [\log \sec x]_0^{\frac{\pi}{4}} + [\log \sin x]_{\frac{\pi}{4}}^{\frac{\pi}{2}}$$

$$= \log \sec \frac{\pi}{4} - \log \sec 0 + \log \sin \frac{\pi}{2} - \log \sin \frac{\pi}{4}$$

$$= \log \sqrt{2} - 0 - 0 - \log \frac{1}{\sqrt{2}}$$

$$= \log \sqrt{2} - (\log 1 - \log \sqrt{2})$$

$$= \log \sqrt{2} - (0 - \log \sqrt{2})$$

$$= \log \sqrt{2} - (-\log \sqrt{2})$$

$$= \log \sqrt{2} + \log \sqrt{2}$$

$$= 2 \log \sqrt{2}$$

$$= \log (\sqrt{2})^2$$

$$= \log 2$$

7. Find the area of the region bounded by the parabola $y^2 = x$ and the line $y = x - 2$.

Solution: From $y = x - 2$

To solve substitute $x = y + 2$ in

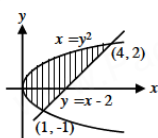
$y^2 = x$, we get

$$y^2 = y + 2$$

$$y^2 - y - 2 = 0$$

$$(y - 2)(y + 1) = 0$$

$$y = 2 \text{ and } y = -1$$



$$\text{Area } A = \int_{-1}^2 (y + 2 - y^2) \, dy$$

$$= \left[\frac{y^2}{2} + 2y - \frac{y^3}{3} \right]_{-1}^2$$

$$= \left[\left(2 + 4 - \frac{8}{3} \right) - \left(\frac{1}{2} - 2 + \frac{1}{3} \right) \right]$$

$$= \left[\left(6 - \frac{8}{3} \right) - \left(-2 + \frac{3+2}{6} \right) \right]$$

$$= \left[\left(6 - \frac{8}{3} \right) - \left(-2 + \frac{5}{6} \right) \right]$$

$$= \left[\left(\frac{18-8}{3} \right) - \left(\frac{-12+5}{6} \right) \right]$$

$$= \left[\left(\frac{10}{3} \right) - \left(-\frac{7}{6} \right) \right]$$

$$= \frac{10}{3} + \frac{7}{6}$$

$$= \frac{20+7}{6}$$

$$= \frac{27}{6}$$

$$= \frac{9}{2}$$

8. Father of a family wishes to divide his square field bounded by $x = 0$, $x = 4$, $y = 4$ and $y = 0$ along the curve $y^2 = 4x$ and $x^2 = 4y$ into three equal parts for his wife, daughter and son. Is it possible to divide? If so, find the area to be divided among them.

Solution: Given $y^2 = 4x$ and $x^2 = 4y$

$$\text{So, } 16y^2 = 64x$$

$$\text{Squaring } x^2 = 4y \text{ we get } x^4 = 16y^2$$

$$\text{Hence } x^4 = 64x$$

$$x^4 - 64x = 0$$

$$x(x^3 - 64) = 0 \text{ which gives}$$

$$x = 0 \text{ and } x^3 = 64 \Rightarrow x = 4$$

The given curves intersect at $(0,0)$, $(4,4)$

Area of square field = $4 \times 4 = 16$

Area of the middle portion, the area between

the given curves $A = \int_0^4 (y_U - y_L) \, dx$

From upper curve $y^2 = 4x$ we get $y = 2\sqrt{x}$

From lower curve $x^2 = 4y$ we get $y = \frac{x^2}{4}$

$$\therefore A = \int_0^4 \left(2\sqrt{x} - \frac{x^2}{4} \right) dx$$

$$\begin{aligned}
&= \left[2 \frac{x^{\frac{3}{2}}}{\frac{3}{2}} - \frac{1}{4} \frac{x^3}{3} \right]_0^4 \\
&= \left[2 \times \frac{2}{3} x^{\frac{3}{2}} - \frac{1}{12} x^3 \right]_0^4 \\
&= \left[\frac{4}{3} 4^{\frac{3}{2}} - \frac{1}{12} 4^3 \right] - 0 \\
&= \left[\frac{4}{3} 4 \times \sqrt{4} - \frac{64}{12} \right] \\
&= \left[\frac{16}{3} \times 2 - \frac{16}{3} \right] \\
&= \left[\frac{32}{3} - \frac{16}{3} \right] \\
&= \frac{32-16}{3} \\
&= \frac{16}{3}
\end{aligned}$$

Intersecting area is $\frac{16}{3}$ square units, the remaining each of two parts also $\frac{16}{3}$ square units. Hence the total area can be divided equally among the three.

9. The curve $y = (x - 2)^2 + 1$ has a minimum point at P . A point Q on the curve is such that the slope of PQ is 2. Find the area bounded by the curve and the chord PQ .

Solution: Given $y = (x - 2)^2 + 1$

$$\frac{dy}{dx} = 2(x - 2)$$

Substituting $\frac{dy}{dx} = 0$, we get $x - 2 = 0$,

$$\text{So, } x = 2$$

$$\frac{d^2y}{dx^2} = 2 > 0$$

Hence the curve is minimum at $x = 2$

When $x = 2$, $y = (x - 2)^2 + 1$ gives $y = 1$

So, the point P is $(2, 1)$

Given : A point Q on the curve is such that the slope of PQ is 2

Equation of the chord PQ is

$$y - y_1 = m(x - x_1)$$

Here (x_1, y_1) is $(2, 1)$ and $m = 2$

Equation of PQ is $y - 1 = 2(x - 2)$

$$y - 1 = 2x - 4$$

$$y = 2x - 4 + 1$$

$$y = 2x - 3$$

Solving the curve $y = (x - 2)^2 + 1$

and the line PQ $y = 2x - 3$ we get,

$$(x - 2)^2 + 1 = 2x - 3$$

$$x^2 - 4x + 4 + 1 = 2x - 3$$

$$x^2 - 4x + 5 = 2x - 3$$

$$x^2 - 4x + 5 - 2x + 3 = 0$$

$$x^2 - 6x + 8 = 0$$

$(x - 4)(x - 2) = 0$ which gives

$$x = 4 \text{ and } x = 2$$

$$\text{Area } A = \int_2^4 (y_1 - y_2) dx$$

Curve $y_1 = 2x - 3$ and $y_2 = x^2 - 4x + 5$

$$y_1 - y_2 = 2x - 3 - (x^2 - 4x + 5)$$

$$= 2x - 3 - x^2 + 4x - 5$$

$$= -x^2 + 6x - 8$$

$$\text{Area } A = \int_2^4 (-x^2 + 6x - 8) dx$$

$$= \left[-\frac{x^3}{3} + 6\frac{x^2}{2} - 8x \right]_2^4$$

$$= \left[-\frac{x^3}{3} + 3x^2 - 8x \right]_2^4$$

$$= \left[-\frac{64}{3} + 3(16) - 32 \right] - \left[-\frac{8}{3} + 3(4) - 16 \right]$$

$$= \left[-\frac{64}{3} + 48 - 32 \right] - \left[-\frac{8}{3} + 12 - 16 \right]$$

$$= \left[-\frac{64}{3} + 16 \right] - \left[-\frac{8}{3} - 4 \right]$$

$$= -\frac{64}{3} + 16 + \frac{8}{3} + 4$$

$$= \frac{-64+8}{3} + 20$$

$$= \frac{-56}{3} + 20$$

$$= \frac{-56+60}{3}$$

$$= \frac{4}{3} \text{ Square units.}$$

10. Find the area of the region common to the

circle $x^2 + y^2 = 16$ and parabola $y^2 = 6x$

Solution: $x^2 + y^2 = 16$ is the circle of

radius 4, with centre at the origin. Parabola

$y^2 = 6x$ is open right through the origin.

Solving, $x^2 + y^2 = 16$ and $y^2 = 6x$ we get

$$x^2 + 6x = 16$$

$$x^2 + 6x - 16 = 0$$

$(x + 8)(x - 2) = 0$ which gives

$$x = -8 \text{ and } x = 2$$

Since, $x \neq -8$, $x = 2$ is the value

So, to find the area between the two curves

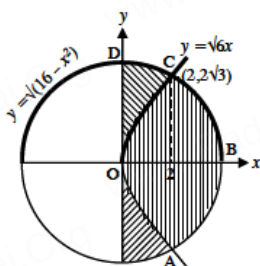
at $x = 2$ and $x = 4$

From $y^2 = 6x$ we get $y = \sqrt{6}\sqrt{x}$ and

From $x^2 + y^2 = 16$

$$y^2 = 16 - x^2$$

$$y = \sqrt{16 - x^2}$$



As the area is symmetrical,

$$A = 2 \int_0^2 (\sqrt{6}\sqrt{x}) dx + 2 \int_2^4 (\sqrt{16 - x^2}) dx$$

$$\text{Let } A_1 = 2 \int_0^2 (\sqrt{6}\sqrt{x}) dx$$

$$= 2\sqrt{6} \int_0^2 (\sqrt{x}) dx$$

$$= 2\sqrt{6} \int_0^2 (\sqrt{x}) dx$$

$$= 2\sqrt{6} \left[\frac{x^{\frac{3}{2}}}{\frac{3}{2}} \right]_0^2$$

$$= 2\sqrt{6} \times \frac{2}{3} \left[x^{\frac{3}{2}} \right]_0^2$$

$$= \frac{4}{3} \sqrt{6} \times \left[2^{\frac{3}{2}} \right]$$

$$= \frac{4}{3} \sqrt{6} \times [2\sqrt{2}]$$

$$= \frac{8}{3} \sqrt{12}$$

$$A_2 = 2 \int_2^4 (\sqrt{16 - x^2}) dx$$

$$= 2 \left[\frac{x}{2} \sqrt{16 - x^2} + \frac{16}{2} \sin^{-1} \left(\frac{x}{4} \right) \right]_2^4$$

$$= 2 \left[\left(0 + \frac{16}{2} \sin^{-1}(1) \right) - \left(\sqrt{12} + \frac{16}{2} \sin^{-1} \left(\frac{1}{2} \right) \right) \right]$$

$$= 2 \left[\left(\frac{16}{2} \times \frac{\pi}{2} \right) - \left(\sqrt{12} + \frac{16}{2} \times \frac{\pi}{6} \right) \right]$$

$$= 2 \left[(4\pi) - \left(\sqrt{12} + \frac{4}{3}\pi \right) \right]$$

$$= 2 \left[4\pi - \sqrt{12} - \frac{4}{3}\pi \right]$$

$$= 8\pi - 2\sqrt{12} - \frac{8}{3}\pi$$

$$= 8\pi - 2\sqrt{12} - \frac{16}{3}$$

So, required area $A = A_1 + A_2$

$$= \frac{8}{3} \sqrt{12} + 8\pi - 2\sqrt{12} - \frac{8}{3}\pi$$

$$= \frac{8\sqrt{12} - 6\sqrt{12}}{3} + \frac{24\pi - 8\pi}{3}$$

$$= \frac{2\sqrt{12}}{3} + \frac{16\pi}{3}$$

$$= \frac{2\sqrt{4 \times 3}}{3} + \frac{16\pi}{3}$$

$$= \frac{4\sqrt{3}}{3} + \frac{16\pi}{3}$$

$$= \frac{4}{3} (\sqrt{3} + 4\pi)$$

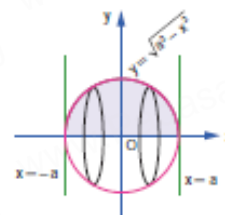
Example 9.62

Find the volume of a sphere of radius a .

Solution:

Volume of sphere can be obtained by revolving the circle. Equation of the circle with radius a is, $x^2 + y^2 = a^2$.

Its parametric form is $x = a \cos \theta$, $y = a \sin \theta$



Volume about y axis $V = \int_a^b \pi x^2 dy$

From $y = a \sin \theta$, $dy = a \cos \theta d\theta$

In the I quadrant, $\theta = 0$ to $\frac{\pi}{2}$

By symmetrical, $V = 2 \int_0^{\frac{\pi}{2}} \pi (a \cos \theta)^2 a \cos \theta d\theta$

$$= 2 \int_0^{\frac{\pi}{2}} \pi a^3 \cos^3 \theta d\theta$$

$$= 2\pi a^3 \int_0^{\frac{\pi}{2}} \cos^3 \theta d\theta$$

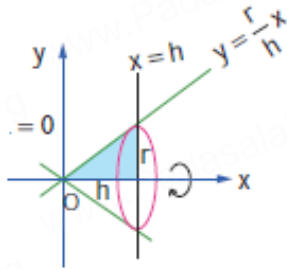
$$= 2\pi a^3 \times \frac{2}{3}$$

$$= \frac{4}{3} \pi a^3 \text{ cubic units.}$$

Example 9.63

Find the volume of a right-circular cone of base radius r and height h .

Solution:



Let $OA = h$, and $AB = r$, then in the triangle OAB when OB revolves about x axis, we the volume of the cone.

Slope of $OB = m = \tan \theta = \frac{\text{opp}}{\text{adj}} = \frac{AB}{OB} = \frac{r}{h}$

Equation of OB is $y = mx + c$ as this line passes through the origin, $c = 0$

Equation of OB is $y = \frac{r}{h}x$ and $x = 0$ to h

Volume about x axis $V = \int_a^b \pi y^2 dx$

$$= \int_0^h \pi \left(\frac{r}{h}x\right)^2 dx$$

$$= \pi \int_0^h \frac{r^2}{h^2} x^2 dx$$

$$= \pi \frac{r^2}{h^2} \int_0^h x^2 dx$$

$$= \pi \frac{r^2}{h^2} \left(\frac{x^3}{3}\right)_0^h$$

$$= \pi \frac{r^2}{h^2} \left(\frac{h^3}{3} - 0\right)$$

$$= \pi \frac{r^2}{h^2} \left(\frac{h^3}{3}\right)$$

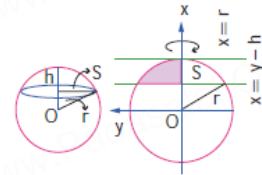
$$= \pi r^2 \left(\frac{h}{3}\right)$$

Volume of cone $V = \frac{1}{3} \pi r^2 h$ cubic units.

Example 9.64

Find the volume of the spherical cap of height h cut of from a sphere of radius r .

Solution:



Solution: If the region in the first quadrant bounded by the circle $x^2 + y^2 = r^2$, the x -axis, the lines $x = r - h$ and $x = r$ is revolved about the x -axis, then the solid generated is a spherical cap of height h cut of from a sphere of radius r . Hence, the required volume is given

by $V = \int_{r-h}^r \pi y^2 dx$

$$= \pi \int_{r-h}^r (r^2 - x^2) dx$$

$$= \pi \left[r^2 x - \frac{x^3}{3} \right]_{r-h}^r$$

$$= \pi \left[\left(r^2 r - \frac{r^3}{3} \right) - \left(r^2 (r-h) - \frac{(r-h)^3}{3} \right) \right]$$

$$= \pi \left[\left(r^3 - \frac{r^3}{3} \right) - \left(\frac{3r^2(r-h) - (r-h)^3}{3} \right) \right]$$

$$= \pi \left[\left(r^3 - \frac{r^3}{3} \right) - \left(\frac{3r^3 - 3r^2 h - (r-h)^3}{3} \right) \right]$$

$$= \pi \left[\left(\frac{3r^3 - r^3}{3} \right) - \left(\frac{3r^3 - 3r^2 h - (r-h)^3}{3} \right) \right]$$

$$= \frac{\pi}{3} [2r^3 - (3r^3 - 3r^2 h - (r^3 - 3r^2 h + 3rh^2 - h^3))]$$

$$= \frac{\pi}{3} [2r^3 - (3r^3 - 3r^2 h - r^3 + 3r^2 h - 3rh^2 + h^3)]$$

$$= \frac{\pi}{3} [2r^3 - (2r^3 - 3rh^2 + h^3)]$$

$$= \frac{\pi}{3} [2r^3 - 2r^3 + 3rh^2 - h^3]$$

$$= \frac{\pi}{3} [3rh^2 - h^3]$$

$$= \frac{\pi h^2}{3} [3r - h]$$

Volume of the cap $V = \frac{1}{3}\pi h^2[3r - h]$

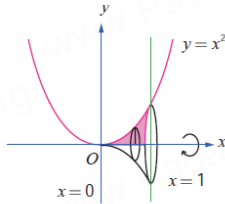
Hence the major axis is x axis.

Example 9.65

Find the volume of the solid formed by revolving the region bounded by the parabola $y = x^2$, x - axis, ordinates $x = 0$ and $x = 1$ about the x -axis.

Solution: The region is revolved about x - axis,

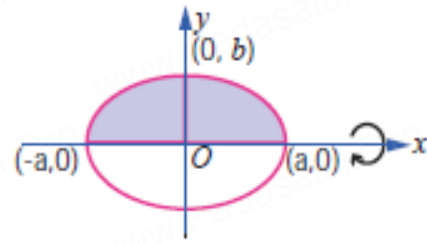
between the ordinates $x = 0$ and $x = 1$



$$\begin{aligned}
 V &= \int_0^1 \pi y^2 dx \\
 &= \pi \int_0^1 (x^2 + 4x + 5)^2 dx = \\
 &\pi \int_0^1 (x^4 + 16x^2 + 25 + 8x^3 + 40x + 10x^2) dx \\
 &= \pi \int_0^1 (x^4 + 16x^2 + 25 + 8x^3 + 40x + 10x^2) dx \\
 &= \pi \int_0^1 (x^4 + 16x^2 + 25 + 8x^3 + 40x + 10x^2) dx \\
 &= \pi \int_0^1 (x^4 + 16x^2 + 25 + 8x^3 + 40x + 10x^2) dx \\
 &= \pi \left(\frac{x^5}{5} + 26\frac{x^3}{3} + 25x + 8\frac{x^4}{4} + 40\frac{x^2}{2} \right)_0^1 \\
 &= \pi \left(\frac{1}{5} + \frac{26}{3} + 25 + 2 + 20 \right) \\
 &= \pi \left(\frac{1}{5} + \frac{26}{3} + 47 \right) \\
 &= \pi \left(\frac{3+130+705}{15} \right) \\
 &= \pi \left(\frac{838}{15} \right)
 \end{aligned}$$

Example 9.66 Find the volume of the solid formed by revolving the region bounded by the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$, $a > b$ about the major axis.

Solution: $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$, $a > b$



$$V = \int_a^b \pi y^2 dx$$

From $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$

$$\begin{aligned}
 \frac{y^2}{b^2} &= 1 - \frac{x^2}{a^2} \\
 &= \frac{a^2 - x^2}{a^2}
 \end{aligned}$$

$$y^2 = \frac{b^2}{a^2} (a^2 - x^2)$$

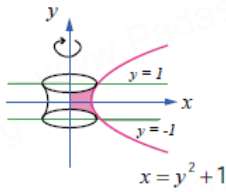
On x axis y is 0, and by symmetry

$$\begin{aligned}
 \therefore V &= 2 \int_0^a \pi \left[\frac{b^2}{a^2} (a^2 - x^2) \right] dx \\
 &= \frac{2\pi b^2}{a^2} \int_0^a (a^2 - x^2) dx \\
 &= \frac{2\pi b^2}{a^2} \left(a^2 x - \frac{x^3}{3} \right)_0^a \\
 &= \frac{2\pi b^2}{a^2} \left(a^2 a - \frac{a^3}{3} \right) \\
 &= \frac{2\pi b^2}{a^2} \left(a^3 - \frac{a^3}{3} \right) \\
 &= \frac{2\pi b^2}{a^2} \left(\frac{3a^3 - a^3}{3} \right) \\
 &= \frac{2\pi b^2}{a^2} \left(\frac{2a^3}{3} \right) = \frac{4}{3} \pi a b^2
 \end{aligned}$$

Example 9.67 Find, by integration, the volume of the solid generated by revolving about y -axis the region bounded between the parabola $x = y^2 + 1$, the y -axis, and the lines $y = 1$ and $y = -1$.

Solution: Given Parabola equation is $x = y^2 + 1$ gives

$$x - 1 = y^2 \Rightarrow y^2 = x - 1$$



It is symmetrical about x axis with vertex $(1,0)$ and focus $(\frac{5}{4}, 0)$. Hence the required volume is given by,

$$\begin{aligned} V &= \int_{-1}^1 \pi x^2 dy \\ &= \pi \int_{-1}^1 (y^2 + 1)^2 dy \\ &= \pi \int_{-1}^1 (y^4 + 2y^2 + 1) dy \\ &= \pi \left(\frac{y^5}{5} + 2\frac{y^3}{3} + y \right)_{-1}^1 \\ &= \pi \left[\left(\frac{1}{5} + \frac{2}{3} + 1 \right) - \left(-\frac{1}{5} - \frac{2}{3} - 1 \right) \right] \\ &= \pi \left[\frac{1}{5} + \frac{2}{3} + 1 + \frac{1}{5} + \frac{2}{3} + 1 \right] \\ &= \pi \left[\frac{2}{5} + \frac{4}{3} + 2 \right] \\ &= \pi \left[\frac{6+20+30}{15} \right] \\ &= \pi \left[\frac{56}{15} \right] \end{aligned}$$

Example 9.68 Find, by integration, the volume of the solid generated by revolving about y -axis the region bounded between the curve $y = \frac{3}{4}\sqrt{x^2 - 16}$, $x \geq 4$, the y -axis, and the lines $y = 1$ and $y = 6$.

Solution: $y = \frac{3}{4}\sqrt{x^2 - 16}$

$$4y = 3\sqrt{x^2 - 16}$$

Squaring, $16y^2 = 9(x^2 - 16)$

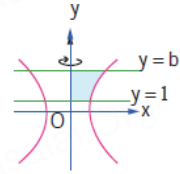
$$= 9x^2 - 144$$

$$9x^2 - 16y^2 = 144$$

Dividing by 144,

$$\frac{9x^2}{144} - \frac{16y^2}{144} = 1$$

$$\frac{x^2}{16} - \frac{y^2}{9} = 1$$



Hence the given equation is hyperbola, between $y = 1$ and $y = 6$ and it lies above the x axis. Since the hyperbola revolves about the y axis, Volume $V = \int_1^6 \pi x^2 dy$

From $9x^2 - 16y^2 = 144$

$$9x^2 = 16y^2 + 144$$

$$x^2 = \frac{16}{9}y^2 + \frac{144}{9}$$

$$x^2 = \frac{16}{9}(y^2 + 9)$$

$$V = \frac{16}{9}\pi \int_1^6 (y^2 + 9) dy$$

$$= \frac{16\pi}{9} \left(\frac{y^3}{3} + 9y \right)_1^6$$

$$= \frac{16\pi}{9} \left[\left(\frac{216}{3} + 54 \right) - \left(\frac{1}{3} + 9 \right) \right]$$

$$= \frac{16\pi}{9} \left(\frac{216}{3} + 54 - \frac{1}{3} - 9 \right)$$

$$= \frac{16\pi}{9} \left(\frac{215}{3} + 45 \right)$$

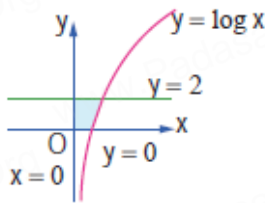
$$= \frac{16\pi}{9} \left(\frac{215+135}{3} \right)$$

$$= \frac{16\pi}{9} \left(\frac{350}{3} \right)$$

$$= \frac{5600}{27}\pi$$

Example 9.69 Find, by integration, the volume of the solid generated by revolving about y -axis the region bounded by the curves $y = \log x$, $y = 0$, $x = 0$ and $y = 2$.

Solution:



The given curve revolves about the y axis,

$$\begin{aligned}\text{Volume } V &= \int_0^2 \pi x^2 dy \\ &= \pi \int_0^2 e^y dy \\ &= \pi (e^y)_0^2 \\ &= \pi (e^2 - e^0) \\ &= \pi (e^2 - 1)\end{aligned}$$

EXERCISE 9.9

1. Find, by integration, the volume of the solid generated by revolving about the x -axis, the region enclosed by $y = 2x^2$, $x = 0$ and $x = 1$.

Solution: Volume $V = \int_0^1 \pi y^2 dx$

$$\begin{aligned}&= \pi \int_0^1 4x^4 dx \\ &= 4\pi \int_0^1 x^4 dx \\ &= 4\pi \left(\frac{x^5}{5} \right)_0^1 \\ &= 4\pi \left(\frac{1}{5} - 0 \right) \\ &= \frac{4\pi}{5}\end{aligned}$$

2. Find, by integration, the volume of the solid generated by revolving about the x -axis, the region enclosed by $y = e^{-2x}$, $y = 0$, $x = 0$ and $x = 1$

Solution: Volume $V = \int_0^1 \pi y^2 dx$

$$\begin{aligned}&= \pi \int_0^1 e^{-4x} dx \\ &= \pi \left(-\frac{e^{-4x}}{4} \right)_0^1 \\ &= -\frac{\pi}{4} (e^{-4x})_0^1 \\ &= -\frac{\pi}{4} (e^{-4} - e^0) \\ &= -\frac{\pi}{4} (e^{-4} - 1) \\ V &= \frac{\pi}{4} (1 - e^{-4})\end{aligned}$$

3. Find, by integration, the volume of the solid generated by revolving about the y -axis, the region enclosed by $x^2 = y + 1$ and $y = 3$.

Solution: On y -axis, $x = 0$

So $x^2 = y + 1$ gives

$$y + 1 = 0 \Rightarrow y = -1$$

Hence $y = -1$ to $y = 3$

Volume $V = \int_{-1}^3 \pi x^2 dy$

$$\begin{aligned}&= \pi \int_{-1}^3 (y + 1) dy \\ &= \pi \left(\frac{y^2}{2} + y \right)_{-1}^3 \\ &= \pi \left[\left(\frac{9}{2} + 3 \right) - \left(\frac{1}{2} - 1 \right) \right] \\ &= \pi \left(\frac{9}{2} + 3 - \frac{1}{2} + 1 \right) \\ &= \pi \left(\frac{8}{2} + 4 \right) \\ &= \pi (4 + 4) \\ &= 8\pi\end{aligned}$$

4. The region enclosed between the graphs of $y = x$ and $y = x^2$ is denoted by R , Find the volume generated when R is rotated through 360° about x -axis.

Solution: Solving $y = x$ and $y = x^2$

$$x^2 = x$$

$$x^2 - x = 0$$

$$x(x - 1) = 0 \text{ gives}$$

$$x = 0 \text{ and } 1$$

When $x = 0$ we get $y = 0$ and

When $x = 1$ we get $y = 1$

Hence point of intersection is $(0, 0), (1, 1)$

To find the volume about x -axis.

$$\text{Volume } V = \int_0^1 \pi \left[(f(x))^2 - (g(x))^2 \right] dx$$

$$= \int_0^1 \pi [(x)^2 - (x^2)^2] dx$$

$$= \pi \int_0^1 (x^2 - x^4) dx$$

$$= \pi \left(\frac{x^3}{3} - \frac{x^5}{5} \right)_0^1$$

$$= \pi \left(\frac{1}{3} - \frac{1}{5} \right)$$

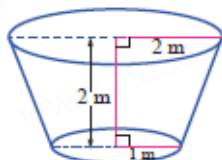
$$= \pi \left(\frac{5-3}{15} \right)$$

$$= \pi \left(\frac{2}{15} \right)$$

$$= \frac{2}{15} \pi$$

5. Find, by integration, the volume of the container which is in the shape of a right circular conical frustum as shown in the Figure.

Solution:



$$\text{Volume of frustum} = \pi \int_0^h \left[\frac{(R-r)(Rh - y)}{h} \right]^2 dy$$

$$\text{Given } r = 1; R = 2; h = 2$$

$$V = \pi \int_0^2 \left[\frac{(2-1)(\frac{4}{2-1} - y)}{2} \right]^2 dy$$

$$= \pi \int_0^2 \left[\frac{(1)(\frac{4}{1} - y)}{2} \right]^2 dy$$

$$= \pi \int_0^2 \left[\frac{4-y}{2} \right]^2 dy$$

$$= \pi \int_0^2 \frac{(4-y)^2}{4} dy$$

$$= \frac{\pi}{4} \int_0^2 (4-y)^2 dy$$

$$= \frac{\pi}{4} \int_0^2 (16 - 8y + y^2) dy$$

$$= \frac{\pi}{4} \left(16y - 8\frac{y^2}{2} + \frac{y^3}{3} \right)_0^2$$

$$= \frac{\pi}{4} \left(16y - 4y^2 + \frac{y^3}{3} \right)_0^2$$

$$= \frac{\pi}{4} \left(32 - 16 + \frac{8}{3} \right)$$

$$= \frac{\pi}{4} \left(16 + \frac{8}{3} \right)$$

$$= \frac{\pi}{4} \left(\frac{48+8}{3} \right)$$

$$= \frac{\pi}{4} \left(\frac{56}{3} \right)$$

$$= \frac{14}{3} \pi$$

6. A watermelon has an ellipsoid shape which can be obtained by revolving an ellipse with major-axis 20 cm and minor-axis 10 cm about its major-axis. Find its volume using integration.

Solution: Given semi major axis $a = 10$ and

semi minor axis $b = 5$

So, equation of the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ gives

$$\frac{x^2}{10^2} + \frac{y^2}{5^2} = 1$$

$$\frac{x^2}{100} + \frac{y^2}{25} = 1$$

$$\frac{y^2}{25} = 1 - \frac{x^2}{100}$$

$$= \frac{100-x^2}{100}$$

$$y^2 = \frac{25}{100}(100 - x^2)$$

On x axis $y=0$ gives $100 - x^2 = 0$
 $x^2 = 100$

$$x = \pm 10$$

By symmetry, $V = 2 \int_0^{10} \pi y^2 dx$

$$= 2\pi \int_0^{10} y^2 dx$$

$$= 2\pi \int_0^{10} \frac{25}{100}(100 - x^2) dx$$

$$= \frac{50}{100} \pi \int_0^{10} (100 - x^2) dx$$

$$= \frac{1}{2} \pi \int_0^{10} (100 - x^2) dx$$

$$= \frac{\pi}{2} \left(100x - \frac{x^3}{3} \right)_0^{10}$$

$$= \frac{\pi}{2} \left(1000 - \frac{1000}{3} \right)$$

$$= \frac{\pi}{2} \left(\frac{3000-1000}{3} \right)$$

$$= \frac{\pi}{2} \left(\frac{2000}{3} \right)$$

$$= \frac{1000}{3} \pi$$

EXERCISE 9.10

Choose the correct or the most suitable answer from the given four alternatives:

1. The value of $\int_0^{\frac{2}{\sqrt{4-x^2}}} \frac{dx}{\sqrt{4-x^2}}$ is

(1) $\frac{\pi}{6}$ (2) $\frac{\pi}{2}$ (3) $\frac{\pi}{4}$ (4) π

2. The value of $\int_{-1}^2 |x| dx$ is

(1) $\frac{1}{2}$ (2) $\frac{3}{2}$ (3) $\frac{5}{2}$ (4) $\frac{7}{2}$

3. For any value of $n \in \mathbb{Z}$,

$$\int_0^{\pi} e^{\cos^2 x} \cos^3[(2n+1)x] dx \text{ is}$$

(1) $\frac{\pi}{2}$ (2) π (3) 0 (4) 2

4. The value of $\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \sin^2 x \cos x dx$ is

(1) $\frac{3}{2}$ (2) $\frac{1}{2}$ (3) 0 (4) $\frac{2}{3}$

5. The value of $\int_{-4}^4 \left[\tan^{-1} \left(\frac{x^2}{x^4+1} \right) + \tan^{-1} \left(\frac{x^4+1}{x^2} \right) \right] dx$

(1) π (2) 2π (3) 3π (4) 4π

6. The value of $\int_{-\frac{\pi}{4}}^{\frac{\pi}{4}} \left(\frac{2x^7-3x^2+7x^3-x+1}{\cos^2 x} \right) dx$ is

(1) 4 (2) 3 (3) 2 (4) 0

7. If $f(x) = \int_0^x t \cos t dt$, then $\frac{df}{dt} =$

(1) $\cos x - x \sin x$ (2) $\sin x + x \cos x$

(3) $x \cos x$ (4) $x \sin x$

8. The area between $y^2 = 4x$ and its latus rectum is

(1) $\frac{2}{3}$ (2) $\frac{4}{3}$ (3) $\frac{8}{3}$ (4) $\frac{5}{3}$

9. The value of $\int_0^1 x(1-x)^{99} dx$ is

(1) $\frac{1}{11000}$ (2) $\frac{1}{10100}$ (3) $\frac{1}{10010}$ (4) $\frac{1}{10001}$

10. The value of $\int_0^x \frac{dx}{1+5^{\cos x}}$ is

(1) $\frac{\pi}{2}$ (2) π (3) $\frac{3\pi}{2}$ (4) 2π

11. If $\frac{\Gamma(n)}{\Gamma(n)} = 90$ then n is

(1) 10 (2) 5 (3) 8 (4) 9

12. The value of $\int_0^6 \cos^3 3x dx$ is

(1) $\frac{2}{3}$ (2) $\frac{2}{9}$ (3) $\frac{1}{9}$ (4) $\frac{1}{3}$

13. The value of $\int_0^{\pi} \sin^4 x dx$ is

(1) $\frac{3\pi}{10}$ (2) $\frac{3\pi}{8}$ (3) $\frac{3\pi}{4}$ (4) $\frac{3\pi}{2}$

14. The value of $\int_0^{\infty} e^{-3x} x^2 dx$ is

(1) $\frac{7}{27}$ (2) $\frac{5}{27}$ (3) $\frac{4}{27}$ (4) $\frac{2}{27}$

15. If $\int_0^a \frac{1}{4+x^2} dx = \frac{\pi}{8}$ then a is

(1) 4 (2) 1 (3) 3 (4) 2

16. The volume of solid of revolution of the region bounded by $y^2 = x(a - x)$ about x -axis is

- (1) πa^3 (2) $\frac{\pi a^3}{4}$ (3) $\frac{\pi a^3}{5}$ (4) $\frac{\pi a^3}{6}$

17. If $f(x) = \int_0^x \frac{e^{\sin u}}{u} du, x > 1$ and

$\int_1^3 \frac{e^{\sin x^2}}{x} = \frac{1}{2} [f(a) - f(1)]$ then one of the possible value of a is

- (1) 3 (2) 6 (3) 9 (5)

18. The value of $\int_0^1 (\sin^{-1} x)^2 dx$ is

- (1) $\frac{\pi^2}{4} - 1$ (2) $\frac{\pi^2}{4} + 2$
(3) $\frac{\pi^2}{4} + 1$ (4) $\frac{\pi^2}{4} - 2$

19. The value of $\int_0^a (\sqrt{a^2 - x^2})^3 dx$ is

- (1) $\frac{\pi a^3}{16}$ (2) $\frac{3\pi a^4}{16}$ (3) $\frac{3\pi a^2}{8}$ (4) $\frac{3\pi a^4}{8}$

20. If $\int_0^x f(t) dt = x + \int_x^1 t f(t) dt$, then the value of $f(1)$ is

- (1) $\frac{1}{2}$ (2) 2 (3) 1 (4) $\frac{3}{4}$



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12. Discrete Mathematics

Example 12.1

Examine the binary operation (closure property) of the following operations on the respective sets (if it is not, make it binary):

(i) $a * b = a + 3ab - 5b^2; \forall a, b \in \mathbb{Z}$

(ii) $a * b = \left(\frac{a-1}{b-1}\right); \forall a, b \in \mathbb{Q}$

Solution:

(i) Since \times is binary operation on \mathbb{Z} ,

$$a, b \in \mathbb{Z} \Rightarrow ab \in \mathbb{Z} \text{ and } b \times b = b^2 \in \mathbb{Z} \dots (1)$$

The fact that $+$ is binary operation on \mathbb{Z}

and (1) $\Rightarrow 3ab = (ab + ab + ab) \in \mathbb{Z}$ and

$$5b^2 = (b^2 + b^2 + b^2 + b^2 + b^2) \in \mathbb{Z} \dots (2)$$

Also $a \in \mathbb{Z}$ and $3ab \in \mathbb{Z}$ implies $a + 3ab \in \mathbb{Z} \dots (3)$

(2), (3), the closure property of $-$ on \mathbb{Z} yield

$$a * b = a + 3ab - 5b^2; \forall a, b \in \mathbb{Z}.$$

Since $a * b$ belongs to \mathbb{Z} , $*$ is a binary operation on \mathbb{Z} .

(ii) In this problem $a * b$ is in the quotient form.

Since the division by 0 is undefined, the denominator $b - 1$ must be nonzero.

It is clear that $b - 1 = 0$ if $b = 1$. As $1 \in \mathbb{Q}$, $*$ is not a binary operation on the whole of \mathbb{Q} .

However it can be found that by omitting 1 from \mathbb{Q} , the output $a * b$ exists in $\mathbb{Q} \setminus \{1\}$.

Hence $*$ is a binary operation on $\mathbb{Q} \setminus \{1\}$.

Example 12.2

Verify the (i) closure property,

(ii) commutative property,

(iii) associative property

(iv) existence of identity and

(v) existence of inverse for the arithmetic operation $+$ on \mathbb{Z}

Solution:

(i) $m + n \in \mathbb{Z}, \forall m, n \in \mathbb{Z}$.

Hence $+$ is a binary operation on \mathbb{Z} .

(ii) Also $m + n = n + m, \forall m, n \in \mathbb{Z}$.

So the commutative property is satisfied

(iii) $\forall m, n, p \in \mathbb{Z}, m + (n + p) = (m + n) + p$.

Hence the associative property is satisfied.

(iv) $m + e = e + m = m \Rightarrow e = 0$.

Thus $\exists 0 \in \mathbb{Z}, \exists (m + 0) = (0 + m) = m$.

Hence the existence of identity is assured.

(v) $m + m' = m' + m = 0 \Rightarrow m' = -m$.

Thus $\forall m \in \mathbb{Z}, \exists -m \in \mathbb{Z}$,

$\Rightarrow m + (-m) = (-m) + m = 0$. Hence, the

existence of inverse property is also

assured. Thus we see that the usual

addition $+$ on \mathbb{Z} satisfies all the above five properties.

Note that the **additive identity** is 0 and the **additive inverse** of any integer m is $-m$.

Example 12.3

Verify the (i) closure property,

(ii) commutative property,

(iii) associative property

(iv) existence of identity and

(v) existence of inverse for the arithmetic operation $-$ on \mathbb{Z}

Solution:

(i) Though $-$ is not binary on \mathbb{N} ; it is binary on \mathbb{Z} . To check the validity of any more properties satisfied by $-$ on \mathbb{Z} , it is better to check them for some particular simple values.

(ii) Take $m = 4, n = 5$ and

$$(m - n) = (4 - 5) = -1 \text{ and}$$

$$(n - m) = (5 - 4) = 1.$$

Hence $(m - n) \neq (n - m)$. So the operation $-$ is not commutative on \mathbb{Z} .

(iii) In order to check the associative property,

let us put $m = 4, n = 5$ and $p = 7$ in both

$$(m - n) - p \text{ and } m - (n - p).$$

$$(m - n) - p = (4 - 5) - 7 = (-1 - 7) = -8 \dots (1)$$

$$m - (n - p) = 4 - (5 - 7) = (4 + 2) = 6 \dots (2)$$

From (1) and (2), it follows that

$$(m - n) - p \neq m - (n - p).$$

Hence $-$ is not associative on \mathbb{Z} .

(iv) Identity does not exist

(v) So, Inverse does not exist .

Example 12.4

Verify the (i) closure property,
(ii) commutative property,
(iii) associative property
(iv) existence of identity and
(v) existence of inverse for the arithmetic operation $+$ on \mathbb{Z}_e = the set of all even integers.

Solution : Consider the set of all even integers

$$\mathbb{Z}_e = \{2k \mid k \in \mathbb{Z}\} = \{\dots, -6, -4, -2, 0, 2, 4, 6, \dots\}.$$

Let us verify the properties satisfied by $+$ on \mathbb{Z}_e

(i) The sum of any two even integers is also an

even integer. Because $x, y \in \mathbb{Z}_e \Rightarrow x = 2m$

and $y = 2n, m, n \in \mathbb{Z}$.

$$\text{So } (x + y) = 2m + 2n$$

$$= 2(m + n) \in \mathbb{Z}_e$$

$$= 2(n + m)$$

$$= 2n + 2m$$

$$= (y + x).$$

Hence $+$ is a binary operation on \mathbb{Z}_e .

(ii) $\forall x, y \in \mathbb{Z}_e$,

$$(x + y) = 2m + 2n$$

$$= 2(m + n)$$

$$= 2(n + m)$$

$$= 2n + 2m$$

$$= (y + x).$$

So $+$ has commutative property.

(iii) Similarly it can be seen that $\forall x, y, z \in \mathbb{Z}_e$,

$$(x + y) + z = x + (y + z).$$

Hence the associative property is true.

(iv) Now take $x = 2k$, then

$$2k + e = e + 2k = 2k \Rightarrow e = 0.$$

$$\text{Thus } \forall x \in \mathbb{Z}_e, \exists 0 \in \mathbb{Z}_e \ni x + 0 = 0 + x = x.$$

So, 0 is the identity element.

(v) Taking $x = 2k$ and x' as its inverse,

$$\text{we have } 2k + x' = 0 = x' + 2k$$

$$\Rightarrow x' = -2k = x' = -x$$

Thus $\forall x \in \mathbb{Z}_e, \exists -x \in \mathbb{Z}_e$,

$$\ni x + (-x) = (-x) + x = 0$$

Hence $-x$ is the inverse of $x \in \mathbb{Z}_e$.

Example 12.5

Verify the (i) closure property,
(ii) commutative property,
(iii) associative property
(iv) existence of identity and
(v) existence of inverse for the arithmetic operation $+$ on \mathbb{Z}_o = the set of all odd integers

Solution:

Consider the set \mathbb{Z}_o of all odd integers

$$\mathbb{Z}_o = \{2k + 1 \mid k \in \mathbb{Z}\}$$

$$= \{\dots, -5, -3, -1, 1, 3, 5, \dots\}. + \text{ is not a}$$

binary operation on \mathbb{Z}_o because when

$$x = 2m + 1, y = 2n + 1, x + y = 2(m + n) + 2$$

is even for all m and n . For instance, consider

the two odd numbers $3, 7 \in \mathbb{Z}_o$.

Their sum $3 + 7 = 10$ is an even number. In

general, if $x, y \in \mathbb{Z}_o, (x + y) \notin \mathbb{Z}_o$. Other

properties need not be checked as it is not a

binary operation.

Example 12.6

Verify (i) closure property
(ii) commutative property, and
(iii) associative property of the following operation on the given set.
 $(a * b) = a^b; \forall a, b \in \mathbb{N}$ (exponentiation property)

Solution :

(i) It is true that $a * b = a^b; \forall a, b \in \mathbb{N}$.

So $*$ is a binary operation on \mathbb{N} .

(ii) $a * b = a^b$ and $b * a = b^a$.

Put, $a = 2$ and $b = 3$.

$$\text{Then } a * b = 2^3 = 8 \text{ but}$$

$$b * a = 3^2 = 9$$

So $a * b$ need not be equal to $b * a$.

Hence $*$ does not have commutative property.

(iii) Next consider

$$a * (b * c) = a * (b^c) = a^{(b^c)}.$$

Take Put, $a = 2$, $b = 3$ and $c = 4$.

$$a * (b * c) = 2 * (3 * 4) = 2 * (3^4) = 2^{(81)}$$

$$\text{But } (a * b) * c = (a^b) * c = (a^b)^c = a^{bc} = 2^{12}$$

$$\text{Hence } a * (b * c) \neq (a * b) * c.$$

So $*$ does not have associative property on \mathbb{N} .

Example 12.7

Verify (i) closure property,

(ii) commutative property,

(iii) associative property,

(iv) existence of identity, and

(v) existence of inverse for following operation on the given set. $m * n = m + n - mn$; $m, n \in \mathbb{Z}$

Solution:

(i) $m + n - mn$ is clearly an integer and hence

$*$ is a binary operation on \mathbb{Z} .

(ii) $m * n = m + n - mn$ and

$$= n + m - nm$$

$$= n * m$$

So, $*$ has commutative property.

(iii) Consider $(m * n) * p = (m + n - mn) * p$

$$= (m + n - mn) + p - (m + n - mn)p$$

$$= m + n + p - mn - mp - np + mnp \dots (1)$$

$$m * (n * p) = m * (n + p - np)$$

$$= m + (n + p - np) - m(n + p - np)$$

$$= m + n + p - np - mn - mp + mnp \dots (2)$$

From (1) and (2) we see that,

$$(m * n) * p = m * (n * p)$$

So, $*$ has associative property.

(iv) An integer e is to be found that

$$m * e = e * m = m, \forall m \in \mathbb{Z}$$

$$\text{So, } m * e = m + e - me = m$$

$$e - me = m - m$$

$$e(1 - m) = 0$$

$$\text{Gives, } e = 0 \text{ and } (1 - m) = 0$$

But m is an arbitrary integer and hence need

not be equal to 1. So the only possibility is

$e = 0$. Also $m * 0 = 0 * m = m$. Hence 0 is the

identity element and hence the existence of identity is assured.

(v) An integer m' is to be found that

$$m * m' = m' * m = e, \forall m \in \mathbb{Z}$$

$$\text{So, } m * m' = m + m' - mm' = 0$$

$$m' - mm' = -m$$

$$m'(1 - m) = -m$$

$$m' = -\frac{m}{(1-m)} \text{ is not defined at}$$

$$(1 - m) = 0, \text{ that is at } m = 1$$

Hence inverse does not exist in \mathbb{Z} .

Example 12.8

Let $A = \begin{bmatrix} 0 & 1 \\ 1 & 1 \end{bmatrix}, B = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$ be any two

Boolean matrices of the same type. Find $A \vee B$

and $A \wedge B$

Solution:

$$\text{Given } A = \begin{bmatrix} 0 & 1 \\ 1 & 1 \end{bmatrix}, B = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$$

$$A \vee B = \begin{bmatrix} 0 & 1 \\ 1 & 1 \end{bmatrix} \vee \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 0 \vee 1 & 1 \vee 1 \\ 1 \vee 0 & 1 \vee 1 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}$$

$$A \wedge B = \begin{bmatrix} 0 & 1 \\ 1 & 1 \end{bmatrix} \wedge \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 0 \wedge 1 & 1 \wedge 1 \\ 1 \wedge 0 & 1 \wedge 1 \end{bmatrix}$$

$$= \begin{bmatrix} 0 & 1 \\ 0 & 1 \end{bmatrix}$$

Example 12.9

Verify (i) closure property,

(ii) commutative property,

(iii) associative property,

(iv) existence of identity, and

(v) existence of inverse for the operation $+_5$ on

\mathbb{Z}_5 using table corresponding to addition modulo 5.

Solution:

We know that $\mathbb{Z}_5 = \{[0], [1], [2], [3], [4]\}$

$+_5$	0	1	2	3	4
0	0	1	2	3	4
1	1	2	3	4	0
2	2	3	4	0	1
3	3	4	0	1	2
4	4	0	1	2	3

(i) Since each box in the table is filled by exactly one element of $+_5$, the output $a +_5 b$ is unique and hence $+_5$ is a binary operation.

(ii) The entries are symmetrically placed with respect to the main diagonal. So $+_5$ has commutative property.

(iii) The table cannot be used directly for the verification of the associative property. So it is to be verified as usual.

For instance, $(2 +_5 3) +_5 4 = 0 +_5 4 = 4 \pmod{5}$

and $2 +_5 (3 +_5 4) = 2 +_5 2 = 4 \pmod{5}$

Hence $(2 +_5 3) +_5 4 = 2 +_5 (3 +_5 4)$.

Proceeding like this one can verify this for all possible triples and ultimately it can be shown that $+_5$ is associative.

(iv) The row headed by 0 and the column headed by 0 are identical. Hence the identity element is 0.

(v) The existence of inverse is guaranteed provided the identity 0 exists in each row and each column.

From the table,

Inverse of 0 is 0

Inverse of 1 is 4

Inverse of 2 is 3

Inverse of 3 is 2

Inverse of 4 is 1

Example 12.10 Verify (i) closure property, (ii) commutative property, (iii) associative property, (iv) existence of identity, and (v) existence of inverse for the operation \times_{11} on a subset $A = \{1, 3, 4, 5, 9\}$ of the set of remainders $\{0, 1, 2, 3, 4, 5, 6, 7, 8, 9, 10\}$.

Solution:

Given $A = \{[1], [3], [4], [5], [9]\}$

\times_{11}	1	3	4	5	9
1	1	3	4	5	9
3	3	9	1	4	5
4	4	1	5	9	3
5	5	4	9	3	1
9	9	5	3	1	4

(i) Since each box in the table is filled by exactly one element of \times_{11} , the output $a \times_{11} b$ is unique and hence \times_{11} is a binary operation.

(ii) The entries are symmetrically placed with respect to the main diagonal. So \times_{11} has commutative property.

(iii) The table cannot be used directly for the verification of the associative property. So it is to be verified as usual.

For instance,

$(3 \times_{11} 4) \times_{11} 5 = 1 \times_{11} 5 = 5 \pmod{11}$

and $3 \times_{11} (4 \times_{11} 5) = 3 \times_{11} 9 = 5 \pmod{11}$

Hence $(3 \times_{11} 4) \times_{11} 5 = 3 \times_{11} (4 \times_{11} 5)$.

Proceeding like this one can verify this for all possible triples and ultimately it can be shown that \times_{11} is associative.

(iv) The row headed by 1 and the column headed by 1 are identical. Hence the identity element is 1.

- (v) The existence of inverse is guaranteed provided the identity 1 exists in each row and each column. From the table,
Inverse of 1 is 1
Inverse of 3 is 4
Inverse of 4 is 3
Inverse of 5 is 9
Inverse of 9 is 5

EXERCISE 12.1

1. Determine whether $*$ is a binary operation on the sets given below
(i) $a * b = a \cdot |b|$ on \mathbb{R}
(ii) $a * b = \min(a, b)$ on $A = \{1, 2, 3, 4, 5\}$
(iii) $(a * b) = a\sqrt{b}$ is binary on \mathbb{R} .

Solution:

- (i) Let $a, b \in \mathbb{R}$

Then $|b| \in \mathbb{R}$

Hence, $a \cdot |b| \in \mathbb{R}, \forall a, b \in \mathbb{R}$

So, $*$ is the binary operation on \mathbb{R} .

- (ii) $a * b = \min(a, b)$ on $A = \{1, 2, 3, 4, 5\}$

Let $a, b \in A$

Then $\min(a, b) \in A$

For Example,

$$\min(3, 4) = 3 \in A,$$

$$\min(2, 5) = 2 \in A$$

So, $a * b = \min(a, b) \in A, \forall a, b \in A$

So, $*$ is the binary operation on A .

- (iii) $(a * b) = a\sqrt{b}$ is binary on \mathbb{R}

Let $a, b \in \mathbb{R}$

If b is positive then $\sqrt{b} \in \mathbb{R}$, but even though $-b \in \mathbb{R}, \sqrt{-b} \notin \mathbb{R}$.

For example $2, -4 \in \mathbb{R}$

$$\text{But } 2\sqrt{-4} = 2\sqrt{2}i \notin \mathbb{R}$$

So, $*$ is not a binary operation on \mathbb{R}

2. On \mathbb{Z} , define \otimes by

$$(m \otimes n) = m^n + n^m: \forall m, n \in \mathbb{Z}.$$

Is \otimes binary on \mathbb{Z} ?

Solution:

Let $m, n \in \mathbb{Z}$.

Take $m = 3, n = -2$, then

$$m^n + n^m = 3^{-2} + (-3)^2$$

$$= \frac{1}{3^2} + 9$$

$$= \frac{1}{9} + 9$$

$$= \frac{1+81}{9}$$

$$= \frac{82}{9} \notin \mathbb{Z}$$

So, $(m \otimes n) \neq \mathbb{Z}$

Hence, \otimes is not a binary operation on \mathbb{Z} .

3. Let $*$ be defined on \mathbb{R} by

$(a * b) = a + b + ab - 7$. Is $*$ binary on \mathbb{R} ?

If so, find $3 * \left(\frac{-7}{15}\right)$.

Solution:

Let $a, b \in \mathbb{R}$

Then $a + b + ab - 7 \in \mathbb{R}$

So, $*$ is the binary operation on \mathbb{R} .

$$3 * \left(\frac{-7}{15}\right) = 3 + \left(\frac{-7}{15}\right) + 3\left(\frac{-7}{15}\right) - 7$$

$$= 3 - \frac{7}{15} - \frac{21}{15} - 7$$

$$= -4 - \frac{28}{15}$$

$$= \frac{-60-28}{15}$$

$$= -\frac{88}{15} \quad \square$$

4. Let $A = \{a + \sqrt{5}b : a, b \in \mathbb{Z}\}$. Check

whether

the usual multiplication is a binary operation on A .

Solution: \square

Let $A = \{x, y\}$

Such that $x = a + \sqrt{5}b$ and $y = c + \sqrt{5}d$

and $a, b, c, d \in \mathbb{Z}$

$$\text{Now, } xy = (a + \sqrt{5}b)(c + \sqrt{5}d)$$

$$= ac + \sqrt{5}ad + \sqrt{5}bc + 5bd$$

$$= (ac + 5bd) + \sqrt{5}(ad + bc) \in A$$

$\therefore xy \in A$. The usual multiplication is a

binary operation on A .

5. (i) Define an operation $*$ on \mathbb{Q} as follows:

$$a * b = \left(\frac{a+b}{2}\right): a, b \in \mathbb{Q}.$$

Examine the closure, commutative, and associative properties satisfied by $*$ on \mathbb{Q} .

(ii) Define an operation $*$ on \mathbb{Q} as follows:

$$a * b = \left(\frac{a+b}{2}\right): a, b \in \mathbb{Q}.$$

Examine the existence of identity and the existence of inverse for the operation $*$ on \mathbb{Q} .

Solution: (i)

(1) Given $a, b \in \mathbb{Q}$

It gives $a + b \in \mathbb{Q}$.

Hence $\frac{a+b}{2}$ also $\in \mathbb{Q}$.

So, $*$ is closure on \mathbb{Q} .

(2) $a * b = \left(\frac{a+b}{2}\right): a, b \in \mathbb{Q}$

$$= \left(\frac{b+a}{2}\right)$$

$$= b * a$$

So, $*$ is commutative on \mathbb{Q} .

(3) $a * (b * c) = a * \left(\frac{b+c}{2}\right)$

$$= \frac{a + \left(\frac{b+c}{2}\right)}{2}$$

$$= \frac{\frac{2a + b + c}{2}}{2}$$

$$= \frac{2a + b + c}{4}$$

$$(a * b) * c = \left(\frac{a+b}{2}\right) * c$$

$$= \frac{\left(\frac{a+b}{2}\right) + c}{2}$$

$$= \frac{\frac{a+b+2c}{2}}{2}$$

$$= \frac{a+b+2c}{4}$$

$$\text{Hence, } a * (b * c) \neq (a * b) * c$$

So, $*$ is not associative on \mathbb{Q} .

(ii)

(1) Given $a \in \mathbb{Q}$. Let e be the identity.

$$\text{Then } a * e = e * a = a$$

$$\frac{a+e}{2} = a$$

$$a + e = 2a$$

$$e = 2a - a$$

$$e = a, \text{ is the identity element.}$$

For every element a belongs to \mathbb{Q} , we get a is the identity. Since identity element is unique, $*$ has no identity.

(2) Since there is no identity, inverse of a that is a^{-1} cannot be defined. That is,

$$a * a^{-1} = a^{-1} * a = e, \text{ cannot be defined.}$$

Hence $*$ has no inverse.

6. Fill in the following table so that the

binary operation $*$ on $A = \{a, b, c\}$ is

commutative.

$*$	a	b	c
a	b		
b	c	b	a
c	a		c

Solution:

(i) $*$ is commutative

From table $b * a = c$

Hence $a * b = c$

(ii) From table $b * c = a$

Hence $c * b = a$

(iii) From table $c * a = a$

Hence $a * c = a$

So, the table is

$*$	a	b	c
a	b	c	a
b	c	b	a
c	a	a	c

7. Consider the binary operation $*$ defined on the set $A = \{a, b, c, d\}$ by the following table:

$*$	a	b	c	d
a	a	c	b	d
b	d	a	b	c
c	c	d	a	a
d	d	b	a	c

Is it commutative and associative?

Solution:

From table $a * b = c$

and $b * a = d$

So, $a * b \neq b * a$

From table $c * a = c$

and $a * c = b$

So, $c * a \neq a * c$

Hence, $*$ is not commutative

From table $a * (b * c) = a * b = c$ and

$(a * b) * c = c * c = a$

Therefore, $a * (b * c) \neq (a * b) * c$

Hence, $*$ is also not associative.

8. Let $A = \begin{pmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ 1 & 0 & 0 & 1 \end{pmatrix}$, $B = \begin{pmatrix} 0 & 1 & 0 & 1 \\ 1 & 0 & 1 & 0 \\ 1 & 0 & 0 & 1 \end{pmatrix}$,

$C = \begin{pmatrix} 1 & 1 & 0 & 1 \\ 0 & 1 & 1 & 0 \\ 1 & 1 & 1 & 1 \end{pmatrix}$ be any three Boolean

matrices of the same type.

Find (i) $A \vee B$ (ii) $A \wedge B$

(iii) $(A \vee B) \wedge C$ (iv) $(A \wedge B) \vee C$.

Solution:

Given $A = \begin{pmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ 1 & 0 & 0 & 1 \end{pmatrix}$, $B = \begin{pmatrix} 0 & 1 & 0 & 1 \\ 1 & 0 & 1 & 0 \\ 1 & 0 & 0 & 1 \end{pmatrix}$

(i) $A \vee B = \begin{pmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ 1 & 0 & 0 & 1 \end{pmatrix} \vee \begin{pmatrix} 0 & 1 & 0 & 1 \\ 1 & 0 & 1 & 0 \\ 1 & 0 & 0 & 1 \end{pmatrix}$
 $= \begin{pmatrix} 1 \vee 0 & 0 \vee 1 & 1 \vee 0 & 0 \vee 1 \\ 0 \vee 1 & 1 \vee 0 & 0 \vee 1 & 1 \vee 0 \\ 1 \vee 1 & 0 \vee 0 & 0 \vee 0 & 1 \vee 1 \end{pmatrix}$

$$= \begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \\ 1 & 0 & 0 & 1 \end{pmatrix}$$

(ii) $A \wedge B = \begin{pmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ 1 & 0 & 0 & 1 \end{pmatrix} \wedge \begin{pmatrix} 0 & 1 & 0 & 1 \\ 1 & 0 & 1 & 0 \\ 1 & 0 & 0 & 1 \end{pmatrix}$
 $= \begin{pmatrix} 1 \wedge 0 & 0 \wedge 1 & 1 \wedge 0 & 0 \wedge 1 \\ 0 \wedge 1 & 1 \wedge 0 & 0 \wedge 1 & 1 \wedge 0 \\ 1 \wedge 1 & 0 \wedge 0 & 0 \wedge 0 & 1 \wedge 1 \end{pmatrix}$
 $= \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 1 \end{pmatrix}$

(iii) $(A \vee B) \wedge C$

$$= \begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \\ 1 & 0 & 0 & 1 \end{pmatrix} \wedge \begin{pmatrix} 1 & 1 & 0 & 1 \\ 0 & 1 & 1 & 0 \\ 1 & 1 & 1 & 1 \end{pmatrix}$$

$$= \begin{pmatrix} 1 \wedge 1 & 1 \wedge 1 & 1 \wedge 0 & 1 \wedge 1 \\ 1 \wedge 0 & 1 \wedge 1 & 1 \wedge 1 & 1 \wedge 0 \\ 1 \wedge 1 & 0 \wedge 1 & 0 \wedge 1 & 1 \wedge 1 \end{pmatrix}$$

$$= \begin{pmatrix} 1 & 1 & 0 & 1 \\ 0 & 1 & 1 & 0 \\ 1 & 0 & 0 & 1 \end{pmatrix}$$

(iv) $(A \wedge B) \vee C$

$$= \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 1 \end{pmatrix} \vee \begin{pmatrix} 1 & 1 & 0 & 1 \\ 0 & 1 & 1 & 0 \\ 1 & 1 & 1 & 1 \end{pmatrix}$$

$$= \begin{pmatrix} 0 \vee 1 & 0 \vee 1 & 0 \vee 0 & 0 \vee 1 \\ 0 \vee 0 & 0 \vee 1 & 0 \vee 1 & 0 \vee 0 \\ 1 \vee 1 & 0 \vee 1 & 0 \vee 1 & 1 \vee 1 \end{pmatrix}$$

$$= \begin{pmatrix} 1 & 1 & 0 & 1 \\ 0 & 1 & 1 & 0 \\ 1 & 1 & 1 & 1 \end{pmatrix}$$

9. (i) Let $M = \left\{ \begin{pmatrix} x & x \\ x & x \end{pmatrix} : x \in R - (0) \right\}$ and let $*$ be the matrix multiplication. Determine whether M is closed under $*$. If so, examine the commutative and associative properties satisfied by $*$ on M .

(ii) Let $M = \left\{ \begin{pmatrix} x & x \\ x & x \end{pmatrix} : x \in R - (0) \right\}$ and let $*$ be the matrix multiplication. Determine whether M is closed under $*$. If so, examine the existence of identity, existence of inverse properties for the operation $*$ on M .

Solution: (i)

(1) Closure

Let $A = \begin{pmatrix} x & x \\ x & x \end{pmatrix}$ and $B = \begin{pmatrix} y & y \\ y & y \end{pmatrix} : x, y \in R - (0)$

Hence $M = \{A, B\}$

$$\begin{aligned}\text{Now, } AB &= \begin{pmatrix} x & x \\ x & x \end{pmatrix} \begin{pmatrix} y & y \\ y & y \end{pmatrix} \\ &= \begin{pmatrix} xy + xy & xy + xy \\ xy + xy & xy + xy \end{pmatrix} \\ &= \begin{pmatrix} 2xy & 2xy \\ 2xy & 2xy \end{pmatrix} \in M\end{aligned}$$

Since, $x, y \in R - (0)$ gives xy also $\in R - (0)$

So, $AB \in M \Rightarrow A * B \in M$

$\therefore *$ is closed on M

(2) Commutative

$$\text{Let } A = \begin{pmatrix} x & x \\ x & x \end{pmatrix}, B = \begin{pmatrix} y & y \\ y & y \end{pmatrix}$$

such that : $x, y \in R - (0)$

Hence $M = \{A, B\}$

$$\begin{aligned}\text{Now, } A * B &= \begin{pmatrix} x & x \\ x & x \end{pmatrix} \begin{pmatrix} y & y \\ y & y \end{pmatrix} \\ &= \begin{pmatrix} xy + xy & xy + xy \\ xy + xy & xy + xy \end{pmatrix} \\ &= \begin{pmatrix} 2xy & 2xy \\ 2xy & 2xy \end{pmatrix} \\ &= \begin{pmatrix} 2yx & 2yx \\ 2yx & 2yx \end{pmatrix} \\ &= \begin{pmatrix} y & y \\ y & y \end{pmatrix} \begin{pmatrix} x & x \\ x & x \end{pmatrix} \\ &= B * A\end{aligned}$$

So, $A * B \in M = B * A \in M$

$\therefore *$ is commutative on M

(3) Associative

Matrix multiplication is always associative.

That is $A * (B * C) = (A * B) * C, \forall A, B, C \in M$.

$\therefore *$ is also associative on M

Solution: (ii)

(1) Closure

$$\text{Let } A = \begin{pmatrix} x & x \\ x & x \end{pmatrix} \text{ and } B = \begin{pmatrix} y & y \\ y & y \end{pmatrix} : x, y \in R - (0)$$

Hence $M = \{A, B\}$

$$\begin{aligned}\text{Now, } AB &= \begin{pmatrix} x & x \\ x & x \end{pmatrix} \begin{pmatrix} y & y \\ y & y \end{pmatrix} \\ &= \begin{pmatrix} xy + xy & xy + xy \\ xy + xy & xy + xy \end{pmatrix} \\ &= \begin{pmatrix} 2xy & 2xy \\ 2xy & 2xy \end{pmatrix} \in M\end{aligned}$$

Since, $x, y \in R - (0)$ gives xy also $\in R - (0)$

So, $AB \in M \Rightarrow A * B \in M$

$\therefore *$ is closed on M

(2) Existence of Identity

$$\text{Let } A = \begin{pmatrix} x & x \\ x & x \end{pmatrix} \text{ and}$$

$$E = \begin{pmatrix} e & e \\ e & e \end{pmatrix} \text{ be the identity,}$$

such that : $a, e \in R - (0)$

Hence $M = \{A, E\}$

Now, $A * E = E * A = A$

$$\begin{pmatrix} x & x \\ x & x \end{pmatrix} \begin{pmatrix} e & e \\ e & e \end{pmatrix} = \begin{pmatrix} x & x \\ x & x \end{pmatrix}$$

$$\begin{pmatrix} 2xe & 2xe \\ 2xe & 2xe \end{pmatrix} = \begin{pmatrix} x & x \\ x & x \end{pmatrix}$$

$$2xe = x$$

$$2e = 1$$

$$e = \frac{1}{2} \in R - (0)$$

$$\therefore E = \begin{pmatrix} \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & \frac{1}{2} \end{pmatrix} \text{ is the identity } \in M$$

$\therefore *$ has identity on M

(3) Existence of Inverse

$$\text{Let } A = \begin{pmatrix} x & x \\ x & x \end{pmatrix} \text{ and}$$

$$A^{-1} = \begin{pmatrix} x^{-1} & x^{-1} \\ x^{-1} & x^{-1} \end{pmatrix} \text{ be the inverse of A.}$$

Then $A * A^{-1} = A^{-1} * A = E$

$$\begin{pmatrix} x & x \\ x & x \end{pmatrix} \begin{pmatrix} x^{-1} & x^{-1} \\ x^{-1} & x^{-1} \end{pmatrix} = \begin{pmatrix} e & e \\ e & e \end{pmatrix}$$

$$\begin{pmatrix} 2xx^{-1} & 2xx^{-1} \\ 2xx^{-1} & 2xx^{-1} \end{pmatrix} = \begin{pmatrix} \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & \frac{1}{2} \end{pmatrix}$$

$$2xx^{-1} = \frac{1}{2}$$

$$x^{-1} = \frac{1}{4x}, \in R - (0)$$

$$\therefore A^{-1} = \begin{pmatrix} \frac{1}{4x} & \frac{1}{4x} \\ \frac{1}{4x} & \frac{1}{4x} \end{pmatrix} \text{ is the inverse of } A \in M$$

$\therefore *$ has inverse on M

10. (i) Let A be $\mathbb{Q} \setminus \{1\}$. Define $*$ on A by

$x * y = x + y - xy$. Is $*$ binary on A ?

If so, examine the commutative and

associative properties satisfied by $*$ on A .

(ii) Let A be $\mathbb{Q} \setminus \{1\}$. Define $*$ on A by

$$x * y = x + y - xy. \text{ Is } * \text{ binary on } A?$$

If so, examine the existence of identity, existence of inverse properties for the operation $*$ on A .

Solution: (i)

(1) Closure

$$\text{Given } A = \{\mathbb{Q} \setminus \{1\}\}$$

Let $x, y \in A$. That is $x \neq 1$, and $y \neq 1$

$*$ is defined on A by

$$x * y = x + y - xy$$

Let us assume that $x * y = 1$. Then,

$$x + y - xy = 1$$

$$y - xy = 1 - x$$

$$y(1 - x) = 1 - x$$

$$y = \frac{1-x}{(1-x)}$$

It Gives, $y = 1$, which is contradiction to our assumption.

Hence, $x * y \neq 1 \in A$

$\therefore *$ is closed on A

(2) Commutative

$$x * y = x + y - xy$$

$$= y + x - yx$$

$$= y * x$$

$\therefore *$ is commutative on A

(3) Associative

$$x * (y * z) = x * (y + z - yz)$$

$$= x + (y + z - yz) - x(y + z - yz)$$

$$= x + y + z - yz - xy - xz + xyz$$

$$(x * y) * z = (x + y - xy) * z$$

$$= (x + y - xy) + z - (x + y - xy)z$$

$$= x + y - xy + z - xz - yz + xyz$$

From the above results,

$$x * (y * z) = (x * y) * z, \forall x, y, z \in A$$

$\therefore *$ is associative on A

Solution: (ii)

(1) Closure

(2) Existence of Identity

Let $x \in A$. That is $x \neq 1$

Let e be the identity.

$$\text{Then } x * e = e * x = x$$

$$x + e - xe = x$$

$$e - xe = x - x$$

$$e(1 - x) = 0$$

It gives, $e = 0$, and $(1 - x) = 0$

If $(1 - x) = 0$, gives $x = 1$, which is not applicable as $x \in A$.

Hence, $e = 0 \in A$, is the identity.

$\therefore *$ has identity on A

(3) Existence of Inverse

Let $x \in A$. That is $x \neq 1$

Let x^{-1} be the inverse.

$$\text{Then } x * x^{-1} = x^{-1} * x = e$$

$$x + x^{-1} - xx^{-1} = 0$$

$$x^{-1} - xx^{-1} = -x$$

$$x^{-1}(1 - x) = -x$$

$$x^{-1} = -\frac{x}{(1-x)} \in A,$$

is the inverse of $x, \forall x \in A$.

$\therefore *$ has inverse on A .

Example 12.11

Identify the valid statements from the following sentences.

(1) Mount Everest is the highest mountain of the world.

It is true. So it is a statement.

(2) $3 + 4 = 8$.

It is false. So it is a statement.

(3) $7 + 5 > 10$.

It is true. So it is a statement.

(4) Give me that book.

It is a command. So it is not a statement.

(5) $(10 - x) = 7$.

It gives $x = 3$. So it is a statement.

(6) How beautiful this flower is!

It is an exclamatory. So it is not a statement.

(7) Where are you going?

It is a question. So it is not a statement.

(8) Wish you all success.

It is expressing wishes. So it is not a statement.

(9) This is the beginning of the end.

It is a paradox. So it is not a statement.

Example 12.12

Write the statements in words corresponding to $\neg p$, $p \wedge q$, $p \vee q$ and $q \vee \neg p$, where p is 'It is cold' and q is 'It is raining.'

Solution:

Sentence p : 'It is cold'

Sentence q : 'It is raining.'

So, $\neg p$ = It is not cold.

$p \wedge q$ = It is cold and raining.

$p \vee q$ = It is cold or raining.

$q \vee \neg p$ = It is raining or it is not cold.

Example 12.13

How many rows are needed for following statement formulae?

(i) $p \vee \neg t \wedge (p \vee \neg s)$

(ii) $((p \wedge q) \vee (\neg r \vee \neg s)) \wedge (\neg t \wedge v)$

Solution:

(i) It contains 3 variables p, t and s

So, number of rows needed = $2^3 = 8$

(ii) It contains 6 variables p, q, r, s, t and v

So, number of rows needed = $2^6 = 64$

Example 12.15

Write down the (i) conditional statement

(ii) converse statement (iii) inverse statement, and (iv) contra positive statement for the two statements p and q given below.

p : The number of primes is infinite.

q : Ooty is in Kerala.

Solution:

Then the four types of conditional statements corresponding to p and q are respectively listed below.

(i) $p \rightarrow q$: (conditional statement)

"If the number of primes is infinite then Ooty is in Kerala".

(ii) $q \rightarrow p$: (converse statement)

"If Ooty is in Kerala then the number of primes is infinite"

(iii) $\neg p \rightarrow \neg q$ (inverse statement)

"If the number of primes is not infinite then Ooty is not in Kerala".

(iv) $\neg q \rightarrow \neg p$ (contra positive statement)

"If Ooty is not in Kerala then the number of primes is not infinite"

LOGIC TRUTH FORMULAE:(1) For NOT (\neg)

p	$\neg p$
T	F
F	T

(2) For AND (\wedge)

p	q	$p \wedge q$
T	T	T
T	F	F
F	T	F
F	F	F

(3) For OR (\vee)

p	q	$p \vee q$
T	T	T
T	F	T
F	T	T
F	F	F

(4) For CONDITIONAL (\rightarrow)

p	q	$p \rightarrow q$
T	T	T
T	F	F
F	T	T
F	F	T

(5) For BI - CONDITIONAL (\leftrightarrow)

p	q	$p \leftrightarrow q$
T	T	T
T	F	F
F	T	F
F	F	T

(6) For EXCLUSIVE p EOR q (\veebar)

p	q	$p \veebar q$
T	T	F
T	F	T
F	T	T
F	F	F

(7) If the last column contains only T, then the statement called as Tautology.

(8) If the last column contains only F, then the statement called as Contradiction.

(9) The statement which is neither Tautology nor Contradiction is called Contingency.

Example 12.16

Construct the truth table for $(p \vee q) \wedge (p \vee \neg q)$ Let $r: (p \vee q)$ and $s: (p \vee \neg q)$ So, $(p \vee q) \wedge (p \vee \neg q) \equiv r \wedge s$

p	q	$p \vee q$	$r: p \vee q$	$\neg q$	$s: p \vee \neg q$	$r \wedge s$
T	T	T	F	F	T	F
T	F	T	T	T	F	F
F	T	T	T	F	F	F
F	F	F	F	T	T	F

Example 12.17

Establish the equivalence property: $p \rightarrow q \equiv$ $\neg p \vee q$

p	q	$\neg p$	$p \rightarrow q$	$\neg p \vee q$
T	T	F	T	T
T	F	F	F	F
F	T	T	T	T
F	F	T	T	T

The entries in the columns corresponding to $p \rightarrow q$ and $\neg p \vee q$ are identical and hence they are equivalent.

Example 12.18 Establish the equivalence property connecting the bi-conditional with conditional:

 $p \leftrightarrow q \equiv (p \rightarrow q) \wedge (q \rightarrow p)$ Solution:

p	q	$p \leftrightarrow q$	$r: p \rightarrow q$	$s: q \rightarrow p$	$r \wedge s$
T	T	T	T	T	T
T	F	F	F	T	F
F	T	F	T	F	F
F	F	T	T	T	T

The entries in the columns corresponding to $p \leftrightarrow q$ and $(p \rightarrow q) \wedge (q \rightarrow p)$ are identical and hence they are equivalent.

Example 12.19

Using the equivalence property, show that: $p \leftrightarrow q \equiv (p \wedge q) \vee (\neg p \wedge \neg q)$.Solution:

LHS

p	q	$p \leftrightarrow q$
T	T	T
T	F	F
F	T	F
F	F	T

RHS

$r: p \wedge q$	$\neg p$	$\neg q$	$s: \neg p \wedge \neg q$	$r \vee s$
T	F	F	F	T
F	F	T	F	F
F	T	F	F	F
F	T	T	T	T

The entries in the columns corresponding to $p \leftrightarrow q$ and $(p \wedge q) \vee (\neg p \wedge \neg q)$ are identical and hence they are equivalent.

EXERCISE 12.2

1. Let p : Jupiter is a planet and q : India is an island be any two simple statements.

Give verbal sentence describing each of the following statements.

(i) $\neg p$ (ii) $p \wedge \neg q$ (iii) $\neg p \vee q$

(iv) $p \rightarrow \neg q$ (v) $p \leftrightarrow q$

Solution:

p : Jupiter is a planet

q : India is an island

(i) $\neg p$: Jupiter is not a planet

(ii) $p \wedge \neg q$: Jupiter is a planet and India is not an island

(iii) $\neg p \vee q$: Jupiter is not a planet or India is an island

(iv) $p \rightarrow \neg q$: Jupiter is a planet then India is not an island

(v) $p \leftrightarrow q$: Jupiter is a planet if and only if India is an island

2. Write each of the following sentences in symbolic form using statement variables p and q .

p : 19 is a prime number

q : all the angles of a triangle are equal.

(i) 19 is not a prime number and all the angles of a triangle are equal.

(ii) 19 is a prime number or all the angles of a triangle are not equal

(iii) 19 is a prime number and all the angles of a triangle are equal

(iv) 19 is not a prime number

Solution:

(i) 19 is not a prime number and all the angles of a triangle are equal : $\neg p \wedge q$

(ii) 19 is a prime number or all the angles of a triangle are not equal : $p \vee \neg q$

(iii) 19 is a prime number and all the angles of a triangle are equal : $p \wedge q$

(iv) 19 is not a prime number : $\neg p$

3. Determine the truth value of each of the following statements

(i) If $6 + 2 = 5$, then the milk is white.

(ii) China is in Europe or 3 is an integer

(iii) It is not true that $5 + 5 = 9$ or Earth is a planet

(iv) 11 is a prime number and all the sides of a rectangle are equal

Solution:

(i) If $6 + 2 = 5$, then the milk is white.

p : $6 + 2 = 5$ F

q : The milk is white. T

Symbolic Form: $p \rightarrow q$ Truth value is T

(ii) China is in Europe or $\sqrt{3}$ is an integer.

p : China is in Europe F

q : $\sqrt{3}$ is an integer. F

Symbolic Form: $p \vee q$ Truth value is F

(iii) It is not true that $5 + 5 = 9$ or Earth is a planet.

$$p: 5 + 5 = 9 \text{ F}$$

q : Earth is a planet. T

Symbolic Form: $\neg p \vee q$

Truth value is $T \vee T = T$

- (iv) 11 is a prime number and all the sides of a rectangle are equal

p : 11 is a prime number T

q : all the sides of a rectangle are equal. F

Symbolic Form: $p \wedge q$

Truth value is $T \wedge F = F$

4. Which one of the following sentences is a proposition?

- (i) $4 + 7 = 12$
- (ii) What are you doing?
- (iii) $3^n \leq 81, n \in N$
- (iv) Peacock is our national bird
- (v) How tall this mountain is!

Solution:

- (i) $4 + 7 = 12$ Proposition
- (ii) What are you doing? Not a Proposition
- (iii) $3^n \leq 81, n \in N$ Proposition
- (iv) Peacock is our national bird. Proposition
- (v) How tall this mountain is! Not Proposition

5. Write the converse, inverse, and contra positive of each of the following implication.

- (i) If x and y are numbers such that $x = y$, then $x^2 = y^2$
- (ii) If a quadrilateral is a square then it is a rectangle

Solution:

- (i) If x and y are numbers such that $x = y$, then $x^2 = y^2$

p : x and y are numbers such that $x = y$

q : $x^2 = y^2$

Symbolic form of the given statement : $p \rightarrow q$

- (a) Converse: $q \rightarrow p$

If x and y are numbers such that $x^2 = y^2$, then $x = y$

- (b) Inverse: $\neg p \rightarrow \neg q$

If x and y are numbers such that $x \neq y$, then $x^2 \neq y^2$

- (c) Contra positive: $\neg q \rightarrow \neg p$

If x and y are numbers such that $x^2 \neq y^2$, then $x \neq y$

- (ii) If a quadrilateral is a square then it is a rectangle

p : A quadrilateral is a square.

q : A quadrilateral is a rectangle.

Symbolic form of the given statement : $p \rightarrow q$

- (a) Converse: $q \rightarrow p$

If a quadrilateral is a rectangle then it is a square.

- (b) Inverse: $\neg p \rightarrow \neg q$

If a quadrilateral is not a square then it is not a rectangle.

- (c) Contra positive: $\neg q \rightarrow \neg p$

If a quadrilateral is not a rectangle then it is not a square.

6. Construct the truth table for the following statements.

- (i) $\neg p \wedge \neg q$

p	q	$\neg p$	$\neg q$	$\neg p \wedge \neg q$
T	T	F	F	F
T	F	F	T	F
F	T	T	F	F
F	F	T	T	T

- (ii) $\neg(p \wedge \neg q)$

p	q	$\neg q$	$p \wedge \neg q$	$\neg(p \wedge \neg q)$
T	T	F	F	T
T	F	T	T	F
F	T	F	F	T
F	F	T	F	T

(iii) $(p \vee q) \vee \neg q$

p	q	$p \vee q$	$\neg q$	$(p \vee q) \vee \neg q$
T	T	T	F	T
T	F	T	T	T
F	T	T	F	T
F	F	F	T	T

p	q	$p \rightarrow q$	$\neg p$	$(\neg p \rightarrow q)$	$(p \rightarrow q) \leftrightarrow (\neg p \rightarrow q)$
T	T	T	F	T	T
T	F	F	F	T	F
F	T	T	T	T	T
F	F	T	T	F	F

The last column contains T and F. Hence it is contingency.

(iv) $(\neg p \rightarrow r) \wedge (p \leftrightarrow q)$

p	q	r	$\neg p$	$(\neg p \rightarrow r)$	$p \leftrightarrow q$	$(\neg p \rightarrow r) \wedge (p \leftrightarrow q)$
T	T	T	F	T	T	T
T	T	F	F	T	T	T
T	F	T	F	T	F	F
T	F	F	F	T	F	F
F	T	T	T	T	F	F
F	T	F	T	F	F	F
F	F	T	T	T	T	T
F	F	F	T	F	T	F

(iv) $((p \rightarrow q) \wedge (q \rightarrow r)) \rightarrow (p \rightarrow r)$

p	q	r	$(p \rightarrow q)$	$(q \rightarrow r)$	$((p \rightarrow q) \wedge (q \rightarrow r))$
T	T	T	T	T	T
T	T	F	T	F	F
T	F	T	F	T	F
T	F	F	F	T	F
F	T	T	T	T	T
F	T	F	T	F	F
F	F	T	T	T	T
F	F	F	T	T	T

7. Verify whether the following compound propositions are tautologies or contradictions or contingency

(i) $(p \wedge q) \wedge \neg(p \vee q)$

p	q	$p \wedge q$	$p \vee q$	$\neg(p \vee q)$	$(p \wedge q) \wedge \neg(p \vee q)$
T	T	T	T	F	F
T	F	F	T	F	F
F	T	F	T	F	F
F	F	F	F	T	F

The last column contains only F. Hence it is contradiction.

(ii) $((p \vee q) \wedge \neg p) \rightarrow q$

p	q	$p \vee q$	$\neg p$	$(p \vee q) \wedge \neg p$	$((p \vee q) \wedge \neg p) \rightarrow q$
T	T	T	F	F	T
T	F	T	F	F	T
F	T	T	T	T	T
F	F	F	T	F	T

The last column contains only T. Hence it is tautology.

(iii) $(p \rightarrow q) \leftrightarrow (\neg p \rightarrow q)$

$(p \rightarrow r) \leftrightarrow ((p \rightarrow q) \wedge (q \rightarrow r)) \rightarrow (p \rightarrow r)$

T	T
F	T
T	T
F	T
T	T
T	T
T	T
T	T

The last column contains only T. Hence it is tautology.

8. Show that

(i) $\neg(p \wedge q) \equiv \neg p \vee \neg q$

p	q	$p \wedge q$	$\neg(p \wedge q)$	$\neg p$	$\neg q$	$\neg p \vee \neg q$
T	T	T	F	F	F	F
T	F	F	T	F	T	T
F	T	F	T	T	F	T
F	F	F	T	T	T	T

From column (4) and (7), it is proved that

$\neg(p \wedge q) \equiv \neg p \vee \neg q$

(ii) $\neg(p \rightarrow q) \equiv p \wedge \neg q$.

p	q	$p \rightarrow q$	$\neg(p \rightarrow q)$	$\neg q$	$p \wedge \neg q$
T	T	T	F	F	F
T	F	F	T	T	T
F	T	T	F	F	F
F	F	T	F	T	F

From column (4) and (6), it is proved that

$$\neg(p \rightarrow q) \equiv p \wedge \neg q.$$

9. Prove that $q \rightarrow p \equiv \neg p \rightarrow \neg q$

p	q	$q \rightarrow p$	$\neg p$	$\neg q$	$\neg p \rightarrow \neg q$
T	T	T	F	F	T
T	F	T	F	T	T
F	T	F	T	F	F
F	F	T	T	T	T

From column (3) and (6), it is proved that

$$q \rightarrow p \equiv \neg p \rightarrow \neg q$$

10. Show that $p \rightarrow q$ and $q \rightarrow p$ are not equivalent

p	q	$p \rightarrow q$	$q \rightarrow p$
T	T	T	T
T	F	F	T
F	T	T	F
F	F	T	T

$p \rightarrow q$ and $q \rightarrow p$ are not equivalent.

11. Show that $\neg(p \leftrightarrow q) \equiv p \leftrightarrow \neg q$

p	q	$p \leftrightarrow q$	$\neg(p \leftrightarrow q)$	$\neg q$	$p \leftrightarrow \neg q$
T	T	T	F	F	F
T	F	F	T	T	T
F	T	F	T	F	T
F	F	T	F	T	F

From column (4) and (6), it is proved that

$$\neg(p \leftrightarrow q) \equiv p \leftrightarrow \neg q.$$

12. Check whether the statement $p \rightarrow (q \rightarrow p)$ is a tautology or a contradiction without using the truth table.

Solution:

$$\begin{aligned} p \rightarrow (q \rightarrow p) &\equiv p \rightarrow (\neg q \vee p) \\ &\equiv \neg p \vee (\neg q \vee p) \\ &\equiv \neg p \vee (p \vee \neg q) \\ &\equiv (\neg p \vee p) \vee \neg q \\ &\equiv T \vee \neg q \\ &\equiv T \end{aligned}$$

Hence, $p \rightarrow (q \rightarrow p)$ is a tautology.

13. Using truth table check whether the

statements $\neg(p \vee q) \vee (\neg p \wedge q)$ and $\neg p$ are logically equivalent.

p	q	$p \vee q$	$\neg(p \vee q)$	$\neg p$	$\neg p \wedge q$	$\neg(p \vee q) \vee (\neg p \wedge q)$
T	T	T	F	F	F	F
T	F	T	F	F	F	F
F	T	T	F	T	T	T
F	F	F	T	T	F	T

From column (5) and (7), it is proved that

$$\neg p \equiv \neg(p \vee q) \vee (\neg p \wedge q).$$

14. Prove $p \rightarrow (q \rightarrow r) \equiv (p \wedge q) \rightarrow r$ without

using truth table.

Solution:

$$\begin{aligned} p \rightarrow (q \rightarrow r) &\equiv p \rightarrow (\neg q \vee r) \\ &\equiv \neg p \vee (\neg q \vee r) \\ &\equiv (\neg p \vee \neg q) \vee r \\ &\equiv \neg(p \wedge q) \vee r \\ &\equiv (p \wedge q) \rightarrow r \text{ Hence proved.} \end{aligned}$$

15. Prove that $p \rightarrow (\neg q \vee r) \equiv \neg p \vee (\neg q \vee r)$
using truth table.

LHS : $p \rightarrow (\neg q \vee r)$

p	q	r	$\neg q$	$(\neg q \vee r)$	$p \rightarrow (\neg q \vee r)$
T	T	T	F	T	T
T	T	F	F	F	F
T	F	T	T	T	T
T	F	F	T	T	T
F	T	T	F	T	T
F	T	F	F	F	T
F	F	T	T	T	T
F	F	F	T	T	T

RHS : $\neg p \vee (\neg q \vee r)$

p	q	r	$\neg q$	$(\neg q \vee r)$	$\neg p$	$\neg p \vee (\neg q \vee r)$
T	T	T	F	T	F	T
T	T	F	F	F	F	F
T	F	T	T	T	F	T
T	F	F	T	T	F	T
F	T	T	F	T	T	T
F	T	F	F	F	T	T
F	F	T	T	T	T	T
F	F	F	T	T	T	T

From the above tables, the last columns are identical. Hence $p \rightarrow (\neg q \vee r) \equiv \neg p \vee (\neg q \vee r)$.

EXERCISE 12.3

Choose the correct or the most suitable answer from the given four alternatives.

1. A binary operation on a set S is a function from

- (1) $S \rightarrow S$ (2) $(S \times S) \rightarrow S$
(3) $S \rightarrow (S \times S)$ (4) $(S \times S) \rightarrow (S \times S)$

2. Subtraction is not a binary operation in

- (1) \mathbb{R} (2) \mathbb{Z} (3) \mathbb{N} (4) \mathbb{Q}

3. Which one of the following is a binary operation on \mathbb{N} ?

- (1) Subtraction (2) Multiplication
(3) Division (4) All the above

4. In the set \mathbb{R} of real numbers '*' is defined as follows. Which one of the following is not a binary operation on \mathbb{R} ?

- (1) $a * b = \min(a, b)$ (2) $a * b = \max(a, b)$
(3) $a * b = a$ (4) $a * b = a^b$

5. The operation * defined by $a * b = \frac{ab}{7}$ is not a binary operation on

- (1) \mathbb{Q}^+ (2) \mathbb{Z} (3) \mathbb{R} (4) \mathbb{C}

6. In the set \mathbb{Q} define $\odot b = a + b + ab$. For what value of y , $3 \odot (y \odot 5) = 7$?

- (1) $y = \frac{2}{3}$ (2) $y = \frac{-2}{3}$
(3) $y = \frac{-3}{2}$ (4) $y = 4$

7. If $a * b = \sqrt{a^2 + b^2}$ on the real numbers then * is

- (1) commutative but not associative
(2) associative but not commutative
(3) both commutative and associative
(4) neither commutative nor associative

8. Which one of the following statements has the truth value T ?

- (1) $\sin x$ is an even function.
(2) Every square matrix is non-singular
(3) The product of complex number and its conjugate is purely imaginary
(4) $\sqrt{5}$ is an irrational number

9. Which one of the following statements has truth value F ?

- (1) Chennai is in India or $\sqrt{2}$ is an integer
(2) Chennai is in India or $\sqrt{2}$ is an irrational number
(3) Chennai is in China or $\sqrt{2}$ is an integer
(4) Chennai is in China or $\sqrt{2}$ is an irrational number

10. If a compound statement involves 3 simple statements, then the number of rows in the truth table is

- (1) 9 (2) 8 (3) 6 (4) 3

11. Which one is the inverse of the statement

$$(p \vee q) \rightarrow (p \wedge q)?$$

- (1) $(p \wedge q) \rightarrow (p \vee q)$
 (2) $\neg(p \vee q) \rightarrow (p \wedge q)$
 (3) $(\neg p \vee \neg q) \rightarrow (\neg p \wedge \neg q)$
 (4) $(\neg p \wedge \neg q) \rightarrow (\neg p \vee \neg q)$

12. Which one is the contra positive of the statement $(p \vee q) \rightarrow r$?

- (1) $\neg r \rightarrow (\neg p \wedge \neg q)$ (2) $\neg r \rightarrow (p \vee q)$
 (3) $r \rightarrow (p \wedge q)$ (4) $p \rightarrow (q \vee r)$

13. The truth table for $(p \wedge q) \vee \neg q$ is given below

p	q	$(p \wedge q) \vee \neg q$
T	T	(a)
T	F	(b)
F	T	(c)
F	F	(d)

Which one of the following is true?

- (a) (b) (c) (d)
 (1) T T T T
 (2) T F T T
 (3) T T F T
 (4) T F F F

14. In the last column of the truth table for $\neg(p \vee \neg q)$ the number of final outcomes of the truth value 'F' are

- (1) 1 (2) 2 (3) 3 (4) 4

15. Which one of the following is incorrect? For any two propositions p and q , we have

- (1) $\neg(p \vee q) \equiv \neg p \wedge \neg q$
 (2) $\neg(p \wedge q) \equiv \neg p \vee \neg q$
 (3) $\neg(p \vee q) \equiv \neg p \vee \neg q$
 (4) $\neg(\neg p) \equiv p$

16. Which one of the following is correct for the truth value of $(p \wedge q) \rightarrow \neg p$?

- (a) (b) (c) (d)
 (1) T T T T
 (2) F T T T
 (3) F F T T
 (4) T T T F

17. The dual of $\neg(p \vee q) \vee [p \vee (p \wedge \neg r)]$ is

- (1) $\neg(p \wedge q) \wedge [p \vee (p \wedge \neg r)]$
 (2) $(p \wedge q) \wedge [p \wedge (p \vee \neg r)]$
 (3) $\neg(p \wedge q) \wedge [p \wedge (p \wedge r)]$
 (4) $\neg(p \wedge q) \wedge [p \wedge (p \vee \neg r)]$

18. The proposition $p \wedge (\neg p \vee q)$ is

- (1) a tautology
 (2) a contradiction
 (3) logically equivalent to $p \wedge q$
 (4) logically equivalent to $p \vee q$

19. Determine the truth value of each of the following statements:

- (a) $4+2=5$ and $6+3=9$
 (b) $3+2=5$ and $6+1=7$
 (c) $4+5=9$ and $1+2=4$
 (d) $3+2=5$ and $4+7=11$

- (a) (b) (c) (d)
 (1) F T F T
 (2) T F T F
 (3) T T F F
 (4) F F T T

20. Which one of the following is not true?

- (1) Negation of a negation of a statement is the statement itself.
 (2) If the last column of the truth table contains only T then it is a tautology.
 (3) If the last column of its truth table contains only F then it is a contradiction
 (4) If p and q are any two statements then $p \leftrightarrow q$ is a tautology.



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