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8. Differentials and Partial Derivatives

Example 8.1

Find the linear approximation for $f(x) = \sqrt{1+x}, x \geq -1$, at $x_0 = 3$. Use the linear approximation to estimate $f(3.2)$.

Solution:

Linear approximation

$$L(x) = f(x_0) + f^{/}(x_0)(x - x_0)$$

Given $x_0 = 3, x = 3.2$

$$f(x_0) = f(3) = \sqrt{1+3} = \sqrt{4} = 2$$

$$f(x) = \sqrt{1+x}$$

$$f^{/}(x) = \frac{1}{2\sqrt{1+x}}$$

$$f^{/}(x_0) = f^{/}(3) = \frac{1}{2\sqrt{1+3}} = \frac{1}{2\sqrt{4}} = \frac{1}{4}$$

Thus, Linear approximation

$$L(x) = f(x_0) + f^{/}(x_0)(x - x_0)$$

$$= 2 + \frac{1}{4}(x-3)$$

$$= 2 + \frac{x}{4} - \frac{3}{4}$$

$$= \frac{8-3}{4} + \frac{x}{4}$$

$$= \frac{5}{4} + \frac{x}{4}$$

And, $f(x) = \sqrt{1+x}$

$$f(3.2) = \sqrt{1+3.2} = \sqrt{4.2}$$

So, $L(3.2) = \frac{5}{4} + \frac{3.2}{4}$

$$= 1.25 + 0.8 = 2.05$$

....

Example 8.2

Use linear approximation to find an approximate value of $\sqrt{9.2}$ without using a calculator.

Solution:

Let $y = \sqrt{x}$, let $x = 9$ and $dx = 0.2$

$$y = \sqrt{x} = \sqrt{9}$$

$$y = 3$$

From $y = \sqrt{x}$

$$dy = \frac{1}{2\sqrt{x}}\,dx = \frac{1}{2\sqrt{9}}(0.2) = \frac{1}{3}(0.1) = 0.033$$

$$f(x) = y + dy$$

$$f(\sqrt{9.2}) = 3 + 0.033 = 3.033$$

....

Example 8.3

Let us assume that the shape of a soap bubble is a sphere. Use linear approximation to approximate the increase in the surface area of a soap bubble as its radius increases from 5 cm to 5.2 cm. Also, calculate the percentage error.

Solution:

Radius of the sphere $r = 5$

Increase radius $dr = 0.2$

Surface area of the sphere $S = 4\pi r^2$

Approximate change in SA $dS = 4\pi(2r)dr$

$$= 4\pi(2 \times 5)(0.2) = 4\pi(10)(0.2) = 4\pi(2)$$

Approximate change in SA $dS = 8\pi$

Actual change in surface area $= S(5.2) - S(5)$

$$S(5.2) - S(5) = 4\pi(5.2)^2 - 4\pi(5)^2$$

$$= 4\pi[(5.2)^2 - (5)^2]$$

$$= 4\pi[(5.2+5) \times (5.2-5)]$$

$$= 4\pi[(10.2) \times (0.2)]$$

$$= 4\pi(2.04)$$

$$= 8.16\pi$$

$$\text{Relative error} = \frac{\text{Actual error} - \text{approximate error}}{\text{Actual error}}$$

$= \frac{8.16\pi - 8\pi}{8.16\pi}$

$= \frac{0.16\pi}{8.16\pi}$

$= \frac{16}{816}$

$= 0.01960$

Percentage error $=$ Relative error $\times 100$

$= 0.01960 \times 100$

$= 01.960\ \%$

....

Example 8.4

A right circular cylinder has radius r =10 cm. and height $h = 20$ cm. Suppose that the radius of the cylinder is increased from 10 cm to 10.1 cm and the height does not change. Estimate the change in the volume of the cylinder. Also, calculate the relative error and percentage error.

Solution:

Height of the cylinder $h = 20$

Radius of the cylinder $r = 10$

Increase radius $dr = 0.1$

Volume of the cylinder $V = \pi r^2 h$

$V = \pi r^2 (20)$

$V = 20\pi r^2$

Approximate change in V $dV = 20\pi(2r)dr$

$= 20\pi(2 \times 10)(0.1)$

$= 20\pi(20)(0.1)$

$= 20\pi(2)$

Approximate change in V $dV = 40\pi$

Actual change in Volume $= V(10.1) - V(10)$

$V(10.1) - V(10) = 20\pi(10.1)^2 - 20\pi(10)^2$

$= 20\pi[(10.1)^2 - (10)^2]$

$= 20\pi[(10.1 + 10) \times (10.1 - 10)]$

$= 20\pi[(20.1) \times (0.1)]$

$= 20\pi(2.01)$

$= 40.2\pi$

Relative error $= \frac{Actual\ error - approximate\ error}{Actual\ error}$

$= \frac{40.2\pi - 40\pi}{40.2\pi}$

$= \frac{0.2\pi}{40.2\pi}$

$= \frac{2}{402}$

$= \frac{1}{201}$

$= 0.00497$

Percentage error $=$ Relative error $\times 100$

$= 0.00497 \times 100$

$= 0.497\ \%$

....

EXERCISE 8.1

1. Let $f(x) = \sqrt[3]{x}$. Find the linear approxi. at $x = 27$. Use the linear approximation to approximate $\sqrt[3]{27.2}$

Solution:

Linear approximation

$L(x) = f(x_0) + f'(x_0)(x - x_0)$

Given $x_0 = 27, x = 27.2$

$f(x) = \sqrt[3]{x}$

$f(x_0) = f(27) = \sqrt[3]{27} = 3$

$f(x) = (x)^{\frac{1}{3}}$

$f'(x) = \frac{1}{3}(x)^{\frac{1}{3}-1}$

$= \frac{1}{3}(x)^{-\frac{2}{3}}$

$f'(x) = \frac{1}{3} \times \frac{1}{(x)^{\frac{2}{3}}}$

$f'(x_0) = \frac{1}{3} \times \frac{1}{(27)^{\frac{2}{3}}}$

$= \frac{1}{3} \times \frac{1}{(3)^{3 \times \frac{2}{3}}}$

$= \frac{1}{3} \times \frac{1}{(3)^2}$

$= \frac{1}{3} \times \frac{1}{9}$

$f'(x_0) = \frac{1}{27}$

Linear approximation

$L(x) = f(x_0) + f'(x_0)(x - x_0)$

$= 3 + \frac{1}{27}(x - 27)$

To find $\sqrt[3]{27.2}$

$$L(27.2) = 3 + \frac{1}{27}(27.2 - 27)$$
$$= 3 + \frac{1}{27}(0.2)$$
$$= 3 + \frac{0.2}{27}$$
$$= 3 + 0.007$$
$$\sqrt[3]{27.2} = 3.007$$

....

2. Use the linear approximation to find approximate values of

(i) $(123)^{\frac{2}{3}}$ (ii) $\sqrt[4]{15}$ (iii) $\sqrt[3]{26}$

Solution: (i) $(123)^{\frac{2}{3}}$

Linear approximation

$$L(x) = f(x_0) + f'(x_0)(x - x_0)$$

Given $x_0 = 125, x = 123$

$$f(x) = (123)^{\frac{2}{3}}$$
$$f(x_0) = f(125) = (125)^{\frac{2}{3}}$$
$$= (5)^{3\times\frac{2}{3}}$$
$$= (5)^2$$
$$= 25$$
$$f(x) = (x)^{\frac{2}{3}}$$
$$f'(x) = \frac{2}{3}(x)^{\frac{2}{3}-1}$$
$$= \frac{2}{3}(x)^{-\frac{1}{3}}$$
$$f'(x) = \frac{2}{3} \times \frac{1}{(x)^{\frac{1}{3}}}$$
$$f'(x_0) = \frac{2}{3} \times \frac{1}{(125)^{\frac{1}{3}}}$$
$$= \frac{2}{3} \times \frac{1}{(5)^{3\times\frac{1}{3}}}$$
$$= \frac{2}{3} \times \frac{1}{(5)^1}$$
$$= \frac{2}{3} \times \frac{1}{5}$$
$$f'(x_0) = \frac{2}{15}$$

Linear approximation

$$L(x) = f(x_0) + f'(x_0)(x - x_0)$$
$$= 25 + \frac{2}{15}(x - 125)$$

To find $(123)^{\frac{2}{3}}$

$$L(123) = 25 + \frac{2}{15}(123 - 125)$$
$$= 25 + \frac{2}{15}(-2)$$
$$= 25 - \frac{4}{15}$$
$$= 25 - 0.266$$
$$(123)^{\frac{2}{3}} = 24.734$$

(ii) $\sqrt[4]{15}$

Linear approximation

$$L(x) = f(x_0) + f'(x_0)(x - x_0)$$

Given $x_0 = 16, x = 15$

$$f(x) = \sqrt[4]{15} = (15)^{\frac{1}{4}}$$
$$f(x_0) = f(16) = (16)^{\frac{1}{4}}$$
$$= (2)^{4\times\frac{1}{4}}$$
$$= (2)^1$$
$$= 2$$
$$f(x) = (x)^{\frac{1}{4}}$$
$$f'(x) = \frac{1}{4}(x)^{\frac{1}{4}-1}$$
$$= \frac{1}{4}(x)^{-\frac{3}{4}}$$
$$f'(x) = \frac{1}{4} \times \frac{1}{(x)^{\frac{3}{4}}}$$
$$f'(x_0) = \frac{1}{4} \times \frac{1}{(16)^{\frac{3}{4}}}$$
$$= \frac{1}{4} \times \frac{1}{(2)^{4\times\frac{3}{4}}}$$
$$= \frac{1}{4} \times \frac{1}{(2)^3}$$
$$= \frac{1}{4} \times \frac{1}{8}$$
$$f'(x_0) = \frac{1}{32}$$

Linear approximation

$$L(x) = f(x_0) + f'(x_0)(x - x_0)$$
$$= 2 + \frac{1}{32}(x - 16)$$

To find $\sqrt[4]{15}$

$$L(15) = 2 + \frac{1}{32}(15 - 16)$$
$$= 2 + \frac{1}{32}(-1)$$
$$= 2 - \frac{1}{32}$$
$$= 2 - 0.03125$$
$$\sqrt[4]{15} = 1.96875$$

(iii) $\sqrt[3]{26}$

Linear approximation

$$L(x) = f(x_0) + f'(x_0)(x - x_0)$$

Given $x_0 = 27, x = 26$

$$f(x) = \sqrt[3]{26} = (26)^{\frac{1}{3}}$$
$$f(x_0) = f(27) = (27)^{\frac{1}{3}}$$
$$= (3)^{3\times\frac{1}{3}}$$
$$= (3)^1$$
$$= 3$$
$$f(x) = (x)^{\frac{1}{3}}$$
$$f'(x) = \frac{1}{3}(x)^{\frac{1}{3}-1}$$
$$= \frac{1}{3}(x)^{-\frac{2}{3}}$$
$$f'(x) = \frac{1}{3} \times \frac{1}{(x)^{\frac{2}{3}}}$$
$$f'(x_0) = \frac{1}{3} \times \frac{1}{(27)^{\frac{2}{3}}}$$
$$= \frac{1}{3} \times \frac{1}{(3)^{3\times\frac{2}{3}}}$$
$$= \frac{1}{3} \times \frac{1}{(3)^2}$$
$$= \frac{1}{3} \times \frac{1}{9}$$
$$f'(x_0) = \frac{1}{27}$$

Linear approximation

$$L(x) = f(x_0) + f'(x_0)(x - x_0)$$
$$= 3 + \frac{1}{27}(x - 27)$$

To find $\sqrt[3]{26}$

$$L(26) = 3 + \frac{1}{27}(26 - 27)$$
$$= 3 + \frac{1}{27}(-1)$$
$$= 3 - \frac{1}{27}$$
$$= 3 - 0.0370$$
$$\sqrt[3]{26} = 2.963$$

....

3. Find a linear approximation for the following functions at the indicated points.

(i) $f(x) = x^3 - 5x + 12, x_0 = 2$

(ii) $g(x) = \sqrt{x^2 + 9}, x_0 = -4$

(iii) $h(x) = \frac{x}{x+1}, x_0 = 1$

Solution:

(i) $f(x) = x^3 - 5x + 12, x_0 = 2$

$$f(x) = x^3 - 5x + 12$$
$$f(x_0) = f(2) = (2)^3 - 5(2) + 12$$
$$= 8 - 10 + 12$$
$$= 20 - 10$$
$$f(2) = 10$$
$$f'(x) = 3x^2 - 5$$
$$f'(x_0) = f'(2) = 3(2)^2 - 5$$
$$= 3(4) - 5$$
$$= 12 - 5$$
$$= 7$$

Linear approximation

$$L(x) = f(x_0) + f'(x_0)(x - x_0)$$
$$= f(2) + f'(2)(x - 2)$$
$$= 10 + 7(x - 2)$$
$$= 10 + 7x - 14$$
$$= 7x - 4$$

(ii) $g(x) = \sqrt{x^2 + 9}, x_0 = -4$

$$g(x) = \sqrt{x^2 + 9}$$
$$g(x_0) = g(-4) = \sqrt{(-4)^2 + 9}$$
$$= \sqrt{16 + 9}$$
$$= \sqrt{25}$$
$$g(-4) = 5$$

$$g'(x) = \frac{1}{2\sqrt{x^2+9}} \times 2x$$

$$= \frac{x}{\sqrt{x^2+9}}$$

$$g'(x_0) = g'(-4) = \frac{(-4)}{\sqrt{(-4)^2+9}}$$

$$= \frac{-4}{\sqrt{16+9}}$$

$$= \frac{-4}{\sqrt{25}}$$

$$= -\frac{4}{5}$$

Linear approximation

$$L(x) = g(x_0) + g'(x_0)(x - x_0)$$

$$= g(-4) + g'(-4)(x+4)$$

$$= 5 - \frac{4}{5}(x+4)$$

$$= 5 - \frac{4x}{5} - \frac{16}{5}$$

$$= 5 - \frac{16}{5} - \frac{4x}{5}$$

$$= \frac{25-16}{5} - \frac{4x}{5}$$

$$= \frac{9}{5} - \frac{4x}{5}$$

$$= \frac{9-4x}{5} = \frac{1}{5}(9-4x)$$

(iii) $h(x) = \frac{x}{x+1}, x_0 = 1$

$$h(x) = \frac{x}{x+1}$$

$$h(x_0) = h(1) = \frac{1}{1+1}$$

$$= \frac{1}{2}$$

$$h'(x) = \frac{(x+1)(x)' - x(x+1)'}{(x+1)^2}$$

$$= \frac{(x+1)(1) - x(1)}{(x+1)^2}$$

$$= \frac{x+1-x}{(x+1)^2}$$

$$= \frac{1}{(x+1)^2}$$

$$h'(x_0) = h'(1) = \frac{1}{(1+1)^2}$$

$$= \frac{1}{(2)^2}$$

$$= \frac{1}{4}$$

Linear approximation

$$L(x) = h(x_0) + h'(x_0)(x - x_0)$$

$$= h(1) + h'(1)(x-1)$$

$$= \frac{1}{2} + \frac{1}{4}(x-1)$$

$$= \frac{1}{2} + \frac{x}{4} - \frac{1}{4}$$

$$= \frac{1}{2} - \frac{1}{4} + \frac{x}{4}$$

$$= \frac{2-1}{4} + \frac{x}{4}$$

$$= \frac{1}{4} + \frac{x}{4}$$

$$= \frac{x+1}{4} = \frac{1}{4}(x+1)$$

....

4. The radius of a circular plate is measured as 12.65 cm instead of the actual length 12.5 cm. find the following in calculating the area of the circular plate:

(i) Absolute error (ii) Relative error

(iii) Percentage error

<u>Solution</u>:

Change in radius $dr = 12.65 - 12.5 = 0.15$

Area of the circular plate $A = \pi r^2$

$$A' = 2\pi r dr$$

$$= 2\pi(12.5)(0.15)$$

$$= \pi(12.5)(0.3)$$

$$= 3.75\pi$$

Actual change in Area $= A(12.65) - A(12.5)$

$$A(12.65) - A(12.5) = \pi(12.65)^2 - \pi(12.5)^2$$

$$= \pi[(12.65)^2 - (12.5)^2]$$

$$= \pi[(12.65) + (12.5)(12.65) - (12.5)]$$

$$= \pi[(25.15)(0.15)]$$

$$= \pi(3.7725)$$

(i) Absolute error

= Actual change in Area – Approximate value

$= \pi(3.7725) - 3.75\pi$

$= 0.0225\pi$

(ii)Relative error $= \frac{Actual\ error - approximate\ error}{Actual\ error}$

$= \frac{\pi(3.7725) - 3.75\pi}{\pi(3.7725)}$

$= \frac{0.0225\pi}{\pi(3.7725)}$

$= \frac{0.0225}{3.7725}$

$= \frac{225}{37725}$

$= 0.00596$

$= 0.006$

(iii) Percentage error = Relative error × 100 %

$= 0.006 \times 100\ \%$

$= 0.6 \times 100\ \%$

....

5. A sphere is made of ice having radius 10 cm. Its radius decreases from 10 cm to 9 8 .cm. Find approximations for the following:

(i) change in the volume

(ii) change in the surface area

Solution:

Change in radius $dr = -0.2$

Volume of the sphere $V = \frac{4}{3}\pi r^3$

$V' = \frac{4}{3} \times 3\pi r^2 dr$

$= 4\pi r^2 dr$

$= 4\pi(10)^2(-0.2)$

$= 4\pi(100)(-0.2)$

$= 400\pi(-0.2)$

$V' = -80\pi$

Volume decreases by $80\pi\ cm^3$

Surface Area of the sphere $S = 4\pi r^2$

$S' = 4 \times 2\pi r dr$

$= 8\pi r dr$

$= 8\pi(10)(-0.2)$

$= 80\pi(-0.2)$

$S' = -16\pi$

Surface Area decreases by $16\pi\ cm^2$

....

6. The time T, taken for a complete oscillation of a single pendulum with length l, is given by the equation $T = 2\pi\sqrt{\frac{l}{g}}$, where g is a constant. Find the approximate percentage error in the calculated value of T corresponding to an error of 2 percent in the value of l.

Solution: $T = 2\pi\sqrt{\frac{l}{g}}$

$T = 2\pi\left(\frac{l}{g}\right)^{\frac{1}{2}}$

Taking log on both sides,

$\log T = \log\left(2\pi\left(\frac{l}{g}\right)^{\frac{1}{2}}\right)$

$= \log(2\pi) + \log\left(\frac{l}{g}\right)^{\frac{1}{2}}$

$= \log(2\pi) + \frac{1}{2}\log\left(\frac{l}{g}\right)$

$= \log(2\pi) + \frac{1}{2}[\log(l) - \log(g)]$

D.w.r.t.l,

$\frac{1}{T}\frac{dT}{dl} = 0 + \frac{1}{2}\left(\frac{1}{l} - 0\right)$

$\frac{dT}{T} = \frac{1}{2}\left(\frac{1}{l}\right)dl$

$\frac{dT}{T} = \frac{1}{2}\left(\frac{dl}{l}\right)$

Percentage error

$\frac{dT}{T} \times 100 = \frac{1}{2}\left(\frac{dl}{l}\right) \times 100$

$= \frac{1}{2}\left(\frac{0.02l}{l}\right) \times 100$

$= \frac{1}{2}(0.02) \times 100$

$= (0.01) \times 100$

$= 1\%$

....

7. Show that the percentage error in the n^{th} root of a number is approximately $\frac{1}{n}$ times the percentage error in the number.

Solution: Let x be the number.

$$\therefore y = (x)^{\frac{1}{n}}$$

Taking log on both sides,

$$\log y = \log(x)^{\frac{1}{n}}$$

$$= \frac{1}{n}\log(x)$$

D.w.r.t.x,

$$\frac{1}{y}\frac{dy}{dx} = \frac{1}{n} \times \frac{1}{x}$$

$$\frac{dy}{y} = \frac{1}{n} \times \frac{dx}{x}$$

$$\frac{dy}{y} \times 100 = \frac{1}{n} \times \frac{dx}{x} \times 100$$

$$= \frac{1}{n} \times (\text{times the percentage of the number})$$

Example 8.5

Let $f, g : (a, b) \to \mathbb{R}$ be differentiable functions.

Show that $d(fg) = fdg + gdf$

Solution:

Let $f, g : (a, b) \to \mathbb{R}$ be differentiable functions.

Then (fg) is also a differentiable function on $\mathbb{R}$

$$\therefore d(fg) = f(dg) + g(df)$$

Example 8.6 Let $g(x) = x^2 + \sin x$.

Calculate the differential dg

Solution:

$$g(x) = x^2 + \sin x$$

$$\therefore \frac{dg}{dx} = 2x + \cos x$$

$$dg = (2x + \cos x)\, dx$$

Example 8.7

If the radius of a sphere, with radius 10 cm, has to decrease by 0.1 cm, approximately how much will its volume decrease?

Solution:

Change in radius $dr = -0.1$

Volume of the sphere $V = \frac{4}{3}\pi r^3$

$$\frac{dV}{dr} = \frac{4}{3} \times 3\pi r^2$$

$$dV = \frac{4}{3} \times 3\pi r^2 dr$$

$$= 4\pi r^2 dr$$

$$= 4\pi(10)^2(-0.1)$$

$$= 4\pi(100)(-0.1)$$

$$= 400\pi(-0.1)$$

$$dV = -40\pi$$

Volume of the sphere decreases by $40\pi\ cm^3$.

EXERCISE 8.2

1. Find differential dy for each of the following functions:

(i) $y = \frac{(1-2x)^3}{3-4x}$

Solution: $y = \frac{(1-2x)^3}{3-4x}$

$$\frac{dy}{dx} = \frac{(3-4x)[(1-2x)^3]' - (1-2x)^3(3-4x)'}{(3-4x)^2}$$

$$= \frac{(3-4x)[3(1-2x)^2(-2)] - (1-2x)^3(-4)}{(3-4x)^2}$$

$$= \frac{(1-2x)^2[(3-4x)(-6) + 4(1-2x)]}{(3-4x)^2}$$

$$= \frac{(1-2x)^2[-18+24x+4-8x]}{(3-4x)^2}$$

$$dy = \frac{(1-2x)^2(16x-14)}{(3-4x)^2}dx$$

(ii) $y = [3 + \sin(2x)]^{2/3}$

Solution: $y = [3 + \sin(2x)]^{2/3}$

$$\frac{dy}{dx} = \frac{2}{3}[3 + \sin(2x)]^{\frac{2}{3}-1}(2\cos 2x)$$

$$= \frac{4}{3}[3 + \sin(2x)]^{-\frac{1}{3}}(\cos 2x)$$

$$dy = \frac{4}{3}[3 + \sin(2x)]^{-\frac{1}{3}}(\cos 2x)\, dx$$

(iii) $y = e^{x^2-5x+7}\cos(x^2-1)$

Solution:

$$y = e^{x^2-5x+7}\cos(x^2-1)$$

$$\frac{dy}{dx} = e^{x^2-5x+7}\sin(x^2-1)\,(2x)$$

$$+ \cos(x^2-1)\left(e^{x^2-5x+7}\right)(2x-5)$$

$$\frac{dy}{dx} = e^{x^2-5x+7}[2x\sin(x^2-1) + \cos(x^2-1)(2x-5)]$$

$$dy = e^{x^2-5x+7}[2x\sin(x^2-1) + \cos(x^2-1)(2x-5)]dx$$

2. Find df for $f(x) = x^2 + 3x$ and evaluate it for (i) $x = 2$ and $dx = 0.1$ (ii) $x = 3$ and $dx = 0.02$

Solution: (i) $x = 2$ and $dx = 0.1$

$f(x) = x^2 + 3x$

$\frac{df}{dx} = 2x + 3$

$df = (2x + 3)dx$

$= [2(2) + 3](0.1)$

$= (4 + 3)(0.1)$

$= (7)(0.1)$

$df = 0.7$

(ii) $x = 3$ and $dx = 0.02$

$f(x) = x^2 + 3x$

$\frac{df}{dx} = 2x + 3$

$df = (2x + 3)dx$

$= [2(3) + 3](0.02)$

$= (6 + 3)(0.02)$

$= (9)(0.02)$

$df = 0.18$

....

3. Find Δf and df for the function f for the indicated values of $x, \Delta x$ and compare

(i) $(x) = x^3 - 2x^2$; $x = 2, \Delta x = dx = 0.5$

(ii) $(x) = x^2 + 2x + 3$; $x = -0.5, \Delta x = dx = 0.1$

Solution:

(i) $(x) = x^3 - 2x^2$; $x = 2, \Delta x = dx = 0.5$

$f(x) = x^3 - 2x^2$

$\frac{df}{dx} = 3x^2 - 4x$

$df = (3x^2 - 4x)dx$

$= [3(2)^2 - 4(2)](0.5)$

$= (12 - 8)(0.5)$

$= (4)(0.5)$

$df = 2$

We know, $\Delta f = f(x + \Delta x) - f(x)$

Given $x = 2, \Delta x = 0.5$

$f(x) = x^3 - 2x^2$

$f(x + \Delta x) = f(2 + 0.5) = f(2.5)$

$f(2.5) = (2.5)^3 - 2(2.5)^2$

$= 15.625 - 2(6.25)$

$= 15.625 - 12.50$

$= 3.125$

$f(x) = x^3 - 2x^2$

$f(2) = (2)^3 - 2(2)^2$

$= 8 - 2(4)$

$= 8 - 8$

$= 0$

$\therefore \Delta f = f(x + \Delta x) - f(x)$

$= 3.125 - 0$

$= 3.125$

(ii) $(x) = x^2 + 2x + 3$; $x = -0.5, \Delta x = dx = 0.1$

$f(x) = x^2 + 2x + 3$

$\frac{df}{dx} = 2x + 2$

$df = (2x + 2)dx$

$= [2(-0.5) + 2](0.1)$

$= (-1 + 2)(0.1)$

$= (1)(0.1)$

$df = 0.1$

We know, $\Delta f = f(x + \Delta x) - f(x)$

Given $x = -0.5, \Delta x = 0.1$

$f(x) = x^2 + 2x + 3$

$f(x + \Delta x) = f(-0.5 + 0.1) = f(-0.4)$

$f(-0.4) = (-0.4)^2 + 2(-0.4) + 3$

$= 0.16 - 0.8 + 3$

$= 3.16 - 0.8$

$= 2.36$

$f(x) = x^2 + 2x + 3$

$f(-0.5) = (-0.5)^2 + 2(-0.5) + 3$

$= 0.25 - 1 + 3$

$= 0.25 + 2$

$= 2.25$

$\therefore \Delta f = f(x + \Delta x) - f(x)$

$= 2.36 - 2.25$

$= 0.11$

---

4. Assuming $\log_{10} e = 0.4343$, find an approximate value of $\log_{10} 1003$.

Solution:

Let $y = \log x$, $x = 1000$, $dx = 3\log_{10} e$

$\frac{dy}{dx} = \frac{1}{x}$

$dy = \frac{1}{x}\, dx$

$= \frac{1}{1000}\,(3\log_{10} e)$

$= \frac{1}{1000}\,(3 \times 0.4343)$

$= \frac{1}{1000}\,(1.3029)$

$= 0.0013029$

$y = \log x$

$= \log 1000$

$= \log(10)^3$

$= 3\log(10)$

$= 3 \times 1 = 3$

$\log_{10} 1003 = y + dy$

$= 3 + 0.0013029$

$= 3.0013029$

---

5. The trunk of a tree has diameter 30 cm. During the following year, the circumference grew 6 cm. (i) Approximately, how much did the tree's diameter grow?

(ii) What is the percentage increase in area of the tree's cross-section?

Solution:

(i) Given diameter $d = 30\ cm$

Circumference $P = 2\pi r$

$\frac{dP}{dr} = 2\pi$

$dP = 2\pi dr$

Given increase in perimeter $dP = 6$

$\therefore 6 = 2\pi dr$

$dr = \frac{6}{2\pi}$

$dr = \frac{3}{\pi}$

Change in radius $dr = \frac{3}{\pi}$

$\therefore$ Change in diameter $2dr = \frac{6}{\pi}$

(ii) Area $A = \pi r^2$

$\frac{dA}{dr} = \pi(2r)$

$dA = 2\pi r dr$

$\frac{dA}{A} = \frac{2\pi r dr}{\pi r^2}$

$= \frac{2dr}{r}$

$= \frac{2\left(\frac{3}{\pi}\right)}{(15)}$

$= \frac{6}{\pi(15)}$

$\frac{dA}{A} = \frac{2}{5\pi}$

So, % error $= \frac{dA}{A} \times 100$

$= \frac{2}{5\pi} \times 100$

$= \frac{40}{\pi}$

---

6. An egg of a particular bird is very nearly spherical. If the radius to the inside of the shell is 5 mm and radius to the outside of the shell is 5.3 mm, find the volume of the shell approximately.

Solution:

Volume of the sphere $V = \frac{4}{3}\pi r^3$

$\frac{dV}{dr} = \frac{4}{3} \times 3\pi r^2$

$dV = \frac{4}{3} \times 3\pi r^2 dr$

Here, $r = 5cm$, and $dr = 0.3$

$dV = \frac{4}{3} \times 3\pi(5)^2(0.3)$

$= 4 \times \pi(25)(0.3)$

$= 100\,\pi(0.3)$

$dV = 30\,\pi$

7. Assume that the cross section of the artery of human is circular. A drug is given to a patient to dilate his arteries. If the radius of an artery is increased from 2 mm to 2.1 mm, how much is cross-sectional area increased approximately?

Solution:

Area of the circle $A = \pi r^2$

$\frac{dA}{dr} = \pi(2r)$

$dA = 2\pi r dr$

Here, $r = 2mm$, and $dr = 0.1$

$dA = 2\pi(2)(0.1)$

$= 4\pi(0.1)$

So, $dA = 0.4\,\pi$

8. In a newly developed city, it is estimated that the voting population (in thousands) will increase according to $V(t) = 30 + 12t^2 - t^3, 0 \le t \le 8$ where t is the time in years. Find the approximate change in voters for the time change from 4 to $4\frac{1}{6}$ year.

Solution: $V(t) = 30 + 12t^2 - t^3$

$\frac{dV}{dt} = 12(2t) - 3t^2$

$dV = (24t - 3t^2)dt$

Here, $t = 4$, and $dt = \frac{1}{6}$

$\therefore dV = [24(t) - 3(t)^2]dt$

$= [24(4) - 3(4)^2]\left(\frac{1}{6}\right)$

$= [96 - 3(16)]\left(\frac{1}{6}\right)$

$= (96 - 48)\left(\frac{1}{6}\right)$

$= (48)\left(\frac{1}{6}\right)$

$= 8$ (in thousands)

9. The relation between the number of words y a person learns in x hours is given by $y = 52\sqrt{x}, 0 \le x \le 9$. What is the approximate number of words learned when x changes from
(i) 1 to 1.1 hour? (ii) 4 to 4.1 hour?

Solution:

$y = 52\sqrt{x}$

$\frac{dy}{dx} = 52\frac{1}{2\sqrt{x}}$

$= \frac{26}{\sqrt{x}}$

$dy = \frac{26}{\sqrt{x}}\,dx$

(i) 1 to 1.1 hour

When $x = 1$, and $dx = 0.1$

$dy = \frac{26}{\sqrt{1}}\,(0.1)$

$= 26\,(0.1)$

$dy = 2.6$

Approximately 3 words.

(ii) 4 to 4.1 hour

When $x = 4$, and $dx = 0.1$

$dy = \frac{26}{\sqrt{4}}\,(0.1)$

$= \frac{26}{2}\,(0.1)$

$= 13\,(0.1)$

$dy = 1.3$

Approximately 1 word.

10. A circular plate expands uniformly under the influence of heat. If it's radius increases from 10.5 cm to 10.75 cm, then find an approximate change in the area and the approximate percentage change in the area.

Solution: Area $A = \pi r^2$

$\frac{dA}{dr} = \pi(2r)$

$dA = 2\pi r dr$

(i) $r = 10.5cm$, and $dr = 0.25$

$$dA = 2\pi(10.5)(0.25)$$
$$= 0.5\pi(10.5)$$
$$dA = 5.25\pi$$
$$\frac{dA}{A} = \frac{2\pi r dr}{\pi r^2}$$
$$= \frac{2dr}{r}$$
$$= \frac{2(0.25)}{10.5}$$
$$= \frac{0.5}{10.5}$$
$$= 0.0476$$

So, % error $= \frac{dA}{A} \times 100$

$$= 0.0476 \times 100$$
$$= 4.76\ \%$$

....

11. A coat of paint of thickness 0.2 cm is applied to the faces of a cube whose edge is 10 cm. Use the differentials to find approximately how many cubic centimeters of paint is used to paint this cube. Also calculate the exact amount of paint used to paint this cube.

Solution:

Volume of the cube $V = a^3$

$$\frac{dV}{da} = 3a^2$$
$$dV = 3a^2\, da$$

$a = 10cm$, and $da = 0.2$

$$dV = 3(10)^2\,(0.2)$$
$$= 0.6\,(100)$$
$$dV = 60$$

Exact amount of paint used $= (10.2)^3 - (10)^3$

$$= 1061.208 - 1000$$
$$= 61.208$$

....

EXERCISE 8.3

1. Evaluate $\lim_{(x,y)\to(1,2)} g(x,y)$, if the limit exists, where $g(x,y) = \frac{3x^2 - xy}{x^2 + y^2 + 3}$

Solution:

$$\lim_{(x,y)\to(1,2)} g(x,y) = \lim_{(x,y)\to(1,2)} \frac{3x^2 - xy}{x^2 + y^2 + 3}$$
$$= \frac{3(1)^2 - (1)(2)}{(1)^2 + (2)^2 + 3}$$
$$= \frac{3 - 2}{1 + 4 + 3}$$
$$= \frac{1}{8}$$

....

2. Evaluate $\lim_{(x,y)\to(0,0)} \cos\left(\frac{x^3 + y^2}{x + y + 2}\right)$. If the limit exists.

Solution:

$$\lim_{(x,y)\to(0,0)} \cos\left(\frac{x^3 + y^2}{x + y + 2}\right)$$
$$= \cos\left(\frac{0 + 0}{0 + 0 + 2}\right)$$
$$= \cos\left(\frac{0}{2}\right)$$
$$= \cos(0)$$
$$= 1$$

....

3. Let $f(x,y) = \frac{y^2 - xy}{\sqrt{x} - \sqrt{y}}$ for $(x,y) \neq (0,0)$. Show that $\lim_{(x,y)\to(0,0)} f(x,y) = 0$.

Solution:

$$\lim_{(x,y)\to(0,0)} f(x,y)$$
$$= \lim_{(x,y)\to(0,0)} \frac{y^2 - xy}{\sqrt{x} - \sqrt{y}} \times \frac{\sqrt{x} + \sqrt{y}}{\sqrt{x} + \sqrt{y}}$$
$$= \lim_{(x,y)\to(0,0)} \frac{y(y - x)(\sqrt{x} + \sqrt{y})}{(x - y)}$$
$$= \lim_{(x,y)\to(0,0)} \frac{-y(x - y)(\sqrt{x} + \sqrt{y})}{(x - y)}$$
$$= \lim_{(x,y)\to(0,0)} -y\left(\sqrt{x} + \sqrt{y}\right)$$
$$= (0 + 0)$$
$$= 0 \text{ Hence proved.}$$

....

4. Evaluate $\lim_{(x,y)\to(0,0)} \cos\left(\frac{e^x \sin y}{y}\right)$, if the limit exists.

Solution:

$$\lim_{(x,y)\to(0,0)} \cos\left(\frac{e^x \sin y}{y}\right)$$

$$= \cos\left(\lim_{(x,y)\to(0,0)} \frac{e^x \sin y}{y}\right)$$

$$= \cos\left(\lim_{(x,y)\to(0,0)} e^x \lim_{(x,y)\to(0,0)} \frac{\sin y}{y}\right)$$

$$= \cos(e^0 \times 1)$$

$$= \cos(1 \times 1)$$

$$= \cos 1$$

....

5. Let $g(x,y) = \frac{x^2 y}{x^4 + y^2}$ for $(x,y) \neq (0,0)$ and $f(0,0) = 0$.

(i) Show that $\lim_{(x,y)\to(0,0)} g(x,y) = 0$ along every line $y = mx, m \in \mathbb{R}$.

Solution:

$$\lim_{(x,y)\to(0,0)} g(x,y) = \lim_{(x,y)\to(0,0)} \frac{x^2 y}{x^4 + y^2}$$

When $y = mx$

$$= \lim_{(x,y)\to(0,0)} \frac{x^2 mx}{x^4 + m^2 x^2}$$

$$= \lim_{(x,y)\to(0,0)} \frac{x^2(mx)}{x^2(x^2+m^2)}$$

$$= \lim_{(x,y)\to(0,0)} \frac{(mx)}{(x^2+m^2)}$$

$$= \frac{0}{0+m^2}$$

$$= 0$$

(ii) Show that $\lim_{(x,y)\to(0,0)} g(x,y) = \frac{k}{1+k^2}$, along every parabola $y = kx^2, k \in \mathbb{R} \setminus \{0\}$

Solution:

$$\lim_{(x,y)\to(0,0)} g(x,y) = \lim_{(x,y)\to(0,0)} \frac{x^2 y}{x^4 + y^2}$$

When $y = kx^2$

$$= \lim_{(x,y)\to(0,0)} \frac{x^2 kx^2}{x^4 + k^2 x^4}$$

$$= \lim_{(x,y)\to(0,0)} \frac{x^4(k)}{x^4(1+k^2)}$$

$$= \frac{k}{1+k^2} \text{ hence proved.}$$

6. Show that $f(x,y) = \frac{x^2 - y^2}{y^2 + 1}$ is continuous at every $(x,y) \in \mathbb{R}^2$.

Solution:

(i) Let $(a,b) \in \mathbb{R}^2$ be an arbitrary point.

$f(a,b) = \frac{a^2 - b^2}{b^2 + 1}$ is defined for $\forall (a,b) \in \mathbb{R}^2$

(ii) $\lim_{(x,y)\to(a,b)} f(x,y)$

$$= \lim_{(x,y)\to(a,b)} \frac{x^2 - y^2}{y^2 + 1}$$

$$= \frac{a^2 - b^2}{b^2 + 1}$$

Limit exists at $(a,b) \in \mathbb{R}^2$

(iii) $\lim_{(x,y)\to(a,b)} f(x,y) = L$

$$= f(a,b)$$

$$= \frac{a^2 - b^2}{b^2 + 1}$$

Hence f is continuous at the point on $\mathbb{R}^2$

....

7. Let $g(x,y) = \frac{e^y \sin x}{x}$, for $x \neq 0$ and $g(0,0) = 1$. Show that g is continuous at (0, 0)

Solution:

$$|g(x,y) - g(0,0)| = \left|\frac{e^y \sin x}{x} - 1\right|$$

$$= \left|\frac{e^y \sin x - x}{x}\right|$$

(i) $g(0,0) = 1$

Hence g is defined.

(ii) $\lim_{(x,y)\to(0,0)} g(x,y)$

$$= \lim_{(x,y)\to(0,0)} \frac{e^y \sin x}{x}$$

$$= \lim_{(x,y)\to(0,0)} e^y \lim_{(x,y)\to(0,0)} \frac{\sin x}{x}$$

$$= (e^0 \times 1)$$

$$= (1 \times 1)$$

$$= 1$$

Hence limit exists at (0,0)

(iii) $\lim_{(x,y)\to(0,0)} g(x,y) = 1 = g(0,0)$

Hence g is continuous at $(0,0)$

Example 8.11

Let $f(x,y) = 0$ if $xy \neq 0$ and $f(x,y) = 1$ if $xy = 0$.

(i) Calculate: $\frac{\partial f}{\partial x}(0,0), \frac{\partial f}{\partial y}(0,0)$

(ii) Show that f is not continuous at $(0,0)$.

Solution :

Note that the function f takes value 1 on the x, y − axes and 0 everywhere else on $\mathbb{R}^2$.

So let us calculate

$$\frac{\partial f}{\partial x}(0,0) = \lim_{h\to 0}\frac{f(0+h,0)-f(0,0)}{h} = \lim_{h\to 0}\frac{1-1}{h} = 0$$

$$\frac{\partial f}{\partial y}(0,0) = \lim_{k\to 0}\frac{f(0+h,0)-f(0,0)}{h} = \lim_{k\to 0}\frac{1-1}{h} = 0$$

This completes (i)

Now for (ii) let us calculate the limit of f as $(x,y)\to(0,0)$ along the line $y = x$. Then $\lim_{(x,y)\to(0,0)} f(x,y) = 0$, because the line $y = x$ when $x \neq 0$, $f(x,y) = 0$, but $f(0,0) = 1 \neq 0$, hence f is not continuous at $(0,0)$.

Example 8.12

Let $F(x,y) = x^3y + y^2x + 7$ for all $(x,y) \in \mathbb{R}^2$. Calculate $\frac{\partial F}{\partial x}(-1,3)$ and $\frac{\partial F}{\partial y}(-2,1)$

Solution: $F(x,y) = x^3y + y^2x + 7$

$$\frac{\partial F}{\partial x} = 3x^2y + y^2$$

$$\text{So, } \frac{\partial F}{\partial x}(-1,3) = 3(-1)^2(3) + (3)^2$$

$$= 3(1)(3) + 9$$

$$= 9 + 9 = 18$$

$$\frac{\partial F}{\partial y} = x^3y + 2yx$$

$$\text{So, } \frac{\partial F}{\partial y}(-2,1) = (-2)^3(1) + 2(1)(-2)$$

$$= -8(1) - 4$$

$$= -8 - 4 = -12$$

Example 8.13

Let $f(x,y) = \sin(xy^2) + e^{x^3+5y}$ for all $(x,y) \in \mathbb{R}^2$. Calculate $\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}, \frac{\partial^2 f}{\partial y\partial x}, \frac{\partial^2 f}{\partial x\partial y}$

Solution: $f(x,y) = \sin(xy^2) + e^{x^3+5y}$

$$\frac{\partial f}{\partial x} = \cos(xy^2)(y^2) + e^{x^3+5y}(3x^2)$$

$$\frac{\partial f}{\partial y} = \cos(xy^2)(x2y) + e^{x^3+5y}(5)$$

$$\frac{\partial^2 f}{\partial y\partial x} = \cos(xy^2)(2y) + (y^2)[-\sin(xy^2)](2xy) + \left(e^{x^3+5y}\right)5(3x^2) + 0$$

$$= 2y\cos(xy^2) - 2xy^3[\sin(xy^2)] + 15x^2\left(e^{x^3+5y}\right)$$

$$\frac{\partial^2 f}{\partial x\partial y} = \cos(xy^2)(2y) + (x2y)[-\sin(xy^2)](y^2) + 5e^{x^3+5y}(3x^2)$$

$$= (2y)\cos(xy^2) - 2xy^3\sin(xy^2) + 15x^2e^{x^3+5y}$$

Example 8.14

Let $W(x,y) = xy + \frac{e^y}{y^2+1}$ for all $(x,y) \in \mathbb{R}^2$.

Calculate $\frac{\partial^2 W}{\partial y\partial x}, \frac{\partial^2 W}{\partial x\partial y}$

Solution: $W(x,y) = xy + \frac{e^y}{y^2+1}$

$$\frac{\partial W}{\partial x} = y$$

$$\frac{\partial^2 W}{\partial y\partial x} = 1 \;\ldots\ldots\ldots (i)$$

$$W(x,y) = xy + \frac{e^y}{y^2+1}$$

$$\frac{\partial W}{\partial y} = x + \frac{(y^2+1)e^y - e^y(2y)}{(y^2+1)^2}$$

$$\frac{\partial^2 W}{\partial x\partial y} = 1 \;\ldots\ldots\ldots (ii)$$

Example 8.15

Let $u(x,y) = e^{-2y}\cos(2x)$ for all $(x,y) \in \mathbb{R}^2$.

Prove that u is a harmonic function in $\mathbb{R}^2$.

Solution:

$$u(x,y) = e^{-2y}\cos(2x)$$

$$u_x = e^{-2y}[-\sin(2x)2]$$

$$= e^{-2y}[-2\sin(2x)]$$
$$u_{xx} = e^{-2y}[-2\cos(2x)\,2]$$
$$= e^{-2y}[-4\cos(2x)]$$
$$u_{xx} = -4\cos(2x)\,e^{-2y}$$
$$u_y = \cos(2x)\,e^{-2y}(-2)$$
$$= -2\cos(2x)\,e^{-2y}$$
$$u_{yy} = -2\cos(2x)\,e^{-2y}(-2)$$
$$u_{yy} = 4\cos(2x)\,e^{-2y}$$

Thus,

$$u_{xx} + u_{yy} = -4\cos(2x)\,e^{-2y} + 4\cos(2x)\,e^{-2y}$$
$$= 0$$

Since, u satisfies the Laplace's equation, it is a harmonic function.

....

EXERCISE 8.4

1. Find the partial derivatives of the following functions at the indicated points.

(i) $f(x,y) = 3x^2 - 2xy + y^2 + 5x + 2, (2,-5)$

(ii) $g(x,y) = 3x^2 + y^2 + 5x + 2, (1,-2)$

(iii) $h(x,y,z) = x\sin(xy) + z^2x, (2, \frac{\pi}{4}, 1)$

(iv) $G(x,y) = e^{x+3y}\log(x^2+y^2), \ (-1,1)$

Solution:

(i) $f(x,y) = 3x^2 - 2xy + y^2 + 5x + 2, (2,-5)$

$$f(x,y) = 3x^2 - 2xy + y^2 + 5x + 2$$
$$\frac{\partial f}{\partial x} = 6x - 2y + 5$$

At $(2,-5)$

$$\frac{\partial f}{\partial x} = 6(2) - 2(-5) + 5$$
$$= 12 + 10 + 5$$
$$= 27$$
$$f(x,y) = 3x^2 - 2xy + y^2 + 5x + 2$$
$$\frac{\partial f}{\partial y} = -2x + 2y$$

At $(2,-5)$

$$\frac{\partial f}{\partial y} = -2(2) + 2(-5)$$
$$= -4 - 10 = -14$$

(ii) $g(x,y) = 3x^2 + y^2 + 5x + 2, (1,-2)$

$$g(x,y) = 3x^2 + y^2 + 5x + 2$$
$$\frac{\partial g}{\partial x} = 6x + 5$$

At $(1,-2)$

$$\frac{\partial g}{\partial x} = 6(1) + 5$$
$$= 6 + 5$$
$$= 11$$
$$g(x,y) = 3x^2 + y^2 + 5x + 2$$
$$\frac{\partial g}{\partial y} = 2y$$

At $(1,-2)$

$$\frac{\partial g}{\partial y} = 2(-2)$$
$$= -4$$

(iii) $h(x,y,z) = x\sin(xy) + z^2x, (2, \frac{\pi}{4}, 1)$

$$h(x,y,z) = x\sin(xy) + z^2x$$
$$\frac{\partial h}{\partial x} = x[\cos(xy)\,y] + \sin(xy) + z^2$$

At $(2, \frac{\pi}{4}, 1)$

$$\frac{\partial h}{\partial x} = (2)\left[\cos\left(\frac{\pi}{2}\right)\frac{\pi}{4}\right] + \sin\left(\frac{\pi}{2}\right) + (1)^2$$
$$= (2)\left[(0)\frac{\pi}{4}\right] + 1 + 1$$
$$= 2$$
$$h(x,y,z) = x\sin(xy) + z^2x$$
$$\frac{\partial h}{\partial y} = x[\cos(xy)\,x] + \sin(xy)(0) + 0$$

At $(2, \frac{\pi}{4}, 1)$

$$\frac{\partial h}{\partial y} = (2)\left[\cos\left(\frac{\pi}{2}\right)2\right] + \sin\left(\frac{\pi}{2}\right)(0) + 0$$
$$= (2)[(0)\,2] + 1(0) + 0$$
$$= 4(0) + 0$$
$$= 0$$
$$h(x,y,z) = x\sin(xy) + z^2x$$
$$\frac{\partial h}{\partial z} = 2zx$$

At $(2, \frac{\pi}{4}, 1)$

$$\frac{\partial h}{\partial z} = 2(1)(2) = 4$$

(iv) $G(x,y) = e^{x+3y}\log(x^2+y^2), \ (-1,1)$

$$G(x,y) = e^{x+3y}\log(x^2+y^2)$$

$$\frac{\partial G}{\partial x} = e^{x+3y}\frac{1}{(x^2+y^2)}(2x) + \log(x^2+y^2)\,e^{x+3y}(1)$$

$$= e^{x+3y}\frac{2x}{(x^2+y^2)} + \log(x^2+y^2)\,e^{x+3y}$$

$$= e^{x+3y}\left[\frac{2x}{(x^2+y^2)} + \log(x^2+y^2)\right]$$

At $(-1,1)$

$$\frac{\partial G}{\partial x} = e^{-1+3}\left[\frac{-2}{(1+1)} + \log(1+1)\right]$$

$$= e^2\left[\frac{-2}{(2)} + \log(2)\right]$$

$$= e^2[\log(2) - 1]$$

$$G(x,y) = e^{x+3y}\log(x^2+y^2)$$

$$\frac{\partial G}{\partial y} = e^{x+3y}\frac{1}{(x^2+y^2)}(2y) + \log(x^2+y^2)\,e^{x+3y}(3)$$

$$= e^{x+3y}\frac{2y}{(x^2+y^2)} + 3\log(x^2+y^2)\,e^{x+3y}$$

$$= e^{x+3y}\left[\frac{2y}{(x^2+y^2)} + 3\log(x^2+y^2)\right]$$

At $(-1,1)$

$$\frac{\partial G}{\partial y} = e^{-1+3}\left[\frac{2}{(1+1)} + 3\log(1+1)\right]$$

$$= e^2\left[\frac{2}{(2)} + 3\log(2)\right]$$

$$= e^2[1+\log 2^3]$$

$$= e^2[1+\log 8]$$

....

2. For each of the following functions find the f_x, f_y, and show that $f_{xy} = f_{yx}$

(i) $f(x,y) = \frac{3x}{y+\sin x}$

(ii) $f(x,y) = tan^{-1}\left(\frac{x}{y}\right)$

(iii) $f(x,y) = \cos(x^2-3xy)$

<u>Solution:</u>

(i) $f(x,y) = \frac{3x}{y+\sin x}$

$$\frac{\partial f}{\partial x} = \frac{(y+\sin x)(3) - 3x(\cos x)}{(y+\sin x)^2}$$

$$= \frac{3y + 3\sin x - 3x\cos x}{(y+\sin x)^2}$$

$$\frac{\partial^2 f}{\partial y\partial x} = \frac{(y+\sin x)^2(3) - (3y+3\sin x - 3x\cos x)2(y+\sin x)}{(y+\sin x)^4}$$

$$= \frac{(y+\sin x)[(y+\sin x)(3) - (3y+3\sin x - 3x\cos x)2]}{(y+\sin x)^4}$$

$$= \frac{(y+\sin x)[(3y+3\sin x) - (6y+6\sin x - 6x\cos x)]}{(y+\sin x)^4}$$

$$= \frac{(y+\sin x)[3y+3\sin x - 6y - 6\sin x + 6x\cos x]}{(y+\sin x)^4}$$

$$= \frac{(-3\sin x - 3y + 6x\cos x)}{(y+\sin x)^3}$$

$$= \frac{-3(\sin x + y - 2x\cos x)}{(y+\sin x)^3} \quad \ldots\ldots\ (1)$$

$$\frac{\partial f}{\partial y} = \frac{(y+\sin x)(0) - 3x(1+0)}{(y+\sin x)^2}$$

$$= \frac{-3x}{(y+\sin x)^2}$$

$$\frac{\partial^2 f}{\partial x\partial y} = \frac{(y+\sin x)^2(-3) - (-3x)2(y+\sin x)(\cos x)}{(y+\sin x)^4}$$

$$= \frac{(y+\sin x)[(y+\sin x)(-3) + 6x(\cos x)]}{(y+\sin x)^4}$$

$$= \frac{[(y+\sin x)(-3) + 6x(\cos x)]}{(y+\sin x)^3}$$

$$= \frac{(-3y - 3\sin x + 6x\cos x)}{(y+\sin x)^3}$$

$$= \frac{-3(y+\sin x - 2x\cos x)}{(y+\sin x)^3} \quad \ldots\ldots\ (2)$$

From (1) and (2)

$f_{xy} = f_{yx}$ is proved.

(ii) $f(x,y) = tan^{-1}\left(\frac{x}{y}\right)$

$$\frac{\partial f}{\partial x} = \frac{1}{1+\left(\frac{x}{y}\right)^2}\left(\frac{1}{y}\right)$$

$$= \frac{1}{1+\left(\frac{x^2}{y^2}\right)}\left(\frac{1}{y}\right)$$

$$= \frac{1}{\frac{x^2+y^2}{y^2}}\left(\frac{1}{y}\right)$$

$$= \frac{y^2}{x^2+y^2}\left(\frac{1}{y}\right)$$

$$= \frac{y}{x^2+y^2}$$

$$\frac{\partial^2 f}{\partial y\partial x} = \frac{(x^2+y^2)(1) - y(2y)}{(x^2+y^2)^2}$$

$$= \frac{x^2+y^2-2y^2}{(x^2+y^2)^2}$$

$$\frac{\partial^2 f}{\partial y\partial x} = \frac{x^2-y^2}{(x^2+y^2)^2} \quad \ldots\ldots\ (1)$$

$$\frac{\partial f}{\partial y} = \frac{1}{1+\left(\frac{x}{y}\right)^2}\left(\frac{-x}{y^2}\right)$$

$$= \frac{1}{1+\left(\frac{x^2}{y^2}\right)}\left(\frac{-x}{y^2}\right)$$

$= \frac{1}{\frac{x^2+y^2}{y^2}}\left(\frac{-x}{y^2}\right)$

$= \frac{y^2}{x^2+y^2}\left(\frac{-x}{y^2}\right)$

$= \frac{-x}{x^2+y^2}$

$\frac{\partial^2 f}{\partial x \partial y} = \frac{(x^2+y^2)(-1)-(-x)(2x)}{(x^2+y^2)^2}$

$= \frac{-x^2-y^2+2x^2}{(x^2+y^2)^2}$

$\frac{\partial^2 f}{\partial x \partial y} = \frac{x^2-y^2}{(x^2+y^2)^2}$ .... (2)

From (1) and (2)

$f_{xy} = f_{yx}$ is proved.

(iii) $f(x,y) = \cos(x^2-3xy)$

$\frac{\partial f}{\partial x} = -\sin(x^2-3xy)(2x-3y)$

$= -(2x-3y)\sin(x^2-3xy)$

$\frac{\partial^2 f}{\partial y \partial x} = -(2x-3y)\cos(x^2-3xy)(-3x)$

$-\sin(x^2-3xy)(-3)$

$= 3x(2x-3y)\cos(x^2-3xy) + 3\sin(x^2-3xy)$ ....(1)

$\frac{\partial f}{\partial y} = -\sin(x^2-3xy)(-3x)$

$= 3x\sin(x^2-3xy)$

$\frac{\partial^2 f}{\partial x \partial y} = 3x\cos(x^2-3xy)(2x-3y)$

$+\sin(x^2-3xy)(3)$

$= 3x(2x-3y)\cos(x^2-3xy) + 3\sin(x^2-3xy)$ ....(2)

From (1) and (2)

$f_{xy} = f_{yx}$ is proved.

....

3. If $U(x,y,z) = \frac{x^2+y^2}{xy} + 3z^2y$,

find $\frac{\partial U}{\partial x}, \frac{\partial U}{\partial y}$ and $\frac{\partial U}{\partial z}$

Solution:

$U(x,y,z) = \frac{x^2+y^2}{xy} + 3z^2y$

$\frac{\partial U}{\partial x} = \frac{(xy)(2x+0)-(x^2+y^2)(y)}{(xy)^2} + 0$

$= \frac{2x^2y - x^2y - y^3}{(xy)^2}$

$= \frac{x^2y - y^3}{(xy)^2}$

$= \frac{y(x^2-y^2)}{x^2y^2}$

$= \frac{(x^2-y^2)}{x^2y}$

$U(x,y,z) = \frac{x^2+y^2}{xy} + 3z^2y$

$\frac{\partial U}{\partial y} = \frac{(xy)(0+2y)-(x^2+y^2)(x)}{(xy)^2} + 3z^2$

$= \frac{2xy^2 - x^3 - xy^2}{(xy)^2} + 3z^2$

$= \frac{xy^2 - x^3}{(xy)^2} + 3z^2$

$= \frac{x(y^2-x^2)}{x^2y^2} + 3z^2$

$= \frac{(y^2-x^2)}{xy^2} + 3z^2$

$U(x,y,z) = \frac{x^2+y^2}{xy} + 3z^2y$

$\frac{\partial U}{\partial z} = 0 + 3y(2z)$

$= 6yz$

....

4. If $U(x,y,z) = \log(x^3+y^3+z^3)$,

find $\frac{\partial U}{\partial x} + \frac{\partial U}{\partial y} + \frac{\partial U}{\partial z}$

Solution:

$U(x,y,z) = \log(x^3+y^3+z^3)$

$\frac{\partial U}{\partial x} = \frac{1}{(x^3+y^3+z^3)} 3x^2$

$= \frac{3x^2}{(x^3+y^3+z^3)}$

$\frac{\partial U}{\partial y} = \frac{1}{(x^3+y^3+z^3)} 3y^2$

$= \frac{3y^2}{(x^3+y^3+z^3)}$

$\frac{\partial U}{\partial z} = \frac{1}{(x^3+y^3+z^3)} 3z^2$

$= \frac{3z^2}{(x^3+y^3+z^3)}$

$\frac{\partial U}{\partial x} + \frac{\partial U}{\partial y} + \frac{\partial U}{\partial z} = \frac{3x^2+3y^2+3z^2}{(x^3+y^3+z^3)}$

$= \frac{3(x^2+y^2+z^2)}{(x^3+y^3+z^3)}$

....

5. For each of the following functions find the g_{xy}, g_{xx}, g_{yy} and g_{yx}.

(i) $g(x,y) = xe^y + 3x^2y$

(ii) $g(x,y) = \log(5x+3y)$

(iii) $g(x,y) = x^2 + 3xy - 7y + \cos(5x)$

Solution:

(i) $g(x,y) = xe^y + 3x^2y$

$$g_x = e^y + 6xy$$

$$g_{xx} = 6y$$

$$g_{yx} = e^y + 6x$$

$$g_y = xe^y + 3x^2$$

$$g_{yy} = xe^y$$

$$g_{xy} = e^y + 6x$$

....

6. Let $w(x,y,z) = \frac{1}{\sqrt{x^2+y^2+z^2}}$.

Show that $\frac{\partial^2 w}{\partial x^2} + \frac{\partial^2 w}{\partial y^2} + \frac{\partial^2 w}{\partial z^2} = 0$

Solution:

$$w(x,y,z) = \frac{1}{\sqrt{x^2+y^2+z^2}}$$

$$= \frac{1}{(x^2+y^2+z^2)^{\frac{1}{2}}}$$

$$= (x^2+y^2+z^2)^{-\frac{1}{2}}$$

$$\frac{\partial w}{\partial x} = -\frac{1}{2}(x^2+y^2+z^2)^{-\frac{3}{2}}(2x)$$

$$= -x(x^2+y^2+z^2)^{-\frac{3}{2}}$$

$$\frac{\partial^2 w}{\partial x^2} = -x\left(-\frac{3}{2}\right)(x^2+y^2+z^2)^{-\frac{5}{2}}(2x)$$

$$+(x^2+y^2+z^2)^{-\frac{3}{2}}(-1)$$

$$= (3x^2)(x^2+y^2+z^2)^{-\frac{3}{2}-1} - (x^2+y^2+z^2)^{-\frac{3}{2}}$$

$$= (x^2+y^2+z^2)^{-\frac{3}{2}}[(3x^2)(x^2+y^2+z^2)^{-1} - 1]$$

$$= (x^2+y^2+z^2)^{-\frac{3}{2}}\left[\frac{3x^2}{(x^2+y^2+z^2)} - 1\right]$$

$$= (x^2+y^2+z^2)^{-\frac{3}{2}}\left[\frac{3x^2 - x^2 - y^2 - z^2}{(x^2+y^2+z^2)}\right]$$

$$\frac{\partial^2 w}{\partial x^2} = (x^2+y^2+z^2)^{-\frac{3}{2}}\left[\frac{2x^2-y^2-z^2}{(x^2+y^2+z^2)}\right]$$

Similarly,

$$\frac{\partial^2 w}{\partial y^2} = (x^2+y^2+z^2)^{-\frac{3}{2}}\left[\frac{2y^2-z^2-x^2}{(x^2+y^2+z^2)}\right]$$

$$\frac{\partial^2 w}{\partial z^2} = (x^2+y^2+z^2)^{-\frac{3}{2}}\left[\frac{2z^2-x^2-y^2}{(x^2+y^2+z^2)}\right]$$

$$\frac{\partial^2 w}{\partial x^2} + \frac{\partial^2 w}{\partial y^2} + \frac{\partial^2 w}{\partial z^2}$$

$$= (x^2+y^2+z^2)^{-\frac{3}{2}}\left[\frac{2x^2-y^2-z^2}{(x^2+y^2+z^2)} + \frac{2y^2-z^2-x^2}{(x^2+y^2+z^2)} + \frac{2z^2-x^2-y^2}{(x^2+y^2+z^2)}\right]$$

$$= (x^2+y^2+z^2)^{-\frac{3}{2}}\left[\frac{2x^2-2x^2+2y^2-2y^2+2z^2-2z^2}{(x^2+y^2+z^2)}\right]$$

$$= (x^2+y^2+z^2)^{-\frac{3}{2}}[0]$$

$$\therefore \frac{\partial^2 w}{\partial x^2} + \frac{\partial^2 w}{\partial y^2} + \frac{\partial^2 w}{\partial z^2} = 0$$

....

7. If $V(x,y) = e^x(x\cos y - y\sin y)$ then prove that $\frac{\partial^2 V}{\partial x^2} + \frac{\partial^2 V}{\partial y^2} = 0$

Solution:

$$V(x,y) = e^x(x\cos y - y\sin y)$$

$$V = xe^x(\cos y) - e^x\, y\sin y$$

$$\frac{\partial V}{\partial x} = (\cos y)[xe^x + e^x] - y\sin y\, e^x$$

$$\frac{\partial^2 V}{\partial x^2} = (\cos y)[xe^x + e^x + e^x] - y\sin y\, e^x$$

$$= (\cos y)[xe^x + 2e^x] - y\sin y\, e^x$$

$$= xe^x\cos y + 2\,e^x cosy - y\sin y\, e^x$$

$$V = xe^x(\cos y) - e^x\, y\sin y$$

$$\frac{\partial V}{\partial y} = xe^x(-\sin y) - e^x(y\cos y + \sin y)$$

$$= -xe^x\sin y - e^x y\cos y - e^x\sin y$$

$$\frac{\partial^2 V}{\partial y^2} = -xe^x\cos y - e^x(-y\sin y + \cos y)$$

$$-e^x\cos y$$

$$= -xe^x\cos y - 2\,e^x cosy + y\sin y\, e^x$$

$$\therefore \frac{\partial^2 V}{\partial x^2} + \frac{\partial^2 V}{\partial y^2} = 0$$

....

8. If $w(x,y) = xy + \sin(xy)$, then prove that $\frac{\partial^2 w}{\partial y \partial x} = \frac{\partial^2 w}{\partial x \partial y}$

Solution:

$$w(x,y) = xy + \sin(xy)$$

$$\frac{\partial w}{\partial x} = y + \cos(xy)y$$

$$= y + y\cos(xy)$$

$$\frac{\partial^2 w}{\partial y \partial x} = 1 + y[-\sin(xy)x] + \cos(xy)(1)$$

$$= 1 - xy\sin(xy) + \cos(xy) \,.....\, (i)$$

$$w(x,y) = xy + \sin(xy)$$

$$\frac{\partial w}{\partial y} = x + \cos(xy)x$$
$$= x + x\cos(xy)$$
$$\frac{\partial^2 w}{\partial x \partial y} = 1 + x[-\sin(xy)y] + \cos(xy)(1)$$
$$= 1 - xy\sin(xy) + \cos(xy) \ldots\ldots (ii)$$

From (i) and (ii) $\frac{\partial^2 w}{\partial y \partial x} = \frac{\partial^2 w}{\partial x \partial y}$ is proved.

....

9. If $V(x,y,z) = x^3 + y^3 + z^3 + 3xyz$, show that $\frac{\partial^2 v}{\partial y \partial z} = \frac{\partial^2 v}{\partial z \partial y}$

Solution:

$$V(x,y,z) = x^3 + y^3 + z^3 + 3xyz$$
$$\frac{\partial V}{\partial z} = 3z^2 + 3xy$$
$$\frac{\partial^2 v}{\partial y \partial z} = 3x \quad \ldots\ldots\ldots\ldots (i)$$
$$\frac{\partial V}{\partial y} = 3y^2 + 3xz$$
$$\frac{\partial^2 v}{\partial z \partial y} = 3x \quad \ldots\ldots\ldots\ldots (i)$$

From (i) and (ii) $\frac{\partial^2 v}{\partial y \partial z} = \frac{\partial^2 v}{\partial z \partial y}$ is proved.

....

10. A firm produces two types of calculators each week, x number of type A and y number of type B. The weekly revenue and cost functions (in rupees) are $R(x,y) = 80x + 90y + 0.04xy - 0.05x^2 - 0.05y^2$ and $C(x,y) = 8x + 6y + 2000$ respectively.
(i) Find the profit function $P(x,y)$,
(ii) Find $\frac{\partial P}{\partial x}$ (1200,1800) and $\frac{\partial P}{\partial y}$ at (1200,1800) and interpret these results.

Solution:

Given Revenue $= R(x,y)$ and

Cost $= C(x,y)$

So, Profit $P(x,y) = R(x,y) - C(x,y)$

$$= (80x + 90y + 0.04xy - 0.05x^2 - 0.05y^2) - (8x + 6y + 2000)$$

(i) $P(x,y) = 72x + 84y + 0.04xy - 0.05x^2 - 0.05y^2 - 2000$

$$\frac{\partial P}{\partial x} = 72 + 0.04y - 0.1x$$

At (1200,1800)

$$\frac{\partial P}{\partial x} = 72 + 0.04(1800) - 0.1(1200)$$
$$= 72 + 72.00 - 120.0$$
$$= 144 - 120$$
$$\frac{\partial P}{\partial x} = 24$$
$$\frac{\partial P}{\partial y} = 84 + 0.04x - 0.1y$$

At (1200,1800)

$$\frac{\partial P}{\partial y} = 84 + 0.04(1200) - 0.1(1800)$$
$$= 84 + 48.00 - 180.0$$
$$= 132 - 180$$
$$\frac{\partial P}{\partial y} = -48$$

(ii) $\frac{\partial P}{\partial x} = 24$ and $\frac{\partial P}{\partial y} = -48$ At (1200,1800), shows Profit increases when keeping y as constant.

....

Example 8.16
If $w(x,y,z) = x^2y + y^2z + z^2x, \quad x,y,z \in \mathbb{R}$, find the differential dw.
Solution:

$$dw = w_x dx + w_y dy + w_z dy$$

Given $w(x,y,z) = x^2y + y^2z + z^2x$

$$w_x = 2xy + z^2$$
$$w_y = x^2 + 2yz$$
$$w_z = y^2 + 2zx$$
$$\therefore\ dw = (2xy + z^2)dx + (x^2 + 2yz)dy + (y^2 + 2zx)dz$$

....

Example 8.17

Let $U(x,y,z) = x^2 - xy + 3\sin z, \quad x,y,z \in \mathbb{R}$.

Find the linear approximation for U at $(2,-1,0)$.

Solution:

Linear approximation $L(x,y,z)$
$= U(x_0,y_0,z_0) + U_x(x_0,y_0,z_0)(x - x_0) + U_y(x_0,y_0,z_0)(y - y_0) + U_z(x_0,y_0,z_0)(z - z_0)$

$$L(x,y,z) = U(x_0,y_0,z_0) + \sum U_x(x_0,y_0,z_0)(x - x_0)$$

$U(x,y,z) = x^2 - xy + 3\sin z$

$U(x_0,y_0,z_0) = U(2,-1,0)$

$= (2)^2 - (2)(-1) + 3\sin(0)$

$= 4 + 2 + 3(0)$

$U(x_0,y_0,z_0) = 6$

$U_x = 2x - y$

$U_y = -x$

$U_z = 3\cos z$

At $(2,-1,0)$, $U_x = 2(2) - (-1)$

$= 4 + 1$

$= 5$

$U_y = -(2)$

$= -2$

$U_z = 3\cos(0)$

$= 3(1)$

$= 3$

$\therefore L(x,y,z) = 6 + 5(x-2) - 2(y+1) + 3(z-0)$

$L(x,y,z) = 6 + 5x - 10 - 2y - 2 + 3z$

$= 5x - 2y + 3z - 6$

The required linear approximation for U at $(x_0,y_0,z_0) = 5x - 2y + 3z - 6$

....

EXERCISE 8.5

1. If $w(x,y) = x^3 - 3xy + 2y^2$, $x,y \in \mathbb{R}$, find the linear approximation for w at $(1,-1)$.

Solution: Linear approximation

$L(x,y) = U(x_0,y_0) + \sum U_x(x_0,y_0)(x-x_0)$

$w(x,y) = x^3 - 3xy + 2y^2$

$w(x_0,y_0) = w(1,-1)$

$= (1)^3 - 3(1)(-1) + 2(-1)^2$

$= 1 + 3 + 2(1)$

$= 4 + 2$

$w(x_0,y_0) = 6$

$w_x = 3x^2 - 3y$

$w_y = -3x + 4y$

At $(1,-1)$

$w_x = 3(1)^2 - 3(-1)$

$= 3 + 3 = 6$

$w_y = -3(1) + 4(-1)$

$= -3 - 4$

$= -7$

$\therefore L(x,y,z) = 6 + 6(x-1) - 7(y+1)$

$= 6 + 6x - 6 - 7y - 7$

$= 6x - 7y - 7$

The required linear approximation for w at $(x_0,y_0) = 6x - 7y - 7$

....

2. Let $z(x,y) = x^2y + 3xy^4$, $x,y \in \mathbb{R}$. Find the linear approximation for z at $(2,-1)$.

Solution: Linear approximation

$L(x,y) = U(x_0,y_0) + \sum U_x(x_0,y_0)(x-x_0)$

$z(x,y) = x^2y + 3xy^4$

$z(x_0,y_0) = z(2,-1)$

$= (2)^2(-1) + 3(2)(-1)^4$

$= 4(-1) + 6(1)$

$= -4 + 6$

$z(x_0,y_0) = 2$

$z_x = 2xy + 3y^4$

$z_y = x^2 + 12xy^3$

At $(2,-1)$

$z_x = 2(2)(-1) + 3(-1)^4$

$= -4 + 3$

$= -1$

$z_y = (2)^2 + 12(2)(-1)^3$

$= 4 - 24$

$= -20$

$\therefore L(x,y,z) = 2 - 1(x-2) - 20(y+1)$

$= 2 - x + 2 - 20y - 20$

$= -x - 20y - 16$

The required linear approximation for z at $(x_0,y_0) = -(x + 20y + 16)$

3. If $v(x,y) = x^2 - xy + \frac{1}{4}y^2 + 7, x, y \in \mathbb{R}$, find the differential dv.

Solution: $dv = v_x dx + v_y dy$

Given $v(x,y) = x^2 - xy + \frac{1}{4}y^2 + 7$

$$v_x = 2x - y$$

$$v_y = -x + \frac{1}{4}(2y)$$

$$= -x + \frac{y}{2}$$

$$\therefore dv = (2x-y)dx + \left(-x + \frac{y}{2}\right)dy$$

4. Let $W(x,y,z) = x^2 - xy + 3\sin z, x, y, z \in \mathbb{R}$. Find the linear approximation at $(2,-1,0)$

Solution: Example:- 8.17

5. Let $V(x,y,z) = xy + yz + zx, \quad x, y, z \in \mathbb{R}$. Find the differential dV

Solution:

$$dV = V_x dx + V_y dy + V_z dy$$

Given $V(x,y,z) = xy + yz + zx$

$$V_x = y + z$$

$$V_y = x + z$$

$$V_z = y + x$$

$$\therefore dV = (y+z)dx + (x+z)dy + (y+x)dz$$

Example 8.18 Verify $\frac{dF}{dt} = \frac{\partial F}{\partial x}\frac{dx}{dt} + \frac{\partial F}{\partial y}\frac{dy}{dt}$ for

$F(x,y) = x^2 - 2y^2 + 2xy$

at $x(t) = \cos t, y(t) = \sin t, t \in [0, 2\pi]$.

Solution: $F(x,y) = x^2 - 2y^2 + 2xy$

At $x(t) = \cos t, y(t) = \sin t$

$$F(t) = (\cos t)^2 - 2(\sin t)^2 + 2(\cos t)(\sin t)$$

$$= \cos^2 t - 2\sin^2 t + 2\sin t\cos t$$

$$\frac{dF}{dt} = 2\cos t(-\sin t) - 2(2\sin t\cos t) + 2[\sin t(-\sin t) + \cos t(\cos t)]$$

$$= -2\cos t\sin t - 4\cos t\sin t - 2\sin^2 t + 2\cos^2 t$$

$$\frac{dF}{dt} = -6\cos t\sin t - 2\sin^2 t + 2\cos^2 t \,..(1)$$

$$F(x,y) = x^2 - 2y^2 + 2xy$$

$$\frac{\partial F}{\partial x} = 2x + 2y$$

$$\frac{\partial F}{\partial y} = -4y + 2x$$

$$x(t) = \cos t$$

$$\frac{dx}{dt} = -\sin t$$

$$y(t) = \sin t$$

$$\frac{dy}{dt} = \cos t$$

$$\frac{\partial F}{\partial x}\frac{dx}{dt} + \frac{\partial F}{\partial y}\frac{dy}{dt} = (2x+2y)(-\sin t) + (-4y+2x)(\cos t)$$

$$= (2\cos t + 2\sin t)(-\sin t) + (-4\sin t + 2\cos t)(\cos t)$$

$$= -2\cos t\sin t - 2\sin^2 t - 4\cos t\sin t + 2\cos^2 t$$

$$= -6\cos t\sin t - 2\sin^2 t + 2\cos^2 t \,.....(2)$$

From (1) and (2)

$\frac{dF}{dt} = \frac{\partial F}{\partial x}\frac{dx}{dt} + \frac{\partial F}{\partial y}\frac{dy}{dt}$ is verified.

Example 8.19

Let $g(x,y) = x^2 - yx + \sin(x+y)$, $x(t) = e^{3t}, \quad y(t) = t^2, t \in \mathbb{R}$ Find $\frac{dg}{dt}$

Solution: $\frac{dg}{dt} = \frac{\partial g}{\partial x}\frac{dx}{dt} + \frac{\partial g}{\partial y}\frac{dy}{dt}$

$$g(x,y) = x^3 - yx + \sin(x+y)$$

$$\frac{\partial g}{\partial x} = 2x - y + \cos(x+y)$$

$$\frac{\partial g}{\partial y} = -x + \cos(x+y)$$

$$x(t) = e^{3t}$$

$$\frac{dx}{dt} = 3e^{3t}$$

$$y(t) = t^2$$

$$\frac{dy}{dt} = 2t$$

$$\therefore \frac{dg}{dt} = \frac{\partial g}{\partial x}\frac{dx}{dt} + \frac{\partial g}{\partial y}\frac{dy}{dt}$$

$$= \big(2x - y + \cos(x+y)\big)(3e^{3t}) + \big(-x + \cos(x+y)\big)(2t)$$

$$= \big(2e^{3t} - t^2 + \cos(e^{3t} + t^2)\big)(3e^{3t})$$

$$+\left(-e^{3t}+cos(e^{3t}+t^2)\right)(2t)$$

$$=6e^{6t}-3t^2e^{3t}+3e^{3t}cos(e^{3t}+t^2)$$

$$-2te^{3t}+2tcos(e^{3t}+t^2)$$

Example 8.20

Let $g(x,y)=2y+x^2$,

$x=2r-s, y=r^2+2s, r,s \in \mathbb{R}$. Find $\frac{\partial g}{\partial r},\frac{\partial g}{\partial s}$

Solution: (i) $\frac{\partial g}{\partial r}=\frac{\partial g}{\partial x}\frac{\partial x}{\partial r}+\frac{\partial g}{\partial y}\frac{\partial y}{\partial r}$

$$g(x,y)=2y+x^2$$

$$\frac{\partial g}{\partial x}=2x$$

$$\frac{\partial g}{\partial y}=2$$

$$x=2r-s$$

$$\frac{\partial x}{\partial r}=2$$

$$y=r^2+2s$$

$$\frac{\partial y}{\partial r}=2r$$

$$\therefore \frac{\partial g}{\partial r}=\frac{\partial g}{\partial x}\frac{\partial x}{\partial r}+\frac{\partial g}{\partial y}\frac{\partial y}{\partial r}$$

$$=(2x)(2)+(2)(2r)$$

$$=4x+4r$$

$$=4(2r-s)+4r$$

$$=8r-4s+4r$$

$$\frac{\partial g}{\partial r}=12r-4s$$

(ii) $\frac{\partial g}{\partial s}=\frac{\partial g}{\partial x}\frac{\partial x}{\partial s}+\frac{\partial g}{\partial y}\frac{\partial y}{\partial s}$

$$g(x,y)=2y+x^2$$

$$\frac{\partial g}{\partial x}=2x$$

$$\frac{\partial g}{\partial y}=2$$

$$x=2r-s$$

$$\frac{\partial x}{\partial s}=-1$$

$$y=r^2+2s$$

$$\frac{\partial y}{\partial s}=2$$

$$\therefore \frac{\partial g}{\partial s}=\frac{\partial g}{\partial x}\frac{\partial x}{\partial s}+\frac{\partial g}{\partial y}\frac{\partial y}{\partial s}$$

$$=(2x)(-1)+(2)(2)$$

$$=-2x+4$$

$$=-2(2r-s)+4$$

$$=-4r+2s+4r$$

$$\frac{\partial g}{\partial s}=-4r+2s+4$$

EXERCISE 8.6

1. If $u(x,y)=x^2y+3xy^4, x=e^t$ and $y=\sin t$, find $\frac{du}{dt}$ and evaluate it at $t=0$

Solution: $\frac{du}{dt}=\frac{\partial u}{\partial x}\frac{dx}{dt}+\frac{\partial u}{\partial y}\frac{dy}{dt}$

$$u(x,y)=x^2y+3xy^4$$

$$\frac{\partial u}{\partial x}=2xy+3y^4$$

$$\frac{\partial u}{\partial y}=x^2+12xy^3$$

$$x(t)=e^t$$

$$\frac{dx}{dt}=e^t$$

$$y(t)=\sin t$$

$$\frac{dy}{dt}=\cos t$$

$$\frac{du}{dt}=\left(\frac{\partial u}{\partial x}\right)\left(\frac{dx}{dt}\right)+\left(\frac{\partial u}{\partial y}\right)\left(\frac{dy}{dt}\right)$$

$$=(2xy+3y^4)(e^t)+(x^2+12xy^3)(\cos t)$$

$$=[2(e^t)(\sin t)+3(\sin t)^4](e^t)$$

$$+[(e^t)^2+12(e^t)(\sin t)^3](\cos t)$$

$$=2e^{2t}\sin t+3e^t sin^4t+e^{2t}\cos t+12e^t\cos t\, sin^3t$$

$$\frac{du}{dt}=e^t(2e^t\sin t+3sin^4t+e^t\cos t+12\cos t\, sin^3t)$$

at $t=0$,

$$\frac{du}{dt}=e^0(2e^0\sin 0+3sin^4 0+e^0\cos 0+12\cos 0\, sin^3 0)$$

$$=1(0+0+1+0)$$

$$=1(1)$$

$$\frac{du}{dt}=1$$

2. If $u(x,y,z)=xy^2z^3$,

$x=\sin t, y=\cos t, z=1+e^{2t}$ find $\frac{du}{dt}$

Solution: $\frac{du}{dt}=\frac{\partial u}{\partial x}\frac{dx}{dt}+\frac{\partial u}{\partial y}\frac{dy}{dt}+\frac{\partial u}{\partial z}\frac{dz}{dt}$

$$u(x,y,z)=xy^2z^3$$

$$\frac{\partial u}{\partial x}=y^2z^3$$

$\frac{\partial u}{\partial y} = x(2y)z^3 = 2xyz^3$

$\frac{\partial u}{\partial z} = xy^2(3z^2) = 3xy^2z^2$

$x = \sin t$

$\frac{dx}{dt} = \cos t$

$x = \cos t$

$\frac{dy}{dt} = -\sin t$

$z = 1 + 2e^{2t}$

$\frac{dz}{dt} = 2(e^{2t}) = 2e^{2t}$

$\frac{du}{dt} = (y^2z^3)(\cos t) + (2xyz^3)(-\sin t)$

$+(3xy^2z^2)(2e^{2t})$

$= y^2z^3 \cos t - 2xyz^3 \sin t + 6xy^2z^2e^{2t}$

$= yz^2(yz\cos t - 2xz\sin t + 6xye^{2t})$

$= \cos t\,(1 + e^{2t})^2$

$[\cos^2 t(1 + e^{2t}) - 2\sin^2 t(1 + e^{2t}) + 6\sin t\cos t\, e^{2t}]$

3. If $w(x, y, z) = x^2 + y^2 + z^3$,

$x = e^t, y = e^t \sin t, z = e^t \cos t$, find $\frac{dw}{dt}$

Solution: $\frac{dw}{dt} = \frac{\partial w}{\partial x}\frac{dx}{dt} + \frac{\partial w}{\partial y}\frac{dy}{dt} + \frac{\partial w}{\partial z}\frac{dz}{dt}$

$w(x, y, z) = x^2 + y^2 + z^2$

$\frac{\partial w}{\partial x} = 2x$

$\frac{\partial w}{\partial y} = 2y$

$\frac{\partial w}{\partial z} = 2z$

$x = e^t$

$\frac{dx}{dt} = e^t$

$y = e^t \sin t$

$\frac{dy}{dt} = e^t\cos t + \sin t\, e^t$

$z = e^t \cos t$

$\frac{dz}{dt} = e^t(-\sin t) + \cos t\, e^t$

$\frac{\partial w}{\partial x}\frac{dx}{dt} = (2x)(e^t)$

$= (2e^t)e^t$

$= 2e^{2t}$

$\frac{\partial w}{\partial y}\frac{dy}{dt} = (2y)(e^t\cos t + \sin t\, e^t)$

$= (2e^t \sin t)(e^t\cos t + \sin t\, e^t)$

$= (2e^{2t}\sin t)(\cos t + \sin t)$

$\frac{\partial w}{\partial z}\frac{dz}{dt} = (2e^t\cos t)(-e^t\sin t + \cos t\, e^t)$

$= (2e^{2t}\cos t)(-\sin t + \cos t)$

$\frac{dw}{dt} = \frac{\partial w}{\partial x}\frac{dx}{dt} + \frac{\partial w}{\partial y}\frac{dy}{dt} + \frac{\partial w}{\partial z}\frac{dz}{dt}$

$= 2e^{2t}(1 + \sin t\cos t + \sin^2 t - \sin t\cos t + \cos^2 t)$

$= 2e^{2t}(1 + \sin^2 t + \cos^2 t)$

$= 2e^{2t}(1 + 1)$

$= 2e^{2t}(2)$

$\frac{dw}{dt} = 4e^{2t}$

4. Let $U(x, y, z) = xyz$,

$x = e^{-t}, y = e^{-t}\cos t, z = \sin t, t \in \mathbb{R}$, find $\frac{dU}{dt}$

Solution: $\frac{dU}{dt} = \frac{\partial U}{\partial x}\frac{dx}{dt} + \frac{\partial U}{\partial y}\frac{dy}{dt} + \frac{\partial U}{\partial z}\frac{dz}{dt}$

$U(x, y, z) = xyz$

$\frac{\partial U}{\partial x} = yz$

$\frac{\partial U}{\partial y} = xz$

$\frac{\partial U}{\partial z} = xy$

$x = e^{-t}$

$\frac{dx}{dt} = -e^{-t}$

$y = e^{-t}\cos t$

$\frac{dy}{dt} = e^{-t}(-\sin t) + \cos t\,(-e^{-t})$

$= e^{-t}(-\sin t - \cos t)$

$= -e^{-t}(\sin t + \cos t)$

$z = \sin t$

$\frac{dz}{dt} = \cos t$

$\frac{\partial U}{\partial x}\frac{dx}{dt} = (yz)(-e^{-t})$

$= (e^{-t}\cos t)(\sin t)(-e^{-t})$

$= -e^{-2t}\cos t\sin t$

$$\frac{\partial U}{\partial y}\frac{dy}{dt} = (xz)[-e^{-t}(\sin t + \cos t)]$$
$$= (e^{-t})(\sin t)[-e^{-t}(\sin t + \cos t)]$$
$$= (-e^{-2t})(\sin^2 t + \sin t\cos t)$$
$$\frac{\partial w}{\partial z}\frac{dz}{dt} = (e^{-t})(e^{-t}\cos t)(\cos t)$$
$$= (e^{-2t})(\cos^2 t)$$
$$\frac{dw}{dt} = \frac{\partial w}{\partial x}\frac{dx}{dt} + \frac{\partial w}{\partial y}\frac{dy}{dt} + \frac{\partial w}{\partial z}\frac{dz}{dt}$$
$$= -e^{-2t}(\sin t\cos t + \sin^2 t + \sin t\cos t - \cos^2 t)$$
$$= -e^{-2t}[2\sin t\cos t - (\cos^2 t - \sin^2 t)]$$
$$\frac{dw}{dt} = -e^{-2t}(\sin 2t - \cos 2t)$$

....

5. If $w(x,y) = 6x^3 - 3xy + 2y^2$, $x = e^s, y = \cos s, s \in \mathbb{R}$, find $\frac{dw}{ds}$, and evaluate at $s = 0$

Solution: $\frac{dw}{ds} = \frac{\partial w}{\partial x}\frac{dx}{ds} + \frac{\partial w}{\partial y}\frac{dy}{ds}$

$$w(x,y) = 6x^3 - 3xy + 2y^2$$
$$\frac{\partial w}{\partial x} = 18x^2 - 3y$$
$$\frac{\partial w}{\partial y} = -3x + 4y$$
$$x = e^s$$
$$\frac{dx}{ds} = e^s$$
$$y = \cos s$$
$$\frac{dy}{ds} = -\sin s$$
$$\frac{\partial w}{\partial x}\frac{dx}{ds} = (18x^2 - 3y)(e^s)$$
$$= [18(e^s)^2 - 3(\cos s)]e^s$$
$$= e^s[18(e^{2s}) - 3(\cos s)]$$
$$\frac{\partial w}{\partial y}\frac{dy}{ds} = (-3x + 4y)(-\sin s)$$
$$= [-3(e^s) + 4(\cos s)](-\sin s)$$
$$= (2\,e^{2t}\sin t)(\cos t + \sin t)$$
$$\frac{dw}{ds} = \frac{\partial w}{\partial x}\frac{dx}{ds} + \frac{\partial w}{\partial y}\frac{dy}{ds}$$
$$= 18e^{3s} - 3\cos s\, e^s + 3e^s \sin s - 4\sin s\cos s$$

at $s = 0$

$$\frac{dw}{dt} = 18e^0 - 3\cos 0\, e^0 + 3e^0\sin 0 - 4\sin 0\cos 0$$
$$= 18 - 3 + 0 - 0$$
$$\frac{dw}{dt} = 15$$

....

6. If $z(x,y) = x\tan^{-1}(xy)$, $x = t^2, y = se^t, s,t \in \mathbb{R}$, find $\frac{\partial z}{\partial s}$ and $\frac{\partial z}{\partial t}$ at $s = t = 1$.

Solution: (i) $\frac{\partial z}{\partial s} = \frac{\partial z}{\partial x}\frac{\partial x}{\partial s} + \frac{\partial z}{\partial y}\frac{\partial y}{\partial s}$

$$z(x,y) = x\tan^{-1}(xy)$$
$$\frac{\partial z}{\partial x} = x\left[\frac{1}{1+(xy)^2}(y)\right] + \tan^{-1}(xy)(1)$$
$$= x\left(\frac{y}{1+x^2y^2}\right) + \tan^{-1}(xy)$$
$$\frac{\partial z}{\partial x} = \frac{xy}{1+x^2y^2} + \tan^{-1}(xy)$$
$$x = t^2$$
$$\frac{\partial x}{\partial s} = 0$$
$$\frac{\partial z}{\partial x}\frac{\partial x}{\partial s} = \left[\frac{xy}{1+x^2y^2} + \tan^{-1}(xy)\right](0)$$
$$\frac{\partial z}{\partial x}\frac{\partial x}{\partial s} = 0$$
$$z(x,y) = x\tan^{-1}(xy)$$
$$\frac{\partial z}{\partial y} = x\left[\frac{1}{1+(xy)^2}(x)\right] + \tan^{-1}(xy)(0)$$
$$= \left(\frac{x^2}{1+(xy)^2}\right)$$
$$y = se^t$$
$$\frac{\partial y}{\partial s} = e^t$$
$$\frac{\partial z}{\partial x}\frac{\partial y}{\partial s} = \left(\frac{x^2}{1+(xy)^2}\right)(e^t)$$

at $x = t^2, y = se^t$.

$$\frac{\partial z}{\partial s} = \left(\frac{t^4}{1+(t^2se^t)^2}\right)(e^t)$$
$$\frac{\partial z}{\partial s} = \frac{t^4e^t}{1+(t^2se^t)^2}$$

at $s = t = 1$.

$$\frac{\partial z}{\partial s} = \frac{e}{1+e^2}$$

(ii) $\frac{\partial z}{\partial t} = \frac{\partial z}{\partial x}\frac{\partial x}{\partial t} + \frac{\partial z}{\partial y}\frac{\partial y}{\partial t}$

$$z(x,y) = xtan^{-1}(xy)$$

$$\frac{\partial z}{\partial x} = x\left[\frac{1}{1+(xy)^2}(y)\right] + tan^{-1}(xy)(1)$$

$$= x\left(\frac{y}{1+x^2y^2}\right) + tan^{-1}(xy)$$

$$\frac{\partial z}{\partial x} = \frac{xy}{1+x^2y^2} + tan^{-1}(xy)$$

$$x = t^2$$

$$\frac{\partial x}{\partial t} = 2t$$

$$\frac{\partial z}{\partial x}\frac{\partial x}{\partial t} = \left[\frac{xy}{1+x^2y^2} + tan^{-1}(xy)\right](2t)$$

at $s = t = 1$ then $x = 1, y = e.$

$$\frac{\partial z}{\partial x}\frac{\partial x}{\partial t} = \left[\frac{e}{1+e^2} + tan^{-1}(e)\right](2)$$

$$= \left[\frac{2e}{1+e^2} + 2tan^{-1}(e)\right]$$

$$z(x,y) = xtan^{-1}(xy)$$

$$\frac{\partial z}{\partial y} = x\left[\frac{1}{1+(xy)^2}(y)\right] + tan^{-1}(xy)(0)$$

$$= \left(\frac{xy}{1+(xy)^2}\right)$$

$$y = se^t$$

$$\frac{\partial y}{\partial t} = se^t$$

$$\frac{\partial z}{\partial y}\frac{\partial y}{\partial t} = \left(\frac{xy}{1+(xy)^2}\right)(se^t)$$

at $s = t = 1$ then $x = 1, y = e$

$$\frac{\partial z}{\partial y}\frac{\partial y}{\partial t} = \left(\frac{1}{1+e^2}\right)(e)$$

$$= \frac{e}{1+e^2}$$

$$\frac{\partial z}{\partial t} = \frac{\partial z}{\partial x}\frac{\partial x}{\partial t} + \frac{\partial z}{\partial y}\frac{\partial y}{\partial t}$$

$$\frac{\partial z}{\partial t} = \frac{2e}{1+e^2} + 2tan^{-1}(e) + \frac{e}{1+e^2}$$

$$\frac{\partial z}{\partial t} = \frac{3e}{1+e^2} + 2tan^{-1}(e)$$

....

7. Let $U(x,y) = e^x siny$ where $x = st^2, y = s^2t,\ s,t \in \mathbb{R}$, find $\frac{\partial U}{\partial s}, \frac{\partial U}{\partial t}$ and evaluate them at $s = t = 1$.

Solution:

$$\frac{\partial U}{\partial s} = \frac{\partial U}{\partial x}\frac{\partial x}{\partial s} + \frac{\partial U}{\partial y}\frac{\partial y}{\partial s}$$

$$U(x,y) = e^x siny$$

$$\frac{\partial U}{\partial x} = \sin y\, e^x$$

$$x = st^2$$

$$\frac{\partial x}{\partial s} = t^2$$

$$\frac{\partial U}{\partial x}\frac{\partial x}{\partial s} = (\sin y\, e^x)(t^2)$$

$s = t = 1$ then $x = 1, y = 1$

$$\frac{\partial U}{\partial x}\frac{\partial x}{\partial s} = [\sin(1)\, e^1](1)$$

$$= e\sin(1)$$

$$\frac{\partial U}{\partial y} = e^x \cos y$$

$$y = s^2t$$

$$\frac{\partial y}{\partial s} = 2st$$

$$\frac{\partial U}{\partial y}\frac{\partial y}{\partial s} = (e^x\cos y)(2st)$$

$s = t = 1$ then $x = 1, y = 1$

$$\frac{\partial U}{\partial y}\frac{\partial y}{\partial s} = [e\cos(1)](2)$$

$$= 2e\cos(1)$$

$$\frac{\partial U}{\partial s} = e\sin(1) + 2e\cos(1)$$

....

8. Let $z(x,y) = x^3 - 3x^2y^3$ where $x = se^t$, $y = se^{-t}, s,t \in \mathbb{R}$, find $\frac{\partial z}{\partial s}$ and $\frac{\partial z}{\partial t}$

Solution:

$$\frac{\partial z}{\partial s} = \frac{\partial z}{\partial x}\frac{\partial x}{\partial s} + \frac{\partial z}{\partial y}\frac{\partial y}{\partial s}$$

$$z(x,y) = x^3 - 3x^2y^3$$

$$\frac{\partial z}{\partial x} = 3x^2 - 6xy^3$$

$$x = se^t$$

$$\frac{\partial x}{\partial s} = e^t$$

$$\frac{\partial z}{\partial x}\frac{\partial x}{\partial s} = (3x^2 - 6xy^3)(e^t)$$

$$= [3(se^t)^2 - 6(se^t)(se^{-t})^3](e^t)$$

$= (3s^2e^{2t} - 6se^ts^3e^{-3t})(e^t)$

$= (3s^2e^{2t} - 6s^4e^{-2t})(e^t)$

$\frac{\partial z}{\partial x}\frac{\partial x}{\partial s} = (3s^2e^t)(e^{2t} - 2s^2e^{-2t})$

$z(x, y) = x^3 - 3x^2y^3$

$\frac{\partial z}{\partial y} = -9x^2y^2$

$y = se^{-t}$

$\frac{\partial y}{\partial s} = e^{-t}$

$\frac{\partial z}{\partial y}\frac{\partial y}{\partial s} = (-9x^2y^2)(e^{-t})$

$= [-9(x)^2(y)^2](e^{-t})$

$= [-9(se^t)^2(se^{-t})^2](e^{-t})$

$= -9s^2e^{2t}s^2e^{-2t}e^{-t}$

$= -9s^4e^{-t}$

$\frac{\partial z}{\partial s} = (3s^2e^t)(e^{2t} - 2s^2e^{-2t}) + -9s^4e^{-t}$

$= (3s^2e^t)(e^{2t} - 2s^2e^{-2t} - 3s^2e^{-2t})$

....

9. $W(x, y, z) = xy + yz + zx,$
$x = u - v, y = uv, z = u + v, u, v \in \mathbb{R}$.
Find $\frac{dw}{du}, \frac{dw}{dv}$ and $\frac{dw}{dz}$ evaluate them at $\left(\frac{1}{2}, 1\right)$

<u>Solution</u>: (i) $\frac{dw}{du} = \frac{\partial w}{\partial x}\frac{\partial x}{\partial u} + \frac{\partial w}{\partial y}\frac{\partial y}{\partial u} + \frac{\partial w}{\partial z}\frac{\partial z}{\partial u}$

$W(x, y, z) = xy + yz + zx$

$\frac{\partial w}{\partial x} = y + z$

$\frac{\partial w}{\partial y} = x + z$

$\frac{\partial w}{\partial z} = y + x$

$x = u - v$

$\frac{\partial x}{\partial u} = 1$

$y = uv$

$\frac{\partial y}{\partial u} = v$

$z = u + v$

$\frac{\partial z}{\partial u} = 1$

$\frac{dw}{du} = \left(\frac{\partial w}{\partial x}\right)\left(\frac{\partial x}{\partial u}\right) + \left(\frac{\partial w}{\partial y}\right)\left(\frac{\partial u}{\partial u}\right) + \left(\frac{\partial w}{\partial z}\right)\left(\frac{\partial z}{\partial u}\right)$

$= (y + z)(1) + (x + z)(v) + (y + x)(1)$

$= (y + z) + (x + z)(v) + (y + x)$

At $x = u - v, y = uv, z = u + v$

$\frac{dw}{du} = (uv + u + v) + (u - v + u + v)(v)$
$+(uv + u - v)$

$= (uv + u + v) + (2u)(v) + (uv + u - v)$

$= uv + u + v + 2uv + uv + u - v$

$\frac{dw}{du} = 4uv + 2u$

at $\left(\frac{1}{2}, 1\right)$

$\frac{dw}{du} = 4\left(\frac{1}{2}\right)(1) + 2\left(\frac{1}{2}\right)$

$= 2 + 1$

$\frac{dw}{du} = 3$

(ii) $\frac{dw}{dv} = \frac{\partial w}{\partial x}\frac{\partial x}{\partial v} + \frac{\partial w}{\partial y}\frac{\partial y}{\partial v} + \frac{\partial w}{\partial z}\frac{\partial z}{\partial v}$

$W(x, y, z) = xy + yz + zx$

$\frac{\partial w}{\partial x} = y + z$

$\frac{\partial w}{\partial y} = x + z$

$\frac{\partial w}{\partial z} = y + x$

$x = u - v$

$\frac{\partial x}{\partial v} = -1$

$y = uv$

$\frac{\partial y}{\partial v} = u$

$z = u + v$

$\frac{\partial z}{\partial v} = 1$

$\frac{dw}{dv} = \left(\frac{\partial w}{\partial x}\right)\left(\frac{\partial x}{\partial v}\right) + \left(\frac{\partial w}{\partial y}\right)\left(\frac{\partial y}{\partial v}\right) + \left(\frac{\partial w}{\partial z}\right)\left(\frac{\partial z}{\partial v}\right)$

$= (y + z)(-1) + (x + z)(u) + (y + x)(1)$

$= -(y + z) + (x + z)(u) + (y + x)$

At $x = u - v, y = uv, z = u + v$

$$\frac{dw}{dv} = -(uv+u+v) + (u-v+u+v)(u) + (uv+u-v)$$

$$= -(uv+u+v) + (2u)(u) + (uv+u-v)$$

$$= -uv-u-v+2u^2+uv+u-v$$

$$\frac{dw}{dv} = 2u^2 - 2v$$

at $\left(\frac{1}{2}, 1\right)$

$$\frac{dw}{dv} = 2\left(\frac{1}{2}\right)^2 - 2(1)$$

$$= 2\left(\frac{1}{4}\right) - 2$$

$$= \frac{1}{2} - 2$$

$$= \frac{1-4}{2}$$

$$\frac{dw}{dv} = -\frac{3}{2}$$

....

Example 8.21

Show that $F(x,y) = \frac{x^2+5xy-10y^2}{3x+7y}$ is a homogeneous function of degree 1.

Solution:

Given $F(x,y) = \frac{x^2+5xy-10y^2}{3x+7y}$

$$\therefore F(tx,ty) = \frac{(tx)^2+5(tx)(ty)-10(ty)^2}{3(tx)+7(ty)}$$

$$= \frac{t^2x^2+5t^2xy-10t^2y^2}{3tx+7ty}$$

$$= \frac{t^2(x^2+5xy-10y^2)}{t(3x+7y)}$$

$$= \frac{t(x^2+5xy-10y^2)}{(3x+7y)}$$

$= t\,F(x,y)$ for all $t \in \mathbb{R}$.

So, F is a homogeneous function of degree 1.

....

Example 8.22 If $u = sin^{-1}\left(\frac{x+y}{\sqrt{x}+\sqrt{y}}\right)$,

Show that $x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} = \frac{1}{2}\tan u$

Solution:

Given $u = sin^{-1}\left(\frac{x+y}{\sqrt{x}+\sqrt{y}}\right)$

$$\sin u\,(x,y) = \left(\frac{x+y}{\sqrt{x}+\sqrt{y}}\right)$$

$$\sin u\,(tx,ty) = \left(\frac{tx+ty}{\sqrt{tx}+\sqrt{ty}}\right)$$

$$= \frac{t(x+y)}{\sqrt{t}(\sqrt{x}+\sqrt{x})}$$

$$= \frac{\sqrt{t}\sqrt{t}(x+y)}{\sqrt{t}(\sqrt{x}+\sqrt{x})}$$

$$= \frac{\sqrt{t}(x+y)}{(\sqrt{x}+\sqrt{x})}$$

$$= \frac{t^{\frac{1}{2}}(x+y)}{(\sqrt{x}+\sqrt{x})}$$

$= t^{\frac{1}{2}}\sin u\,(x,y)$ for all $t \in \mathbb{R}$.

So, $\sin u\,(x,y)$ is a homogeneous function of degree $\frac{1}{2}$

By Euler's theorem,

$$x\frac{\partial f}{\partial x} + y\frac{\partial f}{\partial y} = \frac{1}{2}f$$

Now, substituting $f = \sin u$ in the above equation, we get

$$x\frac{\partial(\sin u)}{\partial x} + y\frac{\partial(\sin u)}{\partial x} = \frac{1}{2}\sin u$$

$$x\cos u\frac{\partial u}{\partial x} + y\cos u\frac{\partial u}{\partial x} = \frac{1}{2}\sin u$$

$$\cos u\left(x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial x}\right) = \frac{1}{2}\sin u$$

$$x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial x} = \frac{1}{2}\frac{\sin u}{\cos u}$$

$$x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial x} = \frac{1}{2}\tan u \quad \text{Proved.}$$

....

EXERCISE 8.7

1. In each of the following cases, determine whether the following function is homogeneous or not.
If it is so, find the degree.

(i) $f(x,y) = x^2y + 6x^3 + 7$

Solution:

$$f(x,y) = x^2y + 6x^3 + 7$$

$$f(tx,ty) = (tx)^2(ty) + 6(tx)^3 + 7$$

$$= t^2x^2ty + 6t^3x^3 + 7$$

$$= t^3x^2y + 6t^3x^3 + 7$$

It is not homogeneous function.

(ii) $h(x,y) = \frac{6x^2y^3 - \pi y^5 + 9x^4y}{2020x^2 + 2019y^2}$

$$h(tx,ty) = \frac{6(tx)^2(ty)^3 - \pi(ty)^5 + 9(tx)^4(ty)}{2020(tx)^2 + 2019(ty)^2}$$

$$= \frac{6t^2x^2t^3y^3 - \pi t^5y^5 + 9t^4x^4ty}{2020t^2x^2 + 2019t^2y^2}$$

$$= \frac{6t^5x^2y^3 - \pi t^5y^5 + 9t^5x^4y}{2020t^2x^2 + 2019t^2y^2}$$

$$= \frac{t^5(6x^2y^3 - \pi y^5 + 9x^4y)}{t^2(2020x^2 + 2019y^2)}$$

$$= \frac{t^3(6x^2y^3 - \pi y^5 + 9x^4y)}{(2020x^2 + 2019y^2)}$$

$$= t^3 h(x,y)$$

It is a homogeneous function of degree 3.

....

(iii) $g(x,y,z) = \frac{\sqrt{3x^2+5y^2+z^2}}{4x+7y}$

$$g(tx,ty,tz) = \frac{\sqrt{3(tx)^2+5(ty)^2+(tz)^2}}{4(tx)+7(ty)}$$

$$= \frac{\sqrt{3t^2x^2+5t^2y^2+t^2z^2}}{4tx+7ty}$$

$$= \frac{\sqrt{t^2(3x^2+5y^2+z^2)}}{t(4x+7y)}$$

$$= \frac{t\sqrt{(3x^2+5y^2+z^2)}}{t(4x+7y)}$$

$$= \frac{t^0\sqrt{(3x^2+5y^2+z^2)}}{(4x+7y)}$$

It is a homogeneous function of degree 0.

(iv) $u(x,y,z) = xy + \sin\left(\frac{y^2-2z^2}{xy}\right)$

$$u(tx,ty,tz) = txty + \sin\left(\frac{(ty)^2-2(tz)^2}{(tx)(ty)}\right)$$

$$= t^2xy + \sin\left(\frac{t^2y^2-2t^2z^2}{t^2xy}\right)$$

$$= t^2xy + \sin\left[\frac{t^2(y^2-2z^2)}{t^2(xy)}\right]$$

$$= t^2xy + \sin\left(\frac{y^2-2z^2}{xy}\right)$$

It is not homogeneous function.

....

2. Prove that $f(x,y) = x^3 - 2x^2y + 3xy^2 + y^3$ is homogeneous; what is the degree? Verify Euler's Theorem for f .

<u>Solution</u>:

$$f(x,y) = x^3 - 2x^2y + 3xy^2 + y^3$$

$$f(tx,ty) = (tx)^3 - 2(tx)^2(ty) + 3(tx)(ty)^2 + (ty)^3$$

$$= t^3x^3 - 2t^2x^2ty + 3txt^2y^2 + t^3y^3$$

$$= t^3x^3 - 2t^3x^2y + 3t^3xy^2 + t^3y^3$$

$$= t^3(x^3 - 2x^2y + 3xy^2 + y^3)$$

It is a homogeneous function of degree 3.

By Euler's theorem, $x\frac{\partial f}{\partial x} + y\frac{\partial f}{\partial y} = 3f$

To verify,

$$f(x,y) = x^3 - 2x^2y + 3xy^2 + y^3$$

$$\frac{\partial f}{\partial x} = 3x^2 - 4xy + 3y^2$$

$$\frac{\partial f}{\partial y} = -2x^2 + 6xy + 3y^2$$

$$x\frac{\partial f}{\partial x} + y\frac{\partial f}{\partial x} = x(3x^2 - 4xy + 3y^2)$$

$$+y(-2x^2 + 6xy + 3y^2)$$

$$= 3x^3 - 4x^2y + 3xy^2 + y^3 - 2x^2y + 6xy^2 + 3y^3$$

$$= 3x^3 - 6x^2y + 9xy^2 + 3y^3$$

$$= 3(x^3 - 2x^2y + 3xy^2 + y^3)$$

$$= 3f$$

Hence Euler's theorem, $x\frac{\partial f}{\partial x} + y\frac{\partial f}{\partial y} = 3f$ is verified.

....

3. Prove that $g(x,y) = x\log\left(\frac{y}{x}\right)$ is homogeneous; what is the degree? Verify Euler's Theorem for g.

<u>Solution</u>:

$$g(x,y) = x\log\left(\frac{y}{x}\right)$$

$$g(tx,ty) = tx\log\left(\frac{ty}{tx}\right)$$

$$= tx\log\left(\frac{y}{x}\right)$$

$$= tg(x,y)$$

It is a homogeneous function of degree 1.

By Euler's theorem, $x\frac{\partial g}{\partial x} + y\frac{\partial g}{\partial y} = g$

$$g(x,y) = x\log\left(\frac{y}{x}\right)$$

$$\frac{\partial g}{\partial x} = x\left[\frac{1}{\left(\frac{y}{x}\right)}y\left(\frac{-1}{x^2}\right)\right] + \log\left(\frac{y}{x}\right)$$

$$= x\left[\left(\frac{x}{y}\right)\left(\frac{-y}{x^2}\right)\right] + \log\left(\frac{y}{x}\right)$$

$$= \left[\left(\frac{x^2}{y}\right)\left(\frac{-y}{x^2}\right)\right] + \log\left(\frac{y}{x}\right)$$

$$\frac{\partial g}{\partial x} = -1 + \log\left(\frac{y}{x}\right)$$

$$g(x,y) = x\log\left(\frac{y}{x}\right)$$

$$\frac{\partial g}{\partial y} = x\left[\frac{1}{\left(\frac{y}{x}\right)}\left(\frac{1}{x}\right)(1)\right] + \log\left(\frac{y}{x}\right)(0)$$

$$= x\left[\left(\frac{x}{y}\right)\left(\frac{1}{x}\right)\right] + 0$$

$$= \frac{x}{y}$$

$$\therefore x\frac{\partial g}{\partial x} + y\frac{\partial g}{\partial y} = x\left[-1 + \log\left(\frac{y}{x}\right)\right] + y\left(\frac{x}{y}\right)$$

$$= -x + x\log\left(\frac{y}{x}\right) + x$$

$$x\frac{\partial g}{\partial x} + y\frac{\partial g}{\partial y} = x\log\left(\frac{y}{x}\right)$$

Hence, $x\frac{\partial g}{\partial x} + y\frac{\partial g}{\partial y} = g$ is verified.

....

4. If $u(x,y) = \frac{x^2+y^2}{\sqrt{x+y}}$, prove that $x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} = \frac{3}{2}u$

Solution:

$$u(x,y) = \frac{x^2+y^2}{\sqrt{x+y}}$$

$$u(tx,ty) = \frac{(tx)^2+(ty)^2}{\sqrt{(tx)+(ty)}}$$

$$= \frac{t^2x^2+t^2y^2}{\sqrt{(tx)+(ty)}}$$

$$= \frac{t^2(x^2+y^2)}{\sqrt{t(x+y)}}$$

$$= \frac{t^2(x^2+y^2)}{\sqrt{t}\sqrt{(x+y)}}$$

$$= \frac{t^{\frac{3}{2}}(x^2+y^2)}{\sqrt{(x+y)}}$$

$$= t^{\frac{3}{2}}u(x,y)$$

It is a homogeneous function of degree $\frac{3}{2}$.

Hence by Euler's theorem $x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} = \frac{3}{2}u$

....

5. If $v(x,y) = \log\left(\frac{x^2+y^2}{x+y}\right)$, prove that $x\frac{\partial v}{\partial x} + y\frac{\partial v}{\partial y} = 1$

Solution:

$$v(x,y) = \log\left(\frac{x^2+y^2}{x+y}\right)$$

$$e^v = \frac{x^2+y^2}{x+y}$$

Let $f = e^v(tx,ty) = \frac{(tx)^2+(ty)^2}{(tx)+(ty)}$

$$= \frac{t^2(x^2+y^2)}{t(x+y)}$$

$$= \frac{t(x^2+y^2)}{(x+y)}$$

It is a homogeneous function of degree 1.

By Euler's theorem, $x\frac{\partial f}{\partial x} + y\frac{\partial f}{\partial y} = f$

But $f = e^v$

So, $x\frac{\partial e^v}{\partial x} + y\frac{\partial e^v}{\partial y} = e^v$

$$xe^v\frac{\partial v}{\partial x} + ye^v\frac{\partial v}{\partial y} = e^v$$

$$e^v\left(x\frac{\partial v}{\partial x} + y\frac{\partial v}{\partial y}\right) = e^v$$

$x\frac{\partial v}{\partial x} + y\frac{\partial v}{\partial y} = \frac{e^v}{e^v} = 1$ is proved.

....

6. If $w(x,y,z) = \log\left(\frac{5x^3y^4+7y^2xz^4-75y^3z^4}{x^2+y^2}\right)$, find $x\frac{\partial w}{\partial x} + y\frac{\partial w}{\partial y} + z\frac{\partial w}{\partial z}$

Solution:

$$w(x,y,z) = \log\left(\frac{5x^3y^4+7y^2xz^4-75y^3z^4}{x^2+y^2}\right)$$

$$e^w = \frac{5x^3y^4+7y^2xz^4-75y^3z^4}{x^2+y^2}$$

Let $f = e^w = \frac{5x^3y^4+7y^2xz^4-75y^3z^4}{x^2+y^2}$

It is a homogeneous function of degree 5.

By Euler's theorem, $x\frac{\partial f}{\partial x} + y\frac{\partial f}{\partial y} + z\frac{\partial f}{\partial z} = 5f$

But $f = e^w$

So, $x\frac{\partial e^w}{\partial x} + y\frac{\partial e^w}{\partial y} + z\frac{\partial e^w}{\partial z} = 5e^w$

$$xe^w\frac{\partial w}{\partial x} + ye^w\frac{\partial w}{\partial y} + ze^w\frac{\partial w}{\partial z} = 5e^w$$

$$e^w\left(x\frac{\partial w}{\partial x} + y\frac{\partial w}{\partial y} + z\frac{\partial w}{\partial z}\right) = 5e^w$$

$$\therefore x\frac{\partial w}{\partial x} + y\frac{\partial w}{\partial y} + z\frac{\partial w}{\partial z} = 5$$

....

EXERCISE 8.8

1. A circular template has a radius of 10 cm. The measurement of radius has an approximate error of 0.02 cm. Then the percentage error in calculating area of this template is

(1) 0.2% **(2) 0.4%**
(3) 0.04% (4) 0.08%

2. The percentage error of fifth root of 31 is approximately how many times the percentage error in 31?

(1) $\frac{1}{31}$ **(2) $\frac{1}{5}$** (3) 5 (4) 31

3. If $u(x,y) = e^{x^2+y^2}$, then $\frac{\partial u}{\partial x}$ is equal to

(1) $e^{x^2+y^2}$ **(2) $2xu$** (3) x^2u (4) y^2u

4. If $v(x,y) = \log(e^x + e^y)$, then $\frac{\partial v}{\partial x} + \frac{\partial v}{\partial y}$ is equal to

(1) $e^x + e^y$ (2) $\frac{1}{e^x+e^y}$ (3) 2 **(4) 1**

5. If $w(x,y) = x^y$, $x > 0$, then $\frac{\partial w}{\partial x}$ is equal to

(1) $x^y \log x$ (2) $y \log x$
(3) yx^{y-1} (4) $x \log y$

6. If $f(x,y) = e^{xy}$, then $\frac{\partial^2 f}{\partial x \partial y}$ is equal to

(1) xye^{xy} **(2) $(1 + xy)e^{xy}$**
(3) $(1 + y)e^{xy}$ (4) $(1 + x)e^{xy}$

7. If we measure the side of a cube to be 4 cm with an error of 0.1 cm, then the error in our calculation of the volume is

(1) 0.4 cu.cm (2) 0.45 cu.cm
(3) 2 cu.cm **(4) 4.8 cu.cm**

8. The change in the surface area $S = 6x^2$ of a cube when the edge length varies from x_0 to $x_0 + dx$ is

(1) $12x_0 + dx$ **(2) $12x_0dx$**
(3) $6x_0dx$ (4) $6x_0 + dx$

9. The approximate change in the volume V of a cube of side x metres caused by increasing the side by 1% is

(1) $0.3xdx\ m^3$ (2) $0.03x\ m^3$
(3) $0.03x^2\ m^3$ (4) $0.03x^3\ m^3$

10. If $g(x,y) = 3x^2 - 5y + 2y^2$, $x(t) = e^t$ and $y(t) = \cos t$, then $\frac{dg}{dt}$ is

(1) $6e^{2t} + 5\sin t - 4\cos t \sin t$
(2) $6e^{2t} - 5\sin t + 4\cos t \sin t$
(3) $3e^{2t} + 5\sin t + 4\cos t \sin t$
(4) $3e^{2t} - 5\sin t + 4\cos t \sin t$

11. If $f(x) = \frac{x}{x+1}$, then its differential is

(1) $\frac{-1}{(x+1)^2}\,dx$ **(2) $\frac{1}{(x+1)^2}\,dx$**
(3) $\frac{1}{x+1}dx$ (4) $\frac{-1}{x+1}dx$

12. If $u(x,y) = x^2 + 3xy + y - 2019$, then $\frac{\partial u}{\partial x}_{(4,-5)}$ is equal to

(1) -4 (2) -3 **(3) -7** (4) 13

13. Linear approximation for $g(x) = \cos x$

at $x = \frac{\pi}{2}$ is

(1) $x + \frac{\pi}{2}$ **(2) $-x + \frac{\pi}{2}$**

(3) $x - \frac{\pi}{2}$ (4) $-x - \frac{\pi}{2}$

14. If $(x, y, z) = x^2(y - z) + y^2(z - x) + z^2(x - y)$,

then $\frac{\partial w}{\partial x} + \frac{\partial w}{\partial y} + \frac{\partial w}{\partial z}$ is

(1) $xy + yz + zx$ (2) $x(y + z)$

(3) $y(z + x)$ **(4) 0**

15. If $f(x, y, z) = xy + yz + zx$,

then $f_x - f_z$ is equal to

(1) $z - x$ (2) $y - z$

(3) $x - z$ (4) $y - x$

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