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Chap 4 Inverse Trigonometric Functions

EX 4.1 Find the Principal value of $\sin^{-1}(-\frac{1}{2})$ (in radians and degrees)

Soln:

$$\text{Let } y = \sin^{-1}(-\frac{1}{2}) \quad \text{--- (1)}$$

$$\text{Then } \sin y = -\frac{1}{2}$$

$$\sin y = -\sin \frac{\pi}{6}$$

$$y = -\frac{\pi}{6} \in [-\frac{\pi}{2}, \frac{\pi}{2}]$$

$$\Rightarrow \sin^{-1}(-\frac{1}{2}) = -\frac{\pi}{6}$$

The Principal value of

$$\sin^{-1}(-\frac{1}{2}) \text{ is } -\frac{\pi}{6}$$

$$\text{and } \sin^{-1}(-\frac{1}{2}) \text{ is } -30^\circ$$

EX 4.2 Find the Principal

value of $\sin^{-1}(2)$, if it exists.

Soln: Let $y = \sin^{-1}x$

Since the domain of $y = \sin^{-1}x$ is $[-1, 1]$

and $2 \notin [-1, 1]$

$\therefore \sin^{-1}(2)$ does not exist

EX 4.3

Find the Principal value of (i) $\sin^{-1}(\frac{1}{\sqrt{2}})$

$$(ii) \sin^{-1}(\sin(-\frac{\pi}{3}))$$

$$(iii) \sin^{-1}(\sin(\frac{5\pi}{6}))$$

Soln:

wkt, $\sin^{-1}: [-1, 1] \rightarrow [-\frac{\pi}{2}, \frac{\pi}{2}]$ is given by

$$\text{Let } y = \sin^{-1}x \Leftrightarrow x = \sin y$$

$$\text{For } -1 \leq x \leq 1 \text{ and } -\frac{\pi}{2} \leq y \leq \frac{\pi}{2}$$

$$(i) \sin^{-1}(\frac{1}{\sqrt{2}}) = \frac{\pi}{4} \in [-\frac{\pi}{2}, \frac{\pi}{2}]$$

$$(ii) \sin^{-1}(\sin(-\frac{\pi}{3})) = -\frac{\pi}{3} \in [-\frac{\pi}{2}, \frac{\pi}{2}]$$

$$(iii) \sin^{-1}(\sin(\frac{5\pi}{6})) = \sin^{-1}(\sin(\pi - \frac{\pi}{6}))$$

$$= \sin^{-1}(\sin \frac{\pi}{6})$$

$$= \frac{\pi}{6} \in [-\frac{\pi}{2}, \frac{\pi}{2}]$$

EX 4.4

Find the domain of $\sin^{-1}(2-3x^2)$

Soln:

$$\text{Since } a \leq |x| \leq b$$

$$\Rightarrow [-b, -a] \cup [a, b]$$

wkt, the domain of

$$\sin^{-1}(x) \text{ is } [-1, 1]$$

$$\text{Given: } \sin^{-1}(2-3x^2)$$

$$\Rightarrow -1 \leq 2-3x^2 \leq 1$$

$$\Rightarrow -1-2 \leq -3x^2 \leq 1-2$$

$$\Rightarrow -3 \leq -3x^2 \leq -1$$

Multiply by -1 ,

$$3 \geq 3x^2 \geq 1$$

$$(ie) 1 \leq 3x^2 \leq 3$$

$$\div \text{ by } 3, \frac{1}{3} \leq x^2 \leq 1$$

$$\frac{1}{\sqrt{3}} \leq |x| \leq 1$$

$$x \in [-1, -\frac{1}{\sqrt{3}}] \cup [\frac{1}{\sqrt{3}}, 1]$$

EXERCISE 4.1

1) Find all the values of x such that

$$(i) -10\pi \leq x \leq 10\pi \text{ and } \sin x = 0$$

$$(ii) -8\pi \leq x \leq 8\pi \text{ and } \sin x = -1$$

Soln:

$$(i) \text{ Given: } \sin x = 0 \text{ and } -10\pi \leq x \leq 10\pi$$

$$\Rightarrow x = n\pi,$$

$$n = 0, \pm 1, \pm 2, \dots, \pm 10$$

(ii) Given: $-8\pi \leq x \leq 8\pi$ and $\sin x = -1$

$$\Rightarrow \sin x = -\sin \frac{\pi}{2}$$

$$\Rightarrow \sin x = \sin\left(-\frac{\pi}{2}\right)$$

$$\Rightarrow x = n\pi + (-1)^n \left(-\frac{\pi}{2}\right)$$

$$x = n\pi + (-1)^{n+1} \frac{\pi}{2},$$

$$n = 0, \pm 1, \pm 2, \pm 3, 4$$

$$x = -\frac{13\pi}{2}, -\frac{9\pi}{2}, -\frac{5\pi}{2}, -\frac{\pi}{2},$$

$$\frac{3\pi}{2}, \frac{7\pi}{2}, \frac{11\pi}{2}, \frac{15\pi}{2}$$

$$\therefore x = (4n-1)\frac{\pi}{2},$$

$$n = 0, \pm 1, \pm 2, \pm 3, 4$$

2) Find the period and amplitude of

(i) $y = \sin 7x$ (ii) $y = -\sin\left(\frac{1}{3}x\right)$

(iii) $y = 4\sin(-2x)$

Soln:

(i) Given: $y = \sin 7x$

Compare $y = a \sin bx$

Amplitude = $|a| = |1| = 1$

period = $\frac{2\pi}{|b|} = \frac{2\pi}{|7|} = \frac{2\pi}{7}$

$\therefore$ Amplitude = 1 and
period = $\frac{2\pi}{7}$

(ii) Given: $y = -\sin\left(\frac{1}{3}x\right)$

Compare $y = a \sin bx$

Amplitude = $|a| = |-1| = 1$

Period = $\frac{2\pi}{|b|} = \frac{2\pi}{|\frac{1}{3}|}$

$$= \frac{2\pi}{\frac{1}{3}} = 2\pi \times 3$$

$$\frac{1}{3} = 6\pi$$

$\therefore$ Amplitude = 1

Period = 6π

(iii) $y = 4\sin(-2x)$

Compare $y = a \sin bx$

Amplitude = $|a| = |4| = 4$

Period = $\frac{2\pi}{|b|} = \frac{2\pi}{|1-2|}$

$$= \frac{2\pi}{1} = \pi$$

$\therefore$ Amplitude = 4
period = π

3) Sketch the graph of $y = \sin\left(\frac{1}{3}x\right)$ for

$0 \leq x < 6\pi$

Soln:

Given: $y = \sin\left(\frac{1}{3}x\right)$,

$0 \leq x < 6\pi$

Put $x = 0$, $y = \sin 0 = 0$

$x = \pi$, $y = \sin \frac{\pi}{3} = \frac{\sqrt{3}}{2} = 0.8$

$x = 2\pi$, $y = \sin \frac{2\pi}{3} = \sin\left(\pi - \frac{\pi}{3}\right)$

$$= \sin \frac{\pi}{3} = \frac{\sqrt{3}}{2} = 0.8$$

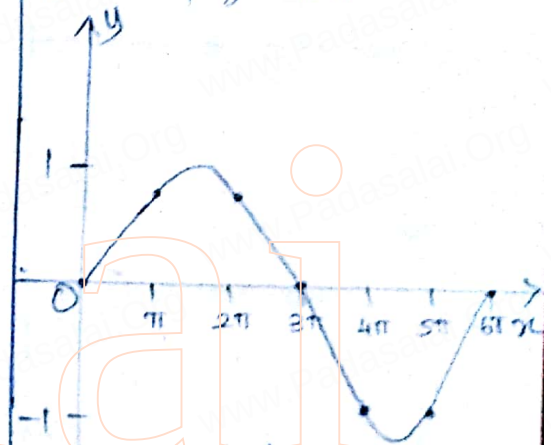
$x = 3\pi$, $y = \sin \pi = 0$

$x = 4\pi$, $y = \sin \frac{4\pi}{3} = \sin\left(\pi + \frac{\pi}{3}\right)$

$y = \sin\left(2\pi - \frac{\pi}{3}\right) = -\sin \frac{\pi}{3}$

$$= -\frac{\sqrt{3}}{2} = -0.8$$

$x = 6\pi$, $y = \sin 2\pi = 0$



$y = \sin\left(\frac{1}{3}x\right)$

Compare $y = a \sin bx$

Amplitude = $|a| = |1| = 1$

Period = $\frac{2\pi}{|b|} = \frac{2\pi}{|\frac{1}{3}|} = \frac{2\pi}{\frac{1}{3}}$

$$= 2\pi \times 3 = 6\pi$$

$\therefore$ Amplitude = 1, period = 6π

4) Find the value of

(i) $\sin^{-1}(\sin(\frac{2\pi}{3}))$

(ii) $\sin^{-1}(\sin(\frac{5\pi}{4}))$

Soln:

(i) $\sin^{-1}(\sin(\frac{2\pi}{3}))$

$= \sin^{-1}(\sin(\pi - \frac{\pi}{3}))$

$= \sin^{-1}(\sin \frac{\pi}{3}) = \frac{\pi}{3} \in [-\frac{\pi}{2}, \frac{\pi}{2}]$

(ii) $\sin^{-1}(\sin(\frac{5\pi}{4}))$

$= \sin^{-1}(\sin(\pi + \frac{\pi}{4}))$

$= \sin^{-1}(-\sin(\frac{\pi}{4}))$

$= \sin^{-1}(\sin(-\frac{\pi}{4}))$

$= -\frac{\pi}{4} \in [-\frac{\pi}{2}, \frac{\pi}{2}]$

5) For what value of x does $\sin x = \sin^{-1}x$?

Soln:

The given two curves intersect only at origin

$\therefore$ The given eqn & soln

is $\sin(0) = 0$ & $\sin^{-1}(0) = 0$

$\therefore \sin x = \sin^{-1}x = 0$

$\therefore x = 0$

6) Find the domain of the following

(i) $f(x) = \sin^{-1}(\frac{x^2+1}{2x})$

(ii) $g(x) = 2\sin^{-1}(2x-1) - \frac{\pi}{4}$

Soln:

(i) Given:

$f(x) = \sin^{-1}(\frac{x^2+1}{2x})$

WKT, the domain of

$\sin^{-1}x$ is $[-1, 1]$

$\therefore -1 \leq \frac{x^2+1}{2x} \leq 1$

Multiply by $2x^2$,
 $-2x^2 \leq x(x^2+1) \leq 2x^2$

$-2x^2 \leq x(x^2+1)$ and

$x(x^2+1) \leq 2x^2$

Let us consider the first inequality

$-2x^2 \leq x(x^2+1)$

$0 \leq x(x^2+1) + 2x^2$

$0 \leq x(x^2+1+2x)$

$0 \leq x(x+1)^2$

(ii) $x(x+1)^2 \geq 0$

$x \geq 0$ and $(x+1)^2 \geq 0$

$x+1 \geq 0$

$x \geq -1$

$\therefore x \in [0, \infty) \cup \{-1\}$

Let us consider the second inequality

$x(x^2+1) \leq 2x^2$

$x(x^2+1) - 2x^2 \leq 0$

$x(x^2+1-2x) \leq 0$

$x(x-1)^2 \leq 0$

$x \leq 0$ and $(x-1)^2 \leq 0$

$(x-1) \leq 0$

$x \leq 1$

$\therefore x \in (-\infty, 0] \cup \{1\}$

Now, we get

$x \in (\{1\} \cup [0, \infty)) \cap ((-\infty, 0] \cup \{1\})$

$\therefore [x \in \{-1, 1\}]$ function

(ii) Given:

$g(x) = 2\sin^{-1}(2x-1) - \frac{\pi}{4}$

WKT, the domain of

$\sin^{-1}x$ is $[-1, 1]$

$\therefore -1 \leq 2x-1 \leq 1$

Add 1, we get

$$0 \leq 2x \leq 2$$

$$\div 2, \text{ we get,}$$

$$0 \leq x \leq 1$$

$$\therefore x \in [0, 1]$$

The domain of the function is $[0, 1]$

1) Find the value of $\sin^{-1}(\sin \frac{5\pi}{9} \cos \frac{\pi}{9} + \cos \frac{5\pi}{9} \sin \frac{\pi}{9})$

Soln:

Given:

$$\sin^{-1}(\sin \frac{5\pi}{9} \cos \frac{\pi}{9} + \cos \frac{5\pi}{9} \sin \frac{\pi}{9})$$

$$= \sin^{-1}(\sin(\frac{5\pi}{9} + \frac{\pi}{9}))$$

$$= \sin^{-1}(\sin(\frac{6\pi}{9}))$$

$$= \sin^{-1}(\sin(\frac{2\pi}{3}))$$

$$= \sin^{-1}(\sin(\pi - \frac{\pi}{3}))$$

$$= \sin^{-1}(\sin \frac{\pi}{3}) = \frac{\pi}{3} \in [-\frac{\pi}{2}, \frac{\pi}{2}]$$

4.4 The Cosine Function and Inverse Cosine Function

Ex 4.5 Find the Principal value of $\cos^{-1}(\frac{\sqrt{3}}{2})$

Soln:

$$\text{Let } y = \cos^{-1}(\frac{\sqrt{3}}{2})$$

$$\text{Then } \cos y = \frac{\sqrt{3}}{2}$$

$$\cos y = \cos \frac{\pi}{6}$$

$$\therefore y = \frac{\pi}{6} \in [0, \pi]$$

$\therefore$ The Principal value of $\cos^{-1}(\frac{\sqrt{3}}{2})$ is $\frac{\pi}{6}$

Ex 4.6

Find (i) $\cos^{-1}(\frac{-1}{\sqrt{2}})$

$$(ii) \cos^{-1}(\cos(\frac{-\pi}{3}))$$

$$(iii) \cos^{-1}(\cos(\frac{\pi}{6}))$$

Soln:
Wkt, $\cos^{-1}: [-1, 1] \rightarrow [0, \pi]$
is given by

Let $y = \cos^{-1}x \Leftrightarrow x = \cos y$
for $-1 \leq x \leq 1$ and $0 \leq y \leq \pi$

$$(i) \cos^{-1}(\frac{-1}{\sqrt{2}})$$

$$\text{Let } y = \cos^{-1}(\frac{-1}{\sqrt{2}}) \text{--- ①}$$

$$\cos y = \frac{-1}{\sqrt{2}}$$

$$\cos y = \cos(\pi - \frac{\pi}{4})$$

$$\cos y = \cos(\frac{3\pi}{4})$$

$$y = \frac{3\pi}{4} \in [0, \pi]$$

$$\text{①} \Rightarrow \cos^{-1}(\frac{-1}{\sqrt{2}}) = \frac{3\pi}{4}$$

$$(ii) \cos^{-1}(\cos(\frac{-\pi}{3}))$$

$$\text{Let } = \cos^{-1}(\cos(\frac{\pi}{3}))$$

$$= \frac{\pi}{3} \in [0, \pi]$$

$$\text{but } -\frac{\pi}{3} \notin [0, \pi]$$

$$= \cos^{-1}(\cos(\frac{-\pi}{3})) = \frac{\pi}{3}$$

$$(iii) \cos^{-1}(\cos(\frac{\pi}{6}))$$

$$= \cos^{-1}(\cos(\pi + \frac{\pi}{6}))$$

$$= \cos^{-1}(-\cos \frac{\pi}{6})$$

$$= \cos^{-1}(-\frac{\sqrt{3}}{2})$$

$$\text{Let } y = \cos^{-1}(-\frac{\sqrt{3}}{2}) \text{--- ①}$$

$$\text{Then } \cos y = -\frac{\sqrt{3}}{2}$$

$$\cos y = \cos(\pi - \frac{\pi}{6})$$

$$\cos y = \cos(\frac{5\pi}{6})$$

$$y = \frac{5\pi}{6} \in [0, \pi]$$

$$\text{①} \Rightarrow \cos^{-1}(-\frac{\sqrt{3}}{2}) = \frac{5\pi}{6}$$

$$\therefore \cos^{-1}(\cos(\frac{\pi}{6})) = \frac{5\pi}{6}$$

Ex 4.7

Find the domain of $\cos^{-1}(\frac{2+\sin x}{3})$

$$\cos^{-1}(\frac{2+\sin x}{3})$$

Soln:

Let $y = \cos^{-1}x$, $-1 \leq x \leq 1$

Given: $\cos^{-1}\left(\frac{2+\sin x}{3}\right)$

$$\therefore -1 \leq \frac{2+\sin x}{3} \leq 1$$

Multiply by 3, we get,

$$-3 \leq 2+\sin x \leq 3$$

Add -2, we get,

$$-3-2 \leq \sin x \leq 3-2$$

$$-5 \leq \sin x \leq 1 \text{ reduces to } -1 \leq \sin x \leq 1$$

$$\Rightarrow -\sin^{-1}(1) \leq x \leq \sin^{-1}(1)$$

$$\Rightarrow -\frac{\pi}{2} \leq x \leq \frac{\pi}{2}$$

$\therefore$ The domain of $\cos^{-1}\left(\frac{2+\sin x}{3}\right)$ is $\left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$

EXERCISE 4.2

1) Find all values of x such that

(i) $-6\pi \leq x \leq 6\pi$ and

(ii) $-5\pi \leq x \leq 5\pi$

and $\cos x = 1$

Soln:

(i) Given: $\cos x = 0$,

$$x = (2n+1)\frac{\pi}{2}, n \in \mathbb{Z}$$

and Given: $-6\pi \leq x \leq 6\pi$

$$\text{So, } \cos x = 0, x = (2n+1)\frac{\pi}{2},$$

$$n = 0, \pm 1, \pm 2, \pm 3, \pm 4, \pm 5$$

(ii) Given: $\cos x = 1$,

$$\cos x = \cos 0$$

$$\text{Compare } \cos x = \cos \alpha,$$

$$x = 2n\pi \pm \alpha, n \in \mathbb{Z}$$

$$x = 2n\pi \pm 0, n \in \mathbb{Z}$$

$$\boxed{x = 2n\pi, n \in \mathbb{Z}}$$

and Given: $-5\pi \leq x \leq 5\pi$

So, $\cos x = 1$,

$$x = 2n\pi, n = 0, \pm 1, \pm 2$$

2) State the reason

$$\text{For } \cos^{-1}\left[\cos\left(-\frac{\pi}{6}\right)\right] \neq -\frac{\pi}{6}$$

Soln:

$$\cos^{-1}(\cos x) = x \text{ if } 0 \leq x \leq \pi$$

$$\text{Since } -\frac{\pi}{6} \notin [0, \pi]$$

$$\text{So, } \cos^{-1}\left[\cos\left(-\frac{\pi}{6}\right)\right] \neq -\frac{\pi}{6}$$

3) Is $\cos^{-1}(-x) = \pi - \cos^{-1}(x)$ true? Justify your answer

Soln:

$$\text{Let } \cos^{-1}(-x) = \alpha \quad \text{--- (1)}$$

$$-x = \cos \alpha$$

$$x = -\cos \alpha$$

$$x = \cos(\pi - \alpha)$$

$$\cos^{-1}(x) = \pi - \alpha$$

$$\boxed{\alpha = \pi - \cos^{-1}(x)}$$

$$\text{--- (1)} \Rightarrow \cos^{-1}(-x) = \pi - \cos^{-1}(x)$$

Yes, It is true

4) Find the Principal value of $\cos^{-1}\left(\frac{1}{2}\right)$

Soln:

$$\text{Let } y = \cos^{-1}\left(\frac{1}{2}\right)$$

$$\text{Then } \cos y = \frac{1}{2}$$

$$\cos y = \cos \frac{\pi}{3}$$

$$\therefore y = \frac{\pi}{3} \in [0, \pi]$$

$\therefore$ The Principal value of $\cos^{-1}\left(\frac{1}{2}\right)$ is $\frac{\pi}{3}$

5) Find the value of

$$(i) 2 \cos^{-1}\left(\frac{1}{2}\right) + \sin^{-1}\left(\frac{1}{2}\right)$$

$$(ii) \cos^{-1}\left(\frac{1}{2}\right) + \sin^{-1}(-1)$$

$$(iii) \cos^{-1}\left(\cos \frac{\pi}{7} \cos \frac{\pi}{17} - \sin \frac{\pi}{7} \sin \frac{\pi}{17}\right)$$

Soln:

$$(i) \text{ Let } y = \cos^{-1}\left(\frac{1}{2}\right) \quad \text{--- (1)}$$

$$\text{Then } \cos y = \frac{1}{2}$$

$$\cos y = \cos \frac{\pi}{3}$$

$$\textcircled{1} \Rightarrow \cos^{-1}\left(\frac{1}{2}\right) = \frac{\pi}{3}$$

Also, Let $y = \sin^{-1}\left(\frac{1}{2}\right)$

Then $\sin y = \frac{1}{2}$ $\textcircled{2}$

$$\sin y = \sin \frac{\pi}{6}$$

$$y = \frac{\pi}{6} \in \left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$$

$$\textcircled{2} \Rightarrow y = \sin^{-1}\left(\frac{1}{2}\right)$$

$$\therefore \sin^{-1}\left(\frac{1}{2}\right) = \frac{\pi}{6}$$

$$2\cos^{-1}\left(\frac{1}{2}\right) + \sin^{-1}\left(\frac{1}{2}\right)$$

$$= 2\left(\frac{\pi}{3}\right) + \frac{\pi}{6}$$

$$= \frac{4\pi + \pi}{6} = \frac{5\pi}{6}$$

$$\therefore 2\cos^{-1}\left(\frac{1}{2}\right) + \sin^{-1}\left(\frac{1}{2}\right) = \frac{5\pi}{6}$$

(ii) Let $y = \cos^{-1}\left(\frac{1}{2}\right)$

Then $\cos y = \frac{1}{2}$ $\textcircled{1}$

$$y = \frac{\pi}{3} \in [0, \pi]$$

$$\textcircled{1} \Rightarrow \cos^{-1}\left(\frac{1}{2}\right) = \frac{\pi}{3}$$

Also, Let $y = \sin^{-1}(-1)$

$\sin y = -1$ $\textcircled{2}$

$$\sin y = -\sin \frac{\pi}{2}$$

$$\sin y = \sin\left(-\frac{\pi}{2}\right)$$

$$y = -\frac{\pi}{2} \in \left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$$

$$\textcircled{2} \Rightarrow \sin^{-1}(-1) = -\frac{\pi}{2}$$

$$\cos^{-1}\left(\frac{1}{2}\right) + \sin^{-1}(-1)$$

$$= \frac{\pi}{3} - \frac{\pi}{2}$$

$$= \frac{2\pi - 3\pi}{6}$$

$$= -\frac{\pi}{6}$$

$$\therefore \cos^{-1}\left(\frac{1}{2}\right) + \sin^{-1}(-1) = -\frac{\pi}{6}$$

$$= \cos^{-1}\left(\cos\left(\frac{\pi}{7} + \frac{\pi}{7}\right)\right)$$

$$= \cos^{-1}\left(\cos\left(\frac{17\pi + \pi}{119}\right)\right)$$

$$= \cos^{-1}\left(\cos\left(-\frac{24\pi}{119}\right)\right)$$

$$= \frac{24\pi}{119}$$

6) Find the domain of

(i) $f(x) = \sin^{-1}\left(\frac{|x|-2}{3}\right)$

$$+ \cos^{-1}\left(\frac{1-|x|}{4}\right)$$

Soln:

Let $y = \sin^{-1}x$, $-1 \leq x \leq 1$

Given: $\sin^{-1}\left(\frac{|x|-2}{3}\right)$

$$-1 \leq \frac{|x|-2}{3} \leq 1$$

Multiply by 3, we get,

$$-3 \leq |x|-2 \leq 3$$

Add 2, we get

...

$$-1 \leq |x| \leq 5$$

But $-1 \leq |x|$ is not possible

$$\therefore |x| \leq 5$$

Also, Let $y = \cos^{-1}x$, $-1 \leq x \leq 1$

Given: $\cos^{-1}\left(\frac{1-|x|}{4}\right)$

$$-1 \leq \frac{1-|x|}{4} \leq 1$$

Multiply by 4, we get,

$$-4 \leq 1-|x| \leq 4$$

Add -1, we get,

$$-4-1 \leq -|x| \leq 4-1$$

$$-5 \leq -|x| \leq 3$$

Multiply by (-1), we get

$$5 \geq |x| \geq -3$$

(i.e) $-3 \leq |x| \leq 5$

But $-3 \leq |x|$ is not possible,

$$\therefore |x| \leq 5$$

∴ The domain of given $f(x)$ is $[-5, 5]$

(c) $-5 \leq x \leq 5$

∴ The domain is $[-5, 5]$

(ii) $g(x) = \sin^{-1}x + \cos^{-1}x$

Soln:

Wkt, The domain of $\sin^{-1}x$ and $\cos^{-1}x$ is $[-1, 1]$

∴ The domain of the given function is $[-1, 1]$

7) For what value of x , the inequality

$\frac{\pi}{2} < \cos^{-1}(3x-1) < \pi$ holds?

Soln:

Given: $\frac{\pi}{2} < \cos^{-1}(3x-1) < \pi$

This inequality lies in II quadrant

Wkt, $\cos^{-1}x$ give values in II quadrant, if

$-1 < x < 0$. This

leads to

$-1 < 3x-1 < 0$, Add 1, negat

$-1+1 < 3x < 1$

$0 < 3x < 1$

$\div 3$, we get,

$0 < x < \frac{1}{3}$

∴ $x \in (0, \frac{1}{3})$

8) Find the value of

(i) $\cos\left(\cos^{-1}\left(\frac{4}{5}\right) + \sin^{-1}\left(\frac{4}{5}\right)\right)$

(ii) $\cos^{-1}\left(\cos\left(\frac{4\pi}{3}\right)\right) + \cos^{-1}\left(\cos\left(\frac{5\pi}{4}\right)\right)$

Soln:

(i) Let $y = \sin^{-1}\left(\frac{4}{5}\right)$

Then $\sin y = \frac{4}{5}$

$\cos\left(\frac{\pi}{2} - y\right) = \frac{4}{5}$

$\frac{\pi}{2} - y = \cos^{-1}\left(\frac{4}{5}\right)$

$\frac{\pi}{2} = y + \cos^{-1}\left(\frac{4}{5}\right)$

$\frac{\pi}{2} = \sin^{-1}\left(\frac{4}{5}\right) + \cos^{-1}\left(\frac{4}{5}\right)$

∴ Given:

$\cos\left(\cos^{-1}\left(\frac{4}{5}\right) + \sin^{-1}\left(\frac{4}{5}\right)\right)$

$= \cos\frac{\pi}{2} = 0$

∴ $\cos\left(\cos^{-1}\left(\frac{4}{5}\right) + \sin^{-1}\left(\frac{4}{5}\right)\right) = 0$

(ii) $\cos^{-1}\left(\cos\left(\frac{4\pi}{3}\right)\right) + \cos^{-1}\left(\cos\left(\frac{5\pi}{4}\right)\right)$

$= \cos^{-1}\left(\cos\left(\pi + \frac{\pi}{3}\right)\right) + \cos^{-1}\left(\cos\left(\pi + \frac{\pi}{4}\right)\right)$

$= \cos^{-1}\left(-\cos\left(\frac{\pi}{3}\right)\right) + \cos^{-1}\left(-\cos\left(\frac{\pi}{4}\right)\right)$

$= \cos^{-1}\left(\cos\left(\pi - \frac{\pi}{3}\right)\right) + \cos^{-1}\left(\cos\left(\pi - \frac{\pi}{4}\right)\right)$

$= \cos^{-1}\left(\cos\left(\frac{2\pi}{3}\right)\right) + \cos^{-1}\left(\cos\left(\frac{3\pi}{4}\right)\right)$

$= \frac{2\pi}{3} + \frac{3\pi}{4}$

$= \frac{8\pi + 9\pi}{12} = \frac{17\pi}{12}$

4.5 The Tangent Function and the Inverse Tangent Function

value of $\tan^{-1}(\sqrt{3})$
Soln:

Let $y = \tan^{-1}(\sqrt{3})$

Then $\tan y = \sqrt{3}$

$\tan y = \tan \frac{\pi}{3}$

$y = \frac{\pi}{3} \in (-\frac{\pi}{2}, \frac{\pi}{2})$

$\therefore \tan^{-1}(\sqrt{3}) = \frac{\pi}{3}$

Ex 4.9

Find (i) $\tan^{-1}(-\sqrt{3})$

ii) $\tan^{-1}(\tan \frac{3\pi}{5})$

iii) $\tan(\tan^{-1}(2019))$

Soln:

(i) Let $y = \tan^{-1}(-\sqrt{3})$

Then $\tan y = -\sqrt{3}$

$\tan y = -\tan \frac{\pi}{3}$

$y = -\frac{\pi}{3} \in (-\frac{\pi}{2}, \frac{\pi}{2})$

$\therefore \tan^{-1}(-\sqrt{3}) = -\frac{\pi}{3}$

(ii) $\tan^{-1}(\tan \frac{3\pi}{5})$

Find $\theta \in (-\frac{\pi}{2}, \frac{\pi}{2})$

$\therefore \tan \theta = \tan \frac{3\pi}{5}$

$\tan \frac{3\pi}{5} = \tan(\frac{3\pi}{5} - \pi)$
 $= \tan(-\frac{2\pi}{5})$

$\tan \frac{3\pi}{5} = \tan(-\frac{2\pi}{5})$

$\tan^{-1}(\tan \frac{3\pi}{5})$

$= \tan^{-1}(\tan(-\frac{2\pi}{5}))$

$= -\frac{2\pi}{5} \in (-\frac{\pi}{2}, \frac{\pi}{2})$

$x \in \mathbb{R}$

Given: $\tan(\tan^{-1}(2019))$
 $= 2019$

Ex 4.10

Find the value of

$\tan^{-1}(-1) + \cos^{-1}(\frac{1}{2}) + \sin^{-1}(-\frac{1}{2})$

Soln:

Let $y = \tan^{-1}(-1)$

Then $\tan y = -1$

$\tan y = -\tan \frac{\pi}{4}$

$\tan y = \tan(-\frac{\pi}{4})$

$y = -\frac{\pi}{4} \in (-\frac{\pi}{2}, \frac{\pi}{2})$

$\therefore \tan^{-1}(-1) = -\frac{\pi}{4}$

Let $y = \cos^{-1}(\frac{1}{2})$

Then $\cos y = \frac{1}{2}$

$y = \frac{\pi}{3} \in [0, \pi]$

$\therefore \cos^{-1}(\frac{1}{2}) = \frac{\pi}{3}$

Let $y = \sin^{-1}(-\frac{1}{2})$

Then $\sin y = -\frac{1}{2}$

$\sin y = -\sin \frac{\pi}{6}$

$\sin y = \sin(-\frac{\pi}{6})$

$y = -\frac{\pi}{6} \in [-\frac{\pi}{2}, \frac{\pi}{2}]$

$\therefore \sin^{-1}(-\frac{1}{2}) = -\frac{\pi}{6}$

$\therefore \tan^{-1}(-1) + \cos^{-1}(\frac{1}{2}) +$

$\sin^{-1}(-\frac{1}{2}) = -\frac{\pi}{4} + \frac{\pi}{3} - \frac{\pi}{6}$

$= \frac{-3\pi + 4\pi - 2\pi}{12}$

$= -\frac{\pi}{12}$

EX 4.11

Prove that

$$\tan(\sin^{-1}x) = \frac{x}{\sqrt{1-x^2}},$$

$$-1 < x < 1$$

Soln: Given: $\tan(\sin^{-1}x) = \frac{x}{\sqrt{1-x^2}}$

If $x=0$ then both sides are equal to 0 ——— ①

Assume that $-1 < x < 0$,

Then $\theta = \sin^{-1}x$, $-\frac{\pi}{2} < \theta < 0$

$$\sin \theta = x \Rightarrow \sin \theta = \frac{x}{1}$$

$$\Rightarrow \tan \theta = \frac{\sin \theta}{\cos \theta} = \frac{\sin \theta}{\sqrt{1-\sin^2 \theta}}$$

$$= \frac{\sin \theta}{\sqrt{1-x^2}} = \frac{x}{\sqrt{1-x^2}}$$

$$\therefore \tan \theta = \frac{x}{\sqrt{1-x^2}}$$

$$\tan(\sin^{-1}x) = \frac{x}{\sqrt{1-x^2}} \quad \text{--- ②}$$

Assume that $0 < x < 1$

Let $\theta = \sin^{-1}x$ Then $0 < \theta < \frac{\pi}{2}$

$$\sin \theta = x \Rightarrow \sin \theta = \frac{x}{1}$$

$$\Rightarrow \tan \theta = \frac{\sin \theta}{\cos \theta}$$

$$= \frac{\sin \theta}{\sqrt{1-\sin^2 \theta}} = \frac{\sin \theta}{\sqrt{1-x^2}}$$

$$\tan \theta = \frac{x}{\sqrt{1-x^2}} \quad \text{--- ③}$$

$$\tan(\sin^{-1}x) = \frac{x}{\sqrt{1-x^2}}$$

From ①, ② and ③

we get

$$\tan(\sin^{-1}x) = \frac{x}{\sqrt{1-x^2}},$$

$$-1 < x < 1$$

EXERCISE 4.3

1) Find the domain

of the following functions:

(i) $\tan^{-1}(\sqrt{9-x^2})$

(ii) $\frac{1}{2} \tan^{-1}(1-x^2) - \frac{\pi}{4}$

Soln:

(i) Let $y = \tan^{-1}(\sqrt{9-x^2})$

Wkt, The domain of $\tan^{-1}x$ is $(-\infty, \infty)$

$\therefore \sqrt{9-x^2}$ is must be a real.

$$9-x^2 \geq 0 \Rightarrow 9 \geq x^2$$

$$(ii) x^2 \leq 9$$

$$\Rightarrow -3 \leq x \leq 3$$

$$\therefore x \in [-3, 3]$$

The domain of the given function is $[-3, 3]$

(ii) Let $y = \frac{1}{2} \tan^{-1}(1-x^2) - \frac{\pi}{4}$

Wkt, The domain of $\tan^{-1}x$ is $(-\infty, \infty)$

$\therefore 1-x^2$ is must be a real for all values of x .

$$(ii) x \in (-\infty, \infty)$$

$$\therefore x \in \mathbb{R}$$

The domain of the given function is $\mathbb{R}$

2) Find the value of

(i) $\tan^{-1}(\tan \frac{5\pi}{4})$

(ii) $\tan^{-1}(\tan(\frac{-\pi}{6}))$

Soln: Wkt, $\tan^{-1}(\tan x) = x$

(i) $\tan^{-1}(\tan \frac{5\pi}{4})$

$= \tan^{-1}(\tan(\pi + \frac{\pi}{4}))$

$$= \frac{\pi}{4} \in \left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$$

(ii) $\tan^{-1}\left(\tan\left(-\frac{\pi}{6}\right)\right)$ wkt $\tan^{-1}(\tan x) = x$

$$= -\frac{\pi}{6} \in \left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$$

3) Find the value of (i) $\tan\left(\tan^{-1}\left(\frac{7\pi}{4}\right)\right)$

(ii) $\tan\left(\tan^{-1}(1947)\right)$

(iii) $\tan\left(\tan^{-1}(-0.2021)\right)$

Soln: wkt, $\tan(\tan^{-1}x) = x, x \in \mathbb{R}$

(i) $\tan\left(\tan^{-1}\left(\frac{7\pi}{4}\right)\right)$

$$= \frac{7\pi}{4}$$

(ii) wkt, $\tan(\tan^{-1}x) = x, x \in \mathbb{R}$

$$\tan\left(\tan^{-1}(1947)\right) = 1947$$

(iii) $\tan\left(\tan^{-1}(-0.2021)\right)$

$$= -0.2021$$

4) Find the value of

(i) $\tan\left(\cos^{-1}\left(\frac{1}{2}\right) - \sin^{-1}\left(-\frac{1}{2}\right)\right)$

(ii) $\sin\left(\tan^{-1}\left(\frac{1}{2}\right) - \cos^{-1}\left(\frac{4}{5}\right)\right)$

(iii) $\cos\left(\sin^{-1}\left(\frac{4}{5}\right) - \tan^{-1}\left(\frac{3}{4}\right)\right)$

Soln:

(i) Let $y = \cos^{-1}\left(\frac{1}{2}\right)$

Then $\cos y = \frac{1}{2}$

$$\cos y = \cos \frac{\pi}{3}$$

$$y = \frac{\pi}{3} \in [0, \pi]$$

$\therefore \cos^{-1}\left(\frac{1}{2}\right) = \frac{\pi}{3}$

Also Let $y = \sin^{-1}\left(-\frac{1}{2}\right)$

Then $\sin y = -\frac{1}{2}$

$\sin y = \sin\left(-\frac{\pi}{6}\right)$

$$y = -\frac{\pi}{6} \in \left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$$

$\therefore \sin^{-1}\left(-\frac{1}{2}\right) = -\frac{\pi}{6}$

$\therefore \tan\left(\cos^{-1}\left(\frac{1}{2}\right) - \sin^{-1}\left(-\frac{1}{2}\right)\right)$

$$= \tan\left(\frac{\pi}{3} - \left(-\frac{\pi}{6}\right)\right)$$

$$= \tan\left(\frac{\pi}{3} + \frac{\pi}{6}\right)$$

$$= \tan\left(\frac{2\pi + \pi}{6}\right)$$

$$= \tan\left(\frac{3\pi}{6}\right)$$

$$= \tan \frac{\pi}{2}$$

$$= \infty$$

(ii) $\sin\left(\tan^{-1}\left(\frac{1}{2}\right) - \cos^{-1}\left(\frac{4}{5}\right)\right)$

Soln:

Let $\alpha = \tan^{-1}\left(\frac{1}{2}\right)$

α lies in I & III quadrant

Let $\beta = \cos^{-1}\left(\frac{4}{5}\right)$

$\cos \beta = \frac{4}{5} > 0$

$\therefore \beta$ lies in I & IV quadrant

$\sin\left(\tan^{-1}\left(\frac{1}{2}\right) - \cos^{-1}\left(\frac{4}{5}\right)\right)$

$$= \sin(\alpha - \beta)$$

$$= \sin \alpha \cos \beta - \cos \alpha \sin \beta$$

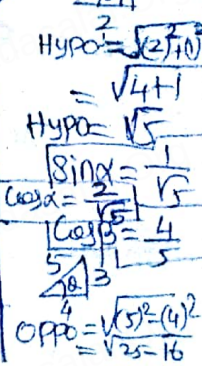
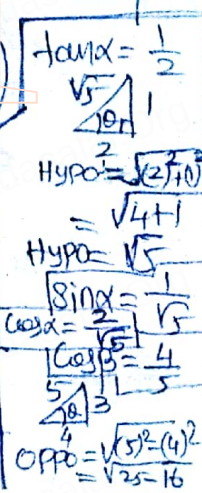
$$= \left(\frac{1}{\sqrt{5}}\right)\left(\frac{4}{5}\right) - \left(\frac{2}{\sqrt{5}}\right)\left(\frac{3}{5}\right)$$

$$= \frac{4}{5\sqrt{5}} - \frac{6}{5\sqrt{5}}$$

$$= \frac{-2}{5\sqrt{5}}$$

$$= \frac{-2}{5\sqrt{5}} \times \frac{\sqrt{5}}{\sqrt{5}} = \frac{-2\sqrt{5}}{25}$$

$$= \frac{-2\sqrt{5}}{25}$$



$$(iii) \cos(\sin^{-1}(\frac{4}{5}) - \tan^{-1}(\frac{3}{4}))$$

Soln:

$$\text{Let } \alpha = \sin^{-1}(\frac{4}{5})$$

$$\sin \alpha = \frac{4}{5} > 0$$

α lies in the I & II Quadrant

$$\text{Let } \beta = \tan^{-1}(\frac{3}{4})$$

$$\tan \beta = \frac{3}{4} > 0$$

$$\text{lies in I & III Quadrant}$$

$$\cos(\sin^{-1}(\frac{4}{5}) - \tan^{-1}(\frac{3}{4}))$$

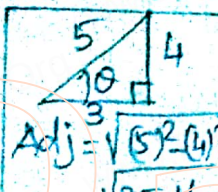
$$= \cos(\alpha - \beta)$$

$$= \cos \alpha \cos \beta + \sin \alpha \sin \beta$$

$$= (\frac{3}{5})(\frac{4}{5}) + (\frac{4}{5})(\frac{3}{5})$$

$$= \frac{12}{25} + \frac{12}{25}$$

$$= \frac{24}{25}$$

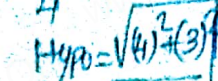


$$\text{Adj} = \sqrt{5^2 - 4^2}$$

$$= \sqrt{25 - 16}$$

$$= \sqrt{9} = 3$$

$$\cos \alpha = \frac{3}{5}$$



$$\text{Hyp} = \sqrt{4^2 + 3^2}$$

$$= \sqrt{16 + 9}$$

$$= \sqrt{25}$$

$$\text{Hyp} = 5$$

$$\cos \beta = \frac{4}{5}$$

$$\sin \beta = \frac{3}{5}$$

4.6 The Cosecant Function and the Inverse Cosecant Function

EX 4.12

Find the Principal value of (i) $\csc^{-1}(-1)$

(ii) $\csc^{-1}(-2)$

Soln:

(i) Let $y = \csc^{-1}(-1)$

Then $\csc y = -1$

The range of Principal value of $y = \csc^{-1}x$ is $[-\frac{\pi}{2}, \frac{\pi}{2}] \setminus \{0\}$

$\csc y = -\csc \frac{\pi}{2}$

$\csc y = \csc(-\frac{\pi}{2})$

$\therefore y = -\frac{\pi}{2} \in [-\frac{\pi}{2}, \frac{\pi}{2}] \setminus \{0\}$

$\therefore \csc^{-1}(-1) = -\frac{\pi}{2}$

The Principal value of $\csc^{-1}(-1)$ is $-\frac{\pi}{2}$

(ii) Let $y = \csc^{-1}(-2)$

Then $\csc y = -2$

(i) Let $y = \csc^{-1}(-2)$

Then $\csc y = -2$, The range of principal value of $y = \csc^{-1}x$ is $[-\frac{\pi}{2}, \frac{\pi}{2}] \setminus \{0\}$

$\frac{1}{\cos y} = -2$

$\cos y = -\frac{1}{2} \Rightarrow \cos y = -\cos \frac{\pi}{3}$

$\cos y = \cos(\pi - \frac{\pi}{3})$

$\cos y = \cos \frac{2\pi}{3}$

$y = \frac{2\pi}{3} \in [0, \pi] \setminus \{\frac{\pi}{2}\}$

$\therefore \csc^{-1}(-2) = \frac{2\pi}{3}$

$\therefore$ The Principal value of $\csc^{-1}(-2)$ is $\frac{2\pi}{3}$

EX 4.13

Find the value of $\csc^{-1}(-\frac{2\sqrt{3}}{3})$

Soln: Let $y = \csc^{-1}(-\frac{2\sqrt{3}}{3})$

Then $\csc y = -\frac{2\sqrt{3}}{3}$

$\frac{1}{\sin y} = -\frac{2\sqrt{3}}{3}$

$\sin y = -\frac{\sqrt{3}}{2}$

$\sin y = -\sin \frac{\pi}{3}$

$\sin y = \sin(\pi - \frac{\pi}{3})$

$\sin y = \sin \frac{5\pi}{6}$

Then $\csc y = -\frac{2\sqrt{3}}{3}$

$\csc y = -\frac{2}{\sqrt{3}}$

$\frac{1}{\sin y} = -\frac{2}{\sqrt{3}}$

$\sin y = -\frac{\sqrt{3}}{2}$

$\Rightarrow \csc y = -\csc \frac{\pi}{3}$

$\csc y = \csc(\pi - \frac{\pi}{3})$

$\csc y = \csc \frac{5\pi}{6}$

$y = \frac{5\pi}{6} \in [0, \pi] \setminus \{\frac{\pi}{2}\}$

$\therefore \csc^{-1}(-\frac{2\sqrt{3}}{3}) = \frac{5\pi}{6}$

$\therefore$ The value of $\csc^{-1}(-\frac{2\sqrt{3}}{3})$ is $\frac{5\pi}{6}$

EX 4.14

If $\cot^{-1}(\frac{1}{7}) = \theta$, find the value of $\cos \theta$

$$\text{Let } \theta = \cot^{-1}(7)$$

$$\Rightarrow \theta \in (0, \pi)$$

$$\cot \theta = \frac{1}{7}$$

$$\frac{1}{\tan \theta} = \frac{1}{7}$$

$$\tan \theta = 7$$

$$\Rightarrow \tan \theta = \frac{7}{1}$$

From the diagram,

$$\cos \theta = \frac{1}{5\sqrt{2}}$$

$\therefore$ The value of $\cos \theta$ is $\frac{1}{5\sqrt{2}}$

EX 4.15

Show that

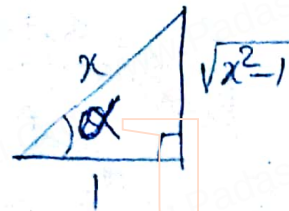
$$\cot^{-1}\left(\frac{1}{\sqrt{x^2-1}}\right) = \sec^{-1}x, \quad |x| > 1$$

$$\text{Let } \cot^{-1}\left(\frac{1}{\sqrt{x^2-1}}\right) = \alpha$$

$$\text{Then } \cot \alpha = \frac{1}{\sqrt{x^2-1}}$$

$$\frac{1}{\tan \alpha} = \frac{1}{\sqrt{x^2-1}}$$

$$\Rightarrow \tan \alpha = \sqrt{x^2-1}$$



$$\Rightarrow \tan \alpha = \frac{\sqrt{x^2-1}}{1}$$

From the diagram,

$$\sec \alpha = \frac{x}{1}$$

$$\sec \alpha = x$$

$$\alpha = \sec^{-1}(x)$$

$$\therefore \cot^{-1}\left(\frac{1}{\sqrt{x^2-1}}\right) = \sec^{-1}(x)$$

$$\therefore \cot^{-1}\left(\frac{1}{\sqrt{x^2-1}}\right) = \sec^{-1}x, \quad |x| > 1$$

1) Find the Principal value of

$$(i) \sec^{-1}\left(\frac{2}{\sqrt{3}}\right) \quad (ii) \cot^{-1}(\sqrt{3})$$

$$(iii) \operatorname{cosec}^{-1}(-\sqrt{2})$$

Soln:

$$(i) \text{ Let } y = \sec^{-1}\left(\frac{2}{\sqrt{3}}\right)$$

Then $\sec y = \frac{2}{\sqrt{3}}$, The range of Principal value of $y = \sec^{-1}x$ is $[0, \pi] \setminus \{\pi/2\}$

$$\frac{1}{\cos y} = \frac{2}{\sqrt{3}}$$

$$\Rightarrow \cos y = \frac{\sqrt{3}}{2}$$

$$\cos y = \cos \frac{\pi}{6}$$

$$y = \frac{\pi}{6} \in [0, \pi] \setminus \{\pi/2\}$$

$$\sec^{-1}\left(\frac{2}{\sqrt{3}}\right) = \frac{\pi}{6}$$

$\therefore$ The Principal value of $\sec^{-1}\left(\frac{2}{\sqrt{3}}\right)$ is $\frac{\pi}{6}$

$$(ii) \text{ Let } y = \cot^{-1}(\sqrt{3})$$

Then $\cot y = \sqrt{3}$, The range of Principal value of $y = \cot^{-1}x$ is $(0, \pi)$

$$\frac{1}{\tan y} = \sqrt{3}$$

$$\Rightarrow \tan y = \frac{1}{\sqrt{3}}$$

$$\tan y = \tan \frac{\pi}{6}$$

$$y = \frac{\pi}{6}$$

$$\therefore \cot^{-1}(\sqrt{3}) = \frac{\pi}{6}$$

$\therefore$ The Principal value of $\cot^{-1}(\sqrt{3})$ is $\frac{\pi}{6}$

Soln:

$$\text{Let } y = \cot^{-1}(\sqrt{3})$$

Then $\cot y = \sqrt{3}$, The range of Principal value of $y = \cot^{-1}x$ is $(0, \pi)$

$$\frac{1}{\tan y} = \sqrt{3}$$

$$\Rightarrow \tan y = \frac{1}{\sqrt{3}}$$

$$\tan y = \tan \frac{\pi}{6}$$

$$y = \frac{\pi}{6}$$

$$\therefore \cot^{-1}(\sqrt{3}) = \frac{\pi}{6}$$

$\therefore$ The Principal value of $\cot^{-1}(\sqrt{3})$ is $\frac{\pi}{6}$

$$(iii) \text{ Let } y = \operatorname{cosec}^{-1}(-\sqrt{2})$$

Then $\operatorname{cosec} y = -\sqrt{2}$

$$\frac{1}{\sin y} = -\sqrt{2}$$

$$\Rightarrow \sin y = -\frac{1}{\sqrt{2}}$$

$$\sin y = -\sin \frac{\pi}{4}$$

$$y = -\frac{\pi}{4}$$

$$\therefore \operatorname{cosec}^{-1}(-\sqrt{2}) = -\frac{\pi}{4}$$

$\therefore$ The Principal value of $\operatorname{cosec}^{-1}(-\sqrt{2})$ is $-\frac{\pi}{4}$

$$(iv) \text{ Let } y = \tan^{-1}\left(\frac{1}{\sqrt{3}}\right)$$

Then $\tan y = \frac{1}{\sqrt{3}}$, The range of Principal value of $y = \tan^{-1}x$ is $(-\frac{\pi}{2}, \frac{\pi}{2})$

$$\frac{1}{\tan y} = \sqrt{3}$$

The range of Principal value branch of

$$y = \operatorname{cosec}^{-1} x \text{ is } \left[-\frac{\pi}{2}, \frac{\pi}{2}\right] \setminus \{0\}$$

$$\frac{1}{\sin y} = -\sqrt{2}$$

$$\Rightarrow \sin y = -\frac{1}{\sqrt{2}}$$

$$\sin y = -\sin \frac{\pi}{4}$$

$$\sin y = \sin\left(-\frac{\pi}{4}\right)$$

$$y = -\frac{\pi}{4} \in \left[-\frac{\pi}{2}, \frac{\pi}{2}\right] \setminus \{0\}$$

$$\therefore \operatorname{cosec}^{-1}(-\sqrt{2}) = -\frac{\pi}{4}$$

$\therefore$ The Principal value of $\operatorname{cosec}^{-1}(-\sqrt{2})$ is $-\frac{\pi}{4}$

2) Find the value of
(i) $\tan^{-1}(\sqrt{3}) - \sec^{-1}(-2)$

$$(ii) \sin^{-1}(-1) + \cos^{-1}\left(\frac{1}{2}\right) + \cot^{-1}(2)$$

$$(iii) \cot^{-1}(1) + \sin^{-1}\left(-\frac{\sqrt{3}}{2}\right) - \sec^{-1}(-\sqrt{2})$$

Soln:

$$(i) \text{ Let } \alpha = \tan^{-1}(\sqrt{3})$$

$$\text{Then } \tan \alpha = \sqrt{3}$$

$$\tan \alpha = \tan \frac{\pi}{3}$$

$$\alpha = \frac{\pi}{3} \in \left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$$

$$\boxed{\tan^{-1}(\sqrt{3}) = \frac{\pi}{3}}$$

$$\text{Let } \beta = \sec^{-1}(-2)$$

$$\text{Then } \sec \beta = -2$$

$$\frac{1}{\cos \beta} = -2$$

$$\Rightarrow \cos \beta = -\frac{1}{2}$$

$$\cos \beta = -\cos \frac{\pi}{3}$$

$$\cos \beta = \cos\left(\pi - \frac{\pi}{3}\right)$$

$$\cos \beta = \cos \frac{2\pi}{3}$$

$$\beta = \frac{2\pi}{3} \in [0, \pi] \setminus \left\{\frac{\pi}{2}\right\}$$

$$\therefore \boxed{\sec^{-1}(-2) = \frac{2\pi}{3}}$$

$$\tan^{-1}(\sqrt{3}) - \sec^{-1}(-2)$$

$$= \frac{\pi}{3} - \frac{2\pi}{3} = -\frac{\pi}{3}$$

$$\therefore \tan^{-1}(\sqrt{3}) - \sec^{-1}(-2) = -\frac{\pi}{3}$$

$$(ii) \text{ Let } \alpha = \sin^{-1}(-1)$$

$$\text{Then } \sin \alpha = -1$$

$$\sin \alpha = -\sin \frac{\pi}{2}$$

$$\sin \alpha = \sin\left(-\frac{\pi}{2}\right)$$

$$\alpha = -\frac{\pi}{2} \in \left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$$

$$\boxed{\sin^{-1}(-1) = -\frac{\pi}{2}}$$

$$\text{Let } \beta = \cos^{-1}\left(\frac{1}{2}\right)$$

$$\text{Then } \cos \beta = \frac{1}{2}$$

$$\cos \beta = \cos \frac{\pi}{3}$$

$$\beta = \frac{\pi}{3} \in [0, \pi]$$

$$\boxed{\cos^{-1}\left(\frac{1}{2}\right) = \frac{\pi}{3}}$$

$$\sin^{-1}(-1) + \cos^{-1}\left(\frac{1}{2}\right) + \cot^{-1}(2)$$

$$= -\frac{\pi}{2} + \frac{\pi}{3} + \cot^{-1}(2)$$

$$= \frac{-3\pi + 2\pi}{6} + \cot^{-1}(2)$$

$$= -\frac{\pi}{6} + \cot^{-1}(2) = \cot^{-1}(2) - \frac{\pi}{6}$$

$$\therefore \sin^{-1}(-1) + \cos^{-1}\left(\frac{1}{2}\right)$$

$$+ \cot^{-1}(2) = \cot^{-1}(2) - \frac{\pi}{6}$$

(iii)

$$\text{Let } \alpha = \cot^{-1}(1)$$

$$\text{Then } \cot \alpha = 1$$

$$\Rightarrow \tan \alpha = 1$$

$$\tan \alpha = \tan \frac{\pi}{4}$$

$$\alpha = \frac{\pi}{4} \in (0, \pi)$$

$$\boxed{\cot^{-1}(1) = \frac{\pi}{4}}$$

$$\text{Let } \beta = \sin^{-1}\left(-\frac{\sqrt{3}}{2}\right)$$

$$\text{Then } \sin \beta = -\frac{\sqrt{3}}{2}$$

$$\sin \beta = -\sin \frac{\pi}{3}$$

$$\sin \beta = \sin\left(-\frac{\pi}{3}\right)$$

$$\beta = -\frac{\pi}{3} \in \left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$$

$$\boxed{\sin^{-1}\left(-\frac{\sqrt{3}}{2}\right) = -\frac{\pi}{3}}$$

$$\text{Let } \gamma = \sec^{-1}(-\sqrt{2})$$

$$\text{Then } \sec \gamma = -\sqrt{2}$$

$$\sec \gamma = -\frac{1}{\sqrt{2}}$$

$$\cos \gamma = -\cos \frac{\pi}{4}$$

$$\cos \gamma = \cos\left(\pi - \frac{\pi}{4}\right)$$

$$\cos \gamma = \cos \frac{3\pi}{4}$$

$$\gamma = \frac{3\pi}{4} \in [0, \pi] \setminus \left\{\frac{\pi}{2}\right\}$$

$$\boxed{\sec^{-1}(-\sqrt{2}) = \frac{3\pi}{4}}$$

$$\therefore \cot^{-1}(1) + \sin^{-1}\left(-\frac{\sqrt{3}}{2}\right)$$

$$= \sec^{-1}(-\sqrt{2})$$

$$= \frac{\pi}{4} - \frac{\pi}{3} - \frac{3\pi}{4}$$

$$= \frac{3\pi - 4\pi - 9\pi}{12}$$

$$= -\frac{10\pi}{12} = -\frac{5\pi}{6}$$

$$= \sec^{-1}(-\sqrt{2}) = -\frac{5\pi}{6}$$

4.10 Properties of Inverse Trigonometrical Functions

Ex 4.16

Prove that

$$\frac{\pi}{2} \leq \sin^{-1}x + 2\cos^{-1}x \leq \frac{3\pi}{2}$$

Soln:

$$\sin^{-1}x + 2\cos^{-1}x = \sin^{-1}x + \cos^{-1}x + \cos^{-1}x$$

$$\sin^{-1}x + 2\cos^{-1}x = \frac{\pi}{2} + \cos^{-1}x$$

$$\text{wkt, } 0 \leq \cos^{-1}x \leq \pi$$

$$\text{Add } \frac{\pi}{2},$$

$$\frac{\pi}{2} + 0 \leq \cos^{-1}x + \frac{\pi}{2} \leq \pi + \frac{\pi}{2}$$

$$\frac{\pi}{2} \leq \sin^{-1}x + 2\cos^{-1}x \leq \frac{3\pi}{2}$$

from ①

$$(i) \cos^{-1}\left(\cos\left(\frac{13\pi}{3}\right)\right)$$

$$(ii) \tan^{-1}\left(\tan\left(\frac{3\pi}{4}\right)\right)$$

$$(iii) \sec^{-1}\left(\sec\left(\frac{5\pi}{3}\right)\right)$$

$$(iv) \sin^{-1}[\sin 10]$$

$$\text{Soln: } \cos^{-1}\left(\cos\left(\frac{13\pi}{3}\right)\right)$$

(i) The range of principal values of $\cos^{-1}x$ is $[0, \pi]$, since $\frac{13\pi}{3} \notin [0, \pi]$

$$\cos^{-1}\left(\cos\left(\frac{13\pi}{3}\right)\right)$$

$$= \cos^{-1}\left(\cos\left(4\pi + \frac{\pi}{3}\right)\right)$$

$$= \cos^{-1}\left(\cos \frac{\pi}{3}\right)$$

$$= \frac{\pi}{3} \in [0, \pi]$$

(ii) The principal range of $\tan^{-1}x$ is $\left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$

$$\text{Given: } \tan^{-1}\left(\tan\left(\frac{3\pi}{4}\right)\right)$$

Since $\frac{3\pi}{4} \notin (-\frac{\pi}{2}, \frac{\pi}{2}) = \frac{\pi}{3} \in [0, \pi] \setminus \{\frac{\pi}{2}\}$

$$\tan^{-1}(\tan(\frac{3\pi}{4}))$$

$$= \tan^{-1}(\tan(\pi - \frac{\pi}{4}))$$

$$= \tan^{-1}(-\tan \frac{\pi}{4})$$

$$= \tan^{-1}(\tan(-\frac{\pi}{4}))$$

$$= -\frac{\pi}{4} \in (-\frac{\pi}{2}, \frac{\pi}{2})$$

(iii) $\sec^{-1}(\sec(\frac{5\pi}{3}))$

The Principal range of $\sec^{-1}x$ is $[0, \pi] \setminus \{\frac{\pi}{2}\}$

Since $\frac{5\pi}{3} \notin [0, \pi] \setminus \{\frac{\pi}{2}\}$

$$\sec^{-1}(\sec(\frac{5\pi}{3}))$$

$$= \sec^{-1}(\sec(2\pi - \frac{\pi}{3}))$$

$$= \sec^{-1}(\sec \frac{\pi}{3})$$

(iv) $\sin^{-1}[\sin 10]$

The Principal range of $\sin^{-1}x$ is $[-\frac{\pi}{2}, \frac{\pi}{2}]$

Since $10 \notin [-\frac{\pi}{2}, \frac{\pi}{2}]$

$$\sin^{-1}[\sin 10]$$

$$= \sin^{-1}[\sin(3\pi + 10 - 3\pi)]$$

$$= \sin^{-1}[\sin(2\pi + \pi + (10 - 3\pi))]$$

$$= \sin^{-1}[\sin(2\pi + (\pi + (10 - 3\pi)))]$$

$$= \sin^{-1}[\sin(\pi + (10 - 3\pi))]$$

$$= \sin^{-1}[-\sin(10 - 3\pi)]$$

$$= \sin^{-1}[\sin(3\pi - 10)]$$

$$= 3\pi - 10 \in [-\frac{\pi}{2}, \frac{\pi}{2}]$$

Ex 4.18

Find the value of

(i) $\sin[\frac{\pi}{3} - \sin^{-1}(-\frac{1}{2})]$

(ii) $\cos[\frac{1}{2} \cos^{-1}(\frac{1}{8})]$

(iii) $\tan[\frac{1}{2} \sin^{-1}(\frac{2a}{1+a^2}) + \frac{1}{2} \cos^{-1}(\frac{1-a^2}{1+a^2})]$

Soln:

(i) $\sin[\frac{\pi}{3} - \sin^{-1}(-\frac{1}{2})]$

$$= \sin[\frac{\pi}{3} - (-\sin^{-1}(\frac{1}{2}))]$$

$$= \sin[\frac{\pi}{3} - (-\frac{\pi}{6})]$$

$$= \sin[\frac{\pi}{3} + \frac{\pi}{6}]$$

$$= \sin[\frac{2\pi + \pi}{6}] = \sin[\frac{3\pi}{6}]$$

$$= \sin[\frac{\pi}{2}] = 1$$

(ii) $\cos[\frac{1}{2} \cos^{-1}(\frac{1}{8})]$

Let $\theta = \cos^{-1}(\frac{1}{8})$

Then $\cos \theta = \frac{1}{8}$ and

$\theta \in [0, \pi]$

$$2 \cos^2 \frac{\theta}{2} - 1 = \frac{1}{8}$$

$$2 \cos^2 \frac{\theta}{2} = 1 + \frac{1}{8}$$

$$2 \cos^2 \frac{\theta}{2} = \frac{9}{8}$$

$$\cos^2 \frac{\theta}{2} = \frac{9}{16}$$

$$\cos \frac{\theta}{2} = \frac{3}{4} \quad \text{--- (1) because } \cos \frac{\theta}{2} \text{ is positive}$$

$$\cos[\frac{1}{2} \cos^{-1}(\frac{1}{8})]$$

$$= \cos[\frac{1}{2}(\theta)]$$

$$= \cos(\frac{\theta}{2})$$

$$= \frac{3}{4}$$

Let $a = \tan \theta$
 $\tan \left[\frac{1}{2} \sin^{-1} \left(\frac{2a}{1+a^2} \right) + \frac{1}{2} \cos^{-1} \left(\frac{1-a^2}{1+a^2} \right) \right]$
 $= \tan \left[\frac{1}{2} \sin^{-1} \left(\frac{2 \tan \theta}{1 + \tan^2 \theta} \right) + \frac{1}{2} \cos^{-1} \left(\frac{1 - \tan^2 \theta}{1 + \tan^2 \theta} \right) \right]$
 $= \tan \left[\frac{1}{2} \sin^{-1} (\sin 2\theta) + \frac{1}{2} \cos^{-1} (\cos 2\theta) \right]$
 $= \tan \left[\frac{1}{2} (2\theta) + \frac{1}{2} (2\theta) \right]$
 $= \tan [0 + 0]$
 $= \tan 2\theta$
 $= \frac{2 \tan \theta}{1 - \tan^2 \theta} = \frac{2a}{1 - a^2}$

for $|x| < 1$
 Soln:
 let $\sin^{-1} x = \theta$
 Then $x = \sin \theta, -1 \leq x \leq 1$
 $\tan(\sin^{-1} x) = \tan \theta$
 $= \frac{\sin \theta}{\cos \theta}$
 $= \frac{\sin \theta}{\sqrt{\cos^2 \theta}}$
 $= \frac{\sin \theta}{\sqrt{1 - \sin^2 \theta}}$
 $\tan(\sin^{-1} x) = \frac{x}{\sqrt{1 - x^2}}, |x| < 1$

EX 4.20

Evaluate:

$$\sin \left[\sin^{-1} \left(\frac{3}{5} \right) + \sec^{-1} \left(\frac{5}{4} \right) \right]$$

Soln:

Then $\sec \theta = \frac{5}{4}$

$$\therefore \cos \theta = \frac{4}{5}$$

$$\begin{aligned} \sin \theta &= \sqrt{\sin^2 \theta} = \sqrt{1 - \cos^2 \theta} \\ &= \sqrt{1 - \left(\frac{4}{5} \right)^2} = \sqrt{1 - \frac{16}{25}} \\ &= \sqrt{\frac{9}{25}} = \frac{3}{5} \end{aligned}$$

$$\sin \theta = \frac{3}{5}$$

$$\theta = \sin^{-1} \left(\frac{3}{5} \right)$$

from (1), $\sec^{-1} \left(\frac{5}{4} \right) = \sin^{-1} \left(\frac{3}{5} \right)$

$$\sin \left[\sin^{-1} \left(\frac{3}{5} \right) + \sec^{-1} \left(\frac{5}{4} \right) \right]$$

$$= \sin \left[\sin^{-1} \left(\frac{3}{5} \right) + \sin^{-1} \left(\frac{3}{5} \right) \right]$$

$$= \sin \left[2 \sin^{-1} \left(\frac{3}{5} \right) \right] \text{--- (2)}$$

$$\left[\because \sin^{-1} (2x\sqrt{1-x^2}) = 2 \sin^{-1} x \right]$$

$$\text{if } |x| \leq \frac{1}{\sqrt{2}}$$

$$\begin{aligned} &\sin \left[\sin^{-1} \left(\frac{3}{5} \right) + \sec^{-1} \left(\frac{5}{4} \right) \right] \\ &= \sin \left[2 \sin^{-1} \left(\frac{3}{5} \right) \right] \\ &= \sin \left[\sin^{-1} \left(2 \left(\frac{3}{5} \right) \sqrt{1 - \left(\frac{3}{5} \right)^2} \right) \right] \\ &= \sin \left[\sin^{-1} \left(\frac{6}{5} \sqrt{1 - \frac{9}{25}} \right) \right] \\ &= \sin \left[\sin^{-1} \left(\frac{6}{5} \sqrt{\frac{16}{25}} \right) \right] \\ &= \sin \left[\sin^{-1} \left(\frac{6}{5} \times \frac{4}{5} \right) \right] \\ &= \sin \left[\sin^{-1} \left(\frac{24}{25} \right) \right] \\ &= \frac{24}{25} \in [-1, 1] \end{aligned}$$

EX 4.19

Prove that

Ex 4.21

Prove that

$$(i) \tan^{-1} \frac{1}{2} + \tan^{-1} \frac{1}{3} = \frac{\pi}{4}$$

$$(ii) 2 \tan^{-1} \frac{1}{2} + \tan^{-1} \frac{1}{7} = \tan^{-1} \frac{31}{17}$$

Soln:

$$\text{WKT, } \tan^{-1} x + \tan^{-1} y = \tan^{-1} \left(\frac{x+y}{1-xy} \right) \quad \text{if } xy < 1$$

$$\tan^{-1} \frac{1}{2} + \tan^{-1} \frac{1}{3} = \tan^{-1} \left(\frac{\frac{1}{2} + \frac{1}{3}}{1 - (\frac{1}{2})(\frac{1}{3})} \right)$$

$$= \tan^{-1} \left(\frac{\frac{5}{6}}{1 - \frac{1}{6}} \right)$$

$$= \tan^{-1} \left(\frac{5/6}{5/6} \right)$$

$$= \tan^{-1}(1)$$

$$= \frac{\pi}{4}$$

$$\therefore \tan^{-1} \frac{1}{2} + \tan^{-1} \frac{1}{3} = \frac{\pi}{4}$$

(ii) WKT,

$$2 \tan^{-1} x = \tan^{-1} \left(\frac{2x}{1-x^2} \right), \quad -1 < x < 1$$

$$2 \tan^{-1} \frac{1}{2} + \tan^{-1} \frac{1}{7}$$

$$= \tan^{-1} \left(\frac{2(\frac{1}{2})}{1 - (\frac{1}{2})^2} \right) + \tan^{-1} \frac{1}{7}$$

$$= \tan^{-1} \left(\frac{1}{1 - \frac{1}{4}} \right) + \tan^{-1} \frac{1}{7}$$

$$= \tan^{-1} \left(\frac{4}{3} \right) + \tan^{-1} \frac{1}{7}$$

$$= \tan^{-1} \left[\frac{\frac{4}{3} + \frac{1}{7}}{1 - (\frac{4}{3})(\frac{1}{7})} \right]$$

$$= \tan^{-1} \left[\frac{\frac{28+3}{21}}{1 - \frac{4}{21}} \right]$$

$$= \tan^{-1} \left[\frac{31}{17} \right]$$

$$\therefore \tan^{-1} \left[\frac{31}{17} \right] = \tan^{-1} \left(\frac{x+y}{1-xy} \right) \quad \text{if } xy < 1$$

$$= \tan^{-1} \left(\frac{31}{17} \right)$$

$$\therefore 2 \tan^{-1} \frac{1}{2} + \tan^{-1} \frac{1}{7} = \tan^{-1} \frac{31}{17}$$

Ex 4.22If $\cos^{-1} x + \cos^{-1} y + \cos^{-1} z = \pi$ and $0 < x, y, z < 1$, Show that $x^2 + y^2 + z^2 + 2xyz = 1$ Soln:Let $\cos^{-1} x = \alpha$, $\cos^{-1} y = \beta$ Then $x = \cos \alpha$, $y = \cos \beta$ Given: $\cos^{-1} x + \cos^{-1} y + \cos^{-1} z = \pi$

$$\alpha + \beta + \cos^{-1} z = \pi$$

$$\alpha + \beta = \pi - \cos^{-1} z \quad \text{--- (1)}$$

Now,

$$\cos(\alpha + \beta) = \cos \alpha \cos \beta - \sin \alpha \sin \beta$$

$$= \cos \alpha \cos \beta - \sqrt{1 - \cos^2 \alpha} \sqrt{1 - \cos^2 \beta}$$

$$= \cos \alpha \cos \beta - \sqrt{1 - \cos^2 \alpha} \sqrt{1 - \cos^2 \beta}$$

$$= xy - \sqrt{1-x^2} \sqrt{1-y^2}$$

$$\cos(\alpha + \beta) = xy - \sqrt{1-x^2} \sqrt{1-y^2}$$

$$\cos(\pi - \cos^{-1} z) = xy - \sqrt{1-x^2} \sqrt{1-y^2}$$

$$-\cos(\cos^{-1} z) = xy - \sqrt{1-x^2} \sqrt{1-y^2}$$

$$-z - xy = -\sqrt{1-x^2} \sqrt{1-y^2}$$

$$+(z + xy) = +\sqrt{1-x^2} \sqrt{1-y^2}$$

Squaring on both sides, we get,

$$(z + xy)^2 = (1-x^2)(1-y^2)$$

$$z^2 + x^2 y^2 + 2xyz = 1 - y^2 - x^2 + x^2 y^2$$

$$= 1 - y^2 - x^2 + x^2 y^2$$

$$\Rightarrow x^2 + y^2 + z^2 + 2xyz = 1$$

Ex 4.23If $a_1, a_2, a_3, \dots, a_n$ is an arithmetic Progressionon with common difference d ,

Prove that

$$+ \dots + \tanh\left(\frac{d}{1+a_n a_{n-1}}\right) \Big] \\ = \frac{a_n - a_1}{1+a_1 a_n}$$

Soln:

$$\tanh\left(\frac{d}{1+a_1 a_2}\right) = \tanh\left(\frac{a_2 - a_1}{1+a_1 a_2}\right) = \tanh\left[\tanh^{-1}\left(\frac{a_2 - a_1}{1+a_1 a_2}\right)\right] \\ = \tanh^{-1} a_2 - \tanh^{-1} a_1$$

$$\tanh\left(\frac{d}{1+a_2 a_3}\right) = \tanh\left(\frac{a_3 - a_2}{1+a_2 a_3}\right) = \tanh^{-1} a_3 - \tanh^{-1} a_2$$

Continuing inductively we get,

$$\tanh\left(\frac{d}{1+a_n a_{n-1}}\right) = \tanh^{-1} a_n - \tanh^{-1} a_{n-1}$$

Adding we get,

$$\tanh\left(\frac{d}{1+a_1 a_2}\right) + \tanh\left(\frac{d}{1+a_2 a_3}\right) + \dots + \tanh\left(\frac{d}{1+a_n a_{n-1}}\right) = \tanh^{-1} a_n - \tanh^{-1} a_1$$

$$\tanh\left(\frac{d}{1+a_n a_{n-1}}\right) = \tanh^{-1} a_n - \tanh^{-1} a_1$$

$$= \tanh^{-1} a_n - \tanh^{-1} a_1$$

$$= \frac{a_n - a_1}{1+a_1 a_n}$$

Ex 4.24

Solve $\tanh^{-1}\left(\frac{1-x}{1+x}\right) = \frac{1}{2} \tanh^{-1} x$ for $x > 0$

Soln: Given

$$\tanh^{-1}\left(\frac{1-x}{1+x}\right) = \frac{1}{2} \tanh^{-1} x$$

$$\tanh^{-1} 1 - \tanh^{-1} x = \frac{1}{2} \tanh^{-1} x$$

$$\frac{\pi}{4} = \frac{1}{2} \tanh^{-1} x + \tanh^{-1} x$$

$$\frac{\pi}{4} = \frac{3}{2} \tanh^{-1} x$$

$$\frac{\pi}{2} = 3 \tanh^{-1} x$$

$$x = \tanh^{-1} \frac{\pi}{6}$$

$$x = \frac{1}{\sqrt{3}}$$

Ex 4.25

Solve $\sin^{-1} x > \cos^{-1} x$

Soln:

Given: $\sin^{-1} x > \cos^{-1} x$, $-1 \leq x \leq 1$

Add $\sin^{-1} x$ on both sides, we get,

$$\sin^{-1} x + \sin^{-1} x > \sin^{-1} x + \cos^{-1} x$$

$$2 \sin^{-1} x > \frac{\pi}{2}$$

$$\sin^{-1} x > \frac{\pi}{4} \in \left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$$

$$x > \sin \frac{\pi}{4}$$

$$x > \frac{1}{\sqrt{2}}$$

The sine function is increases in the interval

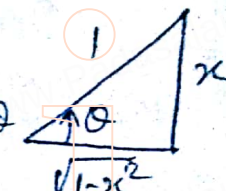
the interval $\left(\frac{1}{\sqrt{2}}, 1\right]$
(i.e) $\frac{1}{\sqrt{2}} < x \leq 1$

Ex 4.26

Show that $\cot(\sin^{-1} x) = \frac{\sqrt{1-x^2}}{x}$, $-1 \leq x \leq 1$ and $x \neq 0$

Soln:

Let $\sin^{-1} x = \theta$



Then $x = \sin \theta \Rightarrow \sin \theta = \frac{x}{1}$

$$\cos \theta = \sqrt{1 - \sin^2 \theta}$$

$$= \sqrt{1 - x^2}$$

$$\therefore \cot \theta = \frac{\cos \theta}{\sin \theta} = \frac{\sqrt{1-x^2}}{x}$$

$$\therefore \cot(\sin^{-1} x) = \cot \theta = \frac{\cos \theta}{\sin \theta} = \frac{\sqrt{1-x^2}}{x}, \quad -1 \leq x \leq 1 \text{ and } x \neq 0$$

$$\therefore \cot(\sin^{-1}x) = \frac{\sqrt{1-x^2}}{x},$$

$$|x| \leq 1 \text{ and } x \neq 0$$

Ex 4.27

Solve: $\tan^{-1}2x + \tan^{-1}3x = \frac{\pi}{4}$
if $6x^2 < 1$

Soln:

Given: $\tan^{-1}2x + \tan^{-1}3x = \frac{\pi}{4}$
if $6x^2 < 1$

$$\tan^{-1}2x + \tan^{-1}3x = \tan^{-1}\left(\frac{2x+3x}{1-6x^2}\right)$$

$$\tan^{-1}2x + \tan^{-1}3x = \tan^{-1}\left(\frac{5x}{1-6x^2}\right)$$

$$\frac{\pi}{4} = \tan^{-1}\left(\frac{5x}{1-6x^2}\right)$$

$$\tan \frac{\pi}{4} = \frac{5x}{1-6x^2}$$

$$1 = \frac{5x}{1-6x^2}$$

$$1-6x^2 = 5x$$

$$6x^2 + 5x - 1 = 0$$

$$6x^2 + 6x - x - 1 = 0$$

$$6x(x+1) - 1(x+1) = 0$$

$$(6x-1)(x+1) = 0$$

$$6x-1=0, x+1=0$$

$$x = \frac{1}{6}, x = -1 \text{ does}$$

$$\text{not satisfy } 6x^2 < 1$$

$$\therefore x = -1 \text{ is not a}$$

Soln

$$\therefore x = \frac{1}{6} \text{ is the only}$$

Solution of the eqn.

Ex 4.28

Solve $\tan^{-1}\left(\frac{x-1}{x-2}\right) + \tan^{-1}\left(\frac{x+1}{x+2}\right) = \frac{\pi}{4}$

Soln:

Given:

$$\tan^{-1}\left(\frac{x-1}{x-2}\right) + \tan^{-1}\left(\frac{x+1}{x+2}\right) = \frac{\pi}{4}$$

$$\tan^{-1}\left[\frac{\frac{x-1}{x-2} + \frac{x+1}{x+2}}{1 - \left(\frac{x-1}{x-2}\right)\left(\frac{x+1}{x+2}\right)}\right] = \frac{\pi}{4}$$

$$\left[\frac{(x-1)(x+2) + (x+1)(x-2)}{(x-2)(x+2) - (x-1)(x+1)}\right] = \tan \frac{\pi}{4}$$

$$\frac{x^2 + x - 2 + x^2 - x - 2}{(x^2 - 4) - (x^2 - 1)} = 1$$

$$\frac{2x^2 - 4}{x^2 - 4 - x^2 + 1} = 1$$

$$\frac{2x^2 - 4}{-3} = 1$$

$$2x^2 - 4 = -3$$

$$2x^2 = -3 + 4$$

$$2x^2 = 1 \Rightarrow x^2 = \frac{1}{2}$$

$$\Rightarrow x = \pm \frac{1}{\sqrt{2}}$$

Ex 4.29

Solve: $\cos\left(\sin^{-1}\left(\frac{x}{\sqrt{1+x^2}}\right)\right)$
 $= \sin\left(\cot^{-1}\left(\frac{3}{4}\right)\right)$

Soln:

Wkt, $\sin^{-1}\left(\frac{x}{\sqrt{1+x^2}}\right) = \cos^{-1}\left(\frac{1}{\sqrt{1+x^2}}\right)$

(Property ix(vi))

$$\cos\left(\sin^{-1}\left(\frac{x}{\sqrt{1+x^2}}\right)\right) = \frac{1}{\sqrt{1+x^2}} \quad \text{--- (1)}$$

From the diagram,

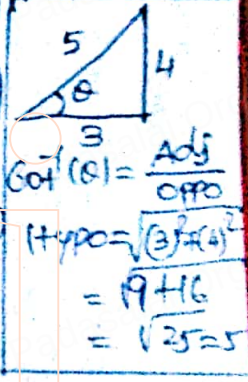
we have, Property

$$\cot^{-1}\left(\frac{3}{4}\right) = \sin^{-1}\left(\frac{4}{5}\right)$$

$$\therefore \sin\left(\cot^{-1}\left(\frac{3}{4}\right)\right)$$

$$= \sin\left(\sin^{-1}\left(\frac{4}{5}\right)\right)$$

$$= \frac{4}{5} \quad \text{--- (2)}$$



from (1) we get,

$$\textcircled{1} = \textcircled{2}$$

$$\frac{1}{\sqrt{1+x^2}} = \frac{4}{5}$$

$$\Rightarrow \sqrt{1+x^2} = \frac{5}{4}$$

On Squaring, $1+x^2 = \frac{25}{16}$

$$x^2 = \frac{25}{16} - 1$$

$$x^2 = \frac{9}{16} \Rightarrow x = \pm \frac{3}{4}$$

EXERCISE 4.5

1) Find the value, if it exists, If not, give the reason for non-existence

(i) $\sin^{-1}(\cos \pi)$

(ii) $\tan^{-1}(\sin(-\frac{5\pi}{2}))$

(iii) $\sin^{-1}[\sin 5]$

(i) $\sin^{-1}(\cos \pi) = \sin^{-1}(-1)$

$$= -\sin^{-1}(1)$$

$$= -\frac{\pi}{2}$$

(ii) $\tan^{-1}(\sin(-\frac{5\pi}{2}))$

$$= \tan^{-1}(-\sin(\frac{5\pi}{2}))$$

$$= \tan^{-1}(-\sin(2\pi + \frac{\pi}{2}))$$

$$= \tan^{-1}(-\sin \frac{\pi}{2})$$

$$= \tan^{-1}(-1)$$

$$= -\tan^{-1}(1) = -\frac{\pi}{4}$$

(iii) $\sin^{-1}[\sin 5]$

$$= \sin^{-1}[\sin(2\pi - 2\pi + 5)]$$

$$= \sin^{-1}[\sin(2\pi - (2\pi - 5))]$$

$$= \sin^{-1}[-\sin(2\pi - 5)]$$

$$= -\sin^{-1}[\sin(2\pi - 5)]$$

2) Find the value of the expression in terms of x , with the help of a reference triangle,

(i) $\sin(\cos^{-1}(1-x))$

(ii) $\cos(\tan^{-1}(3x-1))$

(iii) $\tan(\sin^{-1}(x + \frac{1}{2}))$

Soln:

(i) Let $\theta = \cos^{-1}(1-x)$

$$\cos \theta = 1-x$$

$$1 - \cos^2 \theta = 1 - (1-x)^2$$

$$\sin^2 \theta = 1 - (1-2x+x^2)$$

$$\sin^2 \theta = 1 - 1 + 2x - x^2$$

$$\sin^2 \theta = 2x - x^2$$

$$\sin \theta = \sqrt{2x - x^2}$$

(ii) Let $\theta = \tan^{-1}(3x-1)$

$$\tan \theta = 3x-1$$

$$1 + \tan^2 \theta = 1 + (3x-1)^2$$

$$\sec^2 \theta = 1 + 9x^2 - 6x + 1$$

$$\sec^2 \theta = 9x^2 - 6x + 2$$

$$\sec \theta = \sqrt{9x^2 - 6x + 2}$$

$$\frac{1}{\cos \theta} = \sqrt{9x^2 - 6x + 2}$$

$$\cos \theta = \frac{1}{\sqrt{9x^2 - 6x + 2}}$$

$$\cos(\tan^{-1}(3x-1)) = \frac{1}{\sqrt{9x^2 - 6x + 2}}$$

$$\cos(\tan^{-1}(3x-1)) = \frac{1}{\sqrt{9x^2 - 6x + 2}}$$

$$\cos(\tan^{-1}(3x-1)) = \frac{1}{\sqrt{9x^2 - 6x + 2}}$$

(iii) Let $\theta = \sin^{-1}(x + \frac{1}{2})$

$$\sin \theta = x + \frac{1}{2}$$

$$1 - \sin^2 \theta = 1 - (x + \frac{1}{2})^2$$

$$\cos^2 \theta = 1 - (x^2 + x + \frac{1}{4})$$

$$\cos^2 \theta = 1 - x^2 - x - \frac{1}{4}$$

$$= \frac{4 - 4x^2 - 4x - 1}{4}$$

$$\cos^2 \theta = \frac{3 - 4x - 4x^2}{4}$$

$$\cos \theta = \frac{\sqrt{3 - 4x - 4x^2}}{2}$$

$$\tan \theta = \frac{\sin \theta}{\cos \theta}$$

$$= \frac{x + \frac{1}{2}}{\frac{\sqrt{3 - 4x - 4x^2}}{2}}$$

$$= \frac{2x + 1}{\sqrt{3 - 4x - 4x^2}}$$

$$= \frac{2x + 1}{\sqrt{3 - 4x - 4x^2}}$$

$$\tan \theta = \frac{2x + 1}{\sqrt{3 - 4x - 4x^2}}$$

$$\tan(\sin^{-1}(x + \frac{1}{2}))$$

$$= \frac{2x + 1}{\sqrt{3 - 4x - 4x^2}}$$

$$B) \text{ Find the value}$$

$$(i) \sin^{-1}(\cos(\sin^{-1}(\frac{\sqrt{3}}{2})))$$

$$(ii) \cos(\sin^{-1} \frac{3}{5} + \sin^{-1} \frac{4}{5})$$

$$(iii) \tan(\sin^{-1} \frac{3}{5} + \cos^{-1} \frac{3}{5})$$

Soln:

$$(i) \sin^{-1}(\cos(\sin^{-1}(\frac{\sqrt{3}}{2})))$$

$$= \sin^{-1}(\cos(\frac{\pi}{3}))$$

$$= \sin^{-1}(\frac{1}{2})$$

$$= \frac{\pi}{6}$$

$$(ii) \text{ using Property,}$$

$$\sin^{-1} x + \sin^{-1} y$$

$$= \sin^{-1}(x\sqrt{1-y^2} + y\sqrt{1-x^2})$$

Now,

$$\sin^{-1} \frac{3}{5} + \sin^{-1} \frac{4}{5}$$

$$= \sin^{-1}(\frac{3}{5}\sqrt{1-(\frac{4}{5})^2} + \frac{4}{5}\sqrt{1-(\frac{3}{5})^2})$$

$$= \sin^{-1}(\frac{3}{5}\sqrt{1-\frac{16}{25}} + \frac{4}{5}\sqrt{1-\frac{9}{25}})$$

$$= \sin^{-1}(\frac{3}{5}\sqrt{\frac{9}{25}} + \frac{4}{5}\sqrt{\frac{16}{25}})$$

$$= \sin^{-1}(\frac{3}{5} \times \frac{3}{5} + \frac{4}{5} \times \frac{4}{5})$$

$$= \sin^{-1}(\frac{9}{25} + \frac{16}{25})$$

$$= \sin^{-1}(\frac{25}{25})$$

$$= \sin^{-1}(1)$$

$$= \frac{\pi}{2}$$

$$\therefore \cos(\sin^{-1} \frac{3}{5} + \sin^{-1} \frac{4}{5})$$

$$= \cos(\frac{\pi}{2})$$

$$= \frac{1}{\tan \frac{\pi}{2}} = \frac{1}{\infty} = 0$$

(iii)

$$\text{Let } \alpha = \sin^{-1}(\frac{3}{5})$$

$$\left[\sin \alpha = \frac{3}{5} \right] \quad (1)$$

$$1 - \sin^2 \alpha = 1 - (\frac{3}{5})^2$$

$$\cos^2 \alpha = 1 - \frac{9}{25} = \frac{16}{25}$$

$$\left[\cos \alpha = \frac{4}{5} \right]$$

$$\tan \alpha = \frac{\sin \alpha}{\cos \alpha} = \frac{3/5}{4/5}$$

$$\tan \alpha = \frac{3}{4}$$

$$\alpha = \tan^{-1}(\frac{3}{4})$$

$$\sin^{-1}(\frac{3}{5}) = \tan^{-1}(\frac{3}{4})$$

$$\begin{aligned}
 &= \tan\left(\tan^{-1}\frac{3}{4} + \tan^{-1}\frac{2}{3}\right) \\
 &= \tan\left[\tan^{-1}\left(\frac{\frac{3}{4} + \frac{2}{3}}{1 - \left(\frac{3}{4}\right)\left(\frac{2}{3}\right)}\right)\right] \\
 &= \tan\left[\tan^{-1}\left(\frac{\frac{9+8}{12}}{\frac{12-6}{12}}\right)\right] \\
 &= \tan\left[\tan^{-1}\left(\frac{17}{6}\right)\right] \\
 &= \frac{17}{6}
 \end{aligned}$$

4) Prove that

(i) $\tan^{-1}\frac{2}{11} + \tan^{-1}\frac{7}{24} = \tan^{-1}\frac{1}{2}$

(ii) $\sin^{-1}\frac{3}{5} - \cos^{-1}\frac{12}{13} = \sin^{-1}\frac{16}{65}$

Soln:

(i) $\tan^{-1}\frac{2}{11} + \tan^{-1}\frac{7}{24}$

$$\begin{aligned}
 &= \tan^{-1}\left[\frac{\frac{2}{11} + \frac{7}{24}}{1 - \left(\frac{2}{11}\right)\left(\frac{7}{24}\right)}\right] \\
 &= \tan^{-1}\left[\frac{\frac{48+77}{264}}{\frac{264-14}{264}}\right] \\
 &= \tan^{-1}\left[\frac{125}{250}\right] \\
 &= \tan^{-1}\frac{1}{2}
 \end{aligned}$$

(ii) Let $\alpha = \cos^{-1}\frac{12}{13}$ L-①

$$\cos\alpha = \frac{12}{13}$$

$$1 - \cos^2\alpha = 1 - \left(\frac{12}{13}\right)^2$$

$$\sin^2\alpha = 1 - \frac{144}{169}$$

$$\sin^2\alpha = \frac{25}{169}$$

$\sin\alpha = \frac{5}{13}$

from ①, $\cos^{-1}\frac{12}{13} = \sin^{-1}\frac{5}{13}$

$$\begin{aligned}
 &\sin^{-1}\frac{3}{5} - \cos^{-1}\frac{12}{13} \\
 &= \sin^{-1}\frac{3}{5} - \sin^{-1}\frac{5}{13}
 \end{aligned}$$

[using $\sin^{-1}x - \sin^{-1}y$
 $= \sin^{-1}(x\sqrt{1-y^2} - y\sqrt{1-x^2})$]

$$\begin{aligned}
 &= \sin^{-1}\left(\frac{3}{5}\sqrt{1 - \left(\frac{5}{13}\right)^2} - \frac{5}{13}\sqrt{1 - \left(\frac{3}{5}\right)^2}\right) \\
 &= \sin^{-1}\left(\frac{3}{5}\sqrt{1 - \frac{25}{169}} - \frac{5}{13}\sqrt{1 - \frac{9}{25}}\right) \\
 &= \sin^{-1}\left(\frac{3}{5}\sqrt{\frac{144}{169}} - \frac{5}{13}\sqrt{\frac{16}{25}}\right) \\
 &= \sin^{-1}\left(\left(\frac{3}{5} \times \frac{12}{13}\right) - \left(\frac{5}{13} \times \frac{4}{5}\right)\right)
 \end{aligned}$$

$$\begin{aligned}
 &= \sin^{-1}\left(\frac{16}{65}\right) \\
 &\therefore \sin^{-1}\frac{3}{5} - \cos^{-1}\frac{12}{13} = \sin^{-1}\frac{16}{65}
 \end{aligned}$$

5) Prove that

$$\tan^{-1}x + \tan^{-1}y + \tan^{-1}z = \tan^{-1}\left[\frac{x+y+z-xyz}{1-xy-yz-zx}\right]$$

Soln:

$$\begin{aligned}
 &\tan^{-1}x + \tan^{-1}y + \tan^{-1}z \\
 &= \tan^{-1}\left[\frac{x+y}{1-xy}\right] + \tan^{-1}z \\
 &= \tan^{-1}\left[\frac{\frac{x+y}{1-xy} + z}{1 - \left(\frac{x+y}{1-xy}\right)(z)}\right] \\
 &= \tan^{-1}\left[\frac{x+y+z-xyz}{(1-xy) - (xz+yz)}\right] \\
 &= \tan^{-1}\left[\frac{x+y+z-xyz}{1-xy-yz-zx}\right]
 \end{aligned}$$

6) If $\tan^{-1}x + \tan^{-1}y + \tan^{-1}z = \pi$, Show that $x+y+z = xyz$

Soln:

Let $\alpha = \tan^{-1}x \Rightarrow \tan \alpha = x$

$\beta = \tan^{-1}y \Rightarrow \tan \beta = y$

$\gamma = \tan^{-1}z \Rightarrow \tan \gamma = z$

Given:

$$\tan^{-1}x + \tan^{-1}y + \tan^{-1}z = \pi$$

$$\alpha + \beta + \gamma = \pi$$

$$\alpha + \beta = \pi - \gamma$$

$$\tan(\alpha + \beta) = \tan(\pi - \gamma)$$

$$\frac{\tan \alpha + \tan \beta}{1 - \tan \alpha \tan \beta} = -\tan \gamma$$

$$\frac{x+y}{1-xy} = -z$$

$$x+y = -z(1-xy)$$

$$x+y = -z + xyz$$

$$x+y+z = xyz$$

7) Prove that

$$\tan^{-1}x + \tan^{-1}\frac{2x}{1-x^2} = \tan^{-1}\frac{3x-x^3}{1-3x^2},$$

$$|x| < \frac{1}{\sqrt{3}}$$

Soln:

By Property VII (i)

$$2\tan^{-1}x = \tan^{-1}\frac{2x}{1-x^2}$$

Given:

$$\tan^{-1}x + \tan^{-1}\frac{2x}{1-x^2}$$

$$= \tan^{-1}x + 2\tan^{-1}x$$

$$= 3\tan^{-1}x \quad \text{--- (1)}$$

$$\text{Let } \theta = \tan^{-1}x$$

$$\Rightarrow \tan \theta = x$$

WkT,

$$\tan 3\theta = \frac{3\tan \theta - \tan^3 \theta}{1 - 3\tan^2 \theta}$$

$$\tan 3\theta = \frac{3x - x^3}{1 - 3x^2}$$

$$3\theta = \tan^{-1}\frac{3x - x^3}{1 - 3x^2}$$

$$3\tan^{-1}x = \tan^{-1}\frac{3x - x^3}{1 - 3x^2}$$

① $\Rightarrow$

$$\tan^{-1}x + \tan^{-1}\frac{2x}{1-x^2} = \tan^{-1}\frac{3x-x^3}{1-3x^2}, \quad |x| < \frac{1}{\sqrt{3}}$$

8) Simplify:

$$\tan^{-1}\frac{x}{y} - \tan^{-1}\frac{x-y}{x+y}$$

Soln:

$$\tan^{-1}\frac{x}{y} - \tan^{-1}\frac{x-y}{x+y}$$

$$= \tan^{-1}\left[\frac{\frac{x}{y} - \frac{x-y}{x+y}}{1 + \left(\frac{x}{y}\right)\left(\frac{x-y}{x+y}\right)}\right]$$

$$= \tan^{-1}\left[\frac{\frac{x(x+y) - y(x-y)}{y(x+y)}}{1 + \frac{y(x+y) + x(x-y)}{y(x+y)}}\right]$$

$$= \tan^{-1}\left[\frac{x^2 + xy - xy + y^2}{xy + y^2 + x^2 - xy}\right]$$

$$= \tan^{-1}\left[\frac{x^2 + y^2}{x^2 + y^2}\right]$$

$$= \tan^{-1}(1)$$

$$= \frac{\pi}{4}$$

9) Solve:

$$(i) \sin^{-1}\frac{5}{x} + \sin^{-1}\frac{12}{x} = \frac{\pi}{2}$$

Soln:

$$\text{Let } \theta = \sin^{-1}\frac{5}{x}$$

$$\sin \theta = \frac{5}{x}$$

$$1 - \sin^2 \theta = 1 - \frac{25}{x^2}$$

$$\cos^2 \theta = 1 - \frac{25}{x^2}$$

$$\cos \theta = \sqrt{1 - \frac{25}{x^2}}$$

$$\theta = \cos^{-1} \left(\sqrt{1 - \frac{25}{x^2}} \right)$$

Given:

$$\sin^{-1} \frac{5}{x} + \sin^{-1} \frac{12}{x} = \frac{\pi}{2}$$

$$\sin^{-1} \frac{12}{x} = \frac{\pi}{2} - \sin^{-1} \frac{5}{x}$$

$$\frac{12}{x} = \sin \left(\frac{\pi}{2} - \sin^{-1} \frac{5}{x} \right)$$

$$\frac{12}{x} = \sin \left(\frac{\pi}{2} - \theta \right)$$

$$\frac{12}{x} = \cos \theta$$

$$\frac{12}{x} = \cos \left[\cos^{-1} \sqrt{1 - \frac{25}{x^2}} \right]$$

$$\frac{12}{x} = \sqrt{1 - \frac{25}{x^2}}$$

Squaring on both sides,

$$\frac{144}{x^2} = 1 - \frac{25}{x^2}$$

$$\frac{144}{x^2} = \frac{x^2 - 25}{x^2}$$

$$x^2 = 144 + 25$$

$$x^2 = 169 \Rightarrow \boxed{x = 13}$$

9(ii) Solve

$$2 \tan^{-1} x = \cos^{-1} \frac{1-a^2}{1+a^2} - \cos^{-1} \frac{1-b^2}{1+b^2}$$

$a > 0, b > 0$

Soln:

Let $a = \tan \alpha$ and $b = \tan \beta$

Then $\alpha = \tan^{-1} a$ and

$\beta = \tan^{-1} b$

Given:

$$2 \tan^{-1} x = \cos^{-1} \left(\frac{1 - \tan^2 \alpha}{1 + \tan^2 \alpha} \right) - \cos^{-1} \left(\frac{1 - \tan^2 \beta}{1 + \tan^2 \beta} \right)$$

$$2 \tan^{-1} x = \cos^{-1} (\cos 2\alpha) - \cos^{-1} (\cos 2\beta)$$

$$2 \tan^{-1} x = 2\alpha - 2\beta$$

$$\div 2, \tan^{-1} x = \alpha - \beta$$

$$x = \tan(\alpha - \beta)$$

$$x = \frac{\tan \alpha - \tan \beta}{1 + \tan \alpha \tan \beta}$$

$$x = \frac{a - b}{1 + ab}$$

9(iii) Solve:

$$2 \tan^{-1} (\cos x) = \tan^{-1} (2 \operatorname{cosec} x)$$

$$2 \tan^{-1} x = \tan^{-1} \left(\frac{2x}{1-x^2} \right), |x| < 1$$

$$2 \tan^{-1} (\cos x) = \tan^{-1} \left(\frac{2 \cos x}{1 - \cos^2 x} \right)$$

Given:

$$2 \tan^{-1} (\cos x) = \tan^{-1} (2 \operatorname{cosec} x)$$

$$\tan^{-1} \left(\frac{2 \cos x}{1 - \cos^2 x} \right) = \tan^{-1} (2 \operatorname{cosec} x)$$

$$\frac{2 \cos x}{1 - \cos^2 x} = 2 \operatorname{cosec} x$$

$$\frac{\cos x}{\sin^2 x} = \frac{1}{\sin x}$$

$$\sin x \cos x = \sin^2 x$$

$$\sin x \cos x - \sin^2 x = 0$$

$$\sin x (\cos x - \sin x) = 0$$

$$\sin x = 0 \text{ (or) } \cos x - \sin x = 0$$

If $\sin x = 0$ then

$$\boxed{x = n\pi}, n \in \mathbb{Z}$$

If $\cos x - \sin x = 0$
then $\cos x = \sin x$
 $1 = \frac{\sin x}{\cos x}$

$$\tan x = 1$$

$$\tan x = \tan \frac{\pi}{4}$$

Compare $\tan x = \tan \alpha$

$$\therefore \boxed{\alpha = \frac{\pi}{4}}$$

$$\therefore x = n\pi + \alpha, n \in \mathbb{Z}$$

$$x = n\pi + \frac{\pi}{4}, n \in \mathbb{Z}$$

$$\therefore x = n\pi, x = n\pi + \frac{\pi}{4}, n \in \mathbb{Z}$$

9(ii) Solve

$$\cot^{-1} x - \cot^{-1}(x+2) = \frac{\pi}{12}, x > 0$$

Soln:

$$\cot^{-1} x - \cot^{-1}(x+2) = \frac{\pi}{12}$$

$$\tan^{-1}\left(\frac{1}{x}\right) - \tan^{-1}\left(\frac{1}{x+2}\right) = \frac{\pi}{12}$$

$$\tan^{-1}\left[\frac{\frac{1}{x} - \frac{1}{x+2}}{1 + \left(\frac{1}{x}\right)\left(\frac{1}{x+2}\right)}\right] = \frac{\pi}{12}$$

$$\left[\frac{\frac{x+2-x}{x(x+2)}}{\frac{x(x+2)+1}{x(x+2)}}\right] = \tan \frac{\pi}{12}$$

$$\frac{2}{x^2+2x+1} = \tan \frac{180^\circ}{12}$$

$$\frac{2}{x^2+2x+1} = \tan 15^\circ$$

$$\frac{2}{x^2+2x+1} = 2 - \sqrt{3}$$

$$\frac{2}{2 - \sqrt{3}} = x^2 + 2x + 1$$

$$x^2 + 2x + 1 = \frac{2}{2 - \sqrt{3}} \times \frac{2 + \sqrt{3}}{2 + \sqrt{3}}$$

$$(x+1)^2 = \frac{2(2 + \sqrt{3})}{4 - 3}$$

$$= \frac{2(2 + \sqrt{3})}{1}$$

$$= 4 + 2\sqrt{3}$$

$$= 1 + 3 + 2\sqrt{3}$$

$$(x+1)^2 = (1 + \sqrt{3})^2$$

$$x+1 = 1 + \sqrt{3}, \text{ [only positive root]}$$

$$\boxed{x = \sqrt{3}}$$

10) Find the number of solutions of the equation

$$\tan^{-1}(x-1) + \tan^{-1}x + \tan^{-1}(x+1) = \tan^{-1}(3x)$$

Soln:

$$\tan^{-1}(x-1) + \tan^{-1}x + \tan^{-1}(x+1) = \tan^{-1}(3x)$$

$$\tan^{-1}(x-1) + \tan^{-1}(x+1) + \tan^{-1}x = \tan^{-1}(3x)$$

$$\tan^{-1}\left(\frac{x-1+x+1}{1 - [(x-1)(x+1)]}\right) + \tan^{-1}x = \tan^{-1}(3x)$$

$$\tan^{-1}\left(\frac{2x}{1 - (x^2 - 1)}\right) + \tan^{-1}x = \tan^{-1}(3x)$$

$$\tan^{-1}\left(\frac{2x}{1 - x^2 + 1}\right) + \tan^{-1}x = \tan^{-1}(3x)$$

$$\tan^{-1}\left(\frac{2x}{2 - x^2}\right) + \tan^{-1}x = \tan^{-1}(3x)$$

$$\tan^{-1}\left[\frac{\frac{2x}{2 - x^2} + x}{1 - \left(\frac{2x}{2 - x^2}\right)x}\right] = \tan^{-1}(3x)$$

$$\frac{\tan^{-1}\left[\frac{(2-x^2)-2x^2}{2-x^2}\right]}{2-x^2}$$

$$= \tan^{-1}(3x)$$

$$\tan^{-1}\left[\frac{2x+2x-x^3}{2-3x^2}\right] = \tan^{-1}(3x)$$

$$\tan^{-1}\left[\frac{4x-x^3}{2-3x^2}\right] = \tan^{-1}(3x)$$

$$\frac{4x-x^3}{2-3x^2} = 3x$$

$$4x-x^3 = 3x(2-3x^2)$$

$$4x-x^3 = 6x-9x^3$$

$$9x^3-x^3+4x-6x=0$$

$$8x^3-2x=0$$

$$2x(4x^2-1)=0$$

$$2x=0, 4x^2-1=0$$

$$x=0, x^2=\frac{1}{4}$$

$$x=\pm\frac{1}{2}$$

$$\therefore x=0, x=\frac{1}{2} \text{ and } x=-\frac{1}{2}$$

$\therefore$ The number of solutions of the given equation is 3

EXERCISE 4.6

Choose the correct or the most suitable answer from the given four alternatives

1) The value of $\sin^{-1}(\cos x)$, $0 \leq x \leq \pi$ is

$$\sin^{-1}(\cos x) = \sin^{-1}(\sin(\frac{\pi}{2}-x))$$

$$= \frac{\pi}{2} - x$$

$$\text{Ans: (3) } \frac{\pi}{2} - x$$

2) If $\sin^{-1}x + \sin^{-1}y = \frac{2\pi}{3}$

then $\cos^{-1}x + \cos^{-1}y$ is equal to

Soln: Using Property V

$$\sin^{-1}x + \cos^{-1}x = \frac{\pi}{2}$$

$$\sin^{-1}x = \frac{\pi}{2} - \cos^{-1}x$$

$$\text{Also } \sin^{-1}y + \cos^{-1}y = \frac{\pi}{2}$$

$$\sin^{-1}y = \frac{\pi}{2} - \cos^{-1}y$$

Given:

$$\sin^{-1}x + \sin^{-1}y = \frac{2\pi}{3}$$

$$\frac{\pi}{2} - \cos^{-1}x + \frac{\pi}{2} - \cos^{-1}y = \frac{2\pi}{3}$$

$$\cos^{-1}x + \cos^{-1}y = \pi - \frac{2\pi}{3}$$

$$\cos^{-1}x + \cos^{-1}y = \frac{2\pi - 2\pi}{3}$$

$$\cos^{-1}x + \cos^{-1}y = \frac{\pi}{3}$$

$$\text{Ans: (2) } \frac{\pi}{3}$$

3) $\sin^{-1}\frac{3}{5} - \cos^{-1}\frac{12}{13}$
 $+ \sec^{-1}\frac{5}{3} - \operatorname{cosec}^{-1}\frac{13}{12}$ is equal to

Soln:

$$\sin^{-1}\frac{3}{5} - \cos^{-1}\frac{12}{13}$$

$$+ \sec^{-1}\frac{5}{3} - \operatorname{cosec}^{-1}\frac{13}{12}$$

$$= \sin^{-1}\frac{3}{5} - \cos^{-1}\frac{12}{13}$$

$$+ \cos^{-1}\frac{3}{5} - \sin^{-1}\frac{12}{13}$$

$$= (\sin^{-1}\frac{3}{5} + \cos^{-1}\frac{3}{5}) - (\cos^{-1}\frac{12}{13} + \sin^{-1}\frac{12}{13})$$

$$= \frac{\pi}{2} - \frac{\pi}{2} = 0$$

Ans: (3) 0

4) If $\sin^{-1}x = 2\sin^{-1}x$ has a solution then

Soln: Let $-\frac{\pi}{2} \leq \sin^{-1}x \leq \frac{\pi}{2}$

Given: $\sin^{-1}x = 2\sin^{-1}x$

Now,

$$-\frac{\pi}{2} \leq 2\sin^{-1}x \leq \frac{\pi}{2}$$

$$\div 2, -\frac{\pi}{4} \leq \sin^{-1}x \leq \frac{\pi}{4}$$

$$-\sin\frac{\pi}{4} \leq x \leq \sin\frac{\pi}{4}$$

$$-\frac{1}{\sqrt{2}} \leq x \leq \frac{1}{\sqrt{2}}$$

$$|x| \leq \frac{1}{\sqrt{2}}$$

$$\text{Ans: (1) } |x| \leq \frac{1}{\sqrt{2}}$$

5) $\sin^{-1}(\cos x) = \frac{\pi}{2} - x$ is valid for

$$\text{Let } -\frac{\pi}{2} \leq \sin^{-1}x \leq \frac{\pi}{2}$$

$$\text{Put } x = \cos x,$$

$$-\frac{\pi}{2} \leq \sin^{-1}(\cos x) \leq \frac{\pi}{2}$$

$$\text{Given: } \sin^{-1}(\cos x) = \frac{\pi}{2} - x$$

Now,

$$-\frac{\pi}{2} \leq \frac{\pi}{2} - x \leq \frac{\pi}{2}$$

$$\text{Subtract } -\frac{\pi}{2},$$

$$-\frac{\pi}{2} - \frac{\pi}{2} \leq \frac{\pi}{2} - x - \frac{\pi}{2} \leq \frac{\pi}{2} - \frac{\pi}{2}$$

$$-\pi \leq -x \leq 0$$

$$\pi \geq x \geq 0$$

$$\Rightarrow 0 \leq x \leq \pi$$

$$\text{Ans: (2) } 0 \leq x \leq \pi$$

6) If $\sin^{-1}x + \sin^{-1}y + \sin^{-1}z = \frac{3\pi}{2}$ the value of

$$x^{\frac{2017}{19}} + x^{\frac{2018}{19}} + x^{\frac{2019}{19}} - \frac{9}{x^{\frac{101}{19}} + y^{\frac{101}{19}} + z^{\frac{101}{19}}}$$

Soln:

$$\text{Let } \sin^{-1}(1) = \pi/2$$

$$\text{Given: } \sin^{-1}(x) + \sin^{-1}(y) + \sin^{-1}(z) = \frac{3\pi}{2}$$

$$\text{Then } \sin^{-1}(1) + \sin^{-1}(1) + \sin^{-1}(1) = \frac{\pi}{2} + \frac{\pi}{2} + \frac{\pi}{2} = \frac{3\pi}{2}$$

$$\therefore x = y = z = 1$$

Also, Given

$$\begin{aligned} & x^{\frac{2017}{19}} + x^{\frac{2018}{19}} + x^{\frac{2019}{19}} - \frac{9}{x^{\frac{101}{19}} + y^{\frac{101}{19}} + z^{\frac{101}{19}}} \\ &= (1)^{\frac{2017}{19}} + (1)^{\frac{2018}{19}} + (1)^{\frac{2019}{19}} - \frac{9}{(1)^{\frac{101}{19}} + (1)^{\frac{101}{19}} + (1)^{\frac{101}{19}}} \end{aligned}$$

$$= 1 + 1 + 1 - \frac{9}{1 + 1 + 1}$$

$$= 3 - \frac{9}{3}$$

$$= 3 - 3 = 0$$

$$\text{Ans: (1) 0}$$

7) If $\cot^{-1}x = \frac{2\pi}{5}$ for some $x \in \mathbb{R}$, the value of $\tan^{-1}x$ is

Soln: Using Property V

$$\tan^{-1}x + \cot^{-1}x = \frac{\pi}{2}, x \in \mathbb{R}$$

$$\tan^{-1}x + \frac{2\pi}{5} = \frac{\pi}{2}$$

$$\tan^{-1}x = \frac{\pi}{2} - \frac{2\pi}{5}$$

$$\tan^{-1}x = \frac{5\pi - 4\pi}{10}$$

$$\tan^{-1}x = \frac{\pi}{10}$$

$$\text{Ans: (3) } \frac{\pi}{10}$$

8) The domain of the function defined by $f(x) = \sin^{-1}\sqrt{x-1}$ is

Soln:

Given: $f(x) = \sin^{-1}\sqrt{x-1}$
 $\sqrt{x-1}$ must be real

The domain of $\sin^{-1}x$ is $-1 \leq x \leq 1$
 Put $x = x-1$, upto 1
 $-1 \leq x-1 \leq 1$ — ②

From ① & ② we get,

$$0 \leq x-1 \leq 1$$

Add 1, $0+1 \leq x-1+1 \leq 1+1$
 $1 \leq x \leq 2$

$$\therefore x \in [1, 2]$$

$$\text{Ans: (1)} [1, 2]$$

9) If $x = \frac{1}{5}$, the value of $\cos(\cos^{-1}x + 2\sin^{-1}x)$ is

Soln: Given: $x = \frac{1}{5}$ and given

$$\cos(\cos^{-1}x + 2\sin^{-1}x) =$$

$$= \cos\left(\frac{\pi}{2} + \sin^{-1}x\right) \text{ using Property V}$$

$$= -\sin(\sin^{-1}x)$$

$$= -x = -\frac{1}{5}$$

$$\text{Ans: (4)} -\frac{1}{5}$$

10) $\tan^{-1}\left(\frac{1}{4}\right) + \tan^{-1}\left(\frac{2}{9}\right)$ is equal to

Soln:

$$\tan^{-1}\left(\frac{1}{4}\right) + \tan^{-1}\left(\frac{2}{9}\right)$$

$$= \tan^{-1}\left[\frac{\frac{1}{4} + \frac{2}{9}}{1 - \left(\frac{1}{4}\right)\left(\frac{2}{9}\right)}\right]$$

$$= \tan^{-1}\left[\frac{\frac{9+8}{36}}{\frac{36-2}{36}}\right]$$

$$= \tan^{-1}\left[\frac{17}{34}\right]$$

$$\text{Ans: (4)} \tan^{-1}\left(\frac{1}{2}\right)$$

11) If the function $F(x) = \sin^{-1}(x^2-3)$, then x belongs to

Soln:

Let $-1 \leq x \leq 1$ be the domain of $\sin^{-1}x$

$$-1 \leq x^2-3 \leq 1$$

$$\text{Add } 3, -1+3 \leq x^2-3+3 \leq 1+3$$

$$2 \leq x^2 \leq 4$$

$$\pm\sqrt{2} \leq x \leq \pm 2$$

$$\therefore -2 \leq x \leq -\sqrt{2} \text{ and}$$

$$\sqrt{2} \leq x \leq 2$$

$$\therefore x \in [-2, -\sqrt{2}] \cup [\sqrt{2}, 2]$$

$$\text{Ans: (3)} [-2, -\sqrt{2}] \cup [\sqrt{2}, 2]$$

of a triangle, then the third angle is
Soln:

$$\text{Let } A = \cot^{-1} 2$$

$$B = \cot^{-1} 3$$

$$\therefore A+B+C = \pi$$

$$\cot^{-1} 2 + \cot^{-1} 3 + C = \pi$$

$$\tan^{-1}\left(\frac{1}{2}\right) + \tan^{-1}\left(\frac{1}{3}\right) + C = \pi$$

$$\tan^{-1}\left[\frac{\frac{1}{2} + \frac{1}{3}}{1 - \left(\frac{1}{2}\right)\left(\frac{1}{3}\right)}\right] + C = \pi$$

$$\tan^{-1}\left[\frac{\frac{5}{6}}{\frac{5}{6}}\right] + C = \pi$$

$$\tan^{-1}(1) + C = \pi$$

$$\frac{\pi}{4} + C = \pi$$

$$C = \pi - \frac{\pi}{4} \Rightarrow C = \frac{3\pi}{4}$$

Ans: (2) $\frac{3\pi}{4}$

13) $\sin^{-1}(\tan \frac{\pi}{4}) - \sin^{-1}(\sqrt{\frac{3}{x}})$
 $= \frac{\pi}{6}$. Then x is a
 root of the equation

Soln:

Given:

$$\sin^{-1}(\tan \frac{\pi}{4}) - \sin^{-1}(\sqrt{\frac{3}{x}}) = \frac{\pi}{6}$$

$$\sin^{-1}(1) - \frac{\pi}{6} = \sin^{-1}(\sqrt{\frac{3}{x}})$$

$$\frac{\pi}{2} - \frac{\pi}{6} = \sin^{-1}(\sqrt{\frac{3}{x}})$$

$$\frac{3\pi - \pi}{6} = \sin^{-1}(\sqrt{\frac{3}{x}})$$

$$\frac{2\pi}{6} = \sin^{-1}(\sqrt{\frac{3}{x}})$$

$$\frac{\pi}{3} = \sin^{-1}(\sqrt{\frac{3}{x}})$$

$$\sin \frac{\pi}{3} = \sqrt{\frac{3}{x}}$$

$$\frac{\sqrt{3}}{2} = \sqrt{\frac{3}{x}}$$

Squaring on both sides,
 we get,

$$\frac{3}{4} = \frac{3}{x} \Rightarrow 3x = 12 \Rightarrow x = 4$$

$\therefore x = 4$ is a

root of the equation

$$x^2 - x - 12 = 0$$

$$(x-4)(x+3) = 0$$

$$x-4=0, x+3=0$$

$$x=4, x=-3$$

$$\therefore \text{Ans: (2) } x^2 - x - 12 = 0$$

14) $\sin^{-1}(2\cos^2 x - 1)$
 $+ \cos^{-1}(1 - 2\sin^2 x) =$

Soln:

$$\sin^{-1}(2\cos^2 x - 1)$$

$$+ \cos^{-1}(1 - 2\sin^2 x)$$

$$= \sin^{-1}(\cos 2x) + \cos^{-1}(\cos 2x)$$

$$= \frac{\pi}{2} \quad [\text{By Property IV} \quad \sin^{-1} x + \cos^{-1} x = \frac{\pi}{2}]$$

$$\text{Ans: (1) } \pi/2$$

15) If $\cot^{-1}(\sqrt{8}\sin x)$

$$+ \tan^{-1}(\sqrt{8}\sin x) = u$$
 then

$$\cos 2u \text{ is equal to}$$

Soln: Given

$$\cot^{-1}(\sqrt{8}\sin x) + \tan^{-1}(\sqrt{8}\sin x) = u$$

$$\frac{\pi}{2} = u$$

$$\text{Using Property V(ii)}$$

$$\text{Given: } \cos 2u = \cos 2(\frac{\pi}{2})$$

$$= \cos \pi = -1$$

$$\text{Ans: (3) } -1$$

16) If $|x| \leq 1$ then
 $2\tan^{-1} x - \sin^{-1} \frac{2x}{1+x^2}$ is
 equal to

Soln:

Given:

$$2\tan^{-1} x - \sin^{-1} \frac{2x}{1+x^2}$$

$$= 2\tan^{-1} x - 2\tan^{-1} x$$

$$= 0 \quad (\text{Using Property VII(iii)})$$

$$\text{Ans: (3) } 0$$

17) The equation

$$\tan^{-1} x - \cot^{-1} x = \tan^{-1}(\frac{1}{13})$$

$$\text{has}$$

$$\text{Soln: By Property V(ii)}$$

$$\tan^{-1} x + \cot^{-1} x = \frac{\pi}{2}$$

Given $\tan^{-1}x - \cot^{-1}x = \tan^{-1}\left(\frac{1}{\sqrt{3}}\right)$

$$\tan^{-1}x - \left(\frac{\pi}{2} - \tan^{-1}x\right) = \frac{\pi}{6}$$

$$\tan^{-1}x - \frac{\pi}{2} + \tan^{-1}x = \frac{\pi}{6}$$

$$2\tan^{-1}x = \frac{\pi}{2} + \frac{\pi}{6}$$

$$2\tan^{-1}x = \frac{3\pi + \pi}{6}$$

$$2\tan^{-1}x = \frac{4\pi}{6}$$

$$2\tan^{-1}x = \frac{2\pi}{3}$$

$$\tan^{-1}x = \frac{\pi}{3}$$

$$x = \tan \frac{\pi}{3} \Rightarrow \boxed{x = \sqrt{3}}$$

$x = \sqrt{3}$ is a unique soln.
Ans: (2) unique solution

18) If $\sin^{-1}x + \cot^{-1}\left(\frac{1}{2}\right) = \frac{\pi}{2}$
then x is equal to

$$\sin^{-1}x + \cot^{-1}\left(\frac{1}{2}\right) = \frac{\pi}{2}$$

$$\text{Let } \cot^{-1}\left(\frac{1}{2}\right) = \alpha$$

$$\text{Then } \cot \alpha = \frac{1}{2}$$

$$\frac{1}{\tan \alpha} = \frac{1}{2}$$

$$\therefore \tan \alpha = 2$$

$$1 + \tan^2 \alpha = \sec^2 \alpha$$

$$1 + 4 = \sec^2 \alpha$$

$$\therefore \sec^2 \alpha = 5$$

$$\sec \alpha = \sqrt{5}$$

$$\frac{1}{\cos \alpha} = \sqrt{5}$$

$$\cos \alpha = \frac{1}{\sqrt{5}}$$

$$\alpha = \cos^{-1}\left(\frac{1}{\sqrt{5}}\right)$$

$$\text{From (1), } \cot^{-1}\left(\frac{1}{2}\right) = \cos^{-1}\left(\frac{1}{\sqrt{5}}\right)$$

$$\cot^{-1}\left(\frac{1}{2}\right) = \cos^{-1}\left(\frac{1}{\sqrt{5}}\right)$$

$$\Rightarrow \sin^{-1}x + \cos^{-1}\left(\frac{1}{\sqrt{5}}\right) = \frac{\pi}{2}$$

$$\sin^{-1}x + \cos^{-1}\left(\frac{1}{\sqrt{5}}\right) = \frac{\pi}{2}$$

$$\text{Put } x = \frac{1}{\sqrt{5}}$$

$$\sin^{-1}\left(\frac{1}{\sqrt{5}}\right) + \cos^{-1}\left(\frac{1}{\sqrt{5}}\right) = \frac{\pi}{2}$$

$$\therefore x = \frac{1}{\sqrt{5}}$$

$$\text{Ans: (2) } \frac{1}{\sqrt{5}}$$

$$19) \text{ If } \sin^{-1}\frac{x}{5} + \cos^{-1}\frac{5}{4} = \frac{\pi}{2} \text{ then the value}$$

$$\text{of } x \text{ is}$$

$$\text{Soln: Given: } \sin^{-1}\frac{x}{5} + \cos^{-1}\frac{5}{4} = \frac{\pi}{2}$$

$$\sin^{-1}\frac{x}{5} + \cos^{-1}\frac{5}{4} = \frac{\pi}{2}$$

$$\sin^{-1}\frac{x}{5} + \sin^{-1}\frac{4}{5} = \frac{\pi}{2}$$

$$\sin^{-1}\frac{x}{5} + \sin^{-1}\frac{4}{5} = \frac{\pi}{2}$$

$$\sin^{-1}\frac{x}{5} + \sin^{-1}\frac{4}{5} = \frac{\pi}{2}$$

$$\therefore \boxed{x=3} \text{ Ans: (4) 3}$$

$$\text{Then } \cos \alpha = \frac{1}{\sqrt{5}}$$

$$\frac{1}{\sin \alpha} = \frac{5}{4}$$

$$\sin \alpha = \frac{4}{5}$$

$$\sin^2 \alpha = \frac{16}{25}$$

$$1 - \sin^2 \alpha = \cos^2 \alpha$$

$$1 - \frac{16}{25} = \cos^2 \alpha$$

$$\frac{9}{25} = \cos^2 \alpha$$

$$\cos \alpha = \frac{3}{5}$$

$$\alpha = \cos^{-1}\left(\frac{3}{5}\right)$$

$$\cos^{-1}\frac{5}{4} = \cos^{-1}\left(\frac{3}{5}\right)$$

$$\Rightarrow \sin^{-1}\frac{x}{5} + \cos^{-1}\left(\frac{3}{5}\right) = \frac{\pi}{2}$$

$$\text{Put } x = 3,$$

$$\sin^{-1}\frac{3}{5} + \cos^{-1}\frac{3}{5} = \frac{\pi}{2}$$

$$\therefore \boxed{x=3} \text{ Ans: (4) 3}$$

20) $\sin(\tan^{-1}x)$,
 $|x| < 1$ is equal to

Soln:

Let $\tan^{-1}x = \theta$

$$\Rightarrow \boxed{x = \tan \theta}$$

$$1 + \tan^2 \theta = \sec^2 \theta$$

$$1 + x^2 = \sec^2 \theta$$

$$\sec \theta = \sqrt{1 + x^2}$$

$$\frac{1}{\cos \theta} = \sqrt{1 + x^2}$$

$$\Rightarrow \cos \theta = \frac{1}{\sqrt{1 + x^2}}$$

$$\sin \theta = \tan \theta \times \cos \theta$$

$$\sin \theta = x \left(\frac{1}{\sqrt{1 + x^2}} \right)$$

$$\sin(\tan^{-1}x) = \frac{x}{\sqrt{1 + x^2}}$$

$$\text{Ans: (4) } \frac{x}{\sqrt{1 + x^2}}$$

8. Differentials and Partial Derivatives

EXERCISE 8-1

- 1) Let $f(x) = \sqrt[3]{x}$. Find the linear approximation at $x=27$. Use the linear approximation to approximate $\sqrt[3]{27.2}$

Soln:

$$L(x) = f(x_0) + f'(x_0)(x - x_0)$$

Given: $f(x) = \sqrt[3]{x}$, $x_0 = 27$,
 $\Delta x = 0.2$

$$f(x_0) = f(27) = \sqrt[3]{27}$$

$$= (27)^{1/3} = (3^3)^{1/3} = 3$$

$$\therefore \boxed{f(x_0) = 3}$$

Also, $f(x) = x^{1/3}$

$$f'(x) = \frac{1}{3} x^{1/3 - 1}$$

$$f'(x) = \frac{1}{3} \frac{x^{1/3}}{x}$$

$$f'(x_0) = f'(27) = \frac{1}{3} \frac{(27)^{1/3}}{27}$$

$$= \frac{1}{3} \frac{(3^3)^{1/3}}{27} = \frac{1}{3} \times \frac{3}{27}$$

$$\boxed{f'(x_0) = \frac{1}{27}}$$

$$\textcircled{1} \Rightarrow L(x) = f(x_0) + f'(x_0)(x - x_0)$$

$$L(x) = 3 + \frac{1}{27}(x - 27)$$

$$= 3 + \frac{x}{27} - 1$$

$$L(x) = \frac{x}{27} + 2 \text{ which is the required linear approximation.}$$

Also, $f(x) \approx L(x)$

when $x = 27.2$,

$$f(27.2) = \sqrt[3]{27.2} \approx \frac{27.2}{27} + 2$$

$$= 1.0074 + 2$$

$$= 3.0074$$

$$\begin{array}{r} 1.0074 \\ 27 \overline{) 27.2} \\ \underline{27} \\ 0.200 \\ \underline{189} \\ 110 \\ \underline{108} \\ 2 \end{array}$$

- 2) Use the linear approximation to find approximate values of (i) $(123)^{2/3}$

Soln:

$$L(x) = f(x_0) + f'(x_0)(x - x_0)$$

Let $f(x) = x^{2/3}$,

$x_0 = 125$ and $\Delta x = -2$

$$f(x_0) = f(125) = (125)^{2/3}$$

$$= (5^3)^{2/3} = 5^2 = 25$$

$$\therefore \boxed{f(x_0) = 25}$$

$$f'(x) = \frac{2}{3} x^{2/3 - 1}$$

$$f'(x) = \frac{2}{3} \frac{x^{2/3}}{x}$$

$$f'(x_0) = f'(125) = \frac{2}{3} \frac{(125)^{2/3}}{125}$$

$$= \frac{2}{3} \frac{(5^3)^{2/3}}{125}$$

$$= \frac{2}{3} \frac{25}{125} = \frac{2}{15}$$

$$\boxed{f'(x_0) = \frac{2}{15}}$$

$$\textcircled{1} \Rightarrow L(x) = f(x_0) + f'(x_0)(x - x_0)$$

$$\begin{aligned} L(x) &= 25 + \frac{2}{15}(x - 125) \\ &= 25 + \frac{2x}{15} - \frac{2 \times 125}{15} \\ &= 25 + \frac{2x}{15} - \frac{50}{3} \\ &= \frac{2x}{15} + 25 - \frac{50}{3} \\ &= \frac{2x}{15} + \frac{75 - 50}{3} \end{aligned}$$

$$L(x) = \frac{2x}{15} + \frac{25}{3} \text{ which}$$

is the required linear approximation.

$$\text{Also, } f(x) \approx L(x)$$

When $x = 123$,

$$\begin{aligned} f(123) &= (123)^{2/3} \approx \frac{2(123)}{15} + \frac{25}{3} \\ &= \frac{246}{15} + \frac{25}{3} \end{aligned}$$

$$= \frac{246 + 125}{15} = \frac{371}{15}$$

$$= 24.73$$

$$(123)^{2/3} = 24.73$$

$$\begin{array}{r} 24.73 \\ 15 \overline{) 371} \\ \underline{30} \\ 71 \\ \underline{60} \\ 110 \\ \underline{105} \\ 50 \\ \underline{45} \\ 5 \end{array}$$

(ii) $\sqrt[4]{15}$
Soln:

$$L(x) = f(x_0) + f'(x_0)(x - x_0) \quad \textcircled{1}$$

$$\text{Let } f(x) = \sqrt[4]{x} = x^{1/4}, x_0 = 16,$$

$$\text{and } \Delta x = -1$$

$$\begin{aligned} f(x_0) &= f(16) = (16)^{1/4} \\ &= (2^4)^{1/4} = 2 \end{aligned}$$

$$\boxed{f(x_0) = 2}$$

$$f'(x) = \frac{1}{4} x^{-3/4}$$

$$f'(x) = \frac{1}{4} \frac{x^{1/4}}{x}$$

$$f'(x_0) = f'(16) = \frac{1}{4} \frac{(16)^{1/4}}{16}$$

$$= \frac{1}{4} \frac{(2^4)^{1/4}}{16}$$

$$= \frac{2^1}{64 \cdot 32}$$

$$\boxed{f'(x_0) = \frac{1}{32}}$$

$$\textcircled{1} \Rightarrow L(x) = f(x_0) + f'(x_0)(x - x_0)$$

$$= 2 + \frac{1}{32}(x - 16)$$

$$= 2 + \frac{x}{32} - \frac{16}{32}$$

$$= \frac{x}{32} + 2 - \frac{1}{2}$$

$L(x) = \frac{x}{32} + \frac{3}{2}$ which is the required linear approximation.

Also, $f(x) \approx L(x)$

When $x = 15$,

$$f(15) = \sqrt[4]{15} \approx \frac{15}{32} + \frac{3}{2}$$

$$= \frac{15 + 48}{32} = \frac{63}{32}$$

$$\sqrt[4]{15} \approx 1.9688$$

$$\begin{array}{r} 1.9688 \\ 32 \overline{) 63} \\ \underline{32} \\ 310 \\ \underline{288} \\ 220 \\ \underline{192} \\ 280 \\ \underline{256} \\ 240 \\ \underline{224} \\ 160 \\ \underline{160} \\ 0 \end{array}$$

(iii) $\sqrt[3]{26}$

Soln:

$$L(x) = f(x_0) + f'(x_0)(x - x_0) \quad \text{--- (1)}$$

$$\text{Let } f(x) = \sqrt[3]{x} = x^{1/3},$$

$$x_0 = 27 \text{ and } \Delta x = -1$$

$$f'(x_0) = f'(27) = (27)^{1/3}$$

$$= (3^3)^{1/3} = 3$$

$$\therefore \boxed{f(x_0) = 3}$$

$$f'(x) = \frac{1}{3} x^{-2/3}$$

$$f'(x_0) = \frac{1}{3} \frac{x^{-2/3}}{x}$$

$$f'(x_0) = f'(27) = \frac{1}{3} \frac{(27)^{-2/3}}{(27)^{1/3}}$$

$$= \frac{1}{3} \frac{(3^3)^{-2/3}}{(3^3)^{1/3}} = \frac{1}{3} \left(\frac{8}{27} \right)$$

$$\boxed{f'(x_0) = \frac{1}{27}}$$

$$\Rightarrow L(x) = f(x_0) + f'(x_0)(x - x_0)$$

$$= 3 + \frac{1}{27}(x - 27)$$

$$= 3 + \frac{x}{27} - 1$$

$L(x) = \frac{x}{27} + 2$ which is the required linear approximation.

$$\text{Also, } f(x) \approx L(x)$$

$$\text{When } x = 26,$$

$$f(26) = \sqrt[3]{26} \approx \frac{26}{27} + 2$$

$$= 0.9629 + 2$$

$$= 2.9629$$

$$\sqrt[3]{26} \approx 2.963$$

$$\begin{array}{r} 0.9629 \\ 27 \overline{) 260} \\ \underline{243} \\ 170 \\ \underline{162} \\ 80 \\ \underline{54} \\ 260 \\ \underline{243} \\ 17 \end{array}$$

3) Find a linear approximation

tion for the following functions at the indicated points

$$(i) f(x) = x^3 - 5x + 12, x_0 = 2$$

Soln: Given:

$$f(x) = x^3 - 5x + 12, x_0 = 2$$

$$f(x_0) = f(2) = (2)^3 - 5(2) + 12$$

$$f(2) = 8 - 10 + 12$$

$$f(x_0) = f(2) = 10 \Rightarrow \boxed{f(x_0) = 10}$$

$$f'(x) = 3x^2 - 5$$

$$f'(x_0) = f'(2) = 3(2)^2 - 5$$

$$= 3(4) - 5$$

$$= 12 - 5$$

$$f'(x_0) = f'(2) = 7 \Rightarrow \boxed{f'(x_0) = 7}$$

$$L(x) = f(x_0) + f'(x_0)(x - x_0)$$

$$= 10 + 7(x - 2)$$

$$= 10 + 7x - 14$$

$$L(x) = 7x - 4 \text{ which is the required linear approximation}$$

$$(ii) g(x) = \sqrt{x^2 + 9}, x_0 = -4$$

Soln: Given:

$$g(x) = \sqrt{x^2 + 9}, x_0 = -4$$

$$g(x_0) = g(-4) = \sqrt{(-4)^2 + 9} = \sqrt{16 + 9} = \sqrt{25} = 5$$

$$\therefore \boxed{g(x_0) = 5}$$

$$g(x) = \sqrt{x^2 + 9} = (x^2 + 9)^{1/2}$$

$$g'(x) = \frac{1}{2} (x^2 + 9)^{-1/2} (2x)$$

$$g'(x) = \frac{x}{\sqrt{x^2 + 9}}$$

$$g'(x_0) = g'(-4) = \frac{-4}{\sqrt{(-4)^2 + 9}}$$

$$= \frac{-4}{\sqrt{16 + 9}} = \frac{-4}{\sqrt{25}}$$

$$\boxed{g'(x_0) = \frac{-4}{5}}$$

$$L(x) = g(x_0) + g'(x_0)(x - x_0)$$

$$= 5 + \left(\frac{-4}{5} \right) (x + 4)$$

$$= \frac{25 - 4x - 16}{5} = \frac{9 - 4x}{5}$$

$$L(x) = \frac{9 - 4x}{5} \text{ which is the required linear approximation}$$

$$(iii) h(x) = \frac{x}{x+1}, x_0 = 1$$

Soln: Given:

$$h(x) = \frac{x}{x+1}, x_0 = 1$$

$$h(x_0) = h(1) = \frac{1}{1+1} = \frac{1}{2}$$

$$\therefore h(x_0) = \frac{1}{2}$$

$$h'(x) = \frac{(x+1) \cdot 1 - x \cdot 1}{(x+1)^2}$$

$$= \frac{x+1-x}{(x+1)^2} = \frac{1}{(x+1)^2}$$

$$h'(x_0) = h'(1) = \frac{1}{(1+1)^2} = \frac{1}{2^2} = \frac{1}{4}$$

$$\therefore h'(x_0) = \frac{1}{4}$$

$$\therefore L(x) = h(x_0) + h'(x_0)(x - x_0)$$

$$= \frac{1}{2} + \frac{1}{4}(x-1)$$

$$= \frac{2+x-1}{4}$$

$$L(x) = \frac{x+1}{4} \text{ which is the required linear approximation}$$

4) The radius of a circular plate is measured as 12.65 cm instead of the actual length 12.5 cm. Find the following in calculating the area of the circular plate:

(i) Absolute error

(ii) Relative error

(iii) percentage error

Soln:

$$\text{Given: } r = 12.50$$

$$\text{and } dr = \Delta r = 0.15$$

$$\text{Area of Circle } A = \pi r^2$$

$$\frac{dA}{dr} = 2\pi r$$

$$dA = 2\pi r dr$$

$$= 2\pi \times 12.50 \times 0.15$$

$$dA = 3.75\pi \text{ cm}^2$$

$$\therefore \text{Approximate value}$$

$$= 3.75\pi \text{ cm}^2$$

Actual value

$$= A(12.65) - A(12.5)$$

$$= \pi(12.65)^2 - \pi(12.5)^2$$

$$= 160.0225\pi - 156.25\pi$$

$$= 3.7725\pi \text{ cm}^2$$

$$(i) \text{ Absolute error} \\ = \text{Actual value} - \text{Approximate value}$$

$$= 3.7725\pi - 3.75\pi$$

$$= 0.0225\pi \text{ cm}^2$$

$$(ii) \text{ Relative error} \\ = \frac{\text{Actual value} - \text{Approximate value}}{\text{Actual value}}$$

$$= \frac{3.7725\pi - 3.75\pi}{3.7725\pi}$$

$$= \frac{0.0225\pi}{3.7725\pi}$$

$$= \frac{0.0225}{3.7725} \times \frac{10000}{10000}$$

$$= \frac{225}{37725}$$

$= 0.00596 = 0.006 \text{ (app)}$ (iii) percentage error $= \text{Relative error} \times 100$ $= 0.006 \times 100$ $= 0.6 \%$	$= \frac{4}{3} \pi r^3$ $V'(r) = \frac{4}{3} \pi \cdot 3r^2$ $V'(r) = 4\pi r^2$ Change in the volume $= V(9.8) - V(10) \approx V'(r) dr$ $\approx 4\pi (10)^2 (-0.2)$ $= 4\pi (100) (-0.2)$ $= 4\pi (-20) = -80\pi \text{ cm}^3$ $\therefore$ Approximate change in the volume is decreased by $80\pi \text{ cm}^3$	$= 80\pi (-0.2)$ $= -16\pi \text{ cm}^2$ $\therefore$ Approximate change in the Surface Area is decreased by $16\pi \text{ cm}^2$	$T = 2\pi \left(\frac{l}{g}\right)^{1/2}$ Taking log on both sides we get, $\log T = \log \left(2\pi \left(\frac{l}{g}\right)^{1/2}\right)$ $= \log 2\pi + \log \left(\frac{l}{g}\right)^{1/2}$ $= \log 2\pi + \frac{1}{2} \log \left(\frac{l}{g}\right)$ $= \log 2\pi + \frac{1}{2} [\log l - \log g]$ $\log T = \log 2\pi + \frac{1}{2} [\log l - \log g]$ $\frac{1}{T} dT = 0 + \frac{1}{2} \left[\frac{1}{l} dl - 0\right]$ $\frac{dT}{T} = \frac{1}{2l} dl$ Given: $\frac{dl}{l} = 2\% \text{ of } l$ $\frac{dT}{T} = \frac{1}{2l} \left(\frac{2}{100} l\right)$ $\frac{dT}{T} = \frac{1}{100}$
5. A Sphere is made of ice having radius 10 cm. Its radius decreases from 10 cm to 9.8 cm. Find approximations for the following: (i) change in the volume (ii) change in the surface area. Soln: Given: $r = 10$, $dr = \Delta r = 9.8 - 10 = -0.2$ (i) Volume of the sphere V	(ii) Surface Area of the sphere $S = 4\pi r^2$ $S'(r) = 8\pi r$ Change in the Surface Area $= S(9.8) - S(10) \approx S'(r) dr$ $\approx 8\pi (10) (-0.2)$	6) The time T, taken for a complete oscillation of a single pendulum with length l, is given by the equation $T = 2\pi \sqrt{\frac{l}{g}}$, where g is a constant. Find the approximate percentage error in the calculated value of T corresponding to an error of 2 percent in the value of l Soln: Given: $T = 2\pi \sqrt{\frac{l}{g}}$	Find the approximate percentage error in the calculated value of T corresponding to an error of 2 percent in the value of l Soln: Given: $T = 2\pi \sqrt{\frac{l}{g}}$

∴ Relative error

$$\frac{\Delta T}{T} \approx \frac{dT}{T} = \frac{1}{100}$$

∴ percentage error

$$= \frac{\Delta T}{T} \times 100 = \frac{1}{100} \times 100 = 1\%$$

Q Show that the percentage error in the n^{th} root of a number is approximately $\frac{1}{n}$ times the percentage error in the number.

Soln: Let x be the number.

Let $y = f(x) = (x)^{\frac{1}{n}}$
Taking log on both sides

We get,

$$\log y = \log(x)^{\frac{1}{n}}$$

$$\log y = \frac{1}{n} \log x$$

$$\frac{1}{y} dy = \frac{1}{n} \times \frac{1}{x} dx$$

$$\frac{\Delta y}{y} \approx \frac{dy}{y} = \frac{1}{n} \times \frac{1}{x} dx$$

$$\frac{\Delta y}{y} \times 100 \approx \frac{1}{n} \left(\frac{dx}{x} \times 100 \right)$$

$= \frac{1}{n}$ times the percentage error in the number.

EX 8.1

Find the linear approximation for $f(x) = \sqrt{1+x}$, $x \geq -1$, at $x_0 = 3$. Use

the linear approximation to estimate $f(3.2)$

Soln:

$$L(x) = f(x_0) + f'(x_0)(x - x_0) \quad \text{--- (1)}$$

$$\text{Given: } f(x) = \sqrt{1+x},$$

$$x_0 = 3 \text{ and } \Delta x = 0.2$$

$$f(x_0) = f(3) = \sqrt{1+3} = \sqrt{4}$$

$$\boxed{f(x_0) = 2}$$

$$\text{Also, } f(x) = \sqrt{1+x} = (1+x)^{\frac{1}{2}}$$

$$f'(x) = \frac{1}{2} (1+x)^{-\frac{1}{2}}$$

$$f'(x) = \frac{1}{2\sqrt{1+x}}$$

$$f'(x_0) = f'(3) = \frac{1}{2\sqrt{1+3}}$$

$$= \frac{1}{2\sqrt{4}} = \frac{1}{(2)(2)} = \frac{1}{4}$$

$$\therefore \boxed{f'(x_0) = \frac{1}{4}}$$

$$\text{①} \Rightarrow L(x) = f(x_0) + f'(x_0)(x - x_0)$$

$$L(x) = 2 + \frac{1}{4}(x - 3)$$

$$= 2 + \frac{x}{4} - \frac{3}{4}$$

$$L(x) = \frac{x}{4} + \frac{5}{4} \text{ which}$$

is the required linear approximation.

$$\text{Also, } f(x) \approx L(x)$$

$$\text{when } x = 3.2,$$

$$f(3.2) = \sqrt{1+3.2}$$

$$= \sqrt{4.2} \approx \frac{3.2}{4} + \frac{5}{4}$$

$$= \frac{8.2}{4} = 2.050$$

Ex 8.2

Use linear approximation to find an approximate value of $\sqrt{9.2}$ without using a calculator.

Soln:

$$L(x) = f(x_0) + f'(x_0)(x - x_0) \quad \text{①}$$

Given: $f(x) = \sqrt{x}$, $x_0 = 9$
and $\Delta x = 0.2$

$$f(x_0) = f(9) = \sqrt{9} = 3$$

$$\therefore \boxed{f(x_0) = 3}$$

$$\text{Also, } f(x) = \sqrt{x} = x^{1/2}$$

$$f'(x) = \frac{1}{2} x^{-1/2} = \frac{1}{2\sqrt{x}}$$

$$f'(x_0) = f'(9) = \frac{1}{2\sqrt{9}} \\ = \frac{1}{2(3)} = \frac{1}{6}$$

$$\boxed{f'(x_0) = \frac{1}{6}}$$

$$\text{① } L(x) = f(x_0) + f'(x_0)(x - x_0)$$

$$L(x) = 3 + \frac{1}{6}(x - 9) \\ = 3 + \frac{x}{6} - \frac{9}{6} \\ = \frac{x}{6} + 3 - \frac{3}{2}$$

$L(x) = \frac{x}{6} + \frac{3}{2}$ which is the required linear approximation.

Also, $f(x) \approx L(x)$,

$$\text{When } x = 9.2 + \frac{3}{2} \\ = 1.53333 + 1.5 \\ = 3.03333$$

Ex 8.3

Let us assume that the shape of a soap bubble is a sphere. Use linear approximation to approximate the increase in the surface

area of a soap bubble as its radius increases from 5 cm to 5.2 cm.

Also, Calculate the percentage error.

Soln: Given: $r = 5$, $dr = \Delta r = 5.2 - 5 = 0.2$

Surface Area of the Sphere $S(r) = 4\pi r^2$

$$S'(r) = 8\pi r$$

$$S'(5) = 8\pi(5)$$

$$S'(5) = 40\pi$$

Change in the Surface

$$\text{Area} = S(5.2) - S(5) \approx S'(5)dr$$

$$\approx (40\pi)(0.2)$$

$$= 8\pi \text{ cm}^2$$

Also,

$$S(5.2) - S(5)$$

$$= 4\pi(5.2)^2 - 4\pi(5)^2$$

$$= 4\pi(27.04) - 4\pi(25)$$

$$= 108.16\pi - 100\pi \\ = 8.16\pi \text{ cm}^2$$

Relative error

$$= \frac{\text{Actual value} - \text{Approximate value}}{\text{Actual Value}}$$

$$= \frac{8.16\pi - 8\pi}{8.16\pi}$$

$$= \frac{0.16\pi}{8.16\pi} = \frac{0.16}{8.16} \times \frac{100}{100}$$

$$= \frac{16}{816} = 0.019607$$

Percentage error

$$= \text{Relative error} \times 100$$

$$= 0.019607 \times 100$$

$$= 1.9607\%$$

Ex 8.4

A right Circular Cylinder has radius

$r = 10\text{ cm}$ and height $h = 20\text{ cm}$. Suppose that the radius of the cylinder is increased from 10 cm to 10.1 cm and the height does not change. Estimate the change in the volume of the cylinder. Also, calculate the relative error and percentage error.

Sol: Given: $r = 10\text{ cm}$,
 $h = 20\text{ cm}$ and

$dr = \Delta r = 0.1$
Volume of the cylinder $V = \pi r^2 h$
 $V = 20\pi r^2$

$$V'(r) = 40\pi r \, dr$$

$$V'(r) = 40\pi (10) (0.1)$$

$$= (400\pi) (0.1)$$

$$V'(r) = 40\pi \text{ cm}^3$$

$\therefore$ The change in the volume is $40\pi \text{ cm}^3$

$\therefore$ Approximate value
 $= 40\pi \text{ cm}^3$

Actual Value

$$= V(10.1) - V(10)$$

$$= 20\pi (10.1)^2 - 20\pi (10)^2$$

$$= 20\pi (102.01) - 20\pi (100)$$

$$= 2040.2\pi - 2000\pi$$

$$= 40.2\pi \text{ cm}^3$$

$\therefore$ Relative error

$$= \frac{\text{Actual value} - \text{Approximate value}}{\text{Actual value}}$$

$$= \frac{40.2\pi - 40\pi}{40.2\pi}$$

$$= \frac{0.2\pi}{40.2\pi} = \frac{0.2}{40.2} \times \frac{10}{10}$$

$$= \frac{2}{402} = \frac{1}{201} = 0.00497$$

$\therefore$ Percentage error

$$= \text{Relative error} \times 100$$

$$= 0.00497 \times 100$$

$$= 0.497\%$$

8.2.3- Differentials

EXERCISE 8.2

1) Find differential dy for each of the following functions:

(i) $y = \frac{(1-2x)^3}{3-4x}$

Soln:

$$\frac{dy}{dx} = \frac{(3-4x)3(1-2x)^2(-2) - (1-2x)^3(-4)}{(3-4x)^2}$$

$$= \frac{2(1-2x)^2[(3-4x)(-3) - (1-2x)(-2)]}{(3-4x)^2}$$

$$= \frac{2(1-2x)^2[-9+12x+2-4x]}{(3-4x)^2}$$

$= \frac{2(1-2x)^2(8x-7)}{(3-4x)^2}$ $\frac{dy}{dx} = \frac{2(1-2x)^2(8x-7)}{(3-4x)^2}$ $dy = \frac{2(1-2x)^2(8x-7)}{(3-4x)^2} dx$	$\frac{dy}{dx} = e^{x^2-5x+7} [(2x-5)\cos(x^2-1) - 2x \sin(x^2-1)]$ $dy = e^{x^2-5x+7} [(2x-5)\cos(x^2-1) - 2x \sin(x^2-1)] dx$	$df = (2x+3)dx$ when $x=3$, $dx=0.02$ $df = (2(3)+3)(0.02)$ $= 9(0.02)$ $df = 0.18$	$df = 2.0$ $\Delta f = f(x+\Delta x) - f(x)$ $= f(2+0.5) - f(2)$ $= f(2.5) - f(2)$ $= f\left(\frac{5}{2}\right) - f(2)$ $= \left[\left(\frac{5}{2}\right)^3 - 2\left(\frac{5}{2}\right)^2\right] - [2^3 - 2(2)^2]$ $= \left[\frac{125}{8} - 2\left(\frac{25}{4}\right)\right] - [8 - 8]$ $= \left[\frac{125}{8} - 100\right] - [0]$ $= \frac{25}{8} = 3.125$ $\therefore \Delta f = 3.125$
(ii) $y = (3 + \sin(2x))^{2/3}$ Soln: $\frac{dy}{dx} = \frac{2}{3} (3 + \sin(2x))^{-1/3} (\cos(2x) \cdot 2)$ $\frac{dy}{dx} = \frac{4}{3} \frac{\cos 2x}{(3 + \sin 2x)^{1/3}}$ $dy = \frac{4}{3} \frac{\cos 2x}{(3 + \sin 2x)^{1/3}} dx$	2) Find df for $f(x) = x^2 + 3x$ and evaluate it for (i) $x=2$ and $dx=0.1$ (ii) $x=3$ and $dx=0.02$ Soln: (i) $f(x) = x^2 + 3x$ $\frac{df}{dx} = 2x + 3$ $df = (2x+3)dx$ when $x=2$, $dx=0.1$ $df = (2(2)+3)(0.1)$ $= 7(0.1)$ $df = 0.7$	3) Find Δf and df for the function f for the indicated values of x, Δx and compare (i) $f(x) = x^3 - 2x^2$; $x=2$, $\Delta x = dx = 0.5$ Soln: $f(x) = x^3 - 2x^2$ $\frac{df}{dx} = 3x^2 - 4x$ $df = (3x^2 - 4x)dx$ when $x=2$, $dx=0.5$ $df = (3(2)^2 - 4(2))(0.5)$ $= (12 - 8)(0.5)$ $= (4)(0.5)$	(ii) $f(x) = x^2 + 2x + 3$; $x = -0.5$, $\Delta x = dx = 0.1$ Soln: $f(x) = x^2 + 2x + 3$ $\frac{df}{dx} = 2x + 2$
(iii) $y = e^{x^2-5x+7} \cos(x^2-1)$ Soln: $\frac{dy}{dx} = e^{x^2-5x+7} (-\sin(x^2-1)(2x)) + \cos(x^2-1) e^{x^2-5x+7} (2x-5)$	(ii) $f(x) = x^2 + 3x$ $\frac{df}{dx} = 2x + 3$		

$$df = (2x+2)dx$$

when $x = -0.5, dx = 0.1$

$$df = (2(-0.5)+2)(0.1)$$

$$= (-1.0+2)(0.1)$$

$$= (2-1)(0.1)$$

$$= (1)(0.1)$$

$$df = 0.1$$

$$\Delta f = f(x+\Delta x) - f(x)$$

$$= f(-0.5+0.1) - f(-0.5)$$

$$= f(-0.4) - f(-0.5)$$

$$= [(-0.4)^2 + 2(-0.4) + 3]$$

$$- [(-0.5)^2 + 2(-0.5) + 3]$$

$$= [0.16 - 0.8 + 3]$$

$$- [0.25 - 1.0 + 3]$$

$$= [-0.64 + 3] - [-0.75 + 3]$$

$$= 2.36 - 2.25$$

$$\Delta f = 0.11$$

4) Assuming $\log_{10} e = 0.4343$,
find an approximate
value of $\log_{10} 3$

Soln: Given: $\log_{10} e = 0.4343$

Let $y = f(x) = \log_{10} x$,

$x = 1000, \Delta x = dx = 3$

$y = f(1000) = \log_{10} 1000$

$$= \log_{10} 10^3 = 3 \log_{10} 10$$

$$= 3(1)$$

$$\boxed{y = 3}$$

$$y = \log_{10} x$$

$$\frac{dy}{dx} = \frac{1}{x} \log_{10} e$$

$$dy = \frac{\log_{10} e}{x} dx$$

$$= \frac{0.4343}{1000} \times 3$$

$$= \frac{1.3029}{1000}$$

$$dy = 0.0013029$$

$$f(x+\Delta x) \approx y + dy$$

$$f(1003) \approx 3 + 0.0013029$$

$$\log_{10} 1003 \approx 3.0013029$$

5) The trunk of a tree
has diameter 30 cm.

During the following year,

the Circumference grew
6 cm. (i) Approximately,
how much did the tree's
diameter grow?

(ii) what is the percenta
ge increase in area
of the tree's cross-sec
-tion?

Soln: Given:

Diameter = 30 cm

Radius $r = 15$ cm

Circumference

$$= 2\pi r = 2\pi \times 15 = 30\pi$$

Increase in Circumferen

$$-ce = 6$$

$$2\pi r_2 - 2\pi r_1 = 36 - 30 = 6$$

$$2\pi(r_2 - r_1) = 6$$

$$r_2 - r_1 = \frac{K3}{2\pi}$$

$$\therefore \Delta r = dr = \frac{3}{\pi}$$

Area of the circle

$$A = \pi r^2$$

$$\frac{dA}{dr} = 2\pi r$$

$$dA = 2\pi r dr$$

$$dA = 2\pi(15) \frac{3}{\pi}$$

$$dA = 90 \text{ cm}$$

(ii)

Percentage of increasing

$$= \frac{\text{Increasing area}}{\text{actual area}} \times 100$$

$$= \frac{90}{\pi(15)^2} \times 100$$

$$= \frac{9000}{225\pi} = \frac{40}{\pi} \%$$

6) An egg of a particular bird is very nearly spherical. If the radius to the inside of the shell is 5 mm and radius to the outside of the shell is 5.3 mm. Find the volume of the shell approximately.

Soln: Given: $r = 5 \text{ mm}$
 $\Delta r = dr = 5.3 - 5 = 0.3$
 $\therefore dr = 0.3$

Volume of the sphere

$$V = \frac{4}{3} \pi r^3$$

$$\frac{dV}{dr} = \frac{4}{3} \pi \cdot 3r^2$$

$$dV = 4\pi r^2 dr$$

Volume of Shell = $V(5.3) - V(5)$

$$\approx dV = 4\pi(5)^2(0.3)$$

Volume of Shell $\approx 30\pi \text{ mm}^3$

7) Assume that the cross section of the artery of human is circular. A drug is given to a patient to dilate his arteries. If the radius of an artery is increased from 2 mm to 2.1 mm, how much is cross-sectional area increased approximately?

Soln: Given: $r = 2 \text{ mm}$
 $\Delta r = dr = 2.1 - 2 = 0.1$
 Area of Circular artery
 $A = \pi r^2$

$$\frac{dA}{dr} = 2\pi r$$

$$dA = 2\pi r dr$$

$\therefore$ Cross-sectional area increased

$$= A(2.1) - A(2) \approx dA = 2\pi(2)(0.1)$$

$$= 0.4\pi \text{ mm}^2$$

$\therefore$ Cross-sectional area increased is $0.4\pi \text{ mm}^2$ (approximately)

8) In a newly developed city, it is estimated that the voting population (in thousands) will increase according to $V(t) = 30 + 12t^2 - t^3$, $0 \leq t \leq 8$ where t is the time in years. Find the approximate

Change in Voters for the time change from 4 to $4\frac{1}{6}$ year

Soln: Given:

$$V(t) = 30 + 12t^2 - t^3, \\ 0 \leq t \leq 8, t = 4$$

$$\Delta t = dt = 4\frac{1}{6} - 4$$

$$dt = \frac{1}{6}$$

$$\frac{dv}{dt} = 24t - 3t^2$$

$$dv = (24t - 3t^2) dt$$

Change in voters $\Delta v = v(4\frac{1}{6}) - v(4)$

$$\approx dv = (24t - 3t^2) dt$$

$$= (24(4) - 3(4)^2) \frac{1}{6}$$

$$= (96 - 48) \frac{1}{6}$$

$$= (48) \frac{1}{6}$$

$$= 8 \text{ (thousand)}$$

$\therefore$ Change in voters is 8000 (approximately)

9) The relation between the number of words y a person learns in x hours is given by $y = 52\sqrt{x}$, $0 \leq x \leq 9$. What is the approximate number of words learned when x changes from (i) 1 to 1.1 hour? (ii) 4 to 4.1 hour?

Soln: Given: $y = 52\sqrt{x}$, $0 \leq x \leq 9$

(i) $x = 1$, $\Delta x = dx = 1.1 - 1 = 0.1$

$$y = 52x^{1/2}$$

$$\frac{dy}{dx} = 52 \times \frac{1}{2} x^{-1/2}$$

$$\frac{dy}{dx} = 26 \times \frac{1}{x^{1/2}}$$

$$dy = 26 \times \frac{1}{\sqrt{x}} dx$$

$$\Delta y \approx dy = 26 \times \frac{1}{\sqrt{1}} (0.1)$$

$$= 26 \times \frac{1}{1} (0.1)$$

$$= 26 \times (0.1)$$

$$= 2.6$$

$$\approx 3 \text{ words}$$

$\therefore$ The number of words learned is 3 words (approximately)

(ii) $x = 4$, $\Delta x = dx = 4.1 - 4 = 0.1$

$$dy = 26 \times \frac{1}{\sqrt{x}} dx$$

$$= 26 \times \frac{1}{\sqrt{4}} (0.1)$$

$$= \frac{13}{2} \times \frac{1}{2} (0.1)$$

$$= 13 \times 0.1$$

$$= 1.3$$

$$\approx 1 \text{ word}$$

$\therefore$ The number of words learned is 1 word (approximately)

10) A circular plate expands uniformly under the influence of heat. If its radius increases from 10.5 cm to 10.75 cm, then

Find an approximate change in the area and the approximate percentage change in the area.

Soln: Given: $r = 10.5$

$$\Delta r = dr = 10.75 - 10.5 = 0.25$$

Area of Circular plate

$$A = \pi r^2$$

$$\frac{dA}{dr} = 2\pi r$$

$$dA = 2\pi r dr$$

(i) Approximate change

$$\text{in Area } \Delta A \approx dA = 2\pi(10.5)(0.25)$$

$$= 2\pi \times 2.625$$

$$dA = 5.25\pi \text{ cm}^2$$

$\therefore$ Approximate Change

$$\text{in the Area} = 5.25\pi \text{ cm}^2$$

(ii) Approximate Percentage

change in the Area

$$= \frac{dA}{A} \times 100$$

$$= \frac{5.25\pi}{\pi(10.5)^2} \times 100$$

$$= \frac{525}{110.25} \times \frac{100}{100}$$

$$= \frac{52500}{11025}$$

$$= 4.76\%$$

11) A Coat of paint of thickness 0.2 cm is applied to the faces of a cube

whose edge is 10 cm. Use the differentials to find approximately how many Cubic Centimeters of Paint is used to paint this cube. Also calculate

the exact amount of paint used to paint this cube

Soln: Given:

Edge of cube $a = 10$ cm

and $\Delta a = da = 0.2$ cm

Volume of cube $V = a^3$

$$\frac{dV}{da} = 3a^2$$

$$dV = 3a^2 da$$

$$\text{Volume of Paint } \Delta V \approx dV = 3(10)^2(0.2)$$

$$= 300 \times 0.2$$

$$= 60.0$$

$$= 60 \text{ cm}^3$$

$\therefore$ The paint used to cube is 60 cm^3 (approximately)

Exact volume of paint

$$= V(10.2) - V(10)$$

$$= (10.2)^3 - (10)^3$$

$$= 1061.208 - 1000$$

$$= 61.208$$

$$= 61.2 \text{ cm}^3$$

$\therefore$ The Exact amount of paint used to paint of cube is 61.2 cm^3

Ex 8-5.

Let $f, g: (a, b) \rightarrow \mathbb{R}$ be differentiable functions. Show that $d(fg) = fdg + gdf$

Soln:

Let $f, g: (a, b) \rightarrow \mathbb{R}$ be differentiable functions

and $h(x) = f(x)g(x)$

$$d(h(x)) = d(f(x)g(x))$$

$$\therefore dh = h'(x)$$

$$h'(x)dx = (f(x)g'(x) + g(x)f'(x))dx$$

$$h'(x)dx = f(x)g'(x)dx + g(x)f'(x)dx$$

$$dh = f(x)dg + g(x)df$$

$$d(fg) = f dg + g df$$

EX 8.6

Let $g(x) = x^2 + \sin x$.

Calculate the differential

of dg

Soln: Given: $g(x) = x^2 + \sin x$

$$\frac{d(g(x))}{dx} = 2x + \cos x$$

$$d(g(x)) = (2x + \cos x)dx$$

$$\therefore dg = (2x + \cos x)dx$$

EX 8.7

If the radius of a sphere, with radius 10 cm, has to decrease by 0.1 cm, approximately how much will its volume decrease?

Soln: Given: $r = 10$ cm

and $dr = 9.9 - 10 = -0.1$

where $r > 0$

Volume of a sphere

$$V = \frac{4}{3} \pi r^3$$

$$dV = \frac{4}{3} \pi 3r^2 dr$$

$$\therefore \Delta V \approx dV = 4\pi r^2 dr$$

$$= 4\pi (10)^2 (-0.1)$$

$$= 4\pi (100) (-0.1)$$

$$= 400\pi (-0.1)$$

$$= -40.0\pi$$

$$= -40\pi \text{ cm}^3$$

8.3 Functions of Several Variables 8.3.1 and 8.3.2

limit and continuity

of Functions of one variable and two variables

EXERCISE 8.3

1) Evaluate $\lim_{(x,y) \rightarrow (1,2)} g(x,y)$,

if the limit exists, where $g(x,y) = \frac{3x^2 - xy}{x^2 + y^2 + 3}$

Soln:

$$\lim_{(x,y) \rightarrow (1,2)} g(x,y) = \lim_{(x,y) \rightarrow (1,2)} \frac{3x^2 - xy}{x^2 + y^2 + 3}$$

$$= \frac{3(1)^2 - (1)(2)}{(1)^2 + (2)^2 + 3}$$

$$= \frac{3 - 2}{1 + 4 + 3} = \frac{1}{8}$$

2) Evaluate

$$\lim_{(x,y) \rightarrow (0,0)} \cos\left(\frac{x^3 + y^2}{x + y + 2}\right)$$

If the limit exists

Soln: $\lim_{(x,y) \rightarrow (0,0)} \cos\left(\frac{x^3 + y^2}{x + y + 2}\right)$

$$= \cos\left(\frac{0+0}{0+0+2}\right)$$

$$= \cos\left(\frac{0}{2}\right) = \cos 0 = 1$$

3) Let $f(x, y) = \frac{y^2 - xy}{\sqrt{x} - \sqrt{y}}$
for $(x, y) \neq (0, 0)$. Show
that $\lim_{(x, y) \rightarrow (0, 0)} f(x, y) = 0$

Soln:

$$\lim_{(x, y) \rightarrow (0, 0)} f(x, y) = \lim_{(x, y) \rightarrow (0, 0)} \frac{y^2 - xy}{\sqrt{x} - \sqrt{y}} \cdot \frac{\sqrt{x} + \sqrt{y}}{\sqrt{x} + \sqrt{y}}$$

$$= \lim_{(x, y) \rightarrow (0, 0)} \frac{y(y-x)(\sqrt{x} + \sqrt{y})}{(\sqrt{x})^2 - (\sqrt{y})^2}$$

$$= \lim_{(x, y) \rightarrow (0, 0)} \frac{y(-1)(\sqrt{x} + \sqrt{y})}{(x-y)}$$

$$= \lim_{(x, y) \rightarrow (0, 0)} y(-1)(\sqrt{x} + \sqrt{y})$$

$$= (0)(-1)(\sqrt{0} + \sqrt{0})$$

$$= (0)(-1)(0+0) = (0)(-1)(0) = 0$$

4) Evaluate $\lim_{(x, y) \rightarrow (0, 0)} \cos\left(\frac{e^x \sin y}{y}\right)$
if the limit exists.

Soln:

By Composite Function
theorem,

$$\lim_{(x, y) \rightarrow (0, 0)} \cos\left(\frac{e^x \sin y}{y}\right) = \cos\left(\lim_{(x, y) \rightarrow (0, 0)} \frac{e^x \sin y}{y}\right)$$

$$= \cos\left(\lim_{(x, y) \rightarrow (0, 0)} e^x \cdot \lim_{(x, y) \rightarrow (0, 0)} \frac{\sin y}{y}\right)$$

$$= \cos(e^0 \cdot 1) = \cos(1 \cdot 1) = \cos(1)$$

5) Let $g(x, y) = \frac{x^2 y}{x^4 + y^2}$ for $(x, y) \neq (0, 0)$ and $f(0, 0) = 0$

(i) Show that $\lim_{(x, y) \rightarrow (0, 0)} g(x, y) = 0$

along every line $y = mx, m \in \mathbb{R}$

(ii) Show that $\lim_{(x, y) \rightarrow (0, 0)} g(x, y) = \frac{k}{1+k^2}$

along every parabola $y = kx^2, k \in \mathbb{R} \setminus \{0\}$

along every parabola $y = kx^2, k \in \mathbb{R} \setminus \{0\}$

Soln: (i)

$$\lim_{(x, y) \rightarrow (0, 0)} g(x, y) = \lim_{(x, y) \rightarrow (0, 0)} \frac{x^2 y}{x^4 + y^2}$$

$$= \lim_{(x, y) \rightarrow (0, 0)} \frac{x^2 mx}{x^4 + m^2 x^2}$$

$$= \lim_{(x, y) \rightarrow (0, 0)} \frac{x^2 mx}{x^2(x^2 + m^2)}$$

$$= \lim_{(x, y) \rightarrow (0, 0)} \frac{mx}{(x^2 + m^2)}$$

$$= \frac{m(0)}{0 + m^2} = \frac{0}{m^2} = 0$$

$$(ii) \lim_{(x, y) \rightarrow (0, 0)} g(x, y) = \lim_{(x, y) \rightarrow (0, 0)} \frac{x^2 y}{x^4 + y^2}$$

$$= \lim_{(x, y) \rightarrow (0, 0)} \frac{x^2 k x^2}{x^4 + (k x^2)^2}$$

$$= \lim_{(x, y) \rightarrow (0, 0)} \frac{x^4 k}{x^4 + k^2 x^4}$$

$$= \lim_{(x, y) \rightarrow (0, 0)} \frac{x^4 k}{x^4(1 + k^2)}$$

$$= \lim_{(x, y) \rightarrow (0, 0)} \frac{k}{1 + k^2}$$

$$= \frac{k}{1 + k^2}$$

6) Show that $f(x, y) = \frac{x^2 - y^2}{y^2 + 1}$ is continuous at every $(x, y) \in \mathbb{R}^2$

Soln:

Let $(a, b) \in \mathbb{R}^2$ be an arbitrary point

(i) $f(a, b) = \frac{a^2 - b^2}{b^2 + 1}$ is defined for all $(a, b) \in \mathbb{R}^2$

(ii) $\lim_{(x, y) \rightarrow (a, b)} f(x, y) = \lim_{(x, y) \rightarrow (a, b)} \frac{x^2 - y^2}{y^2 + 1}$

$$= \frac{a^2 - b^2}{b^2 + 1} = f(a, b)$$

$$= L$$

$\therefore$ limit exists at $(a, b) \in \mathbb{R}^2$

(ii) $\lim_{(x, y) \rightarrow (a, b)} f(x, y) = L = f(a, b)$

$$= \frac{a^2 - b^2}{b^2 + 1}$$

$\therefore f$ is continuous at every point of $\mathbb{R}^2$

7) Let $g(x,y) = \frac{e^y \sin x}{x}$, For $x \neq 0$ and $g(0,0) = 1$. Show that g is continuous at $(0,0)$ <u>Soln</u>: Given: $g(x,y) = \frac{e^y \sin x}{x}$ and $g(0,0) = 1$ $g(x,y) - g(0,0) = \left \frac{e^y \sin x}{x} - 1 \right$ $= \left \frac{e^y \sin x - x}{x} \right$ (i) $g(0,0) = 1$, g is defined at $(0,0)$ (ii) $\lim_{(x,y) \rightarrow (0,0)} g(x,y) = \lim_{(x,y) \rightarrow (0,0)} \frac{e^y \sin x}{x}$ $= \lim_{(x,y) \rightarrow (0,0)} e^y \lim_{(x,y) \rightarrow (0,0)} \frac{\sin x}{x}$ $= e^0 \cdot 1 = 1 \cdot 1 = 1 = L$	$\therefore$ limit exists at $(0,0)$ (iii) $\lim_{(x,y) \rightarrow (0,0)} g(x,y) = 1 = g(0,0)$ $\therefore g$ is continuous at $(0,0)$ <u>Ex 8.8</u> Let $F(x,y) = \frac{3x-5y+8}{x^2+y^2+1}$ For all $(x,y) \in \mathbb{R}^2$. Show that F is continuous on $\mathbb{R}^2$ <u>Soln</u>: Let $(a,b) \in \mathbb{R}^2$ be an arbitrary point (i) $F(a,b) = \frac{3a-5b+8}{a^2+b^2+1}$ is defined for all $(a,b) \in \mathbb{R}^2$	(ii) $\lim_{(x,y) \rightarrow (a,b)} F(x,y) = \lim_{(x,y) \rightarrow (a,b)} \frac{3x-5y+8}{x^2+y^2+1}$ $= \frac{3a-5b+8}{a^2+b^2+1} = F(a,b)$ $\therefore$ limit exists at $(a,b) \in \mathbb{R}^2$ (iii) $\lim_{(x,y) \rightarrow (a,b)} F(x,y) = L = F(a,b)$ $= \frac{3a-5b+8}{a^2+b^2+1}$ $\therefore F$ is continuous at every point of $\mathbb{R}^2$ <u>Ex 8.9</u> Consider $f(x,y) = \frac{xy}{x^2+y^2}$ if $(x,y) \neq (0,0)$ and $f(0,0) = 0$. Show that f is not continuous at $(0,0)$ and continuous at all other points of $\mathbb{R}^2$	at all other points of $\mathbb{R}^2$ <u>Soln</u>: f is defined for every $(x,y) \in \mathbb{R}^2$ (i) f is continuous at $(a,b) \neq (0,0)$ Let $(a,b) = (2,5)$ $\lim_{(x,y) \rightarrow (2,5)} f(x,y) = \lim_{(x,y) \rightarrow (2,5)} \frac{xy}{x^2+y^2}$ $= \frac{(2)(5)}{(2)^2+(5)^2}$ $= \frac{10}{4+25}$ $= \frac{10}{29} = L$ $\therefore$ limit exists at $(2,5)$
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$\therefore f$ is continuous at $(2,5)$
 $\therefore f$ is continuous at every point $(a,b) \neq (0,0)$

(ii) Let $y = mx$
 $\lim_{(x,y) \rightarrow (0,0)} f(x,y) = \lim_{(x,y) \rightarrow (0,0)} \frac{xy}{x^2+y^2}$

$$= \lim_{(x,y) \rightarrow (0,0)} \frac{x(mx)}{x^2+m^2x^2}$$

$$= \lim_{(x,y) \rightarrow (0,0)} \frac{mx^2}{x^2(1+m^2)}$$

$$= \lim_{(x,y) \rightarrow (0,0)} \frac{m}{1+m^2}$$

$$= \frac{m}{1+m^2} \neq f(0,0),$$

if $m \neq 0$ So for different values of m

$$\therefore \lim_{(x,y) \rightarrow (0,0)} f(x,y) = \lim_{(x,y) \rightarrow (0,0)} \frac{xy}{x^2+y^2}$$

does not exist
 $\therefore f$ cannot be continuous

at $(0,0)$

Ex 8.10

Consider $g(x,y) = \frac{2x^2y}{x^2+y^2}$

if $(x,y) \neq (0,0)$ and $g(0,0) = 0$

Show that g is continuous on $\mathbb{R}^2$

Soln: Given: $g(0,0) = 0$

g is defined for all

$(x,y) \in \mathbb{R}^2$

$\therefore g$ is continuous at all point $(x,y) \neq (0,0)$

$$\lim_{(x,y) \rightarrow (0,0)} |g(x,y) - g(0,0)|$$

$$= \lim_{(x,y) \rightarrow (0,0)} \left| \frac{2x^2y}{x^2+y^2} - 0 \right|$$

$$= \lim_{(x,y) \rightarrow (0,0)} \left| \frac{2x^2y}{x^2+y^2} \right|$$

$$= \lim_{(x,y) \rightarrow (0,0)} \frac{2|xyx|}{|x^2+y^2|}$$

$$= \lim_{(x,y) \rightarrow (0,0)} \frac{2|xyx|}{|x^2+y^2|}$$

$$= \lim_{(x,y) \rightarrow (0,0)} \frac{2|xyx|}{|x^2+y^2|}$$

$$= \lim_{(x,y) \rightarrow (0,0)} \frac{2|xyx|}{x^2+y^2}$$

$$\leq \lim_{(x,y) \rightarrow (0,0)} \frac{(x^2+y^2)|x|}{(x^2+y^2)} \quad \left[\begin{array}{l} 0 \leq (x-y)^2 \\ \forall x,y \in \mathbb{R} \end{array} \right]$$

$$\leq \lim_{(x,y) \rightarrow (0,0)} |x| = |0| = 0$$

$$\therefore \lim_{(x,y) \rightarrow (0,0)} \frac{2x^2y}{x^2+y^2} = 0 = g(0,0)$$

$\therefore g$ is continuous at $(0,0)$

So g is continuous at every point of $\mathbb{R}^2$

8.5 partial Derivatives

EXERCISE 8.4

1) Find the partial derivatives of the following functions at the indicated points.

(i) $f(x,y) = 3x^2 - 2xy + y^2 + 5x + 2$

$(2, -5)$

Soln: Given:

$$f(x,y) = 3x^2 - 2xy + y^2 + 5x + 2$$

$$\frac{\partial f}{\partial x}(x,y) = 6x - 2y + 5$$

$$\frac{\partial f}{\partial x}(2, -5) = 6(2) - 2(-5) + 5$$

$$= 12 + 10 + 5$$

$$= 27$$

$$\frac{\partial f}{\partial y}(x,y) = -2x + 2y$$

$$\frac{\partial f}{\partial y}(2, -5) = -2(2) + 2(-5)$$

$$= -4 - 10$$

$$= -14$$

(ii) $g(x,y) = 3x^2 + y^2 + 5x + 2$

Soln: Given: $g(x,y) = 3x^2 + y^2 + 5x + 2$

$$\frac{\partial g}{\partial x}(x,y) = 6x + 5$$

$$\frac{\partial g}{\partial x}(1, -2) = 6(1) + 5 = 6 + 5 = 11$$

$$\frac{\partial g}{\partial y}(x,y) = 2y$$

$$\frac{\partial g}{\partial y}(1, -2) = 2(-2) = -4$$

(iii) $h(x, y, z) = x \sin(\pi y) + z^2 x$, $(2, \frac{\pi}{4}, 1)$ Soln: Given: $h(x, y, z) = x \sin(\pi y) + z^2 x$ $\frac{\partial h}{\partial x}(x, y, z) = x \cos(\pi y) y + \sin(\pi y) \cdot 1 + z^2$ $\frac{\partial h}{\partial x}(2, \frac{\pi}{4}, 1) = 2 \cos\left(2\left(\frac{\pi}{4}\right)\right) \frac{\pi}{4} + \sin\left(2\left(\frac{\pi}{4}\right)\right) + (1)^2$ $= 2 \cos\left(\frac{\pi}{2}\right) \frac{\pi}{4} + \sin\frac{\pi}{2} + 1$ $= 2(0) \frac{\pi}{4} + 1 + 1$ $= 0 + 2 = 2$ $\frac{\partial h}{\partial y}(x, y, z) = x \cos(\pi y) \pi + \sin(\pi y)(0) + 0$ $\frac{\partial h}{\partial y}(x, y, z) = x^2 \cos(\pi y) + 0$	$\frac{\partial h}{\partial y}(2, \frac{\pi}{4}, 1) = (2)^2 \cos\left(2\left(\frac{\pi}{4}\right)\right)$ $= 4 \cos\frac{\pi}{2}$ $= 4(0) = 0$ $\frac{\partial h}{\partial z}(x, y, z) = 2zx$ $\frac{\partial h}{\partial z}(2, \frac{\pi}{4}, 1) = 2(1)(2)$ $= 4$ (iv) $G(x, y) = e^{x+3y} \log(x^2+y^2)$, $(-1, 1)$ Soln: Given: $G(x, y) = e^{x+3y} \log(x^2+y^2)$ $\frac{\partial G}{\partial x}(x, y) = e^{x+3y} \frac{1}{x^2+y^2} (2x) + \log(x^2+y^2) e^{x+3y} \cdot 1$ $\frac{\partial G}{\partial x}(x, y) = e^{x+3y} \left[\frac{2x}{x^2+y^2} + \log(x^2+y^2) \right]$	$\frac{\partial G}{\partial x}(-1, 1) = e^{-1+3(1)} \left[\frac{2(-1)}{(-1)^2+(1)^2} + \log((-1)^2+(1)^2) \right]$ $= e^{-1+3} \left[\frac{-2}{1+1} + \log(1+1) \right]$ $= e^2 \left[\frac{-2}{2} + \log 2 \right]$ $= e^2 [\log 2 - 1]$ $\frac{\partial G}{\partial y}(x, y) = e^{x+3y} \frac{1}{x^2+y^2} (2y) + \log(x^2+y^2) e^{x+3y} (3)$ $\frac{\partial G}{\partial y}(x, y) = e^{x+3y} \left[\frac{2y}{x^2+y^2} + 3 \log(x^2+y^2) \right]$ $\frac{\partial G}{\partial y}(-1, 1) = e^{-1+3(1)} \left[\frac{2(1)}{(-1)^2+(1)^2} + 3 \log((-1)^2+(1)^2) \right]$ $= e^{-1+3} \left[\frac{2}{1+1} + 3 \log(1+1) \right]$	$= e^2 \left[\frac{2}{2} + 3 \log 2 \right]$ $= e^2 [1 + \log 2^3]$ $= e^2 [1 + \log 8]$ 2) For each of the following functions find the f_x, f_y and show that $f_{xy} = f_{yx}$ (i) $f(x, y) = \frac{3x}{y + \sin x}$ Soln: Given: $f(x, y) = \frac{3x}{y + \sin x}$ $f_x = \frac{(y + \sin x)(3) - 3x(\cos x)}{(y + \sin x)^2}$
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$$f_x = \frac{3y + 2\sin x - 3x \cos x}{(y + \sin x)^2}$$

$$f_y = \frac{(y + \sin x)(0) - 3x(1)}{(y + \sin x)^2}$$

$$f_y = \frac{-3x}{(y + \sin x)^2}$$

$$f_{xy} = (f_y)_x = \left(\frac{-3x}{(y + \sin x)^2} \right)_x$$

$$= \frac{-(y + \sin x)^2(-3) - (-3x)2(y + \sin x)(\cos x)}{((y + \sin x)^2)^2}$$

$$= \frac{(y + \sin x)[(y + \sin x)(-3) + 6x \cos x]}{(y + \sin x)^4}$$

$$f_{xy} = \frac{3(2x \cos x - (y + \sin x))}{(y + \sin x)^3} \quad \text{--- ①}$$

$$f_{yx} = (f_x)_y = \left(\frac{3y + 2\sin x - 3x \cos x}{(y + \sin x)^2} \right)_y$$

$$= \frac{(y + \sin x)^2(3) - (3y + 2\sin x - 3x \cos x)2(y + \sin x)(\cos x) \cdot 1}{((y + \sin x)^2)^2}$$

$$= \frac{3(y + \sin x)[(y + \sin x) - 2(y + \sin x - x \cos x)]}{(y + \sin x)^4}$$

$$= \frac{3[y + \sin x - 2y - 2\sin x + 2x \cos x]}{(y + \sin x)^3}$$

$$= \frac{3[2x \cos x - y - \sin x]}{(y + \sin x)^3}$$

$$f_{yx} = \frac{3(2x \cos x - (y + \sin x))}{(y + \sin x)^3} \quad \text{--- ②}$$

From ① & ② we get,
 $f_{xy} = f_{yx}$

$$(ii) f(x, y) = \tan^{-1}\left(\frac{x}{y}\right)$$

Soln: Given:

$$f(x, y) = \tan^{-1}\left(\frac{x}{y}\right)$$

$$f_x = \frac{1}{1 + \left(\frac{x}{y}\right)^2} \left(\frac{1}{y}\right)$$

$$= \frac{1}{1 + \frac{x^2}{y^2}} \left(\frac{1}{y}\right)$$

$$= \frac{1}{\frac{x^2 + y^2}{y^2}} \left(\frac{1}{y}\right)$$

$$= \frac{y^2}{x^2 + y^2} \left(\frac{1}{y}\right)$$

$$f_x = \frac{y}{x^2 + y^2}$$

$$f_y = \frac{1}{1 + \left(\frac{x}{y}\right)^2} \left(\frac{-x}{y^2}\right)$$

$$= \frac{1}{1 + \frac{x^2}{y^2}} \left(\frac{-x}{y^2}\right)$$

$$= \frac{1}{\frac{x^2 + y^2}{y^2}} \left(\frac{-x}{y^2}\right)$$

$$= \frac{y^2}{x^2 + y^2} \left(\frac{-x}{y^2}\right)$$

$$f_y = \frac{-x}{x^2 + y^2}$$

$$f_{xy} = (f_y)_x = \left(\frac{-x}{x^2 + y^2} \right)_x$$

$$= \frac{(x^2 + y^2)(-1) - (-x)(2x)}{(x^2 + y^2)^2}$$

$$= \frac{-x^2 - y^2 + 2x^2}{(x^2 + y^2)^2}$$

$$f_{xy} = \frac{x^2 - y^2}{(x^2 + y^2)^2} \quad \text{--- ①}$$

$$f_{yx} = (f_x)_y = \left(\frac{y}{x^2 + y^2} \right)_y$$

$$= \frac{(x^2 + y^2) \cdot 1 - y(2y)}{(x^2 + y^2)^2}$$

$$= \frac{x^2 + y^2 - 2y^2}{(x^2 + y^2)^2}$$

$$f_{yx} = \frac{x^2 - y^2}{(x^2 + y^2)^2} \quad \text{--- ②}$$

From ① & ② we get,
 $f_{xy} = f_{yx}$

(iii) $f(x, y) = \cos(x^2 - 3xy)$	$= [-\sin(x^2 - 3xy)(-3) + (2x - 3y)(-\cos(x^2 - 3xy)(-3))]$	Soln: Given:	4) If $U(x, y, z) = \log(x^3 + y^3 + z^3)$
Soln: Given:		$U(x, y, z) = \frac{x^2 + y^2}{xy} + 3z^2y$	Find $\frac{\partial U}{\partial x} + \frac{\partial U}{\partial y} + \frac{\partial U}{\partial z}$
$f(x, y) = \cos(x^2 - 3xy)$		$U(x, y, z) = \frac{x^2}{xy} + \frac{y^2}{xy} + 3z^2y$	Soln: Given:
$f_x = -\sin(x^2 - 3xy)(2x - 3y)$	$= [3\sin(x^2 - 3xy) + 3x(2x - 3y)\cos(x^2 - 3xy)]$	$U(x, y, z) = \frac{x}{y} + \frac{y}{x} + 3z^2y$	$U(x, y, z) = \log(x^3 + y^3 + z^3)$
$f_y = -\sin(x^2 - 3xy)(-3x)$	$= 3[\sin(x^2 - 3xy) + (2x^2 - 3xy)\cos(x^2 - 3xy)]$	$\frac{\partial U}{\partial x} = \frac{1}{y} - \frac{y}{x^2}$	$\frac{\partial U}{\partial x} = \frac{1}{x^3 + y^3 + z^3} (3x^2)$
$f_y = 3x\sin(x^2 - 3xy)$	$f_{yx} = 3[(2x^2 - 3xy)\cos(x^2 - 3xy) + \sin(x^2 - 3xy)]$	$\frac{\partial U}{\partial x} = \frac{x^2 - y^2}{x^2y}$	$\frac{\partial U}{\partial x} = \frac{3x^2}{x^3 + y^3 + z^3}$
$f_{xy} = (f_y)_x = (3x\sin(x^2 - 3xy))_x$	$= 3[x\cos(x^2 - 3xy)(2x - 3y) + \sin(x^2 - 3xy) \cdot 1]$	$\frac{\partial U}{\partial y} = -\frac{x}{y^2} + \frac{1}{x} + 3z^2$	$\frac{\partial U}{\partial y} = \frac{3y^2}{x^3 + y^3 + z^3}$
$f_{xy} = 3[(2x^2 - 3xy)\cos(x^2 - 3xy) + \sin(x^2 - 3xy)]$	$f_{xy} = f_{yx}$	$= \frac{-x^2 + y^2}{y^2x} + 3z^2$	$\frac{\partial U}{\partial z} = \frac{3z^2}{x^3 + y^3 + z^3}$
$f_{yx} = (f_x)_y = (-\sin(x^2 - 3xy))_y$	$= -\cos(x^2 - 3xy)(-3x)$	$\frac{\partial U}{\partial y} = \frac{y^2 - x^2}{y^2x} + 3z^2$	$\frac{\partial U}{\partial x} + \frac{\partial U}{\partial y} + \frac{\partial U}{\partial z} = \frac{3x^2 + 3y^2 + 3z^2}{x^3 + y^3 + z^3}$
$f_{yx} = 3x\cos(x^2 - 3xy)$		$\frac{\partial U}{\partial z} = 6zy = 6yz$	
	3) If $U(x, y, z) = \frac{x^2 + y^2}{xy} + 3z^2y$, find $\frac{\partial U}{\partial x}$, $\frac{\partial U}{\partial y}$ and $\frac{\partial U}{\partial z}$		

$$\frac{\partial u}{\partial x} + \frac{\partial u}{\partial y} + \frac{\partial u}{\partial z} = \frac{3(x^2 + y^2 + z^2)}{x^3 + y^3 + z^3}$$

5) For each of the following functions find the g_{xy} , g_{xx} , g_{yy} and g_{yx}

(i) $g(x, y) = xe^y + 3x^2y$

Soln: Given:

$$g(x, y) = xe^y + 3x^2y$$

$$g_x = e^y + 6xy \text{ and}$$

$$g_y = xe^y + 3x^2$$

$$g_{xy} = (g_y)_x = (xe^y + 3x^2)_x$$

$$g_{xy} = e^y + 6x$$

$$g_{xx} = (g_x)_x = (e^y + 6xy)_x$$

$$g_{xx} = 6y$$

$$g_{yy} = (g_y)_y = (xe^y + 3x^2)_y$$

$$g_{yy} = xe^y$$

$$g_{yx} = (g_x)_y = (e^y + 6xy)_y$$

$$g_{yx} = e^y + 6x$$

(ii) $g(x, y) = \log(5x + 3y)$

Soln: Given:

$$g(x, y) = \log(5x + 3y)$$

$$g_x = \frac{1}{5x + 3y} (5) = \frac{5}{5x + 3y}$$

$$g_y = \frac{1}{5x + 3y} (3) = \frac{3}{5x + 3y}$$

$$g_{xy} = (g_y)_x = \left(\frac{3}{5x + 3y}\right)_x = 3 \left(\frac{-1}{(5x + 3y)^2} (5)\right)$$

$$g_{xy} = \frac{-15}{(5x + 3y)^2}$$

$$g_{xx} = (g_x)_x = \left(\frac{5}{5x + 3y}\right)_x = 5 \left(\frac{-1}{(5x + 3y)^2} (5)\right)$$

$$g_{xx} = \frac{-25}{(5x + 3y)^2}$$

$$g_{yy} = (g_y)_y = \left(\frac{3}{5x + 3y}\right)_y$$

$$= 3 \left(\frac{-1}{(5x + 3y)^2} (3)\right)$$

$$g_{yy} = \frac{-9}{(5x + 3y)^2}$$

$$g_{yx} = (g_x)_y = \left(\frac{5}{5x + 3y}\right)_y$$

$$= 5 \left(\frac{-1}{(5x + 3y)^2} (3)\right)$$

$$g_{yx} = \frac{-15}{(5x + 3y)^2}$$

(iii) $g(x, y) = x^2 + 3xy - 7y + \cos(5x)$

Soln: Given:

$$g(x, y) = x^2 + 3xy - 7y + \cos(5x)$$

$$g_x = 2x + 3y - \sin(5x)(5)$$

$$g_x = 2x + 3y - 5 \sin(5x)$$

$$g_y = 3x - 7$$

$$g_{xy} = (g_y)_x$$

$$g_{xy} = (3x - 7)_x = 3$$

$$g_{xy} = 3$$

$$g_{xx} = (g_x)_x = (2x + 3y - 5 \sin(5x))_x$$

$$g_{xx} = 2 - 5 \cos(5x)(5)$$

$$g_{xx} = 2 - 25 \cos(5x)$$

$$g_{yy} = (g_y)_y = (3x - 7)_y$$

$$g_{yy} = 0$$

$$g_{yx} = (g_x)_y$$

$$= (2x + 3y - 5 \sin(5x))_y$$

$$g_{yx} = 3$$

b) Let $W(x, y, z) = \frac{1}{\sqrt{x^2 + y^2 + z^2}}$

$(x, y, z) \neq (0, 0, 0)$. Show that $\frac{\partial^2 W}{\partial x^2} + \frac{\partial^2 W}{\partial y^2} + \frac{\partial^2 W}{\partial z^2} = 0$

Soln:

Given:

$$W(x, y, z) = \frac{1}{\sqrt{x^2 + y^2 + z^2}}$$

$$W(x, y, z) = (x^2 + y^2 + z^2)^{-1/2}$$

$$\frac{\partial W}{\partial x} = -\frac{1}{2}(x^2 + y^2 + z^2)^{-3/2} (2x)$$

$$\frac{\partial W}{\partial x} = -x(x^2 + y^2 + z^2)^{-3/2}$$

$$\frac{\partial^2 W}{\partial x^2} = -x \left[-\frac{3}{2}(x^2 + y^2 + z^2)^{-5/2} (2x) \right]$$

$$+ [x^2 + y^2 + z^2]^{-3/2} (-1)$$

$$= \frac{3x^2}{(x^2 + y^2 + z^2)^{5/2}} - \frac{1}{(x^2 + y^2 + z^2)^{3/2}}$$

$$= \frac{3x^2}{(x^2 + y^2 + z^2)^{5/2}} - \frac{(x^2 + y^2 + z^2)}{(x^2 + y^2 + z^2)^{5/2}}$$

$$= \frac{3x^2 - x^2 - y^2 - z^2}{(x^2 + y^2 + z^2)^{5/2}}$$

$$\frac{\partial^2 W}{\partial x^2} = \frac{2x^2 - y^2 - z^2}{(x^2 + y^2 + z^2)^{5/2}} \quad \text{--- (1)}$$

$$\frac{\partial^2 W}{\partial y^2} = \frac{2y^2 - x^2 - z^2}{(x^2 + y^2 + z^2)^{5/2}} \quad \text{--- (2)}$$

$$\frac{\partial^2 W}{\partial z^2} = \frac{2z^2 - x^2 - y^2}{(x^2 + y^2 + z^2)^{5/2}} \quad \text{--- (3)}$$

$$\text{①} + \text{②} + \text{③} \Rightarrow \frac{\partial^2 W}{\partial x^2} + \frac{\partial^2 W}{\partial y^2} + \frac{\partial^2 W}{\partial z^2} = \frac{2x^2 - y^2 - z^2 + 2y^2 - x^2 - z^2 + 2z^2 - x^2 - y^2}{(x^2 + y^2 + z^2)^{5/2}}$$

$$= \frac{0}{(x^2 + y^2 + z^2)^{5/2}}$$

$$\frac{\partial^2 W}{\partial x^2} + \frac{\partial^2 W}{\partial y^2} + \frac{\partial^2 W}{\partial z^2} = 0$$

7) If $V(x, y) = e^x(x \cos y - y \sin y)$, then prove that $\frac{\partial^2 V}{\partial x^2} + \frac{\partial^2 V}{\partial y^2} = 0$

Soln: Given: $V(x, y) = e^x(x \cos y - y \sin y)$

$$\frac{\partial V}{\partial x} = e^x [\cos y] + (x \cos y - y \sin y) [e^x]$$

$$\frac{\partial V}{\partial x} = e^x [\cos y + x \cos y - y \sin y]$$

$$\frac{\partial^2 V}{\partial x^2} = e^x [\cos y] + (\cos y + x \cos y - y \sin y) [e^x]$$

$$\frac{\partial^2 V}{\partial x^2} = e^x [\cos y + \cos y + x \cos y - y \sin y]$$

$$\frac{\partial^2 V}{\partial x^2} = e^x [2 \cos y + x \cos y - y \sin y]$$

$$\frac{\partial V}{\partial y} = e^x [-x \sin y - (y \cos y + \sin y)]$$

$$+ [x \cos y - y \sin y] (0)$$

$$\frac{\partial V}{\partial y} = e^x [-x \sin y - y \cos y - \sin y]$$

$$\frac{\partial^2 V}{\partial y^2} = e^x [-x \cos y - (y(-\sin y) + \cos y) - \cos y]$$

$$\frac{\partial^2 V}{\partial y^2} = e^x [-x \cos y + y \sin y - \cos y - \cos y]$$

$$\frac{\partial^2 V}{\partial y^2} = e^x [-2 \cos y - x \cos y + y \sin y]$$

$$\text{①} + \text{②} \Rightarrow \frac{\partial^2 V}{\partial x^2} + \frac{\partial^2 V}{\partial y^2}$$

$$= e^x [2 \cos y - 2 \cos y + x \cos y - x \cos y + y \sin y - y \sin y]$$

$$= e^x [0] = 0$$

$$\therefore \frac{\partial^2 V}{\partial x^2} + \frac{\partial^2 V}{\partial y^2} = 0$$

$$\frac{\partial^2 V}{\partial x^2} + \frac{\partial^2 V}{\partial y^2} = 0$$

$$\frac{\partial^2 V}{\partial x^2} + \frac{\partial^2 V}{\partial y^2} = 0$$

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$$\frac{\partial^2 V}{\partial x^2} + \frac{\partial^2 V}{\partial y^2} = 0$$

$$\frac{\partial^2 V}{\partial x^2} + \frac{\partial^2 V}{\partial y^2} = 0$$

Soln: Given:

$$W(x, y) = xy + \sin(xy)$$

$$\frac{\partial W}{\partial x} = y + \cos(xy)y$$

$$\frac{\partial W}{\partial x} = y(1 + \cos(xy))$$

$$\frac{\partial^2 W}{\partial y \partial x} = \frac{\partial}{\partial y} \left(\frac{\partial W}{\partial x} \right)$$

$$= \frac{\partial}{\partial y} [y(1 + \cos(xy))]$$

$$= y(0 - \sin(xy)(x)) + (1 + \cos(xy))$$

$$\frac{\partial^2 W}{\partial y \partial x} = 1 + \cos(xy) - xy \sin(xy)$$

$$\frac{\partial W}{\partial y} = x + \cos(xy)x$$

$$\frac{\partial W}{\partial y} = x(1 + \cos(xy))$$

$$\frac{\partial^2 W}{\partial x \partial y} = \frac{\partial}{\partial x} \left(\frac{\partial W}{\partial y} \right)$$

$$= \frac{\partial}{\partial x} [x(1 + \cos(xy))]$$

$$= x(0 - \sin(xy)(y)) + (1 + \cos(xy))$$

$$\frac{\partial^2 W}{\partial x \partial y} = 1 + \cos(xy) - xy \sin(xy)$$

from ① & ② we get,

$$\frac{\partial^2 W}{\partial y \partial x} = \frac{\partial^2 W}{\partial x \partial y}$$

9) If $V(x, y, z) = x^3 + y^3 + z^3 + 3xyz$
Show that $\frac{\partial^2 V}{\partial y \partial z} = \frac{\partial^2 V}{\partial z \partial y}$

Soln: Given:

$$V(x, y, z) = x^3 + y^3 + z^3 + 3xyz$$

$$\frac{\partial V}{\partial z} = 3z^2 + 3xy$$

$$\frac{\partial^2 V}{\partial y \partial z} = \frac{\partial}{\partial y} \left(\frac{\partial V}{\partial z} \right)$$

$$= \frac{\partial}{\partial y} (3z^2 + 3xy)$$

$$\frac{\partial^2 V}{\partial y \partial z} = 3x \quad \text{--- ①}$$

$$\frac{\partial V}{\partial y} = 3y^2 + 3xz$$

$$\frac{\partial^2 V}{\partial z \partial y} = \frac{\partial}{\partial z} \left(\frac{\partial V}{\partial y} \right)$$

$$= \frac{\partial}{\partial z} (3y^2 + 3xz)$$

$$\frac{\partial^2 V}{\partial z \partial y} = 3x \quad \text{--- ②}$$

from ① & ② we get,

$$\frac{\partial^2 V}{\partial y \partial z} = \frac{\partial^2 V}{\partial z \partial y}$$

10) A firm produces two types of calculators each week, x number of type A and y number of type B. The weekly revenue and cost functions (in rupees) are $R(x, y) = 80x + 90y + 0.04xy - 0.05x^2 - 0.05y^2$ and $C(x, y) = 8x + 6y + 2000$ respectively.

(i) Find the profit function $P(x, y)$

(ii) Find $\frac{\partial P}{\partial x}(1200, 1800)$

and $\frac{\partial P}{\partial y}(1200, 1800)$ and

interpret these results.

Soln:

(i) Profit = Revenue - Cost

$$P(x, y) = R(x, y) - C(x, y)$$

$$= 80x + 90y + 0.04xy - 0.05x^2 - 0.05y^2 - 8x - 6y - 2000$$

$$= 72x + 84y + 0.04xy - 0.05x^2 - 0.05y^2 - 2000$$

$$P(x, y) = 72x + 84y + 0.04xy - 0.05x^2 - 0.05y^2 - 2000$$

$$- 0.05x^2 - 0.05y^2 - 2000$$

$$(ii) \frac{\partial P}{\partial x} = 72 + 0.04y - 0.10x$$

$$\frac{\partial P}{\partial x}(1200, 1800) = 72 + 0.04(1800) - 0.10(1200)$$

$$= 72 + 72 - 120$$

$$= 24$$

$$\frac{\partial P}{\partial y} = 84 + 0.04x - 0.10y$$

$$\frac{\partial P}{\partial y}(1200, 1800) = 84 + 0.04(1200) - 0.10(1800)$$

$$= 84 + 48 - 180$$

$$= -48$$

$$\text{At } (1200, 1800), \frac{\partial P}{\partial x} = 24, \frac{\partial P}{\partial y} = -48$$

$\therefore$ Keeping y constant and increasing x $\therefore$ Profit increases	$= \lim_{h \rightarrow 0} \frac{1-1}{h} = 0$ $\frac{\partial F}{\partial y}(0,0) = \lim_{k \rightarrow 0} \frac{F(0,0+k) - F(0,0)}{k}$ $= \lim_{k \rightarrow 0} \frac{1-1}{k} = 0$	$\frac{\partial F}{\partial y}(-2,1)$ Soln: Given: $F(x,y) = x^3y + y^2x + 7$ $\frac{\partial F}{\partial x}(x,y) = 3x^2y + y^2$ $\frac{\partial F}{\partial x}(-1,3) = 3(-1)^2(3) + (3)^2$ $= 3(1)(3) + (3)^2$ $= 9 + 9 = 18$ Also, $\frac{\partial F}{\partial y}(x,y) = x^3 + 2yx$ $\frac{\partial F}{\partial y}(-2,1) = (-2)^3 + 2(1)(-2)$ $= -8 - 4$ $= -12$	$\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}, \frac{\partial^2 f}{\partial y \partial x} \text{ and } \frac{\partial^2 f}{\partial x \partial y}$ Soln: Given: $f(x,y) = \sin(xy^2) + e^{x^3+5y}$ $\frac{\partial f}{\partial x} = \cos(xy^2)y^2 + e^{x^3+5y}(3x^2)$ $\frac{\partial f}{\partial y} = \cos(xy^2)2xy + e^{x^3+5y}(5)$ $\frac{\partial f}{\partial y} = \cos(xy^2)2xy + 5e^{x^3+5y}$ $\frac{\partial^2 f}{\partial y \partial x} = \frac{\partial}{\partial y} \left(\frac{\partial f}{\partial x} \right)$ $= \frac{\partial}{\partial y} (\cos(xy^2)y^2 + 3x^2 e^{x^3+5y})$ $= (\cos(xy^2)2y + y^2(-\sin(xy^2)2xy) + 3x^2 e^{x^3+5y}(5))$
<u>EX 8.11</u> Let $f(x,y) = 0$ if $xy \neq 0$ and $f(x,y) = 1$ if $xy = 0$ (i) Calculate: $\frac{\partial f}{\partial x}(0,0)$, $\frac{\partial f}{\partial y}(0,0)$. (ii) Show that f is not continuous at $(0,0)$ Soln: (i) f takes value 1 on the x, y axes and 0 every where else on $\mathbb{R}^2$ $\frac{\partial f}{\partial x}(0,0) = \lim_{h \rightarrow 0} \frac{f(0+h,0) - f(0,0)}{h}$	(ii) $\lim_{(x,y) \rightarrow (0,0)} f(x,y) = 0$; because along the line $y=x$ when $x \neq 0$, $f(x,y) = 0$ But $f(0,0) = 1 \neq 0$ $\therefore f$ is not continuous at $(0,0)$	<u>EX 8.12</u> Let $F(x,y) = x^3y + y^2x + 7$ for all $(x,y) \in \mathbb{R}^2$. Calculate $\frac{\partial F}{\partial x}(-1,3)$ and	<u>EX 8.13</u> Let $f(x,y) = \sin(xy^2) + e^{x^3+5y}$ for all $(x,y) \in \mathbb{R}^2$. Calculate

$$= 2y(\cos(xy^2) - 2xy^3 \sin(xy^2)) + 15x^2 e^{x^3+5y}$$

Also,

$$\frac{\partial^2 f}{\partial x \partial y} = \frac{\partial}{\partial x} \left(\frac{\partial f}{\partial y} \right)$$

$$= \frac{\partial}{\partial x} (\cos(xy^2) 2xy + 5e^{x^3+5y})$$

$$= (\cos(xy^2) 2y + 2xy(-\sin(xy^2)) + 5e^{x^3+5y} (3x^2))$$

$$= 2y(\cos(xy^2) - 2xy^3 \sin(xy^2)) + 15x^2 e^{x^3+5y}$$

EX 8.14

Let $W(x, y) = xy + \frac{e^y}{y^2+1}$

For all $(x, y) \in \mathbb{R}^2$ Calculate

$$\frac{\partial^2 W}{\partial y \partial x} \text{ and } \frac{\partial^2 W}{\partial x \partial y}$$

Soln: Given:

$$W(x, y) = xy + \frac{e^y}{y^2+1}$$

$$\frac{\partial W}{\partial x}(x, y) = y + 0 = y$$

$$\frac{\partial^2 W}{\partial y \partial x}(x, y) = \frac{\partial}{\partial y} \left(\frac{\partial W}{\partial x} \right)$$

$$= \frac{\partial}{\partial y} (y) = 1$$

$$\frac{\partial W}{\partial y}(x, y) = x + \frac{(y^2+1)^y - e^y(2y)}{(y^2+1)^2}$$

$$\frac{\partial^2 W}{\partial x \partial y}(x, y) = \frac{\partial}{\partial x} \left(\frac{\partial W}{\partial y} \right)$$

$$= \frac{\partial}{\partial x} \left[x + \frac{(y^2+1)^y - 2ye^y}{(y^2+1)^2} \right]$$

$$\frac{\partial^2 W}{\partial x \partial y} = 1$$

EX 8.15

Let $u(x, y) = e^{-2y} \cos(2x)$

For all $(x, y) \in \mathbb{R}^2$ Prove that u is a harmonic function in $\mathbb{R}^2$

Soln: Given:

$$u(x, y) = e^{-2y} \cos(2x)$$

$$u_x(x, y) = e^{-2y} (-\sin(2x)(2))$$

$$u_x(x, y) = -2e^{-2y} \sin(2x)$$

$$u_{xx}(x, y) = -2e^{-2y} \cos(2x)(2)$$

$$u_{xx}(x, y) = -4e^{-2y} \cos(2x)$$

Also,

$$u_y(x, y) = e^{-2y} (-2) \cos(2x)$$

$$u_y(x, y) = -2e^{-2y} \cos(2x)$$

$$u_{yy}(x, y) = -2e^{-2y} (-2) \cos(2x)$$

$$u_{yy}(x, y) = 4e^{-2y} \cos(2x)$$

$$u_{xx} + u_{yy} = -4e^{-2y} \cos(2x)$$

$$+ 4e^{-2y} \cos(2x)$$

$$u_{xx} + u_{yy} = 0$$

$\therefore u$ is a harmonic function in $\mathbb{R}^2$

8.6 Linear Approximation

and Differential of a function of several variables

EXERCISE 8.5

1) IF $W(x, y) = x^3 - 3xy + 2y^2$, $x, y \in \mathbb{R}$, find the linear approximation for W at $(1, -1)$

Soln: Given:

$$W(x, y) = x^3 - 3xy + 2y^2$$

$$W(1, -1) = (1)^3 - 3(1)(-1) + 2(-1)^2$$

$$= 1 + 3 + 2 = 6$$

$$W(1, -1) = 6$$

$$\frac{\partial W}{\partial x}(x, y) = 3x^2 - 3y$$

$$\frac{\partial W}{\partial x}(1, -1) = 3(1)^2 - 3(-1)$$

$$= 3 + 3 = 6$$

$$\frac{\partial W}{\partial x}(1, -1) = 6$$

$$\frac{\partial W}{\partial y}(x, y) = -3x + 4y$$

$$\frac{\partial W}{\partial y}(1, -1) = -3(1) + 4(-1)$$

$$= -3 - 4 = -7$$

$$\frac{\partial W}{\partial y}(1, -1) = -7$$

$$\frac{\partial W}{\partial y}(1, -1) = -7$$

$$\text{linear approximation is}$$

$$L(x, y) = W(x_0, y_0) + \frac{\partial W}{\partial x} \bigg|_{(x_0, y_0)} (x - x_0)$$

$$+ \frac{\partial W}{\partial y} \bigg|_{(x_0, y_0)} (y - y_0)$$

$$+ \frac{\partial W}{\partial y} \bigg|_{(x_0, y_0)} (y - y_0)$$

$L(x, y) = 6 + 6(x-1) + (-7)(y+1)$ $= 6 + 6x - 6 - 7y - 7$ $L(x, y) = 6x - 7y - 7$ is the required linear approximation for w at $(1, -1)$	$\frac{\partial z}{\partial x}(x, y) = 2xy + 3y^4$ $\frac{\partial z}{\partial x}(2, -1) = 2(2)(-1) + 3(-1)^4$ $= -4 + 3(1)$ $= -4 + 3$ $\frac{\partial z}{\partial x}(2, -1) = -1$	$L(x, y) = 2 + (-1)(x-2) + (-20)(y+1)$ $= 2 - x + 2 - 20y - 20$ $= -x - 20y - 16$ $L(x, y) = -(x + 20y + 16)$ is the required linear approximation for z at $(2, -1)$	$\frac{\partial v}{\partial y}(x, y) = -x + \frac{1}{2}y$ $dv = \frac{\partial v}{\partial x}(x, y)dx + \frac{\partial v}{\partial y}(x, y)dy$ $dv = (2x - y)dx + (-x + \frac{1}{2}y)dy$
2) Let $z(x, y) = x^2y + 3xy^4$, $x, y \in \mathbb{R}$. Find the linear approximation for z at $(2, -1)$ <u>Soln:</u> Given: $z(x, y) = x^2y + 3xy^4$ $z(2, -1) = (2)^2(-1) + 3(2)(-1)^4$ $= 4(-1) + 3(2)(1)$ $= -4 + 6$ $z(2, -1) = 2$	$\frac{\partial z}{\partial y}(x, y) = x^2 + 12xy^3$ $\frac{\partial z}{\partial y}(2, -1) = (2)^2 + 12(2)(-1)^3$ $= 4 + 24(-1)$ $= 4 - 24$ $\frac{\partial z}{\partial y}(2, -1) = -20$ linear Approximation is $L(x, y) = z(x_0, y_0) + \frac{\partial z}{\partial x} \bigg _{(x_0, y_0)}(x - x_0)$ $+ \frac{\partial z}{\partial y} \bigg _{(x_0, y_0)}(y - y_0)$	3) If $v(x, y) = x^2 - xy + \frac{1}{4}y^2 + 7$, $x, y \in \mathbb{R}$, find the differential dv. <u>Soln:</u> Given: $v(x, y) = x^2 - xy + \frac{1}{4}y^2 + 7$ $\frac{\partial v}{\partial x}(x, y) = 2x - y$ $\frac{\partial v}{\partial y}(x, y) = -x + \frac{1}{2}y$	4) Let $W(x, y, z) = x^2 - xy + 3\sin z$, $x, y, z \in \mathbb{R}$. Find the linear approximation at $(2, -1, 0)$ <u>Soln:</u> Given: $W(x, y, z) = x^2 - xy + 3\sin z$ $W(2, -1, 0) = (2)^2 - (2)(-1) + 3\sin 0$ $= 4 + 2 + 3(0)$ $W(2, -1, 0) = 6$ $\frac{\partial W}{\partial x}(x, y, z) = 2x - y$ $\frac{\partial W}{\partial x}(2, -1, 0) = 2(2) - (-1)$ $= 4 + 1 = 5$

$\frac{\partial W}{\partial x}(2, -1, 0) = 5$ $\frac{\partial W}{\partial y}(x, y, z) = -x$ $\frac{\partial W}{\partial y}(2, -1, 0) = -2$ $\frac{\partial W}{\partial z}(x, y, z) = 3 \cos z$ $\frac{\partial W}{\partial z}(2, -1, 0) = 3 \cos 0 = 3(1) = 3$ linear Approximation is $L(x, y, z) = W(x_0, y_0, z_0)$ $+ \frac{\partial W}{\partial x} \Big _{(x_0, y_0, z_0)} (x - x_0) + \frac{\partial W}{\partial y} \Big _{(x_0, y_0, z_0)} (y - y_0)$ $+ \frac{\partial W}{\partial z} \Big _{(x_0, y_0, z_0)} (z - z_0)$ $L(x, y, z) = 6 + 5(x - 2) + (-2)(y + 1) + 3(z - 0)$ $= 6 + 5x - 10 - 2y - 2 + 3z$ $L(x, y, z) = 5x - 2y + 3z - 6$ is the required linear	approximation for W at $(2, -1, 0)$ 5) Let $V(x, y, z) = xy + yz + zx$, $x, y, z \in \mathbb{R}$. Find the differential dV Soln: Given: $V(x, y, z) = xy + yz + zx$ $\frac{\partial V}{\partial x}(x, y, z) = y + z$ $\frac{\partial V}{\partial y}(x, y, z) = x + z$ $\frac{\partial V}{\partial z}(x, y, z) = y + x$ $dV = \frac{\partial V}{\partial x}(x, y, z) dx + \frac{\partial V}{\partial y}(x, y, z) dy + \frac{\partial V}{\partial z}(x, y, z) dz$ $dV = (y + z)dx + (x + z)dy + (y + x)dz$ Ex 8.16 If $W(x, y, z) = x^2y + y^2z$	$+ z^2x$, $x, y, z \in \mathbb{R}$, find the differential dW . Soln: Given: $W(x, y, z) = x^2y + y^2z + z^2x$ $\frac{\partial W}{\partial x}(x, y, z) = 2xy + z^2$ $\frac{\partial W}{\partial y}(x, y, z) = x^2 + 2yz$ $\frac{\partial W}{\partial z}(x, y, z) = y^2 + 2zx$ $dW = \frac{\partial W}{\partial x}(x, y, z) dx + \frac{\partial W}{\partial y}(x, y, z) dy + \frac{\partial W}{\partial z}(x, y, z) dz$ $dW = (2xy + z^2)dx + (x^2 + 2yz)dy + (y^2 + 2zx)dz$ EX 8.17 Let $U(x, y, z) = x^2 - xy + 3 \sin z$, $x, y, z \in \mathbb{R}$. Find the linear approximation for U at $(2, -1, 0)$	Soln: Given: $U(x, y, z) = x^2 - xy + 3 \sin z$ $U(2, -1, 0) = (2)^2 - (2)(-1) + 3 \sin 0$ $= 4 + 2 + 3(0)$ $U(2, -1, 0) = 6$ $\frac{\partial U}{\partial x}(x, y, z) = 2x - y$ $\frac{\partial U}{\partial x}(2, -1, 0) = 2(2) - (-1)$ $= 4 + 1 = 5$ $\frac{\partial U}{\partial y}(x, y, z) = -x$ $\frac{\partial U}{\partial y}(2, -1, 0) = -2$ $\frac{\partial U}{\partial z}(x, y, z) = 3 \cos z$ $\frac{\partial U}{\partial z}(2, -1, 0) = 3 \cos 0$ $= 3(1) = 3$ linear Approximation is $L(x, y, z) = U(x_0, y_0, z_0)$ $+ \frac{\partial U}{\partial x} \Big _{(x_0, y_0, z_0)} (x - x_0) + \frac{\partial U}{\partial y} \Big _{(x_0, y_0, z_0)} (y - y_0)$ $+ \frac{\partial U}{\partial z} \Big _{(x_0, y_0, z_0)} (z - z_0)$
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$$L(x, y, z) = 6 + 5(x-2) + (-2)(y+1) + 3(z-0)$$

$$= 6 + 5x - 10 - 2y - 2 + 3z$$

$$L(x, y, z) = 5x - 2y + 3z - 6$$

is the required linear approximation for U at $(2, -1, 0)$

8.6.1 Function of Function Rule

EXERCISE 8.6

1) If $U(x, y) = xy + 3xy^4$, $x = e^t$ and $y = \sin t$, find $\frac{du}{dt}$ and evaluate it at $t=0$

Soln:

Given: $U(x, y) = xy + 3xy^4$,
 $x = e^t$ and $y = \sin t$

$$\frac{du}{dt} = \frac{\partial u}{\partial x} \cdot \frac{dx}{dt} + \frac{\partial u}{\partial y} \cdot \frac{dy}{dt}$$

$$= (xy + 3y^4) \cdot e^t + (x^2 + 12xy^3) \cdot \cos t$$

$$\frac{du}{dt} = (2e^t \sin t + 3 \sin^4 t) \cdot e^t + (e^{2t} + 12e^t \sin^3 t) \cdot \cos t$$

$$\frac{du}{dt} = e^t [2e^t \sin t + 3 \sin^4 t + e^t \cos t + 12 \sin^3 t \cos t]$$

put $t=0$,

$$\frac{du}{dt} = e^0 [2e^0 \sin 0 + 3 \sin^4 0 + e^0 \cos 0 + 12 \sin^3 0 \cos 0]$$

$$= (1) [2(1)(0) + 3(0) + (1)(1) + 12(0)(1)]$$

$$\frac{du}{dt} = 1 \text{ at } t=0$$

2) If $U(x, y, z) = xy^2z^3$, $x = \sin t$, $y = \cos t$, $z = 1 + e^{2t}$, find $\frac{du}{dt}$

Soln: Given: $U(x, y, z) = xy^2z^3$,
 $x = \sin t$, $y = \cos t$, $z = 1 + e^{2t}$

$$\frac{du}{dt} = \frac{\partial u}{\partial x} \cdot \frac{dx}{dt} + \frac{\partial u}{\partial y} \cdot \frac{dy}{dt} + \frac{\partial u}{\partial z} \cdot \frac{dz}{dt}$$

$$= (y^2z^3) \cdot \cos t + 2xy \cdot z^3 \cdot (-\sin t) + 3xy^2z^2 \cdot 2e^{2t}$$

$$= z^2 [y^2 \cos t - 2xyz \sin t + 6xy^2 e^{2t}]$$

$$= (1 + e^{2t})^2 [\cos^2 t (1 + e^{2t}) \cos t - 2 \sin t \cos t (1 + e^{2t}) \sin t + 6 \sin t \cos^2 t e^{2t}]$$

$$= (1 + e^{2t})^2 [\cos^3 t (1 + e^{2t}) - \sin t \cos 2t (1 + e^{2t}) + 6 e^{2t} \sin t \cos^2 t]$$

3) If $W(x, y, z) = x^2 + y^2 + z^2$, $x = e^t$, $y = e^t \sin t$ and $z = e^t \cos t$, find $\frac{dw}{dt}$

Soln: Given: $W(x, y, z) = x^2 + y^2 + z^2$, $x = e^t$, $y = e^t \sin t$ and $z = e^t \cos t$

$$\frac{dw}{dt} = \frac{\partial w}{\partial x} \cdot \frac{dx}{dt} + \frac{\partial w}{\partial y} \cdot \frac{dy}{dt} + \frac{\partial w}{\partial z} \cdot \frac{dz}{dt}$$

$$= 2x \cdot e^t + 2y \cdot (e^t \cos t + \sin t e^t) + 2z \cdot (e^t (-\sin t) + \cos t e^t)$$

$$\begin{aligned}
 &= 2e^t e^t + 2e^t \sin t \cdot e^t (\cos t + \sin t) \\
 &+ 2e^t (\cos t \cdot e^t (\cos t - \sin t)) \\
 &= 2e^{2t} + 2e^{2t} \sin t (\cos t + \sin t) \\
 &+ 2e^{2t} \cos t (\cos t - \sin t) \\
 &= 2e^{2t} [1 + \sin t \cos t + \sin^2 t \\
 &\quad + \cos^2 t - \sin t \cos t] \\
 &= 2e^{2t} [1 + 1] \\
 &= 2e^{2t} [2] = 4e^{2t}
 \end{aligned}$$

4) Let $U(x, y, z) = xyz$,
 $x = e^{-t}$, $y = e^{-t} \cos t$, $z = \sin t$,
 $t \in \mathbb{R}$. Find $\frac{dU}{dt}$

Soln: Given:

$$U(x, y, z) = xyz, \quad x = e^{-t},$$

$$y = e^{-t} \cos t \text{ and } z = \sin t$$

$$\begin{aligned}
 \frac{dU}{dt} &= \frac{\partial U}{\partial x} \cdot \frac{dx}{dt} + \frac{\partial U}{\partial y} \cdot \frac{dy}{dt} \\
 &+ \frac{\partial U}{\partial z} \cdot \frac{dz}{dt}
 \end{aligned}$$

$$\begin{aligned}
 &= yz \cdot (-e^{-t}) + xz \cdot (e^{-t}(-\sin t)) \\
 &+ \cos t (-e^{-t}) + xy \cdot \cos t \\
 &= e^{-t} \cos t \sin t \cdot (-e^{-t}) \\
 &+ e^{-t} \sin t \cdot (-e^{-t}(\sin t + \cos t)) \\
 &+ e^{-t} e^{-t} \cos t \cdot \cos t \\
 &= -e^{-2t} \sin t \cos t - e^{-2t} (\sin^2 t \\
 &\quad + \sin t \cos t) + e^{-2t} \cos^2 t \\
 &= -e^{-2t} [\sin t \cos t + \sin^2 t \\
 &\quad + \sin t \cos t - \cos^2 t] \\
 &= -e^{-2t} [2 \sin t \cos t - (\cos^2 t - \sin^2 t)] \\
 &= -e^{-2t} [\sin 2t - \cos 2t]
 \end{aligned}$$

5) If $W(x, y) = 6x^3 - 3xy + 2y^2$,
 $x = e^s$, $y = \cos s$, $s \in \mathbb{R}$, find
 $\frac{dW}{ds}$, and evaluate at
 $s = 0$

Soln: Given:

$$W(x, y) = 6x^3 - 3xy + 2y^2,$$

$$x = e^s, \quad y = \cos s$$

$$\begin{aligned}
 \frac{dW}{ds} &= \frac{\partial W}{\partial x} \cdot \frac{dx}{ds} + \frac{\partial W}{\partial y} \cdot \frac{dy}{ds} \\
 &= (18x^2 - 3y) \cdot e^s + (-3x + 4y) \cdot (-\sin s) \\
 &= (18e^{2s} - 3\cos s) \cdot e^s + (-3e^s + 4\cos s) \cdot (-\sin s) \\
 &= 18e^{3s} - 3e^s \cos s + 3e^s \sin s \\
 &\quad - 4 \sin s \cos s
 \end{aligned}$$

Put $s = 0$,

$$\begin{aligned}
 \frac{dW}{ds} &= 18e^0 - 3e^0 \cos 0 \\
 &\quad + 3e^0 \sin 0 - 4 \sin 0 \cos 0 \\
 &= 18(1) - 3(1)(1) + 3(1)(0) - 4(0)(1) \\
 &= 18 - 3 = 15 \\
 \frac{dW}{ds} &= 15
 \end{aligned}$$

6) If $z(x, y) = x \tan^{-1}(xy)$,
 $x = t^2$, $y = se^t$, $s, t \in \mathbb{R}$
Find $\frac{\partial z}{\partial s}$ and $\frac{\partial z}{\partial t}$ at
 $s = t = 1$

Soln: Given:

$$z(x, y) = x \tan^{-1}(xy),$$

$$x = t^2, \quad y = se^t$$

$$\begin{aligned}
 \frac{\partial z}{\partial s} &= \frac{\partial z}{\partial x} \cdot \frac{\partial x}{\partial s} + \frac{\partial z}{\partial y} \cdot \frac{\partial y}{\partial s} \\
 &= \left[x \cdot \frac{1}{1+(xy)^2} (y) + \tan^{-1}(xy) \cdot x \right] \cdot 0 \\
 &\quad + \left[x \cdot \frac{1}{1+(xy)^2} (x) + \tan^{-1}(xy) \cdot (0) \right] e^t \\
 &= 0 + \left[\frac{x^2}{1+x^2y^2} \right] e^t \\
 &= \frac{x^2 e^t}{1+x^2y^2} \\
 &= \frac{(t^2)^2 e^t}{1+(t^2)^2 (se^t)^2} = \frac{t^4 e^t}{1+t^4 s^2 e^{2t}}
 \end{aligned}$$

Put $s = t = 1$

$$\frac{\partial z}{\partial s} = \frac{(1)^4 e^{(1)}}{1 + (1)^4 e^{2(1)} (1)^2}$$

$$= \frac{(1)e}{1 + (1)e^2(1)}$$

$$\frac{\partial z}{\partial s} = \frac{e}{1 + e^2}$$

Also,

$$\frac{\partial z}{\partial t} = \frac{\partial z}{\partial x} \cdot \frac{\partial x}{\partial t} + \frac{\partial z}{\partial y} \cdot \frac{\partial y}{\partial t}$$

$$= \left[x \frac{1}{1 + (xy)^2} y + \tan^{-1}(xy) \right] \cdot 2t$$

$$+ \left[x \frac{1}{1 + (xy)^2} x \right] \cdot se^t$$

$$= \left[\frac{xy}{1 + x^2 y^2} + \tan^{-1}(xy) \right] \cdot 2t$$

$$+ \left[\frac{x^2}{1 + x^2 y^2} \right] \cdot se^t$$

$$= \left[\frac{t^2 se^t}{1 + t^4 s^2 e^t} + \tan^{-1}(t^2 se^t) \right] 2t$$

$$+ \left[\frac{t^4}{1 + t^4 s^2 e^t} \right] \cdot se^t$$

Put $s = t = 1$

$$= \left[\frac{(1)^2 (1) e^{(1)}}{1 + (1)^4 (1)^2 e^{2(1)}} + \tan^{-1}((1)^2 (1) e^{(1)}) \right] \times 2(1)$$

$$+ \left[\frac{(1)^4}{1 + (1)^4 (1)^2 e^{2(1)}} \right] \cdot (1) e^{(1)}$$

$$= \left[\frac{e}{1 + e^2} + \tan^{-1} e \right] 2$$

$$+ \left[\frac{1}{1 + e^2} \right] e$$

$$= \frac{2e}{1 + e^2} + 2 \tan^{-1} e + \frac{e}{1 + e^2}$$

$$\frac{\partial z}{\partial t} = \frac{3e}{1 + e^2} + 2 \tan^{-1} e$$

7) Let $U(x, y) = e^x \sin y$,
where $x = st^2$, $y = s^2 t$,
 $s, t \in \mathbb{R}$. Find $\frac{\partial U}{\partial s}$,

$\frac{\partial U}{\partial t}$ and evaluate them at $s = t = 1$

Soln: Given:

$$U(x, y) = e^x \sin y,$$

$$x = st^2, y = s^2 t$$

$$\frac{\partial U}{\partial s} = \frac{\partial U}{\partial x} \cdot \frac{\partial x}{\partial s} + \frac{\partial U}{\partial y} \cdot \frac{\partial y}{\partial s}$$

$$= e^x \sin y \cdot t^2 + e^x \cos y \cdot 2st$$

$$= t e^x [t \sin y + 2s \cos y]$$

$$= t e^{st^2} [t \sin(s^2 t) + 2s \cos(s^2 t)]$$

Put $s = t = 1$,

$$\frac{\partial U}{\partial s} = (1) e^{(1)(1)^2} [(1) \sin((1)^2 (1)) + 2(1) \cos((1)^2 (1))]$$

$$\frac{\partial U}{\partial s} = e [\sin(1) + 2 \cos(1)]$$

Also,

$$\frac{\partial U}{\partial t} = \frac{\partial U}{\partial x} \cdot \frac{\partial x}{\partial t} + \frac{\partial U}{\partial y} \cdot \frac{\partial y}{\partial t}$$

$$= e^x \sin y \cdot (2st) + e^x \cos y \cdot (s^2)$$

$$= s e^x [2t \sin y + s \cos y]$$

$$\frac{\partial U}{\partial t} = s e^{st^2} [2t \sin(s^2 t) + s \cos(s^2 t)]$$

Put $s = t = 1$

$$\frac{\partial U}{\partial t} = (1) e^{(1)(1)^2} [2(1) \sin((1)^2 (1)) + (1) \cos((1)^2 (1))]$$

$\frac{\partial v}{\partial t} = e[2\sin(1) + \cos(1)]$	$= (3x^2 - 6xy^3) \cdot se^t + (-9x^2y^2) \cdot (-se^{-t})$	Given: $W(x, y, z) = xy + yz + zx$ $x = u - v, y = uv$ and $z = u + v$	$\therefore \frac{\partial W}{\partial u} \left(\frac{1}{2}, 1 \right) = 3$
8) Let $z(x, y) = x^3 - 3x^2y^3$ where $x = se^t, y = se^{-t}$ $s, t \in \mathbb{R}$. Find $\frac{\partial z}{\partial s}$ and $\frac{\partial z}{\partial t}$	$= (3s^2e^{2t} - 6set s^3e^{-3t})se^t + (-9s^2e^{2t}s^2e^{-2t}) \cdot (-se^{-t})$	$\frac{\partial W}{\partial u} = \frac{\partial W}{\partial x} \cdot \frac{\partial x}{\partial u} + \frac{\partial W}{\partial y} \cdot \frac{\partial y}{\partial u} + \frac{\partial W}{\partial z} \cdot \frac{\partial z}{\partial u}$	Also, $\frac{\partial W}{\partial v} = \frac{\partial W}{\partial x} \cdot \frac{\partial x}{\partial v} + \frac{\partial W}{\partial y} \cdot \frac{\partial y}{\partial v} + \frac{\partial W}{\partial z} \cdot \frac{\partial z}{\partial v}$
Soln: Given: $z(x, y) = x^3 - 3x^2y^3$ $x = se^t, y = se^{-t}$ $\frac{\partial z}{\partial s} = \frac{\partial z}{\partial x} \cdot \frac{\partial x}{\partial s} + \frac{\partial z}{\partial y} \cdot \frac{\partial y}{\partial s}$ $= (3x^2 - 6xy^3) \cdot e^t + (-9x^2y^2) \cdot e^{-t}$ $= (3s^2e^{2t} - 6set s^3e^{-3t}) \cdot e^t + (-9s^2e^{2t}s^2e^{-2t}) \cdot e^{-t}$ $= 3s^2e^{2t} - 6s^4e^{-2t} - 9s^4e^{-2t}$ $= 3s^2e^{2t} - 15s^4e^{-2t}$	$= (3s^2e^{2t} - 6s^4e^{-2t})se^t + 9s^5e^{-t}$ $= 3s^3e^{3t} - 6s^5e^{-t} + 9s^5e^{-t}$ $= 3s^3e^{3t} + 3s^5e^{-t}$ $= 3s^3(e^{3t} + s^2e^{-t})$ $\therefore \frac{\partial z}{\partial t} = 3s^3(e^{3t} + s^2e^{-t})$	$= (y+z) \cdot 1 + (x+z) \cdot v + (y+x) \cdot 1$ $= (y+z) + (x+z) \cdot v + (y+x)$ $= (uv + u + v) + (u - v + u + v)v + (u + v + u - v)$ $= uv + u + v + uv - v + uv + u - v$ $= 4uv + 2u$	$= (y+z) \cdot (-1) + (x+z) \cdot u + (y+x) \cdot 1$ $= -y - z + (x+z) \cdot u + y + x$ $= -z + (x+z) \cdot u + x$ $\frac{\partial W}{\partial v} = -(u+v) + (u-v + u+v)u + u - v$ $= -u - v + 2u^2 + u - v$ $= 2u^2 - 2v$
Also, $\frac{\partial z}{\partial t} = \frac{\partial z}{\partial x} \cdot \frac{\partial x}{\partial t} + \frac{\partial z}{\partial y} \cdot \frac{\partial y}{\partial t}$	9) $W(x, y, z) = xy + yz + zx$ $x = u - v, y = uv, z = u + v$ $u, v \in \mathbb{R}$. Find $\frac{\partial W}{\partial u}$, $\frac{\partial W}{\partial v}$ and evaluate them at $\left(\frac{1}{2}, 1 \right)$ Soln:	$\frac{\partial W}{\partial u} = 2u(2v+1)$ $\frac{\partial W}{\partial u} \left(\frac{1}{2}, 1 \right) = 2 \left(\frac{1}{2} \right) (2(1)+1)$ $= (2+1) = 3$	$\frac{\partial W}{\partial v} = 2(u^2 - v)$ $\frac{\partial W}{\partial v} \left(\frac{1}{2}, 1 \right) = 2 \left(\left(\frac{1}{2} \right)^2 - 1 \right)$ $= 2 \left(\frac{1}{4} - 1 \right) = 2 \left(-\frac{3}{4} \right)$ $= -\frac{3}{2}$

Ex 8.18

Verify the above theorem for $F(x, y) = x^2 - 2y^2 + 2xy$ and $x(t) = \cos t$, $y(t) = \sin t$, $t \in [0, 2\pi]$

Soln:

Given: $F(x, y) = x^2 - 2y^2 + 2xy$ and $x(t) = \cos t$, $y(t) = \sin t$

$$F(x, y) = \cos^2 t - 2\sin^2 t + 2\cos t \sin t$$

$$\frac{dF}{dt} = 2\cos t(-\sin t) - 4\sin t \cos t + 2(\cos t \cos t + \sin t(-\sin t))$$

$$= -2\sin t \cos t - 4\sin t \cos t + 2(\cos^2 t - \sin^2 t)$$

$$= -6\sin t \cos t + 2(\cos^2 t - \sin^2 t)$$

$$\frac{dF}{dt} = -6\sin t \cos t + 2(\cos^2 t - \sin^2 t) \quad \text{--- (1)}$$

$$\frac{\partial F}{\partial x} \cdot \frac{dx}{dt} + \frac{\partial F}{\partial y} \cdot \frac{dy}{dt}$$

$$= (2x + 2y) \cdot (-\sin t) + (-4y + 2x) \cdot (\cos t)$$

$$= (2\cos t + 2\sin t) \cdot (-\sin t) + (-4\sin t + 2\cos t) \cdot (\cos t)$$

$$= -2\sin t \cos t - 2\sin^2 t + (-4\sin t \cos t + 2\cos^2 t)$$

$$= -2\sin t \cos t - 2\sin^2 t - 4\sin t \cos t + 2\cos^2 t$$

$$= -6\sin t \cos t + 2(\cos^2 t - \sin^2 t) \quad \text{--- (2)}$$

$$\text{From (1) & (2) we get}$$

$$\frac{dF}{dt} = \frac{\partial F}{\partial x} \cdot \frac{dx}{dt} + \frac{\partial F}{\partial y} \cdot \frac{dy}{dt}$$

Ex 8.19

Let $g(x, y) = x^2 - yx + \sin(x+y)$, $x(t) = e^{3t}$, $y(t) = t^2$, $t \in \mathbb{R}$

Find $\frac{dg}{dt}$

Soln:

$$\frac{dg}{dt} = \frac{\partial g}{\partial x} \cdot \frac{dx}{dt} + \frac{\partial g}{\partial y} \cdot \frac{dy}{dt}$$

$$= (2x - y + \cos(x+y)) \cdot 3e^{3t} + (-x + \cos(x+y)) \cdot 2t$$

$$= (2e^{3t} - t^2 + \cos(e^{3t} + t^2)) \cdot 3e^{3t} + (-e^{3t} + \cos(e^{3t} + t^2)) \cdot 2t$$

$$= 6e^{6t} - 3t^2 e^{3t} + 3e^{3t} \cos(e^{3t} + t^2) - 2te^{3t} + 2t \cos(e^{3t} + t^2)$$

$$= 6e^{6t} - 3t^2 e^{3t} + 3e^{3t} \cos(e^{3t} + t^2) - 2te^{3t} + 2t \cos(e^{3t} + t^2)$$

$$= 6e^{6t} - 3t^2 e^{3t} + 3e^{3t} \cos(e^{3t} + t^2) - 2te^{3t} + 2t \cos(e^{3t} + t^2)$$

$$= 6e^{6t} - 3t^2 e^{3t} + 3e^{3t} \cos(e^{3t} + t^2) - 2te^{3t} + 2t \cos(e^{3t} + t^2)$$

$$= 6e^{6t} - 3t^2 e^{3t} + 3e^{3t} \cos(e^{3t} + t^2) - 2te^{3t} + 2t \cos(e^{3t} + t^2)$$

$$= 6e^{6t} - 3t^2 e^{3t} + 3e^{3t} \cos(e^{3t} + t^2) - 2te^{3t} + 2t \cos(e^{3t} + t^2)$$

$$= 6e^{6t} - 3t^2 e^{3t} + 3e^{3t} \cos(e^{3t} + t^2) - 2te^{3t} + 2t \cos(e^{3t} + t^2)$$

$$= 6e^{6t} - 3t^2 e^{3t} + 3e^{3t} \cos(e^{3t} + t^2) - 2te^{3t} + 2t \cos(e^{3t} + t^2)$$

$$= 6e^{6t} - 3t^2 e^{3t} + 3e^{3t} \cos(e^{3t} + t^2) - 2te^{3t} + 2t \cos(e^{3t} + t^2)$$

$$= 6e^{6t} - 3t^2 e^{3t} + 3e^{3t} \cos(e^{3t} + t^2) - 2te^{3t} + 2t \cos(e^{3t} + t^2)$$

$$= 6e^{6t} - 3t^2 e^{3t} + 3e^{3t} \cos(e^{3t} + t^2) - 2te^{3t} + 2t \cos(e^{3t} + t^2)$$

$$= 6e^{6t} - 3t^2 e^{3t} + 3e^{3t} \cos(e^{3t} + t^2) - 2te^{3t} + 2t \cos(e^{3t} + t^2)$$

$$= 6e^{6t} - 3t^2 e^{3t} + 3e^{3t} \cos(e^{3t} + t^2) - 2te^{3t} + 2t \cos(e^{3t} + t^2)$$

$$= 6e^{6t} - 3t^2 e^{3t} + 3e^{3t} \cos(e^{3t} + t^2) - 2te^{3t} + 2t \cos(e^{3t} + t^2)$$

$$= 6e^{6t} - 3t^2 e^{3t} + 3e^{3t} \cos(e^{3t} + t^2) - 2te^{3t} + 2t \cos(e^{3t} + t^2)$$

$x = 2r - s$, $y = r^2 + 2s$, $r, s \in \mathbb{R}$. Find $\frac{\partial g}{\partial r}$, $\frac{\partial g}{\partial s}$

Soln:

$$\frac{\partial g}{\partial r} = \frac{\partial g}{\partial x} \cdot \frac{\partial x}{\partial r} + \frac{\partial g}{\partial y} \cdot \frac{\partial y}{\partial r}$$

$$= (2x) \cdot (2) + (2) \cdot (2r)$$

$$= 2(2r - s) \cdot 2 + 2 \cdot (2r)$$

$$= 4(2r - s) + 4r$$

$$= 8r - 4s + 4r$$

$$= 12r - 4s$$

$$\frac{\partial g}{\partial s} = \frac{\partial g}{\partial x} \cdot \frac{\partial x}{\partial s} + \frac{\partial g}{\partial y} \cdot \frac{\partial y}{\partial s}$$

$$= (2x) \cdot (-1) + 2 \cdot 2$$

$$= (2(2r - s)) \cdot (-1) + 4$$

$$= (4r - 2s) \cdot (-1) + 4$$

$$= 2s - 4r + 4$$

8.6.2 Homogeneous Functions and Euler's Theorem

EXERCISE 8.7

1) In each of the following cases, determine whether the following function is homogeneous or not. If it is so, find the degree.

(i) $f(x, y) = x^2y + 6x^3 + 7$

Soln: Given:

$$\begin{aligned} f(x, y) &= x^2y + 6x^3 + 7 \\ f(\lambda x, \lambda y) &= (\lambda x)^2(\lambda y) + 6(\lambda x)^3 + 7 \\ &= (\lambda^2 x^2)(\lambda y) + 6(\lambda^3 x^3) + 7 \\ &= \lambda^3 x^2y + \lambda^3 6x^3 + 7 \end{aligned}$$

We cannot take λ^3 out side as common

$\therefore f(x, y)$ is not a homogeneous function.

(ii) $h(x, y) = \frac{6x^2y^3 - \pi y^5 + 9x^4y}{2020x^2 + 2019y^2}$

Soln: Given:

$$\begin{aligned} h(x, y) &= \frac{6x^2y^3 - \pi y^5 + 9x^4y}{2020x^2 + 2019y^2} \\ h(\lambda x, \lambda y) &= \frac{6(\lambda x)^2(\lambda y)^3 - \pi(\lambda y)^5 + 9(\lambda x)^4(\lambda y)}{2020(\lambda x)^2 + 2019(\lambda y)^2} \end{aligned}$$

$$= \frac{6\lambda^2 x^2 \lambda^3 y^3 - \pi \lambda^5 y^5 + 9\lambda^4 x^4 \lambda y}{2020 \lambda^2 x^2 + 2019 \lambda^2 y^2}$$

$$= \frac{\lambda^5 6x^2y^3 - \lambda^5 \pi y^5 + \lambda^5 9x^4y}{\lambda^2 2020x^2 + \lambda^2 2019y^2}$$

$$= \frac{\lambda^3 [6x^2y^3 - \pi y^5 + 9x^4y]}{\lambda^2 [2020x^2 + 2019y^2]}$$

$$h(\lambda x, \lambda y) = \lambda^3 h(x, y)$$

$\therefore h(x, y)$ is a homogeneous function of degree = 3

(iii) $g(x, y, z) = \frac{\sqrt{3x^2 + 5y^2 + z^2}}{4x + 7y}$

Soln:

Given: $g(x, y, z) = \frac{\sqrt{3x^2 + 5y^2 + z^2}}{4x + 7y}$

$$g(\lambda x, \lambda y, \lambda z)$$

$$= \frac{\sqrt{3(\lambda x)^2 + 5(\lambda y)^2 + (\lambda z)^2}}{4(\lambda x) + 7(\lambda y)} = \frac{\sqrt{3\lambda^2 x^2 + 5\lambda^2 y^2 + \lambda^2 z^2}}{4\lambda x + 7\lambda y}$$

$$= \frac{\sqrt{\lambda^2(3x^2 + 5y^2 + z^2)}}{\lambda(4x + 7y)}$$

$$= \frac{\lambda \sqrt{3x^2 + 5y^2 + z^2}}{\lambda(4x + 7y)}$$

$$= \lambda^{\cancel{1}-1} \frac{\sqrt{3x^2 + 5y^2 + z^2}}{4x + 7y}$$

$$= \lambda^{1-1} \frac{\sqrt{3x^2 + 5y^2 + z^2}}{4x + 7y}$$

$$= \lambda^0 \frac{\sqrt{3x^2 + 5y^2 + z^2}}{4x + 7y}$$

$$g(\lambda x, \lambda y, \lambda z) = \lambda^0 g(x, y, z)$$

$\therefore g(x, y, z)$ is a homogeneous function of degree = 0

(iv) $U(x, y, z) = xy + \sin\left(\frac{y^2 - z^2}{xy}\right)$

Soln:

Given: $U(x, y, z) = xy + \sin\left(\frac{y^2 - z^2}{xy}\right)$

$$U(\lambda x, \lambda y, \lambda z) = (\lambda x)(\lambda y) + \sin\left(\frac{(\lambda y)^2 - z^2}{(\lambda x)(\lambda y)}\right)$$

$$= \lambda^2 xy + \sin\left(\frac{\lambda^2 y^2 - z^2}{\lambda^2 xy}\right)$$

$$= \lambda^2 xy + \sin\left(\frac{\lambda^2 (y^2 - z^2)}{\lambda^2 xy}\right)$$

$$= \lambda^2 xy + \sin\left(\frac{y^2 - z^2}{xy}\right)$$

We cannot take λ^2 outside the function as common

$\therefore U(x, y, z)$ is not a homogeneous function

2) Prove that $f(x, y) = x^3 - 2x^2y + 3xy^2 + y^3$ is homogeneous; what is the degree? verify Euler's Theorem for f

Soln: Given:

$$\begin{aligned} f(x, y) &= x^3 - 2x^2y + 3xy^2 + y^3 \\ f(\lambda x, \lambda y) &= (\lambda x)^3 - 2(\lambda x)^2(\lambda y) + 3(\lambda x)(\lambda y)^2 + (\lambda y)^3 \\ &= \lambda^3 x^3 - 2(\lambda^2 x^2)(\lambda y) + 3(\lambda x)(\lambda^2 y^2) + \lambda^3 y^3 \\ &= \lambda^3 x^3 - 2\lambda^3 x^2 y + 3\lambda^3 x y^2 + \lambda^3 y^3 \\ &= \lambda^3 (x^3 - 2x^2 y + 3x y^2 + y^3) \\ f(\lambda x, \lambda y) &= \lambda^3 f(x, y) \\ f &\text{ is a homogeneous function of degree 3} \end{aligned}$$

∴ By Euler's Theorem,

$$x \frac{\partial f}{\partial x} + y \frac{\partial f}{\partial y} = 3f$$

Verification:

$$x \frac{\partial f}{\partial x} + y \frac{\partial f}{\partial y}$$

$$= x[3x^2 - 4xy + 3y^2]$$

$$+ y[-2x^2 + 6xy + 3y^2]$$

$$= 3x^3 - 4x^2y + 3xy^2 - 2x^2y + 6xy^2 + 3y^3$$

$$= 3x^3 - 6x^2y + 9xy^2 + 3y^3$$

$$= 3(x^3 - 2x^2y + 3xy^2 + y^3)$$

$$= 3f$$

$$\therefore x \frac{\partial f}{\partial x} + y \frac{\partial f}{\partial y} = 3f$$

∴ Euler's Theorem is verified

3) Prove that $g(x, y) = x \log\left(\frac{y}{x}\right)$ is homogeneous; what is the degree? Verify Euler's Theorem for g

Soln: Given:

$$g(x, y) = x \log\left(\frac{y}{x}\right)$$

$$g(\lambda x, \lambda y) = (\lambda x) \log\left(\frac{\lambda y}{\lambda x}\right)$$

$$g(\lambda x, \lambda y) = \lambda (x \log\left(\frac{y}{x}\right))$$

∴ g is a homogeneous function of degree 1

∴ By Euler's Theorem,

$$x \frac{\partial g}{\partial x} + y \frac{\partial g}{\partial y} = (1)g$$

$$x \frac{\partial g}{\partial x} + y \frac{\partial g}{\partial y} = g$$

Verification:

$$x \frac{\partial g}{\partial x} + y \frac{\partial g}{\partial y}$$

$$= x \left[x \frac{1}{y} \left(\frac{-y}{x^2} \right) + \log\left(\frac{y}{x}\right) \right]$$

$$+ y \left[x \frac{1}{y} \left(\frac{1}{x} \right) \right]$$

$$= x \left[\frac{x}{y} \left(\frac{-y}{x^2} \right) + \log\left(\frac{y}{x}\right) \right]$$

$$+ y \left[\frac{x}{y} \right]$$

$$= -x + x \log\left(\frac{y}{x}\right) + x$$

$$= x \log\left(\frac{y}{x}\right)$$

$$= (1) x \log\left(\frac{y}{x}\right)$$

$$= (1)g$$

$$x \frac{\partial g}{\partial x} + y \frac{\partial g}{\partial y} = g$$

∴ Euler's Theorem is verified

4) If $u(x, y) = \frac{x^2 + y^2}{\sqrt{x + y}}$,

Prove that $x \frac{\partial u}{\partial x} + y \frac{\partial u}{\partial y} = \frac{3}{2} u$

Soln: Given:

$$u(x, y) = \frac{x^2 + y^2}{\sqrt{x + y}}$$

$$u(\lambda x, \lambda y) = \frac{(\lambda x)^2 + (\lambda y)^2}{\sqrt{\lambda x + \lambda y}}$$

$$= \frac{\lambda^2 x^2 + \lambda^2 y^2}{\sqrt{\lambda(x + y)}}$$

$$= \frac{\lambda^2(x^2 + y^2)}{\sqrt{\lambda} \sqrt{x + y}}$$

$$= \frac{\lambda^2(x^2 + y^2)}{\lambda^{1/2} \sqrt{x + y}}$$

$$= \lambda^2 \cdot \lambda^{-1/2} \frac{x^2 + y^2}{\sqrt{x + y}}$$

$$u(\lambda x, \lambda y) = \lambda^{3/2} u(x, y)$$

∴ u is a homogeneous function of degree $= \frac{3}{2}$

∴ By Euler's Theorem,

$$x \frac{\partial f}{\partial x} + y \frac{\partial f}{\partial y} = \frac{3}{2} f$$

Substitute $f = u$,

$$x \frac{\partial u}{\partial x} + y \frac{\partial u}{\partial y} = \frac{3}{2} u$$

5) If $v(x, y) = \log\left(\frac{x^2 + y^2}{x + y}\right)$,
Prove that $x \frac{\partial v}{\partial x} + y \frac{\partial v}{\partial y} = 1$

Soln: Given:

$$v(x, y) = \log\left(\frac{x^2 + y^2}{x + y}\right)$$

v is not a homogeneous function

Taking exponential on both sides we get,

$$e^v = \log e\left(\frac{x^2 + y^2}{x + y}\right)$$

$$e^v = \frac{x^2 + y^2}{x + y} = f(x, y)$$

$$\therefore f(x, y) = \frac{x^2 + y^2}{x + y}$$

$$f(\lambda x, \lambda y) = \frac{(\lambda x)^2 + (\lambda y)^2}{\lambda x + \lambda y}$$

$$= \frac{\lambda^2 x^2 + \lambda^2 y^2}{\lambda x + \lambda y}$$

$$= \frac{\lambda^2(x^2 + y^2)}{\lambda(x + y)}$$

$$= \lambda \frac{x^2 + y^2}{x + y}$$

$$f(\lambda x, \lambda y) = \lambda f(x, y)$$

∴ f is a homogeneous function of degree $= 1$

∴ By Euler's Theorem,

$$x \frac{\partial f}{\partial x} + y \frac{\partial f}{\partial y} = 1 f$$

Substitute $f = e^v$,

$$x \frac{\partial(e^v)}{\partial x} + y \frac{\partial(e^v)}{\partial y} = (1)e^v$$

$$x e^v \frac{\partial v}{\partial x} + y e^v \frac{\partial v}{\partial y} = e^v$$

dividing both sides by e^v we obtain,

$$x \frac{\partial v}{\partial x} + y \frac{\partial v}{\partial y} = 1$$

6) If $w(x, y, z) = \log\left(\frac{5x^3y^4 + 7y^2xz^4 - 75y^3z^4}{x^2 + y^2}\right)$,

find $x \frac{\partial w}{\partial x} + y \frac{\partial w}{\partial y} + z \frac{\partial w}{\partial z}$

Soln: Given:

$$w(x, y, z) = \log\left(\frac{5x^3y^4 + 7y^2xz^4 - 75y^3z^4}{x^2 + y^2}\right)$$

w is not a homogeneous function.

Taking exponential on both sides we get,

$$e^w = \log_e \left(\frac{5x^3y^4 + 7y^2xz^4 - 15y^3z^4}{x^2 + y^2} \right)$$

$$e^w = \frac{5x^3y^4 + 7y^2xz^4 - 15y^3z^4}{x^2 + y^2}$$

$$e^w = \frac{5x^3y^4 + 7y^2xz^4 - 15y^3z^4}{x^2 + y^2} = f(x, y, z)$$

$$f(x, y, z) = \frac{5x^3y^4 + 7y^2xz^4 - 15y^3z^4}{x^2 + y^2}$$

$$f(\lambda x, \lambda y, \lambda z) = \frac{5\lambda^3x^3\lambda^4y^4 + 7\lambda^2y^2\lambda x\lambda^4z^4 - 15\lambda^3y^3\lambda^4z^4}{\lambda^2x^2 + \lambda^2y^2}$$

$$= \frac{\lambda^7(5x^3y^4 + 7y^2xz^4 - 15y^3z^4)}{\lambda^2(x^2 + y^2)}$$

$$= \frac{\lambda^5(5x^3y^4 + 7y^2xz^4 - 15y^3z^4)}{\lambda^2(x^2 + y^2)}$$

$$= \frac{\lambda^5(5x^3y^4 + 7y^2xz^4 - 15y^3z^4)}{\lambda^2(x^2 + y^2)}$$

$$F(\lambda x, \lambda y, \lambda z) = \lambda^5 F(x, y, z)$$

$\therefore f$ is a homogeneous function of degree = 5

$\therefore$ By Euler's Theorem,

$$x \frac{\partial f}{\partial x} + y \frac{\partial f}{\partial y} + z \frac{\partial f}{\partial z} = 5f$$

Substitute $f = e^w$

$$x \frac{\partial (e^w)}{\partial x} + y \frac{\partial (e^w)}{\partial y} + z \frac{\partial (e^w)}{\partial z} = 5e^w$$

$$x e^w \frac{\partial w}{\partial x} + y e^w \frac{\partial w}{\partial y} + z e^w \frac{\partial w}{\partial z} = 5e^w$$

Dividing both sides by e^w we obtain,

$$x \frac{\partial w}{\partial x} + y \frac{\partial w}{\partial y} + z \frac{\partial w}{\partial z} = 5$$

Ex 8.21

Show that $F(x, y) = \frac{x^2 + 5xy - 10y^2}{3x + 7y}$ is a homo

geneous function of degree 1

Soln: Given:

$$F(x, y) = \frac{x^2 + 5xy - 10y^2}{3x + 7y}$$

$$F(\lambda x, \lambda y) = \frac{\lambda^2 x^2 + 5\lambda x \lambda y - 10\lambda^2 y^2}{3\lambda x + 7\lambda y}$$

$$= \frac{\lambda^2(x^2 + 5xy - 10y^2)}{\lambda(3x + 7y)}$$

$$= \frac{\lambda^2[x^2 + 5xy - 10y^2]}{\lambda[3x + 7y]}$$

$$= \lambda \frac{x^2 + 5xy - 10y^2}{3x + 7y}$$

$$F(\lambda x, \lambda y) = \lambda F(x, y)$$

$\therefore F$ is a homogeneous function of degree 1

Ex 8.22

If $u = \sin^{-1} \left(\frac{x+y}{\sqrt{x} + \sqrt{y}} \right)$, Show that

$$x \frac{\partial u}{\partial x} + y \frac{\partial u}{\partial y} = \frac{1}{2} \tan u$$

Soln:

u is not homogeneous function.

$$F(x, y) = \sin u = \frac{x+y}{\sqrt{x} + \sqrt{y}}$$

$\therefore f$ is homogeneous function.

$$F(\lambda x, \lambda y) = \frac{\lambda x + \lambda y}{\sqrt{\lambda x} + \sqrt{\lambda y}}$$

$$= \frac{\sqrt{\lambda}(x+y)}{\sqrt{\lambda}(\sqrt{x} + \sqrt{y})}$$

$$= \frac{x+y}{\sqrt{x} + \sqrt{y}}$$

$$= \frac{x+y}{\sqrt{x} + \sqrt{y}}$$

$$F(\lambda x, \lambda y) = \lambda^{1/2} F(x, y), \quad \forall x, y, \lambda \geq 0$$

$\therefore f$ is a homogeneous function of degree $= \frac{1}{2}$ $\therefore$ By Euler's Theorem, $x \frac{\partial f}{\partial x} + y \frac{\partial f}{\partial y} = \frac{1}{2} f$ Substitute $f = \sin u$, $x \frac{\partial (\sin u)}{\partial x} + y \frac{\partial (\sin u)}{\partial y} = \frac{1}{2} \sin u$ $x \cos u \frac{\partial u}{\partial x} + y \cos u \frac{\partial u}{\partial y} = \frac{1}{2} \sin u$ Dividing both sides by $\cos u$ we obtain, $x \frac{\partial u}{\partial x} + y \frac{\partial u}{\partial y} = \frac{1}{2} \tan u$	The measurement of radius has an approximate error of 0.02 cm. Then the percentage error in calculating area of this template is <u>Soln:</u> Area of Circular template $A = \pi r^2$ $\frac{dA}{dr} = 2\pi r$ $dA = 2\pi r dr$ Approximate change in area $\Delta A \approx 2\pi r dr$ when $r = 10$ cm and $dr = 0.02$ cm $\Delta A = 2\pi (10) (0.02)$ $\Delta A \approx 0.4\pi \text{ cm}^2$ Percentage error $\approx \frac{\Delta A}{A} \times 100$ $\approx \frac{0.4\pi}{\pi(10)^2} \times 100 = 0.4\%$	Ans: (2) 0.4%. 2) The percentage error of fifth root of 31 is approximately how many times the percentage error in 31? <u>Soln:</u> The percentage error in the nth root of any number is approximately $\frac{1}{n}$ times the percentage error in that number $\therefore \frac{1}{5}$ times Ans: (2) $\frac{1}{5}$	Ans: (2) $2xu$ 4) If $v(x, y) = \log(e^x + e^y)$, then $\frac{\partial v}{\partial x} + \frac{\partial v}{\partial y}$ is equal to <u>Soln:</u> Given: $v(x, y) = \log(e^x + e^y)$ $\frac{\partial v}{\partial x} + \frac{\partial v}{\partial y} = \frac{1}{e^x + e^y} (e^x) + \frac{1}{e^x + e^y} (e^y)$ $= \frac{e^x}{e^x + e^y} + \frac{e^y}{e^x + e^y}$ $= \frac{e^x + e^y}{e^x + e^y} = 1$ Ans: (4) 1
<u>EXERCISE 8.8</u> Choose the correct or the most suitable answer from the given four alternatives 1) A Circular template has a radius of 10 cm.		3) If $u(x, y) = e^{x^2 + y^2}$, then $\frac{\partial u}{\partial x}$ is equal to <u>Soln:</u> Given: $u(x, y) = e^{x^2 + y^2}$ $\frac{\partial u}{\partial x} = e^{x^2 + y^2} (2x) = u(2x) = 2xu$	5) If $w(x, y) = x^y, x > 0$, then $\frac{\partial w}{\partial x}$ is equal to <u>Soln:</u> Treat y as constant $\frac{\partial w}{\partial x} = y x^{y-1}$ Ans: (3) $y x^{y-1}$

6) If $f(x,y) = e^{xy}$, then $\frac{\partial^2 f}{\partial x \partial y}$ is equal to

Soln: Given: $f(x,y) = e^{xy}$

$$\frac{\partial f}{\partial y} = x e^{xy}$$

$$\begin{aligned} \frac{\partial^2 f}{\partial x \partial y} &= \frac{\partial}{\partial x} \left(\frac{\partial f}{\partial y} \right) \\ &= \frac{\partial}{\partial x} (x e^{xy}) \\ &= x e^{xy} (y) + e^{xy} (1) \\ &= xy e^{xy} + e^{xy} \\ &= e^{xy} + xy e^{xy} \\ &= (1 + xy) e^{xy} \\ \text{Ans: (2)} (1 + xy) e^{xy} \end{aligned}$$

7) If we measure the side of a cube to be 4 cm with an error of 0.1 cm then

the error in our calculation of the volume is

Soln:

The volume of the cube of side $x = 4$ cm and $dx = 0.1$ cm

$$\therefore V = x^3$$

$$\frac{dV}{dx} = 3x^2$$

$$dV = 3x^2 dx$$

$$= 3(4)^2 (0.1) \text{ cm}^3$$

$$= 3(16) (0.1) \text{ cm}^3$$

$$= 48 (0.1) \text{ cm}^3$$

$$= 4.8 \text{ cm}^3$$

$$= 4.8 \text{ cu. cm}$$

$$\text{Ans: (4)} 4.8 \text{ cu. cm}$$

8) The Change in the surface area $S = 6x^2$ of a cube when the edge length varies from x_0 to $x_0 + dx$ is

Soln: Surface area of the cube $S = 6x^2$

$$\frac{dS}{dx} = 12x$$

Approximate change in the surface area

$$\Delta S = 12x dx$$

$$\text{Put } x = x_0,$$

$$\Delta S = 12x_0 dx$$

$$\text{Ans: (2)} 12x_0 dx$$

9) The approximate change in the volume V of a

cube of side x metres caused by increasing the side by 1% is

Soln:

The volume of the cube of side x is

$$V = x^3$$

$$dV = 3x^2 dx \quad \text{--- (1)}$$

$$\text{When } dx = 1\% \text{ of } x$$

$$= \frac{1}{100} x$$

$$dx = 0.01x$$

$$\text{(1)} \Rightarrow dV = 3x^2 (0.01x)$$

$$= 0.03x^3 \text{ m}^3$$

$$\text{Ans: (4)} 0.03x^3 \text{ m}^3$$

10) If $g(x,y) = 3x^2 - 5y + 2y^2$, $x(t) = e^t$ and $y(t) = \cos t$, then $\frac{dg}{dt}$ is equal to

<u>Soln:</u> Given: $g(x,y) = 3x^2 - 5y + 2y^2$ $x(t) = e^t$ and $y(t) = \cos t$ $\frac{dg}{dt} = \frac{\partial g}{\partial x} \cdot \frac{dx}{dt} + \frac{\partial g}{\partial y} \cdot \frac{dy}{dt}$ $= 6x \cdot e^t + [(-5) + 4y] \cdot (-\sin t)$ $= 6e^t \cdot e^t + 5 \sin t - 4y \sin t$ $= 6e^{2t} + 5 \sin t - 4 \cos t \sin t$ <u>Ans:</u> (1) $6e^{2t} + 5 \sin t - 4 \cos t \sin t$	<u>Ans:</u> (2) $\frac{1}{(x+1)^2} dx$ 12) If $u(x,y) = x^2 + 3xy + y - 2019$, then $\frac{\partial u}{\partial x} \Big _{(4,-5)}$ is equal to <u>Soln:</u> Given: $u(x,y) = x^2 + 3xy + y - 2019$ $\frac{\partial u}{\partial x} = 2x + 3y$ $\frac{\partial u}{\partial x} \Big _{(4,-5)} = 2(4) + 3(-5)$ $= 8 - 15 = -7$ <u>Ans:</u> (3) -7	$g'(x_0) = -\sin \frac{\pi}{2} = -1$ $\boxed{g'(x_0) = -1}$, $g(x_0) = \cos \frac{\pi}{2} = 0$ linear approximation is given by $L(x) = g(x_0) + g'(x_0)(x - x_0)$ $= 0 + (-1)(x - \frac{\pi}{2})$ $L(x) = -x + \frac{\pi}{2}$ <u>Ans:</u> (2) $-x + \frac{\pi}{2}$	$+(-x^2 + y^2 + 2zx - 2zy)$ $= 2xy - 2xz - y^2 + z^2 + x^2$ $+ 2yz - 2xy - z^2 - x^2 + y^2$ $+ 2xz - 2yz$ $= 0$ $\therefore \frac{\partial W}{\partial x} + \frac{\partial W}{\partial y} + \frac{\partial W}{\partial z} = 0$ <u>Ans:</u> (4) 0
11) If $f(x) = \frac{x}{x+1}$, then its differential is given by <u>Soln:</u> Given: $f(x) = \frac{x}{x+1}$ $\frac{df}{dx} = \frac{(x+1) \cdot 1 - x \cdot 1}{(x+1)^2}$ $= \frac{x+1-x}{(x+1)^2}$ $\frac{df}{dx} = \frac{1}{(x+1)^2}$ $df = \frac{1}{(x+1)^2} dx$	13) Linear approximation for $g(x) = \cos x$ at $x = \frac{\pi}{2}$ is <u>Soln:</u> Given: $g(x) = \cos x$ and $x_0 = \frac{\pi}{2}$ $g'(x) = -\sin x$ $g'(x_0) = -\sin x_0$	14) If $W(x,y,z) = x^2(y-z) + y^2(z-x) + z^2(x-y)$, then $\frac{\partial W}{\partial x} + \frac{\partial W}{\partial y} + \frac{\partial W}{\partial z}$ is <u>Soln:</u> Given: $W(x,y,z) = x^2(y-z) + y^2(z-x) + z^2(x-y)$ $= x^2y - x^2z + y^2z - y^2x + z^2x - z^2y$ $\frac{\partial W}{\partial x} + \frac{\partial W}{\partial y} + \frac{\partial W}{\partial z} = (2xy - 2xz - y^2 + z^2) + (x^2 + 2yz - 2yx - z^2)$	15) If $f(x,y,z) = xy + yz + zx$, then $f_x - f_z$ is equal to <u>Soln:</u> Given: $f(x,y,z) = xy + yz + zx$ $f_x - f_z = (y+z) - (y+x)$ $= y+z-y-x$ $= z-x$ <u>Ans:</u> (1) $z-x$

10. Ordinary Differential Equations [10.2 Differential Equation, order and degree]

EXERCISE 10.1

1) For each of the following differential equations, determine its order, degree (if exists)

(i) $\frac{dy}{dx} + xy = \cot x$

Soln:

$$\frac{dy}{dx} + xy = \cot x$$

(order, degree) = (1, 1)

(ii) $\left(\frac{d^3y}{dx^3}\right)^{2/3} - 3\frac{d^2y}{dx^2} + 5\frac{dy}{dx} + 4 = 0$

Soln:

$$\left(\frac{d^3y}{dx^3}\right)^{2/3} - 3\frac{d^2y}{dx^2} + 5\frac{dy}{dx} + 4 = 0$$

$$\left(\frac{d^3y}{dx^3}\right)^{2/3} = 3\frac{d^2y}{dx^2} - 5\frac{dy}{dx} - 4$$

$$\left[\left(\frac{d^3y}{dx^3}\right)^{2/3}\right]^3 = \left[3\frac{d^2y}{dx^2} - 5\frac{dy}{dx} - 4\right]^3$$

$$\left(\frac{d^3y}{dx^3}\right)^2 = \left(3\frac{d^2y}{dx^2} - 5\frac{dy}{dx} - 4\right)^3$$

$\therefore$ (order, degree) = (3, 2)

(iii) $\left(\frac{d^2y}{dx^2}\right)^2 + \left(\frac{dy}{dx}\right)^2 = x \sin\left(\frac{d^2y}{dx^2}\right)$

Soln:

$$\left(\frac{d^2y}{dx^2}\right)^2 + \left(\frac{dy}{dx}\right)^2 = x \sin\left(\frac{d^2y}{dx^2}\right)$$

The given differential equation can not be expressed as polynomial equation with $\frac{d^2y}{dx^2}$ as the leading term degree is not defined.

$\therefore$ (order = 2, but degree is not defined)

(iv) $\sqrt{\frac{dy}{dx}} - 4\frac{dy}{dx} - 7x = 0$

Soln:

$$\sqrt{\frac{dy}{dx}} - 4\frac{dy}{dx} - 7x = 0$$

$$\sqrt{\frac{dy}{dx}} = 4\frac{dy}{dx} + 7x$$

Squaring on both sides we get,

$$\frac{dy}{dx} = \left(4\frac{dy}{dx} + 7x\right)^2$$

$$\frac{dy}{dx} = 16\left(\frac{dy}{dx}\right)^2 + 56x\frac{dy}{dx} + 49x^2$$

$$16\left(\frac{dy}{dx}\right)^2 + 56x\frac{dy}{dx} - \frac{dy}{dx} + 49x^2 = 0$$

(order, degree) = (1, 2)

(v) $y\left(\frac{dy}{dx}\right) = \frac{x}{\left(\frac{dy}{dx}\right) + \left(\frac{dy}{dx}\right)^3}$

Soln:

$$y\left(\frac{dy}{dx}\right) = \frac{x}{\left(\frac{dy}{dx}\right) + \left(\frac{dy}{dx}\right)^3}$$

$$y\left(\frac{dy}{dx}\right)^2 + y\left(\frac{dy}{dx}\right)^4 = x$$

$$y\left(\frac{dy}{dx}\right)^2 + y\left(\frac{dy}{dx}\right)^4 - x = 0$$

(order, degree) = (1, 4)

(vi) $x^2 \frac{d^2y}{dx^2} + \left[1 + \left(\frac{dy}{dx}\right)^{7/2}\right] = 0$

Soln:

$$x^2 \frac{d^2y}{dx^2} + \left[1 + \left(\frac{dy}{dx}\right)^{7/2}\right] = 0$$

$$x^2 \frac{d^2y}{dx^2} = -\left[1 + \left(\frac{dy}{dx}\right)^{7/2}\right]$$

$$x^2 \frac{d^2y}{dx^2} = -\sqrt{1 + \left(\frac{dy}{dx}\right)^2}$$

Squaring on both sides,

$x^4 \left(\frac{d^2y}{dx^2} \right)^2 = 1 + \left(\frac{dy}{dx} \right)^2$ (order, degree) = (2, 2)	The given differential equation cannot be expressed as polynomial equation with $\frac{d^2y}{dx^2}$ as the leading term degree is not defined. $\therefore$ (order = 2, but degree is not defined)	$\frac{d^3y}{dx^3} + 5 \frac{d^2y}{dx^2} + y - 3x^2 = 0$ (order, degree) = (3, 1)	Soln: $\frac{dy}{dx} = x + y + 5$ (order, degree) = (1, 1)
(vii) $\left(\frac{d^2y}{dx^2} \right)^3 = \sqrt{1 + \left(\frac{dy}{dx} \right)^2}$ Soln: $\left(\frac{d^2y}{dx^2} \right)^3 = \sqrt{1 + \left(\frac{dy}{dx} \right)^2}$ Squaring on both sides we get, $\left(\frac{d^2y}{dx^2} \right)^6 = 1 + \frac{dy}{dx}$ (order, degree) = (2, 6)	(ix) $\frac{d^2y}{dx^2} + 5 \frac{dy}{dx} + \int y dx = x^3$ Soln: $\frac{d^2y}{dx^2} + 5 \frac{dy}{dx} + y dx = x^3$	(x) $x = e^{xy \left(\frac{dy}{dx} \right)}$ Soln: $x = e^{xy \left(\frac{dy}{dx} \right)}$ This cannot be expressed as polynomial equation with $\frac{dy}{dx}$ $\therefore$ (order = 1, but degree is not defined)	(ii) $\left(\frac{d^4y}{dx^4} \right)^3 + 4 \left(\frac{dy}{dx} \right)^7 + 6y$ = $5 \cos 3x$ Soln: $\left(\frac{d^4y}{dx^4} \right)^3 + 4 \left(\frac{dy}{dx} \right)^7 + 6y$ = $5 \cos 3x$ (order, degree) = (4, 3)
(viii) $\frac{d^2y}{dx^2} = xy + \cos \left(\frac{dy}{dx} \right)$ Soln: $\frac{d^2y}{dx^2} = xy + \cos \left(\frac{dy}{dx} \right)$	differentiate with respect to x on both sides, we get $\frac{d^3y}{dx^3} + 5 \frac{d^2y}{dx^2} + y = 3x^2$	Ex 10-1 Determine the order and degree (if exists) of the following differential equations: (i) $\frac{dy}{dx} = x + y + 5$	(iii) $\frac{d^2y}{dx^2} + 3 \left(\frac{dy}{dx} \right)^2$ = $x^2 \log \left(\frac{d^2y}{dx^2} \right)$ Soln: The given differential equation is not a Polynomial equation $\therefore$ (order = 2, but degree is not defined)

$$(iv) 3\left(\frac{d^2y}{dx^2}\right) = \left[4 + \left(\frac{dy}{dx}\right)^2\right]^{3/2}$$

Soln:

$$3\left(\frac{d^2y}{dx^2}\right) = \left[4 + \left(\frac{dy}{dx}\right)^2\right]^{3/2}$$

$$3\left(\frac{d^2y}{dx^2}\right) = \left[4 + \left(\frac{dy}{dx}\right)^2\right]^{3/2}$$

$$3\left(\frac{d^2y}{dx^2}\right) = \left[4 + \left(\frac{dy}{dx}\right)^2\right]^{3/2}$$

Squaring on both sides, we get,

$$9\left(\frac{d^2y}{dx^2}\right)^2 = \left[4 + \left(\frac{dy}{dx}\right)^2\right]^3$$

(order, degree) = (2, 2)

$$(v) dy + (xy - \cos x) dx = 0$$

$$\text{Soln: } dy + (xy - \cos x) dx = 0$$

$$dy = -(xy - \cos x) dx$$

$$\frac{dy}{dx} = -(xy - \cos x)$$

$$\frac{dy}{dx} + (xy - \cos x) = 0$$

$\therefore$ (order, degree) = (1, 1)

10.4 Formation of differential Equations from Physical Situation

EXERCISE 10.2

① Express each of the following physical statements in the form of differential equation

(i) Radium decays at a rate proportional to the amount Q present

Soln:

Let Q be the amount

of radium at any time t

$$\therefore \frac{dQ}{dt} \propto Q$$

$$\frac{dQ}{dt} = kQ$$

(ii) The population P of a city increases at a rate proportional to the product of population and the difference between 5,00,000 and the population

Soln: Let P be the population at time t

$$\therefore \frac{dP}{dt} \propto P(500000 - P)$$

$$\frac{dP}{dt} = kP(500000 - P)$$

(iii) For a certain substance, the rate of change of vapor

pressure P with respect to temperature T is proportional to the vapor pressure and inversely proportional to the square of the temperature

Soln:

Let P be the vapor pressure

$$\frac{dP}{dT} \propto \frac{P}{T^2}$$

$$\frac{dP}{dT} = \frac{kP}{T^2}$$

(iv) A saving amount pays 8% interest per year, compounded continuously. In addition,

the income from another investment is credited to the amount continuously at the rate of ₹400 per year

Soln: Let x be the amount at any time t

$$\frac{dx}{dt} \propto x$$

$$\frac{dx}{dt} = kx, \text{ Since } k = 8\% \\ k = \frac{8}{100}$$

$$\therefore \frac{dx}{dt} = \frac{8}{100}x$$

Given: The income from another investment is credited to ₹400

$$\therefore \frac{dx}{dt} = \frac{82}{100}x + 400$$

$$\frac{dx}{dt} = \frac{2x}{25} + 400$$

2) Assume that a spherical

rain drop evaporates at a rate proportional to its surface area. Form a differential equation involving the rate of change of the radius of the rain drop

Soln:

Let A be the Surface area of Spherical rain drop

Let v be the volume of the Sphere and Let r be the radius of the Sphere

$\therefore$ volume of Sphere $v = \frac{4}{3}\pi r^3$ and Surface area of Sphere $= 4\pi r^2$

$$\therefore \frac{dv}{dt} \propto A$$

$$\frac{dv}{dt} = -kA$$

$$\frac{d(\frac{4}{3}\pi r^3)}{dt} = -k4\pi r^2$$

$$\frac{4}{3}\pi \cancel{3}r^2 \frac{dr}{dt} = -k4\pi r^2$$

$$\therefore \frac{dr}{dt} = -k \text{ which}$$

is the required Differential Equation

10.4.2. Formation of Differential Equations from Geometrical Problems

EXERCISE 10.3

1) Find the differential equation of the family of (i) all non-vertical lines in a plane (ii) all non-horizontal lines in a plane

Soln:

(i) The equation of any non-vertical family of line in a plane is $ax+by+c=0$, $a \in \mathbb{R}$, $b \neq 0$

$$a+b\frac{dy}{dx}=0$$

$$\frac{dy}{dx} = -\frac{a}{b}$$

$$\frac{d^2y}{dx^2} = 0$$

(ii) The equation of all non-horizontal family of line in a plane is $ax+by=c$, $a \neq 0$

$$a\frac{dx}{dy}+b=0$$

$$\frac{dx}{dy} = -\frac{b}{a}$$

$$\frac{d^2x}{dy^2} = 0$$

2) Form the differential equation of all straight lines touching the circle $x^2 + y^2 = r^2$

Soln: $c^2 = r^2(1+m^2)$

$$c = \pm r\sqrt{1+m^2}$$

Equation of tangent to the circle $x^2 + y^2 = r^2$ is $y = mx + c$

$$y = mx \pm r\sqrt{1+m^2} \quad \text{--- ①}$$

$$\frac{dy}{dx} = m$$

$$\text{①} \Rightarrow y = \frac{dy}{dx}(x) \pm r\sqrt{1+\left(\frac{dy}{dx}\right)^2}$$

$$y - x \frac{dy}{dx} = \pm r\sqrt{1+\left(\frac{dy}{dx}\right)^2}$$

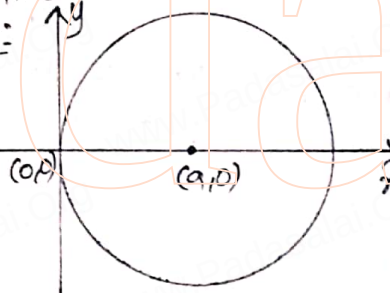
Squaring on both sides we get,

$$\left(y - x \frac{dy}{dx}\right)^2 = r^2 \left(1 + \left(\frac{dy}{dx}\right)^2\right)$$

$$\begin{aligned} -\left(x \frac{dy}{dx} - y\right)^2 &= r^2 \left(1 + \left(\frac{dy}{dx}\right)^2\right) \\ \left(x \frac{dy}{dx} - y\right)^2 &= r^2 \left(1 + \left(\frac{dy}{dx}\right)^2\right) \end{aligned}$$

3) Find the differential equation of the family of circles passing through the origin and having their centres on the x-axis

Soln:



Equation of circle is

$$(x-a)^2 + (y-0)^2 = a^2$$

$$(x-a)^2 + y^2 = a^2 \quad \text{--- ①}$$

$$2(x-a) + 2y \frac{dy}{dx} = 0$$

$$\div 2, (x-a) + y \frac{dy}{dx} = 0$$

$$(x-a) = -y \frac{dy}{dx}$$

Also, $a = x + y \frac{dy}{dx}$

$$\text{①} \Rightarrow \left(-y \frac{dy}{dx}\right)^2 + y^2 = \left(x + y \frac{dy}{dx}\right)^2$$

$$y^2 \left(\frac{dy}{dx}\right)^2 + y^2 = x^2 + y^2 \left(\frac{dy}{dx}\right)^2$$

$$+ 2xy \frac{dy}{dx}$$

$$x^2 + 2xy \frac{dy}{dx} - y^2 = y^2 \left(\frac{dy}{dx}\right)^2 - y^2 \left(\frac{dy}{dx}\right)^2$$

$$x^2 + 2xy \frac{dy}{dx} - y^2 = 0$$

4) Find the differential equation of the family of all the parabolas with latus rectum $4a$ and whose axes are parallel to the

x-axis

Soln:

The equation of parabola is $(y-k)^2 = 4a(x-h)$ [h and k are arbitrary constants] --- ①

$$2(y-k)y' = 4a(1)$$

$$\div 2, (y-k)y' = 2a \quad \text{--- ②}$$

$$(y-k)y'' + y'y' = 0$$

$$(y-k)y'' + (y')^2 = 0$$

$$(y-k)y'' = -(y')^2$$

$$(y-k) = \frac{-(y')^2}{y''}$$

$$\text{②} \Rightarrow \frac{-(y')^2}{y''} y' = 2a$$

$$-(y')^3 = 2ay''$$

$$2ay'' + (y')^3 = 0$$

5) Find the differential equation of the family of parabolas with vertex at $(0, -1)$ and having axis along the y -axis

Soln: Given: $(h, k) = (0, -1)$

The Equation of parabola

$$\text{is } (x-h)^2 = 4a(y-k)$$

$$(x-0)^2 = 4a(y+1)$$

$$x^2 = 4a(y+1) \quad \text{--- ①}$$

[a is an arbitrary constant]

$$2x = 4ay'$$

$$x = 2ay' \Rightarrow a = \frac{x}{2y'}$$

$$\text{①} \Rightarrow x^2 = 4 \left(\frac{x}{2y'} \right) (y+1)$$

$$x = \frac{2}{y'} (y+1)$$

$$xy' = 2(y+1)$$

$$xy' = 2y+2$$

$$xy' - 2y - 2 = 0$$

6) Find the differential equations of the family of all the ellipses having foci on the y -axis and Centre at the origin

Soln: Given: foci on the y -axis

The Equation of ellipse is $\frac{x^2}{b^2} + \frac{y^2}{a^2} = 1$

$$\frac{a^2 x^2 + b^2 y^2}{a^2 b^2} = 1$$

$$a^2 x^2 + b^2 y^2 = a^2 b^2$$

$$a^2 2x + b^2 2y y' = 0$$

$$\div 2, a^2 x + b^2 y y' = 0 \quad \text{--- ①}$$

$$a^2(1) + b^2(y y'' + y' y') = 0$$

$$a^2 + b^2(y y'' + y'^2) = 0 \quad \text{--- ②}$$

Eliminating a^2 and b^2 from ① and ② we get,

$$\begin{vmatrix} x & y y' \\ 1 & y y'' + y'^2 \end{vmatrix} = 0$$

$$x(y y'' + y'^2) - y y' = 0$$

$$x y y'' + x y'^2 - y y' = 0$$

$$x y'^2 + x y y'' - y y' = 0$$

7) Find the differential equation corresponding to the family of curves represented by the equation $y = A e^{8x} + B e^{-8x}$, where A and B are arbitrary constants

Soln:

$$y = A e^{8x} + B e^{-8x}$$

$$\frac{dy}{dx} = 8A e^{8x} - 8B e^{-8x}$$

$$\frac{d^2 y}{dx^2} = 64A e^{8x} + 64B e^{-8x}$$

$$\frac{d^2 y}{dx^2} = 64(A e^{8x} + B e^{-8x})$$

$$\frac{d^2 y}{dx^2} = 64y$$

8) Find the differential equation of the curve represented by

$$xy = a e^x + b e^{-x} + x^2$$

Soln: Given:

$$xy = a e^x + b e^{-x} + x^2$$

$$xy - x^2 = a e^x + b e^{-x} \quad \text{--- ①}$$

[a and b are arbitrary constants]

$$xy' + y(1) - 2x = a e^x - b e^{-x}$$

$$xy' + y - 2x = a e^x - b e^{-x}$$

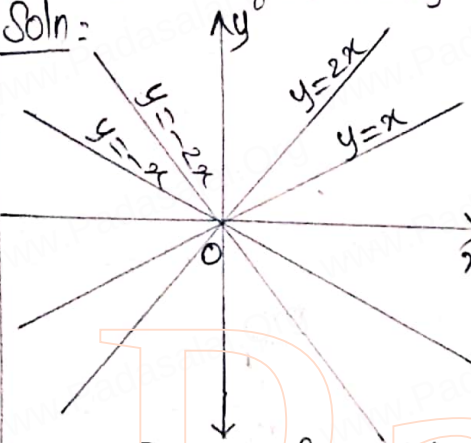
$$xy'' + y'(1) + y' - 2 = a e^x + b e^{-x}$$

$$xy'' + y' + y' - 2 = xy - x^2 \quad \text{from ①}$$

$$xy'' + 2y' + x^2 - xy - 2 = 0$$

Ex 10.2 Find the differential equation for the family of all straight lines

Passing through the origin
Soln:



The family of straight lines passing through the origin is $y = mx$, where m is an arbitrary constant.

$$\boxed{\frac{dy}{dx} = m}$$

$$\textcircled{1} \Rightarrow y = \frac{dy}{dx} x$$

$$\Rightarrow y = x \frac{dy}{dx} \text{ which is}$$

the required differential equation. This equation order is one

EX 10.3

Form the differential equation by eliminating the arbitrary constants A and B from $y = A \cos x + B \sin x$

Soln: Given:

$$y = A \cos x + B \sin x \text{ --- } \textcircled{1}$$

$$\frac{dy}{dx} = -A \sin x + B \cos x$$

$$\frac{d^2y}{dx^2} = -A \cos x - B \sin x$$

$$= -(A \cos x + B \sin x)$$

$$\frac{d^2y}{dx^2} = -y \text{ (from } \textcircled{1})$$

$$\frac{d^2y}{dx^2} + y = 0 \text{ which is}$$

the required differential equation

EX 10.4

Find the differential equation of the family of circles passing through the points $(a, 0)$ and $(-a, 0)$

Soln:

Centre on y-axis

Let $(0, b)$ be the centre of the circle

A circle passing through $\textcircled{1} \Rightarrow$ the points $(a, 0)$ and $(-a, 0)$

$$\therefore r = \sqrt{a^2 + b^2} \Rightarrow r^2 = a^2 + b^2$$

The equation of circle is

$$(x-0)^2 + (y-b)^2 = r^2$$

$$x^2 + (y-b)^2 = a^2 + b^2 \text{ --- } \textcircled{1}$$

[b is an arbitrary constant]

$$2x + 2(y-b) \frac{dy}{dx} = 0$$

$$\div 2, x + (y-b) \frac{dy}{dx} = 0$$

$$\boxed{y-b = -\frac{x}{\frac{dy}{dx}}}$$

$$\boxed{b = \frac{x}{\frac{dy}{dx}} + y}$$

$$x^2 + \left(-\frac{x}{\frac{dy}{dx}}\right)^2 = a^2 + \left(\frac{x}{\frac{dy}{dx}} + y\right)^2$$

$$x^2 + \frac{x^2}{\left(\frac{dy}{dx}\right)^2} = a^2 + \frac{x^2}{\left(\frac{dy}{dx}\right)^2} + y^2 + \frac{2xy}{\frac{dy}{dx}}$$

Multiply by $\frac{dy}{dx}$,

$$x^2 \frac{dy}{dx} = a^2 \frac{dy}{dx} + y^2 \frac{dy}{dx} + 2xy$$

$$x^2 \frac{dy}{dx} - a^2 \frac{dy}{dx} - y^2 \frac{dy}{dx} - 2xy = 0$$

$$(x^2 - a^2 - y^2) \frac{dy}{dx} - 2xy = 0$$

EX 10.5

Find the differential equation of the family of parabolas $y^2 = 4ax$, where a is an arbitrary constant

Soln: Given: $y^2 = 4ax$ — ①

$$2y \frac{dy}{dx} = 4a$$

$$\text{①} \Rightarrow y^2 = (2y \frac{dy}{dx})x$$

$$y = \frac{2x}{y} \frac{dy}{dx}$$

$$\Rightarrow \frac{dy}{dx} = \frac{y}{2x}$$

EX 10.6

Find the differential equation of the family of all ellipses having foci on the x -axis and Centre at the origin

Soln: Given: Foci on the x -axis

The equation of ellipse is $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$, $a > b$, a, b

$$\frac{b^2 x^2 + a^2 y^2}{a^2 b^2} = 1$$

$$b^2 x^2 + a^2 y^2 = a^2 b^2$$

$$b^2 2x + a^2 2y y' = 0$$

$$\div 2, b^2 x + a^2 y y' = 0 \text{ — ①}$$

$$b^2(1) + a^2[y y'' + y' y'] = 0$$

$$b^2 + a^2[y' y'' + y y''] = 0 \text{ — ②}$$

Eliminating a^2 and b^2 from ① and ② we get,

$$\begin{vmatrix} x & y y' \\ 1 & y' y'' + y y'' \end{vmatrix} = 0$$

$$x(y' y'' + y y'') - y y' = 0$$

$$x y' y'' + x y y'' - y y' = 0$$

$$x y \frac{d^2 y}{dx^2} + x \left(\frac{dy}{dx} \right)^2 - y \frac{dy}{dx} = 0$$

10.5 Solution of ordinary Differential Equations

EXERCISE 10.4

1) Show that each of the following expressions is a solution of the corresponding given differential equation

(i) $y = 2x^2$; $xy' = 2y$

Soln: Given:

$$y = 2x^2 \text{ — ①}$$

$$y' = 4x$$

Multiply by x ,

$$xy' = 4x^2$$

$$xy' = 2(2x^2)$$

$$xy' = 2y \text{ [from ①]}$$

(ii) $y = ae^x + be^{-x}$; $y'' - y = 0$

Soln:

Given: $y = ae^x + be^{-x}$ — ①
 a and b are arbitrary constants

$$y' = ae^x - be^{-x}$$

$$y'' = ae^x + be^{-x} \text{ — ②}$$

From ① & ②,

$$\text{②} = \text{①}$$

$$y'' = y$$

$$y'' - y = 0$$

2) Find value of m so that the function $y = e^{mx}$ is a solution of the given differential equation

(i) $y' + 2y = 0$ (ii) $y'' - 5y' + 6y = 0$

Soln: Given: $y' + 2y = 0$ — ①

and $y = e^{mx}$

$y' = m e^{mx}$

$y' = my$

$y' - my = 0$ — ②

Compare from ① & ② we get

$-m = 2 \Rightarrow m = -2$

(ii) Given: $y'' - 5y' + 6y = 0$ — ①

and $y = e^{mx}$

$y' = m e^{mx}$

$y' = my$ — ②

$y' - my = 0$

$y'' - my' = 0$

$y'' = my'$

$y'' = m(my)$ from ②

$y'' = m^2 y$

① $\Rightarrow m^2 y - 5my + 6y = 0$

$y(m^2 - 5m + 6) = 0$

$y = 0, m^2 - 5m + 6 = 0$

$(m-2)(m-3) = 0$

$m-2=0, m-3=0$

$m=2, m=3$

$\therefore m = 2, 3$

3) The slope of the tangent to the curve at any point is the reciprocal of four times the ordinate at that point. The curve passes through $(2, 5)$. Find the equation of the curve

Soln:

Given:

$m = \frac{dy}{dx} = \frac{1}{4y}$

$4y dy = dx$

$4 \int y dy = \int dx$

$2 \times \frac{y^2}{2} = x + c$

$2y^2 = x + c$ — ①

It passes through $(2, 5)$, we get,

$2(5)^2 = 2 + c$

$2(25) = 2 + c$

$50 - 2 = c \Rightarrow c = 48$

① $\Rightarrow 2y^2 = x + 48$ which is the required equation of the curve

4) Show that $y = e^{-x} + mx + n$ is a solution of the differential equation $e^x \left(\frac{d^2 y}{dx^2} \right) - 1 = 0$

Soln:

Given:

$y = e^{-x} + mx + n, \{m, n\}$

$\frac{dy}{dx} = -e^{-x} + m$

$\frac{d^2 y}{dx^2} = e^{-x}$

$\frac{d^2 y}{dx^2} = \frac{1}{e^x}$

$e^x \left(\frac{d^2 y}{dx^2} \right) = 1$

$e^x \left(\frac{d^2 y}{dx^2} \right) - 1 = 0$ which is the required solution of the differential equation

5) Show that $y = ax + \frac{b}{x}$, $x \neq 0$ is a solution of the differential equation $x^2 y'' + xy' - y = 0$

Soln: Given: $y = ax + \frac{b}{x^2}$, {a, b}

$$y' = a - \frac{b}{x^3} \quad \text{--- (2)}$$

$$y'' = \frac{2b}{x^4} \quad \text{--- (3)}$$

$$x^3 y'' = 2b \Rightarrow \boxed{b = \frac{x^3 y''}{2}}$$

$$\textcircled{2} \Rightarrow y' = a - \frac{x^3 y''}{2} \times \frac{1}{x^2}$$

$$y' = a - \frac{x y''}{2}$$

$$\boxed{a = y' + \frac{x y''}{2}}$$

$$\textcircled{1} \Rightarrow y = \left(y' + \frac{x y''}{2} \right) x + \frac{x^3 y''}{2} \times \frac{1}{x^2}$$

$$y = x y' + \frac{x^2 y''}{2} + \frac{x^2 y''}{2}$$

$$y = x y' + \frac{x^2 y''}{2}$$

$x^2 y'' + x y' - y = 0$ which is the required Solution of the differential equation

6) Show that $y = a e^{-3x} + b$, where a and b are arbitrary Constants, is a Solution of the differential equation $\frac{d^2 y}{dx^2} + 3 \frac{dy}{dx} = 0$

Soln:

Given: $y = a e^{-3x} + b$, {a, b}

$$\boxed{\frac{dy}{dx} = -3a e^{-3x}} \quad \text{--- (1)}$$

$$\frac{d^2 y}{dx^2} = -3(-3a e^{-3x})$$

$$\frac{d^2 y}{dx^2} = -3 \frac{dy}{dx} \quad (\text{from } \textcircled{1})$$

$$\frac{d^2 y}{dx^2} + 3 \frac{dy}{dx} = 0 \text{ which}$$

is the required Solution of the differential equation

7) Show that the differential equation representing the family of curves

$y^2 = 2a(x + a^{2/3})$, where a is a positive parameter, is $(y^2 - 2xy \frac{dy}{dx})^3 = 8(y \frac{dy}{dx})^5$

Soln: Given:

$$y^2 = 2a(x + a^{2/3}) \quad \text{--- (1)}$$

$$2y \frac{dy}{dx} = 2a(1 + 0) = 2a \Rightarrow a = y \frac{dy}{dx}$$

$$\div 2, \quad \boxed{\frac{y dy}{dx} = a}$$

$$\textcircled{1} \Rightarrow y^2 = 2y \frac{dy}{dx} (x + (\frac{y dy}{dx})^{2/3})$$

$$y^2 = 2xy \frac{dy}{dx} + 2y \frac{dy}{dx} (\frac{y dy}{dx})^{2/3}$$

$$y^2 - 2xy \frac{dy}{dx} = 2(\frac{y dy}{dx}) (\frac{y dy}{dx})^{2/3}$$

$$y^2 - 2xy \frac{dy}{dx} = 2(\frac{y dy}{dx})^{5/3}$$

$$(\frac{y^2 - 2xy \frac{dy}{dx}}{2})^3 = (\frac{y dy}{dx})^5$$

$$(\frac{y^2 - 2xy \frac{dy}{dx}}{2})^3 = 8(\frac{y dy}{dx})^5$$

8) Show that $y = a \cos bx$ is a Solution of the differential equation $\frac{d^2 y}{dx^2} + b^2 y = 0$

Soln: Given:

$$y = a \cos bx \quad \text{--- (1) } \{a\}$$

$$\frac{dy}{dx} = -a \sin bx (b)$$

$$\frac{d^2 y}{dx^2} = -a \cos bx (b^2)$$

$$\frac{d^2 y}{dx^2} = -y (b^2) \quad [\text{from } \textcircled{1}]$$

$$\frac{d^2 y}{dx^2} = -b^2 y$$

$$\frac{d^2 y}{dx^2} + b^2 y = 0 \text{ which}$$

is the required Solution of the differential equation

Ex 10.7

Show that $x^2 + y^2 = r^2$, where r is a constant, is a solution of the differential equation

$$\frac{dy}{dx} = -\frac{x}{y}$$

Soln: Given: $x^2 + y^2 = r^2$, for

$$2x + 2y \frac{dy}{dx} = 0$$

$$\div 2, x + y \frac{dy}{dx} = 0$$

$$\frac{dy}{dx} = -\frac{x}{y}$$

$\therefore x^2 + y^2 = r^2$ is a solution of the differential equation $\frac{dy}{dx} = -\frac{x}{y}$

Ex 10.8

Show that $y = mx + \frac{7}{m}$, $m \neq 0$ is a solution of the differential equation $xy' + 7\frac{1}{y'} - y = 0$

Soln:

Given: $y = mx + \frac{7}{m}$, $\{m\}$

$$y' = m$$

$$\textcircled{1} \Rightarrow y = y'x + \frac{7}{y'}$$

$$xy' + 7\frac{1}{y'} - y = 0$$

$\therefore$ The given function is a solution of the differential equation $xy' + 7\frac{1}{y'} - y = 0$

Ex 10.9

Show that $y = 2(x^2 - 1) + ce^{-x^2}$ is a solution of the differential equation

$$\frac{dy}{dx} + 2xy - 4x^3 = 0$$

Soln: Given:

$$y = 2(x^2 - 1) + ce^{-x^2}, \{c\}$$

$$\frac{dy}{dx} = 2(2x) + (-2x)e^{-x^2}$$

$$\frac{dy}{dx} = 4x - 2x e^{-x^2} \textcircled{2}$$

$$\textcircled{1} \Rightarrow y - 2(x^2 - 1) = ce^{-x^2}$$

$$\textcircled{2} \Rightarrow \frac{dy}{dx} = 4x - 2x(y - 2(x^2 - 1))$$

$$\frac{dy}{dx} = 4x - 2xy + 4x(x^2 - 1)$$

$$\frac{dy}{dx} = 4x - 2xy + 4x^3 - 4x$$

$$\frac{dy}{dx} = -2xy + 4x^3$$

$$\frac{dy}{dx} + 2xy - 4x^3 = 0$$

$\therefore$ The given function is a solution of the differential equation

$$\frac{dy}{dx} + 2xy - 4x^3 = 0$$

Ex 10.10

Show that $y = a \cos(\log x) + b \sin(\log x)$, $x > 0$ is a solution of the differential equation $x^2 y'' + xy' + y = 0$

Soln: Given:

$$y = a \cos(\log x) + b \sin(\log x), \{a, b\}$$

$$y' = -a \sin(\log x) \frac{1}{x} + b \cos(\log x) \frac{1}{x}$$

Multiple by x ,

$$xy' = -a \sin(\log x) + b \cos(\log x)$$

$$x(y'' + y'(1)) = -a \cos(\log x) \frac{1}{x} - b \sin(\log x) \frac{1}{x}$$

$$xy'' + y' = -\frac{1}{x} (a \cos(\log x) + b \sin(\log x))$$

$$xy'' + y' = -\frac{1}{x} y$$

Multiply by x ,
 $x^2 y'' + xy' = -y$
 $x^2 y'' + xy' + y = 0$

$\therefore y = a \cos(\log x) + b \sin(\log x)$
 is a solution of the given differential equation

10.6 Solution of First Order and First Degree Differential Equations

10.6.1 Variables Separable Method

EXERCISE 10.5

1) If F is the constant force generated by the motor of an automobile of mass M , its velocity v is given by $M \frac{dv}{dt} = F - kv$ where k is a constant. Express v in terms of t

given that $v = 0$ when $t = 0$

Soln: Given:

$$M \frac{dv}{dt} = F - kv$$

$$\int \frac{dv}{F - kv} = \frac{1}{M} \int dt$$

Multiply by $-k$ on both sides,

$$\int \frac{-k dv}{F - kv} = \frac{-k}{M} \int dt$$

$$\log(F - kv) = \frac{-k}{M} t + c \quad \text{--- (1)}$$

When $t = 0, v = 0$

$$\text{(1)} \Rightarrow \log F = 0 + c$$

$$\boxed{c = \log F}$$

$$\text{(1)} \Rightarrow \log(F - kv) = \frac{-k}{M} t + \log F$$

$$\frac{kt}{M} = \log F - \log(F - kv)$$

$$\frac{kt}{M} = \log \left(\frac{F}{F - kv} \right)$$

$$\Rightarrow \log \left(\frac{F}{F - kv} \right) = \frac{kt}{M}$$

$$\frac{F}{F - kv} = e^{\frac{kt}{M}}$$

$$F = (F - kv) e^{\frac{kt}{M}}$$

2) The velocity v , of a Parachute falling vertically satisfies the equation

$$v \frac{dv}{dx} = g \left(1 - \frac{v^2}{k^2} \right) \text{ where}$$

g and k are constants. If v and x are both initially zero. Find v in terms of x .

Soln: Given:

$$v \frac{dv}{dx} = g \left(1 - \frac{v^2}{k^2} \right)$$

$$\frac{v}{\left(1 - \frac{v^2}{k^2} \right)} dv = g dx$$

Multiply by $\frac{-2}{k^2}$ on both sides,

$$\int \frac{-2v}{\left(1 - \frac{v^2}{k^2} \right)} dv = -\frac{2g}{k^2} \int dx$$

$$\log \left(1 - \frac{v^2}{k^2} \right) = -\frac{2gx}{k^2} + c$$

given: $v = 0, x = 0$ --- (1)

$$\text{(1)} \Rightarrow \log(1 - 0) = 0 + c$$

$$\log 1 = c$$

$$\boxed{c = 0}$$

$$\text{(1)} \Rightarrow \log \left(1 - \frac{v^2}{k^2} \right) = -\frac{2gx}{k^2} + 0$$

$$\log \left(1 - \frac{v^2}{k^2} \right) = -\frac{2gx}{k^2}$$

$$\left(1 - \frac{v^2}{k^2} \right) = e^{-\frac{2gx}{k^2}}$$

$1 - e^{-\frac{2gx}{k^2}} = \frac{v^2}{k^2}$ $k^2 \left(1 - e^{-\frac{2gx}{k^2}} \right) = v^2$	$\log(y-1) = \log\left(\frac{x}{x+1}\right)c$ $\log(y-1) = \log\left(\frac{xc}{x+1}\right)$	exponential equations: (i) $\frac{dy}{dx} = \sqrt{\frac{1-y^2}{1-x^2}}$ Soln: Given:	$\frac{1}{\tan^{-1}x} \frac{dx}{1+x^2} = \frac{dy}{y}$ Put $t = \tan^{-1}x$ $\frac{dt}{1+t^2} = \frac{1}{y} dy$ $\int \frac{1}{1+t^2} dt = \int \frac{1}{y} dy$ $\tan^{-1}t = \log y + \log c$
3) Find the equation of the curve whose slope is $\frac{y-1}{x^2+x}$ and which passes through the point (1,0) Soln: Given: Slope = $\frac{dy}{dx} = \frac{y-1}{x^2+x}$ $\frac{dy}{y-1} = \frac{dx}{x^2+x}$	$y-1 = \frac{xc}{x+1} \quad \text{--- (1)}$ It passes through (1,0), $0-1 = \frac{c}{1+1}$ $-1 = \frac{c}{2}$ $[c = -2]$ $(1) \Rightarrow y-1 = \frac{-2x}{x+1}$	$\frac{dy}{dx} = \sqrt{\frac{1-y^2}{1-x^2}}$ $\frac{dy}{\sqrt{1-y^2}} = \frac{dx}{\sqrt{1-x^2}}$ $\sin^{-1}y = \sin^{-1}x + C$	$\log t = -\log y + \log c$ $\log t + \log y = \log c$ $\log ty = \log c$ $ty = c$ $y \tan^{-1}x = c$
$\int \frac{dy}{y-1} = \int \frac{dx}{(x+\frac{1}{2})^2 - (\frac{1}{2})^2}$ $\log(y-1) = \log\left(\frac{x+\frac{1}{2}-\frac{1}{2}}{x+\frac{1}{2}+\frac{1}{2}}\right) + \log c$ $\log(y-1) = \log\left(\frac{x}{x+1}\right) + \log c$	$y = 1 - \frac{2x}{x+1}$ $= \frac{x+1-2x}{x+1}$ $y = \frac{1-x}{1+x}$ 4) Solve the following diff	(ii) $y dx + (1+x^2) \tan^{-1}x dy = 0$ Soln: Given: $y dx + (1+x^2) \tan^{-1}x dy = 0$ $y dx = -(1+x^2) \tan^{-1}x dy$ $\frac{dx}{(1+x^2) \tan^{-1}x} = -\frac{dy}{y}$	(iii) $\sin \frac{dy}{dx} = a, y(0)=1$ Soln: Given: $\sin \frac{dy}{dx} = a$ $\frac{dy}{dx} = \sin^{-1}(a)$ $\int dy = \int \sin^{-1}(a) dx$

$$y = (\sin^{-1}(a))x + c \quad \text{--- ①}$$

when $x=0$, $y=1$

$$1 = (\sin^{-1}(a))(0) + c$$

$$1 = 0 + c$$

$$\boxed{c=1}$$

$$\text{①} \Rightarrow y = (\sin^{-1}(a))x + 1$$

$$y-1 = (\sin^{-1}(a))x$$

$$\frac{y-1}{x} = \sin^{-1}(a)$$

$$\sin\left(\frac{y-1}{x}\right) = a$$

$$\text{(iv)} \quad \frac{dy}{dx} = e^{x+y} + x^3 e^y$$

Soln: Given:

$$\frac{dy}{dx} = e^{x+y} + x^3 e^y$$

$$\frac{dy}{dx} = e^x \cdot e^y + x^3 e^y$$

$$\frac{dy}{dx} = (e^x + x^3) e^y$$

$$\frac{dy}{e^y} = (e^x + x^3) dx$$

$$\int \frac{dy}{e^y} = \int (e^x + x^3) dx$$

$$\frac{e^{-y}}{-1} = e^x + \frac{x^4}{4} + c$$

$$-e^{-y} = e^x + \frac{x^4}{4} + c$$

$$e^x + e^{-y} + \frac{x^4}{4} = -c$$

$$e^x + e^{-y} + \frac{x^4}{4} = c$$

$$\text{(v)} \quad (e^y + 1) \cos x dx + e^y \sin x dy = 0$$

Soln: Given:

$$(e^y + 1) \cos x dx + e^y \sin x dy = 0$$

$$(e^y + 1) \cos x dx = -e^y \sin x dy$$

$$\int \frac{\cos x}{\sin x} dx = - \int \frac{e^y}{e^y + 1} dy$$

$$\log(\sin x) = -\log(e^y + 1) + \log c$$

$$\log(\sin x) + \log(e^y + 1) = \log c$$

$$\log((e^y + 1) \sin x) = \log c$$

$$(e^y + 1) \sin x = c$$

$$\text{(vi)} \quad (y dx - x dy) \cot\left(\frac{x}{y}\right) = n y^2 dx$$

Soln: Given:

$$(y dx - x dy) \cot\left(\frac{x}{y}\right) = n y^2 dx$$

$$\frac{y dx - x dy}{y^2} \cot\left(\frac{x}{y}\right) = n dx$$

$$d\left(\frac{x}{y}\right) \cot\left(\frac{x}{y}\right) = n dx$$

$$\int \cot\left(\frac{x}{y}\right) d\left(\frac{x}{y}\right) = n \int \frac{dx}{x}$$

$$\log \sin\left(\frac{x}{y}\right) = n x + c$$

$$\sin\left(\frac{x}{y}\right) = e^{n x + c}$$

$$\text{(vii)} \quad \frac{dy}{dx} - x \sqrt{25 - x^2} = 0$$

Soln: Given:

$$\frac{dy}{dx} - x \sqrt{25 - x^2} = 0$$

$$\frac{dy}{dx} = x \sqrt{25 - x^2}$$

$$dy = x \sqrt{25 - x^2} dx$$

$$dy = \sqrt{t} \left(-\frac{dt}{2}\right)$$

$$dy = t^{1/2} \left(-\frac{dt}{2}\right)$$

$$\int dy = -\frac{1}{2} \int t^{1/2} dt$$

$$y = -\frac{1}{2} \frac{t^{3/2}}{3/2} + c$$

$$y = -\frac{1}{2} \times \frac{2}{3} t^{3/2} + c$$

$$y = -\frac{1}{3} (25 - x^2)^{3/2} + c$$

$$y = \frac{-(25 - x^2)^{3/2} + 3c}{3}$$

$$3y = -(25 - x^2)^{3/2} + 3c$$

$$\text{(viii)} \quad x \cos y dy = e^x (x \log x + 1) dx$$

Soln: Given:

$$x \cos y dy = e^x (x \log x + 1) dx$$

$$\cos y dy = \frac{e^x (x \log x + 1)}{x} dx$$

Put $t = 25 - x^2$

$$dt = -2x dx$$

$$-dx = x dx$$

$$\frac{-dt}{2} = x dx$$

$$\begin{aligned} \cos y dy &= e^x \left(\frac{x \log x}{x} + \frac{1}{x} \right) dx \\ \cos y dy &= e^x \left(\log x + \frac{1}{x} \right) dx \\ \cos y dy &= (e^x \log x + e^x \frac{1}{x}) dx \\ \cos y dy &= e^x \frac{1}{x} dx + e^x \log x dx \\ \int \cos y dy &= \int d(e^x \log x) \\ \sin y &= e^x \log x + c \end{aligned}$$

$$\begin{aligned} \text{(ix) } \tan y \frac{dy}{dx} &= \cos(x+y) + \cos(x-y) \\ \text{Soln: Given:} \\ \tan y \frac{dy}{dx} &= \cos(x+y) + \cos(x-y) \\ \tan y \frac{dy}{dx} &= \cos x \cos y - \sin x \sin y \\ &\quad + \cos x \cos y + \sin x \sin y \\ \tan y \frac{dy}{dx} &= 2 \cos x \cos y \\ \frac{1}{\cos y} \tan y dy &= 2 \cos x dx \\ \int \sec y \tan y dy &= 2 \int \cos x dx \\ \sec y &= 2 \sin x + c \end{aligned}$$

$$\begin{aligned} \text{(x) } \frac{dy}{dx} &= \tan^2(x+y) \\ \text{Soln: Given:} \\ \frac{dy}{dx} &= \tan^2(x+y) \quad \text{--- ①} \\ \text{Put } x+y &= z \\ 1 + \frac{dy}{dx} &= \frac{dz}{dx} \\ \frac{dy}{dx} &= \frac{dz}{dx} - 1 \\ \text{①} \Rightarrow \frac{dz}{dx} - 1 &= \tan^2 z \\ \frac{dz}{dx} &= 1 + \tan^2 z \\ \frac{dz}{dx} &= \sec^2 z \\ \frac{1}{\sec^2 z} dz &= dx \\ \cos^2 z dz &= dx \end{aligned}$$

$$\begin{aligned} \left(\frac{1 + \cos 2z}{2} \right) dz &= dx \\ \frac{1}{2} \int (1 + \cos 2z) dz &= \int dx \\ \frac{1}{2} \left[z + \frac{\sin 2z}{2} \right] &= x + c \\ \frac{1}{2} \left[z + \frac{2 \sin z \cos z}{2} \right] &= x + c \\ \frac{1}{2} [(x+y) + \sin(x+y) \cos(x+y)] &= x + c \end{aligned}$$

$$\begin{aligned} \text{Ex 10.11} \\ \text{Solve } (1+x^2) \frac{dy}{dx} &= 1+y^2 \\ \text{Soln: Given:} \\ (1+x^2) \frac{dy}{dx} &= 1+y^2 \\ \int \frac{dy}{1+y^2} &= \int \frac{dx}{1+x^2} \\ \tan^{-1} y &= \tan^{-1} x + c \\ \tan^{-1} y - \tan^{-1} x &= c \end{aligned}$$

$$\begin{aligned} \tan^{-1} \left(\frac{y-x}{1+xy} \right) &= c \\ \frac{y-x}{1+xy} &= \tan c = a(\text{say}) \\ \frac{y-x}{1+xy} &= a \\ y-x &= a(1+xy) \end{aligned}$$

$$\begin{aligned} \text{Ex 10.12} \\ \text{Find the particular} \\ \text{Solution of } (1+x^3) dy - x^2 y dx &= 0 \\ \text{Satisfying the Condition} \\ y(1) &= 2 \\ \text{Soln: Given:} \\ (1+x^3) dy - x^2 y dx &= 0 \\ (1+x^3) dy &= x^2 y dx \\ \frac{dy}{y} &= \frac{x^2}{1+x^3} dx \\ \text{Multiple by 3 on both} \\ \text{Sides,} \end{aligned}$$

$$3 \int \frac{dy}{y} = \int \frac{3x^2}{1+x^3} dx$$

$$3 \log y = \log(1+x^3) + \log c$$

$$\log y^3 = \log(1+x^3) + \log c$$

$$\log y^3 = \log(c(1+x^3))$$

$$y^3 = c(1+x^3) \text{ --- ①}$$

When $x=1, y=2$

$$(2)^3 = c(1+(1)^3)$$

$$4 = c(2)$$

$$\therefore \boxed{c=4}$$

$$\text{①} \Rightarrow y^3 = 4(1+x^3)$$

Ex 10.13

Solve $y' = \sin^2(x-y+1)$

Soln: Given:

$$y' = \sin^2(x-y+1)$$

$$\frac{dy}{dx} = \sin^2(x-y+1) \text{ --- ①}$$

Put $z = x-y+1$

$$\frac{dz}{dx} = 1 - \frac{dy}{dx}$$

$$\frac{dy}{dx} = 1 - \frac{dz}{dx}$$

$$\text{①} \Rightarrow 1 - \frac{dz}{dx} = \sin^2 z$$

$$\frac{dz}{dx} = 1 - \sin^2 z = \cos^2 z$$

$$\frac{dz}{\cos^2 z} = dx$$

$$\int \sec^2 z dz = \int dx$$

$$\tan z = x + c$$

$$\tan(x-y+1) = x + c$$

Ex 10.14

Solve: $\frac{dy}{dx} = \sqrt{4x+2y-1}$

Soln: Given:

$$\frac{dy}{dx} = \sqrt{4x+2y-1} \text{ --- ①}$$

Put $z = 4x+2y-1$

$$\frac{dz}{dx} = 4 + 2 \frac{dy}{dx}$$

$$\frac{dz}{dx} - 4 = 2 \frac{dy}{dx}$$

$$\frac{dy}{dx} = \frac{1}{2} \left(\frac{dz}{dx} - 4 \right)$$

$$\text{①} \Rightarrow \frac{1}{2} \left(\frac{dz}{dx} - 4 \right) = \sqrt{z}$$

$$\frac{dz}{dx} - 4 = 2\sqrt{z}$$

$$\frac{dz}{dx} = 4 + 2\sqrt{z}$$

$$\frac{dz}{4+2\sqrt{z}} = dx$$

Put $u = \sqrt{z}$

$$u^2 = z$$

$$2u du = dz$$

$$\frac{z du}{z(2+u)} = dx$$

$$\frac{u du}{2+u} = dx$$

$$\int \left(1 - \frac{2}{u+2} \right) du = \int dx$$

$$u - 2 \ln|u+2| = x + c$$

Put $u = \sqrt{z}$

$$\sqrt{z} - 2 \ln[\sqrt{z}+2] = x + c$$

Put $z = 4x+2y-1$

$$\sqrt{4x+2y-1} - 2 \ln[\sqrt{4x+2y-1}+2] = x + c$$

Ex 10.15

Solve: $\frac{dy}{dx} = \frac{x-y+5}{2(x-y)+7}$

Soln:

Given: $\frac{dy}{dx} = \frac{x-y+5}{2(x-y)+7}$ --- ①

Put $z = x-y$

$$\frac{dz}{dx} = 1 - \frac{dy}{dx}$$

$$\frac{dy}{dx} = 1 - \frac{dz}{dx}$$

$$\text{①} \Rightarrow 1 - \frac{dz}{dx} = \frac{z+5}{2z+7}$$

$$\frac{dz}{dx} = 1 - \frac{z+5}{2z+7}$$

$\frac{dz}{dx} = \frac{2z+7-z-5}{2z+7}$ $\frac{dz}{dx} = \frac{z+2}{2z+7}$ $\left[\frac{2z+7}{z+2} \right] dz = dx$ $\left[\frac{2z+4+3}{z+2} \right] dz = dx$ $\left[\frac{2(z+2)+3}{z+2} \right] dz = dx$ $\left[\frac{2(z+2)}{(z+2)} + \frac{3}{(z+2)} \right] dz = dx$ $\int \left[2 + \frac{3}{z+2} \right] dz = \int dx$ $2z + 3 \log z+2 = x + C$ $2(x-y) + 3 \log x-y+2 = x + C$	Soln: Given: $\frac{dy}{dx} = (3x+y+4)^2 \quad \text{①}$ Put $z = 3x+y+4$ $\frac{dz}{dx} = 3 + \frac{dy}{dx}$ $\frac{dy}{dx} = \frac{dz}{dx} - 3$ $\text{①} \Rightarrow \frac{dz}{dx} - 3 = z^2$ $\frac{dz}{dx} = z^2 + 3$ $\frac{dz}{z^2+3} = dx$ $\frac{1}{\sqrt{3}} \tan^{-1}\left(\frac{z}{\sqrt{3}}\right) = x + C$ $\frac{1}{\sqrt{3}} \tan^{-1}\left(\frac{3x+y+4}{\sqrt{3}}\right) = x + C$	10.6.3. Homogeneous Form or Homogeneous Differential Equation EXERCISE 10.6 Solve the following differential equations: 1) $\left[x + y \cos\left(\frac{y}{x}\right) \right] dx = x \cos\left(\frac{y}{x}\right) dy$ Soln: Given: $\left[x + y \cos\left(\frac{y}{x}\right) \right] dx = x \cos\left(\frac{y}{x}\right) dy$ $\frac{dy}{dx} = \frac{x + y \cos\left(\frac{y}{x}\right)}{x \cos\left(\frac{y}{x}\right)}$ This is a homogeneous differential equation Put $y = vx$ $\frac{d(vx)}{dx} = \frac{x + vx \cos\left(\frac{vx}{x}\right)}{x \cos\left(\frac{vx}{x}\right)}$ $v + x \frac{dv}{dx} = \frac{x(1 + v \cos v)}{x \cos v}$	$v + x \frac{dv}{dx} = \frac{1 + v \cos v}{\cos v}$ $x \frac{dv}{dx} = \frac{1 + v \cos v}{\cos v} - v$ $= \frac{1 + v \cos v - v \cos v}{\cos v}$ $x \frac{dv}{dx} = \frac{1}{\cos v}$ $\int \cos v dv = \int \frac{dx}{x}$ $\sin v = \log x + \log c $ $\sin v = \log cx , \text{ Replace } v \text{ by } \frac{y}{x}$ $\sin \frac{y}{x} = \log cx $
Ex 10.16 Solve: $\frac{dy}{dx} = (3x+y+4)^2$		2) Solve: $(x^3+y^3)dy - x^2ydx = 0$ Soln: Given: $(x^3+y^3)dy - x^2ydx = 0$ $(x^3+y^3)dy = x^2ydx$ $\frac{dy}{dx} = \frac{x^2y}{x^3+y^3}$	

This is a homogeneous differential equation

put $y = vx$

$$\frac{d(vx)}{dx} = \frac{x^2 vx}{x^3 + v^3 x^3}$$

$$v + x \frac{dv}{dx} = \frac{x^3 v}{x^3(1+v^3)}$$

$$x \frac{dv}{dx} = \frac{v}{1+v^3} - v$$

$$= \frac{v - v - v^4}{1+v^3}$$

$$x \frac{dv}{dx} = -\frac{v^4}{1+v^3}$$

$$\left[\frac{1+v^3}{v^4} \right] dv = -\frac{dx}{x}$$

$$\left[\frac{1}{v^4} + \frac{v^3}{v^4} \right] dv = -\frac{dx}{x}$$

$$\int \left[\frac{1}{v^4} + \frac{1}{v} \right] dv = -\int \frac{dx}{x}$$

$$\frac{-\frac{1}{3}}{-3} + \log|v| = -\log|x| + \log|c|$$

$$-\frac{1}{3v^3} + \log|v| = -\log|x| + \log|c|$$

$$\log|v| + \log|x| - \log|c| = \frac{1}{3v^3}$$

$$\log|vx| - \log|c| = \frac{1}{3v^3}$$

$$\log\left|\frac{vx}{c}\right| = \frac{1}{3v^3}$$

Replace v by $\frac{y}{x}$

$$\log\left|\frac{\frac{y}{x}x}{c}\right| = \frac{1}{3\frac{y^3}{x^3}}$$

$$\log\left|\frac{y}{c}\right| = \frac{x^3}{3y^3}$$

$$\left|\frac{y}{c}\right| = e^{\frac{x^3}{3y^3}}$$

$$\frac{y}{c} = \pm e^{\frac{x^3}{3y^3}}$$

$$y = \pm c e^{\frac{x^3}{3y^3}}$$

$$y = c e^{\frac{x^3}{3y^3}} \text{ where } c = \pm c$$

3) Solve: $y e^{\frac{x}{y}} dx = (x e^{\frac{x}{y}} + y) dy$

Soln: Given:

$$y e^{\frac{x}{y}} dx = (x e^{\frac{x}{y}} + y) dy$$

$$\frac{dx}{dy} = \frac{x e^{\frac{x}{y}} + y}{y e^{\frac{x}{y}}}$$

This is a homogeneous differential Equation

put $x = vy$

$$\frac{d(vy)}{dy} = \frac{vy e^{\frac{vy}{y}} + y}{y e^{\frac{vy}{y}}}$$

$$v + y \frac{dv}{dy} = \frac{vy e^v + y}{y e^v}$$

$$v + y \frac{dv}{dy} = \frac{y(v e^v + 1)}{y e^v}$$

$$y \frac{dv}{dy} = \frac{v e^v + 1}{e^v} - v$$

$$y \frac{dv}{dy} = \frac{e^v}{e^v} - \frac{v e^v}{e^v}$$

$$y \frac{dv}{dy} = \frac{1}{e^v}$$

$$\int e^v dv = \int \frac{dy}{y}$$

$$e^v = \log|y| + \log|c|$$

$$e^v = \log|cy|$$

$$e^{\frac{x}{y}} = \log|cy|$$

4) Solve: $2xy dx + (x^2 + 2y^2) dy = 0$

Soln: Given:

$$2xy dx + (x^2 + 2y^2) dy = 0$$

$$(x^2 + 2y^2) dy = -2xy dx$$

$$\frac{dy}{dx} = -\frac{2xy}{x^2 + 2y^2}$$

This is a homogeneous differential Equation

put $y = vx$

$$\frac{d(vx)}{dx} = -\frac{2x vx}{x^2 + 2v^2 x^2}$$

$$v + x \frac{dv}{dx} = - \frac{x^2 2v}{x^2(1+2v^2)}$$

$$x \frac{dv}{dx} = - \frac{2v}{1+2v^2} - v$$

$$x \frac{dv}{dx} = \frac{-2v - v - 2v^3}{1+2v^2}$$

$$x \frac{dv}{dx} = - \frac{3v + 2v^3}{1+2v^2}$$

$$x \frac{dv}{dx} = - \left[\frac{3v + 2v^3}{1+2v^2} \right]$$

$$\left[\frac{1+2v^2}{3v+2v^3} \right] dv = - \frac{dx}{x}$$

Multiply by 3 on both sides,

$$3 \left[\frac{1+2v^2}{3v+2v^3} \right] dv = -3 \frac{dx}{x}$$

$$\int \left[\frac{3+6v^2}{3v+2v^3} \right] dv = -3 \int \frac{dx}{x}$$

$$\log |3v+2v^3| = -3 \log |x| + \log |c|$$

$$\log |3v+2v^3| + 3 \log |x| = \log |c|$$

$$\log |3v+2v^3| + \log |x^3| = \log |c|$$

$$\log |(3v+2v^3)x^3| = \log |c|$$

$$|(3v+2v^3)x^3| = |c|$$

$$\left| \left(3 \frac{y}{x} + 2 \frac{y^3}{x^3} \right) x^3 \right| = |c|$$

$$|3x^2y + 2y^3| = |c|$$

$$3x^2y + 2y^3 = \pm c$$

$$3x^2y + 2y^3 = c \text{ where } c = \pm c$$

5) Solve: $(y^2 - 2xy)dx = (x^2 - 2xy)dy$
Soln: Given:

$$(y^2 - 2xy)dx = (x^2 - 2xy)dy$$

$$\frac{dy}{dx} = \frac{y^2 - 2xy}{x^2 - 2xy}$$

This is a homogeneous differential equation
Put $y = vx$

$$\frac{d(vx)}{dx} = \frac{v^2x^2 - 2xvx}{x^2 - 2xvx}$$

$$v + x \frac{dv}{dx} = \frac{v^2x^2 - 2vx^2}{x^2 - 2vx^2}$$

$$v + x \frac{dv}{dx} = \frac{x^2(v^2 - 2v)}{x^2(1 - 2v)}$$

$$x \frac{dv}{dx} = \frac{v^2 - 2v}{1 - 2v} - v$$

$$= \frac{v^2 - 2v - v + 2v^2}{1 - 2v}$$

$$x \frac{dv}{dx} = \frac{3v^2 - 3v}{1 - 2v}$$

$$\left[\frac{1 - 2v}{3v^2 - 3v} \right] dv = \frac{dx}{x}$$

$$\left[\frac{1 - 2v}{3(v^2 - v)} \right] dv = \frac{dx}{x}$$

Multiply by 3 on both sides,

$$\left[\frac{1 - 2v}{v^2 - v} \right] dv = 3 \frac{dx}{x}$$

$$\left[\frac{1 - 2v}{v(v-1)} \right] dv = 3 \frac{dx}{x} \text{ --- (1)}$$

Using partial fraction,

$$\frac{1 - 2v}{v(v-1)} = \frac{A}{v} + \frac{B}{v-1} \text{ --- (2)}$$

$$\frac{1 - 2v}{v(v-1)} = \frac{A(v-1) + Bv}{v(v-1)}$$

$$1 - 2v = A(v-1) + Bv$$

Put $v=0$,

$$1 = -A \Rightarrow \boxed{A = -1}$$

Put $v=1$,

$$-1 = B \Rightarrow \boxed{B = -1}$$

$$\text{(2)} \Rightarrow \frac{1 - 2v}{v(v-1)} = -\frac{1}{v} - \frac{1}{v-1}$$

$$\text{(1)} \Rightarrow \left[-\frac{1}{v} - \frac{1}{v-1} \right] dv = 3 \frac{dx}{x}$$

$$\int \left[-\frac{1}{v} + \frac{1}{v-1} \right] dv = -3 \int \frac{dx}{x}$$

$$\log|v| + \log|v-1| = -3\log|x|$$

$$\log|v(v-1)| + 3\log|x| = \log|c|$$

$$\log|v^2-v| + \log|x^3| = \log|c|$$

$$\log|x^3(v^2-v)| = \log|c|$$

$$|x^3(v^2-v)| = |c|$$

$$|x^3(\frac{y^2}{x^2} - \frac{y}{x})| = |c|$$

$$|\frac{x^3y^2}{x^2} - \frac{x^3y}{x}| = |c|$$

$$|xy^2 - x^2y| = |c|$$

$$xy^2 - x^2y = \pm c$$

$$xy^2 - x^2y = C \text{ where } C = \pm c$$

6) Solve: $x \frac{dy}{dx} = y - x \cos^2(\frac{y}{x})$

Soln: Given:

$$x \frac{dy}{dx} = y - x \cos^2(\frac{y}{x})$$

$$\frac{dy}{dx} = \frac{y - x \cos^2(\frac{y}{x})}{x}$$

This is a homogeneous differential equation

Put $y = vx$

$$\frac{d(vx)}{dx} = \frac{vx - x \cos^2(\frac{vx}{x})}{x}$$

$$v + x \frac{dv}{dx} = \frac{vx - x \cos^2 v}{x}$$

$$= \frac{x(v - \cos^2 v)}{x}$$

$$v + x \frac{dv}{dx} = v - \cos^2 v$$

$$x \frac{dv}{dx} = v - v - \cos^2 v$$

$$x \frac{dv}{dx} = -\cos^2 v$$

$$\frac{dv}{\cos^2 v} = -\frac{dx}{x}$$

$$\int \sec^2 v dv = -\int \frac{dx}{x}$$

$$\tan v = -(\log|x| + \log|c|)$$

$$\tan v = \log|c| - \log|x|$$

$$\tan v = \log\left[\frac{|c|}{|x|}\right]$$

$$\tan v = \log\left[\left|\frac{c}{x}\right|\right]$$

$$\tan\left(\frac{y}{x}\right) = \log\left|\frac{c}{x}\right|$$

$$\therefore \log\left|\frac{c}{x}\right| = \tan\left(\frac{y}{x}\right)$$

$$\left|\frac{c}{x}\right| = e^{\tan\left(\frac{y}{x}\right)}$$

$$\pm \frac{c}{x} = e^{\tan\left(\frac{y}{x}\right)}$$

$$\Rightarrow x e^{\tan\left(\frac{y}{x}\right)} = \pm c$$

$$\therefore x e^{\tan\left(\frac{y}{x}\right)} = c$$

7) Solve:

$$(1 + 3e^{\frac{y}{x}})dy + 3e^{\frac{y}{x}}(1 - \frac{y}{x})dx = 0$$

given that $y=0$

when $x=1$

Soln:

Given: $(1 + 3e^{\frac{y}{x}})dy + 3e^{\frac{y}{x}}(1 - \frac{y}{x})dx = 0$

$$(1 + 3e^{\frac{y}{x}})dy = -3e^{\frac{y}{x}}(1 - \frac{y}{x})dx$$

$$\frac{dy}{dx} = \frac{-3e^{\frac{y}{x}}(1 - \frac{y}{x})}{(1 + 3e^{\frac{y}{x}})}$$

This is a homogeneous differential equation

Put $y = vx$

$$\frac{d(vx)}{dx} = \frac{-3e^{\frac{vx}{x}}(1 - \frac{vx}{x})}{1 + 3e^{\frac{vx}{x}}}$$

$$v + x \frac{dv}{dx} = \frac{-3e^v(1-v)}{1 + 3e^v}$$

$$x \frac{dv}{dx} = \frac{-3e^v + 3ve^v}{1 + 3e^v} - v$$

$$= \frac{-3e^v + 3ve^v - v - 3ve^v}{1 + 3e^v}$$

$$= \frac{-3e^v - v}{1 + 3e^v}$$

$$x \frac{dv}{dx} = \frac{-(3e^v + v)}{1 + 3e^v}$$

$$\int \left[\frac{1+3e^v}{v+3e^v} \right] dv = - \int \frac{dx}{x}$$

$$\log|v+3e^v| = -\log|x| + \log|c|$$

$$\log|v+3e^v| + \log|x| = \log|c|$$

$$\log|x(v+3e^v)| = \log|c|$$

$$|x(v+3e^v)| = |c|$$

$$|x\left(\frac{y}{x} + 3e^{\frac{y}{x}}\right)| = |c|$$

$$|y + 3xe^{\frac{y}{x}}| = |c|$$

$$y + 3xe^{\frac{y}{x}} = \pm c$$

$$y + 3xe^{\frac{y}{x}} = c \quad \text{--- (1)}$$

When $x=1, y=0$

$$(1) \Rightarrow 0 + 3(1)e^0 = c$$

$$3(1) = c$$

$$\Rightarrow \boxed{c=3}$$

$$(1) \Rightarrow y + 3xe^{y/x} = 3$$

8) Solve: $(x^2+y^2)dy = xydx$.

It is given that $y(1)=1$

and $y(x_0)=e$. Find the value of x_0

Soln: Given:

$$(x^2+y^2)dy = xydx$$

$$\frac{dy}{dx} = \frac{xy}{x^2+y^2}$$

This is a homogeneous Differential Equation

Put $y=vx$

$$\frac{d(vx)}{dx} = \frac{vx}{x^2+v^2x^2}$$

$$v + x \frac{dv}{dx} = \frac{x^2v}{x^2(1+v^2)}$$

$$x \frac{dv}{dx} = \frac{v}{1+v^2} - v$$

$$x \frac{dv}{dx} = \frac{v - v - v^3}{1+v^2}$$

$$x \frac{dv}{dx} = - \frac{v^3}{1+v^2}$$

$$\int \left[\frac{1+v^2}{v^3} \right] dv = - \int \frac{dx}{x}$$

$$\int \left[\frac{1}{v^3} + \frac{v^2}{v^3} \right] dv = - \int \frac{dx}{x}$$

$$\int \left[-\frac{1}{2v^2} + \frac{1}{v} \right] dv = - \int \frac{dx}{x}$$

$$\frac{-1}{2v^2} + \log|v| = -\log|x| + \log|c|$$

$$\frac{-1}{2v^2} + \log|v| = -\log|x| + \log|c|$$

$$\log|v| + \log|x| - \log|c| = \frac{1}{2v^2}$$

$$\log|vx| - \log|c| = \frac{1}{2v^2}$$

$$\log \left| \frac{vx}{c} \right| = \frac{1}{2v^2}$$

$$\left| \frac{vx}{c} \right| = e^{\frac{1}{2v^2}}$$

$$\left| \frac{y}{c} \right| = e^{\frac{1}{2\frac{y^2}{x^2}}}$$

$$\left| \frac{y}{c} \right| = e^{\frac{x^2}{2y^2}}$$

$$\frac{y}{c} = \pm e^{\frac{x^2}{2y^2}}$$

$$y = \pm c e^{\frac{x^2}{2y^2}}$$

$$\therefore y = c e^{\frac{x^2}{2y^2}} \quad \text{--- (1)}$$

when $x=1, y=1$

$$(1) \Rightarrow \frac{(1)^2}{2(1)^2}$$

$$1 = c e^{\frac{1}{2}}$$

$$1 = c e^{\frac{1}{2}} \Rightarrow \boxed{c = e^{-\frac{1}{2}}}$$

$$(1) \Rightarrow y = e^{-\frac{1}{2}} e^{\frac{x^2}{2y^2}} \quad \text{--- (2)}$$

when $x=x_0, y=e$

$$\begin{aligned} ② \Rightarrow e &= e^{-\frac{1}{2}} e^{\frac{x_0^2}{2e^2}} \\ e \cdot e^{\frac{1}{2}} &= e^{\frac{x_0^2}{2e^2}} \\ e^{\frac{3}{2}} &= e^{\frac{x_0^2}{2e^2}} \\ \text{Taking log on both sides} \\ \text{we get,} \\ \log e^{\frac{3}{2}} &= \log e^{\frac{x_0^2}{2e^2}} \\ \frac{3}{2} &= \frac{x_0^2}{2e^2} \\ \Rightarrow x_0^2 &= 3e^2 \\ \Rightarrow x_0 &= \pm \sqrt{3} e \end{aligned}$$

Ex 10.17

Solve $(x^2 - 3y^2)dx + 2xydy = 0$

Soln: Given:

$$(x^2 - 3y^2)dx + 2xydy = 0$$

$$2xydy = -(x^2 - 3y^2)dx$$

$$\frac{dy}{dx} = -\frac{(x^2 - 3y^2)}{2xy}$$

This is a homogeneous Differential Equation

Put $y = vx$

$$\frac{d(vx)}{dx} = \frac{-(x^2 - 3v^2x^2)}{2xvx}$$

$$v + x \frac{dv}{dx} = -\frac{x^2(1 - 3v^2)}{2x^2v}$$

$$x \frac{dv}{dx} = -\frac{(1 - 3v^2)}{2v} - v$$

$$x \frac{dv}{dx} = \frac{-1 + 3v^2 - 2v^2}{2v}$$

$$x \frac{dv}{dx} = \frac{-1 + v^2}{2v} = \frac{v^2 - 1}{2v}$$

$$\int \frac{2v}{v^2 - 1} dv = \int \frac{dx}{x}$$

$$\log|v^2 - 1| = \log|x| + \log|c|$$

$$\log|v^2 - 1| = \log|cx|$$

$$|v^2 - 1| = |cx|$$

$$\left| \frac{y^2}{x^2} - 1 \right| = |cx|$$

$$\left| \frac{y^2 - x^2}{x^2} \right| = |cx|$$

$$\frac{|y^2 - x^2|}{|x^2|} = |cx|$$

$$|y^2 - x^2| = |x^2||cx|$$

$$|y^2 - x^2| = |cx^3|$$

$$y^2 - x^2 = \pm cx^3$$

$$\Rightarrow y^2 - x^2 = kx^3 \text{ where } k = \pm c$$

Ex 10.18

Solve $(y + \sqrt{x^2 + y^2})dx - xdy = 0$ $y(1) = 0$

Soln: Given:

$$(y + \sqrt{x^2 + y^2})dx - xdy = 0$$

$$(y + \sqrt{x^2 + y^2})dx = xdy$$

$$\frac{dy}{dx} = \frac{y + \sqrt{x^2 + y^2}}{x}$$

This is a homogeneous Differential Equation

Put $y = vx$

$$\frac{d(vx)}{dx} = \frac{vx + \sqrt{x^2 + v^2x^2}}{x}$$

$$v + x \frac{dv}{dx} = \frac{vx + \sqrt{x^2(1 + v^2)}}{x}$$

$$v + x \frac{dv}{dx} = \frac{vx + x\sqrt{1 + v^2}}{x}$$

$$v + x \frac{dv}{dx} = \frac{x(v + \sqrt{1 + v^2})}{x}$$

$$x \frac{dv}{dx} = v + \sqrt{1 + v^2} - v$$

$$x \frac{dv}{dx} = \sqrt{1 + v^2}$$

$$\int \frac{dv}{\sqrt{1 + v^2}} = \int \frac{dx}{x}$$

$$\log|v + \sqrt{1 + v^2}| = \log|x| + \log|c|$$

$$\log|v + \sqrt{v^2 + 1}| = \log|cx|$$

$$|v + \sqrt{v^2 + 1}| = |cx|$$

$\left \frac{y}{x} + \sqrt{\frac{y^2}{x^2} + 1} \right = cx $ $\left \frac{y}{x} + \sqrt{\frac{x^2+y^2}{x^2}} \right = cx $ $\left \frac{y}{x} + \frac{\sqrt{x^2+y^2}}{x} \right = cx $ $\left \frac{y + \sqrt{x^2+y^2}}{x} \right = cx $ $\frac{ y + \sqrt{x^2+y^2} }{ x } = cx $ $ y + \sqrt{x^2+y^2} = cx^2 $ $y + \sqrt{x^2+y^2} = \pm cx^2$ $y + \sqrt{x^2+y^2} = cx^2, \text{ where } \pm c = c$ when $x=1, y=0$ $0 + \sqrt{(1)^2 + 0} = c(1)^2$ $1 = c \Rightarrow \boxed{c=1}$ $\textcircled{1} \Rightarrow y + \sqrt{x^2+y^2} = (1)x^2$	$y + \sqrt{x^2+y^2} = x^2$ Ex 10.19 Solve: $(2x+3y)dx + (y-x)dy = 0$ Soln: $(y-x)dy = -(2x+3y)dx$ $\frac{dy}{dx} = -\frac{(2x+3y)}{y-x}$ $\frac{dy}{dx} = \frac{2x+3y}{x-y}$ This is a homogeneous Differential equation Put $y=vx$ $\frac{d(vx)}{dx} = \frac{2x+3vx}{x-vx}$ $v + x \frac{dv}{dx} = \frac{x(2+3v)}{x(1-v)}$ $x \frac{dv}{dx} = \frac{2+3v}{1-v} - v$	$x \frac{dv}{dx} = \frac{2+3v-v+v^2}{1-v}$ $x \frac{dv}{dx} = \frac{2+2v+v^2}{1-v}$ $x \frac{dv}{dx} = \frac{1+1+2v+v^2}{1-v}$ $x \frac{dv}{dx} = \frac{1+(1+v)^2}{1-v}$ $\left[\frac{1-v}{1+(1+v)^2} \right] dv = \frac{dx}{x}$ $-\frac{1}{2} \left[\frac{2v+2-4}{1+(1+v)^2} \right] dv = \frac{dx}{x}$ $-\frac{1}{2} \left[\frac{2v+2}{1+(1+v)^2} - \frac{4}{1+(1+v)^2} \right] dv = \frac{dx}{x}$ $-\frac{1}{2} \left[\frac{2v+2}{v^2+2v+2} - \frac{4}{1+(1+v)^2} \right] dv = \int \frac{dx}{x}$ $\frac{-1}{2} \log v^2+2v+2 + 2 \tan^{-1}(v+1) = \log x + \log c $	Multiple by -2 on both sides we get, $\log v^2+2v+2 - 4 \tan^{-1}(v+1) = -2 \log x - 2 \log c $ $\log v^2+2v+2 - 4 \tan^{-1}(v+1) = -\log x^2 - 2 \log c $ $\log v^2+2v+2 + \log x^2 - 4 \tan^{-1}(v+1) = -2 \log c $ $\log (v^2+2v+2)x^2 - 4 \tan^{-1}(v+1) = -2 \log c $ $\log\left \left(\frac{y^2}{x^2} + 2\frac{y}{x} + 2\right)x^2\right - 4 \tan^{-1}\left(\frac{y}{x} + 1\right) = -2 \log c $ $\log y^2 + 2xy + 2x^2 - 4 \tan^{-1}\left(\frac{x+y}{x}\right) = K$ Where $K = -2 \log c$ Ex 10.20
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Solve: $y^2 + x^2 \frac{dy}{dx} = xy \frac{dy}{dx}$

Soln: Given:

$$y^2 + x^2 \frac{dy}{dx} = xy \frac{dy}{dx}$$

$$\dots y^2 = xy \frac{dy}{dx} - x^2 \frac{dy}{dx}$$

$$y^2 = (xy - x^2) \frac{dy}{dx}$$

$$\Rightarrow \frac{dy}{dx} = \frac{y^2}{xy - x^2}$$

This is a homogeneous differential equation

Put $y = vx$

$$\frac{d(vx)}{dx} = \frac{v^2 x^2}{xvx - x^2}$$

$$v + x \frac{dv}{dx} = \frac{v^2 x^2}{x^2(v-1)}$$

$$x \frac{dv}{dx} = \frac{v^2}{v-1} - v$$

$$x \frac{dv}{dx} = \frac{v^2 - v^2 + v}{v-1}$$

$$x \frac{dv}{dx} = \frac{v}{v-1}$$

$$\left[\frac{v-1}{v} \right] dv = \frac{dx}{x}$$

$$\left[\frac{v}{v} - \frac{1}{v} \right] dv = \frac{dx}{x}$$

$$\int \left[1 - \frac{1}{v} \right] dv = \int \frac{dx}{x}$$

$$v - \log|v| = \log|x| + \log|c|$$

$$v = \log|v| + \log|x| + \log|c|$$

$$v = \log|vxc|$$

$$\frac{y}{x} = \log \left| \frac{y}{x} xc \right|$$

$$\frac{y}{x} = \log|cy|$$

$$\Rightarrow \log|cy| = \frac{y}{x}$$

$$|cy| = e^{\frac{y}{x}}$$

$$cy = \pm e^{\frac{y}{x}}$$

$$y = \pm \frac{1}{c} e^{\frac{y}{x}}$$

$$y = K e^{\frac{y}{x}} \text{ where } K = \pm \frac{1}{c}$$

Ex 10-21

Solve: $(1+2e^{\frac{x}{y}})dx + 2e^{\frac{x}{y}}(1-\frac{x}{y})dy = 0$

Soln: Given:

$$(1+2e^{\frac{x}{y}})dx + 2e^{\frac{x}{y}}(1-\frac{x}{y})dy = 0$$

$$(1+2e^{\frac{x}{y}})dx = -2e^{\frac{x}{y}}(1-\frac{x}{y})dy$$

$$\frac{dx}{dy} = \frac{-2e^{\frac{x}{y}}(1-\frac{x}{y})}{(1+2e^{\frac{x}{y}})}$$

This is a homogeneous differential equation

Put $x = vy$

$$\frac{d(vy)}{dy} = \frac{-2e^{\frac{vy}{y}}(1-\frac{vy}{y})}{(1+2e^{\frac{vy}{y}})}$$

$$v + y \frac{dv}{dy} = \frac{-2e^v(1-v)}{(1+2e^v)}$$

$$y \frac{dv}{dy} = \frac{-2e^v + 2ve^v}{1+2e^v} - v$$

$$y \frac{dv}{dy} = \frac{-2e^v + 2ve^v - v(1+2e^v)}{1+2e^v}$$

$$y \frac{dv}{dy} = -\frac{(v+2e^v)}{1+2e^v}$$

$$\int \frac{1+2e^v}{v+2e^v} dv = -\int \frac{dy}{y}$$

$$\log(v+2e^v) = -\log|y| + \log|c|$$

$$\log|v+2e^v| + \log|y| = \log|c|$$

$$\log \left| \left(\frac{x}{y} + 2e^{\frac{x}{y}} \right) y \right| = \log|c|$$

$$\log \left| x + 2ye^{\frac{x}{y}} \right| = \log|c|$$

$$|x + 2ye^{\frac{x}{y}}| = |c|$$

10.7 First order Linear Differential Equations

EXERCISE 10.7

Solve the following Linear differential equations:

1) Solve: $\cos x \frac{dy}{dx} + y \sin x = 1$

Soln: Given:

$$\cos x \frac{dy}{dx} + y \sin x = 1$$

$$\div \cos x, \quad \frac{dy}{dx} + \frac{\sin x}{\cos x} y = \frac{1}{\cos x}$$

$$\frac{dy}{dx} + \tan x y = \frac{1}{\cos x}$$

This is a linear differential equation in y

Here $P = \tan x$, $Q = \frac{1}{\cos x}$

$$I.F = e^{\int P dx} = e^{\int \tan x dx} = e^{\log \sec x}$$

$$\boxed{I.F = \sec x}$$

$\therefore$ The Solution is

$$y e^{\int P dx} = \int Q e^{\int P dx} dx + C$$

$$y \sec x = \int \frac{1}{\cos x} \sec x dx + C$$

$$y \sec x = \int \sec x \sec x dx + C$$

$$y \sec x = \int \sec^2 x dx + C$$

$$y \sec x = \tan x + C$$

$$y \left(\frac{1}{\cos x} \right) = \frac{\sin x}{\cos x} + C$$

Multiply by $\cos x$,

$$y = \sin x + C \cos x$$

2) Solve: $(1-x^2) \frac{dy}{dx} - xy = 1$

Soln: Given:

$$(1-x^2) \frac{dy}{dx} - xy = 1$$

$$\div (1-x^2), \quad \frac{dy}{dx} - \frac{x}{1-x^2} y = \frac{1}{1-x^2}$$

This is a linear differential equation in y

Here $P = \frac{-x}{1-x^2}$, $Q = \frac{1}{1-x^2}$

$$I.F = e^{\int P dx} = e^{\int \frac{-x}{1-x^2} dx}$$

$$= e^{\frac{1}{2} \int \frac{-2x}{1-x^2} dx}$$

$$= e^{\frac{1}{2} \log(1-x^2)}$$

$$= e^{\log(1-x^2)^{1/2}}$$

$$= (1-x^2)^{1/2}$$

$$\boxed{I.F = \sqrt{1-x^2}}$$

$\therefore$ The Solution is

$$y e^{\int P dx} = \int Q e^{\int P dx} dx + C$$

$$y \sqrt{1-x^2} = \int \frac{1}{1-x^2} \sqrt{1-x^2} dx + C$$

$$y \sqrt{1-x^2} = \int \frac{1}{\sqrt{1-x^2}} dx + C$$

$$y \sqrt{1-x^2} = \sin^{-1} x + C$$

$$\div \sqrt{1-x^2},$$

$$y = \frac{\sin^{-1} x}{\sqrt{1-x^2}} + \frac{C}{\sqrt{1-x^2}}$$

$$y = \frac{\sin^{-1} x}{\sqrt{1-x^2}} + \frac{C}{(1-x^2)^{1/2}}$$

$$y = \frac{\sin^{-1} x}{\sqrt{1-x^2}} + C(1-x^2)^{-1/2}$$

3) Solve: $\frac{dy}{dx} + \frac{y}{x} = \sin x$

Soln: Given:

$$\frac{dy}{dx} + \frac{1}{x} y = \sin x$$

This is a linear differential equation in y

Here $P = \frac{1}{x}$, $Q = \sin x$

$$I \cdot F = e^{\int P dx} = e^{\int \frac{1}{x} dx} = e^{\log x} = x$$

$$\therefore \boxed{I \cdot F = x}$$

$\therefore$ The Solution is

$$y e^{\int P dx} = \int Q e^{\int P dx} dx + C$$

$$y x = \int (\sin x) x dx + C$$

$$xy = x(-\cos x) - \int (-\cos x) dx + C$$

$$xy = -x \cos x + \int \cos x dx + C$$

$$xy = -x \cos x + \sin x + C$$

$$xy + x \cos x = \sin x + C$$

$$(y + \cos x)x = \sin x + C$$

4) Solve: $(x^2+1) \frac{dy}{dx} + 2xy = \sqrt{x^2+4}$

Soln: $(x^2+1) \frac{dy}{dx} + 2xy = \sqrt{x^2+4}$

$$\div (x^2+1), \frac{dy}{dx} + \frac{2x}{x^2+1} y = \frac{\sqrt{x^2+4}}{x^2+1}$$

This is a linear differential equation in y

Here $P = \frac{2x}{x^2+1}$, $Q = \frac{\sqrt{x^2+4}}{x^2+1}$

$$I \cdot F = e^{\int P dx} = e^{\int \frac{2x}{x^2+1} dx} = e^{\log(x^2+1)} = (x^2+1)$$

$$\therefore \boxed{I \cdot F = (x^2+1)}$$

$\therefore$ The Solution is

$$y e^{\int P dx} = \int Q e^{\int P dx} dx + C$$

$$y(x^2+1) = \int \frac{\sqrt{x^2+4}}{(x^2+1)} (x^2+1) dx + C$$

$$y(x^2+1) = \int \sqrt{x^2+2^2} dx + C$$

$$y(x^2+1) = \frac{x}{2} \sqrt{x^2+2^2} + \frac{4}{2} \log|x+\sqrt{x^2+2^2}| + C$$

$$y(x^2+1) = \frac{x}{2} \sqrt{x^2+4} + 2 \log|x+\sqrt{x^2+4}| + C$$

5) Solve: $(2x-10y^3)dy + ydx = 0$

Soln: Given:

$$(2x-10y^3)dy + ydx = 0$$

$$ydx = -(2x-10y^3)dy$$

$$\frac{dx}{dy} = -\frac{(2x-10y^3)}{y}$$

$$\frac{dx}{dy} = -\frac{2x}{y} + \frac{10y^2}{y}$$

$$\frac{dx}{dy} + \frac{2x}{y} = 10y^2$$

This is a linear differential equation in x

Here $P = \frac{2}{y}$, $Q = 10y^2$

$$I \cdot F = e^{\int P dy} = e^{\int \frac{2}{y} dy} = e^{2 \log y} = e^{\log y^2} = y^2$$

$$\therefore \boxed{I \cdot F = y^2}$$

$\therefore$ The Solution is

$$x e^{\int P dy} = \int Q e^{\int P dy} dy + C$$

$$xy^2 = \int 10y^2(y^2) dy + C$$

$$= 10 \int y^4 dy + C$$

$$= 10 \left[\frac{y^5}{5} \right] + C$$

$$xy^2 = 2y^5 + C$$

6) Solve:

$$x \sin x \frac{dy}{dx} + (x \cos x + \sin x)y = \sin x$$

Soln: Given:

$$x \sin x \frac{dy}{dx} + (x \cos x + \sin x)y = \sin x$$

$$\div x \sin x,$$

$$\frac{dy}{dx} + \frac{x \cos x + \sin x}{x \sin x} y = \frac{\sin x}{x \sin x}$$

$$\frac{dy}{dx} + \frac{x \cos x + \sin x}{x \sin x} y = \frac{1}{x}$$

This is a linear differential equation in y

Here $P = \frac{x \cos x + \sin x}{x \sin x}$, $Q = \frac{1}{x}$

$$I.F = e^{\int P dx} = e^{\int \frac{x \cos x + \sin x}{x \sin x} dx}$$

$$= e^{\log(x \sin x)} = x \sin x$$

$$\therefore \boxed{I.F = x \sin x}$$

$\therefore$ The Solution is

$$y e^{\int P dx} = \int Q e^{\int P dx} dx + c$$

$$y(x \sin x) = \int \frac{1}{x} (x \sin x) dx + c$$

$$x y \sin x = \int \sin x dx + c$$

$$x y \sin x = -\cos x + c$$

$$x y \sin x + \cos x = c$$

7) Solve: $(y - e^{\sin^{-1} x}) \frac{dx}{dy} + \sqrt{1-x^2} = 0$

Soln: Given:

$$(y - e^{\sin^{-1} x}) \frac{dx}{dy} + \sqrt{1-x^2} = 0$$

$$(y - e^{\sin^{-1} x}) \frac{dx}{dy} = -\sqrt{1-x^2}$$

$$(y - e^{\sin^{-1} x}) = -\sqrt{1-x^2} \frac{dy}{dx}$$

$$\sqrt{1-x^2} \frac{dy}{dx} + y = e^{\sin^{-1} x}$$

$$\div \sqrt{1-x^2}, \frac{dy}{dx} + \frac{1}{\sqrt{1-x^2}} y = \frac{e^{\sin^{-1} x}}{\sqrt{1-x^2}}$$

This is a linear differential equation in y

Here $P = \frac{1}{\sqrt{1-x^2}}, Q = \frac{e^{\sin^{-1} x}}{\sqrt{1-x^2}}$

$$I.F = e^{\int P dx} = e^{\int \frac{1}{\sqrt{1-x^2}} dx}$$

$$= e^{\sin^{-1} x}$$

$$\therefore \boxed{I.F = e^{\sin^{-1} x}}$$

The Solution is

$$y e^{\int P dx} = \int Q e^{\int P dx} dx + c$$

$$y e^{\sin^{-1} x} = \int \frac{e^{\sin^{-1} x}}{\sqrt{1-x^2}} e^{\sin^{-1} x} dx + c$$

$$y e^{\sin^{-1} x} = \int t e^t dt + c$$

Put $t = \sin^{-1} x$
 $dt = \frac{1}{\sqrt{1-x^2}} dx$

$$y e^{\sin^{-1} x} = \int e^{2t} dt + c$$

$$y e^{\sin^{-1} x} = \frac{e^{2t}}{2} + c$$

$$y e^{\sin^{-1} x} = \frac{e^{2 \sin^{-1} x}}{2} + c$$

8) Solve: $\frac{dy}{dx} + \frac{y}{(1-x)\sqrt{x}} = 1 - \sqrt{x}$

Soln: Given:

$$\frac{dy}{dx} + \frac{y}{(1-x)\sqrt{x}} = 1 - \sqrt{x}$$

This is a linear differential equation in y

Here $P = \frac{1}{(1-x)\sqrt{x}}, Q = 1 - \sqrt{x}$

$$I.F = e^{\int P dx} = e^{\int \frac{1}{(1-x)\sqrt{x}} dx}$$

$$= e^{\int \frac{1}{1-t^2} 2t dt}$$

Put $t = \sqrt{x}$
 $t^2 = x$
 $dt = \frac{1}{2\sqrt{x}} dx$
 $\Rightarrow \frac{1}{\sqrt{x}} dx = 2dt$

$$= e^{\int \frac{2}{(1+t)(1-t)} dt}$$

$$= e^{\log(1+t) - \log(1-t)}$$

$$= e^{\log\left(\frac{1+t}{1-t}\right)}$$

$$= \frac{1+t}{1-t}$$

$$= \frac{1+\sqrt{x}}{1-\sqrt{x}}$$

$$\therefore \boxed{I.F = \frac{1+\sqrt{x}}{1-\sqrt{x}}}$$

$\therefore$ The Solution is

$$y e^{\int P dx} = \int Q e^{\int P dx} dx + c$$

$$y \left(\frac{1+\sqrt{x}}{1-\sqrt{x}} \right) = \int \left(\frac{1-\sqrt{x}}{1-\sqrt{x}} \right) \left(\frac{1+\sqrt{x}}{1-\sqrt{x}} \right) dx + c$$

$$\frac{2}{(1+t)(1-t)} = \frac{A}{1+t} + \frac{B}{1-t}$$

$$\frac{2}{(1+t)(1-t)} = \frac{A(1-t) + B(1+t)}{(1+t)(1-t)}$$

$$2 = A(1-t) + B(1+t)$$

Put $t = 1$
 $2 = A(1-1) + B(1+1)$
 $2 = 2B \Rightarrow B = 1$

Put $t = -1$
 $2 = A(1-(-1)) + B(1+(-1))$
 $2 = 2A \Rightarrow A = 1$

$$\frac{2}{(1+t)(1-t)} = \frac{1}{1+t} + \frac{1}{1-t}$$

$$y\left(\frac{1+\sqrt{x}}{1-\sqrt{x}}\right) = \int (1+x^{1/2}) dx + c$$

$$= x + \frac{x^{3/2}}{3/2} + c$$

$$= x + \frac{2}{3} x^{3/2} + c$$

$$y\left(\frac{1+\sqrt{x}}{1-\sqrt{x}}\right) = x + \frac{2}{3} x\sqrt{x} + c$$

9) Solve:

$$(1+x+xy^2)\frac{dy}{dx} + (y+y^3) = 0$$

Soln: Given:

$$(1+x+xy^2)\frac{dy}{dx} + (y+y^3) = 0$$

$$(1+x+xy^2)\frac{dy}{dx} = -(y+y^3)$$

$$(1+x+xy^2) = -(y+y^3)\frac{dx}{dy}$$

$$1+x(1+y^2) = -y(1+y^2)\frac{dx}{dy}$$

$$y(1+y^2)\frac{dx}{dy} + (1+y^2)x = -1$$

$$\div y(1+y^2), \frac{dx}{dy} + \frac{x}{y} = \frac{-1}{y(1+y^2)}$$

This is a linear differential equation in x

$$\text{Here } P = \frac{1}{y}, Q = \frac{-1}{y(1+y^2)}$$

$$I.F = e^{\int P dy} = e^{\int \frac{1}{y} dy}$$

$$= e^{\log y} = y$$

$$\therefore \boxed{I.F = y}$$

$\therefore$ The Solution is

$$x e^{\int P dy} = \int Q e^{\int P dy} dy + c$$

$$xy = \int \frac{-1}{y(1+y^2)} y dy + c$$

$$xy = -\int \frac{1}{1+y^2} dy + c$$

$$xy = -\tan^{-1} y + c$$

$$xy + \tan^{-1} y = c$$

10) Solve:

$$\frac{dy}{dx} + \frac{y}{x \log x} = \frac{\sin 2x}{\log x}$$

Soln: Given:

$$\frac{dy}{dx} + \frac{1}{x \log x} y = \frac{\sin 2x}{\log x}$$

This is a linear differential equation in y

$$\text{Here } P = \frac{1}{x \log x}, Q = \frac{\sin 2x}{\log x}$$

$$I.F = e^{\int P dx} = e^{\int \frac{1/x}{\log x} dx}$$

$$= e^{\log(\log x)} = \log x$$

$$\therefore \boxed{I.F = \log x}$$

$\therefore$ The Solution is

$$y e^{\int P dx} = \int Q e^{\int P dx} dx + c$$

$$y \log x = \int \frac{\sin 2x (\log x)}{(\log x)} dx + c$$

$$y \log x = \int \sin 2x dx + c$$

$$y \log x = -\frac{\cos 2x}{2} + c$$

$$y \log x + \frac{\cos 2x}{2} = c$$

11) Solve:

$$(x+a)\frac{dy}{dx} - 2y = (x+a)^4$$

Soln: Given:

$$(x+a)\frac{dy}{dx} - 2y = (x+a)^4$$

$$\div (x+a), \frac{dy}{dx} - \frac{2}{x+a} y = (x+a)^3$$

This is a linear differential equation in y

$$\text{Here } P = \frac{-2}{x+a}, Q = (x+a)^3$$

$$I.F = e^{\int P dx} = e^{\int \frac{-2}{x+a} dx}$$

$$= e^{-2 \log(x+a)} = e^{\log(x+a)^{-2}}$$

$$= (x+a)^{-2}$$

$$\therefore \boxed{I.F = \frac{1}{(x+a)^2}}$$

The Solution is $y e^{\int P dx} = \int Q e^{\int P dx} dx + C$ $y \frac{1}{(x+a)^2} = \int \frac{1}{(x+a)^3} \frac{1}{(x+a)^2} dx + C$ $\frac{y}{(x+a)^2} = \int \frac{1}{(x+a)^3} dx + C$ $\frac{y}{(x+a)^2} = \frac{(x+a)^{-2}}{-2} + C$ $\frac{y}{(x+a)^2} = \frac{(x+a)^{-2}}{-2} + C$ $2y = (x+a)^4 + 2C(x+a)^2$	$\frac{dy}{dx} + \frac{3x^2}{1+x^3} y = \frac{\sin^2 x}{1+x^3}$ This is a linear differential equation in y Here $P = \frac{3x^2}{1+x^3}$, $Q = \frac{\sin^2 x}{1+x^3}$ $I.F = e^{\int P dx} = e^{\int \frac{3x^2}{1+x^3} dx}$ $= e^{\log(1+x^3)} = (1+x^3)$ The Solution is $y e^{\int P dx} = \int Q e^{\int P dx} dx + C$ $y(1+x^3) = \int \frac{\sin^2 x}{(1+x^3)} dx + C$ $y(1+x^3) = \int \sin^2 x dx + C$ $y(1+x^3) = \int \left(\frac{1 - \cos 2x}{2} \right) dx + C$ $y(1+x^3) = \int \left(\frac{1}{2} - \frac{\cos 2x}{2} \right) dx + C$	$y(1+x^3) = \frac{x}{2} - \frac{\sin 2x}{4} + C$ 13) Solve: $x \frac{dy}{dx} + y = x \log x$ Soln: Given: $x \frac{dy}{dx} + y = x \log x$ $\div x, \frac{dy}{dx} + \frac{1}{x} y = \log x$ This is a linear differential equation in y Here $P = \frac{1}{x}$, $Q = \log x$ $I.F = e^{\int P dx} = e^{\int \frac{1}{x} dx}$ $= e^{\log x} = x$ $\therefore \boxed{I.F = x}$ The Solution is $y e^{\int P dx} = \int Q e^{\int P dx} dx + C$	$yx = \int (\log x) x dx + C$ $= (\log x) \frac{x^2}{2} - \int \frac{x^2}{2} \frac{1}{x} dx + C$ $u = \log x \quad \frac{du}{dx} = \frac{1}{x}$ $du = \frac{1}{x} dx \quad v = \frac{x^2}{2}$ $\int u dv = uv - \int v du$ $= \frac{x^2}{2} \log x - \frac{1}{2} \int x dx + C$ $= \frac{x^2}{2} \log x - \frac{1}{2} \left(\frac{x^2}{2} \right) + C$ $= \frac{x^2}{2} \log x - \frac{x^2}{4} + C$ $yx = \frac{2x^2 \log x - x^2 + 4C}{4}$ $4yx = 2x^2 \log x - x^2 + 4C$
12) Solve: $\frac{dy}{dx} = \frac{\sin^2 x}{1+x^3} - \frac{3x^2}{1+x^3} y$ Soln: Given: $\frac{dy}{dx} = \frac{\sin^2 x}{1+x^3} - \frac{3x^2}{1+x^3} y$	$y(1+x^3) = \int \frac{\sin^2 x}{(1+x^3)} dx + C$ $y(1+x^3) = \int \sin^2 x dx + C$ $y(1+x^3) = \int \left(\frac{1 - \cos 2x}{2} \right) dx + C$ $y(1+x^3) = \int \left(\frac{1}{2} - \frac{\cos 2x}{2} \right) dx + C$	14) Solve: $x \frac{dy}{dx} + 2y - x^2 \log x = 0$ Soln: Given: $x \frac{dy}{dx} + 2y - x^2 \log x = 0$	14) Solve: $x \frac{dy}{dx} + 2y - x^2 \log x = 0$ Soln: Given: $x \frac{dy}{dx} + 2y - x^2 \log x = 0$

$$x \frac{dy}{dx} + 2y = x^2 \log x$$

$$\div x, \frac{dy}{dx} + \frac{2}{x}y = x \log x$$

This is a linear differential equation in y

$$\text{Here } P = \frac{2}{x}, Q = x \log x$$

$$I.F = e^{\int P dx} = e^{\int \frac{2}{x} dx}$$

$$= e^{2 \log x} = e^{\log x^2} = x^2$$

$$\therefore \boxed{I.F = x^2}$$

$\therefore$ The Solution is

$$y e^{\int P dx} = \int Q e^{\int P dx} dx + C$$

$$y x^2 = \int (x \log x) x^2 dx + C$$

$$x^2 y = \int (\log x) x^3 dx + C$$

$$\int u dv = uv - \int v du \quad \begin{matrix} u = \log x & \int dv = \int x^3 dx \\ du = \frac{1}{x} dx & v = \frac{x^4}{4} \end{matrix}$$

$$= (\log x) \frac{x^4}{4} - \int \frac{x^4}{4} \frac{1}{x} dx + C$$

$$= \frac{x^4}{4} \log x - \int \frac{x^3}{4} dx + C$$

$$= \frac{x^4}{4} \log x - \frac{x^4}{16} + C$$

$$x^2 y = \frac{x^4}{4} \log x - \frac{x^4}{16} + C$$

15) Solve:

$$\frac{dy}{dx} + \frac{3y}{x} = \frac{1}{x^2}, \text{ given that } y=2 \text{ when } x=1$$

Soln: Given:

$$\frac{dy}{dx} + \frac{3}{x}y = \frac{1}{x^2}$$

This is a linear differential equation in y

$$\text{Here } P = \frac{3}{x}, Q = \frac{1}{x^2}$$

$$I.F = e^{\int P dx} = e^{\int \frac{3}{x} dx}$$

$$= e^{3 \log x} = e^{\log x^3} = x^3$$

$$\therefore \boxed{I.F = x^3}$$

The Solution is

$$y e^{\int P dx} = \int Q e^{\int P dx} dx + C$$

$$y x^3 = \int \frac{1}{x^2} x^3 dx + C$$

$$x^3 y = \int x dx + C$$

$$x^3 y = \frac{x^2}{2} + C \quad \text{--- (1)}$$

$$\text{When } x=1, y=2$$

$$(1)^3 (2) = \frac{(1)^2}{2} + C$$

$$2 = \frac{1}{2} + C$$

$$C = 2 - \frac{1}{2}$$

$$\boxed{C = \frac{3}{2}}$$

$$\textcircled{1} \Rightarrow x^3 y = \frac{x^2}{2} + \frac{3}{2}$$

$$x^3 y = \frac{x^2 + 3}{2}$$

$$2x^3 y = x^2 + 3$$

EX 10.22

$$\text{Solve: } \frac{dy}{dx} + 2y = e^{-x}$$

Soln: Given:

$$\frac{dy}{dx} + 2y = e^{-x}$$

This is a linear differential equation in y

$$\text{Here } P = 2, Q = e^{-x}$$

$$I.F = e^{\int P dx} = e^{\int 2 dx} = e^{2x}$$

$$\therefore \boxed{I.F = e^{2x}}$$

The Solution is

$$y e^{\int P dx} = \int Q e^{\int P dx} dx + C$$

$$y e^{2x} = \int e^{-x} e^{2x} dx + C$$

$$ye^{2x} = \int e^{2x-x} dx + c$$

$$ye^{2x} = \int e^x dx + c$$

$$ye^{2x} = e^x + c$$

$$\div e^{2x}, y = \frac{e^x}{e^{2x}} + \frac{c}{e^{2x}}$$

$$y = e^x e^{-2x} + c e^{-2x}$$

$$y = e^{x-2x} + c e^{-2x}$$

$y = e^{-x} + c e^{-2x}$ is the required solution

Ex 10.23

Solve: $[y(1-x\tan x) + x^2 \cos x] dx - x dy = 0$

Soln: Given:

$$[y(1-x\tan x) + x^2 \cos x] dx - x dy = 0$$

$$x dy = [y(1-x\tan x) + x^2 \cos x] dx$$

$$\frac{dy}{dx} = \frac{y(1-x\tan x) + x^2 \cos x}{x}$$

$$\frac{dy}{dx} = -\frac{(x\tan x - 1)y}{x} + \frac{x^2 \cos x}{x}$$

$$\frac{dy}{dx} + \frac{(x\tan x - 1)y}{x} = x \cos x$$

This is a linear differential equation in y

Here $P = \frac{(x\tan x - 1)}{x}$,

$Q = x \cos x$

I.F. = $e^{\int P dx}$

$$= e^{\int \frac{(x\tan x - 1)}{x} dx}$$

$$= e^{\int \left(\frac{x\tan x}{x} - \frac{1}{x} \right) dx}$$

$$= e^{\int (\tan x - \frac{1}{x}) dx}$$

$$= e^{-\log|\cos x| - \log|x|}$$

$$= e^{-[\log|\cos x| + \log|x|]}$$

$$= e^{-\log|x \cos x|} = \frac{1}{x \cos x}$$

$$= \frac{1}{(x \cos x)}$$

$$= \frac{1}{(x \cos x)} = \frac{1}{x \cos x}$$

$$\therefore \text{I.F.} = \frac{1}{x \cos x}$$

The solution is

$$y \cdot \text{I.F.} = \int Q \cdot \text{I.F.} dx + c$$

$$y \cdot \frac{1}{x \cos x} = \int \frac{x^2 \cos x}{x \cos x} dx + c$$

$$\frac{y}{x \cos x} = \int dx + c$$

$$\frac{y}{x \cos x} = x + c$$

$$y = x^2 \cos x + c x \cos x \text{ is}$$

the required solution

Ex 10.24

Solve:

$$\frac{dy}{dx} + 2y \cot x = 3x^2 \operatorname{cosec}^2 x$$

Soln: Given:

$$\frac{dy}{dx} + 2y \cot x = 3x^2 \operatorname{cosec}^2 x$$

This is a linear differential equation in y

Here $P = 2 \cot x$,

$Q = 3x^2 \operatorname{cosec}^2 x$

I.F. = $e^{\int P dx}$

$$= e^{\int 2 \cot x dx}$$

$$= e^{2 \log \sin x}$$

$$= e^{\log(\sin x)^2}$$

$$= e^{\log \sin^2 x} = \sin^2 x$$

$$\therefore \text{I.F.} = \sin^2 x$$

The Solution is
 $y e^{\int P dx} = \int Q e^{\int P dx} dx + C$

$$y \sin^2 x = \int 3x^2 \csc^2 x \sin^2 x dx + C$$

$$y \sin^2 x = \int 3x^2 \frac{1}{\sin^2 x} \sin^2 x dx + C$$

$$= \int 3x^2 dx + C$$

$$= 3 \left[\frac{x^3}{3} \right] + C$$

$y \sin^2 x = x^3 + C$ is the required Solution

Ex 10.25

Solve: $(1+x^3) \frac{dy}{dx} + 6x^2 y = 1+x^2$

Soln: Given:

$$(1+x^3) \frac{dy}{dx} + 6x^2 y = 1+x^2$$

$$\div (1+x^3), \quad \frac{dy}{dx} + \frac{6x^2}{1+x^3} y = \frac{1+x^2}{1+x^3}$$

This is a linear differential equation in y

Here $P = \frac{6x^2}{1+x^3}$, $Q = \frac{1+x^2}{1+x^3}$

$$I.F = e^{\int P dx}$$

$$= e^{\int \frac{6x^2}{1+x^3} dx}$$

$$= e^{2 \int \frac{3x^2}{1+x^3} dx}$$

$$= e^{2 \log(1+x^3)}$$

$$= e^{\log(1+x^3)^2} = (1+x^3)^2$$

$$\therefore I.F = (1+x^3)^2$$

The Solution is

$$y e^{\int P dx} = \int Q e^{\int P dx} dx + C$$

$$y(1+x^3)^2 = \int \frac{(1+x^2)}{(1+x^3)^2} (1+x^3)^2 dx + C$$

$$y(1+x^3)^2 = \int (1+x^2)(1+x^3) dx + C$$

$$= \int (1+x^3+x^2+x^5) dx + C$$

$$= x + \frac{x^4}{4} + \frac{x^3}{3} + \frac{x^6}{6} + C$$

$$y(1+x^3)^2 = x + \frac{x^3}{3} + \frac{x^4}{4} + \frac{x^6}{6} + C$$

$$y = \frac{1}{(1+x^3)^2} \left[x + \frac{x^3}{3} + \frac{x^4}{4} + \frac{x^6}{6} + C \right]$$

is the required Solution

Ex 10.26

Solve: $y e^x dx = (y^3 + 2x e^y) dy$

Soln: Given:

$$y e^y dx = (y^3 + 2x e^y) dy$$

$$\frac{dx}{dy} = \frac{y^3 + 2x e^y}{y e^y} = \left[\frac{y^3}{y} + \frac{2x e^y}{y e^y} \right] e^{-y}$$

$$\frac{dx}{dy} = \left[\frac{y^2}{y} + \frac{2x e^y}{y} \right] e^{-y}$$

$$\frac{dx}{dy} = \left[\frac{2e^y}{y} x + y^2 \right] e^{-y}$$

$$\frac{dx}{dy} = \frac{2e^y e^{-y}}{y} x + y^2 e^{-y}$$

$$\frac{dx}{dy} = \frac{2e^{y(1)}}{y} x + y^2 e^{-y}$$

$$\frac{dx}{dy} = \frac{2}{y} x + y^2 e^{-y}$$

$$\frac{dx}{dy} - \frac{2}{y} x = y^2 e^{-y}$$

This is a linear differential equation in x

Here $P = -\frac{2}{y}$, $Q = y^2 e^{-y}$

$$I.F = e^{\int P dy} = e^{\int -\frac{2}{y} dy}$$

$$= e^{-2 \int \frac{1}{y} dy} = e^{-2 \log y} = e^{\log(y)^{-2}} = \frac{1}{y^2}$$

The Solution is

$$x e^{\int P dy} = \int Q e^{\int P dy} dy + c$$

$$x \left(\frac{1}{y^2} \right) = \int y e^{-y} \left(\frac{1}{y^2} \right) dy + c$$

$$\frac{x}{y^2} = \int e^{-y} dy + c$$

$$\frac{x}{y^2} = \frac{e^{-y}}{-1} + c$$

$$\frac{x}{y^2} = -e^{-y} + c$$

$x = -y^2 e^{-y} + c y^2$ is the required Solution

10.8 Applications of First Order Ordinary Differential Equations

EXERCISE 10.8

1) The rate of increase

in the number of bacteria in a certain bacteria culture is proportional to the number present.

Given that the number triples in 5 hours, find how many bacteria will be present after 10 hours?

Soln:

Let A be the number of bacteria at any time t

$$\frac{dA}{dt} \propto A$$

$$\frac{dA}{dt} = kA$$

$$\int \frac{dA}{A} = \int k dt$$

$$\log A = kt + c$$

$$A = e^{kt+c}$$

$$A = e^{kt} \cdot e^c$$

$$A = c e^{kt} \quad \text{--- ①}$$

when $t=0$, $A=A_0$

$$\text{①} \Rightarrow A_0 = c e^0$$

$$A_0 = c(1) \Rightarrow \boxed{c = A_0}$$

$$\text{①} \Rightarrow A = A_0 e^{kt} \quad \text{--- ②}$$

when $t=5$, $A=3A_0$

$$\text{②} \Rightarrow 3A_0 = A_0 e^{5k}$$

$$\boxed{e^{5k} = 3}$$

When $t=10$, $A=?$

$$\text{②} \Rightarrow A = A_0 e^{10k}$$

$$= A_0 (e^{5k})^2$$

$$= A_0 (3)^2 = 9A_0$$

$$\therefore A = 9A_0$$

$\therefore$ After 10 hours the number of bacteria is 9 times the original number of bacteria

2) Find the population of a city at any time t , given that the rate of increase of population is proportional to the population at that instant and that in a period of 40 years the population increased from 3,00,000 to 4,00,000

Soln:

Let P be the population at time t

$$\frac{dP}{dt} \propto P$$

$$\frac{dP}{dt} = kP$$

$$\int \frac{dP}{P} = \int k dt$$

$$\log P = kt + c$$

$$P = e^{kt+c}$$

$$P = C e^{kt} \quad \text{--- ①}$$

When $t=0$, $P=3,00,000$

$$\text{①} \Rightarrow 3,00,000 = C e^0$$

$$3,00,000 = C(1)$$

$$\boxed{C = 3,00,000}$$

$$\text{①} \Rightarrow P = 3,00,000 e^{kt} \quad \text{--- ②}$$

When $t=40$, $P=4,00,000$

$$\text{②} \Rightarrow 4,00,000 = 3,00,000 e^{40k}$$

$$4 = 3 e^{40k}$$

$$e^{40k} = \frac{4}{3}$$

$$(e^k)^{40} = \frac{4}{3}$$

$$[(e^k)^{40}]^{\frac{1}{40}} = \left(\frac{4}{3}\right)^{\frac{1}{40}}$$

$$\boxed{e^k = \left(\frac{4}{3}\right)^{\frac{1}{40}}}$$

② $\Rightarrow$

$$P = 3,00,000 (e^k)^t$$

$$= 3,00,000 \left(\left(\frac{4}{3}\right)^{\frac{1}{40}}\right)^t$$

$$P = 3,00,000 \left(\frac{4}{3}\right)^{\frac{t}{40}}$$

3) The equation of electromotive force for an electric circuit containing resistance and self-inductance is $E = Ri + L \frac{di}{dt}$. Where E is the electromotive force is given to the

circuit, R the resistance and L , the coefficient of induction. Find the current i at time t when $E=0$

Soln: Given:

$$L \frac{di}{dt} + Ri = E$$

$$\div L, \frac{di}{dt} + \frac{R}{L}i = \frac{E}{L}$$

This is a linear differential Equation in i

$$\text{Here } p = \frac{R}{L}, q = \frac{E}{L}$$

$$I.F = e^{\int p dt} = e^{\int \frac{R}{L} dt}$$

$$= e^{\frac{R}{L} t} = e^{\frac{Rt}{L}}$$

$$\therefore \boxed{I.F = e^{\frac{Rt}{L}}}$$

The Solution is

$$i e^{\frac{Rt}{L}} = \int q e^{\frac{Rt}{L}} dt + c$$

$$i e^{\frac{Rt}{L}} = \int \frac{E}{L} e^{\frac{Rt}{L}} dt + c$$

$$i e^{\frac{Rt}{L}} = \frac{E}{L} \int e^{\frac{Rt}{L}} dt + c$$

$$i e^{\frac{Rt}{L}} = \frac{E}{L} \left[\frac{e^{\frac{Rt}{L}}}{\frac{R}{L}} \right] + c$$

$$= \frac{E}{L} \left[\frac{L e^{\frac{Rt}{L}}}{R} \right] + c$$

$$i e^{\frac{Rt}{L}} = \frac{E}{R} e^{\frac{Rt}{L}} + c \quad \text{--- ①}$$

When $E=0$,

$$\text{①} \Rightarrow i e^{\frac{Rt}{L}} = 0 + c$$

$$\Rightarrow c = i e^{\frac{Rt}{L}}$$

$$\Rightarrow i = c e^{-\frac{Rt}{L}}$$

4) The engine of a motor boat moving at 10m/s

is shut off. Given that the retardation at any subsequent time (after shutting off the engine) equal to the velocity at that time. Find the velocity after 2 seconds of switching off the engine.

Soln: Let v be the velocity at time t

$$\text{Given: } -\frac{dv}{dt} = v$$

$$-\frac{dv}{v} = dt$$

$$\int \frac{dv}{v} = \int dt$$

$$\log v = -t + \log c$$

$$\log v - \log c = -t$$

$$\log\left(\frac{v}{c}\right) = -t$$

$$\frac{v}{c} = e^{-t}$$

$$v = ce^{-t} \quad \text{--- (1)}$$

$$\text{When } t=0, v=10$$

$$\text{(1)} \Rightarrow 10 = ce^0$$

$$10 = c(1) \Rightarrow \boxed{c=10}$$

$$\text{(1)} \Rightarrow v = 10e^{-t} \quad \text{--- (2)}$$

$$\text{When } t=2, v=?$$

$$\text{(2)} \Rightarrow v = 10e^{-2}$$

$$v = \frac{10}{e^2}$$

5) Suppose a person deposits 10,000 Indian rupees in a bank account at the rate of 5% per annum compounded continuously. How much money will be in his bank account 18 months later?

Soln:

Let p be the Amount at time t

$$\frac{dp}{dt} \propto p$$

$$\frac{dp}{dt} = kp, \text{ where } k=5\%$$

$$k = \frac{5}{100}$$

$$k=0.05$$

$$\frac{dp}{dt} = 0.05p$$

$$\int \frac{dp}{p} = \int 0.05 dt$$

$$\log p = 0.05t + c$$

$$p = e^{0.05t+c}$$

$$p = ce^{0.05t} \quad \text{--- (1)}$$

$$\text{When } t=0, p=10,000$$

$$\text{(1)} \Rightarrow 10,000 = ce^0 = c(1) = c$$

$$\therefore \boxed{c=10,000}$$

$$\text{(1)} \Rightarrow p = 10,000 e^{0.05t} \quad \text{--- (2)}$$

$$\text{When } t = 1\frac{1}{2} = \frac{3}{2} = 1.5$$

$$\text{(2)} \Rightarrow p = 10,000 e^{0.05 \times 1.5}$$

$$p = 10,000 e^{0.075}$$

6) Assume that the rate at which radioactive nuclei decay is proportional to the number of such nuclei that are present in a given sample. In a certain sample 10% of the original number of radioactive nuclei have undergone disintegration in a period of 100 years. What percentage of the original radioactive nuclei will remain after 1000 years?

Soln:

Let A be the Amount of radioactive nuclei at time t

$$\frac{dA}{dt} \propto A$$

$$\frac{dA}{dt} = kA$$

$$\int \frac{dA}{A} = k \int dt$$

$$\log A = kt + c$$

$$A = e^{kt+c}$$

$$A = Ce^{kt} \quad \text{--- (1)}$$

When $t=0$, $A=A_0$

$$\textcircled{1} \Rightarrow A_0 = Ce^0 = C(1) = C$$

$$\therefore \boxed{C = A_0}$$

$$\textcircled{1} \Rightarrow A = A_0 e^{kt} \quad \text{--- (2)}$$

When $t=100$, $A = \frac{9}{10} A_0$

$$\textcircled{2} \Rightarrow \frac{9}{10} A_0 = A_0 e^{100k}$$

$$\boxed{e^{100k} = \frac{9}{10}}$$

When $t=1000$, $A=?$

$$\textcircled{2} \Rightarrow A = A_0 e^{1000k}$$

$$= A_0 (e^{100k})^{10}$$

$$= A_0 \left(\frac{9}{10}\right)^{10}$$

$$= A_0 \left(\frac{9^{10}}{10^{10}} \times 100\%\right)$$

$$= A_0 \left(\frac{9^{10}}{10^{10}} \times 10^2\%\right)$$

$$A = A_0 \left(\frac{9^{10}}{10^8}\right)\%$$

$\therefore$ After 1000 years will remain radioactive nuclei is $\frac{9^{10}}{10^8}\%$.

7) Water at temperature 100°C cools in 10 minutes to 80°C in a room temperature

of 25°C Find

(i) The temperature of water after 20 minutes
(ii) The time when the temperature is 40°C

$$\left[\log_e \frac{11}{15} = -0.3101;\right.$$

$$\left.\log_e 5 = 1.6094\right]$$

Soln:

Let T be the temperature of water at time t and T_m be the room temperature

$$\frac{dT}{dt} \propto T - T_m$$

$$\frac{dT}{dt} = k(T - T_m)$$

$$\int \frac{dT}{T - T_m} = k \int dt$$

$$\log(T - T_m) = kt + c$$

$$T - T_m = e^{kt+c}$$

$$T - T_m = Ce^{kt}$$

Given: $T_m = 25$

$$T - 25 = Ce^{kt} \quad \text{--- (1)}$$

When $t=0$, $T=100$

$$\textcircled{1} \Rightarrow 100 - 25 = Ce^0 = C(1) = C$$

$$\therefore \boxed{C = 75}$$

$$\textcircled{1} \Rightarrow T - 25 = 75e^{kt} \quad \text{--- (2)}$$

When $t=10$, $T=80$

$$\textcircled{2} \Rightarrow 80 - 25 = 75e^{10k}$$

$$\frac{11}{5} = \frac{15}{1} e^{10k}$$

$$\therefore \boxed{e^{10k} = \frac{11}{15}} \quad \text{--- (3)}$$

(i) when $t=20$, $T=?$

$$\textcircled{2} \Rightarrow T - 25 = 75e^{20k}$$

$$T - 25 = 75(e^{10k})^2$$

$$= 75 \left(\frac{11}{15}\right)^2 = \frac{75}{15} \left(\frac{11}{15} \times \frac{11}{15}\right)$$

$$T - 25 = \frac{121}{3} = 40.33$$

$$T - 25 = 40.33$$

$$T = 25 + 40.33$$

$$T = 65.33^\circ\text{C}$$

$$(ii) \textcircled{3} \Rightarrow e^{10k} = \frac{11}{15}$$

$$\log_e e^{10k} = \log_e \frac{11}{15}$$

$$10k = -0.3101$$

$$k = -\frac{0.3101}{10}$$

$$k = -0.03101$$

when $T = 40$, $t = ?$

$$\textcircled{2} \Rightarrow T - 25 = 75 e^{kt}$$

$$40 - 25 = 75 e^{-0.03101t}$$

$$15 = 75 e^{-0.03101t}$$

$$e^{0.03101t} = 5$$

$$\log_e e^{0.03101t} = \log_e 5$$

$$0.03101t = 1.6094$$

$$t = \frac{1.6094}{0.03101} \times \frac{100000}{100000}$$

$$= \frac{160940}{3101}$$

$$t = 51.89 \text{ minutes}$$

8) At 10.00 A.M. a woman took a cup of hot instant coffee from her microwave oven and placed it on a nearby kitchen counter to cool. At this instant the temperature of the coffee was 180°F , and 10 minutes later it was 160°F . Assume that constant temperature of the kitchen was 70°F

(i) what was the temperature

of the coffee at 10.15 A.M.?

(ii) The woman likes to drink coffee when its temperature is between 130°F and 140°F between what times should she have drunk the coffee?

Soln:

Let T be the temperature of coffee at time t

$$\frac{dT}{dt} \propto T - T_m$$

$$\frac{dT}{dt} = k(T - T_m)$$

$$\int \frac{dT}{T - T_m} = k \int dt$$

$$\log(T - T_m) = kt + c$$

$$T - T_m = e^{kt+c}$$

$$T - T_m = C e^{kt}$$

$$\text{Given: } T_m = 70$$

$$T - 70 = C e^{kt} \quad \textcircled{1}$$

$$\text{When } t = 0, T = 180^\circ\text{F}$$

$$\textcircled{1} \Rightarrow 180 - 70 = C e^0$$

$$110 = C(1) = C$$

$$\therefore C = 110$$

$$\textcircled{1} \Rightarrow T - 70 = 110 e^{kt} \quad \textcircled{2}$$

$$\text{When } t = 10, T = 160^\circ\text{F}$$

$$\textcircled{2} \Rightarrow 160 - 70 = 110 e^{10k}$$

$$90 = 110 e^{10k}$$

$$e^{10k} = \frac{9}{11} \quad \textcircled{3}$$

$$(i) \text{ when } t = 15, T = ?$$

$$\textcircled{2} \Rightarrow T - 70 = 110 e^{15k}$$

$$\begin{aligned}
 T - T_0 &= 110 e^{\frac{30}{2}k} \\
 &= 110 (e^{10k})^{3/2} \\
 &= 110 \left(\frac{9}{11}\right)^{3/2} \\
 &= 110 \left(\frac{9}{11}\right) \left(\frac{9}{11}\right)^{1/2} \\
 &= 90 \sqrt{\frac{9}{11}}
 \end{aligned}$$

$$= 90 \times 0.9$$

$$T - T_0 = 81$$

$$T = T_0 + 81$$

$$\boxed{T = 151^\circ \text{F (app)}}$$

$$(ii) \textcircled{3} \Rightarrow e^{10k} = \frac{9}{11}$$

$$\log_e e^{10k} = \log_e \frac{9}{11}$$

$$10k = \log_e (0.8181)$$

$$10k = -0.0871$$

$$k = \frac{-0.0871}{10}$$

$$\boxed{k = -0.00871}$$

$$\text{When } T = 130^\circ \text{F}, t = ?$$

$$\textcircled{2} \Rightarrow 130 - 70 = 110 e^{kt}$$

$$60 = 110 e^{0.00871t}$$

$$\frac{6}{11} = e^{0.00871t} \Rightarrow \log \frac{6}{11} = \log e^{-0.00871t}$$

$$\log 0.5454 = -0.00871t$$

$$+ 0.26328 = -0.00871t$$

$$t = \frac{0.26328}{0.00871} \Rightarrow \boxed{t = 30.227 \text{ min}}$$

$$\text{When } T = 140^\circ \text{F}, t = ?$$

$$\textcircled{2} \Rightarrow 140 - 70 = 110 e^{kt}$$

$$70 = 110 e^{0.00871t}$$

$$\frac{7}{11} = e^{0.00871t}$$

$$\log \frac{7}{11} = \log e^{0.00871t}$$

$$\log 0.6363 = -0.00871t$$

$$+ 0.1963 = -0.00871t$$

$$t = \frac{0.1963}{0.00871}$$

$$\boxed{t = 22.523 \text{ min}}$$

$\therefore$ She drank the coffee between 10.22 and 10.30 approximately

9) A pot of boiling water at 100°C is removed from a stove at time $t=0$ and left to cool in the kitchen. After 5 minutes, the water temperature has decreased to 80°C , and another 5 minutes later it has dropped to 65°C . Determine the temperature of the kitchen.

Soln:

Let T be the temperature of boiling water at time t

Let T_m be the temperature of kitchen room

$$\frac{dT}{dt} \propto T - T_m$$

$$\frac{dT}{dt} = k(T - T_m)$$

$$\int \frac{dT}{T - T_m} = \int k dt$$

$$\log(T - T_m) = kt + c$$

$$T - T_m = e^{kt+c}$$

$$T - T_m = C e^{kt} \textcircled{1}$$

$$\text{When } t=0, T=100^\circ \text{C}$$

$$\textcircled{1} \Rightarrow 100 - T_m = C e^0 = C \textcircled{2}$$

$$\boxed{C = 100 - T_m}$$

$$\textcircled{1} \Rightarrow T - T_m = (100 - T_m) e^{kt} \textcircled{3}$$

$$\text{When } t=5, T=80^\circ \text{C}$$

$$\textcircled{2} \Rightarrow 80 - T_m = (100 - T_m) e^{5k}$$

$$\boxed{e^{5k} = \frac{80 - T_m}{100 - T_m}}$$

$$\text{When } t=10, T=65^\circ \text{C}$$

$$\textcircled{2} \Rightarrow 65 - T_m = (100 - T_m) e^{10k}$$

$$65 - T_m = (100 - T_m)(e^{5K})^2$$

$$= (100 - T_m)\left(\frac{80 - T_m}{100 - T_m}\right)^2$$

$$= (100 - T_m) \frac{(80 - T_m)^2}{(100 - T_m)^2}$$

$$65 - T_m = \frac{6400 - 160T_m + T_m^2}{100 - T_m}$$

$$(65 - T_m)(100 - T_m) = 6400 - 160T_m + T_m^2$$

$$6500 - 165T_m + T_m^2 = 6400 - 160T_m + T_m^2$$

$$6500 - 6400 = 165T_m - 160T_m$$

$$100 = 5T_m$$

$$\therefore T_m = 20^\circ\text{C}$$

$\therefore$ The temperature of the kitchen room is 20°C

10) A tank initially contains

50 litres of pure water. Starting at time $t=0$ a brine containing with 2 grams of dissolved salt per litre flows into the tank at the rate of 3 litres per minute. The mixture is kept uniform by stirring and the well stirred mixture simultaneously flows out of the tank at the same rate. Find the amount of salt present in the tank at any time $t > 0$

Soln:

Let $x(t)$ denote the amount of salt in the tank at time t

$$\frac{dx}{dt} = \text{inflow rate} - \text{outflow rate}$$

$$\text{inflow rate} = 2 \times 3 = 6$$

$$\text{outflow} = \frac{3}{50}x$$

$$\frac{dx}{dt} = 6 - \frac{3}{50}x$$

$$= -\frac{3}{50}\left(x - \frac{2 \times 50}{3}\right)$$

$$\frac{dx}{dt} = -\frac{3}{50}(x - 100)$$

$$\int \frac{dx}{x - 100} = \int -\frac{3}{50} dt$$

$$\log(x - 100) = -\frac{3t}{50} + C$$

$$x - 100 = e^{-\frac{3t}{50} + C}$$

$$x - 100 = C e^{-\frac{3t}{50}} \quad \text{--- (1)}$$

$$\text{When } t=0, x=0$$

$$0 - 100 = C e^0 = C(1) = C$$

$$\therefore \boxed{C = -100}$$

$$\text{(1)} \Rightarrow x - 100 = -100 e^{-\frac{3t}{50}}$$

$$x = 100 - 100 e^{-\frac{3t}{50}}$$

$$x = 100\left(1 - e^{-\frac{3t}{50}}\right) \text{ which is the amount of salt in the tank}$$

Ex 10-27

The growth of a population is proportional to the number present. If the population of a colony doubles in 50 years, in how many years will the population become triple?

Soln:

Let $x(t)$ be the population at time t

$$\frac{dx}{dt} \propto x$$

$$\frac{dx}{dt} = kx$$

$$\frac{dx}{x} = k dt$$

$$\log x = kt + c$$

$$x = e^{kt+c}$$

$$x = ce^{kt} \quad \text{--- ①}$$

When $x = x_0$, $t = 0$

$$\text{①} \Rightarrow x_0 = ce^0 = c(1) = c$$

$$\therefore \boxed{c = x_0}$$

$$\text{①} \Rightarrow x = x_0 e^{kt} \quad \text{--- ②}$$

When $x = 2x_0$, $t = 50$

$$\text{②} \Rightarrow 2x_0 = x_0 e^{50k}$$

$$e^{50k} = 2$$

$$\log e^{50k} = \log 2$$

$$50k = \log 2$$

$$\boxed{k = \frac{1}{50} \log 2}$$

When $x = 3x_0$, $t = t_1$

$$\text{②} \Rightarrow 3x_0 = x_0 e^{kt_1}$$

$$3 = e^{kt_1}$$

$$\log 3 = \log e^{kt_1}$$

$$\log 3 = kt_1$$

$$\log 3 = \frac{1}{50} \log 2 t_1$$

$$50 \log 3 = \log 2 t_1$$

$$\Rightarrow t_1 = 50 \left(\frac{\log 3}{\log 2} \right)$$

$\therefore$ The population is tripled in $50 \left(\frac{\log 3}{\log 2} \right)$ years

Ex 10.28

A radioactive isotope has an initial mass 200mg, which two years later is 50 mg. Find the expression for the amount of the isotope remaining at any time. What is its half-life? (Half-life means

the time taken for the radioactivity of a specified isotope to fall to half its original value)

Soln:

Let A be the mass of the isotope remaining after t years.

$$\frac{dA}{dt} \propto A$$

$$\frac{dA}{dt} = -KA, \quad K > 0$$

$$\int \frac{dA}{A} = -K \int dt$$

$$\log A = -kt + c$$

$$A = e^{-kt+c}$$

$$A = ce^{-kt} \quad \text{--- ①}$$

When $t = 0$, $A = 200$

$$\text{①} \Rightarrow 200 = ce^0 = c(1) = c$$

$$\boxed{c = 200}$$

$$\text{①} \Rightarrow A = 200 e^{-kt} \quad \text{--- ②}$$

When $t = 2$, $A = 150$

$$\text{②} \Rightarrow 150 = 200 e^{-2k}$$

$$3 = 4 e^{-2k}$$

$$3 e^{2k} = 4$$

$$e^{2k} = \frac{4}{3}$$

$$\log e^{2k} = \log \left(\frac{4}{3} \right)$$

$$2k = \log \left(\frac{4}{3} \right)$$

$$\boxed{k = \frac{1}{2} \log \left(\frac{4}{3} \right)}$$

$$\text{②} \Rightarrow -\frac{1}{2} \log \left(\frac{4}{3} \right) t$$

$$A = 200 e^{-\frac{t}{2} \log \left(\frac{4}{3} \right)}$$

$A(t) = 200 e^{-\frac{t}{2} \log \left(\frac{4}{3} \right)}$ is the mass of isotope remaining after t years when $A = 100$, $t = t_h$

$$1 = 2 e^{-\frac{t_h}{2} \log(\frac{4}{3})}$$

$$1 = 2 e^{-\frac{t_h}{2} \log(\frac{4}{3})}$$

$$\log 1 = \log(2 e^{-\frac{t_h}{2} \log(\frac{4}{3})})$$

$$\log 1 = \log 2 + \log e^{-\frac{t_h}{2} \log(\frac{4}{3})}$$

$$\log 1 - \log 2 = \log e^{-\frac{t_h}{2} \log(\frac{4}{3})}$$

$$\log(\frac{1}{2}) = -\frac{t_h}{2} \log(\frac{4}{3})$$

$$2 \log(\frac{1}{2}) = -t_h \log(\frac{4}{3})$$

$$2 \log(\frac{1}{2}) = t_h \log(\frac{4}{3})$$

$$2 \log(\frac{1}{2}) = t_h \log(\frac{4}{3})$$

$$2 \log(\frac{1}{2}) = t_h \log(\frac{3}{4})$$

$$t_h = \frac{2 \log(\frac{1}{2})}{\log(\frac{3}{4})}$$

a corpse was found by a detective at exactly 8 p.m. Being alert, the detective also measured the body temperature and found it to be 70°F. Two hours later, the detective measured the body temperature again and found it to be 60°F. If the room temperature is 50°F, and assuming that the body temperature of the person before death was 98.6°F, at what time did the murder occur?

$$[\log(2.43) = 0.88789; \log(0.5) = -0.69315]$$

Soln:

Let T be the temperature of the body at any time t

By Newton's law of cooling,

$$\frac{dT}{dt} \propto T - T_m$$

$$\frac{dT}{dt} = k(T - T_m)$$

When $T_m = 50^\circ\text{F}$

$$\frac{dT}{dt} = k(T - 50)$$

$$\int \frac{dT}{T - 50} = k \int dt$$

$$\log(T - 50) = kt + c$$

$$T - 50 = e^{kt+c}$$

$$T - 50 = C e^{kt} \quad \text{--- ①}$$

When $t=0$, $T=70^\circ\text{F}$

① $\Rightarrow$

$$70 - 50 = C e^0 = C(1) = C$$

$$[C = 20]$$

$$\text{①} \Rightarrow T - 50 = 20 e^{kt} \quad \text{--- ②}$$

When $t=2$, $T=60^\circ\text{F}$

② $\Rightarrow$

$$60 - 50 = 20 e^{2k}$$

$$10 = 20 e^{2k}$$

$$1 = 2 e^{2k}$$

$$\frac{1}{2} = e^{2k}$$

$$\log(\frac{1}{2}) = \log e^{2k}$$

$$\therefore 2k = \log(\frac{1}{2})$$

$$[k = \frac{1}{2} \log(\frac{1}{2})]$$

EX 10.29

In a murder investigation,

$$\frac{t}{2} = \frac{\log\left(\frac{48.6}{20}\right)}{\log\left(\frac{1}{2}\right)}$$

It appears that the per
-son was murdered at
about 5.30 P.m

A tank contains 1000 litres of water in which 100 grams of salt is dissolved. Brine (Brine is a high-concentration solution of salt (usually sodium chloride in water)) runs in a rate

Soln:

Let $x(t)$ denote the amount of salt in the tank at time t .

$$\frac{dx}{dt} = -0.01(x - 5000)$$

when $t=0$, $x=100$

$$C = -4900$$

$$x = 5000 - 4900 \text{ €}$$

$\therefore$ The amount of the salt in the tank at time t is $x = 5000 - 1000e^{-0.01t}$

EXERCISE 10.9

Choose the correct or the most suitable answer from the given four alternatives:

1) The order and degree of the differential equation $n \frac{d^2y}{dx^2} + \left(\frac{dy}{dx}\right)^{1/3} + x^{1/4} = 0$ are respectively

Soln: Given:

$$\frac{d^2y}{dx^2} + \left(\frac{dy}{dx}\right)^{1/3} + x^{1/4} = 0$$

$$\frac{d^2y}{dx^2} + x^{1/4} = -\left(\frac{dy}{dx}\right)^{1/3}$$

$$\left(\frac{d^2y}{dx^2} + x^{1/4}\right)^3 = \left(-\left(\frac{dy}{dx}\right)^{1/3}\right)^3$$

$$\left(\frac{d^2y}{dx^2} + x^{1/4}\right)^3 = -\left(\frac{dy}{dx}\right)$$

$$\left(\frac{d^2y}{dx^2}\right)^3 + 3\left(\frac{d^2y}{dx^2}\right)^2 x^{1/4} + 3\frac{d^2y}{dx^2} x^{1/4}^2 + \left(x^{1/4}\right)^3 + \frac{dy}{dx} = 0$$

$\therefore$ (order, degree) = (2, 3)

Ans: (1) (2, 3)

2) The differential equation representing the family of curves $y = A \cos(x+B)$ where A and B are parameters, is

Soln: Given:

$$y = A \cos(x+B), \{A, B\}$$

$$\frac{dy}{dx} = -A \sin(x+B)$$

$$\frac{d^2y}{dx^2} = -A \cos(x+B)$$

$$\frac{d^2y}{dx^2} = -y \Rightarrow \frac{d^2y}{dx^2} + y = 0$$

Ans: (2) $\frac{d^2y}{dx^2} + y = 0$

3) The order and degree of the differential equation $\sqrt{\sin x} (dx+dy) = \sqrt{\cos x} (dx-dy)$ is

Soln: Given:

$$\sqrt{\sin x} (dx+dy) = \sqrt{\cos x} (dx-dy)$$

$$\sqrt{\sin x} dx + \sqrt{\sin x} dy = \sqrt{\cos x} dx - \sqrt{\cos x} dy$$

$$\sqrt{\sin x} dy + \sqrt{\cos x} dy = \sqrt{\cos x} dx - \sqrt{\sin x} dx$$

$$dy (\sqrt{\sin x} + \sqrt{\cos x}) = dx (\sqrt{\cos x} - \sqrt{\sin x})$$

$$\frac{dy}{dx} = \frac{\sqrt{\cos x} - \sqrt{\sin x}}{\sqrt{\sin x} + \sqrt{\cos x}}$$

$\therefore$ (order, degree) = (1, 1)

4) The order of the differential equation of all circles

with centre at (h, k) and radius 'a' is

Soln: Given: Centre is (h, k) and radius is a

$(x-h)^2 + (y-k)^2 = a^2$ is the eqn of circle. ①

$$2(x-h) + 2(y-k)y' = 0$$

$$\div 2, (x-h) + (y-k)y' = 0 \quad \text{②}$$

$$1 + (y-k)y'' + y'y' = 0$$

$$(y-k)y'' + (1+y'^2) = 0$$

$$(y-k)y'' = -(1+y'^2) \quad \text{③}$$

$$\frac{(y-k)}{y''} = -\frac{(1+y'^2)}{y'}$$

$$\text{②} \Rightarrow (x-h) + (y-k)y' = 0$$

$$(x-h) = -(y-k)y'$$

$$(x-h) = -\left[-\frac{(1+y'^2)}{y''}\right]y'$$

$$(x-h) = \frac{(1+y'^2)y'}{y''}$$

$$\textcircled{1} \Rightarrow \left[\frac{(1+y'^2)y'}{y''} \right]^2 + \left[\frac{-(1+y'^2)}{y''^2} \right]^2 = a^2$$

$$\frac{(1+y'^2)^2(y')^2}{(y'')^2} + \frac{(1+y'^2)^2}{(y'')^2} = a^2$$

So Again differentiate,

So order is 3

Ans: (2) 3

5) The differential equation of the family of curves $y = Ae^x + Be^{-x}$ where A and B are arbitrary constants is

Soln: Given:

$$y = Ae^x + Be^{-x} \quad \{A, B\}$$

$$y' = Ae^x - Be^{-x}$$

$$y'' = Ae^x + Be^{-x}$$

$$y'' = y$$

$$\frac{d^2y}{dx^2} = y$$

$$\frac{d^2y}{dx^2} - y = 0$$

$$\text{Ans: (2) } \frac{d^2y}{dx^2} - y = 0$$

6) The general solution of the differential equation

$$\frac{dy}{dx} = \frac{y}{x} \text{ is}$$

$$\text{Soln: Given: } \frac{dy}{dx} = \frac{y}{x}$$

$$\int \frac{dy}{y} = \int \frac{dx}{x}$$

$$\log y = \log x + \log k$$

$$\log y = \log kx$$

$$y = kx$$

$$\text{Ans: (3) } y = kx$$

7) The Solution of the differential equation $2x \frac{dy}{dx} - y = 3$ represents

Soln: Given:

$$2x \frac{dy}{dx} - y = 3$$

$$2x \frac{dy}{dx} = y + 3$$

$$2 \int \frac{dy}{y+3} = \int \frac{dx}{x}$$

$$2 \log(y+3) = \log x + \log c$$

$$\log(y+3)^2 = \log(cx)$$

$$(y+3)^2 = cx \text{ This is}$$

a parabola

$$\text{Ans: (3) Parabola}$$

8) The Solution of $\frac{dy}{dx} + P(x)y = 0$ is

$$\text{Soln: Given: } \frac{dy}{dx} + P(x)y = 0$$

This is a linear differential equation in y

Here $P = P(x)$, $Q = 0$

$$I.F = e^{\int P(x) dx} = e^{\int P(x) dx}$$

$$I.F = e^{\int P(x) dx}$$

The Solution is

$$y e^{\int P(x) dx} = \int a e^{\int P(x) dx} dx + c$$

$$y e^{\int P(x) dx} = \int (0) e^{\int P(x) dx} dx + c$$

$$y e^{\int P(x) dx} = 0 + c$$

$$y e^{\int P(x) dx} = c$$

$$y = c e^{-\int P(x) dx}$$

$$\text{Ans: (2) } y = c e^{-\int P(x) dx}$$

9) The integrating factor of the differential equation $\frac{dy}{dx} + y = \frac{1+y}{x}$ is

$$\text{Soln: } \frac{dy}{dx} + y = \frac{1+y}{x}$$

Given: $\frac{dy}{dx} + y = \frac{1+y}{x}$ $\frac{dy}{dx} + y = \frac{1}{x} + \frac{y}{x}$ $\frac{dy}{dx} + y - \frac{y}{x} = \frac{1}{x}$ $\frac{dy}{dx} + (1 - \frac{1}{x})y = \frac{1}{x}$ This is a linear differential equation in y Here $p = 1 - \frac{1}{x}$, $Q = \frac{1}{x}$ $I.F = e^{\int p dx} = e^{\int (1 - \frac{1}{x}) dx}$ $= e^{x - \log x} = e^x \cdot e^{-\log x}$ $= \frac{e^x}{e^{\log x}}$ $= \frac{e^x}{x}$ Ans: (2) $\frac{e^x}{x}$	10) The integrating factor of the differential equation $\frac{dy}{dx} + p(x)y = Q(x)$ is x, then $P(x)$ Soln: Given: $\frac{dy}{dx} + p(x)y = Q(x) \text{ and}$ $I.F = x$ Here $p = p(x)$, $Q = Q(x)$ $I.F = e^{\int p dx}$ $x = e^{\int p dx}$ Taking log on both sides We get, $\log e^{\int p dx} = \log x$ $\int p dx = \log x$ $d(\int p dx) = d(\log x)$	$P = \frac{1}{x} \Rightarrow P(x) = \frac{1}{x}$ Ans: (3) $\frac{1}{x}$ 11) The degree of the differential equation $y(x) = 1 + \frac{dy}{dx} + \frac{1}{1.2} \left(\frac{dy}{dx}\right)^2 + \frac{1}{1.2.3} \left(\frac{dy}{dx}\right)^3 + \dots$ is Soln: Given: $y(x) = 1 + \frac{dy}{dx} + \frac{1}{1.2} \left(\frac{dy}{dx}\right)^2 + \frac{1}{1.2.3} \left(\frac{dy}{dx}\right)^3 + \dots$ $y(x) = 1 + y' + \frac{1}{2!} (y')^2 + \frac{1}{3!} (y')^3 + \dots$ $y(x) = 1 + \frac{y'}{1!} + \frac{(y')^2}{2!} + \frac{(y')^3}{3!} + \dots$	$y(x) = e^{y'}$ Taking log on both sides, $\log y(x) = \log e^{y'}$ $\log y(x) = y'$ $\therefore \frac{dy}{dx} = \log y$ $\therefore \text{order} = 1$ Ans: (3) 1 12) If p and q are the order and degree of the differential equation $y \frac{dy}{dx} + x^3 \left(\frac{d^2y}{dx^2}\right) + xy = \cos x$, when Soln: Given: $y \frac{dy}{dx} + x^3 \left(\frac{d^2y}{dx^2}\right) + xy = \cos x$ $P = \text{order} = 2$ — ①
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$q = \text{degree} = 1$ — ②
from ① & ②

$2 > 1 \Rightarrow P > q$

Ans: (3) $P > q$

13) The Solution of the differential equation

$$\frac{dy}{dx} + \frac{1}{\sqrt{1-x^2}} = 0 \text{ is}$$

Soln: Given:

$$\frac{dy}{dx} + \frac{1}{\sqrt{1-x^2}} = 0$$

$$\frac{dy}{dx} = -\frac{1}{\sqrt{1-x^2}}$$

$$\int dy = -\int \frac{dx}{\sqrt{1-x^2}}$$

$$y = -\sin^{-1}x + c$$

$$y + \sin^{-1}x = c$$

Ans: (1) $y + \sin^{-1}x = c$

14) The Solution of the differential equation

$$\frac{dy}{dx} = 2xy \text{ is}$$

Soln: Given:

$$\frac{dy}{dx} = 2xy$$

$$\int \frac{dy}{y} = \int 2x dx$$

$$\log y = \cancel{x} \frac{x^2}{\cancel{x}} + \log c$$

$$\log y - \log c = x^2$$

$$\log\left(\frac{y}{c}\right) = x^2$$

$$\frac{y}{c} = e^{x^2}$$

$$y = ce^{x^2}$$

Ans: (1) $y = ce^{x^2}$

15) The general Solution of the differential equation

$\log\left(\frac{dy}{dx}\right) = x+y$ is

Soln: Given:

$$\log\left(\frac{dy}{dx}\right) = x+y$$

$$\frac{dy}{dx} = e^{x+y}$$

$$\frac{dy}{dx} = e^x \cdot e^y$$

$$\frac{dy}{e^y} = e^x dx$$

$$\bar{e}^y dy = e^x dx$$

$$\int e^x dx = \int \bar{e}^y dy$$

$$e^x = -\bar{e}^y + c$$

$$e^x + \bar{e}^y = c$$

Ans: (2) $e^x + \bar{e}^y = c$

16) The Solution of $\frac{dy}{dx} = 2^{y-x}$ is

Soln: Given:

$$\frac{dy}{dx} = 2^{y-x}$$

$$\frac{dy}{dx} = 2^y \cdot 2^{-x}$$

$$\frac{dy}{2^y} = 2^{-x} dx$$

$$\int \bar{2}^y dy = \int \bar{2}^x dx$$

$$\frac{\bar{2}^y}{\log 2} = \frac{\bar{2}^x}{\log 2} + \frac{c}{\log 2}$$

Multiple by $\log 2$,

$$\bar{2}^y = \bar{2}^x + c$$

$$\bar{2}^x - \bar{2}^y = c$$

$$\frac{1}{2^x} - \frac{1}{2^y} = c$$

Ans: (3) $\frac{1}{2^x} - \frac{1}{2^y} = c$

17) The Solution of the

differential equation
 $\frac{dy}{dx} = \frac{y}{x} + \frac{\phi(\frac{y}{x})}{\phi'(\frac{y}{x})}$ is

Soln: Given:

$$\frac{dy}{dx} = \frac{y}{x} + \frac{\phi(\frac{y}{x})}{\phi'(\frac{y}{x})}$$

This is a homogeneous differential equation

Put $y = vx$

$$\frac{d(vx)}{dx} = \frac{vx}{x} + \frac{\phi(\frac{vx}{x})}{\phi'(\frac{vx}{x})}$$

$$v + x \frac{dv}{dx} = v + \frac{\phi(v)}{\phi'(v)}$$

$$x \frac{dv}{dx} = \frac{\phi(v)}{\phi'(v)} - v$$

$$x \frac{dv}{dx} = \frac{\phi(v)}{\phi'(v)}$$

$$\int \frac{\phi'(v)}{\phi(v)} = \int \frac{dx}{x}$$

$$\log \phi(v) = \log x + \log k$$

$$\log \phi(\frac{y}{x}) = \log kx$$

$$\phi(\frac{y}{x}) = kx$$

$$\text{Ans: (2) } \phi(\frac{y}{x}) = kx$$

18) If $\sin x$ is the integrating factor of the linear differential equation $\frac{dy}{dx} + py = Q$,

then p is

Soln: Given:

$$\frac{dy}{dx} + py = Q \text{ and}$$

$$\text{I.F} = \sin x$$

$$\therefore \text{I.F} = e^{\int p dx}$$

$$\sin x = e^{\int p dx}$$

Taking log on both sides,

$$\log \sin x = \log e^{\int p dx}$$

$$\log \sin x = \int p dx$$

$$d(\log \sin x) = d(\int p dx)$$

$$\frac{1}{\sin x} \cos x = p$$

$$\therefore p = \frac{\cos x}{\sin x} = \cot x$$

$$\therefore p = \cot x$$

$$\text{Ans: (4) } \cot x$$

19) The number of arbitrary constants in the general solutions of order n and $n+1$ are respectively

Soln:

n is a arbitrary constants
 $= n$ order

$n+1$ is a arbitrary constants
 $= n+1$ order

$\therefore n$ and $n+1$ are arbitrary constants
 $\Rightarrow n$ and $n+1$ are order
 Ans: (2) $n, n+1$

20) The number of arbitrary constants in the particular solution of a differential equation of third order is

Soln:

order = 3 $\Rightarrow$ number of arbitrary constants

But there is no arbitrary constants in the particular solution.

$\therefore$ Third order is 0
 Ans: (4) 0

21) Integrating factor of the differential equation $-n \frac{dy}{dx} = \frac{x+y+1}{x+1}$ is

Soln: Given:

$$\frac{dy}{dx} = \frac{x+y+1}{x+1} = \frac{(x+1)+y}{(x+1)}$$

$$\frac{dy}{dx} = \frac{(x+1)}{(x+1)} + \frac{y}{(x+1)}$$

$$\frac{dy}{dx} = 1 + \frac{y}{x+1}$$

$$\frac{dy}{dx} - \frac{1}{x+1} y = 1$$

This is a linear differential equation in y

Here $P = -\frac{1}{x+1}$, $Q = 1$

$$I.F = e^{\int P dx} = e^{\int \frac{-1}{x+1} dx} = e^{-\log(x+1)}$$

$$= e^{\log(x+1)^{-1}} = (x+1)^{-1} = \frac{1}{x+1}$$

Ans: (1) $\frac{1}{x+1}$

22) The population P in any year t is such that the rate of increase in the population is proportional to the population. Then

Soln: Given:

$$\frac{dP}{dt} \propto P \Rightarrow \frac{dP}{dt} = kP$$

$$\int \frac{dP}{P} = \int k dt$$

$$\log P = kt + c$$

$$P = e^{kt+c}$$

$$P = ce^{kt}$$

Ans: (1) ce^{kt}

23) P is the amount of certain substance left in after time t. If the rate of evaporation of the substance is proportional to the amount remaining, then

Soln: $\frac{dP}{dt} = -kP$

$$\int \frac{dP}{P} = -k dt$$

$$\log P = -kt + c$$

$$P = e^{-kt+c}$$

$$P = ce^{-kt}$$

Ans: (2) $P = ce^{-kt}$

24) If the solution of the differential equation $\frac{dy}{dx} = \frac{ax+3}{2y+f}$ represents

a circle, then the value of a is

Soln: Given:

$$\frac{dy}{dx} = \frac{ax+3}{2y+f}$$

$$(2y+f)dy = (ax+3)dx$$

$$\frac{2y^2}{2} + fy = \frac{ax^2}{2} + 3x + c$$

$$\frac{2y^2 + 2fy}{2} = \frac{ax^2 + 6x + 2c}{2}$$

$$ax^2 - 2y^2 + 6x - 2fy + 2c = 0$$

This represents a circle

Coeff of x^2 = Coeff of y^2

$$a = -2$$

Ans: (2) -2

25) The slope at any point of a curve $y = f(x)$ is given by $\frac{dy}{dx} = 3x^2$ and it passes through $(-1, 1)$. Then the

equation of the curve is

Soln: Given: $\frac{dy}{dx} = 3x^2$

$$\int dy = \int 3x^2 dx$$

$$y = \cancel{3} \frac{x^3}{\cancel{3}} + c$$

$$y = x^3 + c \text{ --- ①}$$

If passes through $(-1, 1)$,
we get,

$$1 = (-1)^3 + c$$

$$1 = -1 + c$$

$$c = 1 + 1$$

$$\boxed{c = 2}$$

$$\text{①} \Rightarrow y = x^3 + 2$$

$$\text{Ans: (1) } y = x^3 + 2$$

11. Probability Distributions

EXERCISE 11-1

1) Suppose X is the number of tails occurred when three fair coins are tossed once simultaneously. Find the values of the random variable X and number of points in its inverse images.

Soln:

Sample Space $S = \{HHH, HTH, TTH, TTT, HHT, HTT, THT, TTT\}$

$$\therefore n(S) = 8$$

Let x be the random variable that takes on the values 0, 1, 2 and 3

Let $X(w)$ denotes the number of tails

$$X(w) = \begin{cases} 3 & \text{if } w = TTT \Rightarrow X(3) = 1 \\ 2 & \text{if } w = TTH, HTH, THT \Rightarrow X(2) = 3 \\ 1 & \text{if } w = HTH, THT, HHT, HTT \Rightarrow X(1) = 4 \\ 0 & \text{if } w = HHH \Rightarrow X(0) = 1 \end{cases}$$

values of Random variable	0	1	2	3	Total
Number of Points in inverse image	1	3	3	1	8

2) In a pack of 52 playing cards, two cards are drawn at random simultaneously. If the number of black cards drawn is a random variable, find the values of the random variable and number of points in its inverse images.

Soln:

The number of Sample

$$n(S) = {}^{52}C_2 = \frac{52 \times 51}{2 \times 1} = 1326$$

black Card	Red Card	Total
26	26	52

$$\therefore n(S) = 1326$$

Let x be the random variable that takes on the values 0, 1 and 2

Let $X(w)$ denotes the number of black cards.

$$\begin{aligned} X(2) &= {}^{26}C_2 \times {}^{26}C_0 = 325 \times 1 = 325 \\ X(1) &= {}^{26}C_1 \times {}^{26}C_1 = 26 \times 26 = 676 \\ X(0) &= {}^{26}C_0 \times {}^{26}C_2 = 1 \times 325 = 325 \end{aligned}$$

$$X(w) = \begin{cases} 2 & \text{if two black cards} \Rightarrow X(2) = 325 \\ 1 & \text{if one black card} \Rightarrow X(1) = 676 \\ 0 & \text{if no black card} \Rightarrow X(0) = 325 \end{cases}$$

values of Random variable	0	1	2	Total
Number of Points in inverse image	325	676	325	1326

3) An urn contains 5 mangoes and 4 apples. Three fruits are taken at random. If the number of apples taken is a random variable, then find the values of the random variable and number of points in its inverse images

Soln:

Apple Fruits	Mango Fruits	Total Fruits
4	5	9

The number of Sample Space $n(S) = {}^9C_3 = \frac{9 \times 8 \times 7}{3 \times 2 \times 1} = 84$

$$\therefore n(S) = 84$$

Let x be the random variable that takes on the values 0, 1, 2 and 3

Let $X(w)$ denotes the number of Apple Fruits

$$\begin{aligned} X(3) &= {}^4C_3 \times {}^5C_0 = 4 \times 1 = 4 \\ X(2) &= {}^4C_2 \times {}^5C_1 = 6 \times 5 = 30 \\ X(1) &= {}^4C_1 \times {}^5C_2 = 4 \times 10 = 40 \\ X(0) &= {}^4C_0 \times {}^5C_3 = 1 \times 10 = 10 \end{aligned}$$

$$X(w) = \begin{cases} 3 & \text{if 3 Apples} \Rightarrow X(3) = 4 \\ 2 & \text{if 2 Apples} \Rightarrow X(2) = 30 \\ 1 & \text{if 1 Apple} \Rightarrow X(1) = 40 \\ 0 & \text{if no Apple} \Rightarrow X(0) = 10 \end{cases}$$

values of Random variable	0	1	2	3	Total
Number of Points in inverse image	10	40	30	4	84

4) Two balls are chosen randomly from an urn containing 6 red and 8 black balls. Suppose that we win ₹15 for each red ball selected and we lose ₹10 for each black ball selected. X denotes the winning amount, then find the values of X and number of points in its inverse images.

Soln:

Red balls	Black balls	Total balls
6	8	14

The number of Sample space
 $n(S) = {}^{14}C_2 = \frac{14 \times 13}{2 \times 1} = 91$

$$\therefore n(S) = 91$$

$$X(\text{both are red balls}) = ₹2(15) = ₹30$$

$$X(\text{one red and one black ball}) = ₹15 - ₹10 = ₹5$$

$$X(\text{both are black balls}) = ₹2(-10) = -₹20$$

Let X be the random variable that takes on the values 30, 5, -20

Let $X(w)$ denotes the number of red and black balls

$$X(30) = 6C_2 \times 8C_0 = 15 \times 1 = 15$$

$$X(5) = 6C_1 \times 8C_1 = 6 \times 8 = 48$$

$$X(-20) = 6C_0 \times 8C_2 = 1 \times 28 = 28$$

$$X(w) = \begin{cases} 30 & \text{if both red balls} \Rightarrow X(30) = 15 \\ 5 & \text{if one red and one black ball} \Rightarrow X(5) = 48 \\ -20 & \text{if both black balls} \Rightarrow X(-20) = 28 \end{cases}$$

values of Random variables	30	5	-20	Total
Number of points in inverse image	15	48	28	91

5) A Six Sided die is marked '2' on one face, '3' on two of its faces, and '4' on remaining three faces. The die is thrown twice. If X denotes the total score in two throws, find the

values of the random variable and number of points in its inverse images

Soln:

Sample Space $S = \{2, 3, 4\} \times \{2, 3, 4\}$

$$S = \{(2, 2), (2, 3), (2, 4), (3, 2), (3, 3), (3, 4), (4, 2), (4, 3), (4, 4)\}$$

$$n(S) = 9$$

X is assigned to each point (α, β) the sum of the numbers on the dice

$$(\text{i.e.}) X(\alpha, \beta) = \alpha + \beta$$

$$X(2, 2) = 4$$

$$X(2, 3) = X(3, 2) = 5$$

$$X(2, 4) = X(3, 3) = X(4, 2) = 6$$

$$X(3, 4) = X(4, 3) = 7$$

$$X(4, 4) = 8$$

The random variable X takes on the values are 4, 5, 6, 7, 8

values of Random variables	4	5	6	7	8	Total
Number of points in inverse image	1	2	3	2	1	9

Ex 11.1

Suppose two coins are tossed once. If X denotes the number of tails, (i) write down the sample space (ii) find the inverse image of 1 (iii) the values of the random variable and number of elements in its inverse images

Soln:

(i) The Sample Space
 $S = \{HH, HT, TH, TT\}$

$$n(S) = 4$$

(ii) Let X be the random variable that takes on the values 0, 1 and 2

Let $X(w)$ denotes the number of tails

$$X(w) = \begin{cases} 2 & \text{if } w = TT \Rightarrow X(2) = 1 \\ 1 & \text{if } w = HT, TH \Rightarrow X(1) = 2 \\ 0 & \text{if } w = HH \Rightarrow X(0) = 1 \end{cases}$$

The inverse images of 1 is

$\{TH, HT\}$

$$(ii) X^{-1}(\{1\}) = \{TH, HT\}$$

(iii)

values of the Random Variable	0	1	2	Total
Number of elements in inverse image	1	2	1	4

Ex 11.2

Suppose a pair of unbiased dice is rolled once. If X denotes the total score of two dice. Write down (i) the Sample Space (ii) the values taken by the random variable X (iii) the inverse image of 10, and

(iv) the number of elements in inverse image of X

Sol: (i) Sample Space

$$S = \begin{matrix} (1,1) & (1,2) & (1,3) & (1,4) & (1,5) & (1,6) \\ (2,1) & (2,2) & (2,3) & (2,4) & (2,5) & (2,6) \\ (3,1) & (3,2) & (3,3) & (3,4) & (3,5) & (3,6) \\ (4,1) & (4,2) & (4,3) & (4,4) & (4,5) & (4,6) \\ (5,1) & (5,2) & (5,3) & (5,4) & (5,5) & (5,6) \\ (6,1) & (6,2) & (6,3) & (6,4) & (6,5) & (6,6) \end{matrix}$$

$$n(S) = 36$$

X is assigned to each point

(α, β) the sum of the numbers on the dice

$$(ii) X(\alpha, \beta) = \alpha + \beta$$

$$X(1,1) = 2$$

$$X(1,2) = X(2,1) = 3$$

$$X(1,3) = X(2,2) = X(3,1) = 4$$

$$X(1,4) = X(2,3) = X(3,2) = X(4,1) = 5$$

$$X(1,5) = X(2,4) = X(3,3) = X(4,2) = X(5,1) = 6$$

$$X(1,6) = X(2,5) = X(3,4) = X(4,3) = X(5,2)$$

$$= X(6,1) = 7$$

$$X(2,6) = X(3,5) = X(4,4) = X(5,3) = X(6,2) = 8$$

$$X(3,6) = X(4,5) = X(5,4) = X(6,3) = 9$$

$$X(4,6) = X(5,5) = X(6,4) = 10$$

$$X(5,6) = X(6,5) = 11$$

$$X(6,6) = 12$$

(ii) The random variable X takes on the values 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12

(iii) The inverse image of 10 is

$$\{(4,6), (5,5), (6,4)\}$$

(iv) The number of inverse image are given below

values of the random variable	2	3	4	5	6	7	8	9	10	11	12	Total
Number of elements in inverse image	1	2	3	4	5	6	5	4	3	2	1	36

Ex 11.3 An urn contains 2 white balls and 3 red balls. A Sample of

3 balls are chosen at random from the urn. If X denotes the number of red balls chosen, find the values taken by the random variable X and its number of inverse images.

Sol:

White balls	Red balls	Total balls
2	3	5

The number of Sample Space $n(S) = {}^5C_3 = \frac{5 \times 4 \times 3}{3 \times 2 \times 1} = 10$

$$\therefore n(S) = 10$$

Let X be the random variable that takes on the values 1, 2 and 3

Let $X(w)$ denotes the number of red balls chosen

$$X(3) = {}^3C_3 \times {}^2C_0 = 1 \times 1 = 1$$

$$X(2) = {}^3C_2 \times {}^2C_1 = 3 \times 2 = 6$$

$$X(1) = {}^3C_1 \times {}^2C_2 = 3 \times 1 = 3$$

$$X(w) = \begin{cases} 3 & \text{if 3 red balls} \Rightarrow X(3) = 1 \\ 2 & \text{if 2 red balls} \Rightarrow X(2) = 6 \\ 1 & \text{if 1 red ball} \Rightarrow X(1) = 3 \end{cases}$$

values of the Random variable	1	2	3	Total
Number of elements in inverse images	3	6	1	10

EX 11.4

Two balls are chosen randomly from an urn containing 6 white and 4 black balls. Suppose that we win ₹30 for each black ball selected and we lose ₹20 for each white ball selected. If X denotes the winning amount, then find the values of X and number of points in its inverse images.

Soln:

white ball	black ball	Total balls
6	4	10

The number of Sample Space
 $n(S) = {}^{10}C_2 = \frac{10 \times 9}{2 \times 1} = 45$

$$\therefore n(S) = 45$$

X (both are black balls)

$$= ₹2(30) = ₹60$$

X (one black and one white ball)

$$= ₹30 - ₹20 = ₹10$$

X (both are white balls)

$$= ₹2(-20) = ₹40$$

Let X be the random variable that takes on the values 60, 10, -40

Let $X(w)$ denotes the number of white and black balls

$$X(60) = {}^6C_0 \times {}^4C_2 = 1 \times 6 = 6$$

$$X(10) = {}^6C_1 \times {}^4C_1 = 6 \times 4 = 24$$

$$X(-40) = {}^6C_2 \times {}^4C_0 = 15 \times 1 = 15$$

$$X(w) = \begin{cases} 60 & \text{if both black balls} \\ & \Rightarrow X(60) = 6 \\ 10 & \text{if one black and one white ball} \\ & \Rightarrow X(10) = 24 \\ -40 & \text{if both white balls} \\ & \Rightarrow X(-40) = 15 \end{cases}$$

values of the Random variable	60	10	-40	Total
Number of elements in inverse images	6	24	15	45

11.3 Types of Random variable

EXERCISE 11.2

1) Three fair coins are tossed simultaneously. Find the probability mass function for number of heads occurred

Soln:

Sample Space $S = \{HHH, HTH, THH, TTH, HHT, HTT, THT, TTT\}$

$$\therefore n(S) = 8$$

Let X be the random variable that takes on the values 0, 1, 2 and 3

Let $X(w)$ denotes the number of heads

$$X(w) = \begin{cases} 3 & \text{if } w = HHH \Rightarrow X(3) = 1 \\ 2 & \text{if } w = HTH, THH, HHT \\ & \Rightarrow X(2) = 3 \\ 1 & \text{if } w = TTH, HTT, THT \Rightarrow X(1) = 3 \\ 0 & \text{if } w = TTT \Rightarrow X(0) = 1 \end{cases}$$

values of the Random variable	0	1	2	3	Total
Number of elements in inverse images	1	3	3	1	8

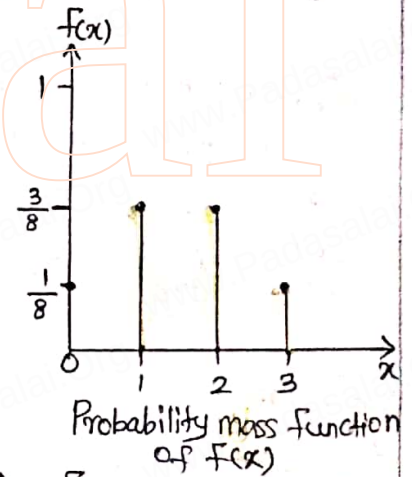
The probabilities are given by

$$f(0) = P(X=0) = \frac{1}{8}$$

$$f(1) = P(X=1) = \frac{3}{8}$$

$$f(2) = P(X=2) = \frac{3}{8}$$

$$\text{and } f(3) = P(X=3) = \frac{1}{8}$$



The function $f(x)$ satisfies the condition
 (i) $f(x) \geq 0$, for $x = 0, 1, 2, 3$

$$(ii) \sum_x f(x) = \sum_{x=0}^3 f(x) = f(0) + f(1) + f(2) + f(3)$$

$$= \frac{1}{8} + \frac{3}{8} + \frac{3}{8} + \frac{1}{8}$$

$$= \frac{8}{8} = 1$$

$\therefore f(x)$ is a probability mass function.

The probability mass function is given by

x	0	1	2	3
f(x)	$\frac{1}{8}$	$\frac{3}{8}$	$\frac{3}{8}$	$\frac{1}{8}$

$$(or) f(x) = \begin{cases} \frac{1}{8} & \text{for } x=0,3 \\ \frac{3}{8} & \text{for } x=1,2 \end{cases}$$

2) A six sided die is marked '1' on one face, '3' on two of its faces, and '5' on remaining three faces. The die is thrown twice. If X denotes the total score in two throws. find (i) the

Probability mass function (ii) the cumulative distribution function (iii) $P(4 \leq x < 10)$ (iv) $P(x \geq 6)$

Soln: Sample Space S

II \ I	1	3	3	5	5	5
1	2	4	4	6	6	6
3	4	6	6	8	8	8
3	4	6	6	8	8	8
5	6	8	8	10	10	10
5	6	8	8	10	10	10
5	6	8	8	10	10	10

Since X denotes the total score in two throws, it takes on the values 2, 4, 6, 8 and 10

Values of the Random Variable	2	4	6	8	10	Total
Number of elements in inverse images	1	4	10	12	9	36

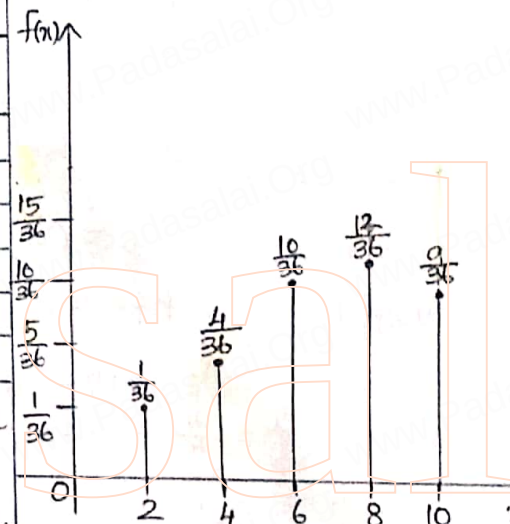
$$P(X=2) = \frac{1}{36}, P(X=4) = \frac{4}{36}$$

$$P(X=6) = \frac{10}{36}, P(X=8) = \frac{12}{36},$$

$$P(X=10) = \frac{9}{36}$$

(i) probability mass function is

X	2	4	6	8	10
f(x)	$\frac{1}{36}$	$\frac{4}{36}$	$\frac{10}{36}$	$\frac{12}{36}$	$\frac{9}{36}$



(ii) Cumulative distribution function:

$$F(x) = P(X \leq x)$$

$$F(x) = P(X \leq x) = 0 \text{ for } x < 2$$

$$F(2) = P(X \leq 2) = \frac{1}{36}$$

$$F(4) = P(X \leq 4) = \frac{5}{36}$$

$$F(6) = P(X \leq 6) = \frac{15}{36}$$

$$F(8) = P(X \leq 8) = \frac{27}{36}$$

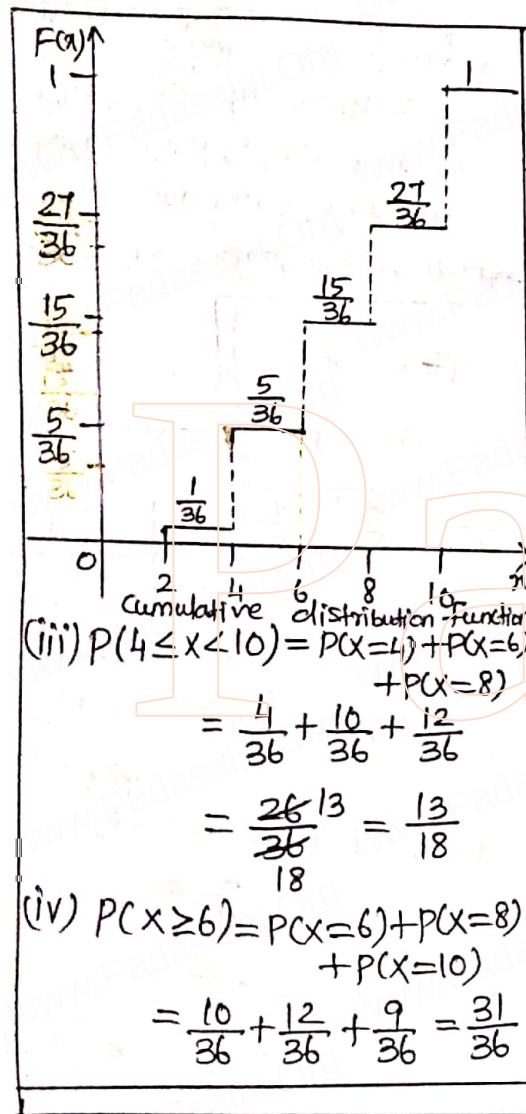
$$F(10) = P(X \leq 10) = \frac{36}{36} = 1$$

$\therefore$ The cumulative distribution function is

$$F(x) = \begin{cases} 0 & \text{for } x < 2 \\ \frac{1}{36} & \text{for } x \leq 2 \\ \frac{5}{36} & \text{for } x \leq 4 \\ \frac{15}{36} & \text{for } x \leq 6 \\ \frac{27}{36} & \text{for } x \leq 8 \\ 1 & \text{for } x \leq 10 \end{cases}$$

(or)

x	2	4	6	8	10
F(x)	$\frac{1}{36}$	$\frac{5}{36}$	$\frac{15}{36}$	$\frac{27}{36}$	1



3) Find the Probability mass function and Cumulative distribution function of number of girl child in families with 4 children, assuming equal probabilities for boys and girls

Soln:

$$\text{Sample space } (S) = 2^{n(4)} = 2^4 = 16$$

$$\therefore n(S) = 16$$

Let X be the random variable denoting the number of girl child

X can take values are 0, 1, 2, 3 and 4

$$X(0) = {}^4C_0 = 1$$

$$X(1) = {}^4C_1 = 4$$

$$X(2) = {}^4C_2 = 6$$

$$X(3) = {}^4C_3 = 4$$

$$X(4) = {}^4C_4 = 1$$

Values of the Random variable	0	1	2	3	4	Total
Number of elements in inverse image	1	4	6	4	1	16

$$P(X=0) = \frac{1}{16}$$

$$P(X=1) = \frac{4}{16} = \frac{1}{4}$$

$$P(X=2) = \frac{6}{16} = \frac{3}{8}$$

$$P(X=3) = \frac{4}{16} = \frac{1}{4}$$

$$P(X=4) = \frac{1}{16}$$

(i) Probability mass function is

$$f(x) = \begin{cases} \frac{1}{4} & \text{for } x=1, 3 \\ \frac{1}{16} & \text{for } x=0, 4 \\ \frac{3}{8} & \text{for } x=2 \end{cases}$$

[OR]

x	0	1	2	3	4
$f(x)$	$\frac{1}{16}$	$\frac{4}{16}$	$\frac{6}{16}$	$\frac{4}{16}$	$\frac{1}{16}$

(ii) Cumulative distribution function:

$$F(x) = P(X \leq x)$$

$$F(0) = P(X \leq 0) = \frac{1}{16}$$

$$F(1) = P(X \leq 1) = \frac{5}{16}$$

$$F(2) = P(X \leq 2) = \frac{11}{16}$$

$$F(3) = P(X \leq 3) = \frac{15}{16}$$

$$F(4) = P(X \leq 4) = \frac{16}{16} = 1$$

$\therefore$ Cumulative distribution function is

x	0	1	2	3	4
$F(x)$	$\frac{1}{16}$	$\frac{5}{16}$	$\frac{11}{16}$	$\frac{15}{16}$	1

$$F(x) = \begin{cases} \frac{1}{16} & \text{for } x \leq 0 \\ \frac{5}{16} & \text{for } x \leq 1 \\ \frac{11}{16} & \text{for } x \leq 2 \\ \frac{15}{16} & \text{for } x \leq 3 \\ 1 & \text{for } x \leq 4 \end{cases}$$

4) Suppose a discrete random variable can only take the values 0, 1 and 2. The probability mass function is defined by $f(x) = \begin{cases} \frac{x^2+1}{k}, & \text{for } x=0,1,2 \\ 0, & \text{otherwise} \end{cases}$

Find (i) the value of k
(ii) Cumulative distribution function

(iii) $P(X \geq 1)$

Soln: (i) Since $f(x)$ is a Probability mass function

$$\therefore \sum_x f(x) = 1$$

$$\frac{(0)^2+1}{k} + \frac{(1)^2+1}{k} + \frac{(2)^2+1}{k} = 1$$

$$\frac{1}{k} + \frac{2}{k} + \frac{5}{k} = 1$$

$$\frac{8}{k} = 1 \Rightarrow \boxed{k=8}$$

$$\therefore f(x) = \begin{cases} \frac{x^2+1}{8}, & \text{for } x=0,1,2 \\ 0, & \text{otherwise} \end{cases}$$

x	0	1	2
$f(x)$	$\frac{1}{8}$	$\frac{2}{8}$	$\frac{5}{8}$

(ii) Cumulative distribution function:

$$F(x) = P(X \leq x)$$

$$F(0) = P(X \leq 0) = 0$$

$$F(0) = P(X \leq 0) = \frac{1}{8}$$

$$F(1) = P(X \leq 1) = \frac{3}{8}$$

$$F(2) = P(X \leq 2) = \frac{8}{8} = 1$$

$$\therefore F(x) = \begin{cases} 0 & \text{for } x < 0 \\ \frac{1}{8} & \text{for } x \leq 0 \\ \frac{3}{8} & \text{for } x \leq 1 \\ 1 & \text{for } x \leq 2 \end{cases}$$

$$(iii) P(X \geq 1) = P(X=1) + P(X=2) \\ = \frac{2}{8} + \frac{5}{8} = \frac{7}{8}$$

5) The cumulative distribution function of a discrete random variable is given by

$$F(x) = \begin{cases} 0 & -\infty < x < -1 \\ 0.15 & -1 \leq x < 0 \\ 0.35 & 0 \leq x < 1 \\ 0.60 & 1 \leq x < 2 \\ 0.85 & 2 \leq x < 3 \\ 1 & 3 \leq x < \infty \end{cases}$$

Find (i) the probability mass function (ii) $P(X < 1)$ and

(iii) $P(X \geq 2)$

Soln: Since x is a discrete random variable

x can take on the values $-1, 0, 1, 2$ and 3

$$\therefore \boxed{f(x) = F(x) - F(x^-)}$$

$$f(-1) = F(-1) - F(-1^-) = 0.15 - 0$$

$$\boxed{f(-1) = 0.15}$$

$$f(0) = F(0) - F(-1) = 0.35 - 0.15$$

$$\boxed{f(0) = 0.20}$$

$$f(1) = F(1) - F(0) = 0.60 - 0.35$$

$$\boxed{f(1) = 0.25}$$

$$f(2) = F(2) - F(1) = 0.85 - 0.60$$

$$\boxed{f(2) = 0.25}$$

$$f(3) = F(3) - F(2) = 1 - 0.85$$

$$\boxed{f(3) = 0.15}$$

(i) Probability mass function is

x	-1	0	1	2	3
$f(x)$	0.15	0.20	0.25	0.25	0.15

$$(ii) P(X < 1) = P(X = -1) + P(X = 0) \\ = f(-1) + f(0) \\ = 0.15 + 0.20 \\ = 0.35$$

$$(iii) P(X \geq 2) = P(X = 2) + P(X = 3) \\ = f(2) + f(3) \\ = 0.25 + 0.15 \\ = 0.40$$

6) A random variable x has

the following Probability mass Function

x	1	2	3	4	5
$f(x)$	k^2	$2k^2$	$3k^2$	$2k$	$3k$

Find (i) the value of k
(ii) $P(2 \leq x < 5)$ (iii) $P(3 < x)$

Soln:
(i) Since $f(x)$ is a Probability mass function

$$\therefore \sum_{x=1}^5 f(x) = 1$$

$$k^2 + 2k^2 + 3k^2 + 2k + 3k = 1$$

$$6k^2 + 5k - 1 = 0 \quad \begin{matrix} -6 \\ \wedge \\ 6-1 \end{matrix}$$

$$6k^2 + 6k - k - 1 = 0$$

$$6k(k+1) - 1(k+1) = 0$$

$$(6k-1)(k+1) = 0$$

$$6k-1=0, k+1=0$$

$$k = \frac{1}{6}, k = -1 \text{ is not possible}$$

$$\therefore \boxed{k = \frac{1}{6}}$$

$\therefore$ The probability mass function is

x	1	2	3	4	5
$f(x)$	$\frac{1}{36}$	$\frac{2}{36}$	$\frac{3}{36}$	$\frac{2}{6}$	$\frac{3}{6}$

$$(ii) P(2 \leq x < 5) = P(x=2) + P(x=3) + P(x=4)$$

$$= f(2) + f(3) + f(4)$$

$$= \frac{2}{36} + \frac{3}{36} + \frac{2}{6}$$

$$= \frac{2+3+12}{36} = \frac{17}{36}$$

$$(iii) P(3 < x) = P(x > 3)$$

$$= P(x=4) + P(x=5)$$

$$= f(4) + f(5)$$

$$= \frac{2}{6} + \frac{3}{6} = \frac{5}{6}$$

(i) The cumulative distribution function of a discrete random variable is given by

$$F(x) = \begin{cases} 0 & \text{for } -\infty < x < 0 \\ \frac{1}{2} & \text{for } 0 \leq x < 1 \\ \frac{3}{5} & \text{for } 1 \leq x < 2 \\ \frac{4}{5} & \text{for } 2 \leq x < 3 \\ \frac{9}{10} & \text{for } 3 \leq x < 4 \\ 1 & \text{for } 4 \leq x < \infty \end{cases}$$

Find (i) the probability mass function (ii) $P(x < 3)$ and

$$(iii) P(x \geq 2)$$

Soln: Since x is a discrete random variable

x can take on the values 0, 1, 2, 3 and 4

$$\therefore \boxed{f(x) = F(x) - F(x^-)}$$

$$f(0) = F(0) - F(0^-) = \frac{1}{2} - 0$$

$$\boxed{f(0) = \frac{1}{2}}$$

$$f(1) = F(1) - F(1^-) = \frac{3}{5} - \frac{1}{2} = \frac{6-5}{10} = \frac{1}{10}$$

$$\boxed{f(1) = \frac{1}{10}}$$

$$f(2) = F(2) - F(2^-) = \frac{4}{5} - \frac{3}{5} = \frac{1}{5}$$

$$\boxed{f(2) = \frac{1}{5}}$$

$$f(3) = F(3) - F(3^-) = \frac{9}{10} - \frac{4}{5} = \frac{9-8}{10} = \frac{1}{10}$$

$$\boxed{f(3) = \frac{1}{10}}$$

$$f(4) = F(4) - F(4^-) = 1 - \frac{9}{10} = \frac{1}{10}$$

$$\boxed{f(4) = \frac{1}{10}}$$

(i) The probability mass function is

x	0	1	2	3	4
$f(x)$	$\frac{1}{2}$	$\frac{1}{10}$	$\frac{1}{5}$	$\frac{1}{10}$	$\frac{1}{10}$

$$(ii) P(X < 3) = P(X=0) + P(X=1) + P(X=2)$$

$$= f(0) + f(1) + f(2)$$

$$= \frac{1}{2} + \frac{1}{10} + \frac{1}{5}$$

$$= \frac{5+1+2}{10} = \frac{8}{10} = \frac{4}{5}$$

$$\therefore P(X < 3) = \frac{4}{5}$$

$$(iii) P(X \geq 2) = P(X=2) + P(X=3) + P(X=4)$$

$$= \frac{1}{5} + \frac{1}{10} + \frac{1}{10}$$

$$= \frac{2+1+1}{10} = \frac{4}{10} = \frac{2}{5}$$

$$\therefore P(X \geq 2) = \frac{2}{5}$$

EX 11.5

Two fair Coins are tossed Simultaneously (equivalent to a fair Coin is tossed twice). Find the probability mass function for number of heads occurred.

Soln:-

Sample Space $S = \{HH, HT, TH, TT\}$
 $n(S) = 4$

Let x be the random variable denoting the number of heads and x takes on the values 0, 1 and 2

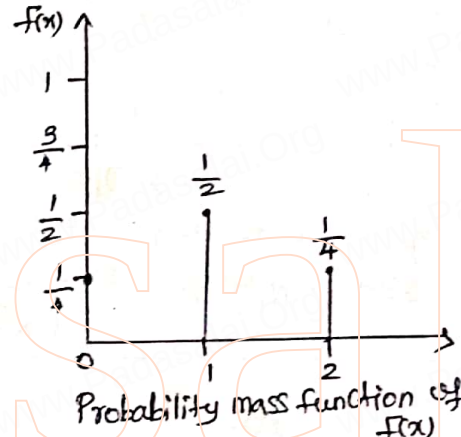
$$X(\omega) = \begin{cases} 2 & \text{if } \omega = HH \Rightarrow X(2) = 1 \\ 1 & \text{if } \omega = HT, TH \Rightarrow X(1) = 2 \\ 0 & \text{if } \omega = TT \Rightarrow X(0) = 1 \end{cases}$$

value of the Random Variable	0	1	2	Total
Number of elements in inverse image	1	2	1	4

The Probabilities are given by $f(0) = P(X=0) = \frac{1}{4}$

$$f(1) = P(X=1) = \frac{2}{4} = \frac{1}{2}$$

$$f(2) = P(X=2) = \frac{1}{4}$$



The function $f(x)$ satisfies the conditions

$$(i) f(x) \geq 0, \text{ for } x = 0, 1, 2$$

$$(ii) \sum_x f(x) = \sum_{x=0}^2 f(x) = f(0) + f(1) + f(2) = \frac{1}{4} + \frac{1}{2} + \frac{1}{4} = 1$$

$\therefore f(x)$ is a probability mass function

The Probability mass function is given by

x	0	1	2
$f(x)$	$\frac{1}{4}$	$\frac{1}{2}$	$\frac{1}{4}$

$$(or) f(x) = \begin{cases} \frac{1}{4} & \text{for } x=0 \\ \frac{1}{2} & \text{for } x=1 \\ \frac{1}{4} & \text{for } x=2 \end{cases}$$

Ex 11.6

A pair of fair dice is rolled once. Find the probability mass function to get the number of fours

Soln:-

Let x be a random variable whose values x are the number of fours $\therefore$ Sample Space

$S = \begin{matrix} (1,1) & (1,2) & (1,3) & (1,4) & (1,5) & (1,6) \\ (2,1) & (2,2) & (2,3) & (2,4) & (2,5) & (2,6) \\ (3,1) & (3,2) & (3,3) & (3,4) & (3,5) & (3,6) \\ (4,1) & (4,2) & (4,3) & (4,4) & (4,5) & (4,6) \\ (5,1) & (5,2) & (5,3) & (5,4) & (5,5) & (5,6) \\ (6,1) & (6,2) & (6,3) & (6,4) & (6,5) & (6,6) \end{matrix}$

$n(S) = 36$
 X takes on the values are 0, 1 and 2
 (i) $X=0$ if (i,j) for $i \neq 4, j \neq 4$
 (ii) $X=1$ if $(1,4) (2,4) (3,4) (5,4) (6,4) (4,1) (4,2) (4,3) (4,5) (4,6)$
 (iii) $X=2$ if $(4,4)$

Values of the Random Variable	0	1	2	Total
Number of elements in inverse images	25	10	1	36

$\therefore f(0) = P(X=0) = \frac{25}{36}$

$f(1) = P(X=1) = \frac{10}{36}$
 $f(2) = P(X=2) = \frac{1}{36}$

Probability mass function of $f(x)$
 $f(x)$ satisfies the condition
 (i) $f(x) \geq 0$ for $x=0, 1, 2$
 (ii) $\sum_x f(x) = \sum_{x=0}^2 f(x) = f(0) + f(1) + f(2)$
 $= \frac{25}{36} + \frac{10}{36} + \frac{1}{36}$
 $= \frac{36}{36} = 1$

$\therefore$ The probability mass function is

x	0	1	2
$f(x)$	$\frac{25}{36}$	$\frac{10}{36}$	$\frac{1}{36}$

OR
 $f(x) = \begin{cases} \frac{25}{36} & \text{for } x=0 \\ \frac{10}{36} & \text{for } x=1 \\ \frac{1}{36} & \text{for } x=2 \end{cases}$

Ex 11.7
 If the probability mass function $f(x)$ of a random variable X is

x	1	2	3	4
$f(x)$	$\frac{1}{12}$	$\frac{5}{12}$	$\frac{5}{12}$	$\frac{1}{12}$

find (i) its cumulative distribution function, hence find
 (ii) $P(X \leq 3)$ and (iii) $P(X \geq 2)$

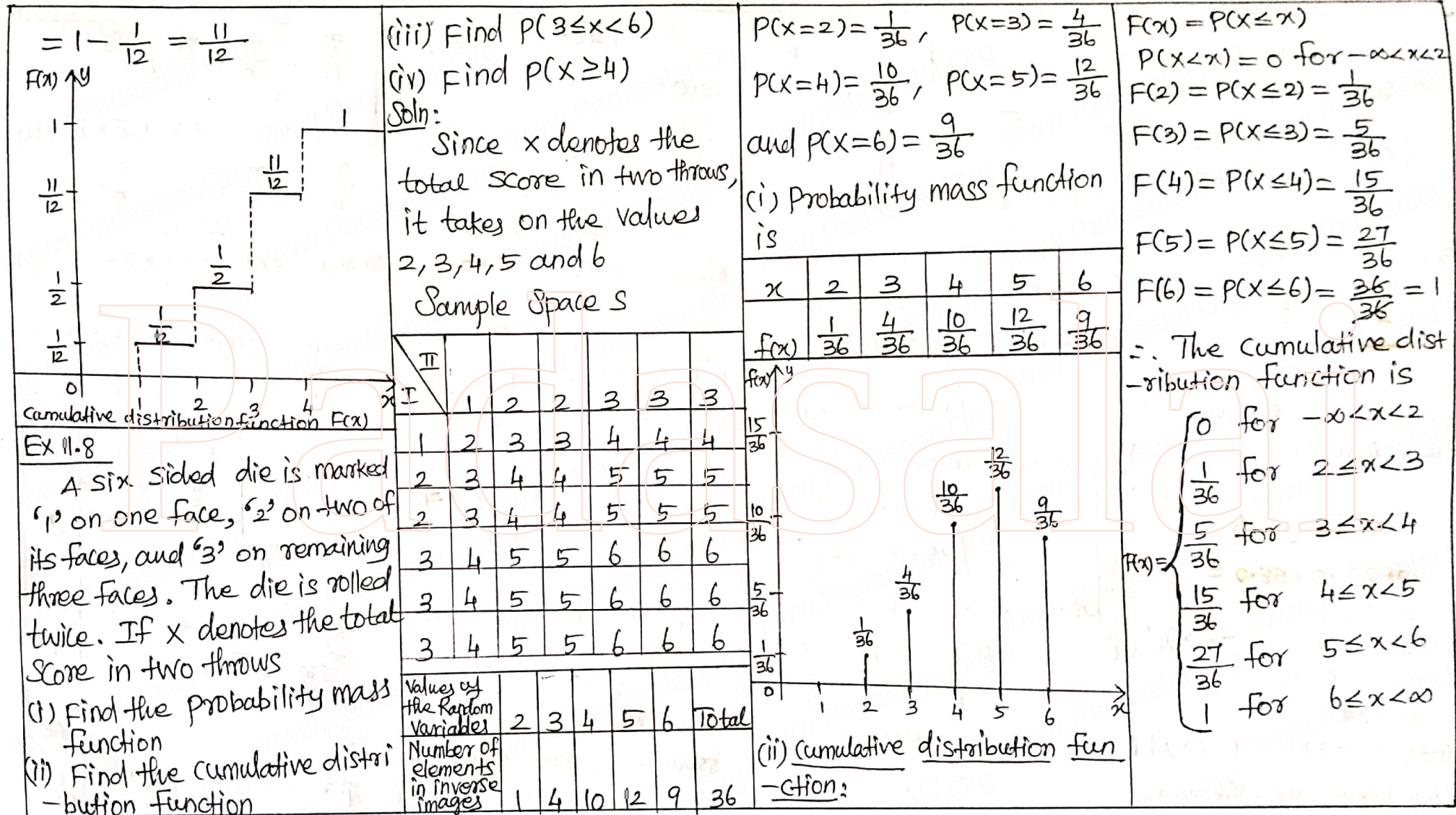
Soln:

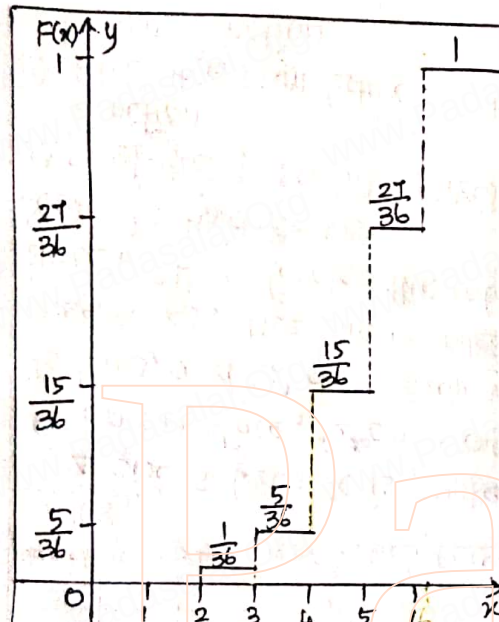
(i) $F(x) = P(X \leq x)$
 $P(X < 1) = 0$ for $-\infty < x < 1$
 $F(1) = P(X \leq 1) = \frac{1}{12}$
 $F(2) = P(X \leq 2) = \frac{1}{12} + \frac{5}{12} = \frac{6}{12} = \frac{1}{2}$
 $F(3) = P(X \leq 3) = \frac{11}{12}$
 $F(4) = P(X \leq 4) = \frac{12}{12} = 1$

$\therefore$ The cumulative distribution function is

$$F(x) = \begin{cases} 0, & -\infty < x < 1 \\ \frac{1}{12}, & 1 \leq x < 2 \\ \frac{1}{2}, & 2 \leq x < 3 \\ \frac{11}{12}, & 3 \leq x < 4 \\ 1, & 4 \leq x < \infty \end{cases}$$

(ii) $P(X \leq 3) = F(3) = \frac{11}{12}$
 (iii) $P(X \geq 2) = 1 - P(X < 2)$





Cumulative distribution Function

$$(iii) P(3 \leq x < 6) = P(x=3) + P(x=4) + P(x=5)$$

$$= \frac{4}{36} + \frac{10}{36} + \frac{12}{36} = \frac{26}{36}$$

$$(iv) P(x \geq 4) = P(x=4) + P(x=5) + P(x=6)$$

$$= \frac{10}{36} + \frac{12}{36} + \frac{9}{36} = \frac{31}{36}$$

Ex 11.9

Find the probability mass function $f(x)$ of the discrete random variable x whose cumulative distribution function $F(x)$ is given by

$$F(x) = \begin{cases} 0 & -\infty < x < -2 \\ 0.25 & -2 \leq x < -1 \\ 0.60 & -1 \leq x < 0 \\ 0.90 & 0 \leq x < 1 \\ 1 & 1 \leq x < \infty \end{cases}$$

Also find (i) $P(x < 0)$ and (ii) $P(x \geq -1)$

Soln: Since x is a discrete random variable x takes on the values $-2, -1, 0$ and 1

$$f(-2) = F(-2) - F(-2^-)$$

$$= 0.25 - 0 = 0.25$$

$$f(-2) = 0.25$$

$$f(-1) = F(-1) - F(-1^-)$$

$$= 0.60 - 0.25 = 0.35$$

$$f(-1) = 0.35$$

$$f(0) = F(0) - F(0^-)$$

$$= 0.90 - 0.60 = 0.30$$

$$f(0) = 0.30$$

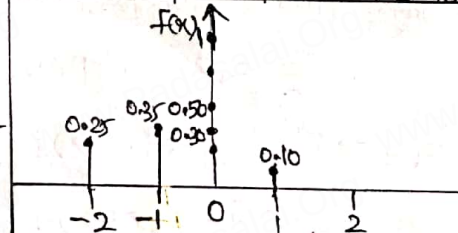
$$f(1) = F(1) - F(1^-)$$

$$= 1 - 0.90 = 0.10$$

$$f(1) = 0.10$$

$\therefore$ The Probability mass function is

x	-2	-1	0	1
$f(x)$	0.25	0.35	0.30	0.10



Probability mass function $f(x)$

$$(i) P(x < 0) = P(x=-2) + P(x=-1)$$

$$= 0.25 + 0.35$$

$$= 0.60$$

$$(ii) P(x \geq -1) = P(x=-1) + P(x=0)$$

$$+ P(x=1)$$

$$= 0.35 + 0.30 + 0.10$$

$$= 0.75$$

Ex 11.10

A random variable x has the following probability mass function

x	1	2	3	4	5	6
$f(x)$	k	$2k$	$6k$	$5k$	$6k$	$10k$

Find (i) $P(2 < x < 6)$

(ii) $P(2 \leq x < 5)$

(iii) $P(x \leq 4)$ (iv) $P(3 < x)$

Soln: Since $f(x)$ is a probability mass function

$$\sum_x f(x) = 1$$

$$k + 2k + 6k + 5k + 6k + 10k = 1$$

$$30k = 1$$

$$k = \frac{1}{30}$$

$\therefore$ The probability mass function is

x	1	2	3	4	5	6
f(x)	$\frac{1}{30}$	$\frac{2}{30}$	$\frac{6}{30}$	$\frac{5}{30}$	$\frac{6}{30}$	$\frac{10}{30}$

$$(i) P(2 < X < 6) = f(3) + f(4) + f(5) \\ = \frac{6}{30} + \frac{5}{30} + \frac{6}{30} = \frac{17}{30}$$

$$(ii) P(2 \leq X < 5) = f(2) + f(3) + f(4) \\ = \frac{2}{30} + \frac{6}{30} + \frac{5}{30} = \frac{13}{30}$$

$$(iii) P(X \leq 4) = f(1) + f(2) + f(3) + f(4) \\ = \frac{1}{30} + \frac{2}{30} + \frac{6}{30} + \frac{5}{30} = \frac{14}{30}$$

$$(iv) P(3 < X) = f(4) + f(5) + f(6) \\ = \frac{5}{30} + \frac{6}{30} + \frac{10}{30} = \frac{21}{30}$$

11.4 Continuous Distributions

EXERCISE 11.3

1) The probability density function of X is given by

$$f(x) = \begin{cases} kx e^{-2x} & \text{for } x > 0 \\ 0 & \text{for } x \leq 0 \end{cases}$$

Find the value of k

Soln: Since f(x) is a Probability density function

$$\therefore \int_{-\infty}^{\infty} f(x) dx = 1$$

$$\int_0^{\infty} kx e^{-2x} dx = 1$$

$$k \left[\frac{1!}{(2)^{1+1}} \right] = 1$$

$$k \left[\frac{1}{(2)^2} \right] = 1$$

$$\text{Since } \int_0^{\infty} x^n e^{-ax} dx = \frac{n!}{a^{n+1}}$$

$$k \left[\frac{1}{4} \right] = 1$$

$$k = 4$$

2) The probability density function of X is

$$f(x) = \begin{cases} x, & 0 < x < 1 \\ 2-x, & 1 \leq x < 2 \\ 0, & \text{otherwise} \end{cases}$$

Find (i) $P(0.2 \leq x < 0.6)$

(ii) $P(1.2 \leq x < 1.8)$

(iii) $P(0.5 \leq x < 1.5)$

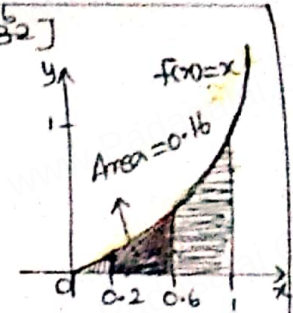
Soln:

$$(i) P(0.2 \leq x < 0.6) \\ = \int_{0.2}^{0.6} f(x) dx = \int_{0.2}^{0.6} x dx$$

$$= \left[\frac{x^2}{2} \right]_{0.2}^{0.6}$$

$$= \frac{1}{2} [(0.6)^2 - (0.2)^2] \\ = \frac{1}{2} [0.36 - 0.04]$$

$$= \frac{1}{2} [0.32] \\ = 0.16$$

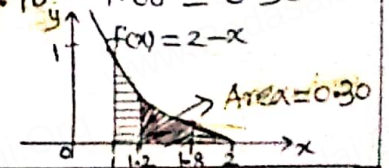


(ii) $P(1.2 \leq x < 1.8)$

$$= \int_{1.2}^{1.8} f(x) dx = \int_{1.2}^{1.8} (2-x) dx$$

$$= \left[2x - \frac{x^2}{2} \right]_{1.2}^{1.8}$$

$$= \left[\left(2(1.8) - \frac{(1.8)^2}{2} \right) - \left(2(1.2) - \frac{(1.2)^2}{2} \right) \right] \\ = \left[3.6 - \frac{1.62}{2} \right] - \left[2.4 - \frac{0.72}{2} \right] \\ = [3.6 - 0.81] - [2.4 - 0.36] \\ = 2.79 - 2.04 = 0.75$$



(iii) $P(0.5 \leq x < 1.5)$

$$= \int_{0.5}^{1.5} f(x) dx = \int_{0.5}^1 f(x) dx + \int_1^{1.5} f(x) dx$$

$$= \int_{0.5}^1 x dx + \int_1^{1.5} (2-x) dx$$

$$= \left[\frac{x^2}{2} \right]_{0.5}^1 + \left[2x - \frac{x^2}{2} \right]_1^{1.5}$$

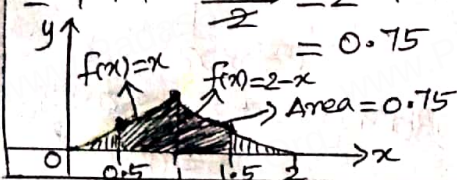
$$= \left[\frac{(1)^2}{2} - \frac{(0.5)^2}{2} \right] + \left[2(1.5) - \frac{(1.5)^2}{2} \right] - \left[2(1) - \frac{(1)^2}{2} \right]$$

$$= \left[\frac{1}{2} - \frac{0.25}{2} \right] + \left[3 - \frac{2.25}{2} \right] - \left[2 - \frac{1}{2} \right]$$

$$= \frac{1}{2} - \frac{0.25}{2} + 3 - \frac{2.25}{2} - 2 + \frac{1}{2}$$

$$= 1 + 1 - \frac{2.50}{2} = 2 - 1.25$$

$$= 0.75$$



3) Suppose the amount of milk sold daily at a milk booth is distributed with a minimum of 200 litres and a maximum of 600 litres with probability density function

$$f(x) = \begin{cases} k, & 200 \leq x \leq 600 \\ 0, & \text{otherwise} \end{cases}$$

Find (i) the value of k

(ii) the distribution function

(iii) the probability that daily sales will fall between 300 litres and 500 litres?

Soln: (i) Since $f(x)$ is a probability density function

$$\int_{-\infty}^{\infty} f(x) dx = 1$$

$$\int_{200}^{600} k dx = 1 \Rightarrow k[x]_{200}^{600} = 1$$

$$k[600 - 200] = 1$$

$$k[400] = 1$$

$$k = \frac{1}{400}$$

$$\therefore f(x) = \begin{cases} \frac{1}{400}, & 200 \leq x \leq 600 \\ 0, & \text{otherwise} \end{cases}$$

(ii) The Distribution Function:

$$F(x) = P(X \leq x) = \int_{-\infty}^x f(u) du$$

a) When $x \in (-\infty, 200)$

$$F(x) = \int_{-\infty}^x f(u) du = \int_{-\infty}^{200} (0) du = 0$$

b) When $x \in [200, 600]$

$$F(x) = \int_{-\infty}^x f(u) du = \int_{-\infty}^{200} f(u) du + \int_{200}^x f(u) du$$

$$= \int_{-\infty}^{200} (0) du + \int_{200}^x \frac{1}{400} du$$

$$= 0 + \frac{1}{400} [u]_{200}^x$$

$$= \frac{1}{400} [x - 200]$$

$$= \frac{x}{400} - \frac{200}{400}$$

$$F(x) = \frac{x}{400} - \frac{1}{2}$$

c) When $x \in (600, \infty)$

$$F(x) = \int_{-\infty}^x f(u) du$$

$$= \int_{-\infty}^{200} f(u) du + \int_{200}^{600} f(u) du + \int_{600}^x f(u) du$$

$$= \int_{-\infty}^{200} (0) du + \int_{200}^{600} \frac{1}{400} du + \int_{600}^x (0) du$$

$$= 0 + \int_{200}^{600} \frac{1}{400} du + 0$$

$$= \frac{1}{400} [u]_{200}^{600}$$

$$= \frac{1}{400} [600 - 200]$$

$$= \frac{1}{400} [400] = 1$$

$$\therefore F(x) = \begin{cases} 0 & \text{for } x < 200 \\ \frac{x}{400} - \frac{1}{2} & \text{for } 200 \leq x \leq 600 \\ 1 & \text{for } x > 600 \end{cases}$$

$$(iii) P(300 < x < 500)$$

$$\begin{aligned} &= F(500) - F(300) \\ &= \left[\frac{500}{400} - \frac{1}{2} \right] - \left[\frac{300}{400} - \frac{1}{2} \right] \\ &= \frac{500}{400} - \frac{1}{2} - \frac{300}{400} + \frac{1}{2} \\ &= \frac{500 - 300}{400} = \frac{200}{400} = \frac{1}{2} \end{aligned}$$

$$\therefore P(300 < x < 500) = \frac{1}{2}$$

4) The probability density function of x is given by

$$f(x) = \begin{cases} k e^{-\frac{x}{3}} & \text{for } x > 0 \\ 0 & \text{for } x \leq 0 \end{cases}$$

Find (i) the value of k

(ii) the distribution function

$$(iii) P(x < 3) \quad (iv) P(5 \leq x)$$

$$(v) P(x \leq 4)$$

Soln: (i)

Since $f(x)$ is a probability density function is

$$\int_{-\infty}^{\infty} f(x) dx = 1$$

$$\int_{-\infty}^{\infty} k e^{-\frac{x}{3}} dx = 1$$

$$k \left[-\frac{e^{-x/3}}{1/3} \right]_0^{\infty} = 1$$

$$-3k \left[e^{-x/3} \right]_0^{\infty} = 1$$

$$-3k \left[e^{-\infty/3} - e^{-0/3} \right] = 1$$

$$-3k \left[e^{-\infty} - e^0 \right] = 1$$

$$-3k \left[\frac{1}{e^{\infty}} - \frac{1}{e^0} \right] = 1$$

$$-3k \left[\frac{1}{\infty} - \frac{1}{1} \right] = 1$$

$$-3k [0 - 1] = 1$$

$$-3k [-1] = 1$$

$$3k = 1$$

$$\boxed{k = \frac{1}{3}}$$

(ii) The distribution function

$$F(x) = P(x \leq x) = \int_{-\infty}^x f(u) du$$

a) when $x \in (-\infty, 0]$,

$$F(x) = \int_{-\infty}^x f(u) du = \int_{-\infty}^0 (0) du = 0$$

b) when $x \in (0, \infty)$

$$F(x) = \int_{-\infty}^x f(u) du = \int_{-\infty}^0 f(u) du + \int_0^x f(u) du$$

$$= \int_{-\infty}^0 (0) du + \int_0^x \frac{1}{3} e^{-\frac{u}{3}} du$$

$$= 0 + \frac{1}{3} \left[-\frac{e^{-u/3}}{1/3} \right]_0^x$$

$$f(x) = \begin{cases} \frac{1}{3} e^{-\frac{x}{3}} & \text{for } x > 0 \\ 0 & \text{for } x \leq 0 \end{cases}$$

$$= - \left[e^{-\frac{x}{3}} - e^{-\frac{0}{3}} \right]$$

$$= - \left[e^{-\frac{x}{3}} - e^0 \right]$$

$$= - \left[e^{-\frac{x}{3}} - \frac{1}{e^0} \right]$$

$$= - \left[e^{-\frac{x}{3}} - \frac{1}{1} \right]$$

$$= - \left[e^{-\frac{x}{3}} - 1 \right]$$

$$F(x) = 1 - e^{-\frac{x}{3}} \therefore F(x) = \begin{cases} 0, & \text{for } x \leq 0 \\ 1 - e^{-\frac{x}{3}}, & \text{for } x > 0 \end{cases}$$

$$(iii) P(x < 3) = P(x \leq 3)$$

$$= F(3)$$

$$= 1 - e^{-\frac{3}{3}}$$

$$= 1 - e^{-1}$$

$$(iv) P(5 \leq x) = P(x \geq 5)$$

$$= 1 - P(x < 5)$$

$$= 1 - P(x \leq 5)$$

$$= 1 - F(5)$$

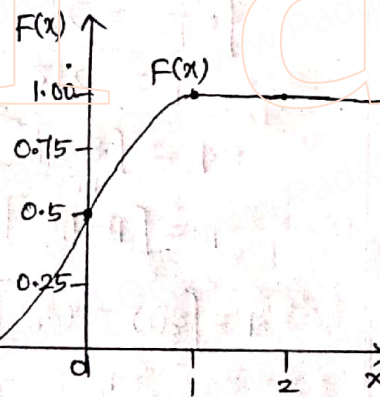
$$= 1 - \left[1 - e^{-\frac{5}{3}} \right]$$

$$= 1 - 1 + e^{-\frac{5}{3}}$$

$$= e^{-\frac{5}{3}}$$

$$\begin{aligned}
 & \left[\left[-\frac{(1)^2}{2} + 1 \right] - [0] \right] \\
 &= - \left[\frac{1}{2} - 1 \right] + \left[-\frac{1}{2} + 1 \right] \\
 &= - \left[-\frac{1}{2} \right] + \left[\frac{1}{2} \right] \\
 &= \frac{1}{2} + \frac{1}{2} = \frac{2}{2} = 1
 \end{aligned}$$

$$\therefore F(x) = \begin{cases} 0, & -\infty < x < -1 \\ \frac{x^2}{2} + x + \frac{1}{2}, & -1 \leq x < 0 \\ -\frac{x^2}{2} + x + \frac{1}{2}, & 0 \leq x < 1 \\ 1, & 1 \leq x < \infty \end{cases}$$



Distribution function

$$\begin{aligned}
 \text{(ii) } P(-0.5 \leq x \leq 0.5) &= F(0.5) - F(-0.5) \\
 &= F\left(\frac{1}{2}\right) - F\left(-\frac{1}{2}\right) \\
 &= \left[-\frac{\left(\frac{1}{2}\right)^2}{2} + \frac{1}{2} + \frac{1}{2} \right] - \left[\frac{\left(-\frac{1}{2}\right)^2}{2} + \left(-\frac{1}{2}\right) + \frac{1}{2} \right] \\
 &= \left[-\left(\frac{1}{4} \times \frac{1}{2}\right) + \frac{1}{2} + \frac{1}{2} \right] - \left[\left(\frac{1}{4} \times \frac{1}{2}\right) - \frac{1}{2} + \frac{1}{2} \right] \\
 &= \left[-\frac{1}{8} + 1 \right] - \left[\frac{1}{8} \right] \\
 &= -\frac{1}{8} + 1 - \frac{1}{8} \\
 &= \frac{-1 + 8 - 1}{8} = \frac{6}{8} = 0.75
 \end{aligned}$$

6) If x is the random variable with distribution function $F(x)$ given by

$$F(x) = \begin{cases} 0, & x < 0 \\ \frac{1}{2}(x^2 + x), & 0 \leq x < 1 \\ 1, & x \geq 1 \end{cases}$$

then find (i) the probability density function $f(x)$ (ii)

$$P(0.3 \leq x \leq 0.6)$$

Soln: Given:

$$F(x) = \begin{cases} 0, & x < 0 \\ \frac{1}{2}(x^2 + x), & 0 \leq x < 1 \\ 1, & x \geq 1 \end{cases}$$

$$f(x) = F'(x) = \begin{cases} 0, & x < 0 \\ \frac{1}{2}(2x + 1), & 0 \leq x < 1 \\ 0, & x \geq 1 \end{cases}$$

$$\therefore f(x) \text{ is not continuous at } x=0 \quad x=1 \\
 \therefore f(x) = \begin{cases} \frac{1}{2}(2x + 1), & 0 \leq x < 1 \\ 0, & \text{otherwise} \end{cases}$$

$$\text{(ii) } P(0.3 \leq x \leq 0.6)$$

$$\begin{aligned}
 &= F(0.6) - F(0.3) \\
 &= \frac{1}{2}((0.6)^2 + 0.6) - \frac{1}{2}((0.3)^2 + 0.3) \\
 &= \frac{1}{2}(0.36 + 0.6) - \frac{1}{2}(0.09 + 0.3) \\
 &= \frac{1}{2}(0.96) - \frac{1}{2}(0.39) \\
 &= \frac{1}{2}[0.96 - 0.39]
 \end{aligned}$$

$$\begin{aligned}
 &= \frac{1}{2} [0.57] \\
 &= 0.285
 \end{aligned}$$

Ex 11.11

Find the constant C such that the function $f(x) = \begin{cases} cx^2, & 1 < x < 4 \\ 0, & \text{otherwise} \end{cases}$ is a density function, and compute (i) $P(1.5 < x < 3.5)$

$$\text{(ii) } P(x \leq 2) \quad \text{(iii) } P(3 < x)$$

Soln: Since $f(x)$ is a probability density function, $\int_{-\infty}^{\infty} f(x) dx = 1$

$$\begin{aligned}
 &\int_{-\infty}^{-1} f(x) dx + \int_{-1}^4 f(x) dx + \int_4^{\infty} f(x) dx = 1 \\
 &\int_{-\infty}^{-1} (0) dx + \int_{-1}^4 (cx^2) dx + \int_4^{\infty} (0) dx = 1 \\
 &0 + c \int_{-1}^4 x^2 dx + 0 = 1
 \end{aligned}$$

$$C \left[\frac{x^3}{3} \right]_1^4 = 1$$

$$\frac{C}{3} [(4)^3 - (1)^3] = 1$$

$$\frac{C}{3} [64 - 1] = 1$$

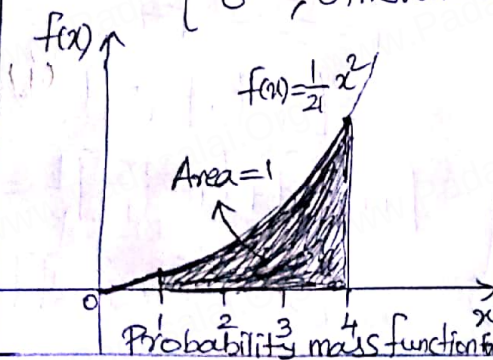
$$\frac{C}{3} [63] = 1$$

$$21C = 1$$

$$C = \frac{1}{21}$$

∴ The probability density function is

$$f(x) = \begin{cases} \frac{1}{21}x^2, & 1 < x < 4 \\ 0, & \text{otherwise} \end{cases}$$



$$(i) P(1.5 < x < 3.5) = \int_{1.5}^{3.5} f(x) dx = \int_{1.5}^{3.5} \frac{1}{21} x^2 dx$$

$$= \frac{1}{21} \left[\frac{x^3}{3} \right]_{1.5}^{3.5}$$

$$= \frac{1}{63} [(3.5)^3 - (1.5)^3]$$

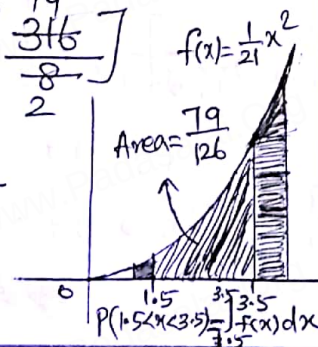
$$= \frac{1}{63} \left[\left(\frac{7}{2}\right)^3 - \left(\frac{3}{2}\right)^3 \right]$$

$$= \frac{1}{63} \left[\frac{343}{8} - \frac{27}{8} \right]$$

$$= \frac{1}{63} \left[\frac{343 - 27}{8} \right]$$

$$= \frac{1}{63} \left[\frac{316}{8} \right]$$

$$= \frac{79}{126}$$



$$(ii) P(x \leq 2) = \int_{-\infty}^2 f(x) dx$$

$$= \int_{-\infty}^1 f(x) dx + \int_1^2 f(x) dx$$

$$= \int_{-\infty}^1 (0) dx + \int_1^2 \frac{1}{21} x^2 dx$$

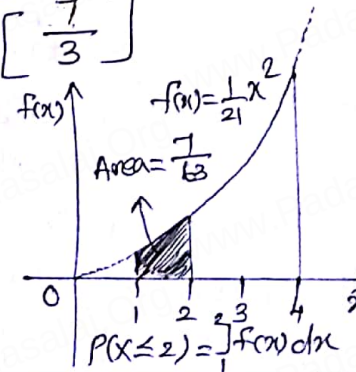
$$= 0 + \frac{1}{21} \left[\frac{x^3}{3} \right]_1^2$$

$$= \frac{1}{21} \left[\frac{(2)^3}{3} - \frac{(1)^3}{3} \right]$$

$$= \frac{1}{21} \left[\frac{8}{3} - \frac{1}{3} \right]$$

$$= \frac{1}{21} \left[\frac{7}{3} \right]$$

$$= \frac{7}{63}$$



$$(iii) P(3 < x) = \int_3^{\infty} f(x) dx$$

$$= \int_3^4 f(x) dx + \int_4^{\infty} f(x) dx$$

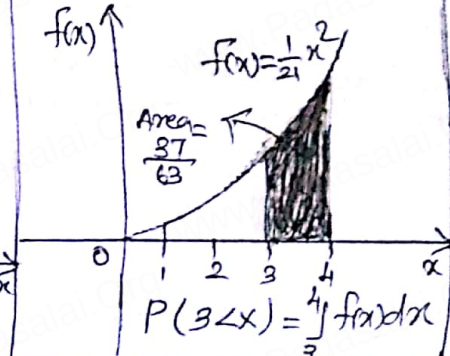
$$= \int_3^4 \frac{1}{21} x^2 dx + \int_4^{\infty} (0) dx$$

$$= \frac{1}{21} \left[\frac{x^3}{3} \right]_3^4 + 0$$

$$= \frac{1}{63} [(4)^3 - (3)^3]$$

$$= \frac{1}{63} [64 - 27]$$

$$= \frac{1}{63} [37] = \frac{37}{63}$$



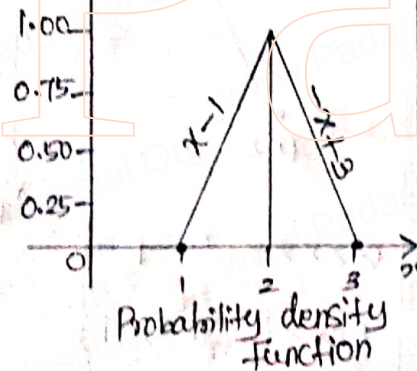
Ex 11.12

If x is the random variable with probability density function $f(x)$ given by $f(x) = \begin{cases} x-1, & 1 \leq x < 2 \\ -x+3, & 2 \leq x < 3 \\ 0, & \text{otherwise} \end{cases}$

find (i) the distribution function $F(x)$

(ii) $p(1.5 \leq x \leq 2.5)$

Soln: $f(x)$



(i) The distribution function:

$$F(x) = P(X \leq x) = \int_{-\infty}^x f(u) du$$

a) When $x \in (-\infty, 1)$

$$F(x) = \int_{-\infty}^x f(u) du = \int_{-\infty}^x (0) du = 0$$

b) When $x \in [1, 2)$

$$\begin{aligned} F(x) &= \int_{-\infty}^x f(u) du \\ &= \int_{-\infty}^1 f(u) du + \int_1^x f(u) du \\ &= \int_{-\infty}^1 (0) du + \int_1^x (u-1) du \\ &= 0 + \left[\frac{u^2}{2} - u \right]_1^x \\ &= \left[\frac{x^2}{2} - x \right] - \left[\frac{1}{2} - 1 \right] \\ &= \left[\frac{x^2}{2} - x \right] - \left[-\frac{1}{2} \right] \\ &= \frac{x^2}{2} - x + \frac{1}{2} \\ &= \frac{x^2 - 2x + 1}{2} = \frac{(x-1)^2}{2} \end{aligned}$$

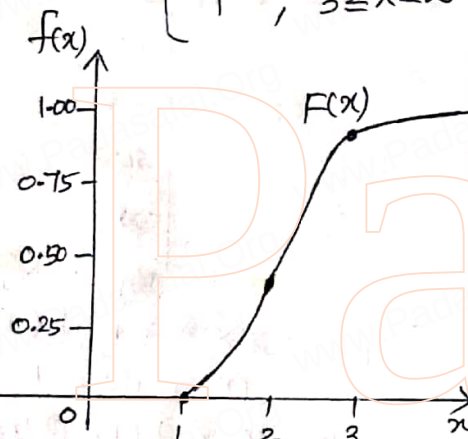
c) When $x \in [2, 3)$

$$\begin{aligned} F(x) &= \int_{-\infty}^x f(u) du \\ &= \int_{-\infty}^1 f(u) du + \int_1^2 f(u) du + \int_2^x f(u) du \\ &= \int_{-\infty}^1 (0) du + \int_1^2 (u-1) du + \int_2^x (3-u) du \\ &= 0 + \left[\frac{(u-1)^2}{2} \right]_1^2 + \left[\frac{(3-u)^2}{2} \right]_2^x \\ &= \left[\frac{(2-1)^2}{2} - \frac{(1-1)^2}{2} \right] + \left[\frac{(3-x)^2}{2} - \frac{(3-2)^2}{2} \right] \\ &= \left[\frac{1}{2} \right] + \left[\frac{(3-x)^2}{2} + \frac{1}{2} \right] \\ &= \frac{1}{2} + \frac{1}{2} - \frac{(3-x)^2}{2} \\ &= 1 - \frac{(3-x)^2}{2} \end{aligned}$$

d) When $x \in [3, \infty)$

$$\begin{aligned} F(x) &= \int_{-\infty}^x f(u) du \\ &= \int_{-\infty}^1 f(u) du + \int_1^2 f(u) du + \int_2^3 f(u) du + \int_3^x f(u) du \\ &= \int_{-\infty}^1 (0) du + \int_1^2 (u-1) du + \int_2^3 (3-u) du + \int_3^x (0) du \\ &= 0 + \left[\frac{(u-1)^2}{2} \right]_1^2 + \left[\frac{(3-u)^2}{2} \right]_2^3 + 0 \\ &= \left[\frac{(2-1)^2}{2} - \frac{(1-1)^2}{2} \right] + \left[\frac{(3-3)^2}{2} - \frac{(3-2)^2}{2} \right] \\ &= \frac{1}{2} + \frac{1}{2} = \frac{2}{2} = 1 \end{aligned}$$

$$\therefore F(x) = \begin{cases} 0, & -\infty < x < 1 \\ \frac{(x-1)^2}{2}, & 1 \leq x < 2 \\ 1 - \frac{(3-x)^2}{2}, & 2 \leq x < 3 \\ 1, & 3 \leq x < \infty \end{cases}$$



Distribution function

$$(ii) P(1.5 \leq x \leq 2.5)$$

$$= F(2.5) - F(1.5) \\ = \left[1 - \frac{(3-2.5)^2}{2} \right] - \left[\frac{(1.5-1)^2}{2} \right]$$

$$= \left[1 - \frac{(0.5)^2}{2} \right] - \left[\frac{(0.5)^2}{2} \right] \\ = 1 - \frac{0.25}{2} - \frac{0.25}{2} \\ = 1 - \frac{0.25}{2} \\ = 1 - 0.25 = 0.75$$

[OR]

$$P(1.5 \leq x \leq 2.5) = \int_{1.5}^{2.5} f(x) dx \\ = \int_{1.5}^2 (x-1) dx + \int_2^{2.5} (3-x) dx \\ = \left[\frac{(x-1)^2}{2} \right]_{1.5}^2 + \left[\frac{(3-x)^2}{-2} \right]_2^{2.5} \\ = \left[\frac{(2-1)^2}{2} - \frac{(1.5-1)^2}{2} \right] + \left[\frac{(3-2.5)^2}{-2} - \frac{(3-2)^2}{-2} \right]$$

$$= \left[\frac{1}{2} - \frac{(0.5)^2}{2} \right] + \left[\frac{(0.5)^2}{-2} + \frac{1}{2} \right] \\ = \frac{1}{2} - \frac{0.25}{2} - \frac{0.25}{2} + \frac{1}{2} \\ = \frac{1}{2} + \frac{1}{2} - \frac{0.25}{2} - \frac{0.25}{2} \\ = \frac{2}{2} - \frac{0.25}{2} \\ = 1 - 0.25 = 0.75$$

Ex 11.13

If x is the random variable with distribution function $F(x)$ given by

$$F(x) = \begin{cases} 0, & x < 0 \\ x, & 0 \leq x < 1 \\ 1, & 1 \leq x \end{cases}$$

then find (i) the probability density function $f(x)$

$$(ii) P(0.2 \leq x \leq 0.7)$$

Soln:

$$(i) f(x) = F'(x) = \begin{cases} 0, & x < 0 \\ 1, & 0 \leq x < 1 \\ 0, & x \geq 1 \end{cases}$$

The p.d.f $f(x)$ is not continuous at $x=0$ (or) at $x=1$

$\therefore$ The probability density function $f(x)$ is $f(x) = \begin{cases} 1, & 0 \leq x < 1 \\ 0, & \text{otherwise} \end{cases}$

$$(ii) P(0.2 \leq x \leq 0.7)$$

$$= F(0.7) - F(0.2) \\ = 0.7 - 0.2 = 0.5$$

[OR]

$$P(0.2 \leq x \leq 0.7) = \int_{0.2}^{0.7} f(x) dx \\ = \int_{0.2}^{0.7} (1) dx = [x]_{0.2}^{0.7} \\ = 0.7 - 0.2 = 0.5$$

Ex 11.14

The probability density function of random variable x is given by $f(x) = \begin{cases} k, & 1 \leq x \leq 5 \\ 0, & \text{otherwise} \end{cases}$

Find (i) Distribution Function

(ii) $P(X < 3)$ (iii) $P(2 < X < 4)$

(iv) $P(3 \leq X)$

Soln: Since $f(x)$ is a Probability density function

$$\int_{-\infty}^{\infty} f(x) dx = 1$$

$$\int_{-\infty}^{\infty} k dx = 1 \Rightarrow k[x]_{-\infty}^{\infty} = 1$$

$$k[5-1] = 1$$

$$4k = 1$$

$$k = \frac{1}{4}$$

$\therefore$ The probability density function is $f(x) = \begin{cases} \frac{1}{4}, & 1 \leq x \leq 5 \\ 0, & \text{otherwise} \end{cases}$

(i) Distribution Function:

$$F(x) = P(X \leq x) = \int_{-\infty}^x f(u) du$$

a) When $x \in (-\infty, 1)$,

$$F(x) = \int_{-\infty}^x f(u) du$$

$$= \int_{-\infty}^x (0) du = 0$$

b) When $x \in [1, 5)$,

$$F(x) = \int_{-\infty}^x f(u) du$$

$$= \int_{-\infty}^1 f(u) du + \int_1^x f(u) du$$

$$= \int_{-\infty}^1 (0) du + \int_1^x \frac{1}{4} du$$

$$= 0 + \frac{1}{4} [u]_1^x = \frac{1}{4} (x-1)$$

c) When $x \in [5, \infty)$,

$$F(x) = \int_{-\infty}^x f(u) du$$

$$= \int_{-\infty}^1 f(u) du + \int_1^5 f(u) du + \int_5^x f(u) du$$

$$= \int_{-\infty}^1 (0) du + \int_1^5 \frac{1}{4} du + \int_5^x (0) du$$

$$= 0 + \frac{1}{4} [u]_1^5 + 0$$

$$= \frac{1}{4} [5-1] = \frac{1}{4} (4) = 1$$

$$\therefore F(x) = \begin{cases} 0, & x < 1 \\ \frac{1}{4}(x-1), & 1 \leq x < 5 \\ 1, & x \geq 5 \end{cases}$$

$$(ii) P(X < 3) = P(X \leq 3) = F(3) = \frac{1}{4} (3-1) = \frac{2}{4} = \frac{1}{2}$$

(iii) $P(2 < X < 4)$

$$= P(2 \leq X \leq 4) = F(4) - F(2)$$

$$= \left[\frac{1}{4} (4-1) \right] - \left[\frac{1}{4} (2-1) \right]$$

$$= \left[\frac{1}{4} (3) \right] - \left[\frac{1}{4} (1) \right]$$

$$= \frac{3}{4} - \frac{1}{4} = \frac{2}{4} = \frac{1}{2}$$

$$(iv) P(3 \leq X) = P(X \geq 3)$$

$$= 1 - P(X < 3)$$

$$= 1 - P(X \leq 3) = 1 - F(3) = 1 - \left[\frac{1}{4} (3-1) \right] = 1 - \left[\frac{1}{4} (2) \right] = 1 - \frac{1}{2} = \frac{1}{2}$$

Ex 11.15

Let x be a random Variable denoting the life time of an electrical equipment having Probability

density function

$$f(x) = \begin{cases} k e^{-2x} & \text{for } x > 0 \\ 0 & \text{for } x \leq 0 \end{cases}$$

Find (i) the value of k

(ii) Distribution function

(iii) $P(X < 2)$ (iv) Calculate the probability that X is at least for four unit of time

(v) $P(X = 3)$

Soln:

(i) Since $f(x)$ is a probability density function is

$$\int_{-\infty}^{\infty} f(x) dx = 1$$

$$\int_{-\infty}^0 0 dx + \int_0^{\infty} k e^{-2x} dx = 1$$

$$0 + k \left[\frac{e^{-2x}}{-2} \right]_0^{\infty} = 1$$

$$-\frac{k}{2} \left[e^{-2(\infty)} - e^{-2(0)} \right] = 1$$

$$-\frac{k}{2} \left[\frac{1}{e^{2(\infty)}} - \frac{1}{e^{2(0)}} \right] = 1$$

$$-\frac{k}{2} \left[\frac{1}{e^{\infty}} - \frac{1}{e^0} \right] = 1$$

$$-\frac{k}{2} \left[\frac{1}{\infty} - \frac{1}{1} \right] = 1$$

$$-\frac{k}{2} [0 - 1] = 1$$

$$\frac{k}{2} = 1$$

$$\boxed{k = 2}$$

$\therefore$ The probability density function is

$$f(x) = \begin{cases} 2 e^{-2x} & \text{for } x > 0 \\ 0 & \text{for } x \leq 0 \end{cases}$$

(ii) The Distribution function

$$F(x) = P(X \leq x) = \int_{-\infty}^x f(u) du$$

a) When $x \in (-\infty, 0]$,

$$F(x) = \int_{-\infty}^x f(u) du$$

$$= \int_{-\infty}^x (0) du = 0$$

b) When $x \in (0, \infty)$,

$$F(x) = \int_{-\infty}^x f(u) du$$

$$= \int_{-\infty}^0 f(u) du + \int_0^x f(u) du$$

$$= \int_{-\infty}^0 (0) du + \int_0^x 2 e^{-2u} du$$

$$= 0 + 2 \left[\frac{e^{-2u}}{-2} \right]_0^x$$

$$= \left[-e^{-2u} \right]_0^x$$

$$= (-e^{-2x}) - (-e^{-2(0)})$$

$$= -e^{-2x} + e^0$$

$$= -e^{-2x} + 1$$

$$= 1 - e^{-2x}$$

$$\therefore F(x) = \begin{cases} 0 & \text{for } x \leq 0 \\ 1 - e^{-2x} & \text{for } x > 0 \end{cases}$$

$$(iii) P(X < 2) = P(X \leq 2)$$

$$= F(2) = 1 - e^{-2(2)}$$

$$= 1 - e^{-4}$$

$$(iv) P(X \geq 4) = 1 - P(X < 4)$$

$$= 1 - P(X \leq 4)$$

$$= 1 - F(4)$$

$$= 1 - [1 - e^{-2(4)}]$$

$$= 1 - 1 + e^{-8} = e^{-8}$$

(v) In the Continuous case, $f(x)$ at $x=a$ is not the probability that X takes the value a .
 (ie) $f(x)$ at $x=a$ is not equal to $P(X=a)$. If X is Continuous type, $P(X=a)=0$ for $a \in R$.
 $\therefore P(X=3)=0$

11.5 Mathematical Expectation

EXERCISE 11.4

1) For the random variable X with the given probability mass function as below, find the mean and variance

$$(i) f(x) = \begin{cases} \frac{1}{10}, & x=2,5 \\ \frac{1}{5}, & x=0,1,3,4 \end{cases}$$

Soln:

x	0	1	2	3	4	5
$f(x)$	$\frac{1}{5}$	$\frac{1}{5}$	$\frac{1}{10}$	$\frac{1}{5}$	$\frac{1}{5}$	$\frac{1}{10}$
$xf(x)$	0	$\frac{1}{5}$	$\frac{2}{10}$	$\frac{3}{5}$	$\frac{4}{5}$	$\frac{5}{10}$
$x^2f(x)$	0	$\frac{1}{5}$	$\frac{4}{10}$	$\frac{9}{5}$	$\frac{16}{5}$	$\frac{25}{10}$

Mean:

$$\begin{aligned} E(X) &= \sum xf(x) \\ &= 0 + \frac{1}{5} + \frac{2}{10} + \frac{3}{5} + \frac{4}{5} + \frac{5}{10} \\ &= \frac{2+2+6+8+5}{10} \\ &= \frac{23}{10} = 2.3 \end{aligned}$$

$$\therefore \text{Mean } E(X) = 2.3$$

Variance:

$$\begin{aligned} E(X^2) &= \sum x^2f(x) \\ &= 0 + \frac{1}{5} + \frac{4}{10} + \frac{9}{5} + \frac{16}{5} + \frac{25}{10} \\ &= \frac{2+4+18+32+25}{10} \end{aligned}$$

$$= \frac{81}{10} = 8.1$$

$$\begin{aligned} \text{Var}(X) &= E(X^2) - [E(X)]^2 \\ &= 8.1 - (2.3)^2 \\ &= 8.1 - 5.29 \end{aligned}$$

$$\text{Var}(X) = 2.81$$

$$\therefore \text{Mean} = 2.3$$

$$\text{Variance} = 2.81$$

$$(ii) f(x) = \begin{cases} \frac{4-x}{6}, & x=1,2,3 \end{cases}$$

Soln:

x	1	2	3
$f(x)$	$\frac{3}{6}$	$\frac{2}{6}$	$\frac{1}{6}$
$xf(x)$	$\frac{3}{6}$	$\frac{4}{6}$	$\frac{3}{6}$
$x^2f(x)$	$\frac{3}{6}$	$\frac{8}{6}$	$\frac{9}{6}$

Mean:

$$\begin{aligned} E(X) &= \sum xf(x) \\ &= \frac{3}{6} + \frac{4}{6} + \frac{3}{6} \\ &= \frac{10}{6} = 1.666 \end{aligned}$$

$$E(X) = 1.67$$

Variance:

$$\begin{aligned} E(X^2) &= \sum x^2f(x) \\ &= \frac{3}{6} + \frac{8}{6} + \frac{9}{6} \\ &= \frac{20}{6} \end{aligned}$$

$$E(X^2) = \frac{20}{6}$$

$$\begin{aligned} \text{Var}(X) &= E(X^2) - [E(X)]^2 \\ &= \frac{20}{6} - \left(\frac{10}{6}\right)^2 \\ &= \frac{20}{6} - \frac{100}{36} \\ &= \frac{120-100}{36} = \frac{20}{36} = \frac{5}{9} \\ &= 0.555 = 0.56 \end{aligned}$$

$$\therefore \text{Mean} = 1.67$$

$$\text{Variance} = 0.56$$

$$(iii) f(x) = \begin{cases} 2(x-1), & 1 < x < 2 \\ 0, & \text{otherwise} \end{cases}$$

Soln: Mean:

$$E(X) = \int_{-\infty}^{\infty} x f(x) dx$$

$$= \int_1^2 x \cdot 2(x-1) dx$$

$$= 2 \int_1^2 (x^2 - x) dx$$

$$= 2 \left[\frac{x^3}{3} - \frac{x^2}{2} \right]_1^2$$

$$= 2 \left[\left[\frac{(2)^3}{3} - \frac{(2)^2}{2} \right] - \left[\frac{(1)^3}{3} - \frac{(1)^2}{2} \right] \right]$$

$$= 2 \left[\frac{8}{3} - \frac{4}{2} - \frac{1}{3} + \frac{1}{2} \right]$$

$$= 2 \left[\frac{7}{3} - \frac{3}{2} \right]$$

$$= \frac{14}{3} - 3$$

$$= \frac{14-9}{3} = \frac{5}{3}$$

$$E(X) = \frac{5}{3}$$

Variance:

$$E(X^2) = \int_{-\infty}^{\infty} x^2 f(x) dx$$

$$= \int_1^2 x^2 \cdot 2(x-1) dx$$

$$= 2 \int_1^2 (x^3 - x^2) dx$$

$$= 2 \left[\frac{x^4}{4} - \frac{x^3}{3} \right]_1^2$$

$$= 2 \left[\left[\frac{(2)^4}{4} - \frac{(2)^3}{3} \right] - \left[\frac{(1)^4}{4} - \frac{(1)^3}{3} \right] \right]$$

$$= 2 \left[\frac{16}{4} - \frac{8}{3} - \frac{1}{4} + \frac{1}{3} \right]$$

$$= 2 \left[\frac{15}{4} - \frac{7}{3} \right]$$

$$= \frac{15}{2} - \frac{14}{3}$$

$$= \frac{45-28}{6} = \frac{17}{6}$$

$$E(X^2) = \frac{17}{6}$$

$$\text{Var}(X) = E(X^2) - [E(X)]^2$$

$$= \frac{17}{6} - \left(\frac{5}{3} \right)^2$$

$$= \frac{17}{6} - \frac{25}{9}$$

$$= \frac{51-50}{18} = \frac{1}{18}$$

$$\text{Var}(X) = \frac{1}{18}$$

$$\therefore \text{Mean} = \frac{5}{3}$$

$$\text{Variance} = \frac{1}{18}$$

$$(iv) f(x) = \begin{cases} \frac{1}{2} e^{-\frac{x}{2}} & \text{for } x > 0 \\ 0, & \text{otherwise} \end{cases}$$

Soln:

using formula,

$$\int_0^{\infty} x^n e^{-ax} dx = \frac{n!}{a^{n+1}}$$

Mean:

$$E(X) = \int_{-\infty}^{\infty} x f(x) dx$$

$$= \int_0^{\infty} x \cdot \frac{1}{2} e^{-\frac{x}{2}} dx$$

$$= \frac{1}{2} \int_0^{\infty} x e^{-\frac{x}{2}} dx$$

$$= \frac{1}{2} \frac{1!}{\left(\frac{1}{2}\right)^{1+1}} \quad [\because n=1, a=\frac{1}{2}]$$

$$= \frac{1}{2} \frac{1}{\left(\frac{1}{2}\right)^2} = \frac{1}{2} \cdot \frac{1}{\frac{1}{4}}$$

$$= \frac{1}{2} \frac{4^2}{1} = \frac{2}{1} = 2$$

$$\therefore E(X) = 2$$

variance:

$$E(X^2) = \int_{-\infty}^{\infty} x^2 f(x) dx$$

$$= \int_0^{\infty} x^2 \frac{1}{2} e^{-\frac{x}{2}} dx$$

$$= \frac{1}{2} \int_0^{\infty} x^2 e^{-\frac{x}{2}} dx$$

$$= \frac{1}{2} \frac{2!}{\left(\frac{1}{2}\right)^{2+1}} \quad \left[\because n=2, a=\frac{1}{2} \right]$$

$$= \frac{1}{2} \frac{2}{\left(\frac{1}{2}\right)^3}$$

$$= \frac{1}{2} \frac{2!}{\frac{1}{8}}$$

$$= \frac{1}{2} \frac{8}{1} = \frac{8}{1} = 8$$

$$E(X^2) = 8$$

$$\begin{aligned} \text{Var}(X) &= E(X^2) - [E(X)]^2 \\ &= 8 - (2)^2 \\ &= 8 - 4 \end{aligned}$$

$$\text{Var}(X) = 4$$

$$\therefore \text{Mean} = 2$$

$$\text{Variance} = 4$$

2) Two balls are drawn in succession without replacement from an urn containing four red balls and three black balls. Let X be the possible outcomes drawing red balls. Find the Probability mass function and mean for X

Soln: Let X be the random variable of drawing number of red balls in two

draws.

$\therefore X$ Can take the values are 0, 1 and 2

Red Ball	Black Ball	Total Balls
4	3	7

$$f(0) = P(X=0) = \frac{{}^4C_0 \times {}^3C_2}{{}^7C_2}$$

$$= \frac{1 \times 3!}{2!1!}$$

$$f(0) = \frac{1}{7}$$

$$f(1) = P(X=1) = \frac{{}^4C_1 \times {}^3C_1}{{}^7C_2}$$

$$= \frac{4 \times 3!}{2!1!}$$

$$f(1) = \frac{4}{7}$$

$$\begin{aligned} f(2) &= P(X=2) = \frac{{}^4C_2 \times {}^3C_0}{{}^7C_2} \\ &= \frac{2!6 \times 1}{2!1!} \end{aligned}$$

$$f(2) = \frac{2}{7}$$

$\therefore$ The Probability mass function is

x	0	1	2
$f(x)$	$\frac{1}{7}$	$\frac{4}{7}$	$\frac{2}{7}$

Mean:

$$E(X) = \sum x f(x)$$

$$= 0 + \frac{4}{7} + \frac{4}{7}$$

$$= \frac{8}{7}$$

$$E(X) = \frac{8}{7}$$

$$\therefore \text{Mean} = \frac{8}{7}$$

3) If μ and σ^2 are the mean and variance of the discrete random variable X , and $E(X+3) = 10$ and $E(X+3)^2 = 116$, find μ and σ^2

Soln: Given:

$$E(X+3) = 10 \text{ and}$$

$$E(X+3)^2 = 116$$

Using Property,

$$E(ax+b) = aE(X) + b$$

$$E(X+3) = 10$$

$$E(X) + 3 = 10$$

$$E(X) = 10 - 3$$

$$\mu = E(X) = 7$$

$$\therefore \mu = 7$$

$$\text{and } E(X+3)^2 = 116$$

$$E(X^2 + 6X + 9) = 116$$

$$E(X^2) + 6E(X) + 9 = 116$$

$$E(X^2) + 6(7) + 9 = 116$$

$$E(X^2) + 42 + 9 = 116$$

$$E(X^2) + 51 = 116$$

$$E(X^2) = 116 - 51$$

$$E(X^2) = 65$$

$$\therefore \sigma^2 = \text{Var}(X) = E(X^2) - [E(X)]^2$$

$$= 65 - (7)^2$$

$$= 65 - 49$$

$$\sigma^2 = 16$$

$$\therefore \mu = 7 \text{ and } \sigma^2 = 16$$

4) Four fair coins are tossed once. Find the Probability mass function, mean and variance for number of heads occurred

Soln:

Let X be the random variable denotes number of heads

$\therefore X$ Can take values are 0, 1, 2, 3 and 4

$$n(s) = 2^4 = 16$$

$$f(0) = P(X=0) = \frac{{}^4C_0}{16} = \frac{1}{16}$$

$$f(1) = P(X=1) = \frac{{}^4C_1}{16} = \frac{4}{16} = \frac{1}{4}$$

$$f(2) = P(X=2) = \frac{{}^4C_2}{16} = \frac{6}{16} = \frac{3}{8}$$

$$f(3) = P(X=3) = \frac{{}^4C_3}{16} = \frac{4}{16} = \frac{1}{4}$$

$$f(4) = P(X=4) = \frac{{}^4C_4}{16} = \frac{1}{16}$$

The probability mass function is

x	0	1	2	3	4
$f(x)$	$\frac{1}{16}$	$\frac{1}{4}$	$\frac{3}{8}$	$\frac{1}{4}$	$\frac{1}{16}$

Also,

x	0	1	2	3	4
$f(x)$	$\frac{1}{16}$	$\frac{1}{4}$	$\frac{3}{8}$	$\frac{1}{4}$	$\frac{1}{16}$
$xf(x)$	0	$\frac{1}{4}$	$\frac{6}{8}$	$\frac{3}{4}$	$\frac{4}{16}$
$x^2f(x)$	0	$\frac{1}{4}$	$\frac{12}{8}$	$\frac{9}{4}$	$\frac{16}{16}$

Mean:

$$E(X) = \sum x f(x)$$

$$= 0 + \frac{1}{4} + \frac{6}{8} + \frac{3}{4} + \frac{4}{16}$$

$$= \frac{4 + 4 + 6 + 3 + 1}{16}$$

$$= \frac{18}{16}$$

$$E(X) = 2$$

Variance:

$$E(X^2) = \sum x^2 f(x)$$

$$= 0 + \frac{1}{4} + \frac{12}{8} + \frac{9}{4} + \frac{16}{16}$$

$$= \frac{4 + 4 + 12 + 9 + 1}{16}$$

$$= \frac{80.5}{16}$$

$$E(x^2) = 5$$

$$\text{Var}(x) = E(x^2) - [E(x)]^2$$

$$\text{Var}(x) = 5 - (2)^2 = 5 - 4 = 1$$

$$\therefore \text{var}(x) = 1$$

$$\therefore \text{Mean} = 2$$

$$\text{variance} = 1$$

5) A Commuter train arrives punctually at a station every half hour. Each morning, a student leaves his house to the train station. Let X denote the amount of time, in minutes, that the student waits for the train from the time he reaches the train station. It is known that

the pdf of X is

$$f(x) = \begin{cases} \frac{1}{30}, & 0 < x < 30 \\ 0, & \text{elsewhere} \end{cases}$$

Obtain and interpret the expected value of the random variable X .

Soln:

Let X be the random variable denotes the waiting time.

X is continuous on $(0, 30)$

Mean:

$$\begin{aligned} E(X) &= \int_{-\infty}^{\infty} x f(x) dx \\ &= \int_0^{30} x \frac{1}{30} dx = \frac{1}{30} \int_0^{30} x dx \\ &= \frac{1}{30} \left[\frac{x^2}{2} \right]_0^{30} \end{aligned}$$

$$= \frac{1}{30} \left[\frac{(30)^2}{2} - 0 \right]$$

$$= \frac{1}{30} \left[\frac{(30)^2}{2} \right]$$

$$= \frac{30 \cdot 15}{2}$$

$$E(X) = 15$$

$\therefore$ The expected value of waiting time is 15 minutes

6) The time to failure in thousands of hours of an electronic equipment used in a manufactured computer has the density function

$$f(x) = \begin{cases} 3e^{-3x}, & x > 0 \\ 0, & \text{elsewhere} \end{cases}$$

Find the expected life

of this electronic equipment

Soln:

Using formula,

$$\int_0^{\infty} x^n e^{-ax} dx = \frac{n!}{a^{n+1}}$$

Mean:

$$E(X) = \int_{-\infty}^{\infty} x f(x) dx$$

$$= \int_0^{\infty} x 3e^{-3x} dx$$

$$= 3 \int_0^{\infty} x e^{-3x} dx$$

$$= 3 \frac{1!}{(3)^{1+1}}$$

$$= 3 \frac{1}{(3)^2} = \frac{1}{3}$$

$$E(X) = \frac{1}{3}$$

Variance:

$$\begin{aligned}
 E(x^2) &= \int_{-\infty}^{\infty} x^2 f(x) dx \\
 &= \int_0^{\infty} x^2 3e^{-3x} dx \\
 &= 3 \int_0^{\infty} x^2 e^{-3x} dx \\
 &= 3 \frac{2!}{(3)^{2+1}} \\
 &= 3 \frac{2}{(3)^3} = \frac{2}{9}
 \end{aligned}$$

$$E(x^2) = \frac{2}{9}$$

$$\begin{aligned}
 \text{Var}(x) &= E(x^2) - [E(x)]^2 \\
 &= \frac{2}{9} - \left(\frac{1}{3}\right)^2 \\
 &= \frac{2}{9} - \frac{1}{9} = \frac{1}{9} \\
 \text{Var}(x) &= \frac{1}{9}
 \end{aligned}$$

[But $E(x^2)$ and variance are not asked in question
So Book answer is given
 $\Rightarrow \frac{1}{3}$ and $\frac{2}{9}$]

1) The Probability density function of the random variable X is given by $f(x) = \begin{cases} 16x e^{-4x} & \text{for } x > 0 \\ 0, & \text{for } x \leq 0 \end{cases}$
Find the mean and variance of x

Soln: Using Formula

$$\int_0^{\infty} x^n e^{-ax} dx = \frac{n!}{a^{n+1}}$$

Mean:

$$E(x) = \int_{-\infty}^{\infty} x f(x) dx$$

$$\begin{aligned}
 &= \int_0^{\infty} x 16x e^{-4x} dx \\
 &= 16 \int_0^{\infty} x^2 e^{-4x} dx
 \end{aligned}$$

$$= 16 \frac{2!}{(4)^{2+1}}$$

$$= 16 \frac{2}{(4)^3} = 16 \frac{2}{16 \times 4} = \frac{1}{2}$$

$$E(x) = \frac{1}{2}$$

Variance:

$$\begin{aligned}
 E(x^2) &= \int_{-\infty}^{\infty} x^2 f(x) dx \\
 &= \int_0^{\infty} x^2 16x e^{-4x} dx \\
 &= 16 \int_0^{\infty} x^3 e^{-4x} dx
 \end{aligned}$$

$$= 16 \frac{3!}{(4)^{3+1}}$$

$$= 16 \frac{6}{(4)^4}$$

$$= 16 \frac{6 \times 3}{16 \times 16 \times 8}$$

$$E(x^2) = \frac{3}{8}$$

$$\text{Var}(x) = E(x^2) - [E(x)]^2$$

$$= \frac{3}{8} - \left(\frac{1}{2}\right)^2$$

$$= \frac{3}{8} - \frac{1}{4}$$

$$= \frac{3-2}{8}$$

$$\text{Var}(x) = \frac{1}{8}$$

$$\therefore \text{Mean} = \frac{1}{2}, \text{Variance} = \frac{1}{8}$$

8) A lottery with 600 tickets gives one prize of ₹200, four prizes of ₹100, and six prizes of ₹50. If the ticket costs is ₹2. find the expected winning amount of a ticket.

Soln: Given:

Total number of lottery tickets = 600

Let X be the random variable and X denote the winning Amount

∴ X can take the values are 200, 100, 50, and 0

$$f(200) = P(X=200) = \frac{1}{600}$$

$$f(100) = P(X=100) = \frac{4}{600}$$

$$f(50) = P(X=50) = \frac{6}{600}$$

$$f(0) = P(X=0) = \frac{589}{600}$$

The Probability mass function is

x	200	100	50	0
$f(x)$	$\frac{1}{600}$	$\frac{4}{600}$	$\frac{6}{600}$	$\frac{589}{600}$

Mean:

$$E(X) = \sum x f(x)$$

$$= 200\left(\frac{1}{600}\right) + 100\left(\frac{4}{600}\right) + 50\left(\frac{6}{600}\right) + 0\left(\frac{589}{600}\right)$$

$$= \frac{200}{600} + \frac{400}{600} + \frac{300}{600}$$

$$= \frac{900}{600} = \frac{3}{2} = 1.50$$

The Expected winning amount = ₹1.50
one lottery ticket cost = ₹2.00

$$\text{Profit} = 1.50 - 2.00$$

$$\text{Profit} = -0.50$$

$$\therefore \text{loss} = ₹0.50$$

EX 11.16

Suppose that $f(x)$ given below represents a probability mass function,

x	1	2	3	4	5	6
$f(x)$	c^2	$2c^2$	$3c^2$	$4c^2$	c	$2c$

Find (i) the value of c

(ii) Mean and variance

Soln: Since $f(x)$ is a Probability mass function is

$$\sum f(x) = 1$$

$$c^2 + 2c^2 + 3c^2 + 4c^2 + c + 2c = 1$$

$$10c^2 + 3c - 1 = 0$$

$$10c^2 + 5c - 2c - 1 = 0$$

$$5c(2c+1) - 1(2c+1) = 0$$

$$(5c-1)(2c+1) = 0$$

$$5c-1=0, 2c+1=0$$

$$c = \frac{1}{5}, c = -\frac{1}{2}$$

Since $f(x) \geq 0 \forall x$
∴ The possible value of c is $\frac{1}{5}$

But $c = -\frac{1}{2}$ is not possible.

The probability mass function is

x	1	2	3	4	5	6
$f(x)$	$\frac{1}{25}$	$\frac{2}{25}$	$\frac{3}{25}$	$\frac{4}{25}$	$\frac{1}{5}$	$\frac{2}{5}$

(ii)

X	1	2	3	4	5	6	Total
f(x)	$\frac{1}{25}$	$\frac{2}{25}$	$\frac{3}{25}$	$\frac{4}{25}$	$\frac{1}{5}$	$\frac{2}{5}$	$\sum f(x) = 1$
$xf(x)$	$\frac{1}{25}$	$\frac{4}{25}$	$\frac{9}{25}$	$\frac{16}{25}$	$\frac{5}{5}$	$\frac{12}{5}$	$\sum xf(x) = \frac{115}{25}$
$x^2f(x)$	$\frac{1}{25}$	$\frac{8}{25}$	$\frac{27}{25}$	$\frac{64}{25}$	$\frac{25}{5}$	$\frac{72}{5}$	$\sum x^2f(x) = \frac{585}{25}$

Mean:

$$E(X) = \sum xf(x) = \frac{115}{25}$$

$$E(X) = 4.6$$

Variance:

$$E(X^2) = \sum x^2f(x) = \frac{585}{25}$$

$$E(X^2) = 23.4$$

$$\text{Var}(X) = E(X^2) - [E(X)]^2$$

$$= 23.4 - (4.6)^2$$

$$= 23.4 - 21.16$$

$$\text{Var}(X) = 2.24$$

$\therefore$ Mean = 4.6
Variance = 2.24

EX 11.17

Two balls are chosen randomly from an urn containing 8 white and 4 black balls. Suppose that we win Rs 20 for each black ball selected and we lose Rs 10 for each white ball selected. Find the expected winning amount and variance.

Soln:

Black Ball	White Ball	Total Ball
4	8	12

Let X be the random variable.

Let X denote the winning amount.

X (both are black balls)
 $= ₹ 2(20) = ₹ 40$
 X (one black and one white ball)
 $= ₹ 20 - ₹ 10 = ₹ 10$
 X (both are white balls)
 $= ₹ (-20) = - ₹ 20$
 $\therefore$ X can take the values are 40, 10 and -20

$$f(40) = P(X=40) = \frac{{}^4C_2}{{}^{12}C_2}$$

$$f(40) = \frac{6}{66}$$

$$f(10) = P(X=10) = \frac{{}^4C_1 \times {}^8C_1}{{}^{12}C_2}$$

$$= \frac{4 \times 8}{66} = \frac{32}{66}$$

$$f(10) = \frac{32}{66}$$

$$f(-20) = P(X=-20) = \frac{{}^8C_2}{{}^{12}C_2}$$

$$f(-20) = \frac{28}{66}$$

Values of Random variable X	40	10	-20	Total
Number of elements in inverse images	6	32	28	66
Probability mass function is				
X	40	10	-20	Total
f(x)	$\frac{6}{66}$	$\frac{32}{66}$	$\frac{28}{66}$	1

Mean:

$$E(X) = \sum xf(x)$$

$$= \frac{240}{66} + \frac{320}{66} - \frac{560}{66}$$

$$= \frac{560 - 560}{66} = \frac{0}{66}$$

$$= 0$$

$\therefore E(X) = 0$
 $\therefore$ The Expected winning amount is 0

Variance:

$$E(x^2) = \sum x^2 f(x)$$

$$= \frac{9600}{66} + \frac{3200}{66} + \frac{11200}{66}$$

$$= \frac{4000}{11}$$

$$E(x^2) = \frac{4000}{11}$$

$$\text{Var}(x) = E(x^2) - [E(x)]^2$$

$$= \frac{4000}{11} - (0)^2$$

$$\text{Var}(x) = \frac{4000}{11}$$

$$\therefore \text{Mean} = 0$$

$$\text{Variance} = \frac{4000}{11}$$

EX 11.18

Find the Mean and variance of a random variable x , whose Probability density

function is $f(x) = \begin{cases} \lambda e^{-\lambda x}, & \text{for } x \geq 0 \\ 0, & \text{otherwise} \end{cases}$

Soln:

Using formula,

$$\int_0^{\infty} x^n e^{-ax} dx = \frac{n!}{a^{n+1}}$$

Mean:

$$E(x) = \int_{-\infty}^{\infty} x f(x) dx$$

$$= \int_{-\infty}^0 x(0) dx + \int_0^{\infty} x \lambda e^{-\lambda x} dx$$

$$= 0 + \lambda \int_0^{\infty} x e^{-\lambda x} dx$$

$$= \lambda \frac{1!}{\lambda^{1+1}} = \lambda \frac{1}{\lambda^2} = \frac{1}{\lambda}$$

$$\therefore E(x) = \frac{1}{\lambda}$$

Variance:

$$E(x^2) = \int_{-\infty}^{\infty} x^2 f(x) dx$$

$$= \int_{-\infty}^0 x^2(0) dx + \int_0^{\infty} x^2 \lambda e^{-\lambda x} dx$$

$$= 0 + \lambda \int_0^{\infty} x^2 e^{-\lambda x} dx$$

$$= \lambda \frac{2!}{\lambda^{2+1}} = \lambda \frac{2}{\lambda^3}$$

$$E(x^2) = \frac{2}{\lambda^2}$$

$$\text{Var}(x) = E(x^2) - [E(x)]^2$$

$$= \frac{2}{\lambda^2} - \left(\frac{1}{\lambda}\right)^2$$

$$= \frac{2}{\lambda^2} - \frac{1}{\lambda^2}$$

$$\text{Var}(x) = \frac{1}{\lambda^2}$$

$$\therefore \text{Mean} = \frac{1}{\lambda} \text{ and variance} = \frac{1}{\lambda^2}$$

11.6 Theoretical Distributions: Some Special Discrete Distributions

EXERCISE 11.5

1) Compute $P(X=k)$ for the binomial distribution $B(n, p)$ where

(i) $n=6, p=\frac{1}{3}, k=3$

(ii) $n=10, p=\frac{1}{5}, k=4$

(iii) $n=9, p=\frac{1}{2}, k=7$

Soln:

(i) The Binomial distribution is

$$f(x) = P(X=x) = \binom{n}{x} p^x (1-p)^{n-x}$$

$$\text{Put } x=k$$

$$P(X=k) = \binom{n}{k} p^k (1-p)^{n-k}$$

Given: $n=6, p=\frac{1}{3}$ and $k=3$

$$P(X=3) = \binom{6}{3} \left(\frac{1}{3}\right)^3 \left(1-\frac{1}{3}\right)^{6-3}$$

$$= 20 \left(\frac{1}{3}\right)^3 \left(\frac{2}{3}\right)^3$$

$= 20 \left(\frac{1}{3^3} \right) \left(\frac{2^3}{3^3} \right)$ $= 20 \left(\frac{1}{27} \right) \left(\frac{8}{27} \right)$ $= \frac{160}{729} \Rightarrow P(X=3) = \frac{160}{729}$	put $x=k$ $P(X=k) = \binom{n}{k} p^k (1-p)^{n-k}$ Given: $n=9$, $P=\frac{1}{2}$ and $K=7$ $P(X=7) = \binom{9}{7} \left(\frac{1}{2} \right)^7 \left(1 - \frac{1}{2} \right)^{9-7}$	(i) exactly 4 times (ii) atleast one time. Soln: Let X be the random variable denote number of hits Given: $P = \frac{1}{4}$ and $n=10$ $q = 1 - P = 1 - \frac{1}{4} = \frac{3}{4}$ $\therefore \boxed{q = \frac{3}{4}}$ X follows the Binomial distribution	(ii) <u>Atleast one time</u> $P(X \geq 1) = 1 - P(X < 1)$ $= 1 - P(X=0)$ $= 1 - \left[\binom{10}{0} \left(\frac{1}{4} \right)^0 \left(\frac{3}{4} \right)^{10-0} \right]$ $= 1 - \left[(1)(1) \left(\frac{3}{4} \right)^{10} \right]$ $= 1 - \left(\frac{3}{4} \right)^{10}$ $P(X \geq 1) = 1 - \frac{3^{10}}{4^{10}}$
(ii) The Binomial distribution is $f(x) = P(X=x) = \binom{n}{x} p^x (1-p)^{n-x}$ put $x=k$ $P(X=k) = \binom{n}{k} p^k (1-p)^{n-k}$ Given: $n=10$, $P=\frac{1}{5}$ and $K=4$ $P(X=4) = \binom{10}{4} \left(\frac{1}{5} \right)^4 \left(1 - \frac{1}{5} \right)^{10-4}$ $P(X=4) = 210 \left(\frac{1}{5} \right)^4 \left(\frac{4}{5} \right)^6$	$P(X=7) = 36 \left(\frac{1}{2} \right)^7 \left(\frac{1}{2} \right)^2$ [OR] $P(X=7) = 36 \left(\frac{1}{2^7} \right) \left(\frac{1}{2^2} \right)$ $= 36 \left(\frac{1}{128} \right) \left(\frac{1}{4} \right)$ $P(X=7) = \frac{9}{128}$	(i) <u>Exactly 4 times</u>: $f(4) = P(X=4) = \binom{10}{4} \left(\frac{1}{4} \right)^4 \left(1 - \frac{1}{4} \right)^{10-4}$ $P(X=4) = \binom{10}{4} \left(\frac{1}{4} \right)^4 \left(\frac{3}{4} \right)^6$	3) Using binomial distribution find the mean and variance of X for the following experiments (i) A fair coin is tossed 100 times, and x denote the number of heads (ii) A fair die is tossed
(iii) The Binomial distribution is $f(x) = P(X=x) = \binom{n}{x} p^x (1-p)^{n-x}$	2) The probability that Mr. Q hits a target at any trial is $\frac{1}{4}$. Suppose he tries at the target 10 times. Find the probability that he hits the target		

240 times, and x denote the number of times that four appeared

Soln:

(i) Given: $n=100$,

$$P = \frac{1}{2}, \quad q = 1 - P = 1 - \frac{1}{2}$$

$$\boxed{q = \frac{1}{2}}$$

$$\therefore \text{Mean} = np = (100)\left(\frac{1}{2}\right)$$

$$\therefore \text{Mean} = 50$$

$$\text{Variance} = npq = (100)\left(\frac{1}{2}\right)\left(\frac{1}{2}\right)$$

$$= \frac{100 \times 25}{4} = 25$$

$$\therefore \text{Variance} = 25$$

$$\therefore \text{Mean} = 50, \text{variance} = 25$$

(ii) Given: $n=240$

$$P = \frac{1}{6}, \quad q = 1 - P = 1 - \frac{1}{6}$$

$$\boxed{q = \frac{5}{6}}$$

$$\therefore \text{Mean} = np = (240)\left(\frac{1}{6}\right)$$

$$\therefore \text{Mean} = 40$$

$$\text{Variance} = npq = (240)\left(\frac{1}{6}\right)\left(\frac{5}{6}\right)$$

$$= \frac{100}{3}$$

$$\therefore \text{variance} = \frac{100}{3}$$

$$\therefore \text{Mean} = 40, \text{Variance} = \frac{100}{3}$$

4) The probability that a certain kind of component will survive a electrical test is $\frac{3}{4}$. Find the probability that exactly 3 of the 5 components tested survive.

Soln:

Let x be the random variable denotes number of survive components

Given: $P = \frac{3}{4}$ and $n=5$

$$q = 1 - P = 1 - \frac{3}{4} = \frac{1}{4}$$

$$\boxed{q = \frac{1}{4}}$$

x follows the Binomial distribution denoted by $X \sim B(5, \frac{3}{4})$

$$f(x) = P(X=x) = \binom{n}{x} P^x (1-P)^{n-x}$$

$$x = 0, 1, 2, \dots, 5$$

$$P(X=3) = \binom{5}{3} \left(\frac{3}{4}\right)^3 \left(1 - \frac{3}{4}\right)^{5-3}$$

$$= 10 \left(\frac{3}{4}\right)^3 \left(\frac{1}{4}\right)^2$$

$$= 10 \left(\frac{27}{64}\right) \left(\frac{1}{16}\right)$$

$$P(X=3) = \frac{270}{1024}$$

5) A retailer purchase a certain kind of electronic device from a manufacturer. The manufacturer indicates that the defective rate of the device is 5%. The

inspector of the retailer randomly picks 10 items from a shipment. What is the probability that there will be (i) atleast one defective item (ii) exactly two defective items.

Soln:

Let x be the random variable denotes number of defective items

Given: $P = 5\%$ and $n=10$

$$P = \frac{5}{100} = 0.05 \Rightarrow \boxed{P = 0.05}$$

$$q = 1 - P = 1 - 0.05 = 0.95$$

$$\boxed{q = 0.95}$$

x follows the Binomial distribution denoted by $X \sim B(10, 0.05)$

$$f(x) = P(X=x) = \binom{n}{x} P^x (1-P)^{n-x}$$

$$x = 0, 1, 2, \dots, 10$$

$$(i) P(X \geq 1) = 1 - P(X < 1)$$

$$= 1 - P(X=0)$$

$$= 1 - \binom{10}{0} (0.05)^0 (0.95)^{10}$$

$$= 1 - (1)(1)(0.95)^{10}$$

$$= 1 - (0.95)^{10}$$

$$(ii) P(X=2) = \binom{10}{2} (0.05)^2 (0.95)^8$$

$$= \binom{10}{2} (0.05)^2 (0.95)^8$$

6) If the Probability that a fluorescent light has a useful life of atleast 600 hours is 0.9. find the probabilities that among 12 such lights

(i) exactly 10 will have a useful life of atleast 600 hours

(ii) atleast 11 will have a useful life of atleast 600 hours

(iii) atleast 2 will not have a useful life of atleast 600 hours

Soln:

Let X be the random variable denotes useful life of atleast 600 hours of a light

Given: $p = 0.9$ and $n = 12$

$$q = 1 - p = 1 - 0.9 = 0.1$$

$$q = 0.1$$

X follows the Binomial distribution denoted by

$$X \sim B(12, 0.9)$$

$$f(x) = P(X=x) = \binom{n}{x} p^x (1-p)^{n-x}, \quad x=0, 1, 2, \dots, 12$$

$$(i) P(X=10) = \binom{12}{10} (0.9)^{10} (0.1)^{2}$$

$$= \binom{12}{10} (0.9)^{10} (0.1)^2$$

$$(ii) P(X \geq 11) = P(X=11) + P(X=12)$$

$$= \binom{12}{11} (0.9)^{11} (0.1)^{1} + \binom{12}{12} (0.9)^{12} (0.1)^0$$

$$= 12(0.9)^{11} (0.1) + (1)(0.9)^{12} (0.1)^0$$

$$= 12(0.9)^{11} (0.1) + (1)(0.9)^{12} (1)$$

$$= 12(0.9)^{11} (0.1) + (0.9)^{12}$$

$$= (0.9)^{11} [12(0.1) + 0.9]$$

$$= (0.9)^{11} [1.2 + 0.9]$$

$$= (0.9)^{11} [2.1]$$

$$P(X \geq 11) = 2.1 (0.9)^{11}$$

$$(iii) P(X < 11) + P(X \geq 11) = 1$$

$$P(X < 11) = 1 - P(X \geq 11)$$

$$= 1 - [P(X=11) + P(X=12)]$$

$$= 1 - \left[\binom{12}{11} (0.9)^{11} (0.1)^{1} + \binom{12}{12} (0.9)^{12} (0.1)^0 \right]$$

$$= 1 - [12(0.9)^{11} (0.1) + (1)(0.9)^{12} (0.1)^0]$$

$$= 1 - [12(0.9)^{11} (0.1) + (1)(0.9)^{12} (1)]$$

$$= 1 - [12(0.9)^{11} (0.1) + (0.9)^{12}]$$

$$= 1 - [12(0.9)^{11} (0.1) + (0.9)^{12}]$$

$$= 1 - [(0.9)^{11} [12(0.1) + 0.9]]$$

$$= 1 - [(0.9)^{11} [1.2 + 0.9]]$$

$$= 1 - [(0.9)^{11} [2.1]]$$

$$= 1 - [2.1 (0.9)^{11}]$$

$$= 1 - [2.1 (0.9)^{11}]$$

$$7) \text{ The mean and Standard deviation of a binomial}$$

Variate X are respectively 6 and 2. Find

(i) the Probability mass function (ii) $P(X=3)$

(iii) $P(X \geq 2)$

Soln: Given:

$$\text{Mean} = np = 6 \text{ and}$$

$$S.D = \sqrt{npq} = 2$$

$$\text{Variance} = npq = 4$$

$$\Rightarrow \frac{npq}{np} = \frac{4}{6}$$

$$\Rightarrow q = \frac{2}{3}$$

$$p = 1 - q = 1 - \frac{2}{3}$$

$$p = \frac{1}{3}$$

$$\Rightarrow np = 6$$

$$n \left(\frac{1}{3} \right) = 6 \Rightarrow n = 18$$

Let x be the random variable.

x follows the Binomial distribution denoted by $X \sim B(18, \frac{1}{3})$

$$f(x) = P(X=x) = \binom{n}{x} p^x (1-p)^{n-x}, \quad x=0, 1, 2, \dots, 18$$

(i) The probability mass function:

$$f(x) = P(X=x) = \binom{18}{x} \left(\frac{1}{3}\right)^x \left(1 - \frac{1}{3}\right)^{18-x}$$

$$f(x) = P(X=x) = \binom{18}{x} \left(\frac{1}{3}\right)^x \left(\frac{2}{3}\right)^{18-x}, \quad x=0, 1, 2, \dots, 18$$

$$(ii) P(X=3) = \binom{18}{3} \left(\frac{1}{3}\right)^3 \left(\frac{2}{3}\right)^{18-3}$$

$$P(X=3) = \binom{18}{3} \left(\frac{1}{3}\right)^3 \left(\frac{2}{3}\right)^{15}$$

$$(iii) P(X \geq 2) = 1 - P(X < 2)$$

$$= 1 - [P(X=0) + P(X=1)]$$

$$= 1 - \left[\binom{18}{0} \left(\frac{1}{3}\right)^0 \left(\frac{2}{3}\right)^{18-0} + \binom{18}{1} \left(\frac{1}{3}\right)^1 \left(\frac{2}{3}\right)^{18-1} \right]$$

$$= 1 - \left[(1)(1) \left(\frac{2}{3}\right)^{18} + 18 \left(\frac{1}{3}\right) \left(\frac{2}{3}\right)^{17} \right]$$

$$= 1 - \left[\left(\frac{2}{3}\right)^{18} + 18 \left(\frac{1}{3}\right) \left(\frac{2}{3}\right)^{17} \right]$$

$$= 1 - \left(\frac{2}{3}\right)^{17} \left[\frac{2}{3} + 6 \right]$$

$$= 1 - \left(\frac{2}{3}\right)^{17} \left(\frac{20}{3}\right)$$

$$= 1 - \frac{20}{3} \left(\frac{2}{3}\right)^{17}$$

8) If $X \sim B(n, p)$ such that $4P(X=4) = P(X=2)$ and $n=6$. Find the distribution, mean and standard deviation.

Soln: Given: $n=6$

Let x be the random variable

x follows the Binomial distribution denoted by $X \sim B(n, p)$

$$f(x) = P(X=x) = \binom{n}{x} p^x (1-p)^{n-x}, \quad x=0, 1, 2, \dots, n$$

$$\text{Given: } 4P(X=4) = P(X=2)$$

$$4 \left[\binom{6}{4} p^4 (1-p)^{6-4} \right] = \left[\binom{6}{2} p^2 (1-p)^{6-2} \right]$$

$$4 \left[\frac{6!}{4!2!} p^4 (1-p)^2 \right] = \left[\frac{6!}{2!4!} p^2 (1-p)^4 \right]$$

$$4p^2 = (1-p)^2$$

$$4p^2 = 1 - 2p + p^2$$

$$4p^2 - p^2 + 2p - 1 = 0$$

$$3p^2 + 2p - 1 = 0$$

$$\frac{-2 \pm \sqrt{2^2 - 4(3)(-1)}}{2(3)}$$

$$3p^2 + 2p - 1 = 0$$

$$3p(p+1) - 1(p+1) = 0$$

$$(3p-1)(p+1) = 0$$

$$3p-1=0, \quad p+1=0$$

$$p = \frac{1}{3}, \quad p = -1 \text{ is not possible}$$

$$\therefore \boxed{p = \frac{1}{3}}$$

$$\text{and } q = 1 - p = 1 - \frac{1}{3}$$

$$\boxed{q = \frac{2}{3}}$$

(i) The Binomial Distribution is

$$P(X=x) = \binom{n}{x} p^x (1-p)^{n-x}, \quad x=0, 1, 2, \dots, n$$

$$P(X=x) = \binom{6}{x} \left(\frac{1}{3}\right)^x \left(1 - \frac{1}{3}\right)^{6-x}$$

$$P(X=x) = \binom{6}{x} \left(\frac{1}{3}\right)^x \left(\frac{2}{3}\right)^{6-x}$$

$$(ii) \text{ Mean} = np = 6 \left(\frac{1}{3}\right) = 2$$

$$\text{Variance} = npq$$

$$\text{Variance} = \cancel{5} \left(\frac{1}{\cancel{5}} \right) \left(\frac{2}{3} \right) = \frac{4}{3}$$

$$\text{S.D} = \sqrt{npq} = \sqrt{\frac{4}{3}} = \frac{2}{\sqrt{3}}$$

$$\therefore \text{Mean} = 2$$

$$\text{variance} = \frac{4}{3} \text{ and }$$

$$\text{S.D} = \frac{2}{\sqrt{3}}$$

9) In a binomial distribution consisting of 5 independent trials, the probability of 1 and 2 successes are 0.4096 and 0.2048 respectively. Find the mean and variance of the distribution

Soln: Given: $n = 5$

Let X be the random variable

X follows the Binomial distribution denoted by

$$X \sim B(n, p)$$

$$f(x) = P(X=x) = \binom{n}{x} p^x (1-p)^{n-x},$$

$$x = 0, 1, 2, \dots, n$$

Given:

$$P(X=1) = 0.4096 \text{ and}$$

$$P(X=2) = 0.2048$$

$$P(X=1) = \binom{5}{1} p^1 (1-p)^{5-1} = 0.4096$$

$$\Rightarrow 5 p (1-p)^4 = 0.4096 \quad \text{--- (1)}$$

$$\text{and } P(X=2) = \binom{5}{2} p^2 (1-p)^{5-2} = 0.2048$$

$$\Rightarrow 10 p^2 (1-p)^3 = 0.2048 \quad \text{--- (2)}$$

$$\frac{(1)}{(2)} \Rightarrow \frac{5 p (1-p)^4}{2 \cdot 10 p^2 (1-p)^3} = \frac{0.4096}{0.2048}$$

$$\frac{1-p}{2p} = 2$$

$$1-p = 4p$$

$$1 = 4p + p \Rightarrow 1 = 5p$$

$$p = \frac{1}{5}, q = 1-p$$

$$q = 1 - \frac{1}{5}$$

$$q = \frac{4}{5}$$

$$\text{Mean} = np = 5 \left(\frac{1}{5} \right) = 1$$

$$\text{Variance} = npq = 5 \left(\frac{1}{5} \right) \left(\frac{4}{5} \right)$$

$$= \frac{4}{5}$$

$$\therefore \text{Mean} = 1$$

$$\text{variance} = \frac{4}{5}$$

Ex 11.19

Find the binomial distribution function for each of the following.

(i) Five fair coins are tossed once and X denotes the number of heads

(ii) A fair die is rolled 10 times and X denotes the number of times 4 appeared.

Soln:

(i) Given: $n = 5$

$$p = \frac{1}{2}, q = 1-p = 1 - \frac{1}{2}$$

$$\therefore q = \frac{1}{2}$$

Let X be the random variable denotes the number of heads

X follows the Binomial distribution denoted by $X \sim B(5, \frac{1}{2})$

$$f(x) = P(X=x) = \binom{n}{x} p^x (1-p)^{n-x},$$

$$x = 0, 1, 2, \dots, 5$$

$$f(x) = P(X=x) = \binom{5}{x} \left(\frac{1}{2}\right)^x \left(\frac{1}{2}\right)^{5-x}$$

$$= \binom{5}{x} \left(\frac{1}{2}\right)^{x+5-x}$$

$$f(x) = \binom{5}{x} \left(\frac{1}{2}\right)^5, x=0,1,2,\dots,5$$

(ii) Given: $n=10$ and
 $P = \frac{1}{6}, q = 1 - P = 1 - \frac{1}{6}$

$$q = \frac{5}{6}$$

Let X be the random variable denotes the number of times 4 appeared

X follows the Binomial distribution denoted by
 $X \sim B(10, \frac{1}{6})$

$$f(x) = P(X=x) = \binom{n}{x} p^x (1-p)^{n-x}$$

$$x=0,1,2,\dots,10$$

$$f(x) = P(X=x) = \binom{10}{x} \left(\frac{1}{6}\right)^x \left(\frac{5}{6}\right)^{10-x}$$

$$x=0,1,2,\dots,10$$

Ex 11.20

A multiple choice examination has ten questions, each question has four distractors with exactly one correct answer. Suppose a student answers by guessing and if x denotes the number of correct answers.

find (i) binomial distribution
 (ii) probability that the student will get seven correct answers (iii) the probability of getting at least one correct answer.

Soln: (i) Given: $n=10$
 let X be the random variable

X follows the Binomial distribution denoted by
 $X \sim B(10, \frac{1}{4})$

$$\therefore p = \frac{1}{4}, q = 1 - p = 1 - \frac{1}{4}$$

$$q = \frac{3}{4}$$

$$f(x) = P(X=x) = \binom{n}{x} p^x (1-p)^{n-x}$$

$$x=0,1,2,\dots,10$$

$$f(x) = P(X=x) = \binom{10}{x} \left(\frac{1}{4}\right)^x \left(\frac{3}{4}\right)^{10-x}$$

$$x=0,1,2,\dots,10$$

$$(ii) P(X=7) = \binom{10}{7} \left(\frac{1}{4}\right)^7 \left(\frac{3}{4}\right)^{10-7}$$

$$= 120 \left(\frac{1}{4}\right)^7 \left(\frac{3}{4}\right)^3 = 120 \left(\frac{1}{4}\right)^7 \left(\frac{3^3}{4^3}\right)$$

$$= 120 \left(\frac{3^3}{4^{10}}\right)$$

$$(iii) P(X \geq 1) = 1 - P(X < 1)$$

$$= 1 - P(X=0)$$

$$= 1 - \left[\binom{10}{0} \left(\frac{1}{4}\right)^0 \left(\frac{3}{4}\right)^{10-0} \right]$$

$$= 1 - \left[(1)(1) \left(\frac{3}{4}\right)^{10} \right]$$

$$= 1 - \left(\frac{3}{4}\right)^{10}$$

Ex 11.21

The mean and variance of a binomial variate X are respectively 2 and 1.5. Find (i) $P(X=0)$

(ii) $P(X=1)$ (iii) $P(X \geq 1)$

Soln: Given:

$$\text{Mean} = np = 2 \quad \text{--- (1)}$$

$$\text{Variance} = npq = 1.5 \quad \text{--- (2)}$$

② $\Rightarrow \frac{npq}{np} = \frac{1.5}{2} = \frac{3}{2}$ $= \frac{3}{2} \times \frac{1}{2}$ $q = \frac{3}{4}$ $P = 1 - q = 1 - \frac{3}{4} = \frac{1}{4}$ $\therefore P = \frac{1}{4}$ ① $\Rightarrow np = 2$ $n\left(\frac{1}{4}\right) = 2 \Rightarrow n = 8$ Let X be the random Variable. X follows the Binomial distribution denoted by $X \sim B\left(8, \frac{1}{4}\right)$ $\therefore$ The Binomial distribution is	$f(x) = P(X=x) = \binom{n}{x} p^x (1-p)^{n-x}$ $x = 0, 1, 2, \dots, 8$ $f(x) = P(X=x) = \binom{8}{x} \left(\frac{1}{4}\right)^x \left(\frac{3}{4}\right)^{8-x}$ $x = 0, 1, 2, \dots, 8$ (i) $P(X=0) = \binom{8}{0} \left(\frac{1}{4}\right)^0 \left(\frac{3}{4}\right)^{8-0}$ $= (1)(1) \left(\frac{3}{4}\right)^8$ $P(X=0) = \left(\frac{3}{4}\right)^8$ (ii) $P(X=1) = \binom{8}{1} \left(\frac{1}{4}\right)^1 \left(\frac{3}{4}\right)^{8-1}$ $= 8 \left(\frac{1}{4}\right) \left(\frac{3}{4}\right)^7$ $P(X=1) = 2 \left(\frac{3}{4}\right)^7$ (iii) $P(X \geq 1) = 1 - P(X < 1)$ $= 1 - P(X=0)$	$= 1 - \left[\binom{8}{0} \left(\frac{1}{4}\right)^0 \left(\frac{3}{4}\right)^{8-0} \right]$ $= 1 - \left[(1)(1) \left(\frac{3}{4}\right)^8 \right]$ $= 1 - \left(\frac{3}{4}\right)^8$ <u>Ex 11.22</u> on the average, 20% of the products manufactured by ABC Company are found to be defective. If we select 6 of these products at random and X denote the number of defective products find the probability that (i) two products are defective (ii) at most one product is	defective (ii) at least two products are defective Soln: Given: $n = 6$ and $p = 20\% = \frac{20}{100} = \frac{1}{5}$ $P = \frac{1}{5}, q = 1 - P$ $= 1 - \frac{1}{5}$ $q = \frac{4}{5}$ Let X be the random Variable denotes the number of defective products. X follows a Binomial distribution denoted by $X \sim B\left(6, \frac{1}{5}\right)$
--	---	---	---

$$f(x) = P(X=x) = \binom{n}{x} p^x (1-p)^{n-x}$$

$$x = 0, 1, 2, \dots, 6$$

$$(i) P(X=2) = \binom{6}{2} \left(\frac{1}{5}\right)^2 \left(\frac{4}{5}\right)^{6-2}$$

$$= 15 \left(\frac{1}{5}\right)^2 \left(\frac{4}{5}\right)^4$$

$$= 15 \left(\frac{1}{5^2}\right) \left(\frac{4^4}{5^4}\right)$$

$$= 15 \left(\frac{4^4}{5^6}\right)$$

$$(ii) P(X \leq 1) = P(X=0) + P(X=1)$$

$$= \binom{6}{0} \left(\frac{1}{5}\right)^0 \left(\frac{4}{5}\right)^{6-0} + \binom{6}{1} \left(\frac{1}{5}\right)^1 \left(\frac{4}{5}\right)^{6-1}$$

$$= (1)(1) \left(\frac{4}{5}\right)^6 + 6 \left(\frac{1}{5}\right) \left(\frac{4}{5}\right)^5$$

$$= \left(\frac{4}{5}\right)^6 + 6 \left(\frac{1}{5}\right) \left(\frac{4}{5}\right)^5$$

$$= \left(\frac{4}{5}\right)^5 \left[\frac{4}{5} + \frac{6}{5} \right]$$

$$= \left(\frac{4}{5}\right)^5 \left[\frac{10}{5} \right]$$

$$= 2 \left(\frac{4}{5}\right)^5$$

$$(iii) P(X \geq 2) = 1 - P(X < 2)$$

$$= 1 - [P(X=0) + P(X=1)]$$

$$= 1 - P(X \leq 1)$$

$$= 1 - 2 \left(\frac{4}{5}\right)^5$$

EXERCISE 11.6

Choose the correct or the most suitable answer from the given four alternatives

1) Let X be random variable with probability density function

$$f(x) = \begin{cases} \frac{2}{x^3}, & x \geq 1 \\ 0, & x < 1 \end{cases}$$

Which of the following Statement is correct
Soln:

12. Discrete Mathematics

EXERCISE 12.1

1) Determine whether $*$ is a binary operation on the sets given below.

(i) $a * b = a \cdot |b|$ on $\mathbb{R}$

(ii) $a * b = \min(a, b)$ on $A = \{1, 2, 3, 4, 5\}$

(iii) $(a * b) = a\sqrt{b}$ is binary on $\mathbb{R}$

Soln: (i) Given:

$$a * b = a \cdot |b| \text{ on } \mathbb{R}$$

Let $a, b \in \mathbb{R}$ and $|b| \in \mathbb{R}$

$$\therefore a \cdot |b| \in \mathbb{R}$$

$$\therefore a * b \in \mathbb{R}, \forall a, b \in \mathbb{R}$$

yes, $*$ is a binary operation on $\mathbb{R}$

(ii) Given:

$$a * b = \min(a, b) \text{ on } A = \{1, 2, 3, 4, 5\}$$

Let $a, b \in A$

Then $\min(a, b) \in A$

$$\therefore a * b \in A, \forall a, b \in A$$

yes, $*$ is a binary operation on A

(iii) Given:

$a * b = a\sqrt{b}$ is binary on $\mathbb{R}$

Let $a, b \in \mathbb{R}$ but

$a\sqrt{b} \notin \mathbb{R}$, Square root of negative number not in $\mathbb{R}$

No, $*$ is not a binary operation on $\mathbb{R}$

2) on $\mathbb{Z}$, define $\otimes$ by $(m \otimes n) = m^n + n^m, \forall m, n \in \mathbb{Z}$. Is $\otimes$ binary on $\mathbb{Z}$?

Soln: Given:

$$(m \otimes n) = m^n + n^m, \forall m, n \in \mathbb{Z}$$

Let $m, n \in \mathbb{Z}$

Put $m = 2, n = -2$

$$\begin{aligned} m^n + n^m &= (2)^{-2} + (-2)^2 \\ &= \frac{1}{(2)^2} + (-2)^2 \\ &= \frac{1}{4} + 4 = \frac{17}{4} \notin \mathbb{Z} \end{aligned}$$

$$\therefore m^n + n^m \notin \mathbb{Z}$$

$$\therefore (m \otimes n) \notin \mathbb{Z}$$

No, $\otimes$ is not binary on $\mathbb{Z}$

3) Let $*$ be defined on $\mathbb{R}$ by $(a * b) = a + b + ab - 7$. Is $*$ binary on $\mathbb{R}$? If

So, find $3 * \left(\frac{-7}{15}\right)$

Soln: Given:

$$(a * b) = a + b + ab - 7$$

Let $a, b \in \mathbb{R}$

Then $a + b + ab - 7 \in \mathbb{R}$

$$\therefore a * b \in \mathbb{R}$$

$\therefore *$ is a binary operation

on $\mathbb{R}$.

Also,

$$\begin{aligned} 3 * \left(\frac{-7}{15}\right) &= 3 - \frac{7}{15} + 3\left(\frac{-7}{15}\right) - 7 \\ &= 3 - \frac{7}{15} - \frac{21}{15} - 7 \\ &= \frac{45 - 7 - 21 - 105}{15} \\ &= \frac{45 - 133}{15} = -\frac{88}{15} \\ \therefore 3 * \left(\frac{-7}{15}\right) &= -\frac{88}{15} \end{aligned}$$

4) Let $A = \{a + \sqrt{5}b; a, b \in \mathbb{R}\}$. Check whether the usual multiplication is a binary operation on A

Soln: Let $x = a + \sqrt{5}b$,

$$y = c + \sqrt{5}d$$

Then $x, y \in A, \forall a, b, c, d \in \mathbb{R}$

$$xy = (a + \sqrt{5}b)(c + \sqrt{5}d)$$

$$xy = (ac + 5bd) + \sqrt{5}(ad + bc)$$

$$xy = (ac + 5bd) + \sqrt{5}(ad + bc) \in A$$

$\therefore xy \in A$ Yes, usual multiplication is binary on A	is closure on Q (ii) <u>Commutative property</u> : Let $a, b \in Q$ $a * b = \frac{a+b}{2}$ $= \frac{b+a}{2}$ $a * b = b * a, \forall a, b \in Q$ $\therefore$ Commutative property is true.	$(a * b) * c = \left(\frac{a+b}{2}\right) * c$ $= \frac{\left(\frac{a+b}{2}\right) + c}{2}$ $= \frac{a+b+2c}{2} \times \frac{1}{2}$ $(a * b) * c = \frac{a+b+2c}{4}$ ①	the operation * on Q Soln: (i) <u>Existence of identity property</u> : Let e be the identity element Let $a, e \in Q$ By the definition of e, $a * e = a$ By the definition of *, $a * e = \frac{a+e}{2}$ $a = \frac{a+e}{2}$ $2a = a+e$ $\Rightarrow e = 2a - a$ $e = a, \forall a \in Q$ This means every element is a identity element This is impossible
5) (i) Define an operation * on Q as follows: $a * b = \left(\frac{a+b}{2}\right); a, b \in Q$ Examine the closure, commutative, and associative properties satisfied by * on Q Soln: (i) <u>closure property</u> : Let $a, b \in Q$ Then $a+b \in Q$ $\therefore \frac{a+b}{2} \in Q$ $\therefore a * b \in Q, \forall a, b \in Q$ $\therefore$ closure property is true $\therefore$ The given operation *	$\therefore$ The given operation * is commutative on Q (iii) <u>Associative property</u> : Let $a, b, c \in Q$ $a * (b * c) = a * \left(\frac{b+c}{2}\right)$ $= \frac{a + \left(\frac{b+c}{2}\right)}{2}$ $= \frac{2a + b + c}{2} \times \frac{1}{2}$ $a * (b * c) = \frac{2a + b + c}{4}$ ②	From ① & ② we get, $a * (b * c) \neq (a * b) * c$ $\therefore$ The given operation * is not associative on Q (ii) Define an operation * on Q as follows: $a * b = \left(\frac{a+b}{2}\right); a, b \in Q$ Examine the existence of identity and the existence of inverse for	

$\therefore$ Identity does not exist
It has no Identity element

(ii) Existence of Inverse Property

The given operation $*$ defined has no identity element

$\therefore a * \bar{a} = \bar{a} * a = e$ is cannot be defined.

The given operation $*$ defined has no inverse

$\therefore$ Inverse does not exist

6) Fill in the following table so that the binary operation $*$ on $A = \{a, b, c\}$ is commutative.

$*$	a	b	c
a	b		
b	c	b	a
c	a		c

Soln:

Given: $*$ is commutative

From the table,

$$b * a = c \Rightarrow a * b = c$$

$$\text{Also, } c * a = a \Rightarrow a * c = a$$

$$b * c = a \Rightarrow c * b = a$$

From the table,

$*$	a	b	c
a	b	c	a
b	c	b	a
c	a	a	c

7) Consider the binary operation $*$ defined on the set $A = \{a, b, c, d\}$ by the following table:

$*$	a	b	c	d
a	a	c	b	d
b	d	a	b	c
c	c	d	a	a
d	d	b	a	c

Is it commutative and associative?

Soln:

(i) Commutative property:

$$\text{Let } a, b, c, d \in A$$

$$a * b = c, b * a = d$$

$$\therefore a * b \neq b * a$$

$$\text{Also, } b * c = b, c * b = d$$

$$\therefore b * c \neq c * b$$

NO. The given operation $*$ is not commutative

(ii) Associative property:

$$\text{Let } a, b, c \in A$$

$$a * (b * c) = a * b = c \text{ and } (a * b) * c = c * c = a$$

$\therefore a * (b * c) \neq (a * b) * c$
NO. The given operation $*$ is not associative.

$$8) \text{ Let } A = \begin{bmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ 1 & 0 & 0 & 1 \end{bmatrix}$$

$$B = \begin{bmatrix} 0 & 1 & 0 & 1 \\ 1 & 0 & 1 & 0 \\ 1 & 0 & 0 & 1 \end{bmatrix}$$

$$C = \begin{bmatrix} 1 & 1 & 0 & 1 \\ 0 & 1 & 1 & 0 \\ 1 & 1 & 1 & 1 \end{bmatrix} \text{ be any three boolean matrices of the same type. Find (i) } A \vee B$$

(ii) $A \wedge B$ (iii) $(A \vee B) \wedge C$

(iv) $(A \wedge B) \vee C$

Soln: (i)

$$A \vee B = \begin{bmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ 1 & 0 & 0 & 1 \end{bmatrix} \vee \begin{bmatrix} 0 & 1 & 0 & 1 \\ 1 & 0 & 1 & 0 \\ 1 & 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \\ 1 & 0 & 0 & 1 \end{bmatrix}$$

(ii) $A \wedge B$

$$A \wedge B = \begin{bmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ 1 & 0 & 0 & 1 \end{bmatrix} \wedge \begin{bmatrix} 0 & 1 & 0 & 1 \\ 1 & 0 & 1 & 0 \\ 1 & 0 & 0 & 1 \end{bmatrix}$$

$$A \wedge B = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 1 \end{bmatrix}$$

(iii) $(A \vee B) \wedge C$

$$(A \vee B) \wedge C = \left[\begin{bmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ 1 & 0 & 0 & 1 \end{bmatrix} \vee \begin{bmatrix} 0 & 1 & 0 & 1 \\ 1 & 0 & 1 & 0 \\ 1 & 0 & 0 & 1 \end{bmatrix} \right] \wedge \begin{bmatrix} 1 & 1 & 0 & 1 \\ 0 & 1 & 1 & 0 \\ 1 & 1 & 1 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \\ 1 & 0 & 0 & 1 \end{bmatrix} \wedge \begin{bmatrix} 1 & 1 & 0 & 1 \\ 0 & 1 & 1 & 0 \\ 1 & 1 & 1 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 1 & 0 & 1 \\ 0 & 1 & 1 & 0 \\ 1 & 0 & 0 & 1 \end{bmatrix}$$

$$(A \vee B) \wedge C = \begin{bmatrix} 1 & 1 & 0 & 1 \\ 0 & 1 & 1 & 0 \\ 1 & 0 & 0 & 1 \end{bmatrix}$$

(iv) $(A \wedge B) \vee C$

$$(A \wedge B) \vee C = \left[\begin{bmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ 1 & 0 & 0 & 1 \end{bmatrix} \vee \begin{bmatrix} 0 & 1 & 0 & 1 \\ 1 & 0 & 1 & 0 \\ 1 & 0 & 0 & 1 \end{bmatrix} \right] \vee \begin{bmatrix} 1 & 1 & 0 & 1 \\ 0 & 1 & 1 & 0 \\ 1 & 1 & 1 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 1 \end{bmatrix} \vee \begin{bmatrix} 1 & 1 & 0 & 1 \\ 0 & 1 & 1 & 0 \\ 1 & 1 & 1 & 1 \end{bmatrix}$$

$$(A \wedge B) \vee C = \begin{bmatrix} 1 & 1 & 0 & 1 \\ 0 & 1 & 1 & 0 \\ 1 & 1 & 1 & 1 \end{bmatrix}$$

9) (i) Let $M = \left\{ \begin{pmatrix} x & x \\ x & x \end{pmatrix}; x \in \mathbb{R} - \{0\} \right\}$

and let $*$ be the matrix multiplication. Determine whether M is closed under $*$. If so, examine the commutative and associative properties satisfied by $*$ on M .

Soln:

Let $M = \left\{ \begin{pmatrix} x & x \\ x & x \end{pmatrix}; x \in \mathbb{R} - \{0\} \right\}$

(i) Closure property:

$$A = \begin{pmatrix} x & x \\ x & x \end{pmatrix} \in M, B = \begin{pmatrix} y & y \\ y & y \end{pmatrix} \in M$$

$$A * B = \begin{pmatrix} x & x \\ x & x \end{pmatrix} * \begin{pmatrix} y & y \\ y & y \end{pmatrix}$$

$$A * B = \begin{pmatrix} 2xy & 2xy \\ 2xy & 2xy \end{pmatrix} \in M$$

$(\because x \neq 0, y \neq 0 \Rightarrow 2xy \neq 0)$
 $\therefore$ Closure property is true
 $\therefore M$ is closed under matrix multiplication $*$

(ii) Commutative property:

$$A = \begin{pmatrix} x & x \\ x & x \end{pmatrix} \in M, B = \begin{pmatrix} y & y \\ y & y \end{pmatrix} \in M$$

$$A * B = \begin{pmatrix} x & x \\ x & x \end{pmatrix} * \begin{pmatrix} y & y \\ y & y \end{pmatrix}$$

$$= \begin{pmatrix} 2xy & 2xy \\ 2xy & 2xy \end{pmatrix}$$

$$= \begin{pmatrix} 2yx & 2yx \\ 2yx & 2yx \end{pmatrix}$$

$$= \begin{pmatrix} y & y \\ y & y \end{pmatrix} * \begin{pmatrix} x & x \\ x & x \end{pmatrix}$$

$$A * B = B * A \quad (\because x \neq 0, y \neq 0 \Rightarrow 2xy \neq 0)$$

$\therefore$ Commutative property is true

$\therefore M$ is commutative under matrix multiplication $*$

(iii) Associative property:

Matrix Multiplication is always associative.

$$(ii) A * (B * C) = (A * B) * C, \quad \forall A, B, C \in M$$

$\therefore$ Associative property is true.

$\therefore M$ is Associative under matrix multiplication $*$

9) (i)

Let $M = \left\{ \begin{pmatrix} x & x \\ x & x \end{pmatrix}; x \in \mathbb{R} - \{0\} \right\}$

and let $*$ be the matrix multiplication. Determine whether M is closed under $*$. If so, examine the existence of identity, existence of inverse properties for the operation $*$ on M

Soln:

Let $M = \left\{ \begin{pmatrix} x & x \\ x & x \end{pmatrix}; x \in \mathbb{R} - \{0\} \right\}$ (i) Closure property: $A = \begin{pmatrix} x & x \\ x & x \end{pmatrix} \in M, B = \begin{pmatrix} y & y \\ y & y \end{pmatrix} \in M$

$$A * B = \begin{pmatrix} x & x \\ x & x \end{pmatrix} * \begin{pmatrix} y & y \\ y & y \end{pmatrix}$$

$$A * B = \begin{pmatrix} 2xy & 2xy \\ 2xy & 2xy \end{pmatrix} \in M$$

$$(\because x \neq 0, y \neq 0 \Rightarrow 2xy \neq 0)$$

 $\therefore$ closure property is true. $\therefore$ M is closed under matrix multiplication *(ii) Existence of Identity PropertyLet $E = \begin{pmatrix} e & e \\ e & e \end{pmatrix} \in M$ and

$$A = \begin{pmatrix} x & x \\ x & x \end{pmatrix} \in M$$

$$A * E = A \Rightarrow \begin{pmatrix} x & x \\ x & x \end{pmatrix} * \begin{pmatrix} e & e \\ e & e \end{pmatrix} = \begin{pmatrix} x & x \\ x & x \end{pmatrix}$$

$$\begin{pmatrix} 2xe & 2xe \\ 2xe & 2xe \end{pmatrix} = \begin{pmatrix} x & x \\ x & x \end{pmatrix}$$

$$2xe = x$$

$$e = \frac{1}{2} (\because x \neq 0)$$

$$\therefore E = \begin{pmatrix} \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & \frac{1}{2} \end{pmatrix} \in M \text{ is}$$

Such that $AE = A$ for every $A \in M$ Similarly $EA = A$ for every $A \in M$

E is the identity element in M

 $\therefore$ Identity property is true $\therefore$ M is identity under Matrix multiplication *(iii) Existence of Inverse propertyLet $A = \begin{pmatrix} x & x \\ x & x \end{pmatrix} \in M$ and

$$A^{-1} = \begin{pmatrix} y & y \\ y & y \end{pmatrix} \in M$$

$$A * A^{-1} = E$$

$$\begin{pmatrix} x & x \\ x & x \end{pmatrix} * \begin{pmatrix} y & y \\ y & y \end{pmatrix} = \begin{pmatrix} \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & \frac{1}{2} \end{pmatrix}$$

$$\begin{pmatrix} 2xy & 2xy \\ 2xy & 2xy \end{pmatrix} = \begin{pmatrix} \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & \frac{1}{2} \end{pmatrix}$$

$$2xy = \frac{1}{2}$$

$$y = \frac{1}{4x}$$

$$\therefore A^{-1} = \begin{pmatrix} \frac{1}{4x} & \frac{1}{4x} \\ \frac{1}{4x} & \frac{1}{4x} \end{pmatrix} \in M \text{ is}$$

Such that $AA^{-1} = E$

$$||^y A^{-1}A = E$$

 $\therefore A^{-1}$ is the inverse of A $\therefore$ Inverse property is true $\therefore$ M is inverse under Matrix multiplication *10) (i) Let A be $\mathbb{Q} \setminus \{1\}$.Define * on A by $x * y = x + y - xy$. Is * binary on A? If so, examine the commutative and associative properties satisfied by * on A

Soln:

Let $A = \mathbb{Q} \setminus \{1\}$ Let $x, y \in A$ Then x and y are rational numbers and $x \neq 1, y \neq 1$ (i) Closure property:

$x * y = x + y - xy$ is a rational number. But to prove $x * y \in A$

To prove that $x * y \neq 1$

Let $x, y \in A, x \neq 1, y \neq 1$

$$x - 1 \neq 0, y - 1 \neq 0$$

$$(x-1)(y-1) \neq 0$$

$$xy - x - y + 1 \neq 0$$

$$1 \neq x + y - xy$$

$$\therefore x + y - xy \neq 1$$

$$\Rightarrow x * y \neq 1$$

$$\Rightarrow x * y \in A, \forall x, y \in A$$

$\therefore$ Closure property is true.

$\therefore A$ is closed under multiplication $*$

(ii) Commutative property:

Let $x, y \in A$

$$x * y = x + y - xy$$

$$= y + x - yx$$

$$x * y = y * x \quad \forall x, y \in A$$

$\therefore$ Commutative property is true.

$\therefore A$ is Commutative under multiplication $*$

(iii) Associative property:

Let $x, y, z \in A$

$$x * (y * z) = x * (y + z - yz)$$

$$= x + (y + z - yz) - x(y + z - yz)$$

$$x * (y * z) = x + y + z - xy$$

$$- yz - zx + xzy \quad \text{--- ①}$$

$$(x * y) * z = (x + y - xy) * z$$

$$= (x + y - xy) + z - (x + y - xy)z$$

$$(x * y) * z = x + y + z - xy - yz$$

$$- zx + xzy \quad \text{--- ②}$$

From ① & ② we get

$$x * (y * z) = (x * y) * z, \forall x, y, z \in A$$

$\therefore$ Associative property is true.

$\therefore A$ is Associative under multiplication $*$

(iv) (ii) Let A be $\mathbb{Q} \setminus \{1\}$.

Define $*$ on A by

$$x * y = x + y - xy. \text{ Is } *$$

binary on A ? If so, exa-

mine the existence of

identity, existence of

inverse properties for

the operation $*$ on A

Soln: Let $A = \mathbb{Q} \setminus \{1\}$

Let $x, y \in A$ Then x and y

are rational numbers and $x \neq 1, y \neq 1$

$x * y = x + y - xy$ is a

rational number. But

to prove $x * y \in A$

To prove that $x * y \neq 1$

Let $x, y \in A, x \neq 1, y \neq 1$

$$x - 1 \neq 0, y - 1 \neq 0$$

$$(x-1)(y-1) \neq 0$$

$$xy - x - y + 1 \neq 0$$

$$1 \neq x + y - xy$$

$$\therefore x + y - xy \neq 1$$

$$\Rightarrow x * y \neq 1$$

$$\Rightarrow x * y \in A, \forall x, y \in A$$

$\therefore$ closure property is true

$\therefore A$ is closed under

multiplication $*$
(ii) Existence of Identity
property:

Let e be the identity element and $x \in A$
By definition of e ,

$$\begin{aligned} x * e &= x \\ \text{By definition of } *, \\ x * e &= x + e - xe \\ x &= x + e - xe \\ x - x &= e - xe \\ 0 &= e(1-x) \\ e &= \frac{0}{1-x} = 0, \text{ since } x \neq 1 \end{aligned}$$

$$\boxed{e = 0 \in A}$$

$\therefore$ Identity element

$$e = 0 \in A$$

$\therefore$ Identity property is true.

$\therefore A$ is identity under multiplication $*$

(iii) Existence of Inverse property

Let $\bar{x}^1$ be the inverse of $x \in A$ and $\bar{x}^1 \in A$
By the definition of

$$\bar{x}^1, x * \bar{x}^1 = e = 0$$

By the definition of $*$,

$$\begin{aligned} x * \bar{x}^1 &= x + \bar{x}^1 - x\bar{x}^1 \\ 0 &= x + \bar{x}^1(1-x) \\ -x &= \bar{x}^1(1-x) \end{aligned}$$

$$\bar{x}^1 = \frac{-x}{1-x}$$

$$\bar{x}^1 = \frac{x}{x-1} \in A, \text{ since } x \neq 1$$

Inverse of x is $\bar{x}^1 = \frac{x}{x-1} \in A$
 $\forall x \in A$
 $\therefore$ Inverse Property is true

$\therefore A$ is Inverse under multiplication $*$

Ex 12.1

Examine the binary operation (closure property) of the following operations on the respective sets (if it is not, make it binary):

$$(i) a * b = a + 3ab - 5b^2;$$

$$\forall a, b \in \mathbb{Z}$$

$$(ii) a * b = \left(\frac{a-1}{b-1} \right), \forall a, b \in \mathbb{Q}$$

Soln:

$$(i) \text{ Let } a, b \in \mathbb{Z}$$

$$\text{Then } 3ab, 5b^2 \in \mathbb{Z}$$

$$\therefore a + 3ab - 5b^2 \in \mathbb{Z}$$

$$\Rightarrow a * b \in \mathbb{Z}$$

$\therefore *$ is a binary operation on $\mathbb{Z}$

(ii) Given a, b in the quotient form. Since

$a * b$ in the quotient form. Since

0 is undefined

$$\therefore b-1 \neq 0$$

$$\Rightarrow b \neq 1$$

It is clear that $b-1=0$ if $b=1$. As $1 \in \mathbb{Q}$
 $*$ is not a binary operation on $\mathbb{Q}$

But omitting 1 from $\mathbb{Q}$

$$\therefore a * b \text{ exists in } \mathbb{Q} \setminus \{1\}$$

$*$ is a binary operation on $\mathbb{Q} \setminus \{1\}$

Ex 12.2

verify the (i) closure property (ii) commutative property (iii) associative property (iv) existence of identity and (v) existence of inverse for the arithmetic operation $+$ on $\mathbb{Z}$

Soln:

(i) <u>closure property</u>: Let $m, n \in \mathbb{Z}$ Then $m+n \in \mathbb{Z}, \forall m, n \in \mathbb{Z}$ $\therefore$ closure property is true $+$ is a binary operation on $\mathbb{Z}$ (ii) <u>Commutative property</u>: Let $m, n \in \mathbb{Z}$ Then $m+n = n+m, \forall m, n \in \mathbb{Z}$ Commutative property is true. $+$ is a binary operation on $\mathbb{Z}$ (iii) <u>Associative property</u>: Let $m, n, p \in \mathbb{Z}$ Then $m+(n+p) = (m+n)+p, \forall m, n, p \in \mathbb{Z}$ Associative property is true (iv) <u>Existence of Identity property</u>	Let e be the identity Let $m, e \in \mathbb{Z}$ $m+e = e+m = m$ $\Rightarrow e = 0$ There exists $e = 0 \in \mathbb{Z}$ Such that $m+0 = 0+m = m$ $\therefore$ Existence of identity is assured. $\therefore$ Identity property is true (v) <u>Existence of inverse property</u> Let $m, -m \in \mathbb{Z}$ $m+(-m) = (-m)+m = 0$ put $m' = -m$ $m+m' = (-m)+m = 0, \forall m, -m \in \mathbb{Z}$ $\therefore$ The Existence of inverse property is also assured. $\therefore$ All the above five properties satisfies	usual addition $+$ on $\mathbb{Z}$ <u>Ex 12.3</u> Verify the (i) closure property (ii) commutative property (iii) associative property (iv) existence of identity and (v) existence of inverse for the arithmetic operation $-$ on $\mathbb{Z}$ <u>Soln:</u> (i) <u>closure property</u>: $-$ is not binary on $\mathbb{N}$ Let $m, n \in \mathbb{Z}$ Then $m-n \in \mathbb{Z}, \forall m, n \in \mathbb{Z}$ $\therefore$ closure property is true $-$ is a binary operation on $\mathbb{Z}$ (ii) <u>Commutative property</u>:	Let $m=4 \in \mathbb{Z}, n=5 \in \mathbb{Z}$ Then $(m-n) = (4-5) = -1 \in \mathbb{Z}$ Also, $(n-m) = (5-4) = 1 \in \mathbb{Z}$ $\therefore (m-n) \neq (n-m)$ $\therefore$ The operation $-$ is not commutative on $\mathbb{Z}$ $\therefore$ Commutative property is not true (iii) <u>Associative property</u>: Let $m=4 \in \mathbb{Z}, n=5 \in \mathbb{Z}$ and $p=7 \in \mathbb{Z}$ Then $(m-n)-p = (4-5)-7 = -1-7 = -8 \in \mathbb{Z}$ $m-(n-p) = 4-(5-7) = 4-(-2) = 4+2 = 6 \in \mathbb{Z}$ $\therefore (m-n)-p \neq m-(n-p)$ $\therefore$ The operation $-$ is not associative on $\mathbb{Z}$
---	--	---	---

$\therefore$ Associative property is not true

(iv) Existence of identity

Property Since operation - is Identity does not exist on $\mathbb{Z}$

(v) Existence of inverse

Property Since operation - is Inverse does not exist on $\mathbb{Z}$

EX 12.4

Verify the (i) closure property (ii) commutative property (iii) associative property (iv) existence of identity and (v) existence of inverse for the arithmetic operation + on $\mathbb{Z}_e =$ the set of all even integers.

Soln:

Consider the set of all even integers

$$\mathbb{Z}_e = \{2k / k \in \mathbb{Z}\}$$

$$= \{\dots, -6, -4, -2, 0, 2, 4, 6, \dots\}$$

(i) closure property:

The sum of any two even integers is also an even integer.

$$\text{Let } x, y \in \mathbb{Z}_e$$

$$\text{Then } x = 2m \text{ and } y = 2n, m, n \in \mathbb{Z}$$

$$x + y = 2m + 2n = 2(m + n) \in \mathbb{Z}_e$$

$\therefore +$ is a binary operation on $\mathbb{Z}_e$

$\therefore$ closure property is true

(ii) Commutative property:

$$\text{Let } x, y \in \mathbb{Z}_e, \text{ Then } x = 2m \text{ and } y = 2n, m, n \in \mathbb{Z}$$

$$\begin{aligned} (x + y) &= (2m + 2n) = 2(m + n) \\ &= 2(n + m) \\ &= 2n + 2m \in \mathbb{Z}_e \end{aligned}$$

$$(x + y) = (y + x)$$

$\therefore +$ has commutative property.

$\therefore$ commutative property is true.

(iii) Associative property:

$$\text{Let } x, y, z \in \mathbb{Z}_e \text{ Then } x = 2m, y = 2n \text{ and } z = 2p, m, n, p \in \mathbb{Z}_e$$

$$(x + y) + z = (2m + 2n) + 2p$$

$$= (2(m + n)) + 2p$$

$$= 2[(m + n) + p]$$

$$(x + y) + z = 2[m + n + p] \text{ --- (1)}$$

$$x + (y + z) = 2m + (2n + 2p)$$

$$= 2m + 2(n + p)$$

$$= 2[m + (n + p)]$$

$$x + (y + z) = 2[m + n + p]$$

From (1) & (2) we get, --- (3)

$$(x + y) + z = x + (y + z) \quad \forall x, y, z \in \mathbb{Z}_e$$

$\therefore$ Associative property is true.

(iv) Existence of identity Property:

Let e be the identity element.

$$\therefore e = 0 \in \mathbb{Z}_e$$

$$\text{Let } x = 2k, \quad \forall x \in \mathbb{Z}_e$$

$$x + e = e + x = x \Rightarrow e = 0$$

$$2k + e = e + 2k = 2k \Rightarrow e = 0$$

$e = 0$ is the identity element

$\therefore$ Identity property is true

(v) Existence of inverse Property:

Let x' be the inverse of x Let $x = 2k, x \in \mathbb{Z}_e$ $x + x' = 0 \Rightarrow x' = -x$ $2k + x' = 0 \Rightarrow x' = -2k$ Also, $x' + x = 0$ $x' + 2k = 0$ $x' = -2k$ $\therefore 2k + x' = x' + 2k = 0$ Let $x, -x \in \mathbb{Z}_e$ $x + (-x) + (-x) + x = 0$ $\therefore -x$ is the inverse of $x \in \mathbb{Z}_e$ $\therefore$ Inverse property is true	Of inverse for the arithmetic operation $+$ on $\mathbb{Z}_o$ = the set of all odd integers <u>Soln:</u> Consider the set of all odd integers $\mathbb{Z}_o = \{2k+1; k \in \mathbb{Z}\}$ $= \{\dots, -5, -3, -1, 1, 3, 5, \dots\}$ (i) <u>Closure property:</u> The sum of any two odd integers is not an odd integer Let $x, y \in \mathbb{Z}_o$ Then $x = 2m+1, y = 2n+1, m, n \in \mathbb{Z}_o$ $x+y = (2m+1) + (2n+1)$ $= 2m+2n+1+1$ $x+y = 2(m+n)+2$ is even $\forall m, n \in \mathbb{Z}_o$	$\therefore x+y \notin \mathbb{Z}_o$ $\therefore +$ is not a binary operation on $\mathbb{Z}_o$ (ii) <u>Commutative property:</u> Let $x, y \in \mathbb{Z}_o$ Then $x = 2m+1, y = 2n+1, m, n \in \mathbb{Z}_o$ $x+y = (2m+1) + (2n+1)$ $= (2n+1) + (2m+1)$ $x+y = y+x$ which is not an odd integer $\therefore +$ is not a commutative property on $\mathbb{Z}_o$ Similarly other properties is not a binary operation on $+$	(ii) Commutative property and (iii) associative property of the following operation on the given set $(a * b) = a^b; \forall a, b \in \mathbb{N}$ (exponentiation property) <u>Soln:</u> (i) <u>Closure property:</u> Let $a, b \in \mathbb{N}$ Then $a * b = a^b \in \mathbb{N}, \forall a, b \in \mathbb{N}$ $\therefore *$ is a binary operation on $\mathbb{N}$ $\therefore$ closure property is true (ii) <u>Commutative property:</u> Let $a, b \in \mathbb{N}$ Then $a * b = a^b$ and $b * a = b^a$ $\therefore a * b \neq b * a$ Put $a = 2, b = 3$ $a * b = 2^3 = 8$ and
Ex 12.5 verify the (i) closure property (ii) commutative property (iii) associative property (iv) existence of identity and (v) existence		EX 12.6 verify (i) closure property	

$b * a = 3^2 = 9$
 $\therefore a * b \neq b * a$
 $\therefore *$ does not have Commutative property

(iii) Associative property:

Let $a, b, c \in \mathbb{N}$
 $a * (b * c) = a * b^c = a^{(b^c)}$
 $(a * b) * c = a^b * c = (a^b)^c$
 $= a^{bc}$

$\therefore a * (b * c) \neq (a * b) * c$
 Take $a = 2, b = 3$ and $c = 4$

$a * (b * c) = 2 * (3 * 4)$
 $= 2 * 3^4 = 2^{3^4} = 2^{81}$

$(a * b) * c = (2 * 3) * 4$
 $= 2^3 * 4 = (2^3)^4 = 2^{12}$

$\therefore a * (b * c) \neq (a * b) * c$
 $\therefore *$ does not have associative property on $\mathbb{N}$

EX 12.7

Verify (i) closure property (ii) Commutative property (iii) associative property (iv) existence of identity and (v) existence of inverse for following operation on the given set

$m * n = m + n - mn; m, n \in \mathbb{Z}$

Soln:

(i) Closure property:

Let $m, n \in \mathbb{Z}$
 Then $m + n - mn \in \mathbb{Z}$

$\therefore m * n \in \mathbb{Z}, \forall m, n \in \mathbb{Z}$
 $\therefore *$ is a binary operation on $\mathbb{Z}$

$\therefore$ closure property is true

(ii) Commutative property:

Let $m, n \in \mathbb{Z}$
 Then $m * n = m + n - mn$

$= n + m - nm$

$= n * m$

$\therefore m * n = n * m, \forall m, n \in \mathbb{Z}$

$*$ has Commutative property on $\mathbb{Z}$

$\therefore$ Commutative property is true.

(iii) Associative property:

Let $m, n, p \in \mathbb{Z}$

$(m * n) * p = m + n + p - mn$

$- np - mp + mnp$

$m * (n * p) = m + n + p - mn$

$- np - mp + mnp$

$\therefore (m * n) * p = m * (n * p), \forall m, n, p \in \mathbb{Z}$

$*$ has associative property on $\mathbb{Z}$

$\therefore$ Associative property is true.

(iv) Existence of identity property:

Let $m, e \in \mathbb{Z}$

Then $m * e = e * m = m, \forall m \in \mathbb{Z}$

$m * e = m + e - me$

$m = m + e - me$

$m - m = e(1 - m)$

$0 = e(1 - m)$

$\Rightarrow e = 0$ (or) $1 - m \neq 0$

$\Rightarrow m \neq 1$

$\therefore \boxed{e = 0}$

$m * 0 = 0 * m = m + 0 - m \cdot 0$

$m * 0 = 0 * m = m, \forall m \in \mathbb{Z}$

$e = 0$ is the identity element

$\therefore$ The existence of identity is assured

$\therefore$ Identity property is true.

(v) Existence of inverse Property

Let $m, m' \in \mathbb{Z}$ Then
 $m * m' = m * m = e = 0,$
 $\forall m, m' \in \mathbb{Z}$

$$\begin{aligned} m * m' &= 0 \\ m + m' - mm' &= 0 \\ m' - mm' &= -m \\ m'(1-m) &= -m \\ m' &= \frac{-m}{1-m} \Rightarrow m' = \frac{m}{m-1} \end{aligned}$$

When $m=1$, $m' = \infty$ is not defined

When $m=2$, $m' = 2 \in \mathbb{Z}$

But except $m=2$, m' need not be an integer for all values of m .

$\therefore$ Inverse does not exist in $\mathbb{Z}$

EX 12.8 Let $A = \begin{bmatrix} 0 & 1 \\ 1 & 1 \end{bmatrix}, B = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$

be any two boolean matrices of the same type. Find $A \vee B$ and $A \wedge B$.

Soln:

$$A \vee B = \begin{bmatrix} 0 & 1 \\ 1 & 1 \end{bmatrix} \vee \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$$

$$A \vee B = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}$$

$$A \wedge B = \begin{bmatrix} 0 & 1 \\ 1 & 1 \end{bmatrix} \wedge \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$$

$$A \wedge B = \begin{bmatrix} 0 & 1 \\ 0 & 1 \end{bmatrix}$$

EX 12.9

verify (i) closure property

(ii) commutative property

(iii) associative property

(iv) existence of identity

and (v) existence of inverse

for the operation $+_5$ on

$\mathbb{Z}_5$ using table corresponding

to addition modulo 5

Soln:

$$\mathbb{Z}_5 = \{[0], [1], [2], [3], [4]\}$$

$+_5$	0	1	2	3	4
0	0	1	2	3	4
1	1	2	3	4	0
2	2	3	4	0	1
3	3	4	0	1	2
4	4	0	1	2	3

(i) Closure Property:

From the table,
 Since each box in the table is filled by exactly one element of $\mathbb{Z}_5$

$\therefore a +_5 b$ is unique

$\therefore +_5$ is a binary operation on

$\therefore$ closure property is true.

(ii) Commutative property

From the table,
 The entries are symmetrically placed with respect to the main diagonal.

$\therefore +_5$ has commutative property.

$\therefore$ commutative property is true

(iii) Associative property

From the table,
 $(2 +_5 3) +_5 4 = 0 +_5 4 = 4$
 (mod 5)
 and

$2 +_5 (3 +_5 4) = 2 +_5 2 = 4$
 (mod 5)

$\therefore (2 +_5 3) +_5 4 = 2 +_5 (3 +_5 4)$

$\therefore$ Associative property is true

(iv) Existence of identity:

Identity element

$$e = 0$$

$\therefore$ Identity property is true

(v) Existence of inverse:

Inverse of 0 is 0

Inverse of 1 is 4

Inverse of 2 is 3

Inverse of 3 is 2

Inverse of 4 is 1

$\therefore$ Inverse exists

$\therefore$ Inverse property is true.

Ex 12.10

Verify: (i) Closure property

(ii) Commutative property (iii)

associative property (iv) existence of identity and (v) existence of inverse for the operation x_{11} on a Subset $A = \{1, 3, 4, 5, 9\}$ of the set of remainders $\{0, 1, 2, 3, 4, 5, 6, 7, 8, 9, 10\}$

Soln:

x_{11}	1	3	4	5	9
1	1	3	4	5	9
3	3	9	1	4	5
4	4	1	5	9	3
5	5	4	9	3	1
9	9	5	3	1	4

(i) Closure property:

From the table,

Since each box has an unique element of A

$\therefore x_{11}$ is a binary operation

on A

$\therefore$ Closure property is true

(ii) Commutative property:

From the table, The entries are Symmetrical about the main diagonal

x_{11} has Commutative property.

$\therefore$ Commutative property is true

(iii) Associative property:

From the table,

As usual, the associative property can be seen to be true.

$\therefore$ Associative property is true

(iv) Existence of identity:

Identity element

$$e = 1$$

$\therefore$ Identity property is true.

(v) Existence of inverse:

Inverse of 1 is 1

Inverse of 3 is 4

Inverse of 4 is 3

Inverse of 5 is 9

Inverse of 9 is 5

$\therefore$ Inverse property is true

Thm 12.1 (Uniqueness of Identity)

In an algebraic structure the identity element (if exists) must be unique

Proof:

Let $(S, *)$ be an algebraic

Structure. Assume that the identity element of S in S

To prove: Identity element is unique.

Suppose that e_1 and e_2 be any two identity elements of S

e_1 is the identity element of S

$$e_1 * e_2 = e_2 * e_1 = e_2 \quad \text{--- ①}$$

e_2 is the identity element of S

$$e_1 * e_2 = e_2 * e_1 = e_1 \quad \text{--- ②}$$

from ① & ② we get,

$$e_1 = e_2$$

$\therefore$ The identity element is unique

Thm 12.2 (Uniqueness of Inverse)

In an algebraic structure the inverse of an element (if exists) must be unique

Proof:

Let $(S, *)$ be an algebraic structure and $a \in S$

Assume that the inverse of a exists in S

To prove: The inverse of a is unique.

Let $a \in S$

Suppose that a has two inverses a_1 and a_2

a_1 is the inverse of a

$$a * a_1 = a_1 * a = e$$

a_2 is the inverse of a

$$a * a_2 = a_2 * a = e$$

$$a_1 = a_1 * e$$

$$= a_1 * (a * a_2)$$

$$= (a_1 * a) * a_2$$

$$= e * a_2$$

$$= a_2$$

$$\therefore a_1 = a_2$$

$\therefore$ The inverse of a is unique

12.3 Mathematical Logic

EXERCISE 12.2

1) Let P : Jupiter is a planet and Q : India is an Island be any two Simple Statements. Give verbal sentence describing each of the following Statements.

(i) $\neg P$ (ii) $P \wedge \neg Q$ (iii) $\neg P \vee Q$

(iv) $P \rightarrow \neg Q$ (v) $P \leftrightarrow Q$

Soln:

Given:

P : Jupiter is a planet

Q : India is an Island

(i) $\neg P$: Jupiter is not a planet

(ii) $P \wedge \neg Q$: Jupiter is a planet and India is not an Island

(iii) $\neg P \vee Q$: Jupiter is not a planet or India is an Island

(iv) $P \rightarrow \neg Q$: If Jupiter is a planet then India is not an Island

(v) Jupiter is a planet if and only if India is an Island

2) write each of the following Sentences in

Symbolic form using statement variables p and q .

(i) 19 is not a prime number and all the angles of a triangle are equal

(ii) 19 is a prime number or all the angles of a triangle are not equal

(iii) 19 is a prime number and all the angles of a triangle are equal

(iv) 19 is not a prime number

Soln: Given:

p : 19 is a prime number

q : All the angles of a triangle are equal

(i) $\neg p \wedge q$

(ii) $p \vee \neg q$

(iii) $p \wedge q$

(iv) $\neg p$

3) Determine the truth value of each of the following statements

(i) If $6+2=5$, then the milk is white

(ii) China is in Europe or $\sqrt{3}$ is an integer

(iii) It is not true that $5+5=9$ or Earth is a planet

(iv) 11 is a prime number and all the sides of a rectangle are equal

Soln:

(i) p : $6+2=5$

q : The milk is white

p	q	$p \rightarrow q$
F	T	T

$\therefore$ The truth value is T

(ii) p : China is in Europe

q : $\sqrt{3}$ is an integer

p	q	$p \vee q$
F	F	F

$\therefore$ The truth value is F

(iii) p : It is true that $5+5=9$

q : Earth is a planet

p	q	$\neg p$	$\neg p \vee q$
F	T	T	T

$\therefore$ The truth value is T

(iv) p : 11 is a prime number

q : All the sides of a triangle are equal

p	q	$p \wedge q$
T	F	F

$\therefore$ The truth value is F

4) Which one of the following sentences is a Proposition?

(i) $4+7=12$ (ii) What are you doing? (iii) $3^n \leq 81$, $n \in \mathbb{N}$ (iv) Peacock is our national bird

(v) How tall this mountain is!

Soln:

(i) $4+7=12 \Rightarrow$ Proposition

(ii) What are you doing? $\Rightarrow$ not a proposition

(iii) $3^n \leq 81$, $n \in \mathbb{N} \Rightarrow$ Proposition

(iv) Peacock is our national bird $\Rightarrow$ Proposition

(v) How tall this mountain is! $\Rightarrow$ not a Proposition

$\therefore$ (i), (iii) and (iv) are Propositions 5) Write the Converse, inverse, and Contrapositive of each of the following implication. (i) If x and y are numbers such that $x=y$, then $x^2=y^2$ (ii) If a quadrilateral is a Square then it is a rectangle. <u>Soln:</u> (i) P: x and y are numbers such that $x=y$ Q: $x^2=y^2$ $\therefore$ Given Statement Symbolic form is $P \rightarrow Q$ (i) <u>Converse</u>: $Q \rightarrow P$ If x and y are numbers such that $x^2=y^2$ then $x=y$	(ii) <u>Inverse</u>: $\neg P \rightarrow \neg Q$ If x and y are numbers such that $x \neq y$ then $x^2 \neq y^2$ (iii) <u>Contrapositive</u>: $\neg Q \rightarrow \neg P$ If x and y are numbers such that $x^2 \neq y^2$ then $x \neq y$ (5)(ii) P: A quadrilateral is a Square Q: A quadrilateral is a rectangle $\therefore$ Given Statement Symbolic form is $P \rightarrow Q$ (i) <u>Converse</u>: $Q \rightarrow P$ If a quadrilateral is a rectangle then it is a Square (ii) <u>Inverse</u>: $\neg P \rightarrow \neg Q$ If a quadrilateral is	not a Square then It is not a rectangle (iii) <u>Contrapositive</u>: If a quadrilateral is not a rectangle then it is not a Square 6) Construct the truth table for the following statements. (i) $\neg P \wedge \neg Q$ <u>Soln:</u> Truth table for $\neg P \wedge \neg Q$ <table border="1"> <tr> <th>P</th> <th>Q</th> <th>$\neg P$</th> <th>$\neg Q$</th> <th>$\neg P \wedge \neg Q$</th> </tr> <tr> <td>T</td> <td>T</td> <td>F</td> <td>F</td> <td>F</td> </tr> <tr> <td>T</td> <td>F</td> <td>F</td> <td>T</td> <td>F</td> </tr> <tr> <td>F</td> <td>T</td> <td>T</td> <td>F</td> <td>F</td> </tr> <tr> <td>F</td> <td>F</td> <td>T</td> <td>T</td> <td>T</td> </tr> </table> (ii) $\neg(P \wedge \neg Q)$ <u>Soln:</u>	P	Q	$\neg P$	$\neg Q$	$\neg P \wedge \neg Q$	T	T	F	F	F	T	F	F	T	F	F	T	T	F	F	F	F	T	T	T	Truth table for $\neg(P \wedge \neg Q)$ <table border="1"> <tr> <th>P</th> <th>Q</th> <th>$\neg Q$</th> <th>$P \wedge \neg Q$</th> <th>$\neg(P \wedge \neg Q)$</th> </tr> <tr> <td>T</td> <td>T</td> <td>F</td> <td>F</td> <td>T</td> </tr> <tr> <td>T</td> <td>F</td> <td>T</td> <td>T</td> <td>F</td> </tr> <tr> <td>F</td> <td>T</td> <td>F</td> <td>F</td> <td>T</td> </tr> <tr> <td>F</td> <td>F</td> <td>T</td> <td>F</td> <td>T</td> </tr> </table> (iii) $(P \vee Q) \vee \neg Q$ <u>Soln:</u> Truth table for $(P \vee Q) \vee \neg Q$ <table border="1"> <tr> <th>P</th> <th>Q</th> <th>$(P \vee Q)$</th> <th>$\neg Q$</th> <th>$(P \vee Q) \vee \neg Q$</th> </tr> <tr> <td>T</td> <td>T</td> <td>T</td> <td>F</td> <td>T</td> </tr> <tr> <td>T</td> <td>F</td> <td>T</td> <td>T</td> <td>T</td> </tr> <tr> <td>F</td> <td>T</td> <td>T</td> <td>F</td> <td>T</td> </tr> <tr> <td>F</td> <td>F</td> <td>F</td> <td>T</td> <td>T</td> </tr> </table> (iv) $(\neg P \rightarrow r) \wedge (P \leftrightarrow Q)$ <u>Soln:</u>	P	Q	$\neg Q$	$P \wedge \neg Q$	$\neg(P \wedge \neg Q)$	T	T	F	F	T	T	F	T	T	F	F	T	F	F	T	F	F	T	F	T	P	Q	$(P \vee Q)$	$\neg Q$	$(P \vee Q) \vee \neg Q$	T	T	T	F	T	T	F	T	T	T	F	T	T	F	T	F	F	F	T	T
P	Q	$\neg P$	$\neg Q$	$\neg P \wedge \neg Q$																																																																										
T	T	F	F	F																																																																										
T	F	F	T	F																																																																										
F	T	T	F	F																																																																										
F	F	T	T	T																																																																										
P	Q	$\neg Q$	$P \wedge \neg Q$	$\neg(P \wedge \neg Q)$																																																																										
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F	T	F	F	T																																																																										
F	F	T	F	T																																																																										
P	Q	$(P \vee Q)$	$\neg Q$	$(P \vee Q) \vee \neg Q$																																																																										
T	T	T	F	T																																																																										
T	F	T	T	T																																																																										
F	T	T	F	T																																																																										
F	F	F	T	T																																																																										

P	q	r	$\neg P(\neg R \rightarrow r)$	$R \rightarrow q$	$(\neg P \rightarrow r) \wedge (R \rightarrow r)$
T	T	T	F	T	T
T	T	F	F	T	T
T	F	T	F	F	F
T	F	F	F	F	F
F	T	T	T	F	F
F	T	F	T	F	F
F	F	T	T	T	T
F	F	F	T	F	F

Since the last Column Contains only F
 $\therefore (P \wedge q) \wedge \neg(P \vee q)$ is a Contradiction

(ii) $((P \vee q) \wedge \neg P) \rightarrow q$

P	q	$P \vee q$	$\neg P$	$((P \vee q) \wedge \neg P) \rightarrow q$
T	T	T	F	T
T	F	T	F	T
F	T	T	T	T
F	F	F	T	T

Since the last Column Contains only T
 $\therefore ((P \vee q) \wedge \neg P) \rightarrow q$ is a Tautology

(iii) $(P \rightarrow q) \leftrightarrow (\neg P \rightarrow q)$

P	q	$P \rightarrow q$	$\neg P$	$\neg P \rightarrow q$	$(P \rightarrow q) \leftrightarrow (\neg P \rightarrow q)$
T	T	T	F	T	T
T	F	F	F	T	F
F	T	T	T	T	T
F	F	T	T	F	F

Since the last Column Contains T and F
 $\therefore$ The given Statement is neither a tautology nor a Contradiction
 $\therefore (P \rightarrow q) \leftrightarrow (\neg P \rightarrow q)$ is a Contingency

(iv) $((P \rightarrow q) \wedge (q \rightarrow r)) \rightarrow (P \rightarrow r)$

P	q	r	$P \rightarrow q$	$q \rightarrow r$	$((P \rightarrow q) \wedge (q \rightarrow r)) \rightarrow (P \rightarrow r)$
T	T	T	T	T	T
T	T	F	F	F	T
T	F	T	F	T	T
T	F	F	F	F	T
F	T	T	T	T	T
F	T	F	F	F	T
F	F	T	T	T	T
F	F	F	T	T	T

Since the last Column Contains only T
 $\therefore ((P \rightarrow q) \wedge (q \rightarrow r)) \rightarrow (P \rightarrow r)$ is a tautology

8) Show that

(i) $\neg(P \wedge q) \equiv \neg P \vee \neg q$

Soln:

P	q	$P \wedge q$	$\neg(P \wedge q)$	$\neg P$	$\neg q$	$\neg P \vee \neg q$
T	T	T	F	F	F	F
T	F	F	T	F	T	T
F	T	F	T	T	F	T
F	F	F	T	T	T	T

L①

L②

From ① and ②

Since the last Columns are identical

$\therefore \neg(P \wedge q) \equiv \neg P \vee \neg q$

(ii) $\neg(P \rightarrow q) \equiv P \wedge \neg q$

P	q	$P \rightarrow q$	$\neg(P \rightarrow q)$	$\neg q$	$P \wedge \neg q$
T	T	T	F	F	F
T	F	F	T	T	T
F	T	T	F	F	F
F	F	T	F	T	F

From ① & ② L①

Since the last Columns are identical
 $\therefore \neg(P \rightarrow q) \equiv P \wedge \neg q$

9) Prove that
 $q \rightarrow p \equiv \neg p \rightarrow \neg q$

Soln:

P	q	$q \rightarrow p$	$\neg p$	$\neg q$	$\neg p \rightarrow \neg q$
T	T	T	F	F	T
T	F	F	F	T	T
F	T	T	T	F	F
F	F	T	T	T	T

L①

L②

From ① & ②
 Since the last Columns are identical
 $\therefore q \rightarrow p \equiv \neg p \rightarrow \neg q$

10) Show that $p \rightarrow q$ and $q \rightarrow p$ are not equivalent

Soln:

P	q	$p \rightarrow q$	$q \rightarrow p$
T	T	T	T
T	F	F	T
F	T	T	F
F	F	T	T

Since the last Columns are not identical

$\therefore p \rightarrow q$ and $q \rightarrow p$ are not equivalent

(ie) $p \rightarrow q \not\equiv q \rightarrow p$

11) Show that

$\neg(p \leftrightarrow q) \equiv p \leftrightarrow \neg q$

Soln:

P	q	$p \leftrightarrow q$	$\neg(p \leftrightarrow q)$	$\neg q$	$p \leftrightarrow \neg q$
T	T	T	F	F	F
T	F	F	T	T	T
F	T	F	T	F	T
F	F	T	F	T	F

L①

L②

From ① & ②

Since the last Columns are identical

$\therefore \neg(p \leftrightarrow q) \equiv p \leftrightarrow \neg q$

12) Check whether the Statement $p \rightarrow (q \rightarrow p)$ is a tautology or a Contradiction

without using the truth table

Soln: Given: $p \rightarrow (q \rightarrow p)$

$$\begin{aligned} p \rightarrow (q \rightarrow p) &\equiv p \rightarrow (\neg q \vee p) \\ &\equiv \neg p \vee (\neg q \vee p) \\ &\equiv \neg p \vee (p \vee \neg q) \end{aligned}$$

[$\therefore$ Commutative Property]

$$\begin{aligned} &\equiv (\neg p \vee p) \vee \neg q \\ &[\therefore \text{Associative Property}] \end{aligned}$$

$$\equiv T \vee \neg q$$

$$\equiv T$$

$\therefore p \rightarrow (q \rightarrow p)$ is a tautology

13) Using truth table check whether the Statements $\neg(p \vee q) \vee (\neg p \wedge q)$ and $\neg p$ are logically equivalent

Soln:

P	q	$p \vee q$	$\neg(p \vee q)$	$\neg p$	$\neg p \wedge q$	$(\neg p \wedge q) \vee (\neg p \vee q)$
T	T	T	F	F	F	F
T	F	T	F	F	F	F
F	T	T	F	T	T	T
F	F	F	T	T	F	T

L①

L②

From ① & ②

Since the last Columns are identical

$\therefore \neg(p \vee q) \vee (\neg p \wedge q) \equiv \neg p$
 Yes. The given Statements are logically equivalent

14) Prove $p \rightarrow (q \rightarrow r) \equiv (p \wedge q) \rightarrow r$ without using truth table

Soln: Given: $p \rightarrow (q \rightarrow r) \equiv (p \wedge q) \rightarrow r$

$$\begin{aligned} p \rightarrow (q \rightarrow r) &\equiv p \rightarrow (\neg q \vee r) \\ &\equiv \neg p \vee (\neg q \vee r) \\ &\equiv (\neg p \vee \neg q) \vee r \\ &[\therefore \text{associative Property}] \\ &\equiv \neg(p \wedge q) \vee r \\ &\equiv (p \wedge q) \rightarrow r \end{aligned}$$

$$\therefore P \rightarrow (q \rightarrow r) \equiv (P \wedge q) \rightarrow r$$

15) prove that

$$P \rightarrow (\neg q \vee r) \equiv \neg P \vee (\neg q \vee r) \text{ using truth table}$$

Soln:

P	q	r	$\neg q$	$\neg q \vee r$	$P \rightarrow (\neg q \vee r)$	$\neg P$	$\neg P \vee (\neg q \vee r)$
T	T	T	F	T	T	F	T
T	T	F	F	F	F	F	F
T	F	T	T	T	T	F	T
T	F	F	T	T	T	F	T
F	T	T	F	T	T	T	T
F	T	F	F	F	T	T	T
F	F	T	T	T	T	T	T
F	F	F	T	T	T	T	T

From ① & ②
Since the last columns are identical.

$$P \rightarrow (\neg q \vee r) \equiv \neg P \vee (\neg q \vee r)$$

Ex 12.11

Identify the valid statements

from the following sentences

(1) Mount Everest is the highest mountain of the world

$$(2) 3+4=8$$

$$(3) 7+5>10$$

(4) Give me that book

$$(5) (10-x)=7$$

(6) How beautiful this flower is!

(7) Where are you going?

(8) Wish you all success

(9) This is the beginning of the end

Soln:

(1) Mount Everest is the highest mountain of the world

It is true statement

$$(2) 3+4=8$$

It is False statement

$$(3) 7+5>10$$

It is true statement

(4) Give me that book

It is a command
 $\therefore$ It is not statement

$$(5) (10-x)=7$$

$$\Rightarrow x = 10-7=3$$

$$\Rightarrow x=3$$

It is true statement for $x=3$

But false statement for $x \neq 3$

(6) How beautiful this flower is!

It is an exclamatory
 $\therefore$ It is not statement

(7) Where are you going?

It is a question
 $\therefore$ It is not statement

(8) Wish you all success

It is an optative

$\therefore$ It is not statement

(9) This is the beginning of the end

It is a paradox
 $\therefore$ It is not statement

Ex 12.12

Write the statements in words corresponding to $\neg P$, $P \wedge q$, $P \vee q$ and $q \vee \neg P$, where P is 'It is Cold' and q is 'It is raining'.

Soln: Given:

P : It is Cold

q : It is raining

(1) $\neg P$: It is not Cold

(2) $P \wedge q$: It is Cold and raining

(3) $P \vee q$: It is Cold or raining

(4) $q \vee \neg P$: It is raining or It is not Cold.

EX 12.13

How many rows are needed for following Statement Formulae?

$$(i) (p \vee \neg t) \wedge (p \vee \neg s)$$

$$(ii) (c \wedge p \wedge q) \vee (\neg r \vee \neg s) \wedge (\neg t \wedge v)$$

Soln:

$$(i) (p \vee \neg t) \wedge (p \vee \neg s)$$

$\Rightarrow$ Which is contains 3 variables p, s and t

$\therefore$ The corresponding truth table will contain $2^3 = 8$ rows

$$(ii) (c \wedge p \wedge q) \vee (\neg r \vee \neg s) \wedge (\neg t \wedge v)$$

$\Rightarrow$ Which is contains 6 variables p, q, r, s, t and v

$\therefore$ The corresponding truth table will contain $2^6 = 64$ rows

EX 12.14

Consider $p \rightarrow q$: If today is Monday, then $4+4=8$ write the truth value.

Soln:

p: Today is Monday

$$q: 4+4=8$$

$\therefore$ The truth value of $p \rightarrow q$ is T because the Conclusion q is T

EX 12.15

write down the (i) Conditional Statement (ii) Converse Statement (iii) Inverse Statement, and (iv) Contrapositive Statement for the two statements p and q given below.

p: The number of primes is infinite

q: Ooty is in Kerala.

Soln: Given:

p: The number of Primes is infinite

q: Ooty is in Kerala

(i) Conditional Statement:

$p \rightarrow q$: If the number of primes is infinite then Ooty is in Kerala

(ii) Converse Statement:

$q \rightarrow p$: If Ooty is in Kerala then the number of primes is infinite

(iii) Inverse Statement:

$\neg p \rightarrow \neg q$: If the number of primes is not infinite then Ooty is not in Kerala

(iv) Contrapositive Statement:

$\neg q \rightarrow \neg p$: If Ooty is not in Kerala then the number of primes is not infinite

EX 12.16

Construct the truth table for $(p \vee q) \wedge (p \vee \neg q)$

Soln:

p	q	$\neg q$	$p \vee q$	$p \vee \neg q$	$(p \vee q) \wedge (p \vee \neg q)$
T	T	F	T	T	T
T	F	T	T	F	F
F	T	F	T	F	F
F	F	T	F	T	F

EX 12.17

Establish the equivalence property: $p \rightarrow q \equiv \neg p \vee q$

Soln:

p	q	$\neg p$	$p \rightarrow q$	$\neg p \vee q$
T	T	F	T	T
T	F	F	F	F
F	T	T	T	T
F	F	T	T	T

From ① & ② Since the last columns are identical

$$\therefore P \rightarrow q \equiv \neg P \vee q$$

Ex 12.18

Establish the equivalence property connecting the bi-conditional with conditional:

$$P \leftrightarrow q \equiv (P \rightarrow q) \wedge (q \rightarrow P)$$

Soln:

P	q	$P \leftrightarrow q$	$P \rightarrow q$	$q \rightarrow P$	$(P \rightarrow q) \wedge (q \rightarrow P)$
T	T	T	T	T	T
T	F	F	F	T	F
F	T	F	T	F	F
F	F	T	T	T	T

From ① & ② we get
Since the last columns are identical

$$\therefore P \leftrightarrow q \equiv (P \rightarrow q) \wedge (q \rightarrow P)$$

Ex 12.19

Using the equivalence property

Show that $P \leftrightarrow q \equiv (P \wedge q) \vee (\neg P \wedge \neg q)$

Soln:

$$P \leftrightarrow q \equiv (\neg P \vee q) \wedge (\neg q \vee P) \quad \text{--- ①}$$

$$\equiv (\neg P \vee q) \wedge (P \vee \neg q) \quad [\text{By Commutative Law}]$$

$$\equiv (\neg P \wedge (P \vee \neg q)) \vee (q \wedge (P \vee \neg q)) \quad [\text{By Distributive Law}]$$

$$\equiv (\neg P \wedge P) \vee (\neg P \wedge \neg q) \vee (q \wedge P) \vee (q \wedge \neg q) \quad [\text{By Distributive Law}]$$

$$\equiv F \vee (\neg P \wedge \neg q) \vee (q \wedge P) \vee F \quad [\text{By Complement Law}]$$

$$\equiv (\neg P \wedge \neg q) \vee (q \wedge P) \quad [\text{By Identity Law}]$$

$$\equiv (P \wedge q) \vee (\neg P \wedge \neg q) \quad [\text{By Commutative Law}]$$

From ①,

$$P \leftrightarrow q \equiv (P \wedge q) \vee (\neg P \wedge \neg q)$$

EXERCISE 12.3

Choose the correct or the most suitable answer from the given four alternatives

1) A binary operation on a set S is a function from

Soln: [page number: 226]

By defn, $(S \times S) \rightarrow S$

Ans: (2) $(S \times S) \rightarrow S$

2) Subtraction is not a binary operation in

Soln:

Let $m=5, n=7, m, n \in \mathbb{N}$

Then $m-n=5-7=-2 \notin \mathbb{N}$

$\therefore m-n \notin \mathbb{N}$

$\therefore$ is not a binary operation on $\mathbb{N}$

Ans: (3) $\mathbb{N}$

3) Which one of the following is a binary operation on $\mathbb{N}$?

Soln:

Let $a, b \in \mathbb{N}$

Then $a * b \in \mathbb{N}, \forall a, b \in \mathbb{N}$

Ex: Let $a=3, b=4 \in \mathbb{N}$

Then $a * b = 3 * 4 = 12 \in \mathbb{N}$

$\therefore a * b \in \mathbb{N}, \forall a, b \in \mathbb{N}$

Ans: (2) Multiplication

4) In the set $\mathbb{R}$ of real numbers '*' is defined as follows. Which one of the following is not a binary operation on $\mathbb{R}$?

Soln: From the option $a * b = a^b$

Let $a=0, b=0$

Then $a * b = 0 * 0 = 0^0 \notin \mathbb{R}$

It is an indeterminate form

$\therefore a * b = a^b$ is not a binary operation on $\mathbb{R}$

Ans: (4) $a * b = a^b$

5) The operation * defined by $a * b = \frac{ab}{7}$ is not a binary operation on

Soln:

Let $a=2, b=3 \in \mathbb{Z}$ Then

$a * b = \frac{ab}{7} \Rightarrow 2 * 3 = \frac{(2)(3)}{7}$

$= \frac{6}{7} \notin \mathbb{Z}$

$\therefore$ Let $a, b \in \mathbb{Z}$ Then

$a * b \notin \mathbb{Z}$

Ans: (2) $\mathbb{Z}$

6) In the set Q define $a \odot b = a + b + ab$, For what value of y , $3 \odot (y \odot 5) = 7$?

Soln: Given: $a \odot b = a + b + ab$

$$3 \odot (y \odot 5) = 7$$

$$3 + (y \odot 5) + 3(y \odot 5) = 7$$

$$3 + y + 5 + 5y + 3(y + 5 + 5y) = 7$$

$$3 + y + 5 + 5y + 3y + 15 + 15y = 7$$

$$23 + 24y = 7$$

$$24y = 7 - 23$$

$$324y = -162$$

$$3y = -2$$

$$y = \frac{-2}{3}$$

$$\text{Ans: (2) } y = \frac{-2}{3}$$

7) If $a * b = \sqrt{a^2 + b^2}$ on the real numbers then $*$ is

Soln: Given: $a * b = \sqrt{a^2 + b^2}$

Let $a, b \in R$ (i) Commutative property:

$$\text{Then } a * b = \sqrt{a^2 + b^2} = \sqrt{b^2 + a^2} = b * a$$

$$\therefore a * b = b * a \quad \forall a, b \in R$$

$\therefore$ Commutative property is true.

Associative property:

Let $a, b, c \in R$ Then

$$a * (b * c) = a * (\sqrt{b^2 + c^2})$$

$$a * (b * c) = \sqrt{a^2 + b^2 + c^2} \quad \text{--- (1)}$$

$$(a * b) * c = (\sqrt{a^2 + b^2}) * c$$

$$(a * b) * c = \sqrt{a^2 + b^2 + c^2} \quad \text{--- (2)}$$

From (1) & (2) we get

$$a * (b * c) = (a * b) * c, \quad \forall a, b, c \in R$$

$\therefore$ Associative property is true.

Ans: (3) both Commutative and Associative

8) Which one of the following statements has the truth value T?

Soln:

Ans: (4) $\sqrt{5}$ is an irrational number

9) Which one of the following statements has truth value F?

Soln: (1) $T \vee F = T$

(2) $T \vee T = T$ (3) $F \vee F = F$

(4) $F \vee T = T$

Ans: (3) Chennai is in China or $\sqrt{2}$ is an integer

10) If a Compound Statement involves 3 simple statements, then the number of rows in the truth table is

Soln:

The number of rows is

$$2^n = 2^3 = 8$$

Ans: (2) 8

11) Which one is the inverse of the statement $(p \vee q) \rightarrow (p \wedge q)$?

Soln:

Inverse of $p \rightarrow q$ is $\neg p \rightarrow \neg q$

Also,

Inverse of $(p \vee q) \rightarrow (p \wedge q)$ is

$$\neg(p \vee q) \rightarrow \neg(p \wedge q) \Rightarrow$$

$$\neg p \wedge \neg q \rightarrow \neg p \vee \neg q$$

$$\text{Ans: (4) } (\neg p \wedge \neg q) \rightarrow (\neg p \vee \neg q)$$

12) Which one is the contrapositive of the statement $(p \vee q) \rightarrow r$?

Soln:

Contrapositive of $p \rightarrow q$ is $\neg q \rightarrow \neg p$

Also,

Contrapositive of $(p \vee q) \rightarrow r$ is

$$\neg r \rightarrow \neg(p \vee q)$$

$$\Rightarrow \neg r \rightarrow (\neg p \wedge \neg q)$$

$$\text{Ans: (1) } \neg r \rightarrow (\neg p \wedge \neg q)$$

13) The truth table for $(p \wedge q) \vee \neg q$ is given below:

p	q	$(p \wedge q) \vee (\neg q)$
T	T	(a)
T	F	(b)
F	T	(c)
F	F	(d)

Which one of the following is true?

P	q	$P \wedge q$	$\neg q$	$(P \wedge q) \vee (\neg q)$
T	T	T	F	T (a)
T	F	F	T	T (b)
F	T	F	F	F (c)
F	F	F	T	T (d)

Ans: (3) T T F T

14) In the last column of the truth table for $\neg(P \vee \neg q)$ the number of final outcomes of the truth value 'F' are

P	q	$\neg q$	$P \vee \neg q$	$\neg(P \vee \neg q)$
T	T	F	T	F
T	F	T	T	F
F	T	F	F	T
F	F	T	T	F

$\therefore$ The number of final outcomes of the truth value 'F' are 3

Ans: (3) 3

15) Which one of the following is incorrect? For any two propositions P and q, we have

Soln:

Ans: (3) $\neg(P \vee q) \equiv \neg P \vee \neg q$

16)

P	q	$(P \wedge q) \rightarrow \neg P$
T	T	(a)
T	F	(b)
F	T	(c)
F	F	(d)

Which one of the following is correct for the truth value of $(P \wedge q) \rightarrow \neg P$?

Soln:

P	q	$P \wedge q$	$\neg P$	$(P \wedge q) \rightarrow \neg P$
T	T	T	F	F (a)
T	F	F	F	T (b)
F	T	F	T	T (c)
F	F	F	T	T (d)

Ans: (2) F T T T

17) The dual of $\neg(P \wedge q) \vee [P \vee (P \wedge \neg r)]$ is

Soln:

The dual of a statement formula is obtained by replacing $\vee$ by $\wedge$ and $\wedge$ by $\vee$

$\therefore$ The dual of

$\neg(P \wedge q) \vee [P \vee (P \wedge \neg r)]$ is

$\neg(P \wedge q) \wedge [P \wedge (P \vee \neg r)]$

Ans: (4) $\neg(P \wedge q) \wedge [P \wedge (P \vee \neg r)]$

18) The proposition $P \wedge (\neg P \vee q)$ is

Soln:

$$P \wedge (\neg P \vee q) \equiv (P \wedge \neg P) \vee (P \wedge q) \quad [\text{By Distributive law}]$$

$$\equiv F \vee (P \wedge q)$$

$$P \wedge (\neg P \vee q) \equiv P \wedge q$$

Ans: (3) logically equivalent to $P \wedge q$

19) Determine the truth value of each of the following statements:

(a) $4+2=5$ and $6+3=9$

(b) $3+2=5$ and $6+1=7$

(c) $4+5=9$ and $1+2=4$

(d) $3+2=5$ and $4+7=11$

Soln:

(a) $F \wedge T = F$ (c) $T \wedge F = F$

(b) $T \wedge T = T$ (d) $T \wedge T = T$

Ans: (1) F T F T

20) Which one of the following is not true?

Soln:

P	q	$P \leftrightarrow q$
T	T	T
T	F	F
F	T	F
F	F	T

Ans: (4) If P and q are any two statements then $P \leftrightarrow q$ is a tautology.