Higher Secondary - First Year – Mathematics

Chapter : 2 – Basic Algebra

π and $\sqrt{p}$, where p is a prime number, are some irrational numbers.

- $|x - a| = r$ if and only if $r \geq 0$ and $x - a = \pm r$.
- $|x - a| \leq r$ if and only if $-r \leq x - a \leq r$ or $a - r \leq x \leq a + r$.
- $|x - a| > r$ implies $x < a - r$ and $x > a + r$ (or) $x \in (-\infty, a - r) \cup (a + r, \infty)$
- Inequalities, in general, have more than one solution.
- The nature of roots of $ax2 + bx + c = 0$ is determined by the discriminant $D = b2 - 4ac$.
- A real number a is a zero of a polynomial function $f(x)$ if and only if $(x - a)$ is a factor of $f(x)$.
- If degree of $f(x)$ is less than the degree of $g(x)$, then $f(x)$ $g(x)$ can be written as sum of its partial fractions.
- In general exponential functions and logarithmic functions are inverse functions to each other

Real Number System

N = {1, 2, 3, . . . } is enough for counting objects. In order to deal with loss or debts, we enlarged N to the set of all integers, Z = {. . . ,−4,−3,−2,−1, 0, 1, 2, . . .} , which consists of the natural numbers, zero, and the negatives of natural numbers. We call {0, 1, 2, 3, · · · } as the setof whole numbers and denote it by W. Note that it differs from N by just one element, namely,zero. Now imagine dividing a cake into five equal parts, which is equivalent to finding a solutionof $5x = 1$. But this equation cannot be solved within Z. Hence we have enlarged Z to the set Q =_ *mn* | *m, n* ∈ Z, *n* _= 0of *ratios*; so we call each $x \in$ Q as a ***rational*** number. Some examples of rational numbers are−5,−73, 0,22/7, 7, 12.

The Number Line

Let us recall "*The Number Line*". It is a horizontal line with the *origin*, to represent 0, and anotherpoint marked to the right of 0 to represent 1. The distance from 0 to 1 defines one unit of length. Now put 2 one unit to the right of 1. Similarly we put any positive rational number x to the right of 0 and$x$ units away. Also, we put a negative rational number $-r$, $r >$ 0, to the left of 0 by r units. Note thatfor any $x, y \in$ Q if $x < y$, then x is to the left of y; also $x < x + y/2 < y$ and hence between any twodistinct rational numbers there is another rational number between them.

2.2.3 Irrational Numbers

Theorem : √2 is not a rational number.

Proof. Suppose that √2 is a rational number. Let$\sqrt{2} = mn$, where m and n are positive integers with no common factors greater than 1. Then, we have $m2 = 2n2$, which implies that $m2$ is even and hence m is even. Let $m = 2k$. Then, we have $2n2 = 4k2$ which gives $n2 = 2k2$. Thus, n is also even. It follows, that m and n are even numbers having a common factor 2.Thus, we arrived at a contradiction. Hence, √2 is an irrational number.

The number π, which is the ratio of the circumference of a circle to its diameter, is an irrational number

Properties of Real Numbers

(i) For any $a, b \in$ R, $a + b \in$ R and $ab \in$ R.
[Sum of two real numbers is again a real number and multiplication of two real numbers isagain a real number.]
(ii) For any $a, b, c \in$ R, $(a + b) + c = a + (b + c)$ and $a(bc) = (ab)c$.
[While adding (or multiplying) finite number of real numbers, we can add (or multiply) by grouping them in any order.]
(iii) For all $a \in$ R, $a + 0 = a$ and $a(1) = a$.
(iv) For every $a \in$ R, $a + (-a) = 0$ and for every $b \in$ R − {0}, b_1b_= 1.
(v) For any $a, b \in$ R, $a + b = b + a$ and $ab = ba$.
(vi) For $a, b, c \in$ R, $a(b + c) = ab + ac$.
(vii) For $a, b \in$ R, $a < b$ if and only if $b - a > 0$.
(viii) For any $a \in$ R, $a2 \geq 0$.
(ix) For any $a, b \in$ R, only one of the following holds: $a = b$ or $a < b$ or $a > b$.
(x) If $a, b \in$ R and $a < b$, then $a + c < b + c$ for all $c \in$ R.
(xi) If $a, b \in$ R and $a < b$, then $ax < bx$ for all $x > 0$.
(xii) If $a, b \in$ R and $a < b$, then $ay > by$ for all $y < 0$.

Absolute Value

Definition and Properties

As we have observed that there is an order preserving one-to-one correspondence between elements of R and points on the number line. Note that for each $x \in$ R, x and $-x$ are equal distance from the origin. The distance of the number $a \in$ R from 0 on the number line is called the ***absolute value*** of that number a and is denoted by $|a|$.

$$|x| = \begin{cases} x & \text{if } x \geq 0, \\ -x & \text{if } x < 0. \end{cases}$$

(i) For any $x \in$ R, we have $|x| = |-x|$ and thus, $|x| = |y|$ if and only if $x = y$ or $x = -y$.
(ii) $|x - a| = r$ if and only if $r \geq 0$ and $x - a = r$ or $x - a = -r$.
Equations Involving Absolute Value
a real number a is said to be a solution of an equation or an inequality, if the statement obtained after replacing the variable by a is true.

(i) If $x, y \in \mathbb{R}$, $|y + x| = |x - y|$, then $xy = 0$.
(ii) For any $x, y \in \mathbb{R}$, $|xy| = |x||y|$.
(iii) $\left|\frac{x}{y}\right| = \frac{|x|}{|y|}$, for all $x, y \in \mathbb{R}$ and $y \neq 0$.
(iv) For any $x, y \in \mathbb{R}$, $|x + y| \leq |x| + |y|$.

Inequalities Involving Absolute Value

solve inequalities involving absolute values. First we analyze very simple inequalities such as (i) $|x| < r$ and (ii) $|x| > r$.

(i) For any $a \in$ R, $|x - a| \leq r$ if and only if $-r \leq x - a \leq r$ if and only if $x \in [a - r, a + r]$.
(ii) For any $a \in$ R, $|x - a| \geq r$ is equivalent to $x - a \leq -r$ or $x - a \geq r$ if and only if $x \in (-\infty, a - r] \cup [a + r, \infty)$.

Linear Inequalities

Quadratic Formula – 5 Mark

$$\begin{aligned}P(x) &= ax^2+bx+c\\ &= a\left(x^2+2x\frac{b}{2a}+\frac{c}{a}\right)\\ &= a\left(x^2+2x\frac{b}{2a}+\left(\frac{b}{2a}\right)^2-\frac{b^2}{4a^2}+\frac{c}{a}\right)\\ &= a\left(x+\frac{b}{2a}\right)^2-a\frac{b^2}{4a^2}+c\\ &= a\left(x+\frac{b}{2a}\right)^2+\left(a\left(\frac{b}{2a}\right)^2-b\frac{b}{2a}+c\right).\end{aligned}$$

$$\text{Thus,}\quad P(x) = a\left(x+\frac{b}{2a}\right)^2+P\left(\frac{b}{2a}\right). \qquad (1)$$

Now, to find the x- intercepts of the curve described by $P(x)$, let us solve for $P(x)=0$.

Considering $P(x)=0$ from (1) it follows that $a\left(x+\frac{b}{2a}\right)^2+P\left(\frac{b}{2a}\right)=0$.

$$\begin{aligned}a\left(x+\tfrac{b}{2a}\right)^2 &= -P\left(\tfrac{b}{2a}\right)\\ &= -\tfrac{(b^2-4ac)}{4a}\\ \left(x+\tfrac{b}{2a}\right)^2 &= \tfrac{b^2-4ac}{4a^2}.\end{aligned}$$

So $x=\frac{\sqrt{b^2-4ac}}{2a}-\frac{b}{2a}$ or $x=-\frac{\sqrt{b^2-4ac}}{2a}-\frac{b}{2a}$.

Hence, $x=\frac{-b\pm\sqrt{b^2-4ac}}{2a}$; which is called the *quadratic formula*.

Discriminant	Nature of roots	Parabola
Positive	real and distinct	intersects x-axis at two points
Zero	real and equal	touches x-axis at one point
Negative	no real	roots does not meet x-axis

Polynomial Functions

expression of the form $a_nx_n + a_n-1x_n-1 + \cdot\cdot\cdot + a0$, where $ai \in$ R, i =0, 1, 2, · · · , n, is called a ***polynomial*** in the variable x. Here n is a non-negative integer. When an _= 0, we say that the polynomial has degree n. The numbers $a_0, a_1, \ldots, an \in$ R are called the coefficients of the polynomial. The number $a0$ is called the constant term and an is called the leading coefficient (when it is non-zero).

A function of the form $P(x) = anxn + an-1xn-1 + \cdot\cdot\cdot + a0$ is called a ***polynomial function*** which isdefined from R to R. We shall treat polynomial and polynomial function as one and the same.
A polynomial with degree 1 is called ***a linear polynomial***. A polynomial with degree 2 is called a ***quadratic polynomial.*** A cubic polynomial is one that has degree three. Likewise, degree 4 and degree 5 polynomials are called ***quartic*** and ***quintic*** polynomials respectively.
Note that any constant a =/ 0 is a polynomial of degree zero!

Division Algorithm

Remainder Theorem

If a polynomial *f*(*x*) is divided by *x*−*a*, then the remainder is *f*(*a*). Thus the remainder $c = f(a) = 0$ if and only if $x - a$ is a factor for *f*(*x*).

Definition 2.1

A real number *a* is said to be a *zero of the polynomial* f(x) if f(a) = 0. If *x* = *a* is a zero of f(x), then *x* − *a* is a *factor* for f(x).

An equation is said to be an ***identity*** if that equation remains valid for all values in its domain. Anequation is called ***conditional equation*** if it is true only for some (not all) of values in its domain.

Important Identities

For all *x, a, b* ∈R we have

1. (*x* + *a*)3 = (*x* + *a*)2(*x* + *a*) = *x*3 + 3*x*2*a* + 3*xa*2 + *a*3 = *x*3 + 3*xa*(*x* + *a*) + *a*3
2. (*x* − *b*)3 = *x*3 − 3*x*2*b* + 3*xb*2 − *b*3 = *x*3 − 3*xb*(*x* − *b*) + *b*3 taking *a* = −*b* in (1)
3. *x*3 + *a*3 = (*x* + *a*)(*x*2 − *xa* + *a*2)
4. *x*3 − *b*3 = (*x* − *b*)(*x*2 + *xb* + *b*2) taking *a* = −*b* in (3)
5. *xn* − *an* = (*x* − *a*)(*xn*−1 + *xn*−2*a* + · · · + *xn*−*k*−1*ak* + · · · + *an*−1), *n* ∈N
6. *xn* + *bn* = (*x* + *b*)(*xn*−1 − *xn*−2*b* + · · · + *xn*−*k*−1(−*b*)*k* + · · · + (−*b*)*n*−1), *n* ∈N

When the root has multiplicity 1, it is called a simple root.

Rational Functions

If the degree of the numerator *P*(*x*) is equal to or larger than that of the denominator *Q*(*x*), then we can write *P*(*x*) = *f*(*x*)*Q*(*x*)+*r*(*x*) where *r*(*x*) = 0 or the degree of *r*(*x*) is less than that of *Q*(*x*). So *P*(*x*)/*Q*(*x*)= *f*(*x*) +*r*(*x*)/*Q*(*x*)

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Partial Fractions

A rational expression $\frac{f(x)}{g(x)}$ is called a proper fraction if the degree of $f(x)$ is less than degree of $g(x)$, where $g(x)$ can be factored into linear factors and quadratic factors without real zeros. Now $\frac{f(x)}{g(x)}$ can be expressed in simpler terms, namely, as a sum of expressions of the form

(i) $\frac{A_1}{(x-a)} + \frac{A_2}{(x-a)^2} + \cdots + \frac{A_k}{(x-a)^k}$ if $x - a$ divides $g(x)$ and

(ii) $\frac{(B_1x + C_1)}{(x^2+ax+b)} + \frac{(B_2x + C_2)}{(x^2+ax+b)^2} + \cdots + \frac{(B_kx + C_k)}{(x^2+ax+b)^k}$ if $x^2 + ax + b$ has no real zeros and $(x^2 + ax + b)$ divides $g(x)$.

The resulting expression of $\frac{f(x)}{g(x)}$ is is called the *partial fraction decomposition*. Such a decomposition is *unique* for a given rational function.

Graphical Representation of Linear Inequalities

A straight line $ax + by = c$ divides the Cartesian plane into two parts. Each part is an half plane. A vertical line $x = c$ will divide the plane in left and right half planes and a horizontal line $y = k$ willdivide the plane into upper and lower half planes.
A point in the cartesian plane which is not on the line $ax + by = c$ will lie in exactly one ofthe two half planes determined by the line and satisfies one of the inequalities $ax + by < c$ or $ax + by > c$.
To identify the half plane represented by $ax + by < c$, choose a point P in any one of the half planes and substitute the coordinates of P in the inequality.
If the inequality is satisfied, then the required half plane is the one that contains P; otherwise therequired half plane is the one that does not contain P. When $c \neq 0$, it is most convenient to take P to be the origin.

Exponents and Radicals

Exponents

Let $n \in N$, $a \in R$. Then $a^n = a \cdot a \cdot \cdot \cdot a$ (n times). If m is a negative integer and the real number
$a \neq 0$, then $a^m = 1/a^{-m}$.
Note that for any $a \neq 0$, we have
$\frac{a}{a} = a^{1-1} = a^0 = 1$..

Properties of Exponents

(i) For $m, n \in Z$ and $a \neq 0$, we have $a^m a^n = a^{m+n}$.
(ii) For $m, n \in Z$ and $a \neq 0$, we have $a^m / a^n = a^{m-n}$.

Definition 2.2

(i) For $n \in N$, n even, and $b > 0$, there is a unique $a > 0$ such that $a^n = b$.
(ii) For $n \in N$, n odd, $b \in R$, there is a unique $a \in R$ such that $a^n = b$. In both cases a is called the nth root of b or radical and is denoted by $b^{1/n}$ or $\sqrt[n]{b}$

Exponential Function

Observe that for any $a > 0$ and $x \in R$, a^x can be defined. If $a = 1$, we define $1^x = 1$. So we shall consider a^x, $x \in R$ for $0 < a \neq 1$. Here a^x is called exponential function with base a. Note that a^x may not be defined if $a < 0$ and $x =$
$1/m$for even $m \in N$. This is why we restrict to $a > 0$. Also,
$a^x > 0$ for all $x \in R$.

Properties of Exponential Function

For $a, b > 0$ and $a \neq 1 \neq b$
(i) $a^{x+y} = a^x a^y$ for all $x, y \in R$,
(ii)$a^x / a^y = a^{x-y}$ for all $x, y \in R$,

(iii) $(ax)y = axy$ for all $x, y \in$ R,
(iv) $(ab)x = axbx$ for all $x \in$ R,
(v) $ax = 1$ if and only if $x = 0$.

A Special Exponential Function

Among all exponential functions, $f(x) = ex, x \in$ R is the most important one as it has applications in many areas like mathematics, science and economics. Then what is this e? The following illustration from compounding interest problem leads to the constant e.

a base $0 < a=/1$, the exponential function $f(x) = ax$ is defined on R having range $(0, \infty)$. We also observed that $f(x)$ is a bijection, hence it has an inverse. We call this inversefunction as ***logarithmic function*** and is denoted by log$a$(.). Let us discuss this function further. Notethat if $f(x)$ takes x to $y = ax$, then log$a$(.) takes y to x.
That is, for $0 < a=/1$, we have
$y = ax$ is equivalent to log$a$ $y = x$.

(i) Note that exponential function ax is defined for all $x \in$ R and $ax > 0$ and so log$a$(·) defined only for positive real numbers.
(ii) Also, $a0 = 1$ for any base a and hence log$a$(1) = 0 for any base a.

2.9.1 Properties of Logarithm

(i) $a^{\log_a x} = x$ for all $x \in (0, \infty)$ and $\log_a(a^y) = y$ for all $y \in \mathbb{R}$.

(ii) For any $x, y > 0$, $\log_a(xy) = \log_a x + \log_a y$. (Product Rule)

(iii) For any $x, y > 0$, $\log_a\left(\frac{x}{y}\right) = \log_a x - \log_a y$. (Quotient Rule)

(iv) For any $x > 0$ and $r \in \mathbb{R}$, $\log_a x^r = r\log_a x$. (Power Rule)

(v) For any $x > 0$, with a and b as bases, $\log_b x = \frac{\log_a x}{\log_a b}$. (Change of base formula.)

Proof. Since exponential function with base a and logarithm function with base a are inverse of each other,

(i) follows by using the definitions.

(ii) For $x, y > 0$ let $\log_a x = u$, $\log_a y = v$, and $\log_a(xy) = w$. Rewriting these in the exponential form we obtain $a^u = x$, $a^v = y$, and, $a^w = xy$. So, $a^w = xy = a^u a^v = a^{u+v}$; thus $w = u + v$. Thus, we obtain $\log_a(xy) = \log_a x + \log_a y$.

(iii) Let $\log_a x = u$, $\log_a y = v$, and $\log_a \frac{x}{y} = w$. Then $a^u = x$, $a^v = y$ and $a^w = \frac{x}{y}$. Hence, $a^w = \frac{x}{y} = \frac{a^u}{a^v} = a^{u-v}$; which implies $w = u - v$.

Thus, we obtain $\log_a\left(\frac{x}{y}\right) = \log_a x - \log_a y$.

(iv) Let $\log_a x = u$. Then $a^u = x$ and therefore, $x^r = (a^u)^r = a^{ru}$. Thus, $\log_a x^r = ru = r\log_a x$.

(v) Let $\log_b x = v$. We have $b^v = x$. Taking logarithm with base a on both sides we get $\log_a b^v = \log_a x$.

On the other hand $\log_a b^v = v\log_a b$ by the Power rule. Therefore, $v\log_a b = \log_a x$.

Hence $\log_b x = \frac{\log_a x}{\log_a b}$, $b > 0$. This completes the proof.

□