SETS,RELATIONS AND FUNCTIONS

Example 1.1 Find the number of subsets of A if $A = \{x : x = 4n + 1; 2 \le n \le 5; n \in N\}$

Example 1.2 In a survey of 5000 persons in a town, it was found that 45% of the persons know Language A; 25% know Language B; 10% know Language C; 5% know Languages A and B; 4% know Languages B and C; and 4% know Languages A and C. If 3% of the persons know all the three Languages, find the number of persons who knows only Language A

Example 1.3 Prove that $((A \cup B' \cup C) \cap (A \cap B' \cap C')) \cup ((A \cup B \cup C') \cap (B' \cap C')) = B' \cap C'$

Example 1.4 If $X = \{1; 2; 3; : : : 10\}$ and $A = \{1; 2; 3; 4; 5\}$ find the number of sets $B \subseteq X$

such that $A - B = \{4\}$

Example 1.5 If A and B are two sets so that $n(B - A) = 2n(A - B) = 4n(A \cap B)$ and if $n(A \cup B) = 14$, then find $n(P(A))$.

Example 1.6 Two sets have m and k elements. If the total number of subsets of the first set is 112 more than that of the second set, find the values of m and k.

Example 1.7 If $n(A) = 10$ and $n(A \cap B) = 3$, find $n((A \cap B)' \cap A)$.

Example 1.8 If $A = \{1; 2; 3; 4\}$ and $B = \{3; 4; 5; 6\}$, find $n((A \cup B) \times (A \cap B) \times (A \triangle B))$.

Example 1.9 IfP(A) denotes the power set of A, then find $n(P(P(P(A))))$.

Exercise - 1.1

1. Write the following in roster form.

(i) $\{x \in N : X^2 < 121$ and x is a prime$\}$.

(ii) the set of all positive roots of the equation $(x - 1)(x + 1)(X^2 - 1) = 0$.

(iii) $\{x \in N : 4x + 9 < 52\}$. (iv) $\{x : \frac{x-4}{x+2} = 3; x \in R - \{-2\}\}$

2. Write the set $\{-1,1\}$ in set builder form.

3. State whether the following sets are finite or infinite.

(i) $\{x \in N : x$ is an even prime number$\}$. (ii) $\{x \in N : x$ is an odd prime number$\}$

(iii) $\{x \in Z : x$ is even and less than 10$\}$ (iv) $\{x \in R : x$ is a rational number$\}$

(v) $\{x \in N : x$ is a rational number$\}$

4. By taking suitable sets A;B;C, verify the following results:

(i) $A \times (B \cap C) = (A \times B) \cap (A \times C)$.

(ii) $A \times (B \cup C) = (A \times B) \cup (A \times C)$.

(iii) $(A \times B) \cap (B \times A) = (A \cap B) \times (B \cap A)$.

(iv) $C - (B - A) = (C \cap A) \cup (C \cap B')$.

(v) $(B - A) \cup C = (B \cup C) - A = B \cap (C - A)$.

(vi) $(B - A) \cup C = (B \cup C) - (A - C)$.

5. Justify the trueness of the statement: "An element of a set can never be a subset of itself."

6. If $n(P(A)) = 1024$; $n(A \cup B) = 15$ and $n(P(B)) = 32$, then find $n(A \cap B)$.

7. If $n(A \cap B) = 3$ and $n(A \cup B) = 10$, then find $n(P(A \triangle B))$.

8. For a set A; $A \times A$ contains 16 elements and two of its elements are (1; 3) and (0; 2): Find the elements of A:

9. Let A and B be two sets such that $n(A) = 3$ and $n(B) = 2$. If (x; 1); (y; 2); (z; 1) are in $A \times B$, find A and B, where x; y; z are distinct elements.

10. If $A \times A$ has 16 elements, $S = \{(a; b) \in A \times A : a < b\}$; (-1; 2) and (0; 1) are two elements of S, then find the remaining elements of S.

Example 1.10 Check the relation $R = \{(1; 1); (2; 2); (3; 3); : : : ; (n; n)\}$ defined on the set $S = \{1; 2; 3; : : : ; n\}$ for the three basic relations.

Example 1.11 Let $S = \{1; 2; 3\}$ and $\rho = \{(1; 1); (1; 2); (2; 2); (1; 3); (3; 1)\}$.

(i) Is ρ reflexive? If not, state the reason and write the minimum set of ordered pairs to be included to ρ so as to make it reflexive.

(ii) Is ρ symmetric? If not, state the reason, write minimum number of ordered pairs to be included to ρ so as to make it symmetric and write minimum number of ordered pairs to be deleted from ρ so as to make it symmetric.

(iii) Is ρ transitive? If not, state the reason, write minimum number of ordered pairs to be included to ρ so as to make it transitive and write minimum number of ordered pairs to be deleted from ρ so as to make it transitive.

(iv) Is ρ an equivalence relation? If not, write the minimum ordered pairs to be included to ρ so as to make it an equivalence relation.

Example 1.12 Let $A = \{0; 1; 2; 3\}$. Construct relations on A of the following types:

(i) not reflexive, not symmetric, not transitive. (ii) not reflexive, not symmetric, transitive.

(iii) not reflexive, symmetric, not transitive. (iv) not reflexive, symmetric, transitive.

(v) reflexive, not symmetric, not transitive. (vi) reflexive, not symmetric, transitive.

(vii) reflexive, symmetric, not transitive. (viii) reflexive, symmetric, transitive.

Example 1.13 In the set Z of integers, define mRn if m- n is a multiple of 12. Prove that R is an equivalence relation.

Exercise - 1.2

1. Discuss the following relations for reflexivity, symmetricity and transitivity:

(i) The relation R defined on the set of all positive integers by "mRn if m divides n".

(ii) Let P denote the set of all straight lines in a plane. The relation R defined by "l R m" if l is perpendicular to m".

(iii) Let A be the set consisting of all the members of a family. The relation R defined by "aRb if a is not a sister of b".

(iv) Let A be the set consisting of all the female members of a family. The relation R defined by "aRb if a is not a sister of b".

(v) On the set of natural numbers the relation R defined by "xRy if $x + 2y = 1$".

2. Let X = {a; b; c; d} and R = {(a; a); (b; b); (a; c)}. Write down the minimum number of ordered pairs to be included to R to make it (i) reflexive (ii) symmetric (iii) transitive (iv) equivalence

3. Let A = {a; b; c} and R = {(a; a); (b; b); (a; c)}. Write down the minimum number of ordered pairs to be included to R to make it (i) reflexive (ii) symmetric (iii) transitive (iv) equivalence

4. Let P be the set of all triangles in a plane and R be the relation defined on P as aRb if a is similar to b. Prove that R is an equivalence relation.

5. On the set of natural numbers let R be the relation defined by aRb if $2a + 3b = 30$. Write down the relation by listing all the pairs. Check whether it is
(i) reflexive (ii) symmetric (iii) transitive (iv) equivalence

6. Prove the relation "friendship" is not an equivalence relation on the set of all people in Chennai.

7. On the set of natural numbers let R be the relation defined by aRb if $a + b = 6$. Write down the relation by listing all the pairs. Check whether it is
(i) reflexive (ii) symmetric (iii) transitive (iv) equivalence

8. Let A = {a; b; c}. What is the equivalence relation of smallest cardinality on A? What is the equivalence relation of largest cardinality on A?

9. In the set Z of integers, define mRn if m - n is divisible by 7. Prove that R is an equivalence relation.

Example 1.14 Check whether the following functions are one-to-one and onto.

(i) $f : N \to N$ defined by $f(n) = n + 2$. (ii) $f : N \cup \{-1,0\} \to N$ defined by $f(n) = n + 2$.

Example 1.15 Check the following functions for one-to-oneness and ontoness.

(i) $f : N \to N$ defined by $f(n) = n^2$: (ii) $f : R \to R$ defined by $f(n) = n^2$

Example 1.16 Check whether the following for one-to-oneness and ontoness.

(i) $f : R \to R$ defined by $f(x) = \frac{1}{x}$ (ii) $f : R - \{0\} \to R$ defined by $f(x) = \frac{1}{x}$

Example 1.17 If $f : R - \{1; 1\} \to R$ is defined by $f(x) = \frac{x}{x^2-1}$ verify whether f is one-to-one or not

Example 1.18 If $f : R \to R$ is defined as $f(x) = 2x^2 - 1$, find the pre-images of 17; 4 and -2.

Example 1.19 If $f : [-2; 2] \to B$ is given by $f(x) = 2x^3$, then find B so that f is onto.

Example 1.20 Check whether the function $f(x) = x|x|$ defined on [-2; 2] is one-to-one or not. If it is one-to-one, find a suitable co-domain so that the function becomes a bijection

Example 1.21 Find the largest possible domain for the real valued function f , $f(x) = \sqrt{x^2 - 5x + 6}$

Example 1.22 Find the domain of $f(x) = \frac{1}{1-2\cos x}$

Example 1.23 Find the range of the function $f(x) = \frac{1}{1-3\cos x}$

Example 1.24 Find the largest possible domain for the real valued function given by $f(x) = \frac{\sqrt{9-x^2}}{\sqrt{x^2-1}}$

Example 1.25 Let f = {(1; 2); (3; 4); (2; 2)} and g = {(2; 1); (3; 1); (4; 2)}. Find g o f and fog.

Example 1.26 Let f = {(1; 4); (2; 5); (3; 5)} and g = {(4; 1); (5; 2); (6; 4)}. Find go f.

Can you find f o g?

Example 1.27 Let f and g be the two functions from R to R defined by f(x) = 3x- 4 and

$g(x) = x^2 + 3$. Find g o f and f o g.

Example 1.28 Show that the statement, "if f and g o f are one-to-one, then g is one-to-one" is not true.

Example 1.29 Let f; g : R → R be defined as f(x) = 2x - |x| and g(x) = 2x + |x|.

Find f o g , g of.

Example 1.30 If f : R → R is defined by f(x) = 2x - 3 prove that f is a bijection and find its Inverse

Theorem 1.1: The number of relations from a set containing m elements to a set containing n elements is 2^{mn}. In particular the number of relations on a set containing n elements is 2^{n^2}

Theorem 1.2: Let f : A → B and g : B → C . If f and g are one-to-one, then g o f is one-to-one

Theorem 1.3: If f and g are real-valued functions, then f(g + h) = fg + fh.

Theorem 1.4: The product of an odd function and an even function is an odd function

Exercise - 1.3

1. Suppose that 120 students are studying in 4 sections of eleventh standard in a school. Let A denote the set of students and B denote the set of the sections. Define a relation from A to B as "x related to y if the student x belongs to the section y". Is this relation a function? What can you say about

the inverse relation? Explain your answer.

2. Write the values of f at -4; 1;-2; 7; 0 if

$$f(x) = \begin{cases} -x+4 & if -\infty < x \le -3 \\ x+4 & if -3 < x < -2 \\ x^2 - x & if -2 \le x < 1 \\ x - x^2 & if \ 1 \le x < 7 \\ 0 & otherwise \end{cases}$$

3. Write the values of f at -3; 5; 2;-1; 0 if

$$f(x)\begin{cases} x^2+x-5 & if\ x\in(-\infty,0) \\ x^2+3x-2 & if x\in(3,\infty) \\ x^2 & if\ \ x\in(0,2) \\ x^2-3 & otherwise \end{cases}$$

4. State whether the following relations are functions or not. If it is a function check for one-to oneness and onto ness. If it is not a function, state why?

(i) If A = {a; b; c} and f = {(a; c); (b; c); (c; b)}; (f : A → A).

(ii) If X = {x; y; z} and f = {(x; y); (x; z); (z; x)}; (f : X →X).

5. Let A = {1; 2; 3; 4} and B = {a; b; c; d}. Give a function from A → B for each of the following:

(i) neither one-to-one nor onto. (ii) not one-to-one but onto.

(iii) one-to-one but not onto. (iv) one-to-one and onto.

6. Find the domain of $f(x)=\frac{1}{1-2sinx}$

7. Find the largest possible domain of the real valued function $f(x)=\frac{\sqrt{4-x^2}}{\sqrt{x^2-9}}$

8. Find the range of the function. $f(x)=\frac{1}{2COSx-1}$

9. Show that the relation xy = -2 is a function for a suitable domain. Find the domain and the range of the function.

10. If f; g : R→ R are defined by f(x) =|x| + x and g(x) = |x| - x, find g o f and f o g.

11. If f; g; h are real valued functions defined on R, then prove that (f +g) o h = f o h+g o h. What can you say about f o (g + h)? Justify your answer.

12. Iff : R → R is defined by f(x) = 3x - 5, prove that f is a bijection and find its inverse.

13. The weight of the muscles of a man is a function of his body weight x and can be expressed as W(x) = 0:35x. Determine the domain of this function.

14. The distance of an object falling is a function of time t and can be expressed as s(t) = -16t^2. Graph the function and determine if it is one-to-one.

15. The total cost of airfare on a given route is comprised of the base cost C and the fuel surcharge S in rupee. Both C and S are functions of the mileage m; C(m) = 0.4m + 50 and S(m) = 0.03m. Determine a function for the total cost of a ticket in terms of the mileage and find the airfare for flying 1600 miles.

16. A salesperson whose annual earnings can be represented by the function A(x) = 30000+0.04x, where x is the rupee value of the merchandise he sells. His son is also in sales and his earnings are represented by the function S(x) = 25 000 + 0.05x. Find (A + S)(x) and determine the total family income if they each sell Rupees 1 50 00 000 worth of merchandise.

17. The function for exchanging American dollars for Singapore Dollar on a given day is $f(x) = 1.23x$, where x represents the number of American dollars. On the same day the function for exchanging Singapore Dollar to Indian Rupee is $g(y) = 50.50y$, where y represents the number of Singapore dollars. Write a function which will give the exchange rate of American dollars in terms of Indian rupee.

18. The owner of a small restaurant can prepare a particular meal at a cost of Rupees 100. He estimates that if the menu price of the meal is x rupees, then the number of customers who will order that meal at that price in an evening is given by the function $D(x) = 200-x$. Express his day revenue, total cost and profit on this meal as functions of x.

19. The formula for converting from Fahrenheit to Celsius temperatures is $y = \frac{5x}{9} - \frac{160}{9}$

Find the inverse of this function and determine whether the inverse is also a function.

20. A simple cipher takes a number and codes it, using the function $f(x) = 3x-4$. Find the inverse of this function, determine whether the inverse is also a function and verify the symmetrical property about the line $y = x$ (by drawing the lines).

Exercise - 1.4

1. For the curve $y = X^3$ given in, draw

(i) $y = -X^3$ (ii) $y = X^3 + 1$ (iii) $y = X^3 - 1$ (iv) $y = (X+1)^3$ with the same scale

2.the curve $y = x^{\frac{1}{3}}$ (i) $y = -x^{\frac{1}{3}}$ (ii) $y = x^{\frac{1}{3}} + 1$ (iii) $y = x^{\frac{1}{3}} - 1$ (iv) $y = (x+1)^{\frac{1}{3}}$

3. Graph the functions $f(x) = X^3$ and $g(x) = \sqrt[3]{x}$ on the same coordinate plane. Find f o g and graph it on the plane as well. Explain your results.

4. Write the steps to obtain the graph of the function $y = 3(x-1)^2 + 5$ from the graph $y=x^2$

5. From the curve $y = \sin x$, graph the functions

(i) $y = \sin(-x)$ (ii) $y = -\sin(-x)$ (iii) $y = \sin(\frac{\pi}{2} + x)$ which is cos x

(iv) $y = \sin(\frac{\pi}{2} - x)$ which is also cos x (refer trigonometry)

6. From the curve $y = x$, draw

(i) $y = -x$ (ii) $y = 2x$ (iii) $y = x + 1$ (iv) $y = \frac{1}{2}x + 1$ (v) $2x + y + 3 = 0$.

7. From the curve $y = |x|$, draw (i) $y = |x - 1| + 1$ (ii) $y = |x + 1| - 1$ (iii) $y = |x + 2| - 3$.

8. From the curve $y = \sin x$, draw $y = \sin |x|$ (Hint: $\sin(-x) = -\sin x$.)

BASIC ALGEBRA

Theorem 2.1: $\sqrt{2}$ is not a rational number.

Exercise - 2.1

1. Classify each element of $\{\sqrt{7}, \frac{1}{4}, 0, 3.14, 4, \frac{22}{7}\}$ as a member of N, Q, R - Q or Z.

2. Prove that $\sqrt{3}$ is an irrational number

3. Are there two distinct irrational numbers such that their difference is a rational number? Justify.

4. Find two irrational numbers such that their sum is a rational number. Can you find two irrational numbers whose product is a rational number.

5. Find a positive number smaller than $\frac{1}{2^{1000}}$ Justify.

Example 2.1 Solve |2x - 17| = 3 for x:

Example 2.2 Solve 3|x - 2| + 7 = 19 for x:

Example 2.3 Solve |2x - 3| = |x - 5|

Example 2.4 Solve |x - 9| < 2 for x:

Example 2.5 Solve $\left|\frac{2}{x-4}\right| > 1, x \neq 4$.

Exercise - 2.2

1. Solve for x: (i) |3 - x| < 7. (ii) |4x - 5|≥-2. (iii) $|3 - \frac{3}{4}x| \leq \frac{1}{4}$ (iv) |xj|- 10 < -3.

2. Solve $\frac{1}{|2x-1|} < 6$ and express the solution using the interval notation.

3. Solve -3|x| + 5 ≤-2 and graph the solution set in a number line.

4. Solve 2|x + 1|- 6 ≤ 7 and graph the solution set in a number line.

5. Solve $\frac{1}{5}|10x - 2| < 1$: 6. Solve |5x - 12| < -2.

Example 2.6 Our monthly electricity bill contains a basic charge, which does not change with number of units used, and a charge that depends only on how many units we use. Let us say Electricity Board charges Rs.110 as basic charge and charges Rs. 4 for each unit we use. If a person wants to keep his electricity bill below Rs.250, then what should be his electricity usage?

Example 2.7 Solve 3x - 5 ≤x + 1 for x.

EXAMPLE2.8 Solve the following system of linear inequalities. 3x - 9 ≥ 0; 4x -10 ≤ 6:

Example 2.9 A girl A is reading a book having 446 pages and she has already finished reading 271 pages. She wants to finish reading this book within a week. What is the minimum number of pages she should read per day to complete reading the book within a week?

Exercise - 2.3

1. Represent the following inequalities in the interval notation:

(i) x ≥-1 and x < 4 (ii) x ≤ 5 and x ≥ -3 (iii) x < -1 or x < 3 (iv) -2x > 0 or 3x - 4 < 11:

2. Solve 23x < 100 when (i) x is a natural number, (ii) x is an integer.

3. Solve -2x ≥ 9 when (i) x is a real number, (ii) x is an integer, (iii) x is a natural number.

4. Solve: (i) $\frac{3(x-2)}{5} \leq \frac{5(2-x)}{3}$. (ii) $\frac{5-x}{3} < \frac{x}{2} - 4$

5. To secure A grade one must obtain an average of 90 marks or more in 5 subjects each of maximum 100 marks. If one scored 84; 87; 95; 91 in first four subjects, what is the minimum mark one scored in the fifth subject to get A grade in the course?

6. A manufacturer has 600 litres of a 12 percent solution of acid. How many litres of a 30 percent acid solution must be added to it so that the acid content in the resulting mixture will be more than 15 percent but less than 18 percent?

7. Find all pairs of consecutive odd natural numbers both of which are larger than 10 and their sum is less than 40.

8. A model rocket is launched from the ground. The height h reached by the rocket after t seconds from lift off is given by $h(t) = -5t^2+100t;\ 0 \le t \le 20$. At what time the rocket is 495 feet above the ground?

9. A plumber can be paid according to the following schemes: In the first scheme he will be paid rupees 500 plus rupees 70 per hour, and in the second scheme he will paid rupees 120 per hour. If he works x hours, then for what value of x does the first scheme give better wages?

10. A and B are working on similar jobs but their annual salaries differ by more than Rs 6000. If B earns rupees 27000 per month, then what are the possibilities of A's salary per month?

Example 2.10 If a and b are the roots of the equation $x^2 - px + q = 0$; find the value of $\frac{1}{a}+\frac{1}{b}$

Example 2.11 Find the complete set of values of a for which the quadratic $x^2 - ax + a + 2 = 0$ has equal roots.

Example 2.12 Find the number of solutions of $x^2 + |x - 1| = 1$:

Exercise - 2.4

1. Construct a quadratic equation with roots 7 and -3:

2. A quadratic polynomial has one of its zeros $1 + \sqrt{5}$ and it satisfies $p(1) = 2$. Find the quadratic polynomial.

3. If α and β are the roots of the quadratic equation $x^2 + \sqrt{2}x + 3 = 0$, form a quadratic polynomial with zeroes $\frac{1}{\alpha}, \frac{1}{\beta}$

4. If one root of $k(x-1)^2 = 5x - 7$ is double the other root, show that $k = 2$ or -25.

5. If the difference of the roots of the equation $2x^2 - (a + 1)x + a - 1 = 0$ is equal to their product, then prove that $a = 2$.

6. Find the condition that one of the roots of $ax^2 + bx + c$ may be (i) negative of the other, (ii) thrice the other, (iii) reciprocal of the other.

7. If the equations $x^2 - ax + b = 0$ and $x^2 - ex + f = 0$ have one root in common and if the second equation has equal roots, then prove that $ae = 2(b + f)$.

8. Discuss the nature of roots of (i) $-x^2 + 3x + 1 = 0$, (ii) $4x^2 - x - 2 = 0$, (iii) $9x^2 + 5x = 0$.

9. Without sketching the graphs, find whether the graphs of the following functions will intersect the x-axis and if so in how many points. (i) $y = x^2 + x + 2$, (ii) $y = x^2 - 3x - 7$, (iii) $y = x^2 + 6x + 9$.

10. Write $f(x) = x^2 + 5x + 4$ in completed square form.

Example 2.13 Solve $3x^2 + 5x - 2 \leq 0$ Example 2.14 Solve $\sqrt{x+14} < x + 2$.

Example 2.15 Solve the equation $\sqrt{6-4x-x^2} = x + 4$

Exercise - 2.5

1. Solve $2x^2 + x - 15 \leq 0$. 2. Solve $-x^2 + 3x - 2 \geq 0$:

Exercise - 2.6

1. Find the zeros of the polynomial function $f(x) = 4x^2 - 25$.
2. If $x = -2$ is one root of $x^3 - x^2 - 17x = 22$, then find the other roots of equation.
3. Find the real roots of $x^4 = 16$:
4. Solve $(2x+1)^2 - (3x+2)^2 = 0$:

Example 2.16 Find a quadratic polynomial f(x) such that, f(0) = 1; f(-2) = 0 and f(1) = 0.

Example 2.17 Construct a cubic polynomial function having zeros at $x = \frac{2}{5}, 1, 1+\sqrt{5}$ such that f(0) = -8:

Example 2.18 Prove that ap+q = 0 if $f(x) = x^3 - 3px + 2q$ is divisible by $g(x) = x^2 + 2ax + a^2$

Example 2.19 Use the method of undetermined coefficients to find the sum of $1 + 2 + 3 + \ldots + (n - 1) + n; n \in N$

Example 2.20 Find the roots of the polynomial equation $(x-1)^3(x+1)^2(x-5) = 0$ and state their multiplicity.

Example 2.21 Solve $x = \sqrt{x+20}$ for $x \in R$.

Example 2.22 The equations $x^2 - 6x + a = 0$ and $x^2 - bx + 6 = 0$ have one root in common. The other root of the first and the second equations are integers in the ratio 4 : 3. Find the common root.

Example 2.23 Find the values of p for which the difference between the roots of the equation $x^2 + px + 8 = 0$ is 2.

Exercise - 2.7

1. Factorize: $x^4 + 1$. (Hint: Try completing the square.)
2. If $x^2 + x + 1$ is a factor of the polynomial $3x^3 + 8x^2 + 8x + a$; then find the value of a.

Example 2.24 Solve $\frac{x+1}{x+3} < 3$

Exercise - 2.8

1. Find all values of x for which $\frac{x^3(x-1)}{(x-2)} > 0$
2. Find all values of x that satisfies the inequality $\frac{2x-3}{(x-2)(x-4)} < 0$.
3. Solve $\frac{x^2-4}{x^2-2x-15} \leq 0$

Example 2.25 Resolve into partial fractions: $\frac{x}{(x+3)(x-4)}$

Example 2.26 Resolve into partial fractions: $\frac{2x}{(x^2+1)(x-1)}$

Example 2.27 Resolve into partial fractions: $\frac{x+1}{x^2(x-1)}$

Exercise - 2.9

Resolve the following rational expressions into partial fractions.

1. $\frac{1}{x^2-a^2}$ 2. $\frac{3x+1}{(x-2)(x+1)}$ 3. $\frac{x}{(x^2+1)(x-1)(x+2)}$ 4. $\frac{x}{(x-1)^3}$ 5. $\frac{1}{x^4-1}$ 6. $\frac{(x-1)^2}{x^3+x}$

7. $\frac{x^2+x+1}{x^2-5x+6}$ 8. $\frac{x^3+2x+1}{x^2+5x+6}$ 9. $\frac{x+12}{(x-1)^2(x-2)}$ 10. $\frac{6x^2-x+1}{x^3+x^2+x+1}$ 11. $\frac{2x^2+5x-11}{x^2+2x-3}$

12. $\frac{7+x}{(1+x)(1+x^2)}$

Example 2.28 Shade the region given by the inequality $x \geq 2$.

Example 2.29 Shade the region given by the linear inequality $x + 2y > 3$

Example 2.30 Solve the linear inequalities and exhibit the solution set graphically:

$x + y \geq 3$; $2x - y \leq 5$; $-x + 2y \leq 3$:

Exercise - 2.10

Determine the region in the plane determined by the inequalities:

(1) $x \leq 3y, x \geq y$. (2) $y \geq 2x, -2x + 3y \leq 6$. (3) $3x + 5y \geq 45, x \geq 0, y \geq 0$.

(4) $2x + 3y \leq 35, y \geq 2, x \geq 5$. (5) $2x + 3y \leq 6$; $x + 4y \leq 4$; $x \geq 0$; $y \geq 0$:

(6) $x - 2y \geq 0$; $2x - y \leq -2$; $x \geq 0$; $y \geq 0$: (7) $2x + y \geq 8$; $x + 2y \geq 8$; $x + y \leq 6$.

Example 2.31 (i) Simplify: $(x^{1/2}y^{-3})^{1/2}$ where x; $y \geq 0$.

(ii) Simplify $\sqrt{x^2 - 10x + 25}$ Example 2.32 Rationalize the denominator of $\frac{\sqrt{5}}{\sqrt{6}+\sqrt{2}}$

Example 2.33 Find the square root of $7 - 4\sqrt{3}$

Exercise - 2.11

1. Simplify: $(i)(125)^{\frac{2}{3}}$ $(ii)16^{-\frac{3}{4}}$ $(iii)(-1000)^{-\frac{2}{3}}$ $(iv)(3^{-6})^{\frac{1}{3}}$ $(v)\ \frac{27^{-\frac{2}{3}}}{27^{-\frac{1}{3}}}$

2. Evaluate $(((256)^{-\frac{1}{2}})^{-\frac{1}{4}})^3$

3. If $(x^{\frac{1}{2}} + x^{-\frac{1}{2}})^4 = \frac{9}{2}$, then find the value of $x^{\frac{1}{2}} - x^{-\frac{1}{2}}$ for $x > 1$.

4.Simplify and hence find the value of n: $3^{2n}9^2 3^{-n} / 3^{3n} = 27$:

5. Find the radius of the spherical tank whose volume is $32\pi/3$ units.

6. Simplify by rationalising the denominator. $\frac{7+\sqrt{6}}{3-\sqrt{2}}$

7. Simplify $\frac{1}{1-\sqrt{8}} - \frac{1}{\sqrt{8}-\sqrt{7}} + \frac{1}{\sqrt{7}-\sqrt{6}} - \frac{1}{\sqrt{6}-\sqrt{5}} + \frac{1}{\sqrt{5}-2}$

8. If $x = \sqrt{2} + \sqrt{3}$ find $\frac{x^2+1}{x^2-2}$

Example 2.34 Find the logarithm of 1728 to the base $2\sqrt{3}$

Example 2.35 If the logarithm of 324 to base a is 4, then find a

Example 2.36 Prove $\log\frac{75}{16} - 2\log\frac{5}{9} + \log\frac{32}{243} = log2$

Example 2.37 If $\log_2 x + \log_4 x + \log_{16} x = \frac{7}{2}$, find the value of x

Example 2.38 Solve $x^{\log_3 x} = 9$:

Example 2.39 Compute $\log_3 5 \log_{25} 27$.

Example 2.40 Given that $\log_{10} 2 = 0:30103$; $\log_{10} 3 = 0:47712$ (approximately), find the number of digits in $2^8 3^{12}$.

Exercise - 2.12

1. Let b > 0 and b ≠1. Express $y = b^x$ in logarithmic form. Also state the domain and range of the logarithmic function.

2. Compute $\log_9 27 - \log_{27} 9$.

3. Solve $\log_8 x + \log_4 x + \log_2 x = 11$:

4. Solve $\log_4 2^{8x} = 2^{\log_2 8}$

5. If $a^2 + b^2 = 7ab$; show that $\log\frac{a+b}{3} = \frac{1}{2}(\log a + \log b)$

6. Prove $\log\frac{a^2}{bc} + \log\frac{b^2}{ca} + \log\frac{c^2}{ab} = 0$:

7. Prove that $\log 2 + 16\log\frac{16}{15} + 12\log\frac{25}{24} + 7\log\frac{81}{80}$

8. Prove llog $\log_{a^2} a \log_{b^2} b \log_{c^2} c = \frac{1}{8}$

9. Prove $\log a + \log a^2 + \log a^3 + \_\_\_ + \log a^n = \frac{n(n+1)}{2}\log a$

10. If $\frac{logx}{y-z} = \frac{logy}{z-x} = \frac{logz}{x-y}$ then prove that xyz = 1.

11. Solve $\log_2 x - 3\log_{1/2} x = 6$.

12. Solve $\log_{5-x}(x^2 - 6x + 65) = 2$:

TRIGONOMETRY

Exercise - 3.11

1. Find the principal value of (i) $\sin^{-1}\frac{1}{\sqrt{2}}$ (ii)$\cos^{-1}\frac{\sqrt{3}}{2}$ (iii) $\text{cosec}^{-1}(-1)$ (iv) $\sec^{-1}(-\sqrt{2})$

2. A man standing directly opposite to one side of a road of width x meter views a circular shaped traffic green signal of diameter a meter on the other side of the road. The bottom of the green signal is b meter height from the horizontal level of viewer's eye. If α denotes the angle subtended by the diameter of the green signal at the viewer's eye, then prove that $\alpha = \tan^{-1}\left(\frac{a+b}{x}\right) - \tan^{-1}\left(\frac{b}{x}\right)$

Example 3.72 Find the principal value of (i) $\sin^{-1}\frac{\sqrt{3}}{2}$ (ii) $\text{cosec}^{-1}(\frac{2}{\sqrt{3}})$ (iii) $tan^{-1}\left(-\frac{1}{\sqrt{3}}\right)$

Exercise - 3.10

1. Determine whether the following measurements produce one triangle, two triangles or no triangle: $\angle B = 88^0, a = 23, b = 2.$ Solve if solution exists.

2. If the sides of a ΔABC are $a = 4, b = 6$ and $c = 8$, then show that $4\cos B + 3\cos C = 2$.

3. In a ΔABC, if $a = \sqrt{3} - 1, b = \sqrt{3} + 1$ and $C = 60^0$, find the other side and other two angles.

4. In any ΔABC, prove that the area $\Delta = \frac{b^2+c^2-a^2}{4\cot A}$

5. In a ΔABC, if $a = 12$ *cm*, $b = 8$ *cm* and $C = 30^0$, then show that its area is 24 *sq.cm.*

6. In a ΔABC, if $a = 18$ *cm*, $b = 24$ *cm* and $c = 30$ *cm*, then show that its area is 216 *sq.cm.*

7. Two soldiers A and B in two different underground bunkers on a straight road, spot an intruder at the top of a hill. The angle of elevation of the intruder from A and B to the ground level in the eastern direction are 30^0 and 45^0 respectively. If A and B stand 5*km* apart, find the distance of the intruder from B.

8. A researcher wants to determine the width of a pond from east to west, which cannot be done by actual measurement. From a point P, he finds the distance to the eastern-most point of the pond to be 8 *km*, while the distance to the western most point from P to be 6 *km*. If the angle between the two lines of sight is 60^0, find the width of the pond.

9. Two Navy helicopters A and B are flying over the Bay of Bengal at same altitude from the sea level to search a missing boat. Pilots of both the helicopters sight the boat at the same time while they are apart 10 *km* from each other. If the distance of the boat from A is 6 *km* and if the line segment AB subtends 60^0 at the boat, find the distance of the boat from B.

10. A straight tunnel is to be made through a mountain. A surveyor observes the two extremities A and B of the tunnel to be built from a point P in front of the mountain. If $AP = 3$*km*, $BP = 5$ *km* and $\angle APB = 120^0$, then find the length of the tunnel to be built.

11. A farmer wants to purchase a triangular shaped land with sides 120 *feet* and 60 *feet* and the angle included between these two sides is 60^0. If the land costs |500 per sq.ft, find the amount he needed to purchase the land. Also find the perimeter of the land.

12. A fighter jet has to hit a small target by flying a horizontal distance. When the target is sighted, the pilot measures the angle of depression to be 30^0. If after 100 *km*, the target has an angle of depression of 45^0, how far is the target from the fighter jet at that instant?

13. A plane is 1 *km* from one landmark and 2 *km* from another. From the planes point of view the land between them subtends an angle of 45^0. How far apart are the landmarks?

14. A man starts his morning walk at a point A reaches two points B and C and finally back to A such that $\angle A = 60^0$ and $\angle B = 45^0$, $AC = 4$*km* in the _ABC. Find the total distance he covered during his morning walk.

15. Two vehicles leave the same place P at the same time moving along two different roads. One vehicle moves at an average speed of 60 km/hr and the other vehicle moves at an average speed of 80 km/hr. After half an hour the vehicle reach the destinations A and B. If AB subtends 60^0 at the initial point P, then find AB.

16. Suppose that a satellite in space, an earth station and the centre of earth all lie in the same plane. Let r be the radius of earth and R be the distance from the centre of earth to the satellite. Let d be the distance from the earth station to the satellite. Let 30^0 be the angle of elevation from the earth station to the satellite. If the line segment connecting earth station and satellite subtends angle α at the centre of earth, then prove that $d = R\sqrt{1+\left(\frac{r}{R}\right)^2 - 2\frac{r}{R}\cos\alpha}$

Example 3.64 In a ΔABC, $a = 3$, $b = 5$ and $c = 7$. Find the values of cos A, cos B and cos C.

Example 3.65 In ΔABC, $A = 30^0$, $B = 60^0$ and $c = 10$, Find a and b.

Example 3.66 In a ΔABC, if $a = 2\sqrt{2}$, $b = 2\sqrt{2}$ and $C = 75^0$, find the other side and the angles.

Example 3.67 Find the area of the triangle whose sides are 13 cm, 14 cm and 15 cm.

Example 3.68 In any ΔABC, prove that $a\cos A + b\cos B + c\cos C = \frac{8\Delta^2}{abc}$

Example 3.69 Suppose that there are two cell phone towers within range of a cell phone. The two towers are located at 6 km apart along a straight highway, running east to west and the cell phone is north of the highway. The signal is 5 km from the first tower and $\sqrt{31}$ km from the second tower. Determine the position of the cell phone north and east of the first tower and how far it is from the highway.

Example 3.70 Suppose that a boat travels 10 km from the port towards east and then turns 60^0 to its left. If the boat travels further 8 km, how far from the port is the boat?

Example 3.71 Suppose two radar stations located 100 km apart, each detect a fighter aircraft between them. The angle of elevation measured by the first station is 30^0, hereas the angle of elevation measured by the second station is 45°. Find the altitude of the aircraft at that instant.

Example 3.56 The Government plans to have a circular zoological park of diameter 8 km. A separate area in the form of a segment formed by a chord of length 4 km is to be allotted exclusively for a veterinary hospital in the park. Find the area of the segment to be allotted for the veterinary hospital.

Example 3.57 In a ΔABC, prove that $b^2\sin 2C + c^2\sin 2B = 2bc\sin A$.

Example 3.58 In a ΔABC, prove that $\sin\left(\frac{B-C}{2}\right) = \frac{b-c}{a}\cos\frac{A}{2}$

Example 3.59 If the three angles in a triangle are in the ratio 1 : 2 : 3, then prove that the corresponding sides are in the ratio $1 : \sqrt{3} : 2$.

Example 3.60 In a ΔABC, prove that $(b + c)\cos A + (c + a)\cos B + (a + b)\cos C = a + b + c$

Example 3.61 In a triangle *ABC*, prove that $\frac{a^2+b^2}{a^2+c^2} = \frac{1+\cos(A-B)\cos C}{1+\cos(A-C)\cos B}$

Example 3.62 Derive cosine formula using the law of sines in a Δ *ABC*.

Example 3.63 Using Heron's formula, show that the equilateral triangle has the aximum area for any fixed perimeter. [Hint: In $xyz \le k$, maximum occurs when $x = y = z$].

Exercise - 3.9

1. In a ΔABC, if $\frac{\sin A}{\sin C} = \frac{\sin(A-B)}{\sin(B-C)}$, prove that a^2, b^2, c^2 are in Arithmetic Progression.

2. The angles of a triangle *ABC*, are in Arithmetic Progression and if $b : c = \sqrt{3} : \sqrt{2}$, find $\angle A$.

3. In a ΔABC, if $\cos C = \frac{\sin A}{2 \sin B}$, show that the triangle is isosceles.

4. In a ΔABC, prove tha $\frac{\sin A}{\sin C} = \frac{c - a\cos B}{b - a\cos C}$

5. In a ΔABC, prove that $a\cos A + b\cos B + c\cos C = 2a\sin B\sin C$.

6. In a ΔABC, $\angle A = 60^\circ$. Prove that $b + c = 2a\cos\left(\frac{B-C}{2}\right)$

7. In a ΔABC, prove the following

(i) $a\sin\left(\frac{A}{2} + B\right) = (b + c)\sin\frac{A}{2}$

(ii) $a(\cos B + \cos C) = 2(b + c)\sin^2\frac{A}{2}$

(iii) $\frac{a^2-c^2}{b^2} = \frac{\sin(A-c)}{\sin(A+C)}$

(iv) $\frac{a\sin(B-C)}{b^2-c^2} = \frac{a\sin(C-A)}{c^2-a^2} = \frac{a\sin(A-B)}{a^2-b^2}$

(v) $\frac{a+b}{a-b} = \tan\left(\frac{A+B}{2}\right)\cot\left(\frac{A-B}{2}\right)$

8. In a ΔABC, prove that $(a^2 - b^2 - c^2)\tan B = (a^2 + b^2 - c^2)\tan C$.

9. An Engineer has to develop a triangular shaped park with a perimeter 120 *m* in a village. The park to be developed must be of maximum area. Find out the dimensions of the park.

10. A rope of length 12 *m* is given. Find the largest area of the triangle formed by this rope and find the dimensions of the triangle so formed.

11. Derive Projection formula from (i) Law of sines, (ii) Law of cosines.

Theorem 3.7: In ΔABC, $\Delta = \sqrt{s(s-a)(s-b)(s-c)}$ c) where *s* is the semi-perimeter of ΔABC.

Theorem 3.6: In ΔABC (i) $\sin\frac{A}{2} = \sqrt{\frac{(s-b)(s-c)}{bc}}$, (ii) $\cos\frac{A}{2} = \sqrt{\frac{s(s-a)}{bc}}$ (iii) $\tan\frac{A}{2} = \sqrt{\frac{(s-b)(s-c)}{s(s-a)}}$

where *s* is the semi-perimeter of _*ABC* given by $s = \frac{a+b+c}{2}$

Theorem 3.5: In ΔABC, area of the triangle is $\Delta = \frac{1}{2}ab\sin c = \frac{1}{2}bc\sin A = \frac{1}{2}ac\sin B$

Theorem 3.4: state and prove projection formulae

Theorem 3.3 (state and proveThe Law of Cosines):

Theorem 3.2: state and prove napier's formulae

Theorem 3.1 (state and proveLaw of Sines):

Exercise - 3.8

1. Find the principal solution and general solutions of the following:

(i) $\sin\theta = -\frac{1}{\sqrt{2}}$ (ii) $\cot\theta = \sqrt{3}$ (iii) $\tan\theta = -\frac{1}{\sqrt{3}}$

2. Solve the following equations for which solutions lies in the interval $0^\circ \le \theta < 360^\circ$

(i) $\sin^4 x = \sin^2 x$ (ii) $2\cos^2 x + 1 = -3\cos x$

(iii) $2\sin^2 x + 1 = 3\sin x$ (iv) $\cos 2x = 1 - 3\sin x$.

3. Solve the following equations:

(i) $\sin 5x - \sin x = \cos 3x$ (ii) $2\cos^2\theta + 3\sin\theta - 3 = 0$

(iii) $\cos\theta + \cos 3\theta = 2\cos 2\theta$ (iv) $\sin\theta + \sin 3\theta + \sin 5\theta = 0$

(v) $\sin 2\theta - \cos 2\theta - \sin\theta + \cos\theta = 0$ (vi) $\sin\theta + \cos\theta = \sqrt{2}$

(vii) $\sin\theta + \sqrt{3}\cos\theta = 1$ (viii) $\cot\theta + \operatorname{cosec}\theta = \sqrt{3}$

(ix) $\tan\theta + \tan\left(\theta + \frac{\pi}{3}\right) + \tan\left(\theta + \frac{2\pi}{3}\right) = \sqrt{3}$ (x) $\cos 2\theta = \frac{\sqrt{5}+1}{4}$

(xi) $2\cos^2 x - 7\cos x + 3 = 0$

Example 3.55 Solve $\sqrt{3}\tan^2\theta + (\sqrt{3} - 1)\tan\theta - 1 = 0$

Example 3.54 Solve $\sqrt{3}\sin\theta - \cos\theta = \sqrt{2}$

Example 3.53 Prove that for any a and b, $-\sqrt{a^2 + b^2} \le a\sin\theta + b\cos\theta \le \sqrt{a^2 + b^2}$

Example 3.52 Solve $2\sin^2 x + \sin^2 2x = 2$

Example 3.51 Solve $\sin x + \cos x = 1 + \sin x\cos x$

Example 3.50 Solve $\sin x - 3\sin 2x + \sin 3x = \cos x - 3\cos 2x + \cos 3x$

Example 3.49 Solve $\tan 2x = -\cot\left(x + \frac{\pi}{3}\right)$

Example 3.48 Solve the equation $\sin 9\theta = \sin\theta$

Example 3.47 Solve $\cos x + \sin x = \cos 2x + \sin 2x$

Example 3.46 Solve $\sin x + \sin 5x = \sin 3x$

Example 3.45 Solve $3\cos^2\theta = \sin^2\theta$

Example 3.44 Find the general solution of (i) $\sec\theta = -2$ (ii) $\tan\theta = \sqrt{3}$

Example 3.43 Find the general solution of $\sin\theta = -\frac{\sqrt{3}}{2}$

Example 3.42 Find the principal SOLN (i) $\sin\theta = \frac{1}{2}$ (ii) $\sin\theta = -\frac{\sqrt{3}}{2}$ (iii) $\operatorname{cosec}\theta = -2$ (iv) $\cos\theta = \frac{1}{2}$

Exercise - 3.7

1. If $A + B + C = 180^0$, prove that

(i) $\sin 2A + \sin 2B + \sin 2C = 4\sin A\sin B\sin C$ (ii) $\cos A + \cos B - \cos C = -1 + 4\cos\frac{A}{2}\cos\frac{B}{2}\cos\frac{C}{2}$

(iii) $\sin^2 A + \sin^2 B + \sin^2 C = 2 + 2\cos A\cos B\cos C$

(iv) $\sin^2 A + \sin^2 B - \sin^2 C = 2\sin A\sin B\cos C$ (v) $\tan\frac{A}{2}\tan\frac{B}{2} + \tan\frac{B}{2}\tan\frac{C}{2} + \tan\frac{C}{2}\tan\frac{A}{2} = 1$

(vi) $\sin A + \sin B + \sin C = 4\cos\frac{A}{2}\cos\frac{B}{2}\cos\frac{C}{2}$

(vii) $\sin(B + C - A) + \sin(C + A - B) + \sin(A + B - C) = 4\sin A\sin B\sin C$.

2. If $A + B + C = 2s$, then prove that $\sin(s - A)\sin(s - B) + \sin s\sin(s - C) = \sin A\sin B$.

3. If $x + y + z = xyz$, then prove that $\frac{2x}{1-x^2} + \frac{2y}{1-y^2} + \frac{2z}{1-z^2} = \frac{2x}{1-x^2}\frac{2y}{1-y^2}\frac{2z}{1-z^2}$

4. If $A + B + C = \frac{\pi}{2}$ prove the following

(i) $\sin 2A + \sin 2B + \sin 2C = 4\cos A\cos B\cos C$ (ii) $\cos 2A + \cos 2B + \cos 2C = 1 + 4\sin A\sin B\cos C$.

5. If ΔABC is a right triangle and if $\angle A = \frac{\pi}{2}$, then prove that

(i) $\cos^2 B + \cos^2 C = 1$ (ii) $\sin^2 B + \sin^2 C = 1$ (iii) $\cos B - \cos C = -1 + 2\sqrt{2}\cos\frac{B}{2}\sin\frac{C}{2}$

Example 3.41 If $A + B + C = \pi$, prove that $\cos^2 A + \cos^2 B + \cos^2 C = 1 - 2\cos A\cos B\cos C$. Example 3.40 Prove that $\sin\frac{A}{2} + \sin\frac{B}{2} + \sin\frac{C}{2} = 1 + 4\sin\left(\frac{\pi-A}{4}\right)\sin\left(\frac{\pi-B}{4}\right)\sin\left(\frac{\pi-C}{4}\right)$

Example 3.39 If $A + B + C = \pi$, prove the following

(i) $\cos A + \cos B + \cos C = 1 + 4\sin\left(\frac{A}{2}\right)\sin\left(\frac{B}{2}\right)\sin\left(\frac{C}{2}\right)$

(ii) $\sin\left(\frac{A}{2}\right)\sin\left(\frac{B}{2}\right)\sin\left(\frac{C}{2}\right) \le \frac{1}{8}$ (iii) $1 < \cos A + \cos B + \cos C \le \frac{3}{2}$

Exercise - 3.6

1. Express each of the following as a sum or difference

(i) $\sin 35^0 \cos 28^0$ (ii) $\sin 4x\cos 2x$ (iii) $2\sin 10\theta\cos 2\theta$ (iv) $\cos 5\theta\cos 2\theta$ (v) $\sin 5\theta\sin 4\theta$.

2. Express each of the following as a product

(i) $\sin 75^0 - \sin 35^0$ (ii) $\cos 65^0 + \cos 15^0$ (iii) $\sin 50^0 + \sin 40^0$ (iv) $\cos 35^0 - \cos 75^0$.

3. Show that $\sin 12^0 \sin 48^0 \sin 54 = 1/8$

4. Show that $\cos\frac{\pi}{15}\cos\frac{2\pi}{15}\cos\frac{3\pi}{15}\cos\frac{4\pi}{15}\cos\frac{5\pi}{15}\cos\frac{6\pi}{15}\cos\frac{7\pi}{15} = \frac{1}{128}$

5. Show that $\frac{\sin 8x\cos x - \sin 6x\cos 3x}{\cos 2x\cos x - \sin 3x\sin 4x} = \tan 2x$ 6. Show that $\frac{(\cos\theta - \cos 3\theta)(\sin 8\theta + \sin 2\theta)}{(\sin 5\theta - \sin\theta)(\cos 4\theta - \cos 6\theta)} = 1$

7. Prove that $\sin x + \sin 2x + \sin 3x = \sin 2x(1 + 2\cos x)$.

8. Prove that $\frac{\sin 4x + \sin 2x}{\cos 4x + \cos 2x} = \tan 3x$

9. Prove that $1 + \cos 2x + \cos 4x + \cos 6x = 4\cos x\cos 2x\cos 3x$.

10. prove that $\sin\frac{\theta}{2}\sin\frac{7\theta}{2} + \sin\frac{3\theta}{2}\sin\frac{11\theta}{2} = \sin 2\theta\sin 5\theta$

11. Prove that $\cos(30^0 - A)\cos(30^0 + A) + \cos(45^0 - A)\cos(45^0 + A) = \cos 2A + \frac{1}{4}$

12. Prove that $\frac{\sin x + \sin 3x + \sin 5x + \sin 7x}{\cos x + \cos 3x + \cos 5x + \cos 7x} = \tan 4x$

13. Prove that $\frac{\sin(4A - 2B) + \sin(4B - 2A)}{\cos(4A - 2B) + \cos(4B - 2A)} = \tan(A + B)$

14. Show that $\cot(A + 15^0) - \tan(A - 15^0) \frac{4\cos 2A}{1 + 2\sin 2A}$

Example 3.38 Show that $\cos 10^0 \cos 30^0 \cos 50^0 \cos 70^0 \frac{3}{16}$

Example 3.37 Simplify $\frac{sin75^0 - sin15^0}{cos75^0 + cos15^0}$

Example 3.36 Show that $\cos 36^0 \cos 72^0 \cos 108^0 \cos 144^0 = \frac{1}{16}$

Example 3.35 Find the value of $\sin 34^0 + \cos 64^0 - \cos 4^0$.

Example 3.34 Express each of the following sum or difference as a product

(i) $\sin 50^0 + \sin 20^0$ (ii) $\cos 6\theta + \cos 2\theta$ (iii) $\cos\frac{3x}{2} - \cos\frac{9x}{2}$

Example 3.33 Express each of the following product as a sum or difference

(i) $\sin 40^0 \cos 30^0$ (ii) $\cos 110^0 \sin 55^0$ (iii) $\sin\frac{x}{2}\cos\frac{3x}{2}$

Identity 3.15: Prove that $\sin(60^0 - A)\sin A\sin(60^0 + A) = \frac{1}{4}\sin 3A$

Exercise - 3.5

1. Find the value of $\cos 2A$, A lies in the first quadrant, when

(i) $\cos A = 15/17$ (ii) $\sin A = 4/5$ (iii) $\tan A = 16/63$

2. If θ is an acute angle, then find (i) $\sin\left(\frac{\pi}{4} - \frac{\theta}{2}\right)$ when $\sin\theta = \frac{1}{25}$ (ii) $\cos\left(\frac{\pi}{4} + \frac{\theta}{2}\right)$ when $\sin\theta = \frac{8}{9}$

3. If $\cos\theta = \frac{1}{2}\left(a + \frac{1}{a}\right)$, show that $\cos 3\theta = \frac{1}{2}\left(a^3 + \frac{1}{a^3}\right)$

4. Prove that $\cos 5\theta = 16\cos^5\theta - 20\cos^3\theta + 5\cos\theta$.

5. Prove that $\sin 4\alpha = 4\tan\alpha\frac{1-\tan^2\alpha}{(1+tan^2\alpha)^2}$.

6. If $A + B = 45^0$, show that $(1 + \tan A)(1 + \tan B) = 2$.

7. Prove that $(1 + \tan 1^0)(1 + \tan 2^0)(1 + \tan 3^0)\ldots(1 + \tan 44^0)$ is a multiple of 4.

8. Prove that $tan\left(\frac{\pi}{4} + \theta\right) - tan\left(\frac{\pi}{4} - \theta\right) = 2\tan 2\theta$.

9. Show that $cot\left(7\frac{1}{2}^0\right) = \sqrt{2} + \sqrt{3} + \sqrt{4} + \sqrt{6}$.

10. Prove that $(1 + \sec 2\theta)(1 + \sec 4\theta)\ldots(1 + \sec 2^n\theta) = \tan 2^n\theta\cot\theta$.

11. Prove that $32\sqrt{3}\sin\frac{\pi}{48}\cos\frac{\pi}{48}\cos\frac{\pi}{24}\cos\frac{\pi}{12}\cos\frac{\pi}{6} = 3$

Example 3.32 Prove that $\cos A\cos 2A\cos 2^2A\cos 2^3A \ldots\ldots\ldots cos2^{n-1}A = \frac{\sin 2^nA}{2^n\sin A}$

Example 3.31 Find the value of $\sqrt{3}\,\text{cosec}\,20^0 - \sec 20^0$

Example 3.30 If $\tan\frac{\theta}{2} = \sqrt{\frac{1-a}{1+a}}\tan\frac{\phi}{2}$, then prove that $\cos\phi = \frac{\cos\theta - a}{1 - a\cos\theta}$

Example 3.29 Find the values of (i) $\sin 18^0$ (ii) $\cos 18^0$ (iii) $\sin 72^0$ (iv) $\cos 36^0$ (v) $\sin 54^0$

Example 3.28 Find x such that $-\pi \le x \le \pi$ and $\cos 2x = \sin x$

Example 3.27 Prove that $1 - \frac{1}{2}\sin 2x = \frac{sin^3x + cos^3x}{\sin x + \cos x}$

Example 3.26 Prove that $\frac{\sin\theta + \sin 2\theta}{1 + \cos\theta + \cos 2\theta} = \tan\theta$

Example 3.25 Prove that $\sin x = 2^{10} \sin\left(\frac{x}{2^{10}}\right) \cos\left(\frac{x}{2}\right) \cos\left(\frac{x}{2^2}\right) \ldots\ldots cos\left(\frac{x}{2^{10}}\right)$

Example 3.24 Prove that $\sin 4A = 4\sin A\cos^3 A - 4\cos A\sin^3 A$

Example 3.23 Find the value of $\sin 2\theta$, when $\sin\theta = 12/13$, θ lies in the first quadrant

Example 3.22 Find the value of $\sin\left(22\frac{1}{2}^{0}\right)$

Example 3.21 A foot ball player can kick a football from ground level with an initial velocity of 80 ft/second. Find the maximum horizontal distance the football travels and at what angle? (Take $g = 32$).

Exercise - 3.4

1. If $\sin x = 15/17$ and $\cos y = 12/13$, $0 < x < \pi/2$, $0 < y < \pi/2$,
find the value of (i) $\sin(x + y)$ (ii) $\cos(x - y)$ (iii) $\tan(x + y)$.

2. If $\sin A = 3/5$ and $\cos B = 9/41$, $0 < A < \pi/2$, $0 < B < \pi/2$,
find the value of (i) $\sin(A + B)$ (ii) $\cos(A - B)$.

3. Find $\cos(x - y)$, given that $\cos x = -4/5$ with $\pi < x < 3\pi/2$ and $\sin y = -24/25$ with $\pi < Y < 3\pi/2$.

4. Find $\sin(x - y)$, given that $\sin x = 8/17$ with $0 < x < \pi/2$ and $\cos y = -24/25$ with $\pi < y < 3\pi/2$.

5. Find the value of (i) $\cos 105^0$ (ii) $\sin 105^0$ (iii) $\tan(7\pi/12)$

6. Prove that

(i) $\cos(30^0 + x) = \frac{\sqrt{3}\cos x - \sin x}{2}$ (ii) $\cos(\pi + \theta) = -\cos\theta$ (iii) $\sin(\pi + \theta) = -\sin\theta$.

7. Find a quadratic equation whose roots are $\sin 15^0$ and $\cos 15^0$.

8. Expand $\cos(A + B + C)$. Hence prove that
$\cos A\cos B\cos C = \sin A\sin B\cos C + \sin B\sin C\cos A + \sin C\sin A\cos B$, if $A + B + C = \frac{\pi}{2}$.

9. Prove that

(i) $\sin(45^0 + \theta) - \sin(45^0 - \theta) = \sqrt{2}\sin\theta$. (ii) $\sin(30^0 + \theta) + \cos(60^0 + \theta) = \cos\theta$.

10. If $a\cos(x + y) = b\cos(x - y)$, show that $(a + b)\tan x = (a - b)\cot y$.

11. Prove that $\sin 105^0 + \cos 105^0 = \cos 45^0$.

12. Prove that $\sin 75^0 - \sin 15^0 = \cos 105^0 + \cos 15^0$.

13. Show that $\tan 75^0 + \cot 75^0 = 4$.

14. Prove that $\cos(A + B)\cos C - \cos(B + C)\cos A = \sin B\sin(C - A)$.

15. Prove that $\sin(n + 1)\theta\sin(n - 1)\theta + \cos(n + 1)\theta\cos(n - 1)\theta = \cos 2\theta$, $n \in Z$.

16. If $x\cos\theta = y\cos\left(\theta + \frac{2\pi}{3}\right) = z\cos\left(\theta + \frac{4\pi}{3}\right)$ find the value of $xy + yz + zx$.

17. Prove that (i) $\sin(A+B)\sin(A-B)=\sin^2 A-\sin^2 B$

(ii) $\cos(A+B)\cos(A-B)=\cos^2 A-\sin^2 B=\cos^2 B-\sin^2 A$

(iii) $\sin^2(A+B)-\sin^2(A-B)=\sin 2A\sin 2B$

(iv) $\cos 8\theta\cos 2\theta=\cos^2 5\theta-\sin^2 3\theta$

18. Show that $\cos^2 A+\cos^2 B-2\cos A\cos B\cos(A+B)=\sin^2(A+B)$.

19. If $\cos(\alpha-\beta)+\cos(\beta-\gamma)+\cos(\gamma-\alpha)=-\frac{3}{2}$,

then prove that $\cos\alpha+\cos\beta+\cos\gamma=\sin\alpha+\sin\beta+\sin\gamma=0$.

20. Show that (i) $\tan(45^0+A)=\frac{1+\tan A}{1-\tan A}$ (ii) $\tan(45^0-A)=\frac{1-\tan A}{1+\tan A}$

21. Prove that $\cot(A+B)=\frac{\cot A\cot B-1}{\cot A+\cot B}$

22. If $\tan x=\frac{n}{n+1}$ and $\tan y=\frac{2n}{2n+1}$, find $\tan(x+y)$.

23. Prove that tan $tan\left(\frac{\pi}{4}+\theta\right)tan\left(\frac{3\pi}{4}-\theta\right)=-1$.

24. Find the values of $\tan(\alpha+\beta)$, given that $\cot\alpha=\frac{1}{2}$ $\alpha\in(\pi,\frac{3\pi}{2})$ and $\sec\beta=-\frac{5}{5}$, $\beta\in(\frac{\pi}{2},\pi)$

25. If $\theta+\varphi=\alpha$ and $\tan\theta=k\tan\varphi$, then prove that $\sin(\theta-\varphi)=\frac{k-1}{k+1}\sin\alpha$

Example 3.20 Expand (i) $\sin(A+B+C)$ (ii) $\tan(A+B+C)$

Example 3.19 A ripple tank demonstrates the effect of two water waves being added together. The two waves are described by $h=8\cos t$ and $h=6\sin t$, where $t\in[0,2\pi)$ is in seconds and h is the height in millimeters above still water. Find the maximum height of the resultant wave and the value of t at which it occurs.

Example 3.18 Point $A(9,12)$ rotates around the origin O in a plane through 60^0 in the anticlockwise direction to a new position B. Find the coordinates of the point B.

Example 3.17 Prove that $cos\left(\frac{3\pi}{4}+x\right)-cos\left(\frac{3\pi}{4}-x\right)=\sqrt{2}\sin x$

Example 3.16 If $\sin x=\frac{4}{5}$ (in I quadrant) and $\cos y=-\frac{12}{13}$(in II quadrant), then find (i) $\sin(x-y)$, (ii) $\cos(x-y)$.

Example 3.15 Find the values of (i) $\cos 15^0$ and (ii) $\tan 165^0$.

Exercise - 3.3

1. Find the values of (i) $\sin(480^0)$ (ii) $\sin(-1110^0)$ (iii) $\cos(300^0)$ (iv) $\tan(1050^0)$

(v) $\cot(660^0)$ (vi) $\tan(\frac{19\pi}{3})$ (vii) $\sin(-\frac{11\pi}{3})$

2. $\left(\frac{5}{7},\frac{2\sqrt{6}}{7}\right)$ is a point on the terminal side of an angle θ in standard position. Determine the trigonometric function values of angle θ.

3. Find the values of other five trigonometric functions for the following:

(i) cos $\theta=-\frac{1}{2}$, θ lies in the III quadrant. (ii) cos $\theta=\frac{2}{3}$ θ lies in the I quadrant.

(iii) sin $\theta=-\frac{2}{3}$, θ lies in the IV quadrant. (iv) tan $\theta=-2$, θ lies in the II quadrant.

(v) sec $\theta=\frac{13}{5}$, θ lies in the IV quadrant.

4. Prove that $\frac{\cot(180+\theta)\sin(90-\theta)\cos(-\theta)}{\sin(270+\theta)\tan(-\theta)\operatorname{cosec}(360+\theta)} = cos^2\theta \cot\theta$

5. Find all the angles between 0^0 and 360^0 which satisfy the equation $\sin^2\theta=\frac{3}{4}$

6. Show that $sin^2\frac{\pi}{18}+sin^2\frac{\pi}{9}+sin^2\frac{7\pi}{18}+sin^2\frac{4\pi}{9} = 2$

Example 3.14 Determine whether the following functions are even, odd or neither.

(i) $\sin^2 x-2\cos^2 x-\cos x$ (ii) $\sin(\cos(x))$ (iii) $\cos(\sin(x))$ (iv) $\sin x+\cos x$

Example 3.13 Prove that $\tan 315^0 \cot(-405^0)+\cot 495^0 \tan(-585^0)=2$

Example 3.12 Find the value of: (i) $\sin 765^0$ (ii) cosec (-1410^0) (iii) $\cot(-\frac{15\pi}{4})$

Example 3.11 Find the value of (i) $\sin 150^0$ (ii) $\cos 135^0$ (iii) $\tan 120^0$.

Example 3.10 Find the values of (i) $\sin(-45^0)$ (ii) $\cos(-45^0)$ (iii) $\cot(-45^0)$

Example 3.9 If sin $\theta=\frac{3}{5}$ and the angle θ is in the second quadrant, then find the values of other five trigonometric functions

Example 3.8 The terminal side of an angle θ in standard position passes through the point $(3,-4)$. Find the six trigonometric function values at an angle θ.:

Exercise - 3.2

1. Express each of the following angles in radian measure:

(i) 30^0 (ii) 135^0 (iii) -205^0 (iv) 150^0 (v) 330^0.

2. Find the degree measure corresponding to the following radian measures

(i) $\pi/3$ (ii) $\pi/9$ (iii) $2\pi/5$ (iv) $7\pi/3$ (v) $10\pi/9$

3. What must be the radius of a circular running path, around which an athlete must run 5 times in order to describe 1 *km*?

4. In a circle of diameter 40 *cm*, a chord is of length 20 *cm*. Find the length of the minor arc of the chord.

5. Find the degree measure of the angle subtended at the centre of circle of radius 100 *cm* by an arc of length 22 *cm*.

6. What is the length of the arc intercepted by a central angle of measure 41^0 in a circle of radius 10 *ft*?

7. If in two circles, arcs of the same length subtend angles 60^0 and 75^0 at the centre, find the ratio of their radii.

8. The perimeter of a certain sector of a circle is equal to the length of the arc of a semi-circle having the same radius. Express the angle of the sector in degrees, minutes and seconds.

9. An airplane propeller rotates 1000 times per minute. Find the number of degrees that a point on the edge of the propeller will rotate in 1 second.

10. A train is moving on a circular track of 1500 *m* radius at the rate of 66 *km/hr*. What angle will it turn in 20 seconds?

11. A circular metallic plate of radius 8 *cm* and thickness 6 *mm* is melted and molded into a pie (a sector of the circle with thickness) of radius 16 *cm* and thickness 4 *mm*. Find the angle of the sector.

Example 3.7 If the arcs of same lengths in two circles subtend central angles 30° and 80°, find the ratio of their radii

Example 3.6 Find the length of an arc of a circle of radius 5 *cm* subtending a central angle measuring 15

Example 3.5 Convert (i) $\pi/5$ radians to degrees (ii) 6 radians to degrees.

Example 3.4 Convert (i) 18^0 to radians (ii) -108^0 to radians.

Exercise - 3.1

1. Identify the quadrant in which an angle of each given measure lies

(i) 25^0 (ii) 825^0 (iii) -55^0 (iv) 328^0 (v) -230^0

2. For each given angle, find a coterminal angle with measure of θ such that $0^\circ \leq \theta < 360^\circ$

(i) 395^0 (ii) 525^0 (iii) 1150^0 (iv) -270^0 (v) -450^0

3. If $a\cos\theta - b\sin\theta = c$, show that $a\sin\theta + b\cos\theta = \pm\sqrt{a^2+b^2-c^2}$.

4. If $\sin\theta + \cos\theta = m$, show that $\cos^6\theta + \sin^6\theta = \frac{4-3(m^2-1)^2}{4}$ where $m^2 \leq 2$.

5. If $\frac{cos^4\alpha}{cos^2\beta} + \frac{sin^4\alpha}{sin^2\beta} = 2sin^2\alpha sin^2\beta$ then ,

prove that (i) $\sin^4\alpha + \sin^4\beta = 2\sin^2\alpha\sin^2\beta$ (ii) $\frac{cos^4\beta}{cos^2\alpha} + \frac{sin^4\beta}{sin^2\alpha} = 1$

6. If $y = \frac{2\ \sin\alpha}{1+\cos\alpha+\sin\alpha}$, then prove that $\frac{1-\cos\alpha+\sin\alpha}{1+\sin\alpha} = y$

7. If $x = \sum_{i=0}^{\infty} cos^{2n}\theta \quad y = \sum_{i=0}^{\infty} sin^{2n}\theta \quad z = \sum_{i=0}^{\infty} cos^{2n}\theta \quad sin^{2n}\theta$, then P.T $xyz = x + y + z$.

8. If $\tan^2\theta = 1 - k^2$, show that $\sec\theta + \tan^3\theta\,\text{cosec}\,\theta = (2-k^2)^{3/2}$. Also, find the values of k for which this result holds.

9. If $\sec\theta + \tan\theta = p$, obtain the values of $\sec\theta$, $\tan\theta$ and $\sin\theta$ in terms of p.

10. If $\cot\theta(1+\sin\theta) = 4m$ and $\cot\theta(1-\sin\theta) = 4n$, then prove that $(m^2-n^2)^2 = mn$

11. If $\text{cosec}\,\theta - \sin\theta = a^3$ and $\sec\theta - \cos\theta = b^3$, then prove that $a^2b^2(a^2+b^2) = 1$

12. Eliminate θ from the equations $a\sec\theta - c\tan\theta = b$ and $b\sec\theta + d\tan\theta = c$.

Example 3.3 Eliminate θ from $a\cos\theta = b$ and $c\sin\theta = d$, where a, b, c, d are constants

Example 3.2 Prove that $(\sec A - \operatorname{cosec} A)(1 + \tan A + \cot A) = \tan A \sec A - \cot A \operatorname{cosec} A$.

Example 3.1 Prove that $\frac{\tan\theta + \sec\theta - 1}{\tan\theta - \sec\theta + 1} = \frac{1+\sin\theta}{\cos\theta}$

COMBINATORICS AND MATHEMATICAL INDUCTION

Example 4.1 Suppose one girl or one boy has to be selected for a competition from a class comprising 17 boys and 29 girls. In how many different ways can this selection be made?

Example 4.2 Consider the 3 cities Chennai, Trichy and Tirunelveli. In order to reach Tirunelveli from Chennai, one has to pass through Trichy. There are 2 roads connecting Chennai with Trichy and there are 3 roads connecting Trichy with Tirunelveli. What are the total number of ways of travelling from Chennai to Tirunelveli?

Example 4.3 A School library has 75 books on Mathematics, 35 books on Physics. A student can choose only one book. In how many ways a student can choose a book on Mathematics or Physics?

Example 4.4 If an electricity consumer has the consumer number say 238:110: 29, then describe the linking and count the number of house connections upto the 29th consumer connection linked to the larger capacity transformer number 238 subject to the condition that each smaller capacity transformer can have a maximal consumer link of say 100.

Example 4.5 A person wants to buy a car. There are two brands of car available in the market and each brand has 3 variant models and each model comes in five different colours as in Figure 4.2 In how many ways she can choose a car to buy?

Example 4.6 A Woman wants to select one silk saree and one sungudi saree from a textile shop located at Kancheepuram. In that shop, there are 20 different varieties of silk sarees and 8 different

varieties of sungudi sarees. In how many ways she can select her sarees?

Example 4.7 In a village, out of the total number of people, 80 percentage of the people own Coconut groves and 65 percent of the people own Paddy fields. What is the minimum percentage of people own both?

Example 4.8

(i) Find the number of strings of length 4, which can be formed using the letters of the word BIRD, without repetition of the letters.

(ii) How many strings of length 5 can be formed out of the letters of the word PRIME taking all the letters at a time without repetition.

Example 4.9 How many strings of length 6 can be formed using letters of the word FLOWER if (i) either starts with F or ends with R? (ii) neither starts with F nor ends with R?

Example 4.10 How many licence plates may be made using either two distinct letters followed by four digits or two digits followed by 4 distinct letters where all digits and letters are distinct?

Example 4.11 Count the number of positive integers greater than 7000 and less than 8000 which are divisible by 5, provided that no digits are repeated.

Example 4.12 How many 4 - digit even numbers can be formed using the digits 0, 1, 2, 3 and 4, if repetition of digits are not permitted?

Example 4.13 Find the total number of outcomes when 5 coins are tossed once.

Example 4.14 In how many ways (i) 5 different balls be distributed among 3 boxes? (ii) 3 different balls be distributed among 5 boxes?

Example 4.15 There are 10 bulbs in a room. Each one of them can be operated independently. Find the number of ways in which the room can be illuminated.

Example 4.16 Find the value of (i) 5! (ii) 6! − 5! (iii) $\frac{8!}{5! \times 2!}$ Example 4.17 Simplify $\frac{7!}{2!}$

Example 4.18 Evaluate $\frac{n!}{r!(n-r)!}$ when (i) $n = 7, r = 5$ (ii) $n = 50, r = 47$ (iii) For any $n, \& r = 3$.

Example 4.19 Let N denote the number of days. If the value of $N!$ is equal to the total number of hours in N days then find the value of N?

Example 4.20 If $\frac{6!}{n!} = 6$ then find the value of n

Example 4.21 If $n! + (n-1)! = 30$, then find the value of n.

Example 4.22 What is the unit digit of the sum 2! + 3! + 4! + ... + 22!?

Example 4.23 If $\frac{1}{7!} + \frac{1}{8!} = \frac{A}{9!}$ then find the value of A.

Example 4.24 Prove that $\frac{(2n)!}{n!} = 2^n(1.3.5 \cdot \cdot \cdot (2n-1))$

Exercise - 4.1

1. (i) A person went to a restaurant for dinner. In the menu card, the person saw 10 Indian and 7 Chinese food items. In how many ways the person can select either an Indian or a Chinese food?

(ii) There are 3 types of toy car and 2 types of toy train available in a shop. Find the number of ways a baby can buy a toy car and a toy train?

(iii) How many two-digit numbers can be formed using 1,2,3,4,5 without repetition of digits?

(iv) Three persons enter in to a conference hall in which there are 10 seats. In how many ways they can take their seats?

(v) In how many ways 5 persons can be seated in a row?

2. (i) A mobile phone has a passcode of 6 distinct digits. What is the maximum number of attempts one makes to retrieve the passcode?

(ii) Given four flags of different colours, how many different signals can be generated if each signal requires the use of three flags, one below the other?

3. Four children are running a race.

(i) In how many ways can the first two places be filled?

(ii) In how many different ways could they finish the race?

4. Count the number of three-digit numbers which can be formed from the digits 2,4,6,8 if

(i) repetitions of digits is allowed. (ii) repetitions of digits is not allowed

5. How many three-digit numbers are there with 3 in the unit place?

(i) with repetition (ii) without repetition.

6. How many numbers are there between 100 and 500 with the digits 0, 1, 2, 3, 4, 5 ? if (i) repetition of digits allowed (ii) the repetition of digits is not allowed.

7. How many three-digit odd numbers can be formed by using the digits 0, 1, 2, 3, 4, 5 ?

if (i) the repetition of digits is not allowed (ii) the repetition of digits is allowed.

8. Count the numbers between 999 and 10000 subject to the condition that there are (i) no restriction. (ii) no digit is repeated. (iii) at least one of the digits is repeated.

9. How many three-digit numbers, which are divisible by 5, can be formed using the digits

0, 1, 2, 3,4, 5 if (i) repetition of digits are not allowed? (ii) repetition of digits are allowed?

10. To travel from a place A to place B, there are two different bus routes $B1, B2$, two different train routes $T1, T2$ and one air route $A1$. From place B to place C there is one bus route say B_1' two different train routes say T_1', T_2' and one air route A_1'. Find the number of routes of commuting from place A to place C via place B without using similar mode of transportation.

11. How many numbers are there between 1 and 1000 (both inclusive) which are divisible neither by 2 nor by 5?

12. How many strings can be formed using the letters of the word LOTUS if the word

(i) either starts with L or ends with S? (ii) neither starts with L nor ends with S?

13. (i) Count the total number of ways of answering 6 objective type questions, each question having 4 choices.

(ii) In how many ways 10 pigeons can be placed in 3 different pigeon holes ?

(iii) Find the number of ways of distributing 12 distinct prizes to 10 students?

14. Find the value of (i) 6! (ii) 4! + 5! (iii) 3! − 2! (iv) 3! × 4! (v) $\frac{12!}{9!\times 3!}$ (vi) $\frac{(n+3)!}{(n+1)!}$

15. Evaluate $\frac{n!}{r!(n-r)!}$ when (i) $n = 6, r = 2$ (ii) $n = 10, r = 3$ (iii) For any n with $r = 2$.

16. Find the value of n if (i) $(n + 1)! = 20(n - 1)!$ (ii) $\frac{1}{8!} + \frac{1}{9!} = \frac{n}{10!}$

Theorem 4.1: If n, r are positive integers and $r \le n$, then the number of permutations of n distinct objects taken r at a time is $n(n-1)(n-2)\cdots(n-r+1)$.

$nPn = n\,Pn{-}1.$, $nPr = n \times n{-}1\ Pr{-}1$, $nPr = n{-}1Pr + r \times n{-}1Pr{-}1$

Example 4.25 Evaluate: (i) $4P4$ (ii) $5P3$ (iii)$8P4$ (iv) $6P5$. Example 4.26 If $(n+2)P4 = 42 \times nP2$, find n.

Example 4.27 If $10Pr = 7\ Pr+2$ find r.

Example 4.28 How many 'letter strings' together can be formed with the letters of the word "VOWELS" so that (i) the strings begin with E (ii) the strings begin with E and end with W.

Example 4.29 A number of four different digits is formed with the use of the digits 1,2,3,4 and 5 in all possible ways. Find the following

(i) How many such numbers can be formed?

(ii) How many of these are even?

(iii) How many of these are exactly divisible by 4?

Example 4.30 How many different strings can be formed together using the letters of the word "EQUATION" so that (i) the vowels always come together? (ii) the vowels never come together?

Example 4.31 There are 15 candidates for an examination. 7 candidates are appearing for mathematics examination while the remaining 8 are appearing for different subjects. In how many ways can they be seated in a row so that no two mathematics candidates are together?

Example 4.32 In how many ways 5 boys and 4 girls can be seated in a row so that no two girls are together.

Example 4.33 4 boys and 4 girls form a line with the boys and girls alternating. Find the number of ways of making this line.

Example 4.34 A van has 8 seats. It has two seats in the front with two rows of three seats behind. The van belongs to a family, consisting of seven members, $F, M, S1, S2, S3, D1, D2$. How many ways can the family sit in the van if (i) There are no restriction? (ii) Either F or M drives the van? (iii) $D1, D2$ sits next to a window and F is driving?

Example 4.35 If the letters of the word TABLE are permuted in all possible ways and the words thus formed are arranged in the dictionary order (alphabetical order), find the ranks of the words (i) TABLE, (ii) BLEAT

Example 4.36 Find the number of ways of arranging the letters of the word BANANA.

Example 4.37 Find the number of ways of arranging the letters of the word RAMANUJAN so that the relative positions of vowels and consonants are not changed.

Example 4.38 Three twins pose for a photograph standing in a line. How many arrangements are Ther(i). when there are no restrictions. (ii). when each person is standing next to his or her twin?

Example 4.39 How many numbers can be formed using the digits 1,2,3,4,2,1 such that, even digits occupies even place?

Example 4.40 How many paths are there from start to end on a 6×4 grid as shown in the picture?

Example 4.41 If the different permutations of all letters of the word BHASKARA are listed as in a dictionary, how many strings are there in this list before the first word starting with B?

Example 4.42 If the letters of the word IITJEE are permuted in all possible ways and the strings thus formed are arranged in the lexicographic order, find the rank of the word IITJEE

Example 4.43 Find the sum of all 4-digit numbers that can be formed using the digits 1, 2, 4, 6, 8.

Exercise - 4.2

1. If $(n-1)P3 : nP4 = 1 : 10$, find n.
2. If $10Pr-1 = 2\times 6\ Pr$, find r.
3. (i) Suppose 8 people enter an event in a swimming meet. In how many ways could the gold, silver and bronze prizes be awarded?

(ii) Three men have 4 coats, 5 waist coats and 6 caps. In how many ways can they wear them?

4. Determine the number of permutations of the letters of the word SIMPLE if all are taken at a time?
5. A test consists of 10 multiple choice questions. In how many ways can the test be answered if

(i) Each question has four choices?

(ii) The first four questions have three choices and the remaining have five choices?

(iii) Question number n has $n+1$ choices?

6. A student appears in an objective test which contain 5 multiple choice questions. Each question has four choices out of which one correct answer.

(i) What is the maximum number of different answers can the students give?

(ii) How will the answer change if each question may have more than one correct answers?

7. How many strings can be formed from the letters of the word ARTICLE, so that vowels occupy the even places?
8. 8 women and 6 men are standing in a line.

(i) How many arrangements are possible if any individual can stand in any position?

(ii) In how many arrangements will all 6 men be standing next to one another?

(iii) In how many arrangements will no two men be standing next to one another?

9. Find the distinct permutations of the letters of the word MISSISSIPPI?
10. How many ways can the product $a^2b^3c^4$ be expressed without exponents?
11. In how many ways 4 mathematics books, 3 physics books, 2 chemistry books and 1 biology book can be arranged on a shelf so that all books of the same subjects are together.
12. In how many ways can the letters of the word SUCCESS be arranged so that all Ss are together?
13. A coin is tossed 8 times,

(i) How many different sequences of heads and tails are possible?

(ii) How many different sequences containing six heads and two tails are possible?

14. How many strings are there using the letters of the word INTERMEDIATE, if

(i) The vowels and consonants are alternative (ii) All the vowels are together

(iii) Vowels are never together (iv) No two vowels are together.

15. Each of the digits 1, 1, 2, 3, 3 and 4 is written on a separate card. The six cards are then laid out in a row to form a 6-digit number.

(i) How many distinct 6-digit numbers are there?

(ii) How many of these 6-digit numbers are even?

(iii) How many of these 6-digit numbers are divisible by 4?

16. If the letters of the word GARDEN are permuted in all possible ways and the strings thus formed are arranged in the dictionary order, then find the ranks of the words (i) GARDEN (ii) DANGER.

17. Find the number of strings that can be made using all letters of the word THING. If these words are written as in a dictionary, what will be the $85th$ string?

18. If the letters of the word FUNNY are permuted in all possible ways and the strings thus formed are arranged in the dictionary order, find the rank of the word FUNNY.

19. Find the sum of all 4-digit numbers that can be formed using digits 1, 2, 3, 4, and 5 repetitions not allowed?

20. Find the sum of all 4-digit numbers that can be formed using digits 0, 2, 5, 7, 8 without repetition?

PROPERTIES OF COMBINATIONS

Property 1: (i) $nC0 = 1$, (ii) $nCn = 1$, (iii) $nCr = \frac{n(n-1)(n-2)\cdots(n-r+1)}{r!}$

Property 4: $nCr + n\,Cr-1 = n+1\,Cr.$

Example 4.44 Evaluate the following: (i) $10C3$ (ii) $15C13$ (iii) $100C99$ (iv) $50C50$.

Example 4.45 Find the value of $5C2$ and $7C3$ using the property 5

Example 4.46 If $nC4 = 495$, What is n?.

Example 4.47 If $nPr = 11880$ and $nCr = 495$, Find n and r.

Example 4.48 Prove that $24C4 + \sum_{i=0}^{4} (28-r)C3 = 29\,C4$

Example 4.49 Prove that $10C2 + 2 \times 10\,C3 + 10\,C4 = 12\,C4$

Example 4.50 If $(n+2)C_7 : (n-1)\,P4 = 13 : 24$ find n.

Example 4.50 If $(n+2)C_7 : (n-1)\,P4 = 13 : 24$ find n.

Example 4.51 A salad at a certain restaurant consists of 4 of the following fruits: apple, banana, guava, pomegranate, grapes, papaya and pineapple. Find the total possible number of fruit salads.

Example 4.52 A Mathes club has 15 members. In that 8 are girls. 6 of the members are to be selected for a competition and half of them should be girls. How many ways of these selections are possible?

Example 4.53 In rating 20 brands of cars, a car magazine picks a first, second, third, fourth and fifth best brand and then 7 more as acceptable. In how many ways can it be done?

Example 4.54 From a class of 25 students, 10 students are to be chosen for an excursion party. There are 4 students who decide that either all of them will join or none of them will join. In how many ways can the excursion party be chosen?\

Example 4.55 A box of one dozen apple contains a rotten apple. If we are choosing 3 apples simultaneously, in how many ways, one can get only good apples.

Example 4.56 An exam paper contains 8 questions, 4 in Part A and 4 in Part B. Examiners are required to answer 5 questions. In how many ways can this be done if

(i) There are no restrictions of choosing a number of questions in either parts.

(ii) At least two questions from Part A must be answered.

Example 4.57 Out of 7 consonants and 4 vowels, how many strings of 3 consonants and 2 vowels can be formed?

Example 4.58 Find the number of strings of 5 letters that can be formed with the letters of the word PROPOSITION.

Example 4.59 If a set of m parallel lines intersect another set of n parallel lines (not parallel to the lines in the first set), then find the number of parallelograms formed in this lattice structure.

Example 4.60 How many diagonals are there in a polygon with n sides?

Exercise - 4.3

1. If $nC12 = nC9$ find $21Cn$.
2. If $15C2r-1 = 15C2r+4$, find r.
3. If $nPr = 720$, and $nCr = 120$, find n, r.
4. Prove that $15C3 + 2 \times 15C4 + 15C5 = 17C5$.
5. Prove that $35C5 + \sum_{i=0}^{4} (29-r)C_4 = 40C5$
6. If $(n+1)C8 : (n-3)P4 = 57 : 16$, find the value of n.
7. Prove that $2nCn = \frac{2^n \times 1 \times 3 \ldots\ldots (2n-1)}{n!}$.
8. Prove that if $1 \le r \le n$ then $n \times (n-1)Cr-1 = (n-r+1)nCr-1$.
9. (i) A Kabaddi coach has 14 players ready to play. How many different teams of 7 players could the coach put on the court?

(ii) There are 15 persons in a party and if each 2 of them shakes hands with each other, how many handshakes happen in the party?

(iii) How many chords can be drawn through 20 points on a circle?

(iv) In a parking lot one hundred , one year old cars, are parked. Out of them five are to be chosen at random for to check its pollution devices. How many different set of five cars can be chosen?

(v) How many ways can a team of 3 boys,2 girls and 1 transgender be selected from 5 boys, 4 girls and 2 transgenders?

10. Find the total number of subsets of a set with (i) 4 elements (ii) 5 elements (iii) n elements.

11. A trust has 25 members.

(i) How many ways 3 officers can be selected?

(ii) In how many ways can a President, Vice President and a Secretary be selected?

12. How many ways a committee of six persons from 10 persons can be chosen along with a chair person and a secretary?

13. How many different selections of 5 books can be made from 12 different books if,

(i) Two particular books are always selected? (ii) Two particular books are never selected?

14. There are 5 teachers and 20 students. Out of them a committee of 2 teachers and 3 students is to be formed. Find the number of ways in which this can be done. Further find in how many of these committees (i) a particular teacher is included? (ii) a particular student is excluded?

15. In an examination a student has to answer 5 questions, out of 9 questions in which 2 are compulsory. In how many ways a student can answer the questions?

16. Determine the number of 5 card combinations out of a deck of 52 cards if there is exactly three aces in each combination.

17. Find the number of ways of forming a committee of 5 members out of 7 Indians and 5 Americans, so that always Indians will be the majority in the committee.

18. A committee of 7 peoples has to be formed from 8 men and 4 women. In how many ways can this be done when the committee consists of

(i) exactly 3 women? (ii) at least 3 women? (iii) at most 3 women?

19. 7 relatives of a man comprises 4 ladies and 3 gentlemen, his wife also has 7 relatives; 3 of them are ladies and 4 gentlemen. In how many ways can they invite a dinner party of 3 ladies and 3 gentlemen so that there are 3 of man's relative and 3 of the wife' s relatives?

20. A box contains two white balls, three black balls and four red balls. In how many ways can three balls be drawn from the box, if at least one black ball is to be included in the draw?

21. Find the number of strings of 4 letters that can be formed with the letters of the word EXAMINATION?.

22. How many triangles can be formed by joining 15 points on the plane, in which no line joining any three points?

23. How many triangles can be formed by 15 points, in which 7 of them lie on one line and the remaining 8 on another parallel line?

24. There are 11 points in a plane. No three of these lies in the same straight line except 4 points, which are collinear. Find,

(i) the number of straight lines that can be obtained from the pairs of these points?

(ii) the number of triangles that can be formed for which the points are their vertices?

25. A polygon has 90 diagonals. Find the number of its sides?

Example 4.61 By the principle of mathematical induction, prove that, for all integers

$$n \geq 1, 1 + 2 + 3 \ldots\ldots + n = \frac{n(n-1)}{2}$$

Example 4.62 Prove that the sum of first n positive odd numbers is n^2

Example 4.63 By the principle of mathematical induction, prove that, for all integers

$$n \geq 1. 1^2 + 2^2 + 3^2 \ldots\ldots\ldots + n^2 = \frac{n(n-1)(2n-1)}{6}$$

Example 4.64 Using the Mathematical induction, show that for any natural number n

$$\frac{1}{1.2} + \frac{1}{2.3} + \frac{1}{3.4} + \cdots \ldots \frac{1}{n(n+1)} = \frac{n}{n+1},$$

Example 4.65 Prove that for any natural number n, $a^n - b^n$ is divisible by $a - b$, where $a > b$

Example 4.66 Prove that $3^{2n+2} - 8n - 9$ is divisible by 8 for all $n \geq 1$.

Example 4.67 Using the Mathematical induction, show that for any integer

$$n \geq 2, 3^n > n^2$$

Example 4.69 By the principle of mathematical induction, prove that, for $n \in \mathbb{N}$,

$$\cos\alpha + \cos(\alpha+\beta) + \cos(\alpha+2\beta) + \cdot\ \cdot\ \cdot + \cos(\alpha + (n-1)\beta) = \cos\left(\alpha + \frac{(n-1)\beta}{2}\right) \times \frac{\sin\left(\frac{n\beta}{2}\right)}{\sin\left(\frac{\beta}{2}\right)}$$

Example 4.70 Using the Mathematical induction, show that for any natural number n, with the assumption $i^2 = -1$, $(r(\cos\theta + i\sin\theta))^n = r^n(\cos n\theta + i\sin n\theta)$,

Exercise - 4.4

1. By the principle of mathematical induction, prove that, for $n \geq 1$

$$1^3 + 2^3 + 3^3 \ldots\ldots n^3 = \left(\frac{n(n-1)}{2}\right)^2$$

2. By the principle of mathematical induction, prove that, for $n \geq 1$

$$1^2 + 3^2 + 5^2 \ldots\ldots\ldots + (2n-1)^2 = \frac{n(2n-1)(2n+1)}{3}$$

3. Prove that the sum of the first n non-zero even numbers is $n^2 + n$.

4. By the principle of Mathematical induction, prove that, for $n \geq 1$

$$1.2 + 2.3 + 3.4 + \cdot\ \cdot\ \cdot + n.(n+1) = \frac{n(n+1)(n+2)}{3}.$$

5. Using the Mathematical induction, show that for any natural number $n \geq 2$,

$$\left(1-\frac{1}{2^2}\right)\left(1-\frac{1}{3^2}\right)\left(1-\frac{1}{4^2}\right)\cdots\cdots\left(1-\frac{1}{n^2}\right)=\frac{n+1}{2n}$$

6. Using the Mathematical induction, show that for any natural number $n \geq 2$,

$$\frac{1}{1+2}+\frac{1}{1+2+3}+\frac{1}{1+2+3+4}+\cdots\cdots\frac{1}{1+2+3\ldots n}=\frac{n-1}{n+1}.$$

7. Using the Mathematical induction, show that for any natural number n,

$$\frac{1}{1.2.3}+\frac{1}{2.3.4}+\frac{1}{3.4.5}+\cdots\ldots+\frac{1}{n.(n+1).(n+2)}=\frac{n(n+3)}{4(n+1)(n+2)}$$

8. Using the Mathematical induction, show that for any natural number n,

$$\frac{1}{2.5}+\frac{1}{5.8}+\frac{1}{8.11}+\cdots\ldots+\frac{1}{(3n-1)(3n+2)}=\frac{n}{6n+4}$$

9. Prove by Mathematical Induction that

$$1!+(2\times 2!)+(3\times 3!)+\ldots+(n\times n!)=(n+1)!-1.$$

10. Using the Mathematical induction, show that for any natural number n, $x^{2n}-y^{2n}$ is divisible by $x+y$.

11. By the principle of Mathematical induction, prove that, for $n \geq 1$

$$1^2+2^2+3^2\ldots\ldots\ldots+n^2>\frac{n^3}{3}$$

12. Use induction to prove that n^3-7n+3 is divisible by 3, for all natural numbers n.

13. Use induction to prove that $5^{n+1}+4\times 6^n$ when divided by 20 leaves a remainder 9, for all natural numbers n.

14. Use induction to prove that $10^n+3\times 4^{n+2}+5$, is divisible by 9, for all natural numbers n.

15. Prove that using the Mathematical induction

$$\sin\alpha+\sin\left(\alpha+\frac{\pi}{6}\right)+\sin\left(\alpha+\frac{2\pi}{6}\right)+\cdots.+\sin\left(\alpha+\frac{(n-1)\pi}{6}\right)=\frac{\sin\left(\alpha+\frac{(n-1)\pi}{12}\right)\sin\left(\frac{n\pi}{12}\right)}{\sin\left(\frac{\pi}{12}\right)}$$

BINOMIAL THEOREM, SEQUENCES AND SERIES

Theorem 5.1 (Binomial theorem for positive integral index): If n is any positive integer, then

$(a+b)^n = nC0\, a^n b^0 + nC1\, a^{n-1}b^1 + \cdot\cdot\cdot + nCr\, a^{n-r}b^r + \cdot\cdot\cdot + nCn\, a^0 b^n$.

Example 5.1 Find the expansion of $(2x+3)^5$.

Example 5.2 Evaluate 98^4.

Example 5.3 Find the middle term in the expansion of $(x+y)^6$.

Example 5.4 Find the middle terms in the expansion of $(x+y)^7$.

Example 5.5 Find the coefficient of x^6 in the expansion of $(3+2x)^{10}$.

Example 5.6 Find the coefficient of x^3 in the expansion of $(2-3x)^7$.

Example 5.7 The $2nd$, $3rd$ and $4th$ terms in the binomial expansion of $(x+a)^n$ are 240, 720 and 1080 for a suitable value of x. Find x, a and n.

Example 5.8 Expand $\left(2x-\frac{1}{2x}\right)^4$

Example 5.9 Expand $\left(x^2+\sqrt{1-x^2}\right)^5+\left(x^2-\sqrt{1-x^2}\right)^5$

Example 5.10 Using Binomial theorem, prove that $6^n - 5^n$ always leaves remainder 1 when divided by 25 for all positive integer n.

Example 5.11 Find the last two digits of the number 7^{400}.

Exercise - 5.1

1. Expand (i) $(2x^2-\frac{3}{x})^3$ (ii) $\left(2x^2-3\sqrt{1-x^2}\right)^4+\left(2x^2+3\sqrt{1-x^2}\right)^4$

2. Compute (i) 102^4 (ii) 99^4 (iii) 9^7.

3. Using binomial theorem, indicate which of the following two number is larger: $(1.01)^{1000000}$,10000.

4. Find the coefficient of x^{15} in $\left(x^2+\frac{1}{x^3}\right)^{10}$

5. Find the coefficient of x^6 and the coefficient of x^2 in $\left(x^2-\frac{1}{x^3}\right)^6$

6. Find the coefficient of x^4 in the expansion of $(1+x^3)^{50}\left(x^2+\frac{1}{x}\right)^5$

7. Find the constant term of $\left(2x^3+\frac{1}{3x^2}\right)^5$

8. Find the last two digits of the number 3^{600}.

9. If n is a positive integer, show that, $9^{n-1}-8n-9$ is always divisible by 64.

10. If n is an odd positive integer, prove that the coefficients of the middle terms in the expansion of $(x+y)^n$ are equal.

11. If n is a positive integer and r is a nonnegative integer, prove that the coefficients of x^r and x^{n-r} in the expansion of $(1+x)^n$ are equal.

12. If a and b are distinct integers, prove that $a-b$ is a factor of a^n-b^n, whenever n is a positive integer. [Hint: write $an=(a-b+b)^n$ and expand]

13. In the binomial expansion of $(a+b)^n$, the coefficients of the $4th$ and $13th$ terms are equal to each other, find n.

14. If the binomial coefficients of three consecutive terms in the expansion of $(a+x)^n$ are in the ratio 1 : 7 : 42, then find n.

15. In the binomial coefficients of $(1+x)^n$, the coefficients of the $5th$, $6th$ and $7th$ terms are in AP. Find all values of n.

16. Prove that $C_0^2+C_1^2+C_2^2+,\dots\dots\dots+C_n^2=\frac{2n!}{(n!)^2}$

Example 5.12 Prove that if *a, b, c* are in HP, if and only if $\frac{a}{c}=\frac{a-b}{b-c}$

Example 5.13 If the 5*th* and 9*th* terms of a harmonic progression are $\frac{1}{19}$ and $\frac{1}{35}$ find the 12*th* term of the sequence.

Example 5.14 Find seven numbers $A1, A2, \ldots, A7$ so that the sequence $4, A1, A2, \ldots, A7, 7$ is in arithmetic progression and also 4 numbers $G1, G2, G3, G4$ so that the sequence $12, G1, G2, G3, G4, \frac{3}{8}$ is in geometric progression.

Example 5.15 If the product of the 4*th*, 5*th* and 6*th* terms of a geometric progression is 4096 and if the product of the 5*th*, 6*th* and 7*th*-terms of it is 32768, find the sum of first 8 terms of the geometric progression.

Theorem 5.2: If *AM* and *GM* denote the arithmetic mean and the geometric mean of two nonnegative numbers, then $AM \geq GM$. The equality holds if and only if the two numbers are equal.

Theorem 5.3: If *GM* and *HM* denote the geometric mean and the harmonic mean of two nonnegative numbers, then $GM \geq HM$. The equality holds if and only if the two numbers are equal.

Exercise - 5.2

1. Write the first 6 terms of the sequences whose *nth* terms are given below and classify them as arithmetic progression, geometric progression, arithmetico-geometric progression, harmonic progression and none of them.

(i) $\frac{1}{2^{n+1}}$ (ii) $\frac{(n+1)(n+2)}{n+3(n+4)}$ (iii) $4\left(\frac{1}{2}\right)^n$ (iv) $\frac{(-1)^n}{n}$ (v) $\frac{2n+3}{3n+4}$ (vi) 2018 (vii) $\frac{3n-2}{3^{n-1}}$

2. Write the first 6 terms of the sequences whose *nth* term *an* is given below.

(i) $a_n = \begin{cases} n+1, \text{if } n \text{ is odd} \\ n \quad, \text{if n is even} \end{cases}$ (ii) $a_n = \begin{cases} 1 & \text{if } n=1 \\ 2 & \text{if } n=2 \\ a_{n-1}+a_{n-2} & \text{if } n>2 \end{cases}$

(iii) $a_n = \begin{cases} n & \text{if n is 1,2 or 3} \\ a_{n-1}+a_{n-2}+a_{n-3} & \text{if } n>3 \end{cases}$

3. Write the *nth* term of the following sequences.

(i) $2, 2, 4, 4, 6, 6, \ldots$ (ii) $\frac{1}{2}, \frac{2}{3}. \frac{3}{4}, \frac{4}{5}, \frac{5}{6} \ldots \ldots\ldots$ (iii) $\frac{1}{2}, \frac{3}{4}, \frac{5}{6}, \frac{7}{8}, \frac{9}{10} \ldots \ldots$ (iv) $6, 10, 4, 12, 2, 14, 0, 16, -2, .$

4. The product of three increasing numbers in5 GP is 5832. If we add 6 to the second number and 9 to the third number, then resulting numbers form an AP. Find the numbers in GP.

5. Write the *nth* term of the sequence $\frac{3}{1^2 2^2}, \frac{5}{2^2 3^2}, \frac{7}{3^2 4^2} \ldots \ldots, \ldots$ as a difference of two terms.

6. If *tk* is the *kth* term of a *GP*, then show that t_{n-k}, t_n, t_{n+k} also form a GP for any positive integer *k*.

7. If *a, b, c* are in geometric progression, and if $a^{\frac{1}{x}} = b^{\frac{1}{y}} = c^{\frac{1}{z}}$, then prove that *x, y, z* are in arithmetic progression.

8. The AM of two numbers exceeds their GM by 10 and HM by 16. Find the numbers.

9. If the roots of the equation $(q-r)x^2+(r-p)x+p-q=0$ are equal, then show that *p, q* and *r* are in AP.

10. If *a, b, c* are respectively the *pth*, *qth* and *rth* terms of a GP, show that $(q-r)\log a+(r-p)\log b+(p-q)\log c=0.$

Example 5.16 Find the sum up to n terms of the series: $1+\frac{6}{7}+\frac{11}{49}+\frac{16}{343}+\cdots$ ...

Example 5.17 Find the sum of the first n terms of the series $\frac{1}{1+\sqrt{3}}+\frac{1}{\sqrt{2}+\sqrt{3}}+\frac{1}{\sqrt{3}+\sqrt{4}}$ ... .

Example 5.18 Find $\sum_{k=0}^{n}\frac{1}{k(k+1)}$

Exercise - 5.3

1. Find the sum of the first 20-terms of the arithmetic progression having the sum of first 10 terms as 52 and the sum of the first 15 terms as 77.

2. Find the sum up to the 17*th* term of the series $\frac{1^3}{1}+\frac{1^3+2^3}{1+3}+\frac{1^3+2^3+3^3}{1+3+5}+\cdots$ ....

3. Compute the sum of first n terms of the following series:

i. $8+88+888+8888+\cdot\ \cdot\ \cdot$ ii. $6+66+666+6666+\cdot\ \cdot\ \cdot$

4. Compute the sum of first n terms of $1+(1+4)+(1+4+4^2)+(1+4+4^2+4^3)+$....

5. Find the general term and sum to n terms of the sequence $1,\frac{4}{3},\frac{7}{9},\frac{10}{27},$ ... ...

6. Find the value of n, if the sum to n terms of the series $\sqrt{3}+\sqrt{75}+\sqrt{243}$ ... . is $435\sqrt{3}$

7. Show that the sum of $(m+n)th$ and $(m-n)th$ term of an AP. is equal to twice the mth term.

8. A man repays an amount of Rs.3250 by paying Rs.20 in the first month and then increases the payment by Rs.15 per month. How long will it take him to clear the amount?

9. In a race, 20 balls are placed in a line at intervals of 4 meters, with the first ball 24 meters away from the starting point. A contestant is required to bring the balls back to the starting place one at a time. How far would the contestant run to bring back all balls?

10. The number of bacteria in a certain culture doubles every hour. If there were 30 bacteria present in the culture originally, how many bacteria will be present at the end of 2nd hour, 4*th* hour and *nth* hour?

11. What will Rs.500 amounts to in 10 years after its deposit in a bank which pays annual interest rate of 10% compounded annually?

12. In a certain town, a viral disease caused severe health hazards upon its people disturbing their normal life. It was found that on each day, the virus which caused the disease spread in Geometric Progression. The amount of infectious virus particle gets doubled each day, being 5 particles on the first day. Find the day when the infectious virus particles just grow over 1,50,000 units?

Example 5.19 Find the sum: $1+\frac{4}{5}+\frac{7}{25}+\frac{16}{125}+\cdots$ ..

Example 5.20 Find$\sum_{i=0}^{\infty}\frac{1}{n^2+5n+6}$

Example 5.21 Expand $(1+x)^{\frac{2}{3}}$ up to four terms for $|x|<1$

Example 5.22 Expand$\frac{1}{(1+3x)^2}$ in powers of x. Find a condition on x for which the expansionis valid.

Example 5.23 Expand $\frac{1}{(3+2x)^2}$ in powers of x. Find a condition on x for which the expansionis valid.

Example 5.24 Find$\sqrt[3]{65}$

Example 5.25 Prove that $\sqrt[3]{x^3+7}-\sqrt[3]{x^3+4}$ is approximately equal to $\frac{1}{x^2}$ when x is large.

Exercise - 5.4

1. Expand the following in ascending powers of x and find the condition on x for which the binomial expansion is valid.

(i)$\frac{1}{5+x}$ (ii)$\frac{2}{(3+4x)^2}$ (iii) $(5+x^2)^{\frac{2}{3}}$ (iv) $(x+2)^{-\frac{2}{3}}$

2. Find $\sqrt[3]{1001}$ approximately (two decimal places).

3. Prove that $\sqrt[3]{x^3+6}-\sqrt[3]{x^3+3}$ is approximately equal to $\frac{1}{x^2}$ when x is sufficiently large.

4. Prove that $\sqrt{\frac{1-x}{1+x}}$ is approximately equal to $1-x+x^2$ when x is very small.

5. Write the first 6 terms of the exponential series (i) e^{5x} (ii) e^{-2x} (iii) $e^{\frac{1}{2}x}$.

6. Write the first 4 terms of the logarithmic series (i) $\log(1+4x)$ (ii) $\log(1-2x)$ (iii) $\log\left(\frac{1+3x}{1-3x}\right)$(iv) $\log\left(\frac{1-2x}{1+2x}\right)$. Find the intervals on which the expansions are valid.

7. If $y=x+\frac{x^2}{2}+\frac{x^3}{3}+\frac{x^4}{4}+\cdots$ ... then show that $x=y-\frac{y^2}{2!}+\frac{y^3}{3!}-\frac{y^4}{4!}+\cdots$..

8. If $p-q$ is small compared to either p or q, then show that $\sqrt[n]{\frac{p}{q}}\simeq\frac{(n+1)p+(n-1)q}{(n-1)p+(n+1)q}$ find $\sqrt[8]{\frac{15}{16}}$

9. Find the coefficient of x^4 in the expansion of $\frac{3-4x+x^2}{e^{2x}}$

10. Find the value $\sum_{i=0}^{\infty}\frac{1}{2n-1}\left(\frac{1}{9^{n-1}}+\frac{1}{9^{2n-1}}\right)$

TWO DIMENSIONAL ANALYTICAL GEOMETRY

Example 6.1 Find the locus of a point which moves such that its distance from the x-axis is equal to the distance from the y-axis.

Example 6.2 Find the path traced out by the point (ct,c/t), here $t \neq 0$ is the parameter and c is a Constant

Example 6.3 Find the locus of a point P moves such that its distances from two fixed points A(1; 0) and B (5; 0) ; are always equal.

Example 6.4 If θis a parameter, find the equation of the locus of a moving point, whose coordinates are (a secθ; b tanθ).

Example 6.5 A straight rod of the length 6 units, slides with its ends A and B always on the x and y axes respectively. If O is the origin, then find the locus of the centroid of ΔOAB .

Example 6.6 If θ is a parameter, find the equation of the locus of a moving point, whose coordinates are (a(θ- sin θ) ; a(1 - cos θ)) .

Exercise - 6.1

1. Find the locus of P, if for all values of α, the co-ordinates of a moving point P is (i) (9 cos α ; 9 sin α) (ii) (9 cos α ; 6 sin α):

2. Find the locus of a point P that moves at a constant distant of (i) two units from the x-axis (ii) three units from the y-axis.

3. If θ is a parameter, find the equation of the locus of a moving point, whose coordinates are $x = acos^3\theta$, $y = a\sin^3\theta$

4. Find the value of k and b, if the points P(-3; 1) and Q(2,b) lie on the locus of $x^2 - 5x + ky = 0$.

5. A straight rod of length 8 units slides with its ends A and B always on the x and y axes respectively. Find the locus of the mid point of the line segmentAB

6. Find the equation of the locus of a point such that the sum of the squares of the distance from the points (3; 5), (1;-1) is equal to 20

7. Find the equation of the locus of the point P such that the line segment AB, joining the points A(1;-6) and B(4;-2), subtends a right angle at P.

8. If O is origin and R is a variable point on $y^2 = 4x$, then find the equation of the locus of the mid-point of the line segment OR.

9. The coordinates of a moving point P are $\left(\frac{a}{2}(cosec\ \theta + \sin\theta), \frac{b}{2}(\text{cosec}\theta - \sin\theta)\right)$, where θ is a variable parameter. Show that the equation of the locus P is $b^2 x^2 - a^2 y^2 = a^2 b^2$.

10. If P(2;-7) is a given point and Q is a point on $2x^2 + 9y^2 = 18$, then find the equations of the locus of the mid-point of PQ.

11. If R is any point on the x-axis and Q is any point on the y-axis and P is a variable point on RQ with RP = b, PQ = a. then find the equation of locus of P.

12. If the points P(6; 2) and Q(-2; 1) and R are the vertices of a ΔPQR and R is the point on the locus $y = x^2 - 3x + 4$, then find the equation of the locus of centroid of ΔPQR

13. If Q is a point on the locus of $x^2 + y^2 + 4x - 3y + 7 = 0$, then find the equation of locus of P which divides segment OQ externally in the ratio 3:4,where O is origin.

14. Find the points on the locus of points that are 3 units from x-axis and 5 units from point (5; 1).

15. The sum of the distance of a moving point from the points (4; 0) and (-4; 0) is always 10 units. Find the equation of the locus of the moving point.

Example 6.7 Find the slope of the straight line passing through the points (5,7) and (7,5). Also find the angle of inclination of the line with the x-axis.

Example 6.8 Find the equation of a straight line cutting an intercept of 5 from the negative direction of the y-axis and is inclined at an angle 150^0 to the x-axis.

Example 6.9 Show the points $\left(0,-\frac{3}{2}\right)$, $(1,-1)$, $\left(2,-\frac{1}{2}\right)$ are collinear.

Example 6.10 The Pamban Sea Bridge is a railway bridge of length about 2065 m constructed on the PalkStrait, which connects the Island town of Rameswaram to Mandapam, the main land of India. The Bridge is restricted to a uniform speed of only 12.5 m/s. If a train of length 560m starts at the entry point of the bridge from Mandapam, then

(i) find an equation of the motion of the train.

(ii) when does the engine touch island

(iii) when does the last coach cross the entry point of the bridge

(iv) what is the time taken by a train to cross the bridge.

Example 6.11 Find the equations of the straight lines, making the y- intercept of 7 and angle between the line and the y-axis is 30^0

Example 6.12 The seventh term of an arithmetic progression is 30 and tenth term is 21.

(i) Find the first three terms of an A.P.

(ii) Which term of the A.P. is zero (if exists)

(iii) Find the relation ship between Slope of the straight line and common difference of A.P.

Example 6.13 The quantity demanded of a certain type of Compact Disk is 22,000 units when a unit price is Rs. 8. The customer will not buy the disk, at a unit price of Rs 30 or higher. On the other side the manufacturer will not market any disk if the price is Rs. 6 or lower. However, if the price Rs.14 the manufacturer can supply 24,000 units. Assume that the quantity demanded and quantity supplied are linearly proportional to the price.

Find (i) the demand equation (ii) supply equation (iii) the market equilibrium quantity and price.

(iv) The quantity of demand and supply when the price is Rs. 10.

Example 6.14 Find the equation of the straight line passing through (- 1, 1) and cutting off equal intercepts, but opposite in signs with the two coordinate axes.

Example 6.15 A straight line L with negative slope passes through the point (9; 4) cuts the positive coordinate axes at the points P and Q. As L varies, find the minimum value of |OP|+|OQ|, where O is the origin.

Example 6.16 The length of the perpendicular drawn from the origin to a line is 12 and makes an angle 150^0 with positive direction of the x-axis. Find the equation of the line.

Example 6.17 Area of the triangle formed by a line with the coordinate axes, is 36 square units. Find the equation of the line if the perpendicular drawn from the origin to the line makes an angle of 45^0 with positive the x-axis.

Example 6.18 Find the equation of the lines make an angle 60^0 with positive x-axis and at a distance $5\sqrt{2}$ units measured from the point (4; 7); along the line $x - y + 3 = 0$:

Example 6.19 Express the equation $\sqrt{3}x - y + 4 = 0$ in the following equivalent form:

(i) Slope and Intercept form (ii) Intercept form (iii) Normal form

Example 6.20 Rewrite $\sqrt{3}x + y + 4 = 0$ in to normal form

Example 6.21 Consider a hollow cylindrical vessel, with circumference 24 cm and height 10 cm. An ant is located on the outside of vessel 4 cm from the bottom. There is a drop of honey at the diagrammatically opposite inside of the vessel, 3 cm from the top.

(i) What is the shortest distance the ant would need to crawl to get the honey drop?.

(ii) Equation of the path traced out by the ant.

(iii) Where the ant enter in to the cylinder?. Here is a picture that illustrates the position of the ant and the honey.

Exercise - 6.2

1. Find the equation of the lines passing through the point (1,1)

(i) with y-intercept (-4) (ii) with slope 3 (iii) and (-2, 3)

(iv) and the perpendicular from the origin makes an angle 60^0 with x- axis.

2. If P(r; c) is mid point of a line segment between the axes, then show that $\frac{x}{r} + \frac{y}{c} = 2$

3. Find the equation of the line passing through the point (1; 5) and also divides the co-ordinate axes in the ratio 3:10.

4. If p is length of perpendicular from origin to the line whose intercepts on the axes are a and b, then show that $\frac{1}{p^2} = \frac{1}{a^2} + \frac{1}{b^2}$

5. The normal boiling point of water is 100^0 C or 212^0 F and the freezing point of water is 0^0 C or 32^0 F. (i) Find the linear relationship between C and F

Find (ii) the value of C for 98.6^0 F and (iii) the value of F for 38^0 C

6. An object was launched from a place P in constant speed to hit a target. At the 15th second it was 1400m away from the target and at the 18th second 800m away. Find (i) the distance between the place and the target (ii) the distance covered by it in 15 seconds.(iii) time taken to hit the target.

7. Population of a city in the years 2005 and 2010 are 1,35,000 and 1,45,000 respectively. Find the approximate population in the year 2015. (assuming that the population is constant)

8. Find the equation of the line, if the perpendicular drawn from the origin makes an angle 30^0 with x-axis and its length is 12.

9. Find the equation of the straight lines passing through (8, 3) and having intercepts sum is 1

10. Show that the points (1, 3), (2, 1) and$(\frac{1}{2}\ ,\ 4)$ are collinear, by using

(i) concept of slope (ii) using a straight line and (iii) any other method

11. A straight line is passing through the point A(1; 2) with slope$\frac{5}{12}$. Find points on the line which are 13 units away from A.

12. A 150m long train is moving with constant velocity of 12.5 m/s.

Find (i) the equation of the motion of the train,

(ii) time taken to cross a pole.

(iii) The time taken to cross the bridge of length 850m is?

13. A spring was hung from a hook in the ceiling. A number of different weights were attached to thespring to make it stretch, and the total length of the spring was measured each time is

Weight, (kg)	2	4	5	8
Length, (cm)	3	4	4.5	6

(i) Draw a graph showing the results.

(ii) Find the equation relating the length of the spring to the weight on it.

(iii) What is the actual length of the spring.

(iv) If the spring has to stretch to 9 cm long, how much weight should be added?

(v) How long will the spring be when 6 kilograms of weight on it?

14. A family is using Liquefied petroleum gas (LPG) of weight 14.2 kg for consumption. (Full weight 29.5kg includes the empty cylinders tare weight of 15.3kg.). If it is use with constant rate then it lasts for 24 days. Then the new cylinder is replaced (i) Find the equation relating the quantity of gas in the cylinder to the days. (ii) Draw the graph for first 96days.

15. In a shopping mall there is a hall of cuboid shape with dimension 800 ×800 × 720 units, which needs to be added the facility of an escalator in the path as shown by the dotted line in the figure.

(i)the minimum total length of the escalator (ii) the heights at which the escalator changes its direction. (iii) the slopes of the escalator at the turning points.

Example 6.22 Find the equations of a parallel line and a perpendicular line passing through the point (1, 2) to the line $3x+ 4y = 7$.

Example 6.23 Find the distance

(i) between two points (5, 4) and (2, 0)   (ii) from a point (1, 2) to the line $5x + 12y - 3 = 0$

(iii) between two parallel lines $3x + 4y = 12$ and $6x + 8y + 1 = 0$.

Example 6.24 Find the nearest point on the line $2x + y = 5$ from the origin.

Example 6.25 Find the equation of the bisector of the acute angle between the lines $3x+4y+2 = 0$ and $5x + 12y - 5 = 0$.

Example 6.26 Find the points on the line $x + y = 5$, that lie at a distance 2 units from the line $4x + 3y - 12 = 0$

Example 6.27 A straight line passes through a fixed point (6; 8): Find the locus of the foot of the perpendicular drawn to it from the origin O.

Example 6.28 Find the equations of the straight lines in the family of the lines $y = mx + 2$, for whichmand the x-coordinate of the point of intersection of the lines with $2x+3y = 10$ are integers.

Example 6.29 Find the equation of the line through the intersection of the lines $3x+2y+5 = 0$ and $3x - 4y + 6 = 0$ and the point (1,1).

Example 6.30 Suppose the Government has decided to erect a new Electrical Power Transmission Substation to provide better power supply to two villages namely A and B. The substation has to to be on the line l. The distances of villages A and B from the foot of the perpendiculars P and Q on the line l are 3 km and 5 km respectively and the distance between P and Q is 6 km. (i) What is the smallest length of cable required to connect the two villages. (ii) Find the equations of the cable lines that connect the power station to two villages. (Using the knowledge in conjunction with the principle of reflection allows for approach to solve this problem.)

Example 6.31 A car rental firm has charges Rs.25 with 1.8 free kilometers, and Rs.12 for every additional kilometer. Find the equation relating the cost y to the number of kilometers x. Also find the cost to travel 15 kilometers

Example 6.32 If a line joining two points (3, 0) and (5, 2) is rotated about the point (3, 0) in counter clockwise direction through an angle 15^0, then find the equation of the line in the new position

Exercise - 6.3

1. Show that the lines are $3x + 2y + 9 = 0$ and $12x + 8y - 15 = 0$ are parallel lines.
2. Find the equation of the straight line parallel to $5x - 4y + 3 = 0$ and having x-intercept 3.
3. Find the distance between the line $4x + 3y + 4 = 0$; and a point (i) (-2; 4) (ii) (7;-3)
4. Write the equation of the lines through the point (1;-1)

(i) parallel to $x + 3y - 4 = 0$ (ii) perpendicular to $3x + 4y = 6$

5. If (-4; 7) is one vertex of a rhombus and if the equation of one diagonal is 5x - y + 7 = 0, then find the equation of another diagonal.

6. Find the equation of the lines passing through the point of intersection lines 4x - y + 3 = 0 and 5x+2y +7 = 0, and

(i) through the point (-1; 2) (ii) Parallel to x-y +5 = 0 (iii) Perpendicular to x - 2y + 1 = 0

7. Find the equations of two straight lines which are parallel to the line 12x + 5y + 2 = 0 and at a unit distance from the point (1, - 1).

8. Find the equations of straight lines which are perpendicular to the line 3x + 4y - 6 = 0 and are at a distance of 4 units from (2, 1).

9. Find the equation of a straight line parallel to 2x + 3y = 10 and which is such that the sum of its intercepts on the axes is 15.

10. Find the length of the perpendicular and the co-ordinates of the foot of the perpendicular from (-10;-2) to the line x + y - 2 = 0

11. If p1 and p2 are the lengths of the perpendiculars from the origin to the straight lines x sec θ + y cosec θ = 2a and x cos θ- y sin θ = a cos 2 θ, then prove that $p_1^2 + p_2^2 = a^2$.

12.Find the distance between the parallel lines

(i) 12x + 5y = 7 and 12x + 5y + 7 = 0 (ii) 3x -4y + 5 = 0 and 6x - 8y - 15 = 0.

13. Find the family of straight lines (i) Perpendicular (ii) Parallel to 3x + 4y - 12 = 0.

14. If the line joining two points A(2,0) and B(3,1) is rotated about A in anticlockwise direction through an angle of 15^0, then find the equation of the line in new position.

15. A ray of light coming from the point (1,2) is reflected at a point A on the x-axis and it passes through the point (5,3). Find the co-ordinates of the point A.

16. A line is drawn perpendicular to 5x = y+7. Find the equation of the line if the area of the triangle formed by this line with co-ordinate axes is 10 sq. units.

17. Find the image of the point (-2; 3) about the line x + 2y - 9 = 0:

18. A photocopy store charges Rs. 1.50 per copy for the first 10 copies and Rs.1.00 per copy after the 10th copy. Let x be the number of copies, and let y be the total cost of photocopying. (i) Draw graph of the cost as x goes from 0 to 50 copies. (ii) Find the cost of making 40 copies

19. Find atleast two equations of the straight lines in the family of the lines y = 5x + b, for which b and the x-coordinate of the point of intersection of the lines with 3x - 4y = 6 are integers

20. Find all the equations of the straight lines in the family of the lines y = mx -3, for which m and the x-coordinate of the point of intersection of the lines with x - y = 6 are integers.

Example 6.33 Separate the equations $5x^2 + 6xy + y^2 = 0$

Example 6.34 If exists, find the straight lines by separating the equations $2x^2 + 2xy + y^2 = 0$.

Example 6.35 Find the equation of the pair of lines through the origin and perpendicular to the pair of lines $ax^2 + 2hxy + by^2 = 0$

Example 6.36 Show that the straight lines $x^2 - 4xy + y^2 = 0$ and $x + y = 3$ form an equilateral triangle.

Example 6.37 If the pair of lines represented by $x^2 - 2cxy - y^2 = 0$ and $x^2 - 2dxy - y^2 = 0$ be such that each pair bisects the angle between the other pair, prove that $cd = -1$.

Example 6.38 If the equation $\lambda x^2 - 10xy + 12y^2 + 5x - 16y - 3 = 0$ represents a pair of straight lines, find

(i) the value of λ and the separate equations of the lines

(ii) point of intersection of the lines

(iii) angle between the lines

Example 6.39 A student when walks from his house, at an average speed of 6 kmph, reaches his school ten minutes before the school starts. When his average speed is 4 kmph, he reaches his school five minutes late. If he starts to school every day at 8.00 A.M, then find (i) the distance between his house and the school (ii) the minimum average speed to reach the school on time and time taken to reach the school (iii) the time the school gate closes (iv) the pair of straight lines of his path of walk.

Example 6.40 If one of the straight lines of $ax^2+2hxy+by^2 = 0$ is perpendicular to $px+qy = 0$, then show that $ap^2 + 2hpq + bq^2 = 0$.

Example 6.41 Show that the straight lines joining the origin to the points of intersection of $3x - 2y + 2 = 0$ and $3x^2 + 5xy - 2y^2 + 4x + 5y = 0$ are at right angles

Exercise - 6.4

1. Find the combined equation of the straight lines whose separate equations are $x - 2y - 3 = 0$ and $x + y + 5 = 0$.
2. Show that $4x^2 + 4xy + y^2 - 6x - 3y - 4 = 0$ represents a pair of parallel lines.
3. Show that $2x^2 + 3xy - 2y^2 + 3x + y + 1 = 0$ represents a pair of perpendicular lines.
4. Show that the equation $2x^2 - xy - 3y^2 - 6x + 19y - 20 = 0$ represents a pair of intersecting lines. Show further that the angle between them is $\tan^{-1}(5)$:
5. Prove that the equation to the straight lines through the origin, each of which makes an angle α with the straight line $y = x$ is $x^2 - 2xy \sec 2\alpha + y^2 = 0$
6. Find the equation of the pair of straight lines passing through the point (1; 3) and perpendicular to the lines $2x - 3y + 1 = 0$ and $5x + y - 3 = 0$
7. Find the separate equation of the following pair of straight lines

(i) $3x^2 + 2xy - y^2 = 0$ (ii) $6(x - 1)^2 + 5(x - 1)(y - 2) - 4(y - 2)^2 = 0$

(iii) $2x^2 - xy - 3y^2 - 6x + 19y - 20 = 0$

8. The slope of one of the straight lines $ax^2 + 2hxy + by^2 = 0$ is twice that of the other,

P.T $8h^2 = 9ab$.

9. The slope of one of the straight lines $ax^2 + 2hxy + by^2 = 0$ is three times the other,

P.T $3h^2 = 4ab$.

10. A ΔOPQ is formed by the pair of straight lines $x^2 - 4xy + y^2 = 0$ and the line PQ. The equation of PQ is $x + y - 2 = 0$. Find the equation of the median of the triangle _OPQ drawn from the origin O.

11. Find p and q, if the following equation represents a pair of perpendicular lines

$6x^2 + 5xy - py^2 + 7x + qy - 5 = 0$

12. Find the value of k, if the following equation represents a pair of straight lines. Further, find whether these lines are parallel or intersecting, $12x^2 + 7xy - 12y^2 - x + 7y + k = 0$

13. For what value of k does the equation $12x^2+2kxy+2y^2+11x-5y+2 = 0$ represent two straight lines.

14. Show that the equation $9x^2-24xy+16y^2-12x+16y-12 = 0$ represents a pair of parallel lines. Find the distance between them.

15. Show that the equation $4x^2 + 4xy + y^2 - 6x - 3y - 4 = 0$ represents a pair of parallel lines. Find the distance between them.

16. Prove that one of the straight lines given by $ax^2 + 2hxy + by^2 = 0$ will bisect the angle between the co-ordinate axes if $(a + b)^2 = 4h^2$

17. If the pair of straight lines $x^2 - 2kxy - y^2 = 0$ bisect the angle between the pair of straight lines $x^2 - 2lxy - y^2 = 0$, Show that the later pair also bisects the angle between the former.

18. Prove that the straight lines joining the origin to the points of intersection of

$3x^2 + 5xy - 3y^2 + 2x + 3y = 0$ and $3x - 2y - 1 = 0$ are at right angles.

MATRICES AND DETERMINANTS

Example 7.1 Suppose that a matrix has 12 elements. What are the possible orders it can have? What if it has 7 elements?

Example 7.2 Construct a 2 × 3 matrix whose (*i, j*)th element is given by $a_{ij} = \frac{\sqrt{3}}{2}|2i - 3j|$ $(1 \le i \le 2), (1 \le j \le 3)$

Example 7.3 Find *x, y, a,* and *b* if $\begin{bmatrix} 3x+4y & 6 & x-2y \\ a+b & 2a-b & -3 \end{bmatrix} = \begin{bmatrix} 2 & 6 & 4 \\ 5 & -5 & -3 \end{bmatrix}$

Example 7.4 Compute *A* + *B* and *A* - *B* if $A = \begin{bmatrix} 4 & \sqrt{5} & 7 \\ -1 & 0 & 0.5 \end{bmatrix}$ and $B = \begin{bmatrix} \sqrt{5} & \sqrt{3} & 7.3 \\ 1 & \frac{1}{3} & \frac{1}{4} \end{bmatrix}$

Example 7.5 Find *A* + *B* + *C* if *A, B, C* are $A = \begin{bmatrix} sin^2\theta & 1 \\ cot^2\theta & 0 \end{bmatrix}$ $B = \begin{bmatrix} cos^2\theta & 0 \\ -cosec^2\theta & 1 \end{bmatrix}$ and $C = \begin{bmatrix} 0 & -1 \\ -1 & 0 \end{bmatrix}$

Example 7.6 Determine 3*B* + 4*C* - *D* if *B, C,* and *D* are given by

$B = \begin{bmatrix} 2 & 3 & 0 \\ 1 & -1 & 5 \end{bmatrix}, C = \begin{bmatrix} -1 & -2 & 3 \\ -1 & 0 & 2 \end{bmatrix}, D = \begin{bmatrix} 0 & 4 & -1 \\ 5 & 6 & -5 \end{bmatrix}$

Example 7.7 Simplify $sec\theta \begin{bmatrix} sec\theta & tan\theta \\ tan\theta & sec\theta \end{bmatrix} - tan\theta \begin{bmatrix} tan\theta & sec\theta \\ sec\theta & tan\theta \end{bmatrix}$

Example 7.8 if $A = \begin{bmatrix} 0 & c & b \\ c & 0 & a \\ b & a & 0 \end{bmatrix}$ **compute** A^2

Example 7.9 Solve for *x* if $\begin{bmatrix} x & 2 & -1 \end{bmatrix} \begin{bmatrix} 1 & 1 & 2 \\ -1 & -4 & 1 \\ -1 & -1 & -2 \end{bmatrix} \begin{bmatrix} x \\ 2 \\ 1 \end{bmatrix} = O$

Example 7.10 if $A = \begin{bmatrix} 1 & -1 & 2 \\ -2 & 1 & 3 \\ 0 & -3 & 4 \end{bmatrix}$ ***and*** $B = \begin{bmatrix} 1 & -3 \\ -1 & 1 \\ 1 & 2 \end{bmatrix}$ find *AB* and *BA* if they exist

Example 7.11 A fruit shop keeper prepares 3 different varieties of gift packages. Pack-I contains 6 apples, 3 oranges and 3 pomegranates. Pack-II contains 5 apples, 4 oranges and 4 pomegranates and Pack –III contains 6 apples, 6 oranges and 6 pomegranates. The cost of an apple, an orange and a pomegranate respectively are Rs.30, Rs. 15 and Rs.45. What is the cost of preparing each package of fruits?

Example 7.12 If $A = \begin{bmatrix} 4 & 6 & 2 \\ 0 & 1 & 5 \\ 0 & 3 & 2 \end{bmatrix}$ ***and*** $B = \begin{bmatrix} 0 & 1 & -1 \\ 3 & -1 & 4 \\ -1 & 2 & 1 \end{bmatrix}$

Verify $(i)(AB)^T = B^TA^T (ii)(A+B)^T = A^T + B^T (iii)(A-B)^T = A^T - B^T (iv)(3A)^T = 3A^T$

Theorem 7.1 For any square matrix *A* with real number entries, $A + A^T$ is a symmetric matrix and $A - A^T$ is a skew-symmetric matrix.

Theorem 7.2 Any square matrix can be expressed as the sum of a symmetric matrix and a skew-symmetric matrix.

Example 7.13

Express the matrix $A = \begin{bmatrix} 1 & 3 & 5 \\ -6 & 8 & 3 \\ -4 & 6 & 5 \end{bmatrix}$ as the sum of a symmetric and a skew-symmetric matrices

EXERCISE 7.1

1. Construct an $m \times n$ matrix $A = [a_{ij}]$ where a_{ij} is given by

   (i)$a_{ij} = \frac{(i-2j)^2}{2}$ $m = 2, n = 3$ (ii) $a_{ij} = \frac{|3i-4j|}{4}, m = 3\ n = 4$

2. Find the values of $p, q, r,$ and s if $\begin{bmatrix} p^2-1 & 0 & -31-q^3 \\ 7 & r+1 & 9 \\ -2 & 8 & s-1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & -4 \\ 7 & \frac{3}{2} & 9 \\ -2 & 8 & -\pi \end{bmatrix}$

3. Determine the value of $x + y$ if $\begin{bmatrix} 2x+y & 4x \\ 5x-7 & 4x \end{bmatrix} = \begin{bmatrix} 7 & 7y-13 \\ y & x+6 \end{bmatrix}$

4. Determine A and B if they satisfy $2A - B + \begin{bmatrix} 6 & -6 & 0 \\ -4 & 2 & 1 \end{bmatrix} = 0$ and $A - 2B = \begin{bmatrix} 3 & 2 & 8 \\ -2 & 1 & -7 \end{bmatrix}$

5. If $A = \begin{bmatrix} 1 & a \\ 0 & 1 \end{bmatrix}$, then compute A^4

6. Consider the matrix $A_\alpha = \begin{bmatrix} \cos\alpha & -\sin\alpha \\ \sin\alpha & \cos\alpha \end{bmatrix}$

   (i)Show that $A_\alpha A_\beta = A_{\alpha+\beta}$

   (ii) Find all possible real values of satisfying the condition $A_\alpha + A_\alpha{}^T = I$

7. If $A = \begin{bmatrix} 4 & 2 \\ -1 & x \end{bmatrix}$ and such that find the value of (A-2I)(A-3I)=O

8. If $A = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ a & b & -1 \end{bmatrix}$, show that A^2 is a unit matrx

9. If $A = \begin{bmatrix} 1 & 0 & 2 \\ 0 & 2 & 1 \\ 2 & 0 & 3 \end{bmatrix}$ $A^3 - A^2 + 7A + kI = O$ find the value of k

10. Give your own examples of matrices satisfying the following conditions in each case:

    (i)A and *B* such that . $AB \neq BA$ *(ii)A* and *B* such that AB=O=BA,A≠O,B≠O

    (iii)A and *B* such that AB=O ,BA≠O

11. Show that where f(x)f(y)=f(x+y) where $f(x) = \begin{bmatrix} \cos x & -\sin x & 0 \\ \sin x & \cos x & 0 \\ 0 & 0 & 1 \end{bmatrix}$

12. If A is a square matrix such that $A^2 = A$, find the value of $7A - (I - A)^3$.

13. Verify the property $A(B + C) = AB + AC$, when the matrices $A, B,$ and C are given by

    $A = \begin{bmatrix} 2 & 0 & -3 \\ 1 & 4 & 5 \end{bmatrix} B = \begin{bmatrix} 3 & 1 \\ -1 & 0 \\ 4 & 2 \end{bmatrix} C = \begin{bmatrix} 4 & 7 \\ 2 & 1 \\ 1 & -1 \end{bmatrix}$

14. Find the matrix A which satisfies The matrix relation $.A\begin{bmatrix}1 & 2 & 3\\4 & 5 & 6\end{bmatrix} = \begin{bmatrix}-7 & -8 & -9\\2 & 4 & 6\end{bmatrix}$

15. If $A^T = \begin{bmatrix}4 & 4\\-1 & 0\\2 & 3\end{bmatrix}$ & $B = \begin{bmatrix}2 & -1 & 1\\7 & 5 & -2\end{bmatrix}$, verify the following

   $\mathbf{(i)(A+B)^T = A^T + B^T = B^T + A^T (ii)(A-B)^T = A^T - B^T (iv)(B^T)^T} = B$

16. If A is a 3×4 matrix and B is a matrix such that both are defined, what is the order of the matrix B?

17. Express the following matrices as the sum of a symmetric matrix and a skew-symmetric matrix:

   (i)$\begin{bmatrix}4 & -2\\3 & -5\end{bmatrix}$ $(ii)\begin{bmatrix}3 & 3 & -1\\-2 & -2 & 1\\-4 & -5 & 2\end{bmatrix}$

18. Find the matrix A such that $\begin{bmatrix}2 & -1\\1 & 0\\-3 & 4\end{bmatrix}A^T = \begin{bmatrix}-1 & -8 & -10\\1 & 2 & -5\\9 & 22 & 15\end{bmatrix}$

19. If $A = \begin{bmatrix}1 & 2 & 2\\2 & 1 & -2\\X & 2 & Y\end{bmatrix}$ is a matrix such that $AA^T = 9I$, find the values of x and y.

20. (i) For what value of x, the matrix $A\begin{bmatrix}0 & 1 & -2\\-1 & 0 & X^3\\2 & -3 & 0\end{bmatrix}$ is skew-symmetric.

   (ii) If $\begin{bmatrix}0 & p & 3\\2 & q^2 & -1\\r & 1 & 0\end{bmatrix}$ is skew-symmetric, find the values of p, q, and r.

21. Construct the matrix $A = [a_{ij}]_{3\times3}$ where a_{ij} . State whether A is symmetric or skew-symmetric.

22. Let A and B be two symmetric matrices. Prove that $AB = BA$ if and only if AB is a symmetric matrix.

23. If A and B are symmetric matrices of same order, prove that

   (i)AB + *BA* is a symmetric matrix. *(ii)AB* - *BA* is a skew-symmetric matrix.

24. A shopkeeper in a Nuts and Spices shop makes gift packs of cashew nuts, raisins and almonds. Pack I contains 100 gm of cashew nuts, 100 gm of raisins and 50 gm of almonds. Pack-II contains 200 gm of cashew nuts, 100 gm of raisins and 100 gm of almonds. Pack-III contains 250 gm of cashew nuts, 250 gm of raisins and 150 gm of almonds. The cost of 50 gm of cashew nuts is Rs. 50, 50 gm of raisins is Rs.10, and 50 gm of almonds is Rs.60. What is the cost of each gift pack?

Example 7.14 Evaluate (i)$\begin{vmatrix}\mathbf{2} & \mathbf{4}\\\mathbf{-1} & \mathbf{2}\end{vmatrix}$ **(ii)**$\begin{vmatrix}\boldsymbol{cos\theta} & \boldsymbol{sin\theta}\\\boldsymbol{-sin\theta} & \boldsymbol{cos\theta}\end{vmatrix}$

Example 7.15 Compute all minors, cofactors of A and hence compute $|A|$ if $A = \begin{vmatrix}1 & 3 & -2\\4 & -5 & 6\\-3 & 5 & 2\end{vmatrix}$

Also check that $|A|$ remains unaltered by expanding along any row or any column

Example 7.16 Find $|A|$ if $A = \begin{vmatrix} 0 & sin\alpha & cos\alpha \\ sin\alpha & 0 & sin\beta \\ cos\alpha & -sin\beta & 0 \end{vmatrix}$

Example 7.17 Compute $|A|$ using Sarrus rule if $A = \begin{vmatrix} 3 & 4 & 1 \\ 0 & -1 & 2 \\ 5 & -2 & 6 \end{vmatrix}$

PROPERTIES (i)The value of a determinant is unaltered by interchanging its rows and columns.

(ii)If any two rows (columns) of a determinant are interchanged the determinant changes its sign but its numerical value is unaltered

(iii)If two rows (columns) of a determinant are identical then the value of the determinant is zero.

(iv)If every element in a row (or column) of a determinant is multiplied by a constant "k" then the value of the determinant is multiplied by k.

Example 7.18 If a, b, c and x are $+VE$, then show that $\begin{vmatrix} (a^x + a^{-x})^2 & (a^x - a^{-x})^2 & 1 \\ (b^x + b^{-x})^2 & (b^x - b^{-x})^2 & 1 \\ (c^x + c^{-x})^2 & (c^{x-}c^{-x})^2 & 1 \end{vmatrix}$ *is zero*

Example 7.19 Without expanding the determinants, show that $|B| = 2|A|$.

Where $B = \begin{bmatrix} b+c & c+a & a+b \\ c+a & a+b & b+c \\ a+b & b+c & c+a \end{bmatrix}$ *and* $A = \begin{bmatrix} a & b & c \\ b & c & a \\ c & a & b \end{bmatrix}$

Example 7.20 Evaluate $\begin{vmatrix} 2014 & 2017 & 0 \\ 2020 & 2023 & 1 \\ 2023 & 2026 & 0 \end{vmatrix}$

EG 7.21i) $\begin{vmatrix} x-1 & x & x-2 \\ 0 & x-2 & x-3 \\ 0 & 0 & x-3 \end{vmatrix} = 0$ **ii)** $\begin{vmatrix} 0 & 1 & 0 \\ x & 2 & x \\ 1 & 3 & x \end{vmatrix} = 0$ **iii)** $\begin{vmatrix} 2 & x & 4 \\ 3 & 2 & 1 \\ 1 & 2 & 3 \end{vmatrix} = -3$ **iv)** $\begin{vmatrix} 4 & 3 & 9 \\ 3 & -2 & 7 \\ 4 & 4 & x \end{vmatrix} = -1$

Example 7.22 Prove that $\begin{vmatrix} 1 & 1 & 1 \\ x & y & z \\ x^2 & y^2 & z^2 \end{vmatrix} = (x-y)(y-z)(z-x)$

EXERCISE 7.2

1. Without expanding the determinant, prove that $\begin{vmatrix} s & a^2 & b^2+c^2 \\ s & b^2 & c^2+a^2 \\ s & c^2 & a^2+b^2 \end{vmatrix} = 0$

2. Show that $\begin{vmatrix} b+c & bc & b^2c^2 \\ c+a & ca & c^2a^2 \\ a+b & ab & a^2b^2 \end{vmatrix}$

3. Prove that . $\begin{vmatrix} a^2 & bc & ac+c^2 \\ a^2+ab & b^2 & ac \\ ab & b^2+bc & c^2 \end{vmatrix} = 4a^2b^2c^2$

4. Prove that $\begin{vmatrix} \mathbf{1+a} & \mathbf{1} & \mathbf{1} \\ \mathbf{1} & \mathbf{1+b} & \mathbf{1} \\ \mathbf{1} & \mathbf{1} & \mathbf{1+c} \end{vmatrix} = \mathbf{abc}\left(\mathbf{1}+\frac{\mathbf{1}}{\mathbf{a}}+\frac{\mathbf{1}}{\mathbf{b}}+\frac{\mathbf{1}}{\mathbf{c}}\right)$

5. Prove that $\begin{vmatrix} sec^2\theta & tan^2\theta & 1 \\ tan^2\theta & sec^2\theta & -1 \\ 38 & 36 & 2 \end{vmatrix} = 0$

6. Show that $\begin{vmatrix} x+2a & y+2b & z+2c \\ x & y & z \\ a & b & c \end{vmatrix} = 0$

7. Write the general form of a 3×3 skew-symmetric matrix and prove that its determinant is 0.

8. If $\begin{vmatrix} a & b & a\alpha+b \\ b & c & b\alpha+c \\ a\alpha+b & b\alpha+c & o \end{vmatrix}$ prove that *a, b, c* are in *G.P.* or α is a root of $ax^2+2bx+c=0$.

9. Prove that $\begin{vmatrix} 1 & a & a^2-bc \\ 1 & b & b^2-ca \\ 1 & c & c^2-ab \end{vmatrix} = 0$

10. If *a, b, c* are *p*th, *q*th and *r*th terms of an *A.P*, find the value of $\begin{vmatrix} a & b & c \\ p & q & r \\ 1 & 1 & 1 \end{vmatrix}$

11. Show that $\begin{vmatrix} \mathbf{a^2+x^2} & \mathbf{ab} & \mathbf{ac} \\ \mathbf{ab} & \mathbf{b^2+x^2} & \mathbf{bc} \\ \mathbf{ac} & \mathbf{bc} & \mathbf{c^2+x^2} \end{vmatrix}$ is divisible by x^4

12. If *a, b, c* are all positive, and are *p*th, *q*th and *r*th terms of a *G.P.*, show that $\begin{vmatrix} loga & p & 1 \\ logb & q & 1 \\ logc & r & 1 \end{vmatrix}$

13. Find the value of if $\begin{vmatrix} 1 & {log_x}^y & {log_x}^z \\ {log_y}^x & 1 & {log_y}^z \\ {log_z}^x & {log_z}^y & 1 \end{vmatrix}$ $if\ x, y, z \neq 1$

14. If $A = \begin{bmatrix} \frac{1}{2} & \alpha \\ 0 & \frac{1}{2} \end{bmatrix}$, prove that $\sum_{i=0}^{n} \det(A^k) = \frac{1}{3}\left(1-\frac{1}{4^n}\right)$

15. Without expanding, evaluate the following determinants :

(i) $\begin{vmatrix} 2 & 3 & 4 \\ 5 & 6 & 8 \\ 6x & 9x & 12x \end{vmatrix}$ (ii) $\begin{vmatrix} x+y & y+z & z+x \\ z & x & y \\ 1 & 1 & 1 \end{vmatrix}$

16. If *A* is a square matrix and $|A| = 2$, find the value of $|AA^T|$

17. If *A* and *B* are square matrices of order 3 such that $|A|$ = -1 and $|B|$ = 3, find the value of $|3AB|$.

18. If λ = - 2, determine the value of $\begin{vmatrix} 0 & 2\lambda & 1 \\ \lambda^2 & 0 & 3\lambda^2+1 \\ -1 & 6\lambda-1 & 0 \end{vmatrix}$

19. Determine the roots of the equation $\begin{vmatrix} 1 & 4 & 20 \\ 1 & -2 & 5 \\ 1 & 2x & 5x^2 \end{vmatrix} = 0$

20. Verify that det(AB) = (det A) (det B) for $A = \begin{bmatrix} 4 & 3 & -2 \\ 1 & 0 & 7 \\ 2 & 3 & -5 \end{bmatrix}$ $B = \begin{bmatrix} 1 & 3 & 3 \\ -2 & 4 & 0 \\ 9 & 7 & 5 \end{bmatrix}$

21. Using cofactors of elements of second row, evaluate $|A|$, where $A = \begin{bmatrix} 5 & 3 & 8 \\ 2 & 0 & 1 \\ 1 & 2 & 3 \end{bmatrix}$

Example 7.23 Using factor $\begin{vmatrix} \mathbf{x+1} & \mathbf{3} & \mathbf{5} \\ \mathbf{2} & \mathbf{x+2} & \mathbf{5} \\ \mathbf{2} & \mathbf{3} & \mathbf{x+4} \end{vmatrix} = \mathbf{(x-1)^2(x+9)}$

P.t. $\begin{vmatrix} \mathbf{1} & \mathbf{a} & \mathbf{a^3} \\ \mathbf{1} & \mathbf{b} & \mathbf{b^3} \\ \mathbf{1} & \mathbf{c} & \mathbf{c^3} \end{vmatrix} = \mathbf{(a-b)(b-c)(c-a)(a+b+c)}$

Example7.24 P.t. $\begin{vmatrix} \mathbf{1} & \mathbf{x^2} & \mathbf{x^3} \\ \mathbf{1} & \mathbf{y^2} & \mathbf{y^3} \\ \mathbf{1} & \mathbf{z^2} & \mathbf{z^3} \end{vmatrix} = \mathbf{(x-y)(y-z)(z-x)(xy+yz+zx)}$

Example 7.25 P.t. $\begin{vmatrix} \mathbf{(q+r)^2} & \mathbf{p^2} & \mathbf{q^2} \\ \mathbf{q^2} & \mathbf{(r+p)^2} & \mathbf{q^2} \\ \mathbf{r^2} & \mathbf{r^2} & \mathbf{(p+q)^2} \end{vmatrix} = \mathbf{2pqr(p+q+r)^2}$

Example7.26 In a triangle ABC, if $\begin{vmatrix} 1 & 1 & 1 \\ 1+\sin A & 1+\sin B & 1+\sin C \\ \sin A(1+\sin A) & \sin B(1+\sin B) & \sin C(1+\sin C) \end{vmatrix} = 0$

prove that ΔABC is an isosceles triangle.

EXERCISE 7.3

1. **Show that** $\begin{vmatrix} \mathbf{b+c} & \mathbf{a-c} & \mathbf{a-b} \\ \mathbf{b-c} & \mathbf{c+a} & \mathbf{b-a} \\ \mathbf{c-b} & \mathbf{c-a} & \mathbf{a+b} \end{vmatrix} = \mathbf{8abc}$

2. **Solve using factor method** $\begin{vmatrix} \mathbf{x+a} & \mathbf{b} & \mathbf{c} \\ \mathbf{a} & \mathbf{x+b} & \mathbf{c} \\ \mathbf{a} & \mathbf{b} & \mathbf{x+c} \end{vmatrix} = \mathbf{0}$

3. **Factorise** $\begin{vmatrix} \mathbf{a} & \mathbf{b} & \mathbf{c} \\ \mathbf{a^2} & \mathbf{b^2} & \mathbf{c^2} \\ \mathbf{bc} & \mathbf{ca} & \mathbf{ab} \end{vmatrix}$

4. **Show that** $\begin{vmatrix} \mathbf{b+c} & \mathbf{a} & \mathbf{a^2} \\ \mathbf{c+a} & \mathbf{b} & \mathbf{b^2} \\ \mathbf{a+b} & \mathbf{c} & \mathbf{c^2} \end{vmatrix} = \mathbf{(a+b+c)(a-b)(b-c)(c-a)}$

5. **Show that** $\begin{vmatrix} \mathbf{x} & \mathbf{a} & \mathbf{a} \\ \mathbf{a} & \mathbf{x} & \mathbf{a} \\ \mathbf{a} & \mathbf{a} & \mathbf{x} \end{vmatrix} = \mathbf{(x-a)^2(x+2a)}$

6. **show that** $\begin{vmatrix} 1 & 1 & 1 \\ x & y & z \\ x^2 & y^2 & z^2 \end{vmatrix} = (x-y)(y-z)(z-x)$

7. *solve* $\begin{vmatrix} 4-x & 4+x & 4+x \\ 4+x & 4-x & 4+x \\ 4+x & 4+x & 4-x \end{vmatrix} = 0$

(i) Show that $\begin{vmatrix} 1 & x & x \\ x & 1 & x \\ x & x & 1 \end{vmatrix}^2 = \begin{vmatrix} 1-2x^2 & -x^2 & -x^2 \\ -x^2 & -1 & x^2-2x \\ -x^2 & x^2-2x & -1 \end{vmatrix}$

(ii) P.T $\begin{vmatrix} 1 & x & x^2 \\ 1 & y & y^2 \\ 1 & z & z^2 \end{vmatrix}\begin{vmatrix} a^2 & 1 & 2a \\ b^2 & 1 & 2b \\ c^2 & 1 & 2c \end{vmatrix} = \begin{vmatrix} (a-x)^2 & (b-x)^2 & (c-x)^2 \\ (a-y)^2 & (b-y)^2 & (c-y)^2 \\ (a-z)^2 & (b-z)^2 & (c-z)^2 \end{vmatrix}$ and

(iii) P.T $\begin{vmatrix} 0 & c & b \\ c & 0 & a \\ b & a & 0 \end{vmatrix}^2 = \begin{vmatrix} b^2+c^2 & ab & ac \\ ab & c^2+a^2 & bc \\ ac & bc & a^2+b^2 \end{vmatrix}$

(iv) Show that $\begin{vmatrix} 2bc-a^2 & c^2 & b^2 \\ c^2 & 2ca-b^2 & a^2 \\ b^2 & a^2 & 2ab-c^2 \end{vmatrix} = \begin{vmatrix} a & b & c \\ b & a & c \\ c & a & b \end{vmatrix}^2$

(v) If A_1, B_1, C_1 are the co factores of a_1, b_1, c_1 in $\Delta = \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}$ Then show that $\begin{vmatrix} A_1 & B_1 & C_1 \\ A_2 & B_2 & C_2 \\ A_3 & B_3 & C_3 \end{vmatrix} = \Delta^2$

(vi) If $A = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}$ And $B = \begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix}$ Show that $|AB| = |A||B|$

(vii) P.t. $\begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix}^2 = \begin{vmatrix} a_1^2+a_2^2 & a_1b_1+a_2b_2 \\ a_1b_1+a_2b_2 & b_1^2+b_2^2 \end{vmatrix}$

Example 7.32 If the area of the triangle with vertices (- 3, 0), (3, 0) and (0, *k*) is 9 square units, find the values of k.

Example 7.33 Find the area of the triangle whose vertices are (- 2, - 3), (3, 2), and (- 1, - 8).

Example 7.34 Show that the points (*a*, *b* + *c*), (*b*, *c* + *a*), and (*c*, *a* + *b*) are collinear.

EXERCISE 7.4

(1) Find the area of the triangle whose vertices are (0, 0), (1, 2) and (4, 3).

(2) If (*k*, 2), (2, 4) and (3, 2) are vertices of the triangle of area 4 square units then determine the value of *k*.

(3) Identify the singular and non-singular matrices: (i) $\begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{bmatrix}$ (ii) $\begin{bmatrix} 2 & -3 & 5 \\ 6 & 0 & 4 \\ 1 & 5 & -7 \end{bmatrix}$ (iii) $\begin{bmatrix} 0 & a-b & k \\ b-a & 0 & 5 \\ -k & -5 & 0 \end{bmatrix}$

(4) Determine the values of *a* and *b* so that the following matrices are singular: (*i*) $\begin{bmatrix} 7 & 3 \\ -2 & a \end{bmatrix}$ (*ii*) $\begin{bmatrix} b-1 & 2 & 3 \\ 3 & 1 & 2 \\ 1 & -2 & 4 \end{bmatrix}$

5) If , determine cos 2θ =0 $\begin{vmatrix} 0 & \cos\theta & \sin\theta \\ \cos\theta & \sin\theta & 0 \\ \sin\theta & 0 & \cos\theta \end{vmatrix}^2$

(6) Find the value of the product $\begin{vmatrix} \log_3 64 & \log_4 3 \\ \log_3 8 & \log_4 9 \end{vmatrix} \times \begin{vmatrix} \log_2 3 & \log_8 3 \\ \log_3 8 & \log_3 4 \end{vmatrix}$

VECTOR ALGEBRA

1. State triangle law of addition

   If two vectors are represented in magnitude and direction by the two sides of a triangle taken in the same order, then their sum is represented by the third side taken in the reverse order In ΔABC $\overrightarrow{AB}+\overrightarrow{BC}+\overrightarrow{CA}=\vec{0}$ Or Let $\vec{a}$, . $\vec{b}$ Be any two vectors then the addition of these two vectors $\vec{a}+\vec{b}$ Can be found by using triangle law of addition
2. Prove vector method vector addition is commutative
3. Prove vector method vector addition is associative
4. For every vector $\vec{a}$,$\vec{a}+\vec{0}=\vec{0}+\vec{a}=\vec{a}$Where $\vec{0}$Is the null vector. [existence of additive identity]
5. For every vector $\vec{a}$There corresponds a vector $\vec{a}$Such that$(-\vec{a})+\vec{a}=\vec{a}+(-\vec{a})=\vec{0}$ [existence of additive inverse]
6. $\overrightarrow{AB}=\overrightarrow{OB}-\overrightarrow{OA}$ Where $\overrightarrow{OA}$ And $\overrightarrow{OB}$ Are the p.vs of A and B respectively.
7. If $\vec{a}$, $\vec{b}$, $\vec{c}$ Be the vectors represented by the three sides of atriangle, taken in order, then prove that $\vec{a}+\vec{b}+\vec{c}=\vec{0}$
8. If $\vec{a}$And $\vec{b}$Represent two adjacent sides and respectively of a paralleogramabcd. Find the diagonals $\overrightarrow{AC}$And $\overrightarrow{BD}$
9. State parallelogram law of addition draw diagram
10. Represent graphically the displacement of

    (i) 30 km west of north 60° (ii) 60 km south of east.50°
11. Represent graphically the displacement of

    north of east. 45 cm 30° (ii) south of west 80 km, 60°
12. The position vectors of the points A, B, C, D are $\vec{a}$, $\vec{b}$, $2\vec{a}+3\vec{b}$, $\vec{a}-2\vec{b}$, find $\overrightarrow{DB}$ AND $\overrightarrow{AC}$
13. Find the position vector of the points which divide the join of the points A and B whose p.vs are $\overrightarrow{2a}-4\vec{b}$ And $2\vec{a}-8\vec{b}$ Internally and externally in the ratio 1 : 3
14. Show that the points with P.V"s $\vec{a}-2\vec{b}+3\vec{c}$, $-2\vec{a}+3\vec{b}+2\vec{c}$And $-8\vec{a}+13\vec{b}$ Are collinear
15. Let$\vec{a}$, $\vec{b}$, $\vec{c}$ Be the P.V s of three distinct points A, B, C. If there exists scalars l, m, n (not all zero) such that $l\vec{a}+m\vec{b}+n\vec{c}=0$ and l+m+n=0 then show that A, B and C lie on a line.
16. Prove that the relation R defined on the set V of all vectors by '$\vec{a}\, R\, \vec{b}$ if $\vec{a}=\vec{b}$ ' is an equivalence relation on V.
17. Let $\vec{a}$&$\vec{b}$ be the position vectors of the points A and B. Prove that the position vectors of the points which trisects the line segment AB are $\frac{\vec{a}+2\vec{b}}{3}$ and $\frac{\vec{b}+2\vec{a}}{3}$

18. If $\vec{a}$&$\vec{b}$ represent a side and a diagonal of a parallelogram, find the other sides and the other diagonal

19. Let A, B, and C be the vertices of a triangle. Let D,E, and F be the midpoints of the sides BC, CA, and AB respectively. Show that $\overrightarrow{AD} + \overrightarrow{BE} + \overrightarrow{CF} = \vec{0}$

20. If $\overrightarrow{PO}+\overrightarrow{OQ}=\overrightarrow{QO} + \overrightarrow{OR}$ Show that the points P,Q,R are collinear.

21. If D is the mid-point of the side BC of a triangle ABC, prove that $\overrightarrow{AB} + \overrightarrow{AC} = 2\overrightarrow{AD}$

22. If G is the centroid of a triangle ABC, prove that $\overrightarrow{GA} + \overrightarrow{GB} + \overrightarrow{GC} = \vec{0}$

23. If ABC and A′B′C′are two triangles and G, G′be their corresponding centroids, prove that $\overrightarrow{AA'} + \overrightarrow{BB'} + \overrightarrow{CC'} = 3\overrightarrow{GG'}$

24. If $\vec{a}$ And $\vec{b}$ Are the vectors determined by two adjacent sides of a regular hexagon, find the vectors determined by the other sides taken in order

25. (i) State and prove internal division (ii)State and prove external division

26. The medians of a triangle are concurrent.

27. By using vectors, prove that a quadrilateral is a parallelogram if and only if the diagonals bisect each other.

28. If ABCD is a quadrilateral and E and F are the mid-points of AC and BD respectively ,P.T $\overrightarrow{AB} + \overrightarrow{AD} + \overrightarrow{CB} + \overrightarrow{CD} = 4\overrightarrow{EF}$

29. Using vector method, prove that the line segments joining the mid-points of the adjacent sides of a quadrilateral taken in order form a parallelogram

30. In a triangle ABC if D and E are the midpoints of AB and AC respectively, show that $\overrightarrow{BE} + \overrightarrow{DC} = \frac{3}{2}\overrightarrow{BC}$

31. Prove that the line segment joining the midpoints of two sides of a triangle is parallel to the third side whose length is half of the length of the third side.

32. Prove that the line segments joining the midpoints of the adjacent sides of a quadrilateral form a parallelogram.

33. Let O be the origin and $p(-2, 4)$ be a point in the xy-plane. $\overrightarrow{OP}$ terms of vectors find $|\overrightarrow{OP}|$

34. Find the components along the coordinates of the position vector of $p(-4, 3)$

35. Express $\overrightarrow{AB}$In terms of unit vectors and the points are $A(-6, 3)$ and $B(-2, -5)$. Find also$|\overrightarrow{AB}|$

Example 8.4 Find a unit vector along the direction of the vector$5\hat{i} - 3\hat{j} + 4\hat{k}$

Example 8.5 Find a direction ratio and direction cosines of the following vectors.

(i)$3\hat{i} + 4\hat{j} - 6\hat{k}$ (ii) $3\hat{i} - 4\hat{k}$

Example 8.6

(i) Find the direction cosines of a vector whose direction ratios are 2, 3, - 6.

(ii)Can a vector have direction angles ? $30^0, 45^0, 60^0$

(iii)Find the direction cosines of $\overrightarrow{\mathbf{AB}}$ where A is (2, 3, 1) and B is (3, - 1, 2). ,

(iv)Find the direction cosines of the line joining (2, 3, 1) and (3, - 1, 2).

(v)The direction ratios of a vector are 2, 3, 6 and it's magnitude is 5. Find the vector.

Example 8.7 Show that the points whose position vectors are

$2\hat{i} + 3\hat{j} - 5\hat{k}$, $3\hat{i} + 4\hat{j} - 2\hat{k}$ and $6\hat{i} - 5\hat{j} + 7\hat{k}$, are collinear.

Example 8.8 Find a point whose position vector has magnitude 5 and parallel to the vector $4\hat{i} - 3\hat{j} + 10\hat{k}$

Example 8:9 Prove that the points whose position vectors

$2\hat{i} + 4\hat{j} + 3\hat{k}$, $4\hat{i} + \hat{j} + 9\hat{k}$ and $10\hat{i} - \hat{j} + 6\hat{k}$ form a right angled triangle.

Example 8.10 Show that the vectors $5\hat{i} + 6\hat{j} + 7\hat{k}$, $7\hat{i} - 8\hat{j} + 9\hat{k}$ and $3\hat{i} + 20\hat{j} + 5\hat{k}$ are coplanar.

EXERCISE 8.2

1. Verify whether the following ratios are direction cosines of some vector or not.
   i) $\frac{1}{5}, \frac{3}{5}, \frac{4}{5}$ (ii) $\frac{1}{\sqrt{2}}, \frac{1}{2}, \frac{1}{2}$ (iii) $\frac{4}{3}, 0, \frac{3}{4}$
2. Find the direction cosines of a vector whose direction ratios are
   (i)1 , 2 , 3 (ii) 3 , - 1 , 3 (iii) 0 , 0 , 7
3. Find the direction cosines and direction ratios for the following vectors.
   (i) $3\hat{i} - 4\hat{j} + 8\hat{k}$ (ii) $3\hat{i} + \hat{j} + \hat{k}$ (iii) $\hat{j}$ (iv) $5\hat{i} - 3\hat{j} - 48\hat{k}$ (v) $3\hat{i} - 3\hat{j} + 4\hat{k}$ (vi) $\hat{i} - \hat{k}$
4. A triangle is formed by joining the points (1, 0, 0), (0, 1, 0) and (0, 0, 1). Find the direction cosines of the medians.
5. If $\frac{1}{2}, \frac{1}{\sqrt{2}}, a$ are the direction cosines of some vector, then find a.
6. If $(a, a+b, a+b+c)$ is one set of direction ratios of the line joining (1, 0, 0) and (0, 1, 0), then find a set of values of a, b, c.
7. Show that the vectors $2\hat{i} - \hat{j} + \hat{k}$, $3\hat{i} - 4\hat{j} - 4\hat{k}$ and $\hat{i} - 3\hat{j} - 5\hat{k}$ form a right angled triangle.
8. Find the value of λ for which the vectors $\vec{a} = 3\hat{i} + 2\hat{j} + 9\hat{k}$ and $\vec{b} = \hat{i} + \lambda\hat{j} + 3\hat{k}$ are parallel.
9. Show that the following vectors are coplanar
   (i) $\hat{i} - 2\hat{j} + 3\hat{k}$, $-2\hat{i} + 3\hat{j} - 4\hat{k}$ $-\hat{j} + 2\hat{k}$
   (ii) $5\hat{i} + 6\hat{j} + 7\hat{k}$, $7\hat{i} - 8\hat{j} + 9\hat{k}$, $3\hat{i} + 20\hat{j} + 5\hat{k}$
10. Show that the points whose position vectors
    $4\hat{i} + 5\hat{j} + \hat{k}$, $-\hat{j} - \hat{k}$, $3\hat{i} + 9\hat{j} + 4\hat{k}$, $-4\hat{i} + 4\hat{j} + 4\hat{k}$ are coplanar.

11. If $\vec{a} = 2\hat{\imath} + 3\hat{\jmath} - 4\hat{k}$, $\vec{b} = 3\hat{\imath} - 4\hat{\jmath} - 5\hat{k}$ and $\vec{c} = -3\hat{\imath} + 2\hat{\jmath} + 3\hat{k}$ find the magnitude and direction cosines of (i) $\vec{a} + \vec{b} + \vec{c}$ (ii) $3\vec{a} - 2\vec{b} + 5\vec{c}$

12. The position vectors of the vertices of a triangle are
$\hat{\imath} + 2\hat{\jmath} + 3\hat{k}$, $3\hat{\imath} - 4\hat{\jmath} + 5\hat{k}$ $-2\hat{\imath} + 3\hat{\jmath} - 7\hat{k}$. Find the perimeter of the triangle.

13. Find the unit vector parallel to) $3\vec{a} - 2\vec{b} + 4\vec{c}$
If $\vec{a} = 3\hat{\imath} - \hat{\jmath} - 4\hat{k}$, $\vec{b} = -2\hat{\imath} + 4\hat{\jmath} - 3\hat{k}$ and $\vec{c} = \hat{\imath} + 2\hat{\jmath} - \hat{k}$

14. The position vectors $\vec{a}$, $\vec{b}$, $\vec{c}$ of three points satisfy the relation $2\vec{a} - 7\vec{b} + 5\vec{c} = \vec{0}$ Are these points collinear?

15. The position vectors of the points *P, Q, R, S* are $\hat{\imath} + \hat{\jmath} + \hat{k}$, $2\hat{\imath} + 5\hat{\jmath}$, $3\hat{\imath} + 2\hat{\jmath} - 3\hat{k}$, $\hat{\imath} - 6\hat{\jmath} - \hat{k}$ and respectively. Prove that the line *PQ* and *RS* are parallel.

16. Find the value or values of *m* for which $m(\hat{\imath} + \hat{\jmath} + \hat{k})$ is a unit vector.

17. Show that the points *A* (1, 1, 1), *B*(1, 2, 3) and *C*(2, - 1, 1) are vertices of an isosceles triangle.

Example 8.11 Find when $\vec{a} . \vec{b}$ when

(i) $\vec{a} = \hat{\imath} - \hat{\jmath} + 5\hat{k}$ and $\vec{b} = 3\hat{\imath} - 2\hat{k}$

(ii) $\vec{a}$ & $\vec{b}$ represent the points (2, 3, - 1) and (- 1, 2, 3)

Example 8.12 Find $(\vec{a} + 3\vec{b})(2\vec{a} - \vec{b})$ if) $\vec{a} = \hat{\imath} + \hat{\jmath} + 2\hat{k}$ and $\vec{b} = 3\hat{\imath} - 2\hat{\imath} - \hat{k}$

Example 8.13 If $\vec{a} = 2\hat{\imath} + 2\hat{\jmath} + 3\hat{k}$, $\vec{b} = -\hat{\imath} + 2\hat{\jmath} + \hat{k}$ and $\vec{c} = 3\hat{\imath} + \hat{\jmath}$ be such that $\vec{a} = \lambda\vec{b}$ is perpendicular to $\vec{c}$ then find λ.

Example 8.14 If $|\vec{a} + \vec{b}| = |\vec{a} - \vec{b}|$ prove that $\vec{a}$ & $\vec{b}$ are perpendicular

Example 8.15 For any vector $\vec{r}$ prove that $\vec{r} = (\vec{r}.\hat{\imath})\hat{\imath} + (\vec{r}.\hat{\imath})\hat{\imath} + (\vec{r}.\hat{k})\hat{k}$

Example 8.16 Find the angle between the vectors $5\hat{\imath} + 3\hat{\jmath} + 4\hat{k}$ and $6\hat{\imath} - 8\hat{\jmath} - \hat{k}$

Example 8.17 Find the projection of $\overrightarrow{AB}$ on $\overrightarrow{CD}$ where *A, B, C, D* are the points (4, - 3, 0), (7, - 5, - 1), (- 2, 1, 3), (0, 2, 5)

Example 8.18 If $\vec{a}$, $\vec{b}$, $\vec{c}$ are three unit vectors satisfying $\vec{a} - \sqrt{3}\vec{b} + \vec{c} = \vec{0}$ then find the angle between $\vec{a}$ and $\vec{c}$

Example 8.19 Show that the points (4, - 3, 1), (2, - 4, 5) and (1, - 1, 0) form a right angled triangle

EXERCISE 8.3

Find when . $\vec{a}.\vec{b}$ (i) $\vec{a} = \hat{\imath} - 2\hat{\jmath} + \hat{k}$, $\vec{b} = 3\hat{\imath} - 4\hat{\jmath} - 2\hat{k}$ and

(ii) $\vec{a} = 2\hat{\imath} + 2\hat{\jmath} - \hat{k}$, $\vec{b} = 6\hat{\imath} - 3\hat{\jmath} + 2\hat{k}$

1. Find the value λ for which the vectors are . $\vec{a}$ & $\vec{b}$ perpendicular, where and
(i) $\vec{a} = 2\hat{\imath} + \lambda\hat{\jmath} + \hat{k}$, $\vec{b} = \hat{\imath} - 2\hat{\jmath} + 3\hat{k}$ (ii) $\vec{a} = \hat{\imath} - 2\hat{\jmath} + \hat{k}$, $\vec{b} = 3\hat{\imath} - 2\hat{\jmath} + \lambda\hat{k}$

2. If $\vec{a}$ & $\vec{b}$ are two vectors such that $|\vec{a}| = 10$, $|\vec{b}| = 15$ $\vec{a}.\vec{b} = 75\sqrt{2}$, find the angle between . . $\vec{a}$ & $\vec{b}$
3. Find the angle between the vectors (i) $2\hat{i} + 3\hat{j} - 6\hat{k}$ and $6\hat{i} - 3\hat{j} + 2\hat{k}$ (ii) $\hat{i} - \hat{j}$ and $\hat{j} - \hat{k}$
4. If are three vectors $\vec{\mathbf{a}}$, $\vec{\mathbf{b}}$, $\vec{\mathbf{c}}$ such that $\vec{\mathbf{a}} + 2\vec{\mathbf{b}} + \vec{\mathbf{c}} = \vec{\mathbf{0}}$ and $|\vec{a}| = 3$, $|\vec{b}| = 4$ $|\vec{c}| = 7$, find the angle between $\vec{a}$ & $\vec{b}$
5. Show that the vectors $\vec{a} = 2\hat{i} + 3\hat{j} + 6\hat{k}$, $\vec{b} = 6\hat{i} + 2\hat{j} - 3\hat{k}$ and $\vec{c} = 3\hat{i} - 6\hat{j} + 2\hat{k}$ are mutually orthogonal,
6. Show that the vectors $-\hat{i} - 2\hat{j} - 6\hat{k}$, $2\hat{i} - \hat{j} + \hat{k}$, $-\hat{i} + 3\hat{j} + 5\hat{k}$ form a right angled triangle.
7. If $|\vec{a}| = 5$, $|\vec{b}| = 6$ $|\vec{c}| = 7$ and $\vec{\mathbf{a}} + \vec{\mathbf{b}} + \vec{\mathbf{c}} = \vec{\mathbf{0}}$, find . $\vec{a}.\vec{b} + \vec{b}.\vec{c} + \vec{c}.\vec{a}$
8. Show that the points (2, - 1, 3), (4, 3, 1) and (3, 1, 2) are collinear.
9. If $\vec{a}, \vec{b}$ are unit vectors and θ is the angle between them, show that

   (i)$\sin\frac{\theta}{2} = \frac{1}{2}|\vec{a} - \vec{b}|$ (ii)$\cos\frac{\theta}{2} = \frac{1}{2}|\vec{a} + \vec{b}|$ (iii) $\tan\frac{\theta}{2} = \frac{|\vec{a}-\vec{b}|}{|\vec{a}+\vec{b}|}$
10. Let $\vec{\mathbf{a}}$, $\vec{\mathbf{b}}$, $\vec{\mathbf{c}}$ be three vectors such that $|\vec{a}| = 3$, $|\vec{b}| = 4$ $|\vec{c}| = 5$ and each one of them being perpendicular to the sum of the other two, find $|\vec{\mathbf{a}} + \vec{\mathbf{b}} + \vec{\mathbf{c}}|$
11. Find the projection of the vector are $\hat{i} + 3\hat{j} + 7\hat{k}$ on are $2\hat{i} + 6\hat{j} + 3\hat{k}$ the vector
12. Find λ, when the projection of $\vec{a} = \lambda\hat{i} + \hat{j} + 4\hat{k}$ on $\vec{b} = 2\hat{i} + 6\hat{j} + 3\hat{k}$ is 4 units
13. Three vectors $\vec{\mathbf{a}}$, $\vec{\mathbf{b}}$, $\vec{\mathbf{c}}$ are such that , $|\vec{a}| = 2$, $|\vec{b}| = 3$ $|\vec{c}| = 4$ and $\vec{\mathbf{a}} + \vec{\mathbf{b}} + \vec{\mathbf{c}} = \vec{\mathbf{0}}$

    find . $4\vec{a}.\vec{b} + 3\vec{b}.\vec{c} + 3\vec{c}.\vec{a}$

Example 8.20 Find $|\vec{a} \times \vec{b}|$, where $\vec{a} = 3\hat{i} + 4\hat{j}$ and $\vec{b} = \hat{i} + \hat{j} + \hat{k}$

Example 8.21 If $\vec{a} = -3\hat{i} + 4\hat{j} - 7\hat{k}$ and $\vec{b} = 6\hat{i} + 2\hat{j} - 3\hat{k}$

verify (i)$\vec{a}$ *and* $\vec{a} \times \vec{b}$ are perpendicular to each other.(ii) $\vec{b}$ *and* $\vec{a} \times \vec{b}$ are perpendicular to each other.

Example 8.22 Find the vectors of magnitude 6 which are perpendicular to both vectors

$\vec{a} = 4\hat{i} - \hat{j} + 3\hat{k}$ and $\vec{b} = -2\hat{i} + \hat{j} - 2\hat{k}$

Example 8.23 Find the cosine and sine angle between the $\vec{a} = 2\hat{i} + \hat{j} + 3\hat{k}$ and $\vec{b} = 4\hat{i} - 2\hat{j} + 2\hat{k}$

Example 8.24 Find the area of the parallelogram whose adjacent sides are

$\vec{a} = 3\hat{i} + \hat{j} + 4\hat{k}$ and $\vec{b} = \hat{i} - \hat{j} + \hat{k}$

Example 8.25 For any two vectors $\vec{\mathbf{a}}$ and $\vec{\mathbf{b}}$ $|\vec{a} \times \vec{b}|^2 + |\vec{a}.\vec{b}|^2 = |\vec{a}|^2 |\vec{b}|^2$

Example 8.26 Find the area of a triangle having the points as its vertices

A(1,0,0),B(0,1,0), and C (0,0,1)

EXERCISE 8.4

1. Find the magnitude of $\vec{a} \times \vec{b}$ if $\vec{a} = 2\hat{\imath} + \hat{\jmath} + 3\hat{k}$ and $\vec{b} = 3\hat{\imath} + 5\hat{\jmath} - 2\hat{k}$
2. Show that $\vec{a} \times (\vec{b} + \vec{c}) + \vec{b} \times (\vec{c} + \vec{a}) + \vec{c} \times (\vec{a} + \vec{b})$ $\hat{\imath} + \hat{\jmath} + \hat{k}$
3. Find the vectors of magnitude $10\sqrt{3}$ that are perpendicular to the plane which contains $\hat{\imath} + 2\hat{\jmath} + \hat{k}$, and $\hat{\imath} + 3\hat{\jmath} + 4\hat{k}$
4. Find the unit vectors perpendicular to each of the vectors $\vec{a} + \vec{b}$ and $\vec{a} - \vec{b}$ where $\vec{a} = \hat{\imath} + \hat{\jmath} + \hat{k}$ and $\vec{b} = \hat{\imath} + 2\hat{\jmath} + 3\hat{k}$
5. Find the area of the parallelogram whose two adjacent sides are determined by the vectors $\hat{\imath} + 2\hat{\jmath} + \hat{k}$, and $3\hat{\imath} - 2\hat{\jmath} + 4\hat{k}$
6. Find the area of the triangle whose vertices are $A(3, -1, 2)$, $B(1, -1, -3)$ and $C(4, -3, 1)$.
7. If $\vec{\mathbf{a}}, \vec{\mathbf{b}}, \vec{\mathbf{c}}$ are position vectors of the vertices *A, B, C* of a triangle *ABC*, show that the area of the triangle *ABC* is $\frac{1}{2}|\vec{a} \times \vec{b} + \vec{b} \times \vec{c} + \vec{c} \times \vec{a}|$. Also deduce the condition for collinearity of the points *A, B,* and *C.*
8. For any $\vec{a}$ vector prove that $|\vec{a} \times \hat{\imath}|^2 + |\vec{a} \times \hat{\imath}|^2 + |\vec{a} \times \hat{k}|^2 = 2\overrightarrow{|a|}^2$
9. Let $\vec{\mathbf{a}}, \vec{\mathbf{b}}, \vec{\mathbf{c}}$ be unit vectors such that $\vec{a}.\vec{b} = \vec{a}.\vec{c}$ and the angle between $\vec{b}$ and $\vec{c}$ $is\ \frac{\pi}{3}$.
   Prove that $\vec{a} = \pm\frac{2}{\sqrt{3}}(\vec{b} \times \vec{c})$
10. Find the angle between the vectors $2\hat{\imath} + \hat{\jmath} - \hat{k}$, and $\hat{\imath} + 2\hat{\jmath} + \hat{k}$ using vector product.

LIMITS AND CONTINUITY

Example 9.1 Calculate . $\lim_{x\to 0}|x|$ $x \geq 0$.

Example 9.2 Consider the function $f(x) = \sqrt{x}$ $x \geq 0$ Does $\lim_{x\to 0} f(x)$ exist?

Example 9.3 Evaluate . $\lim_{x\to 2^-} \lfloor x \rfloor$ **and** $\lim_{x\to 2^+} \lfloor x \rfloor$

Example 9.4 Let $f(x) = \begin{cases} x+1 & ,x > 0 \\ x-1 & ,x < 0 \end{cases}$ Verify the existence of limit as .

Example 9.5 Check if exists or not, where $f(x) = \begin{cases} \frac{|x+5|}{x+5} & for\ x \neq -5 \\ 0 & for\ x = 5 \end{cases}$

Example 9.6 Test the existence of the limit, $\lim_{x\to 1} \frac{4|x-1|+x-1}{|x-1|}$ $x \neq 1$

EXERCISE 9.1

Sketch the graph of f then identify the values of x_0 for which $\lim_{x\to x_0} f(x)$ **exists**

(16) $f(x) = \begin{cases} x^2 & , x \leq 2 \\ 8-2x & ,2 < x < 4 \\ 4 & ,x \geq 4 \end{cases}$ (17) $f(x) = \begin{cases} \sin x & ,x < 0 \\ 1-\cos x & 0 \leq x \leq \pi \\ \cos x & ,x > \pi \end{cases}$

(18) Sketch the graph of a function f that satisfies the given values

(

i) $f(0)$ is undefined

$\lim_{x\to 0} f(x) = 4$

f(2)=6

$\lim_{x\to 2} f(x) = 3$

(ii) $f(-2) = 0$

f(2)=0

$\lim_{x\to -2} f(x) = 0$

$\lim_{x\to 2} f(x)$ **does not exist**

(19) Write a brief description of the meaning of the notation

(20) If $f(2) = 4$, can you conclude anything about the limit of $f(x)$ as x approaches 2?

(21) If the limit of $f(x)$ as x approaches 2 is 4, can you conclude anything about $f(2)$? Explain reasoning.

(22) Evaluate : $\lim_{x\to 3} \frac{x^2-9}{x-3}$ if it exists by finding $f(3^-)$ and $f(3^+)$.

(23) Verify the existence of $\lim_{x\to 1} f(x)$ **where** $f(x) = \begin{cases} \frac{|x-1|}{x-1} & for\ x \neq 1 \\ 0 & for\ x = 1 \end{cases}$

Example 9.7 Calculate $\lim_{x\to 3}(x^3 - 2x + 6)$

Example 9.8 Calculate $\lim_{x\to x_0}(5)$ for any real number x_0

Example 9.9 Compute (i) $\lim\limits_{x\to 8}(5x)$: (ii) $\lim\limits_{x\to -2}\left(-\frac{3}{2}x\right)$

Example 9.10 Compute . $\lim\limits_{x\to 0}\left[\frac{x^2+x}{x}+4x^3+3\right]$

Example 9.11 Calculate . $\lim\limits_{x\to -1}(x^2-3)^{10}$

Example 9.12 Calculate $\lim\limits_{x\to -2}(x^3-3x+6)(-x^2+15)$

Example 9.13 Calculate . $\lim\limits_{x\to 3}\frac{x^2-6x+5}{x^3-8x+7}$

Example 9.14 Compute . $\lim\limits_{x\to 1}\frac{\sqrt{x}-1}{x-1}$

Example 9.15 Find ._ $\lim\limits_{t\to 0}\frac{\sqrt{t^2+9}-3}{t^2}$

Theorem 9.4 $\lim\limits_{x\to a}\frac{x^n-a^n}{x-a}=na^{n-1}$

Example 9.16 Compute . $\lim\limits_{x\to 1}\frac{x^3-1}{x-1}$

Example 9.17 Calculate . $\lim\limits_{t\to 1}\frac{\sqrt{t}-1}{t-1}$

Example 9.18 Find . $\lim\limits_{x\to 0}\frac{(2+x)^5-2^5}{x}$

Example 9.19 Find the positive integer n so that $\lim\limits_{x\to 3}\frac{x^n-3^n}{x-3}=27$

Example 9.20 Find the relation between a and b if exists where $f(x)=\begin{cases} ax+b & if\ x>3 \\ 3ax-4b+1 & if\ x<3 \end{cases}$

EXERCISE 9.2

Evaluate the following limits :

(1) $\lim\limits_{x\to 2}\frac{x^4-16}{x-2}$ (2) $\lim\limits_{x\to 1}\frac{x^m-1}{x^n-1}$ m, n are integers , (3) $\lim\limits_{\sqrt{x}\to 3}\frac{x^2-81}{\sqrt{x}-3}$

(4) $\lim\limits_{h\to 0}\frac{\sqrt{x+h}-\sqrt{x}}{h}$ (5) $\lim\limits_{x\to 5}\frac{\sqrt{x+4}-3}{x-5}$ (6) $\lim\limits_{x\to 2}\frac{\frac{1}{x}-\frac{1}{2}}{x-2}$ (7) $\lim\limits_{x\to 1}\frac{\sqrt{x}-x^2}{1-\sqrt{x}}$

(8) $\lim\limits_{x\to 0}\frac{\sqrt{x^2+1}-1}{\sqrt{x^2+16}-4}$ (9) $\lim\limits_{x\to 0}\frac{\sqrt{1+x}-1}{x}$ (10) $\lim\limits_{x\to 1}\frac{\sqrt[3]{7+x^3}-\sqrt{3+x^2}}{x-1}$ (11) $\lim\limits_{x\to 2}\frac{2-\sqrt{x+2}}{\sqrt[3]{2}-\sqrt[3]{4-x}}$

(12) $\lim\limits_{x\to 0}\frac{\sqrt{1+x^2}-1}{x}$ (13) $\lim\limits_{x\to 0}\frac{\sqrt{1-x}-1}{x^2}$ (14) $\lim\limits_{x\to 5}\frac{\sqrt{x-1}-2}{x-5}$ (15) $\lim\limits_{x\to a}\frac{\sqrt{x-b}-\sqrt{a-b}}{x^2-a^2}$, $a>b$

Example 9.21 Calculate $\lim_{x\to 0} \frac{1}{x^2+x^3}$

Example 9.22 Evaluate $\lim_{x\to 2} \frac{1}{(x-2)^3}$

Example 9.23 Calculate $\lim_{x\to \infty} \frac{x^3+2x+3}{5x^2+1}$

Example 9.24 Calculate . $\lim_{x\to \infty} \frac{1-x^3}{3x+2}$

Example 9.25 Alcohol is removed from the body by the lungs, the kidneys, and by chemical processes in liver. At moderate concentration levels, the majority work of removing the alcohol is done by the liver; less than 5% of the alcohol is eliminated by the lungs and kidneys. The rate r at which the liver processes alcohol from the bloodstream is related to the blood alcohol concentration x by a rational function of the form $r(x) = \frac{\alpha x}{x+\beta}$ for some positive constants a and b. Find the maximum possible rate of removal. **Example 9.26** According to Einstein's theory of relativity, the mass m of a body moving with velocity v is $m = \frac{m_0}{\sqrt{1-\frac{v^2}{c^2}}}$, where $m0$ is the initial mass and c is the speed of light. What happens to m as $v \to c^-$ Why is a left hand limit necessary?

Example 9.27 The velocity in ft/sec of a falling object is modeled by

$r(t) = -\sqrt{\frac{32}{k}\frac{1-e^{-2t\sqrt{32k}}}{1+e^{-2t\sqrt{32k}}}}$, where k is a constant that depends upon the size and shape of the object and the density of the air. Find the limiting velocity of the object, that is, find $\lim_{t\to\infty} r(t)$.

Example 9.28 Suppose that the diameter of an animal's pupils is given by , $f(x) = \frac{160x^{-0.4}+90}{4x^{-0.4}+15}$ where x is the intensity of light and $f(x)$ is in mm. Find the diameter of the pupils with (a) minimum light (b) maximum light.

EXERCISE 9.3

(1) (a) Find the left and right limits of $f(x) = \frac{x^2-4}{(x^2+4x+4)(x+3)}$ at $x = -2$

(b) $f(x) = \tan x$ at $x = \frac{\pi}{2}$

Evaluate the following limits

(2) $\lim_{x\to 3} \frac{x^2-9}{x^2-6x+9}$ (3) $\lim_{x\to\infty} \frac{3}{x-2} - \frac{2x+11}{x^2+x-6}$ (4) $\lim_{x\to\infty} \frac{x^4-5x}{x^2-3x+1}$ (5) $\lim_{x\to\infty} \frac{1+x-3x^2}{1+x^2+3x^3}$

(6) $\lim_{x\to\infty} \frac{x^3}{2x^2-1} - \frac{x^2}{2x+1}$ (7) $\lim_{x\to 0} \frac{x^3+x}{x^4-3x^2+1}$

(8) Show that (i) $\lim\limits_{n\to\infty} \frac{1+2+3+\cdots+n}{3n^2+7n+2} = \frac{1}{6}$ (ii) $\lim\limits_{n\to\infty} \frac{1^3+2^3+\cdots(3n)^3}{1+2+3+\cdots 5n(2n+3)} = \frac{9}{25}$

(iii) $\lim\limits_{n\to\infty} \frac{1}{1.2} + \frac{1}{2.3} + \frac{1}{3.4} + \cdots + \frac{1}{n(n+1)} = 1$

(9) An important problem in fishery science is to estimate the number of fish presently spawning in streams and use this information to predict the number of mature fish or "recruits" that will return to the rivers during the reproductive period. If S is the number of spawners and R the number of recruits, "Beverton-Holt spawner recruit function" is $R(S) = \frac{S}{\alpha S+\beta}$ where a and b are positive constants. Show that this function predicts approximately constant recruitment when the number of spawners is sufficiently large.

(10) A tank contains 5000 litres of pure water. Brine (very salty water) that contains 30 grams of salt per litre of water is pumped into the tank at a rate of 25 litres per minute. The concentration of salt water after t minutes (in grams per litre) is $C(t) = \frac{30t}{200+t}$ What happens to the concentration as $t\to\infty$

Example 9.29 Evaluate . $\lim\limits_{x\to 0} x^2 \sin\left(\frac{1}{x}\right)$

Example 9.30 Prove that . $\lim\limits_{x\to 0} \sin x = 0$

Example 9.31 Show that $\lim\limits_{x\to 0} x\left[\left\lfloor\frac{1}{x}\right\rfloor + \left\lfloor\frac{2}{x}\right\rfloor + \cdots \left\lfloor\frac{15}{x}\right\rfloor\right] = 120$

Result 9.1 (a) $\lim\limits_{x\to 0} \frac{\sin\theta}{\theta} = 1$ (b) $\lim\limits_{x\to 0} \frac{1-\cos\theta}{\theta} = 0.$

Result 9.3 $\lim\limits_{x\to 0} \frac{a^x-1}{x} = \log a$, $a > 0$

Result 9.4 $\lim\limits_{x\to 0} \frac{\log(1+x)}{x} = 1$

Example 9.32 Evaluate : $\lim\limits_{x\to 0} (1+\sin x)^{2\operatorname{cosec} x}$

Example 9.33 Evaluate : . $\lim\limits_{x\to\infty} \left(\frac{x+2}{x-2}\right)^x$

Example 9.34 Evaluate : $\lim\limits_{x\to\frac{\pi}{4}} \frac{4\sqrt{2}-(\cos x+\sin x)^5}{1-\sin 2x}$

Example 9.35 Do the limits of following functions exist as State reasons for your answer.

(i) $\frac{\sin|x|}{x}$ (ii) $\frac{\sin x}{|x|}$ (iii) $\frac{x|x|}{\sin|x|}$ (iv) $\frac{\sin(x-\lfloor x\rfloor)}{x-\lfloor x\rfloor}$.

Example 9.36 Evaluate : . $\lim\limits_{x\to 0} \frac{3^x-1}{\sqrt{1+x}-1}$

EXERCISE 9.4

Evaluate the following limits :

(1) $\lim_{x\to\infty}\left(1+\frac{1}{x}\right)^{7x}$ (2) $\lim_{x\to0}(1+x)^{1/3x}$ (3) $\lim_{x\to\infty}\left(1+\frac{k}{x}\right)^{\frac{m}{x}}$

(4) $\lim_{x\to\infty}\left(\frac{2x^2+3}{2x^2+5}\right)^{8x^2+3}$ (5) $\lim_{x\to\infty}\left(1+\frac{3}{x}\right)^{x+2}$ (6) $\lim_{x\to0}\frac{\sin^3\left(\frac{x}{2}\right)}{x^3}$

(7) $\lim_{x\to0}\frac{\sin\alpha x}{\sin\beta x}$ (8) $\lim_{x\to0}\frac{\tan 2x}{\sin 5x}$ (9) $\lim_{\alpha\to0}\frac{\sin(\alpha^n)}{(\sin\alpha)^n}$

(10) • $\lim_{x\to0}\frac{\sin(a+x)-\sin(a-x)}{x}$ (11) $\lim_{x\to0}\frac{\sqrt{x^2+a^2}-a}{\sqrt{x^2-b^2}-b}$ (12) $\lim_{x\to0}\frac{2\arcsin x}{3x}$

(13) $\lim_{x\to0}\frac{1-\cos x}{x^2}$ (14) $\lim_{x\to0}\frac{\tan 2x}{x}$ (15) $\lim_{x\to0}\frac{2^x-3^x}{x}$

(16) $\lim_{x\to0}\frac{3^x-1}{\sqrt{x+1}-1}$ (17) $\lim_{x\to0}\frac{1-\cos^2 x}{x\sin 2x}$ (18) $\lim_{x\to\infty}\{x[\log(x+a)-\log(x)]\}$

(19) $\lim_{x\to\infty}x\left[3^{\frac{1}{x}}+1-\cos\left(\frac{1}{x}\right)-e^{\frac{1}{x}}\right]$ (20) $\lim_{x\to\pi}\frac{\sin 3x}{\sin 2x}$ (21) $\lim_{x\to\frac{\pi}{2}}(1+\sin x)^{2\operatorname{cosec}x}$

(22) $\lim_{x\to0}\frac{\sqrt{2}-\sqrt{1+\cos x}}{\sin^2 x}$ (23) $\lim_{x\to0}\frac{\sqrt{1+\sin x}-\sqrt{1-\sin x}}{\tan x}$ (24) $\lim_{x\to\infty}\left(\frac{x^2-2x+1}{x^2-4x+2}\right)^x$

(25) $\lim_{x\to0}\frac{e^x-e^{-x}}{\sin x}$ (26) $\lim_{x\to0}\frac{e^{ax}-e^{bx}}{x}$ (27) $\lim_{x\to0}\frac{\sin x(1-\cos x)}{x^3}$

(28) $\lim_{x\to0}\frac{\tan x-\sin x}{x^3}$

Example 9.37

Describe the interval(s) on which each function is continuous.

(i) $f(x)=\tan x$ (ii) $g(x)=\begin{cases}\sin\frac{1}{x} & x\neq0\\ 0 & x=0\end{cases}$ (iii) $h(x)=\begin{cases}x\sin\frac{1}{x} & x\neq0\\ 0 & x=0\end{cases}$

Example 9.38

A tomato wholesaler finds that the price of a newly harvested tomatoes is Rs. 0.16 per kg if he purchases fewer than 100 kgs each day. However, if he purchases at least 100 kgs daily, the price drops to Rs. 0.14 per kg. Find the total cost and discuss the cost when the purchase is 100 kgs.

Example 9.39 Determine if f defined by $f(x)=\begin{cases}\sin\frac{1}{x} & \text{if } x\neq0\\ 0 & \text{if } x=0\end{cases}$ is continuous in R

EXERCISE 9.5

(1) Prove that $f(x) = 2x^2 + 3x - 5$ is continuous at all points in R .

(2) Examine the continuity of the following :

(i) $x+\sin x$ (ii) $x^2\cos x$ (iii) $e^x\tan x$ (iv) $e^{2x}+x^2$ (v) $x.\ln x$

(vi) $\frac{\sin x}{x^2}$ (vii) $\frac{x^2-16}{x+4}$ (viii) $|x+2|+|x-1|$ (ix) $\frac{|x-2|}{|x+1|}$ (x) $\cot x+\tan x$

(3) Find the points of discontinuity of the function f, where

(i) $f(x) = \begin{cases} 4x + 5 \text{ if } x \leq 3 \\ 4x - 5 \text{ if } x > 3 \end{cases}$ (ii) $f(x) = \begin{cases} x + 2 \text{ if } x \geq 2 \\ x^2 \text{ if } x < 2 \end{cases}$,

(iii) $f(x) = \begin{cases} x^3 - 3 \text{ if } x \leq 2 \\ x^2 + 1 \text{ if } x > 2 \end{cases}$ (iv) $f(x) = \begin{cases} \sin x \text{ if } 0 \leq x \leq \frac{\pi}{4} \\ \cos x \text{ if } \frac{\pi}{4} \leq x \leq \frac{\pi}{2} \end{cases}$

(4) At the given point x_0 discover whether the given function is continuous or discontinuous citing the reasons for your answer :

(i) $x_0 = 1$, $f(x) = \begin{cases} \frac{x^2-1}{x-1}, & x \neq 1 \\ 2, & x = 1 \end{cases}$ (ii) $x_0 = 3, f(x) = \begin{cases} \frac{x^2-9}{x-3} & if\ x \neq 3 \\ 5 & if\ x = 3 \end{cases}$

(5) Show that the function $\begin{cases} \frac{x^3-1}{x-1} \text{ if } x \neq 1 \\ 3 \quad \text{ if } x = 1 \end{cases}$ is continuous on $(-\infty, \infty)$, if

(6) For what value of is this function $f(x) = \begin{cases} \frac{x^4-1}{x-1} \text{ if } x \neq 1 \\ \alpha \quad \text{ if } x = 1 \end{cases}$ continuous at $x = 1$?

(7) Let $f(x) = \begin{cases} 0, \text{ if } x < 0 \\ x^2 \text{ if } 0 \leq x < 2 \\ 4 \text{ if } x \geq 2 \end{cases}$. Graph the function. Show that $f(x)$ continuous on $(-\infty, \infty)$.

(8) If f and g are continuous functions with $f(3) = 5$ and $\lim\limits_{x\to 0} [2f(x) - g(x)] = 4$, find $g(3)$.

(9) Find the points at which f is discontinuous. At which of these points f is continuous from the right, from the left, or neither? Sketch the graph of f.

(i) $f(x) = \begin{cases} 2x + 1 & if\ x \leq -1 \\ 3 & if\ -1 < x < 1 \\ 2x - 1 & if\ x \geq 1 \end{cases}$ (ii) $f(x) = \begin{cases} (x-1)^3 & if\ x < 0 \\ (x+1)^3 & if\ x \geq 0 \end{cases}$

(10) A function f is defined as follows : $f(x) = \begin{cases} 0 & for\ x < 0 \\ x^2 & for\ 0 \leq x < 1 \\ 4 - x & for\ x \geq 3 \end{cases}$ Is the function continuous?

(11) Which of the following functions f has a removable discontinuity at $x = x_0$? If the discontinuity is removable, find a function g that agrees with f for $x \neq x_0$ and is continuous on .

(i) $f(x) = \frac{x^2-2x-8}{x+2}$, $x_0 = -2$ (ii) $f(x) = \frac{x^3+64}{x+4}$, $x_0 = -4$ (iii) $f(x) = \frac{3-\sqrt{x}}{9-x}$ $x_0 = 9$

(12) Find the constant b that makes g continuous on $(-\infty,\infty)$ $g(x) = \begin{cases} x^2 - b^2 & if\ x < 4 \\ bx + 20 & if\ x \geq 4 \end{cases}$.

(13) Consider the function $f(x) = x \sin\frac{\pi}{x}$ What value must we give $f(0)$ in order to make the function continuous everywhere?

(14) The function $f(x) = \frac{x^2-1}{x^3-1}$ is not defined at $x = 1$. What value must we give $f(1)$ inorder to make $f(x)$ continuous at $x = 1$?

DIFFERENTIAL CALCULUS

Example 10.1 Find the slope of the tangent line to the graph of $f(x) = 7x + 5$ at any point $(x_0, f(x_0))$.

Example 10.2 Find the slope of tangent line to the graph of $f(x) = -5x2 + 7x$ at $(5, f(5))$.

Example 10.3 Show that the greatest integer function f (x) $= \lfloor x \rfloor$ is not differentiable at any integer?

Theorem 10.1 (Differentiability implies continuity)

If f is differentiable at a point $x = x0$, then f is continuous at $x0$.

EXERCISE10.4

(1) Find the derivatives of the following functions using first principle.

(i) f(x)=6 (ii) $f(x) = -4x + 7$ (iii) $f(x) = -x^2 + 2$

(2) Find the derivatives from the left and from the right at $x = 1$ (if they exist) of the following functions. Are the functions differentiable at $x = 1$?

(i) $f(x) = |x - 1|$ (ii) $f(x) = \sqrt{1 - x^2}$ (iii) $f(x) = f(x) = \begin{cases} x, x \le 1 \\ x^2, x > 1 \end{cases}$

(3) Determine whether the following function is differentiable at the indicated values.

(i) $f(x) = x|x|$ at $x = 0$ (ii) $f(x) = |x^2 - 1|$ at x=1

(iii) $f(x) = |x| + |x - 1|$ at $x = 0, 1$ (iv) f(x)=sin|x| at x=0

(4) Show that the following functions are not differentiable at the indicated value of x.

(i) $f(x) = \begin{cases} -x + 2, x \le 2 \\ 2x - 4, x > \end{cases}$; x = 2 (ii) $f(x) = \begin{cases} 3x, x < 0 \\ -4x, x \ge 0 \end{cases}$; x = 0

(6) If $f(x) = |x + 100| + x^2$, test whether $f'(-100)$ exists.

(7) Examine the differentiability of functions In R by drawing the diagrams.(i) | sinx | (ii) | cos x |

Example 10.7 Differentiate the following with respect to x:

(i) $y = x^3+5x^2+3x+7$ (ii) $y = e^x+\sin x+2$ (iii) $y= 4\operatorname{cosec}x-\log x-2e^x$ (iv) $y = \left(x - \frac{1}{x}\right)^2$

(v) $y= x\, e^x \log x$ (vi) $y = \frac{\cos x}{x^3}$ (vii) $y = \frac{\log x}{e^x}$

(viii) Find $f'(3)$ and $f'(5)$ if $f(x) = |x - 4|$

EXERCISE 10.2

Find the derivatives of the following functions with respect to corresponding independent variables:

(1) $f(x) = x - 3\sin x$ (2) $y = \sin x + \cos x$ (3) $f(x) = x \sin x$

(4) $y = \cos x - 2\tan x$ (5) $g(t) = t^3 \cos t$ (6) $g(t) = 4\sec t + \tan t$

(7) $y = e^x \sin x$ (8) $y = \frac{\tan x}{x}$ (9) $y = \frac{\sin x}{1+\cos x}$

(10) $y = \frac{x}{sinx+cosx}$ (11) $y = \frac{tanx-1}{secx}$ (12) $y = \frac{sinx}{x^3}$

(13) $y = \tan\theta(\sin\theta + \cos\theta)$ (14) $y = \text{cosec}\, x . \cot x$ (15) $y = x \sin x \cos x$

(16) $y = e^{-x}\log x$ (17) $(x^2+5)\log(1+x)\, e^{-3x}$ (18) $y = \sin x^0$ (19) $y = \log_{10} x$

(20) Draw the function $f'(x)$ if $f(x) = 2x^2 - 5x + 3$

Example 10.8 Find .$F'(x) if\ F(x) = \sqrt{x^2+1}$

Example 10.9 Differentiate : (i) $y = \sin(x^2)$ (ii) $y = sin^2 x$

Example 10.10 Differentiate $y = (x^3 - 1)^{100}$

Example 10.11 Find $f'(x)$ if $f(x) = \frac{1}{\sqrt[3]{x^2+x+1}}$

Example 10.12 Find the derivative of the function $g(t) = \left(\frac{t-2}{2t+1}\right)^9$

Example 10.13 Differentiate$(2x+1)^5(x^2-x+1)^4$

Example 10.14 Differentiate : $y = e^{sinx}$ **Example 10.15** Differentiate 2^x

Example 10.16 $if\ y = \tan^{-1}\left(\frac{1+x}{1-x}\right)$ $find\ y'$

EXERCISE 10.3

Differentiate the following :

(1) $y = (x^2+4x+6)^5$ (2) $y = \tan 3x$ (3) $y = \cos(\tan x)$ (4) $y = \sqrt[3]{1+x^2}$ (5) $y = e^{\sqrt{x}}$

(6) $y = \sin(e^x)$ (7) $F(x) = (x^3+4x)^7$ (8) $h(t) = \left(t - \frac{1}{t}\right)^{\frac{3}{2}}$

(9) $f(t) = \sqrt[3]{1+tant}$ (10) $y = \cos(a^3+x^3)$ (11) $y = e^{-mx}$ (12) $y = 4\sec 5x$

(13) $y = (2x-5)^4(8x^2-5)^{-3}$ (14) $y = (x^2+1)\sqrt[3]{x^2+2}$ (15) $y = xe^{-x^2}$

(16) $s(t) = \sqrt[4]{\frac{t^3+1}{t^3-1}}$ (17) $f(x) = \frac{x}{\sqrt{7-3x}}$ (18) y=tan(cosx) (19) $y = \frac{sin^2x}{cosx}$ (20) $y = 5^{\frac{-1}{x}}$

(21) $y = \sqrt{1+2tanx}$ (22) $y = sin^3x + cos^3x$ (23) $y = sin^2(coskx)$

(24) $y = (1+cos^2x)^6$ (25) $y = \frac{e^{3x}}{1+e^x}$ (26) $y = \sqrt{x+\sqrt{x}}$ (27) $y = e^{xcosx}$

(28) $y = \sqrt{x+\sqrt{x+\sqrt{x}}}$ (29) $y = \sin(\tan(\sqrt{sinx}))$ (30) $y = \sin^{-1}\left(\frac{1-x^2}{1+x^2}\right)$

Example 10.17 $find\ \frac{dy}{dx}$ if $x^2+y^2 = 1$

Example 10.18 Find the slopes of the tangent lines to the graph $x^2+y^2 = 1$ at the points corresponding to $x = 1$. **Example 10.19** Find $\frac{dy}{dx}$ $if\ x^4 + x^2y^3 - y^5 = 2x+1$

Example 10.20 Find$\frac{dy}{dx}$ if siny= y cos $2x$

Example 10.21 Find the derivative of $y = \sqrt{x^2+4}\, .sin^2x.2^x$

Example 10.22 Differentiate : $y = \frac{x^{\frac{3}{4}}\sqrt{x^2+1}}{(3x+2)^5}$ **Example 10.23** Differentiate .$y = x^{\sqrt{x}}$

Example 10.24 **$f\ y = \tan^{-1}\left(\frac{1+x}{1-x}\right)$ $find\ y'$**

Example 10.25 Find $f'(x)$ if $f(x) = \cos^{-1}(4x^3-3x)$

Example 10.26 Find $\frac{dy}{dx}$ if $x = at^2$; $y = 2at, t \neq 0$.

Example 10.27 Find $\frac{dy}{dx}$ if $x = a(t - \sin t), y = a(1 - \cos t)$.

Example 10.28 Find the derivative of x^x with respect to $x\log x$

Example 10.29 Find the derivative of $\tan^{-1}(1+x^2)$ *with respect to* $x^2 + x + 1$

Example 10.30 Differentiate $\sin(ax^2 + bx + c)$ with respect to $\cos(lx^2 + mx + n)$

Example 10.31 Find y', y'', y''' if $y = x^3 - 6x^2 - 5x + 3$

Example 10.32 Find y''' *if* $y = \frac{1}{x}$ **Example 10.33** Find f'' if $f(x) = x\cos x$.

Example 10.34 Find y'' if $x^4 + y^4 = 16$

Example 10.35 Find the second order derivative if x and y are given by $x = a\cos t$ $y = a\sin t$.

Example 10.36 Find $\frac{d^2x}{dx^2}$ *if* $x^2 + y^2 = 4$

EXERCISE 10.4

Find the derivatives of the following (1 - 18) :

(1) $y = x\cos x$ *(2)* $y = x^{\log x} + (\log x)^x$ *3)* $xy = e^{(x-y)}$ *(4)* $x^y = y^x$

(5) $(\cos x)^{\log x}$ *(6)* $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ *7)* $\sqrt{x^2 + y^2} = \tan^{-1}\left(\frac{y}{x}\right)$

(8) tan (x + y) + tan (x - y) = x *(9) If cos (xy) = x, show that* $\frac{dy}{dx} = -\frac{1+y\sin(xy)}{x\sin(xy)}$

(10) $\tan^{-1}\sqrt{\frac{1-\cos x}{1+\cos x}}$ *(11)* $\tan^{-1}\left(\frac{6x}{1-9x^2}\right)$ *(12)* $\cos\left(2\tan^{-1}\sqrt{\frac{1-x}{1+x}}\right)$

(13) $x = a\cos^3 t$; $y = a\sin^3 t$ *(14)* $x = a(\cos t + t\sin t)$; $y = a(\sin t - t\cos t)$

(15) $x = \frac{1-t^2}{1+t^2}$; $y = \frac{2t}{1+t^2}$ *(16)* $\cos^{-1}\left(\frac{1-x^2}{1+x^2}\right)$ *(17)* $\tan^{-1}\left(\frac{\cos x+\sin x}{\cos x-\sin x}\right)$

(19) Find the derivative of $\sin x^2$ *with respect to* x^2.

(20) Find the derivative of $\sin^{-1}\left(\frac{2x}{1+x^2}\right)$ *with respect to* $\tan^{-1}x$

(21) If $u = \tan^{-1}\frac{\sqrt{1+x^2}-1}{x}$ *and* $v = \tan^{-1}x$ *find* $\frac{du}{dv}$

(22) Find the derivative with $\tan^{-1}\left(\frac{\sin x}{1+\cos x}\right)$ *with respect to* $\tan^{-1}\left(\frac{\cos x}{1+\sin x}\right)$.

(23) If $y = \sin^{-1}x$ *then find* y''.

(24) If $y = e^{\tan^{-1}x}$ *show that* $(1+x^2)y'' + (2x-1)y' = 0$.

(25) If $y = \frac{\sin^{-1}x}{\sqrt{1-x^2}}$, *show that* $(1-x^2)y_2 - 3xy_1 - y = 0$.

(26) If $x = a(\theta + \sin\theta)$, $y = a(1 - \cos\theta)$ *then prove that at* $\theta = \frac{\pi}{2}, y'' = \frac{1}{a}$ *1*

(27) If $\sin y = x\sin(a+y)$, *then prove that* $\frac{dy}{dx} = \frac{\sin^2(a+y)}{\sin a}$

(28) If $y = (\cos^{-1}x)^2$, *prove that* $(1-x^2)\frac{d^2y}{dx^2} - x\frac{dy}{dx} - 2 = 0$

INTEGRAL CALCULUS

EXERCISE 11.12

Integrate the following functions with respect to x:

(1) (i) $\sqrt{x^2+2x+10}$ (ii) $\sqrt{x^2-2x-3}$ (iii) $\sqrt{(6-x)(x-4)}$

(2) (i) $\sqrt{9-(2x+5)^2}$ (ii) $\sqrt{81+(2x+1)^2}$ (iii) $\sqrt{(x+1)^2-4}$

Examples 11.41 Evaluate the following :

(i) $\int\sqrt{4-x^2}\,dx$ (ii) $\int\sqrt{25x^2-9}\,dx$ (iii) $\int\sqrt{x^2+x+1}\,dx$ (iv) $\int\sqrt{(x-3)(5-x)}\,dx$

EXERCISE 11.11

Integrate the following with respect to x:

(1) (i) $\frac{2x-3}{x^2+4x-12}$ (ii) $\frac{5x-2}{2+2x+x^2}$ (iii) $\frac{3x+1}{2x^2-2x+3}$ (2) (i) $\frac{2x+1}{\sqrt{9+4x-x^2}}$ (ii) $\frac{x+2}{\sqrt{x^2-1}}$ (iii) $\frac{2x+3}{\sqrt{x^2+4x+1}}$

Examples 11.40 Evaluate the following integrals

(i) $\int\frac{3x+5}{x^2+4x+7}dx$ (ii) $\int\frac{x+1}{x^2-3x+1}dx$ (iii) $\int\frac{2x+3}{\sqrt{x^2+x+1}}dx$ (iv) $\int\frac{5x-7}{\sqrt{3x-x^2-2}}dx$

EXERCISE 11.10 Find the integrals of the following :

(1) (i) $\frac{1}{4-x^2}$ (ii) $\frac{1}{25-4x^2}$ (iii) $\frac{1}{9x^2-4}$ (2) (i) $\frac{1}{6x-7-x^2}$ (ii) $\frac{1}{(x-1)^2-25}$ (iii) $\frac{1}{\sqrt{x^2+4x+2}}$

(3) (i) $\frac{1}{\sqrt{(2+x)^2-1}}$ (ii) $\frac{1}{\sqrt{x^2-4x+5}}$ (iii) $\frac{1}{\sqrt{9-8x-x^2}}$

Examples 11.39 Evaluate the integrals (I) $\int\frac{1}{x^2-2x+5}dx$ (ii) $\int\frac{1}{\sqrt{x^2+12x+11}}dx$ (iii) $\int\frac{1}{\sqrt{12+4x-x^2}}dx$

Examples 11.38 Evaluate the following integrals

(i) $\int\frac{1}{(x-2)^2+1}dx$ (ii) $\int\frac{x^2}{1+x^2}dx$ (iii) $\int\frac{1}{\sqrt{1+4x^2}}dx$ (iv) $\int\frac{1}{\sqrt{4x^2-25}}dx$

EXERCISE 11.9 Integrate the following with respect to x:

(1) $e^x(\tan x+\log\sec x)$ (2) $e^x\left(\frac{x-1}{2x^2}\right)$ (3) $e^x\sec x(1+\tan x)$ (4) $e^x\left(\frac{2+\sin 2x}{1+\cos 2x}\right)$

(5) $e^{\tan^{-1}x}\left(\frac{1+x+x^2}{1+x^2}\right)$ (6) $\frac{\log x}{(1+\log x)^2}$

Examples 11.37 Evaluate the following integrals

(i) $\int e^x\left(\frac{1}{x}-\frac{1}{x^2}\right)dx$ (ii) $\int e^x(\sin x+\cos x)dx$ (iii) $\int e^x\left(\frac{1-x}{1+x^2}\right)^2dx$

EXERCISE 11.8 Integrate the following with respect to x

(1) (i) $e^{ax}\cos bx$ -(ii) $e^{2x}\sin x$ (iii) $e^{-x}\cos 2x$ (2) (i) $e^{-3x}\sin 2x$ (ii) $e^{-4x}\sin 2x$ (iii) $e^{-3x}\cos x$

Examples 11.36 Evaluate the following integrals

(i) $\int e^{3x}\cos 2x\,dx$ (ii) $\int e^{-5x}\sin\ 3x\,dx$ (iii) $\int e^{ax}\sin bx\ dx$

EXERCISE 11.7 Integrate the following with respect to x:

(1) (i) $9xe^{3x}$ (ii) $x\sin 3x$ (iii) $25xe^{-5x}$ (iv) $x\sec x\tan x$

(2) (i) $x\log x$ (ii) $27x^2e^{3x}$ (iii) $x^2\cos x$ (iv) $x^3\sin x$

(3) (i) $\frac{x\sin^{-1}x}{\sqrt{1-x^2}}$ (ii) $x^5e^{x^2}$ (iii) $\tan^{-1}\left(\frac{8x}{1-16x^2}\right)$ (iv) $\sin^{-1}\left(\frac{2x}{1+x^2}\right)$

Example 11.35 Integrate the following with respect to x. (i) x^2e^{5x} (ii) $x^3\cos x$ (iii) x^3e^{-x}

Example 11.34 Evaluate : $\int \tan^{-1}\left(\frac{2x}{1-x^2}\right)\,dx$

Example 11.33 Evaluate the following integrals

(i) $\int xe^x dx$ (ii) $\int x\cos x\,dx$ (iii) $\int \log x\,dx$ (iv) $\int \sin^{-1}x\,dx$

EXERCISE 11.6 Integrate the following with respect to x

(1) $\frac{x}{\sqrt{1+x^2}}$ (2) $\frac{x^2}{1+x^6}$ (3) $\frac{e^x-e^{-x}}{e^x+e^{-x}}$ (4) $\frac{10x^9+10^x\log_e 10}{10^x+x^{10}}$ (5) $\frac{\sin\sqrt{x}}{\sqrt{x}}$ (6) $\frac{\cot x}{\log(\sin x)}$

(7) $\frac{\operatorname{cosec} x}{\log(\tan\frac{x}{2})}$ (8) $\frac{\sin 2x}{a^2+b^2\sin^2 x}$ (9) $\frac{\sin^{-1}x}{\sqrt{1-x^2}}$ (10) $\frac{\sqrt{x}}{1+\sqrt{x}}$ (11) $\frac{1}{x\log x\log(\log x)}$

(12) $\alpha\beta x^{\alpha-1}e^{-\beta x^{\alpha}}$ (13) $\tan x\sqrt{\sec x}$ (14) $x(1-x)^{17}$ (15) $\sin^5 x\cos^3 x$ (16) $\frac{\cos x}{\cos(x-a)}$

Example 11.32 Integrate the following with respect to x.

(i) $\int \frac{2x+4}{x^2+4x+6}dx$ (ii) $\int \frac{e^x}{1+e^x}dx$ (iii) $\int \frac{1}{x\log x}dx$ (iv) $\int \frac{\sin x-\cos x}{\sin x+\cos x}dx$ (v) $\int \frac{\cos 2x}{(\sin x+\cos x)^2}dx$

Example 11.31 Integrate the following with respect to x.

(i) $\int \tan x\,dx$ (ii) $\int \cot x\,dx$ (iii) $\int \operatorname{cosec} x\,dx$ (iv) $\int \sec x\,dx$

Example 11.30 Evaluate the following integrals :

(i) $\int 2x\sqrt{1+x^2}\,dx$ (ii) $\int e^{-x^2}x\,dx$ (iii) $\int \frac{\sin x}{1+\cos x}dx$ (iv) $\int \frac{1}{1+x^2}dx$ (v) $\int x(1-x)^8\,dx$

Example 11.15 Integrate the following with respect to x. (i) $(1-x^3)^2$ (ii) $\frac{x^2-x+1}{x^3}$

Example 11.16 Integrate the following with respect to x. (i) $\cos 5x\sin 3x$ (ii) $\cos^3 x$

Example 11.17 Integrate the following with respect to x. (i) $\frac{e^{2x}-1}{e^x}$ (ii) $e^{3x}(e^{2x}-1)$

Example 11.18 Evaluate : $\int \frac{1}{\sin^2 x\cos^2 x}dx$ **Example 11.19** Evaluate : $\int \frac{\sin x}{1+\sin x}dx$

Example 11.20 Evaluate : $\int \sqrt{1+\cos 2x}\,dx$ **Example 11.21** Evaluate : $\int \frac{(x-1)^3}{x^3+x}dx$

Example 11.22 Evaluate : $\int (\tan x+\cot x)^2\,dx$ **Example 11.23** Evaluate : $\int \frac{1-\cos x}{1+\cos x}dx$

Example 11.24 Evaluate : $\int \sqrt{1+\sin 2x}\,dx$ **Example 11.25** Evaluate : $\int \frac{x^2+2}{x-1}dx$

Example 11.26 Evaluate : (i) $\int a^xe^x\,dx$ (ii) $\int e^{x\log 2}e^x\,dx$

Example 11.27 Evaluate : $\int (x-3)\sqrt{x+2}\,dx$ **Example 11.28** Evaluate : $\int \frac{1}{\sqrt{x+1}+\sqrt{x}}dx$

Example 11.29 $\int \frac{3x+7}{x^2-3x+2}dx$ Evaluate : (i) (ii) $\int \frac{x+3}{(x+2)^2(x+1)}dx$

EXERCISE 11.5

(1) $\frac{x^3+4x^2-3x+2}{x^3}$ (2) $\left(\sqrt{x}+\frac{1}{\sqrt{x}}\right)^2$ (3) $(2x-5)(36+4x)$ (4) $\cot^2 x+\tan^2 x$

(5) $\frac{\sin 2x-\cos 2\alpha}{\cos x-\cos\alpha}$ (6) $\frac{\cos 2x}{\sin^2 x+\cos^2 x}$ (7) $\frac{3+\cos x}{\sin^2 x}$ (8) $\frac{\sin^2 x}{1+\cos x}$ (9) $\frac{\sin 4x}{\sin x}$ (10) $\cos 3x\cos 2x$

(11) $\sin^2 5x$

(12) $\frac{1-\cos 4x}{\cot x-\tan x}$ (13) $e^{x\log a}e^{x}$ (14) $(3x+)\sqrt{3x+7}$ (15) $\frac{8^{1+x}+4^{1-x}}{2^x}$ (16) $\frac{1}{\sqrt{x+3}-\sqrt{x-4}}$ (17) $\frac{x+1}{(x+2)(x+3)}$

(18) $\frac{1}{(x-1)(x+2)^2}$ (19) $\frac{3x-9}{(x-1)(x+2)(x^2+1)}$

EXERCISE 11.4

(1) If $f'(x)=4x-5$ *and* $f(2)=1$ find $f(x)$

(2) If$f'9x)=9x^2-6x$ *and* $f(0)-3$ find f(x)

(3) If$f''9x)=12-6$ *and* $f(1)=30$, $f'(1)=5$ find f(x)

(4) A ball is thrown vertically upward from the ground with an initial velocity of 39.2 m/sec. If the only force considered is that attributed to the acceleration due to gravity, find

(i) how long will it take for the ball to strike the ground?

(ii) the speed with which will it strike the ground? and

(iii) how high the ball will rise?

(5) A wound is healing in such a way that t days since Sunday the area of the wound has been decreasing at a rate of $-\frac{3}{(t+2)^2}$ cm^2per day. If on Monday the area of the wound was

(i) What was the area of the wound on Sunday?

(ii) What is the anticipated area of the wound on Thursday if it continues to heal at the same rate?

Example 11.14 At a particular moment, a student needs to stop his speedy bike to avoid a collision with the barrier ahead at a distance 40 metres away from him. Immediately he slows (retardation) the bike under braking at a rate of If the bike is moving at a speed of 24m/s, when the brakes are applied, would it stop before collision?

Example 11.13 A tree is growing so that, after t - years its height is increasing at a rate of $\frac{18}{\sqrt{t}}$ cm per year. Assume that when $t=0$, the height is 5 cm.

(i) Find the height of the tree after 4 years. (ii) After how many years will the height be 149 cm? **Example 11.12** The rate of change of weight of person w in kg with respect to their height h in centimetres is given approximately by $\frac{dw}{dh}=4.364\times10^{-5}h^2$. Find weight as a function of height. Also find the weight of a person whose height is 150 cm.

Example 11.11 A train started from Madurai Junction towards Coimbatore at 3pm (time $t = 0$) with velocity $v(t) = 20t + 50$ kilometre per hour, where t is measured in hours. Find the distance covered by the train at 5pm.

Example 11.10 If $f'(x) = 3x^2 - 4x + 5$ and $f(1) = 3$ then find $f(x)$.

EXERCISE 11.2

(1)$(x+4)^5 + \frac{5}{(2-5x)^4} - cosec^2(3x-1)$ (2)$4\cos(5-2x) + 9\,e^{3x-6} + \frac{24}{6-4x}$

(3)$\sec^2\frac{x}{5} + 18\cos 2x + 10\sec(5x+3)\tan(5x+3)$ (4)$\frac{8}{\sqrt{1-(4x)^2}} + \frac{27}{\sqrt{1-9x^2}} - \frac{15}{1+25x^2}$

(5) $\frac{6}{1+(3x+2)^2} - \frac{12}{\sqrt{1-(3-4x)^2}}$ (6)$\frac{1}{3}\cos\left(\frac{x}{3}-4\right) + \frac{7}{7x+9} + e^{\frac{x}{5}+3}$

Example 11.9 Evaluate the following integrals:

(i) $\frac{12}{(4x-5)^3} + \frac{6}{3x+2} + 16e^{4x+3}$ (ii)$\frac{15}{\sqrt{5x-4}} - 8\cot 4x + 2\,\mathrm{cosec}\,4x + 2$

Example 11.8 Integrate the following with respect to x:

(i)$5x^4$ (ii)$5x^2 - 4 + \frac{7}{x} + \frac{2}{\sqrt{x}}$ (iii)$2\cos x - 4\sin x + 5sec^2x + cosec^2x$

EXERCISE 11.2 Integrate the following functions with respect to x:

(1)	(i)$(x+5)^6$	(ii)$\frac{1}{(2-3x)^4}$	(iii)$\sqrt{3x+2}$
(2)	(i) sin3x	(ii)cos(5-11x)	(iii) $cosec^2(5x-7)$
(3)	(i)e^{3x-6}	(ii)e^{8-7x}	(iii)$\frac{1}{6-4x}$
(4)	(i)$\sec^2\frac{x}{5}$	(ii) cosec(5x+3)cot(5x+3)	(iii) 30 sec(2-15x)tan(2-15x)
(5)	(i)$\frac{1}{\sqrt{1-(4x)^2}}$	(ii)$\frac{1}{\sqrt{1-81x^2}}$	(iii)$\frac{1}{1+36x^2}$

Example 11.7 Integrate the following with respect to x: (i) $\frac{1}{1+(2x)^2}$ (ii)$\frac{1}{\sqrt{1-(9x)^2}}$ (iii)$\frac{1}{\sqrt{1-25x^2}}$

Example 11.6 Integrate the following with respect to x: (i)e^{3x} (ii)e^{4-3x} (iii) $\frac{1}{(3x-2)}$(iv)$\frac{1}{(5-4x)}$

Example 11.5 Integrate the following with respect to x:

(i)sin(2x+4) (ii) $\sec^2(3x+4x)$ (iii)cosec(ax+b)cot(ax+b)

Example 11.4 Evaluate (i)$\int(4x+5)^6\,dx$ (ii) $\int\sqrt{15-2x}\,dx$ (iii) $\int\frac{1}{(3x+7)^4}\,dx$

EXERCISE 11.1

Integrate the following with respect to x:

(1) (i) x^{11} (ii)$\frac{1}{x^7}$ (iii)$\sqrt[3]{x^4}$ (iv)$(x^5)^{\frac{1}{8}}$ (2) (i) $\frac{1}{sin^2x}$(ii)$\frac{tanx}{cosx}$ (iii)$\frac{cosx}{sin^2x}$ (iv)$\frac{1}{cos^2x}$

(3) (i)12^3 (ii)$\frac{x^{24}}{x^{25}}$ (iii)e^x (4) (i)$(1+x^2)^{-1}$ (ii)$(1-x^2)^{-\frac{1}{2}}$

Example 11.3 Integrate the following with respect to x: (i)$\frac{1}{e^{-x}}$ (ii)$\frac{x^2}{x^3}$ (iii) $\frac{1}{x^3}$ (iv)$\frac{1}{1+x^2}$

Example 11.2 Integrate the following with respect to x. (i)$\frac{1}{cos^2x}$ (ii)$\frac{cotx}{sinx}$ (iii)$\frac{sinx}{cos^2x}$ (iv)$\frac{1}{\sqrt{1-x^2}}$

Example 11.1 Integrate the following with respect to x. (i)x^{10} (ii) $\frac{1}{x^{10}}$(iii)$\sqrt{x}$ (iv)$\frac{1}{\sqrt{x}}$

PROBABILITY

EXERCISE 12.4

1) A factory has two Machines-I and II. Machine-I produces 60% of items and Machine-II produces 40% of theitems of the total output. Further 2% of the items produced by Machine-I are defective whereas 4% produced by Machine-II are defective. If an item is drawn at random what is the probability that it is defective?

(2)There are two identical urns containing respectively 6 black and 4 red balls, 2 black and 2 redballs. An urn is chosen at random and a ball is drawn from it.

(i) find the probability that the ball is black

(ii) if the ball is black, what is the probability that it is from the first urn?

(3) A firm manufactures PVC pipes in three plants viz, *X, Y* and *Z.* The daily production volumes from the three firms *X, Y* and *Z* are respectively 2000 units,3000 units and 5000 units. It is known from the past experience that 3% of the output from plant *X*, 4% from plant *Y* and 2% from plant *Z* are defective. A pipe is selected at random from a day's total production,

(i)find the probability that the selected pipe is a defective one.

(ii) if the selected pipe is a defective, then what is the probability that it was produced by plant *Y*?

(4) The chances of *A, B* and *C* becoming manager of a certain company are 5 : 3 : 2. The probabilities that the office canteen will be improved if *A, B*, and *C* become managers are 0.4, 0.5 and 0.3 respectively. If the office canteen has been improved, what is the probability that *B* was appointed as the manager?

(5) An advertising executive is studying television viewing habits of married men and women during prime time hours. Based on the past viewing records he has determined that during prime time wives are watching television 60% of the time. It has also been determined that when the wife is watching television, 40% of the time the husband is also watching. When the wife is not watching the television, 30% of the time the husband is watching the television. Find the probability that (i) the husband is watching the television during the prime time of television (ii) if the husband is watching the television, the wife is also watching the television.

Example 12.29 A consulting firm rents car from three agencies such that 50% from agency *L*, 30% from agency *M* and 20% from agency *N*. If 90% of the cars from *L*, 70% of cars from *M* and 60% of the cars from *N* are in good conditions

(i) what is the probability that the firm will get a car in good condition?

(ii) if a car is in good condition, what is probability that it has come from agency *N*?

Example 12.28 The chances of *X, Y* and *Z* becoming managers of a certain company are 4 : 2 : 3. The probabilities that bonus scheme will be introduced if *X, Y* and *Z* become managers are 0.3, 0.5 and 0.4 respectively. If the bonus scheme has been introduced, what is the probability that *Z* was appointed as the manager?

Example 12.27 A construction company employs 2 executive engineers. Engineer-1 does the work for 60% of jobs of the company. Engineer-2 does the work for 40% of jobs of the company. It is known from the past experience that the probability of an error when engineer-1 does the work is 0.03, whereas the probability of an error in the work of engineer-2 is 0.04. Suppose a serious error occurs in the work, which engineer would you guess did the work?

Example 12.26 A factory has two machines I and II. Machine I produces 40% of items of the output and Machine II produces 60% of the items. Further 4% of items produced by Machine I are defective and 5% produced by Machine II are defective. An item is drawn at random. If the drawn item is defective, find the probability that it was produced by Machine II. (See the previous example, compare the questions).

Example 12.25 A factory has two machines I and II. Machine-I produces 40% of items of the output and Machine-II produces 60% of the items. Further 4% of items produced by Machine-I are defective and 5% produced by Machine-II are defective. If an item is drawn at random, find the probability that it is a defective item.

Example 12.24 Urn-I contains 8 red and 4 blue balls and urn-II contains 5 red and 10 blue balls. One urn is chosen at random and two balls are drawn from it. Find the probability that both balls are red.

EXERCISE 12.3

(1)Can two events be mutually exclusive and independent simultaneously?

(2) If and are two events such that $P(A \cup B) = 0.7$, $P(A \cap B) = 0.2$ and $P(B) = 0.5$, then show that A and B are independent.

(3) If A and B are two independent events such that $P(A \cup B) = 0.6$, $P(A) = 0.2$, find $P(B)$.

(4) If $P(A) = 0.5$, $P(B) = 0.8$ and $P(B/A) = 0.8$, find $P(A/B)$ and $P(A \cup B)$.

(5) If for two events (sample space),$P(A)=\frac{3}{4}$,$P(B)=\frac{2}{5}$ find $A \cup B = S$ the conditional probability $P(A/B)$.

(6)A problem in Mathematics is given to three students whose chances of solving it are $\frac{1}{3}$, $\frac{1}{4}$, $\frac{1}{5}$

(i)What is the probability that the problem is solved?

(ii) What is the probability that exactly one of them will solve it?

(7) The probability that a car being filled with petrol will also need an oil change is 0.30; the probability that it needs a new oil filter is 0.40; and the probability that both the oil and filter need changing is 0.15.

(i) If the oil had to be changed, what is the probability that a new oil filter is needed?

(ii) If a new oil filter is needed, what is the probability that the oil has to be changed?

(8) One bag contains 5 white and 3 black balls. Another bag contains 4 white and 6 black balls. If one ball is drawn from each bag, find the probability that (i)both are white (ii) both are black (iii) one white and one black

(9) Two thirds of students in a class are boys and rest girls. It is known that the probability of a girl getting a first grade is 0.85 and that of boys is 0.70. Find the probability that a student chosen at random will get first grade marks.

(10) Given $P(A) = 0.4$ and = 0.7. Find $P(B)$ if (i) A and B are mutually exclusive (ii) A and B are independent events (iii) $P(A / B) = 0.4$ (iv) $P(B / A) = 0.5$

(11) A year is selected at random. What is the probability that

(i) it contains 53 Sundays (ii) it is a leap year which contains 53 Sundays

(12) Suppose the chances of hitting a target by a person X is 3 times in 4 shots, by Y is 4 times in 5 shots, and by Z is 2 times in 3 shots. They fire simultaneously exactly one time. What is the probability that the target is damaged by exactly 2 hits?

Example 12.23 A main road in a City has 4 crossroads with traffic lights. Each traffic light opens or closes the traffic with the probability of 0.4 and 0.6 respectively. Determine the probability of

(i) a car crossing the first crossroad without stopping (ii) a car crossing first two crossroads without stopping (iii) a car crossing all the crossroads, stopping at third cross.

(iv) a car crossing all the crossroads, stopping at exactly one cross.

Example 12.22 X speaks truth in 70 percent of cases, and Y in 90 percent of cases. What is the probability that they likely to contradict each other in stating the same fact?

Example 12.21 An anti-aircraft gun can take a maximum of four shots at an enemy plane moving away from it. The probability of hitting the plane in the first, second, third, and fourth shot are respectively 0.2, 0.4, 0.2 and 0.1. Find the probability that the gun hits the plane.

Example 12.20 If A & B are two independent events such that $P(A) = 0.4$ & $P(A \cup B) = 0.9$. Find $P(B)$.

Example 12.19 A coin is tossed twice. Events E and F are defined as follows

E= Head on first toss, F= Head on second toss.

Find (i) $P(E \cup F)$ (ii) $P(E/F)$ (iii) $P(\bar{E}/F)$ (iv) Are the events E and F independent?

Example 12.18 Two cards are drawn from a pack of 52 cards in succession. Find the probability that both are Jack when the first drawn card is (i) replaced (ii) not replaced

Example 12.17 A die is rolled. If it shows an odd number, then find the probability of getting 5.

Example 12.16 If $P(A) = 0.6$ $P(B)=0.5$ and $P(A \cap B)=0.2$ find (i)$P(A/B)$ (ii)$P(\bar{A}/B)$ (iii)$P(A/\bar{B})$

EXERCISE 12.2

(1) If and are mutually exclusive events P(A) $= \frac{3}{8}$ and P(B) $= \frac{1}{8}$ then find

(i)$P(\bar{A})$ (ii) P(A∪B) (iii) $P(\bar{A} \cap B)$ (iv)$P(\bar{A} \cup \bar{B})$

(2) If and are two events associated with a random experiment for which *P(A)=* 0.35,

P(A or B)= 0.85, P (A and B)= 0.15 find (i)P(only B) (ii)$P(\bar{A})$ (iii) P(only B)

(3) A die is thrown twice. Let *A* be the event, 'First die shows 5' ,

B be the event, 'second die shows 5'. Find *P(A∪B)*

(4) The probability of an event *A* occurring is 0.5 and *B* occurring is 0.3. If *A* and *B* are mutually exclusive events, then find the probability of (i) *P(A∪B)* *(ii)* $P(A\cap\bar{B})$ *(iii)*$P(\bar{A}\cap B)$

(5) A town has 2 fire engines operating independently. The probability that a fire engine is available when needed is 0.96.

(i)What is the probability that a fire engine is available when needed?

(ii) What is the probability that neither is available when needed?

(6) The probability that a new railway bridge will get an award for its design is 0.48, the probability that it will get an award for the efficient use of materials is 0.36, and that it will get both awards is 0.2. What is the probability, that (i) it will get at least one of the two awards (ii) it will get only one of the awards.

Example 12.15 The probability that a girl, preparing for competitive examination will get a State Government service is 0.12, the probability that she will get a Central Government job is 0.25, and the probability that she will get both is 0.07. Find the probability that (*i*) she will get atleast one of the two jobs (*ii*) she will get only one of the two jobs.

Example 12.14 Given that $P(A) = 0.52$, $P(B) = 0.43$, and $P(A\cap B) = 0.24$,

find (i) $P(A\cap\bar{B})$ *(ii) P(A∪B)(iii)* $P(\bar{A} \cap \bar{B})$ (iv) $P(\bar{A} \cup \bar{B})$

Example 12.13 Nine coins are tossed once, find the probability to get at least two heads.

Example 12.12 Find the probability of getting the number 7, when a usual die is rolled.

EXERCISE 12.1

(1)An experiment has the four possible mutually exclusive and exhaustive outcomes A, B, C, and D. Check whether the following assignments of probability are permissible.

(i) $P(A) = 0.15$, $P(B) = 0.30$, $P(C) = 0.43$, $P(D) = 0.12$

(ii) $P(A) = 0.22$, $P(B) = 0.38$, $P(C) = 0.16$, $P(D) = 0.34$

(iii) $P(A) = \frac{2}{5}$, $P(B) = \frac{3}{5}$, $P(C) = -\frac{1}{5}$, $P(D) = \frac{1}{5}$

(2) If two coins are tossed simultaneously, then find the probability of getting

(i) one head and one tail (ii) at most two tails

(3) Five mangoes and 4 apples are in a box. If two fruits are chosen at random, find the probability that (i) one is a mango and the other is an apple (ii) both are of the same variety.

(4) What is the chance that (i) non-leap year (ii) leap year should have fifty three Sundays?

(5) Eight coins are tossed once, find the probability of getting

(i)exactly two tails (ii) at least two tails (iii) at most two tails

(6) An integer is chosen at random from the first 100 positive integers. What is the probability that the integer chosen is a prime or multiple of 8?

(7) A bag contains 7 red and 4 black balls, 3 balls are drawn at random. Find the probability that

(i) all are red (ii) one red and 2 black.

(8) A single card is drawn from a pack of 52 cards.What is the probability that

(i)the card is an ace or a king (ii) the card will be 6 or smaller (iii) the card is either a queen or 9?

(9) A cricket club has 16 members, of whom only 5 can bowl. What is the probability that in a team of 11 members at least 3 bowlers are selected?

(10) (i) The odds that the event A occurs is 5 to 7, find $P(A)$.

(ii) Suppose $P(B) = \frac{2}{5}$. Express the odds that the event B occurs.

Example 12.11 A man has 2 ten rupee notes, 4 hundred rupee notes and 6 five hundred rupee notes in his pocket. If 2 notes are taken at random, what are the odds in favour of both notes being of hundred rupee denomination and also its probability?

Example 12.10 For a sports meet, a winners' stand comprising of three wooden blocks is in the form as shown in figure. There are six different colours available to choose from and three of the wooden blocks is to be painted such that no two of them has the same colour. Find the probability that the smallest block is to be painted in red, where red is one of the six colours.

Example 12.8 Three letters are written to three different persons and addresses on three envelopes are also written. Without looking at the addresses, what is the probability that (i) exactly one letter goes to the right envelopes (ii) none of the letters go into the right envelopes?

Example 12.7 Three candidates *X, Y,* and *Z* are going to play in a chess competition to win FIDE (World Chess Federation) cup this year. *X* is thrice as likely to win as *Y* and *Y* is twice as likely as to win *Z*. Find the respective probability of *X, Y* and *Z* to win the cup.

Example 12.6 When a pair of fair dice is rolled, what are the probabilities of getting the sum (i)7 (ii) 7 or 9 (iii) 7 or 12?

Example 12.5 Suppose a fair die is rolled. Find the probability of getting (i) an even number (ii) multiple of three.

Example 12.4 Suppose ten coins are tossed. Find the probability to get
(i) exactly two heads (ii) at most two heads (iii) at least two heads

Example 12.3 Three coins are tossed simultaneously, what is the probability of getting
(i) exactly one head (ii) at least one head (iii) at most one head?

Example 12.2 An integer is chosen at random from the first ten positive integers. Find the probability that it is (i) an even number (ii) multiple of three.

Theorem 12.4 If $\bar{A}$ is the complementary event of *A*, then $P(\bar{A})=1-P(A)$

Theorem 12.3 The probability of the impossible event is zero.

Theorem 12.10 (Total Probability of an event)

Theorem 12.11 (Bayes' Theorem)

Theorem 12.6 (Addition theorem on probability)

Theorem 12.5 If A and B are any two events and $\bar{B}$ is the complementary events of B ,
Then $P(A\cap\bar{B}) = P(A) - P(A \cap B)$

Theorem 12.8

(i) $\bar{A}$ *and* $\bar{B}$ are independent. (ii) *A and* $\bar{B}$ are independent. (iii) $\bar{A}$ *and B* are independent.