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- Lesson Plan - Group
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7. Application of Differential Calculus

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- ① A point moves along a straight line in such a way that after t seconds its distance from the origin is $s = 2t^2 + 3t$ m

- (i) Find the average velocity of the points between $t=3$ and $t=6$ seconds.
(ii) Find the instantaneous velocities at $t=3$ and $t=6$ seconds.

$$s(t) = 2t^2 + 3t$$

i)

$$a=3 \quad b=6$$

$$s(3) = 2(3)^2 + 3(3)$$

$$= 18 + 9$$

$$s(3) = 27$$

$$s(6) = 2(6)^2 + 3(6)$$

$$= 72 + 18$$

$$s(6) = 90$$

$$\text{Average velocity} = \frac{s(b) - s(a)}{b - a}$$

$$= \frac{90 - 27}{6 - 3}$$

$$= \frac{63}{3}$$

$$= 21 \text{ m/s}$$

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ii)

$$s(t) = 2t^2 + 3t$$

$$s'(t) = 4t + 3$$

instantaneous rate of change at $t=3$
(velocity)

$$= s'(3)$$

$$= 4(3) + 3 = 12 + 3$$

$$= 15 \text{ m/s}$$

instantaneous (rate of change) velocity at $t=6$

$$= s'(6)$$

$$= 4(6) + 3 = 24 + 3$$

$$= 27 \text{ m/s}$$

- ② A camera is accidentally knocked off an edge of a cliff 400 ft high. The camera falls

a distance of $s = 16t^2$ in t seconds

- i) How long does the camera fall before it hits the ground
- ii) what is the average velocity with which the camera falls during the last 2 seconds?
- iii) what is the instantaneous velocity of the camera when it hits the ground.

$$s(t) = 16t^2$$

- i) when hits the ground $s = 400$

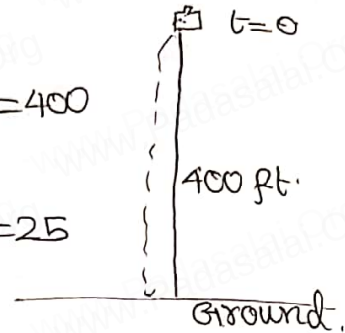
$$16t^2 = 400$$

$$t^2 = \frac{400}{16} = \frac{100}{4} = 25$$

$$t^2 = 25$$

$$t = \pm 5$$

$$t = 5 \text{ sec.}$$



- ii) average velocity in last two seconds
 $t = 3, t = 5$

$$\begin{aligned} \text{average velocity} &= \frac{s(5) - s(3)}{5 - 3} \\ &= \frac{16(5^2) - 16(3^2)}{2} = \frac{8(25 - 9)}{2} \\ &= 8 \times 16 \\ &= 128 \text{ ft/s} \end{aligned}$$

- iii) Instantaneous velocity $= s'(t)$
 $= 16(2t)$
 $= 32t$

when it hits the ground. $t = 5$

$$s'(5) = 32(5) = 160 \text{ ft/s}$$

- ③ A particle moves along a line according to the Law $s(t) = 2t^3 - 9t^2 + 12t - 4$, where $t \geq 0$

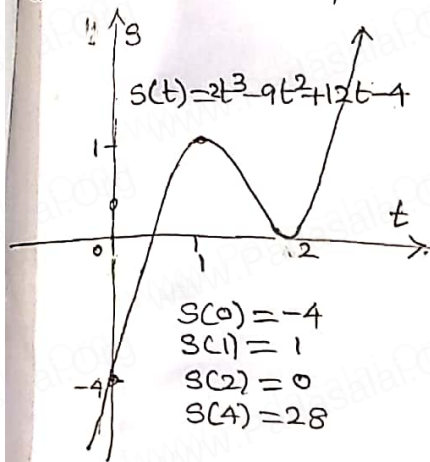
- i) At what time the particle changes direction?
- ii) Find the total distance travelled by the particle in the first 4 seconds.
- iii) Find the particles acceleration each time the velocity is zero.

④.

$$s(t) = 2t^3 - 9t^2 + 12t - 4$$

$$s'(t) = 6t^2 - 18t + 12$$

③ when changes of direction $s'(t)=0$



$$6t^2 - 18t + 12 = 0$$

$$6(t^2 - 3t + 2) = 0$$

$$(t-1)(t-2) = 0$$

$$t=1, t=2$$

$$t=1, 2 \text{ sec}$$

(ii) total distance travelled in first 4 seconds

$$= |s(1) - s(0)| + |s(2) - s(1)|$$

$$+ |s(4) - s(2)|$$

$$= |1 + 4| + |0 - 1| + |28 - 0|$$

$$= 5 + 1 + 28$$

$$= 34 \text{ m}$$

(iii)

$$s'(t) = 6t^2 - 18t + 12$$

$$s''(t) = 12t - 18$$

when velocity is zero $t=1, 2$

$$s''(1) = 12(1) - 18 = -6 \text{ m/s}^2$$

$$s''(2) = 12(2) - 18 = 6 \text{ m/s}^2$$

④ If the volume of a cube of side length x is $V = x^3$, find the rate of change of the volume with respect to x when $x = 5$ units.

$$V = x^3$$

diff. w. r. to x

$$\frac{dV}{dx} = 3x^2$$

$$\left[\frac{dV}{dx} \right]_{x=5} = 3(5)^2 = 3 \times 25 = 75 \text{ units.}$$

⑤ If the mass $m(x)$ (in kgs) of a thin rod of length x (in metres) is given by, $m(x) = \sqrt{3}x$ then what is the rate of change of mass w.r. to the length when it is $x=3$ and $x=27$ mts.

$$m(x) = \sqrt{3}x = \sqrt{3}\sqrt{x}$$

$$a=3 \quad b=27$$

$$\text{rate of change} = m'(x) = \sqrt{3} \cdot \frac{1}{2\sqrt{x}}$$

$$\text{① } x=3 \quad m'(3) = \sqrt{3} \cdot \frac{1}{2\sqrt{3}} = \frac{1}{2} \text{ kg/m}$$

$$\text{② } x=27 \quad m'(27) = \sqrt{3} \cdot \frac{1}{2\sqrt{27}} = \sqrt{3} \cdot \frac{1}{2(3\sqrt{3})} = \frac{1}{6} \text{ kg/m}$$

- 6) A stone is dropped into a pond causing ripples in the concentric circles. The radius r of the outer ripple is increasing at 2 cm/s . When the radius is 5 cm find the rate of changing of the total area of the disturbed water?

Let A be the area at time t .

$$\text{Area of the circle } A = \pi r^2$$

diff w.r to t

$$\frac{dA}{dt} = \pi 2r \frac{dr}{dt} \quad \text{--- ①}$$

When $r = 5$ $\frac{dr}{dt} = 2$

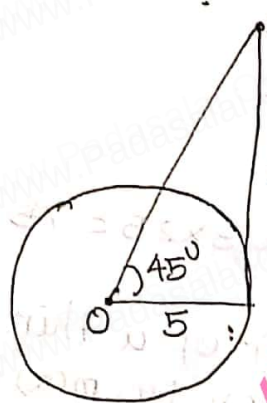
$$\text{①} \Rightarrow \frac{dA}{dt} = \pi 2(5)2$$

$$= 20\pi$$

Area changing rate $= 20\pi \text{ sq cm/s}$

- 7) A beacon makes one revolution every 10 sec. It is located on a ship which is anchored 5 km from a straight shore line. How fast is the beam moving along the shore line when it makes an angle of 45° with the shore.

Let the angular velocity $\frac{d\theta}{dt}$



sea shore

revolution in 10 sec $= 1 = 2\pi$

$\therefore 1 \text{ sec} = \frac{2\pi}{10}$

$$\frac{d\theta}{dt} = \frac{\pi}{5}$$

$$\tan \theta = \frac{\text{opp}}{\text{adj}}$$

$$\tan \theta = \frac{x}{5}$$

$$x = 5 \tan \theta$$

diff. w.r to t

$$\frac{dx}{dt} = 5 \sec^2 \theta \frac{d\theta}{dt}$$

$\theta = 45^\circ$ $x = 5$

$$\frac{dx}{dt} = 5 \sec^2 45^\circ \left(\frac{\pi}{5} \right)$$

$$= (\sqrt{2})^2 \pi$$

$$\frac{dx}{dt} = 2\pi \text{ km/sec}$$

beacon moving rate $2\pi \text{ km/s}$

- 8) A conical water tank with vertex down of 12 metres height has a radius of 5 metres at the top. If water flows into the tank at a rate 10 cubic m/min, how fast is the depth of the water increases when the water is 8 metres deep?

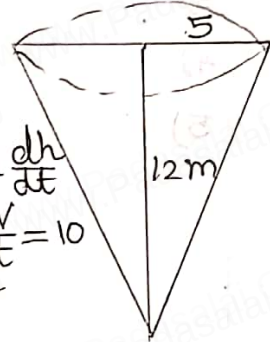
Let V be the volume of tank at any time t

$$r = 5 \quad h = 12$$

$$\frac{r}{h} = \frac{5}{12}$$

$$r = \frac{5h}{12}$$

to find $\frac{dh}{dt}$
when $\frac{dV}{dt} = 10$
 $h = 8$



volume of cone $V = \frac{1}{3} \pi r^2 h$

$$V = \frac{1}{3} \pi \frac{25h^2}{144} h$$

$$V = \frac{25\pi}{3 \times 144} h^3$$

diff. w- r to t

$$\frac{dV}{dt} = \frac{25\pi}{3 \times 144} 3h^2 \frac{dh}{dt}$$

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$$10 = \frac{25\pi}{144} \times 8^2 \times \frac{dh}{dt}$$

$$\frac{10 \times 9}{25\pi \times 4} = \frac{dh}{dt}$$

$$\frac{dh}{dt} = \frac{90}{100\pi}$$

$$\frac{dh}{dt} = \frac{9}{10\pi} \text{ m/min}$$

depth increase rate $\frac{9}{10\pi} \text{ m/min}$.

- 9) A ladder 17 metre long is leaning against the wall. The base of the ladder is pulled away from the wall at a rate of 5 m/s. when the base of the ladder is 8 m. from the wall.
- (i) How fast is the top of the ladder moving down the wall?
- (ii) At what rate, the area of the triangle formed by the ladder, wall, and the floor, is changing?

9)

AB is the ladder
AB = 17 m

Let x, y be the horizontal and vertical movements of ladder

$$\frac{dx}{dt} = 5 \text{ when } x = 8$$

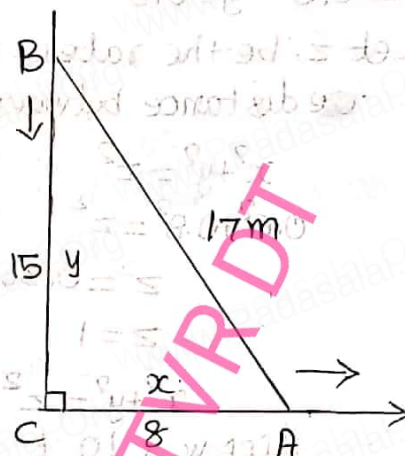
$$AB^2 = BC^2 + AC^2$$

$$17^2 = y^2 + 8^2$$

$$289 - 64 = y^2$$

$$225 = y^2$$

$$y = 15$$



i) $x^2 + y^2 = 17^2$

diff w.r to t

$$2x \frac{dx}{dt} + 2y \frac{dy}{dt} = 0$$

$$2 \div 8(5) + (15) \frac{dy}{dt} = 0$$

$$15 \frac{dy}{dt} = -40$$

$$\frac{dy}{dt} = -\frac{40}{15} = -\frac{8}{3}$$

$$\frac{dy}{dt} = -\frac{8}{3} \text{ m/s}$$

height decreasing rate

$$-\frac{8}{3} \text{ m/s}$$

ii)

area of $\triangle ABC = \Delta = \frac{1}{2} xy$

diff. w.r to t

$$\frac{d\Delta}{dt} = \frac{1}{2} \left(\frac{dx}{dt}(y) + x \frac{dy}{dt} \right)$$

$$= \frac{1}{2} \left(5(15) + 8\left(-\frac{8}{3}\right) \right)$$

$$= \frac{1}{2} \left(\frac{3 \times 75 - 64}{3} \right)$$

$$= \frac{225 - 64}{6} = \frac{161}{6}$$

Area of triangle

changing rate = 26.83 Sq.m/s.

$$\frac{d\Delta}{dt} = 26.83$$

10) A police jeep, approaching an orthogonal intersection from the northern direction is chasing a speeding car that has turned and moving straight east. When the jeep is 0.6 km north of the intersection and the car is 0.8 km to the east. The police determine with a radar that the distance between them and the car is increasing at 20 km/hr. If the jeep is moving at 60 km/hr at the instant of measurement, what is the speed of the car?

Let x, y be the distance moving by car and jeep

at any time t

$$\text{speed of the jeep } \frac{dy}{dt} = -60 \text{ km/hr}$$

(take - for approaching)

$$x=0.6 \quad y=0.8$$

Let z be the rate of change of distance between them.

$$x^2 + y^2 = z^2 \quad \frac{dz}{dt} = 20$$

$$0.6^2 + 0.8^2 = z^2$$

$$z^2 = 0.36 + 0.64 = 1$$

$$z = 1$$

$$x^2 + y^2 = z^2$$

diff w.r to t

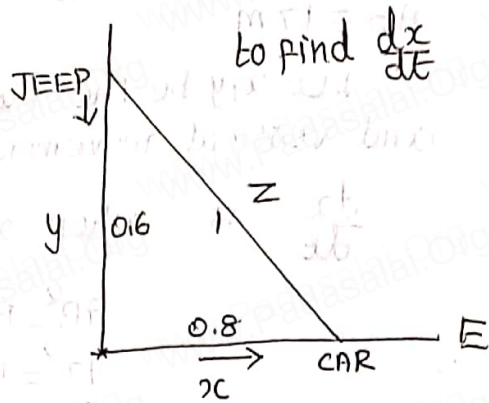
$$2x \frac{dx}{dt} + 2y \frac{dy}{dt} = 2z \frac{dz}{dt}$$

$$2 \div 0.8 \frac{dx}{dt} + 0.6(60) = 1(20)$$

$$0.8 \frac{dx}{dt} - 36 = 20$$

$$0.8 \frac{dx}{dt} = 20 + 36$$

$$\text{Speed of car. } \frac{dx}{dt} = 70 \text{ km/s}$$



$$0.8 \frac{dx}{dt} = 56$$

$$\frac{dx}{dt} = \frac{56 \times 10}{0.8}$$

$$= \frac{560}{8}$$

Equations of tangent and Normal

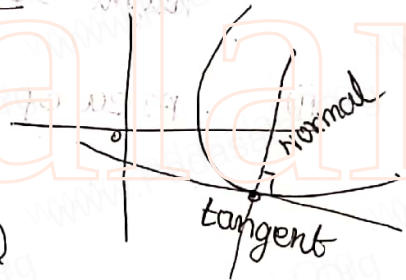
① Equation of tangent

$$y - y_1 = \left(\frac{dy}{dx} \right)_{(x_1, y_1)} (x - x_1)$$

② Equation of Normal

$$y - y_1 = -\frac{1}{\left(\frac{dy}{dx} \right)_{(x_1, y_1)}} (x - x_1)$$

$$\text{or } \left(\frac{dy}{dx} \right)_{(x_1, y_1)} (y - y_1) = -(x - x_1)$$



③ angle between the two curves

$$\tan \theta = \left| \frac{m_1 - m_2}{1 + m_1 m_2} \right|$$

④ two curves are orthogonal if $m_1 m_2 = -1$

Exercise 7.2

① Find the slope of the tangent to the curves at the respective given points,

i) $y = x^4 + 2x^2 - x$ at $x=1$

diff. w.r to x

$$\frac{dy}{dx} = 4x^3 + 4x - 1$$

$$\text{slope} = \left[\frac{dy}{dx} \right]_{x=1} = 4(1) + 4(1) - 1 = 8 - 1 = 7$$

② (11) $x = a \cos^3 t$, $y = b \sin^3 t$ at $t = \pi/2$

$$\frac{dx}{dt} = 3a \cos^2 t (-\sin t) \quad \frac{dy}{dt} = 3b \sin^2 t (\cos t)$$

$$\frac{dy}{dx} = \frac{dy/dt}{dx/dt} = \frac{3b \sin^2 t \cos t}{-3a \cos^2 t \sin t} = -\frac{b}{a} \frac{\sin t}{\cos t}$$

$$\frac{dy}{dx} = -\frac{b}{a} \tan t$$

$$\text{slope} = \left(\frac{dy}{dx}\right)_{t=\pi/2} = -\frac{b}{a} \tan \pi/2 = \infty$$

- ② Find the point on the curve $y = x^2 - 5x + 4$ at which the tangent is parallel to the line $3x + y = 7$

$$y = x^2 - 5x + 4 \quad \text{--- (1)}$$

$$3x + y = 7$$

$$\frac{dy}{dx} = 2x - 5$$

$$\text{slope} = -3$$

Let the tangential point be (x_1, y_1)

$$\text{slope} = \left(\frac{dy}{dx}\right)_{(x_1, y_1)} = 2x_1 - 5$$

$$2x_1 - 5 = -3$$

$$2x_1 = 5 - 3$$

$$2x_1 = 2$$

$$x_1 = 1$$

at (x_1, y_1)

$\Rightarrow$

$$y_1 = x_1^2 - 5x_1 + 4$$

$$y_1 = 1 - 5 + 4$$

$$y_1 = 0$$

$\therefore$ Required point is $(1, 0)$

- 3) Find the point on the curve $y = x^3 - 6x^2 + x + 3$ where the normal is parallel to the line $x + y = 1729$.

$$\text{slope of normal} = -\frac{1}{m}$$

$$\text{slope of tangent} = m = 1$$

Let the tangential point be (x_1, y_1)

$$y = x^3 - 6x^2 + x + 3$$

$$y_1 = x_1^3 - 6x_1^2 + x_1 + 3 \quad \text{--- (1)}$$

$$\frac{dy}{dx} = 3x^2 - 12x + 1$$

$$\text{slope} = \left(\frac{dy}{dx}\right)_{x_1, y_1} = 3x_1^2 - 12x_1 + 1 = 1$$

$$3x_1^2 - 12x_1 = 1 - 1$$

$$3x_1(x_1 - 4) = 0$$

$$x_1 = 0 \quad x_1 = 4$$

$$x_1 = 0 \Rightarrow y_1 = 3 \quad (0, 3)$$

$$x_1 = 4 \Rightarrow y_1 = 4^3 - 6(4)^2 + 4 + 3 = 64 - 96 + 4 + 3 = -25$$

The points are $(0, 3), (4, -25)$

- ④ Find the point on the curve $y^2 - 4xy = x^2 + 5$ for which the tangent is horizontal.

$$y^2 - 4xy = x^2 + 5$$

Let (x_1, y_1) be the point

$$y_1^2 - 4x_1y_1 = x_1^2 + 5 \quad \text{--- (1)}$$

diff w.r to x

$$2y \frac{dy}{dx} - 4(x \frac{dy}{dx} + y(1)) = 2x + 0$$

$$2y \frac{dy}{dx} - 4x \frac{dy}{dx} - 4y = 2x$$

$$2 \frac{dy}{dx} (y - 2x) = 2x + 4y$$

$$\frac{dy}{dx} = \frac{x + 2y}{y - 2x}$$

$$\text{slope at } x_1, y_1 = \left(\frac{dy}{dx} \right)_{(x_1, y_1)} = \frac{x_1 + 2y_1}{y_1 - 2x_1}$$

given tangent is horizontal slope = 0

$$\frac{x_1 + 2y_1}{y_1 - 2x_1} = 0 \quad x_1 + 2y_1 = 0$$

$$x_1 = -2y_1 \quad \text{--- (2)}$$

$$\Rightarrow y_1^2 - 4(-2y_1)y_1 = 4y_1^2 + 5$$

$$y_1^2 + 8y_1^2 = 4y_1^2 + 5$$

$$9y_1^2 - 4y_1^2 = 5$$

$$5y_1^2 = 5$$

$$y_1^2 = 1$$

$$y_1 = \pm 1$$

$$y_1 = -1$$

$$x_1 = -2(-1) = 2$$

$$y_1 = 1$$

$$x_1 = -2(1) = -2$$

The required points are $(2, -1), (-2, 1)$

- ⑤ Find the tangent and normal to the following curves at the given points on the curve.

i) $y = x^2 - x^4$ at $(1, 0)$

$$\frac{dy}{dx} = 2x - 4x^3$$

$$m = \left(\frac{dy}{dx} \right)_{(1, 0)} = 2(1) - 4(1)$$

$$m = -2$$

Equation of tangent

$$y - y_1 = m(x - x_1)$$

$$y - 0 = -2(x - 1)$$

$$y = -2x + 2$$

$$2x + y - 2 = 0$$

Equation of Normal

$$x - 2y + k = 0 \quad \text{--- (1)}$$

This passing through $(1, 0)$

$$1 - 2(0) + k = 0$$

$$k = -1$$

Equation of normal

$$x - 2y - 1 = 0$$

ii) $y = x^4 + 2e^x$ at $(0, 2)$

$$\frac{dy}{dx} = 4x^3 + 2e^x$$

$$m = \left(\frac{dy}{dx} \right)_{(0, 2)} = 4(0) + 2e^0$$

$$= 0 + 2(1)$$

$$m = 2$$

Equation of tangent

$$y - y_1 = m(x - x_1)$$

$$y - 2 = 2(x - 0)$$

$$y - 2 = 2x$$

$$2x - y + 2 = 0$$

$$\text{Equation of normal } x + 2y + k = 0 \quad \text{--- (1)}$$

$$0+2(2)+k=0$$

$$k=-4$$

$$\Rightarrow \text{Equation of normal } x+2y-4=0$$

$$(iii) y=x \sin x \text{ at } (\pi/2, \pi/2)$$

$$\frac{dy}{dx} = x \cos x + \sin x$$

$$m = \left(\frac{dy}{dx} \right)_{(\pi/2, \pi/2)} = \pi/2 \cos \pi/2 + \sin \pi/2$$

$$= \pi/2 (0) + 1$$

$$m=1$$

$$\text{Equation of tangent}$$

$$y-y_1 = m(x-x_1)$$

$$y-\pi/2 = 1(x-\pi/2)$$

$$y-\pi/2 = x-\pi/2$$

$$x-y=0$$

$$\text{Equation of normal}$$

$$x+y+k=0 \quad \text{--- (1)}$$

$$\text{This normal passing through } (\pi/2, \pi/2)$$

$$\frac{\pi}{2} + \frac{\pi}{2} + k = 0$$

$$k = -\pi$$

$$\Rightarrow x+y-\pi=0$$

$$(iv) x=\cos t, y=2\sin^2 t \text{ at } t=\pi/3$$

$$\frac{dx}{dt} = -\sin t \quad \frac{dy}{dt} = 2 \cdot 2 \sin t \cos t$$

$$\frac{dy}{dx} = \frac{4 \sin t \cos t}{-\sin t} = -4 \cos t$$

$$\text{slope } m = \left(\frac{dy}{dx} \right)_{t=\pi/3} = -4 \cos \pi/3 = -4(1/2) = -2$$

$$m = -2$$

$$(x, y) = (\cos \pi/3, 2 \sin^2 \pi/3) = (1/2, 2(\sqrt{3}/2)^2) = (1/2, 2 \cdot \frac{3}{4}) = (1/2, 3/2)$$

$$\text{Equation of tangent}$$

$$y-y_1 = m(x-x_1)$$

$$y-3/2 = -2(x-1/2)$$

$$\frac{2y-3}{2} = -2x + \frac{2}{2}$$

$$2y-3 = -4x+2$$

$$4x+2y=5$$

$$\text{Equation of normal is}$$

$$2x-4y=k \quad \text{--- (1)}$$

$$\text{This passing through } (1/2, 3/2)$$

$$2(\frac{1}{2}) - 4(\frac{3}{2}) = k$$

$$1-6=k$$

$$k=-5$$

$$\Rightarrow 2x-4y=-5$$

6) Find the equations of the tangents to the curve $y=1+x^3$ for which the tangent is orthogonal with the line $x+2y=12$

$$y=1+x^3$$

Let the tangential point be (x_1, y_1)

$$y_1 = 1+x_1^3 \quad \text{--- (1)}$$

$$y=1+x^3$$

$$\frac{dy}{dx} = 0+3x^2$$

$$\text{tangent slope at } (x, y) = 3x_1^2$$

$$m_1 = 3x_1^2$$

$$x+2y=12$$

$$\text{slope} = -\frac{1}{2}$$

$$m_2 = -\frac{1}{2}$$

$$m_1 \times m_2 = -1$$

$$3x_1^2 \left(-\frac{1}{2} \right) = -1$$

$$x_1^2 = \frac{2}{3}$$

$$x_1 = \pm \sqrt{\frac{2}{3}}$$

i)

⑪

$$x_1 = +2 \quad \text{①} \Rightarrow y_1 = 1 + 2^3 = 1 + 8 = 9$$

$$\text{⑫} x_1 = -2 \quad \text{①} \Rightarrow y_1 = 1 - 8 = -7$$

point $(x_1, y_1) = (2, 9)$ slope $m_1 = 3(2)^2 = 12$

point $(x_1, y_1) = (-2, -7)$ slope $m_2 = 3(-2)^2 = 12$

Equation of tangent

$$y - y_1 = m(x - x_1)$$

$$y - 9 = 12(x - 2)$$

$$y - 9 = 12x - 24$$

$$12x - y - 24 + 9 = 0$$

$$12x - y - 15 = 0$$

Equation of tangent

$$y - y_1 = m(x - x_1)$$

$$y + 7 = 12(x + 2)$$

$$y + 7 = 12x + 24$$

$$12x - y + 17 = 0$$

⑦ Find the equations of the tangents to the curve $y = \frac{x+1}{x-1}$ which are parallel to the line $x+2y=6$

Let the tangential point be (x_1, y_1)

$$y_1 = \frac{x_1+1}{x_1-1} \quad \text{--- ①}$$

$$y = \frac{x+1}{x-1}$$

$$\frac{dy}{dx} = \frac{(x-1)(1) - (x+1)(1)}{(x-1)^2}$$

$$= \frac{x-1-x-1}{(x-1)^2} = \frac{-2}{(x-1)^2}$$

$$m_1 = \left(\frac{dy}{dx} \right)_{(x_1, y_1)} = \frac{-2}{(x_1-1)^2}$$

$$\text{slope } m_2 = -\frac{1}{2}$$

$$m_1 = m_2$$

$$\frac{-2}{(x_1-1)^2} = -\frac{1}{2}$$

$$(x_1-1)^2 = 4$$

$$x_1 - 1 = \pm 2$$

$$x_1 = 1 \pm 2$$

$$x_1 = 3, -1$$

$$x_1 = 3 \quad \text{①} \Rightarrow y_1 = \frac{3+1}{3-1} = \frac{4}{2} = 2 \quad (3, 2)$$

$$x_1 = -1 \quad \text{①} \Rightarrow y_1 = \frac{-1+1}{-1-1} = 0 \quad (-1, 0)$$

Equation of tangents $x+2y=k$ --- ①

at $(3, 2)$ ① $\Rightarrow$

$$3 + 2(2) = k_1$$

$$k_1 = 7$$

$$x + 2y = 7 //$$

at $(-1, 0)$ ① $\Rightarrow$

$$-1 + 0 = k_2$$

$$k_2 = -1$$

$$x + 2y = -1 //$$

⑧ Find the equation of tangent and normal to the curve given by $x = 7 \cos t$ and $y = 2 \sin t$, $t \in R$ at any point on the curve.

$$\frac{dx}{dt} = -7 \sin t \quad \frac{dy}{dt} = 2 \cos t$$

$$\frac{dy}{dx} = \frac{2\cos t}{-7\sin t}$$

Equation of tangent

$$y - y_1 = \frac{dy}{dx} (x - x_1)$$

$$y - 2\sin t = \frac{2\cos t}{-7\sin t} (x - 7\cos t)$$

$$-7y\sin t + 14\sin^2 t = 2x\cos t - 14\cos^2 t$$

$$14\sin^2 t + 14\cos^2 t = 2x\cos t + 7y\sin t$$

$$2x\cos t + 7y\sin t = 14(1)$$

$$2x\cos t + 7y\sin t = 14$$

Equation of normal

$$7\sin t x - 2y\cos t = k \quad \text{--- ①}$$

passing through $(7\cos t, 2\sin t)$ then

$$7\sin t \cdot 7\cos t - 2\cos t \cdot 2\sin t = k$$

$$49\sin t \cos t - 4\sin t \cos t = k$$

$$k = 45\sin t \cos t$$

$$\text{①} \Rightarrow 7\sin t x - 2y\cos t = 45\sin t \cos t$$

⑨ Find the angle between the rectangular hyperbola $xy=2$ and the parabola $x^2+4y=0$

$$xy=2 \quad \text{--- ①}$$

$$y = \frac{2}{x}$$

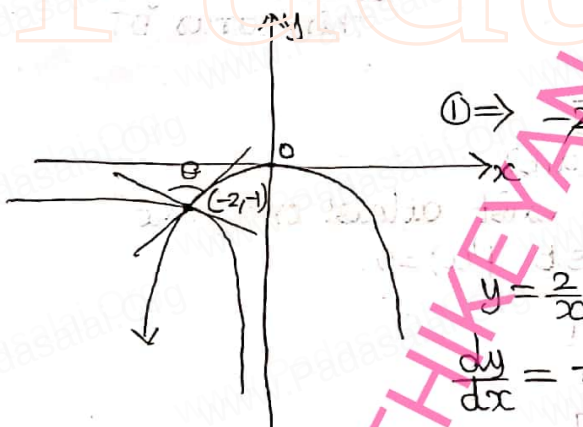
$$x^2 = -4y \quad \text{--- ②}$$

$$x^2 = -4\left(\frac{2}{x}\right)$$

$$x^3 = -8$$

$$x = -2$$

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① $\Rightarrow -2(y) = 2'$ point of intersection is $(-2, -1)$
 $y = -1$

$$y = \frac{2}{x}$$

$$\frac{dy}{dx} = -\frac{2}{x^2}$$

$$m_1 = \left[\frac{dy}{dx} \right]_{(-2,-1)} = \frac{-2}{(-2)^2} = \frac{-2}{4} = -\frac{1}{2}$$

$$m_1 = -\frac{1}{2} \quad m_2 = 1$$

$$4y = -x^2$$

$$4 \frac{dy}{dx} = -2x$$

$$\frac{dy}{dx} = \frac{-2x}{4 \cdot 2}$$

$$m_2 = \left[\frac{dy}{dx} \right]_{(-2,-1)} = \frac{+2}{2} = 1$$

$$\text{angle between the curves, } \tan \theta = \left| \frac{m_1 - m_2}{1 + m_1 m_2} \right|$$

$$= \left| \frac{-\frac{1}{2} - 1}{1 + (-\frac{1}{2})(1)} \right|$$

$$\tan \theta = \left| \frac{-\frac{3}{2}}{\frac{1}{2}} \right|$$

$$\tan \theta = 3$$

$$\theta = \tan^{-1}(3)$$

10) show that the two curves $x^2 - y^2 = r^2$ and $xy = c^2$ where c, r are constants, cut orthogonally,

$$x^2 - y^2 = r^2$$

$$xy = c^2$$

Let the point of intersection be (x_1, y_1)

$$x_1^2 - y_1^2 = r^2 \quad \text{--- (1)} \quad x_1 y_1 = c^2 \quad \text{--- (2)}$$

$$x^2 - y^2 = r^2$$

diff w.r to x

$$2x - 2y \frac{dy}{dx} = 0$$

$$1 - \frac{y}{x} \frac{dy}{dx} = 0$$

$$\frac{dy}{dx} = \frac{x}{y}$$

$$m_1 = \left(\frac{dy}{dx} \right)_{(x_1, y_1)} = \frac{x_1}{y_1}$$

$$y_1 = \frac{c^2}{x_1}$$

diff w.r to x

$$\frac{dy}{dx} = -\frac{c^2}{x^2}$$

$$m_2 = \left(\frac{dy}{dx} \right)_{(x_1, y_1)} = -\frac{c^2}{x_1^2}$$

$$m_1 m_2 = \frac{x_1}{y_1} \times \left(-\frac{c^2}{x_1^2} \right)$$

$$= -\frac{c^2}{x_1 y_1} = -\frac{c^2}{c^2} = -1$$

$$m_1 m_2 = -1$$

$\therefore$ The curves are cut orthogonally.

Mean value Theorem

① Rolle's Theorem

Let $f(x)$ be a function

(i) $f(x)$ is continuous on $[a, b]$

(ii) $f(x)$ is differentiable on (a, b)

(iii) $f(a) = f(b)$, then there exist at least one point $c \in (a, b)$ s.t. $f'(c) = 0$.

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② Lagrange's mean value Theorem.

Let $f(x)$ be a function

(i) $f(x)$ is continuous on $[a, b]$

(ii) $f(x)$ is differentiable in (a, b)

then exist at least one point $c \in (a, b)$ s.t.

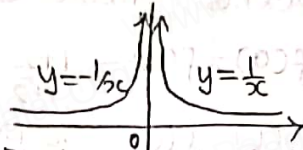
$$f'(c) = \frac{f(b) - f(a)}{b - a}$$

③ If $f(x)$ is continuous in $[a, b]$ and differentiable in (a, b) and if $f'(x) > 0 \quad \forall x \in (a, b)$ then for $x_1, x_2 \in [a, b]$ s.t. $x_1 < x_2$ we have $f(x_1) < f(x_2)$

Exercise 7.3

1) Explain why Rolle's theorem is not applicable to the following function in the respective intervals.

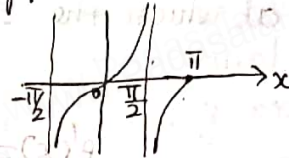
(i) $f(x) = \left| \frac{1}{x} \right|, x \in [-1, 1]$



$f(x)$ is not continuous on $[-1, 1]$, ($f(x)$ is discontinuous at $x=0$)

$\therefore$ Rolle's Theorem cannot be applicable.

(ii) $f(x) = \tan x, x \in [0, \pi]$

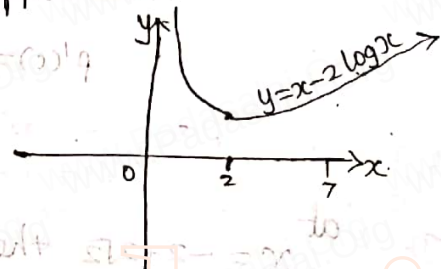


$f(x)$ is not continuous on $[0, \pi]$

$f(x)$ is discontinuous at $x = \pi/2$

Rolle's Theorem cannot be applicable.

(iii) $f(x) = x - 2 \log x, x \in [2, 7]$



(i) $f(x)$ is continuous on $[2, 7]$

(ii) $f(x)$ is differentiable on $(2, 7)$

$$f(a) = f(2) = 2 - 2 \log 2$$

$$f(b) = f(7) = 7 - 2 \log 7$$

$$2 - 2 \log 2 \neq 7 - 2 \log 7$$

$$f(a) \neq f(b)$$

$f(x)$ does not satisfy all the 3 conditions

$\therefore$ Rolle's theorem cannot be applicable.

2) Using the Rolle's theorem, determine the value of x at which the tangent is parallel to the x axis for the following functions.

(i) $f(x) = x^2 - x, x \in [0, 1]$

(i) $f(x)$ is continuous on $[0, 1]$

(ii) $f(x)$ is differentiable on $(0, 1)$

$$(iii) f(a) = f(0) = 0 - 0 = 0$$

$$f(b) = f(1) = 1 - 1 = 0 \quad f(a) = f(b)$$

$\therefore$ By Rolle's theorem there exist $c \in (0, 1)$ such that $f'(c) = 0$ at which the tangent is parallel to the x axis.

$$f(x) = x^2 - x$$

$$f'(x) = 2x - 1$$

$$f'(c) = 0 \Rightarrow 2c - 1 = 0$$

$$c = \frac{1}{2} \in (0, 1)$$

$x = \frac{1}{2}$ the tangent is parallel to x axis.

(ii) $f(x) = \frac{x^2 - 2x}{x+2}, x \in [-1, 6]$

$f(x)$ is continuous on $[-1, 6]$ and differentiable on $(-1, 6)$

$$f(a) = f(-1) = \frac{1+2}{-1+2} = 3$$

$$f(b) = f(6) = \frac{36-12}{6+2} = \frac{24}{8} = 3$$

$$f(a) = f(b)$$

$\therefore$ By Rolle's Theorem, $\exists c \in (-1, 6)$ such that $f'(c) = 0$ at which the tangent is parallel to x axis

$$f(x) = \frac{x^2 - 2x}{x+2}$$

$$f'(x) = \frac{(x+2)(2x-2) - (x^2-2x)(1)}{(x+2)^2} = \frac{2x^2 - 2x + 4x - 4 - x^2 + 2x}{(x+2)^2}$$

$$f'(x) = \frac{x^2 + 4x - 4}{(x+2)^2}$$

$$f'(c) = 0 \Rightarrow c^2 + 4c - 4 = 0$$

$$c^2 + 4c + 4 = +8$$

$$(c+2)^2 = (2\sqrt{2})^2$$

$$c+2 = \pm 2\sqrt{2}$$

$$c = -2 \pm 2\sqrt{2}$$

$$c = -2 + 2\sqrt{2} \in (-1, 6)$$

at

$x = -2 + 2\sqrt{2}$ the tangent is parallel to x axis.

(iii) $f(x) = \sqrt{x} - \frac{x}{3}, x \in [0, 9]$

$f(x)$ is continuous on $[0, 9]$ and differentiable on $(0, 9)$

$$f(a) = f(0) = 0 - \frac{0}{3} = 0$$

$$f(b) = f(9) = \sqrt{9} - \frac{9}{3} = 3 - 3 = 0$$

$$f(a) = f(b)$$

$\therefore$ By Rolle's Theorem $\exists c \in (0, 9)$ such that $f'(c) = 0$ at which the tangent is parallel to x axis

$$f'(x) = \frac{1}{2\sqrt{x}} - \frac{1}{3}$$

$$f'(c) = 0 \Rightarrow \frac{1}{2\sqrt{c}} - \frac{1}{3} = 0$$

$$\frac{1}{2\sqrt{c}} = \frac{1}{3}$$

$$2\sqrt{c} = 3$$

$$\sqrt{c} = \frac{3}{2}$$

$$c = \frac{9}{4} \in (0, 9)$$

$x = \frac{9}{4}$ the tangent parallel to x axis.

③ Explain why Lagrange's mean value theorem is not applicable.

(i) $f(x) = \frac{x+1}{x}, x \in [-1, 2]$

$f(x)$ is not continuous in $[-1, 2]$

when $x=0$ $f(x) = \infty$ not defined

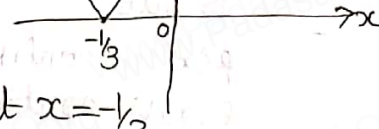
$\therefore$ Lagrange's mean value Theorem cannot be applicable.

(ii) $f(x) = |3x+1|, x \in [-1, 3]$

$f(x)$ is continuous on $[-1, 3]$

but $f(x)$ is ^{not} differentiable on $(-1, 3)$

because $f(x)$ is not different at $x = -1/3$



Lagrange's mean value theorem cannot be applicable,

4) Using the Lagrange's mean value theorem determine the value of x at which the tangent is parallel to the secant line at the end point of the given interval.

i) $f(x) = x^3 - 3x + 2, x \in [-2, 2]$

$f(x)$ is continuous on $[-2, 2]$

$f(x)$ is differentiable on $(-2, 2)$

$\therefore$ By MVT there exist one point $c \in (-2, 2)$

such that $f'(c) = \frac{f(b) - f(a)}{b - a}$

$f'(x) = 3x^2 - 3$

$f(a) = f(-2) = -8 + 6 + 2 = 0$

$f(b) = f(2) = 8 - 6 + 2 = 4$

$\therefore \textcircled{1} \Rightarrow 3c^2 - 3 = \frac{4 - 0}{2 - (-2)}$

$3c^2 - 3 = 1$

$3c^2 = 4$

$c^2 = 4/3$

$c = \pm \frac{2}{\sqrt{3}} \in (-2, 2)$

tangent
At

$x = \pm \frac{2}{\sqrt{3}}$ the secant parallel to secant line

(ii) $f(x) = (x-2)(x-7), x \in [3, 11]$

$f(x) = x^2 - 9x + 14$

$f(x)$ is continuous on $[3, 11]$

$f(x)$ is differentiable on $(3, 11)$

$\therefore$ By MVT

there exist one point $c \in (3, 11)$ such that

$f'(c) = \frac{f(b) - f(a)}{b - a}$ $\textcircled{1}$

$f'(x) = 2x - 9$

$f(b) = f(11) = 121 - 99 + 14 = 36$

$f(a) = f(3) = 9 - 27 + 14 = -4$

$\textcircled{1} \Rightarrow 2c - 9 = \frac{36 - (-4)}{11 - 3}$

$c = 7 \in (3, 11)$

$2c - 9 = 8$

$2c = 17$

$c = 8.5$

$x = 7$ the tangent line is parallel to secant.

5) show that the value in the conclusion of MVT for
 i) $f(x) = \frac{1}{x}$ on a closed interval of positive number $[a, b]$

is $\sqrt{ab}$

$\rightarrow f(x)$ is continuous on $[a, b]$ (a, b) are positive

$f(x)$ is continuous on (a, b)

$\therefore$ By MVT $\exists c \in (a, b)$ such that

$$f'(c) = \frac{f(b) - f(a)}{b - a}$$

$$f(x) = \frac{1}{x}$$

$$f'(x) = -\frac{1}{x^2}$$

$$-\frac{1}{c^2} = \frac{\frac{1}{b} - \frac{1}{a}}{b - a}$$

$$-\frac{1}{c^2} = \frac{\frac{a-b}{ab}}{b-a} \times \frac{1}{-(a-b)}$$

$$-\frac{1}{c^2} = -\frac{1}{ab}$$

$$c^2 = ab \quad c = \sqrt{ab}$$

$$c = \sqrt{ab} \in (a, b)$$

ii) $f(x) = Ax^2 + Bx + C$ on any interval $[a, b]$ is $\frac{a+b}{2}$

$f(x)$ is continuous on $[a, b]$

$f(x)$ is differentiable on (a, b)

$\therefore$ By MVT $\exists c \in (a, b)$ such that

$$f'(c) = \frac{f(b) - f(a)}{b - a}$$

$$2cA + B = \frac{Ab^2 + Bb + C - Aa^2 - Ba - C}{b - a}$$

$$2cA + B = \frac{A(b^2 - a^2) + B(b - a)}{b - a}$$

$$= \frac{A(b+a)(b-a) + B(b-a)}{b-a}$$

$$2cA + B = \frac{(b-a)[A(a+b) + B]}{(b-a)}$$

$$2cA + B = A(a+b) + B$$

$$c = \frac{A(a+b)}{2A}$$

$$\boxed{c = \frac{a+b}{2} \in (a, b)}$$

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6) A race car driver is racing at 20th km If his speed never exceeds 150 km/hr what is the maximum distance he can cover in the next two hours,

Let $s(t)$ be the distance at time t

$$s(t) = 20 \quad s(t_2) = ?$$

$$t_1 = 0 \quad t_2 = 2$$

(18)

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 $s(t)$ is continuous $[0, 2]$ and continuous on $(0, 2)$ $\therefore$ By MVT
 $\exists t \in (0, 2)$

$$s'(t) = \frac{s(t_2) - s(t_1)}{t_2 - t_1}$$

$$s'(t) = \text{speed} \leq 150$$

$$\frac{s(t_2) - 20}{2 - 0} = s'(t) \leq 150$$

$$\therefore \frac{s(t_2) - 20}{2} \leq 150$$

$$s(t_2) - 20 \leq 300$$

$$s(t_2) \leq 300 + 20$$

$$s(t_2) \leq 320$$

maximum distance^{can} cover = 320 km.⑦ suppose that for a function $f(x)$, $f'(x) \leq 1$ for all $1 \leq x \leq 4$
show that $f(4) - f(1) \leq 3$ given that $f'(x) \leq 1$ $\therefore f(x)$ is differentiable on $(1, 4)$ $\Rightarrow f(x)$ is continuous on $[1, 4]$ $\therefore$ By MVT $f'(x) = \frac{f(b) - f(a)}{b - a}$

$$a = 1$$

$$b = 4$$

$$\frac{f(4) - f(1)}{4 - 1} = f'(x) \leq 1$$

$$\frac{f(4) - f(1)}{3} \leq 1$$

$$f(4) - f(1) \leq 3$$

⑧ Does there exist a diff. function $f(x)$ such that
 $f(0) = -1$, $f(2) = 4$ and $f'(x) \leq 2$ for all x justify your answer.given that $f'(x) \leq 2$ $\therefore f(x)$ is differentiable on $(0, 2)$ $\Rightarrow f(x)$ is continuous on $[0, 2]$ $\therefore$ By MVT $\exists$ one point $x \in (0, 2)$ such that

$$f'(x) = \frac{f(b) - f(a)}{b - a}$$

$$a = 0$$

$$b = 2$$

$$f'(x) = \frac{f(2) - f(0)}{2 - 0}$$

$$\frac{f(2) - f(0)}{2} = f'(x) \leq 2$$

$$\frac{f(2) - f(0)}{2} \leq 2$$

$$f(2) - \frac{4 + 1}{2} \leq 2$$

$$2.5 \leq 2$$

contradiction.

$$f'(x) = \frac{4 + 1}{2 - 0}$$

$$f'(x) = \frac{5}{2} \neq 2$$

No,
 $f'(x)$ cannot be 2.5

does not have given values

Q) show that there lies a point on the curve

$f(x) = x(x+3)e^{-\pi/2}$, $-3 \leq x \leq 0$ where tangent drawn is parallel to the x-axis.

$f(x)$ is continuous on $[-3, 0]$

$f(x)$ is differentiable on $(-3, 0)$

$$f(a) = f(-3) = 0$$

$$f(b) = f(0) = 0$$

$$f(a) = f(b)$$

$\therefore$ By Rolle's Theorem $\exists c \in (-3, 0)$ such that $f'(c) = 0$ and that $x = c$ the tangent line parallel to x-axis

$$f(x) = (x^2 + 3x)e^{-\pi/2}$$

$$f'(x) = (2x + 3)e^{-\pi/2}$$

$$f'(c) = 0 \Rightarrow (2c + 3)e^{-\pi/2} = 0$$

$$2c + 3 = 0$$

$$2c = -3$$

$$c = -1.5 \in (-3, 0)$$

$x = -1.5$ the tangent line parallel to x-axis

10) Using mean value theorem prove that for $a, b > 0$

$$|e^{-a} - e^{-b}| < |a - b|$$

$$f(x) = e^{-x}$$

$f(x)$ is continuous on the interval

$f(x)$ is differentiable on the open interval

$\therefore$ By MVT

$$\frac{f(b) - f(a)}{b - a} = f'(c)$$

$$\frac{f(a) - f(b)}{a - b} = -e^{-c}$$

$$\frac{e^{-a} - e^{-b}}{a - b} = -e^{-c}$$

taking modulus on both sides

$$\frac{|e^{-a} - e^{-b}|}{|a - b|} = |e^{-c}| \leq 1$$

$$e^{-x} \leq 1 \text{ for } x > 0$$

$$\frac{|e^{-a} - e^{-b}|}{|a - b|} \leq 1$$

$$|e^{-a} - e^{-b}| \leq |a - b|$$

(a) Taylor's Series

$f(x)$ is infinitely differentiable at $x=a$

$$f(x) = f(a) + \frac{f'(a)}{1!} (x-a) + \dots + \frac{f^n(a)}{n!} (x-a)^n + \dots$$

b) Maclaurin's series

If $a=0$, the expansion takes the form.

$$f(x) = f(0) + \frac{f'(0)}{1!} x + \dots + \frac{f^n(0)}{n!} x^n + \dots$$

Exercise 7.4

1) write the Maclaurin's series expansion

i) e^x

Function	e^x and derivatives	value at $x=0$
$f(x)=e^x$	e^x	$e^0=1$
$f'(x)$	e^x	$e^0=1$
$f''(x)$	e^x	$e^0=1$

Maclaurin's series is

$$f(x) = f(0) + \frac{f'(0)}{1!} x + \frac{f''(0)}{2!} x^2 + \dots$$

$$e^x = 1 + \frac{(1)}{1!} x + \frac{1}{2!} x^2 + \dots$$

$$e^x = 1 + \frac{x}{1!} + \frac{x^2}{2!} + \dots$$

ii) $\sin x$

Function	$\sin x$ and derivatives	value at $x=0$
$f(x)=\sin x$	$\sin x$	$\sin 0 = 0$
$f'(x)$	$\cos x$	$\cos 0 = 1$
$f''(x)$	$-\sin x$	$-\sin 0 = 0$
$f'''(x)$	$-\cos x$	$-\cos 0 = -1$
$f^{(iv)}(x)$	$+\sin x$	$\sin 0 = 0$
$f^{(v)}(x)$	$\cos x$	$\cos 0 = 1$

Maclaurin's series expansion

$$f(x) = f(0) + \frac{f'(0)}{1!} x + \frac{f''(0)}{2!} x^2 + \frac{f'''(0)}{3!} x^3 + \dots$$

$$\sin x = 0 + \frac{1}{1!} x + \frac{0}{2!} x^2 + \frac{-1}{3!} x^3 + \frac{0}{4!} x^4 + \frac{1}{5!} x^5 + \dots$$

$$\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \dots$$

(iii) $\cos x$

function	$\cos x$ and derivatives	value at $x=0$
$f(x) = \cos x$	$\cos x$	$\cos 0 = 1$
$f'(x)$	$-\sin x$	$-\sin 0 = 0$
$f''(x)$	$-\cos x$	$-\cos 0 = -1$
$f'''(x)$	$\sin x$	$\sin 0 = 0$
$f^{IV}(x)$	$\cos x$	$\cos 0 = 1$
$f^V(x)$	$-\sin x$	$-\sin 0 = 0$
$f^{VI}(x)$	$-\cos x$	$-\cos 0 = -1$

Maclaurin's series expansion

$$f(x) = f(0) + \frac{f'(0)}{1!}x + \frac{f''(0)}{2!}x^2 + \frac{f'''(0)}{3!}x^3 + \dots$$

$$\cos x = 1 + \frac{0}{1!}x - \frac{1}{2!}x^2 + \frac{0}{3!}x^3 - \frac{1}{4!}x^4 + \frac{0}{5!}x^5 - \frac{1}{6!}x^6 + \dots$$

$$\cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \dots$$

(iv) $\log(1-x)$; $-1 \leq x \leq 1$

function	$\log(1-x)$ and derivatives	value at $x=0$
$f(x) = \log(1-x)$	$\log(1-x)$	$\log 1 = 0$
$f'(x)$	$-\frac{1}{1-x}$	$-\frac{1}{1-0} = -1$
$f''(x)$	$-\frac{1}{(1-x)^2}$	$-\frac{1}{(1-0)^2} = -1$
$f'''(x)$	$-\frac{2}{(1-x)^3}$	$-\frac{2}{(1-0)^3} = -2$
$f^{IV}(x)$	$-\frac{6}{(1-x)^4}$	$-\frac{6}{(1-0)^4} = -6$

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Maclaurin's series expansion

$$f(x) = f(0) + \frac{f'(0)}{1!}x + \frac{f''(0)}{2!}x^2 + \frac{f'''(0)}{3!}x^3 + \dots$$

$$\log(1-x) = 0 - \frac{1}{1!}x - \frac{1}{2!}x^2 - \frac{2}{3!}x^3 - \frac{6}{4!}x^4 - \dots$$

$$\log(1-x) = -x - \frac{x^2}{2} - \frac{x^3}{3} - \frac{x^4}{4} - \dots$$

$$= -(x + \frac{x^2}{2} + \frac{x^3}{3} + \frac{x^4}{4} + \dots)$$

(v) $\tan^{-1}(x)$; $-1 \leq x \leq 1$

function	$\tan^{-1}x$ and derivatives	value at $x=0$
$f(x) = \tan^{-1}x$	$\tan^{-1}x$	$\tan^{-1}0 = 0$
$f'(x)$	$\frac{1}{1+x^2}$	$\frac{1}{1+0} = 1$

$$f'(x) = \frac{1}{1+x^2} = (1+x^2)^{-1} = 1 - x^2 + x^4 - x^6 + \dots$$

$$f''(x) = 0 - 2x + 4x^3 - 6x^5 + \dots = 0$$

$$f'''(x) = -2 + 12x^2 - 30x^4 + \dots = -2$$

$$f^{(4)}(x) = 0 + 12x - 120x^3 + \dots = 0$$

$$f^{(5)}(x) = 24 - 360x^2 + \dots = 24 = 4!$$

Maclaurin's series expansion is

$$f(x) = f(0) + \frac{f'(0)}{1!}x + \frac{f''(0)}{2!}x^2 + \dots$$

$$\tan^{-1}x = 0 + \frac{1}{1}x + \frac{0}{2!}x^2 - \frac{2}{1 \times 2 \times 3}x^3 + \frac{0}{4!}x^4 + \frac{4!}{5!}x^5 - \dots$$

$$\tan^{-1}x = x - \frac{x^3}{3} + \frac{x^5}{5} - \dots$$

(vi) $\cos^2 x$

Function: $\cos^2 x$ and derivatives value at $x=0$

$$f(x) = \cos^2 x \quad f'(x) = \cos^2 x \quad \cos^2 0 = 1^2 = 1$$

$$f'(x) = 2 \cos x (-\sin x) = -\sin 2x \quad -\sin 0 = 0$$

$$f''(x) = -\cos 2x (2) = -2 \cos 2x \quad -2 \cos 0 = -2(1) = -2$$

$$f'''(x) = -2(-\sin 2x)(2) = 4 \sin 2x \quad 4 \sin 0 = 0$$

$$f^{(4)}(x) = 4 \cos 2x (2) = 8 \cos 2x \quad 8 \cos 0 = 8$$

Maclaurin's series expansion

$$f(x) = f(0) + \frac{f'(0)}{1!}x + \frac{f''(0)}{2!}x^2 + \frac{f'''(0)}{3!}x^3 + \dots$$

$$f(x) = 1 + \frac{0}{1!}x - \frac{2}{2!}x^2 + \frac{0}{3!}x^3 + \frac{8}{4!}x^4 + \dots$$

$$\cos^2 x = 1 - \frac{2x^2}{2!} + \frac{2x^4}{4!} - \dots$$

2) write down the Taylor series expansion of the function $\log x$ about $x=1$ upto three non-zero terms for $x > 0$

Function: $\log x$ and derivatives value at $x=1$

$$f(x) = \log x \quad \log 1 = 0$$

$$f'(x) = \frac{1}{x} \quad \frac{1}{1} = 1$$

$$f''(x) = -\frac{1}{x^2} \quad -\frac{1}{1} = -1$$

$$f'''(x) = \frac{2}{x^3}$$

$$\frac{2}{1^3} = 2$$

$$f^{IV}(x) = \frac{-6}{x^4}$$

$$\frac{-6}{1^4} = -6$$

$$f^V(x) = \frac{24}{x^5}$$

$$\frac{24}{1} = 24$$

Taylor's series expansion is -

$$f(x) = f(a) + \frac{f'(a)}{1!}(x-a) + \frac{f''(a)}{2!}(x-a)^2 + \frac{f'''(a)}{3!}(x-a)^3 + \dots$$

$$a=1$$

$$\log x = f(1) + \frac{f'(1)}{1!}(x-1) + \frac{f''(1)}{2!}(x-1)^2 + \frac{f'''(1)}{3!}(x-1)^3 + \dots$$

$$= 0 + \frac{(x-1)}{1!} + \frac{(x-1)}{2!}(x-1)^2 + \frac{1}{1 \times 2 \times 3}(x-1)^3 + \dots$$

$$\log x = (x-1) - \frac{(x-1)^2}{2} + \frac{(x-1)^3}{3} - \dots$$

③ Expand $\sin x$ in ascending powers $x - \frac{\pi}{4}$ upto 3 non-zero terms

function.

$\sin x$ and derivatives

value at $x = \frac{\pi}{4}$

$$f(x) = \sin x$$

$$\sin x$$

$$\sin \frac{\pi}{4} = \frac{1}{\sqrt{2}}$$

$$f'(x)$$

$$\cos x$$

$$\cos \frac{\pi}{4} = \frac{1}{\sqrt{2}}$$

$$f''(x)$$

$$-\sin x$$

$$-\sin \frac{\pi}{4} = -\frac{1}{\sqrt{2}}$$

Taylor's series expansion at $x = \frac{\pi}{4}$

$$f(x) = f\left(\frac{\pi}{4}\right) + \frac{f'\left(\frac{\pi}{4}\right)}{1!}(x - \frac{\pi}{4}) + \frac{f''\left(\frac{\pi}{4}\right)}{2!}(x - \frac{\pi}{4})^2 + \dots$$

$$= \frac{1}{\sqrt{2}} + \frac{1/\sqrt{2}}{1!}(x - \frac{\pi}{4}) + \frac{-1/\sqrt{2}}{2!}(x - \frac{\pi}{4})^2 + \dots$$

$$= \frac{1/\sqrt{2}}{\sqrt{2}} \left[1 + \frac{(x - \frac{\pi}{4})}{1!} - \frac{(x - \frac{\pi}{4})^2}{2!} + \dots \right]$$

$$= \frac{\sqrt{2}}{2} \left[1 + \frac{(x - \frac{\pi}{4})}{1!} - \frac{(x - \frac{\pi}{4})^2}{2!} + \dots \right]$$

4) Expand the polynomial $f(x) = x^2 - 3x + 2$ in powers of $(x-1)$

function $x^2 - 3x + 2$ and derivatives value at $x=1$

$f(x) = x^2 - 3x + 2$	$x^2 - 3x + 2$	$1 - 3 + 2 = 0$
$f'(x)$	$2x - 3$	$2 - 3 = -1$
$f''(x)$	2	2
$f'''(x)$	0	0

Taylor's series expansion at $x=1$ is

$$f(x) = f(1) + \frac{f'(1)}{1!}(x-1) + \frac{f''(1)}{2!}(x-1)^2 + \dots$$

$$x^2 - 3x + 2 = 0 + \frac{(-1)}{1}(x-1) + \frac{2}{2}(x-1)^2 + 0$$

$$x^2 - 3x + 2 = (x-1)^2 - (x-1)$$

— x —

Indeterminate forms

$$\frac{0}{0}, \frac{\infty}{\infty}, 0 \times \infty, \infty - \infty, 1^\infty, 0^0, \infty^0$$

The L'Hopital's Rule

$f(x)$ and $g(x)$ are diff. and $g'(x) \neq 0$

$$\lim_{x \rightarrow a} \frac{f(x)}{g(x)} = 0 = \lim_{x \rightarrow a} g(x) \text{ then } \lim_{x \rightarrow a} \frac{f(x)}{g(x)} = \lim_{x \rightarrow a} \frac{f'(x)}{g'(x)}$$

$$\lim_{x \rightarrow a} f(x) = \pm \infty = \lim_{x \rightarrow a} g(x) \text{ then } \lim_{x \rightarrow a} \frac{f(x)}{g(x)} = \lim_{x \rightarrow a} \frac{f'(x)}{g'(x)}$$

Exercise 7.5

Evaluate the following limits, if necessary use L'Hopital's Rule.

i) $\lim_{x \rightarrow 0} \frac{1 - \cos x}{x^2}$

$$\lim_{x \rightarrow 0} \frac{1 - \cos x}{x^2} = \frac{1 - \cos 0}{0} = \frac{1 - 1}{0} = \frac{0}{0}$$

This is in indeterminate form. $(\frac{0}{0})$

Use L'Hopital's Rule

$$\lim_{x \rightarrow 0} \frac{1 - \cos x}{x^2} = \lim_{x \rightarrow 0} \frac{0 + \sin x}{2x} = \frac{+\sin 0}{2(0)} = \frac{0}{0}$$

again applying L'Hopital's Rule.

$$\lim_{x \rightarrow 0} \frac{+\sin x}{2x} = \lim_{x \rightarrow 0} \frac{+\cos x}{2} = \frac{+\cos 0}{2} = \frac{+1}{2}$$

$$\textcircled{2} \lim_{x \rightarrow \infty} \frac{2x^2 - 3}{x^2 - 5x + 3} = \lim_{x \rightarrow \infty} \frac{x^2(2 - \frac{3}{x^2})}{x^2(1 - \frac{5}{x} + \frac{3}{x^2})}$$

$$= \frac{2 - \frac{3}{\infty}}{1 - \frac{5}{\infty} + \frac{3}{\infty}} = \frac{2 - 0}{1 - 0} = 2$$

$$\text{II} \lim_{x \rightarrow \infty} \frac{2x^2 - 3}{x^2 - 5x + 3} = \frac{\infty}{\infty}$$

This is in ID form

Applying L'Hopitals Rule. two times

$$\lim_{x \rightarrow \infty} \frac{2x^2 - 3}{x^2 - 5x + 3} = \lim_{x \rightarrow \infty} \frac{4x - 0}{2x - 5}$$

$$= \lim_{x \rightarrow \infty} \frac{4}{2} = \frac{4}{2} = 2$$

$$\textcircled{3} \lim_{x \rightarrow \infty} \frac{x}{\log x} \quad (= \frac{\infty}{\log \infty} = \frac{\infty}{\infty})$$

This is in ID form

Applying L'Hopitals Rule

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$$\lim_{x \rightarrow \infty} \frac{x}{\log x} = \lim_{x \rightarrow \infty} \frac{1}{\frac{1}{x}}$$

$$= \lim_{x \rightarrow \infty} x = \infty$$

$$\textcircled{4} \lim_{x \rightarrow \pi/2^-} \frac{\sec x}{\tan x} \quad (= \frac{\sec \pi/2}{\tan \pi/2} = \frac{\infty}{\infty})$$

This is in ID form

Applying L'Hopitals Rule

$$\lim_{x \rightarrow \pi/2^-} \frac{\sec x}{\tan x} = \lim_{x \rightarrow \pi/2^-} \frac{\sec x \tan x}{\sec^2 x}$$

$$= \lim_{x \rightarrow \pi/2^-} \frac{\sin x}{\cos x} = \lim_{x \rightarrow \pi/2^-} \sin x$$

$$= \sin \pi/2 = 1$$

$$\text{M-II} \lim_{x \rightarrow \pi/2^-} \frac{\sec x}{\tan x} = \lim_{x \rightarrow \pi/2^-} \frac{\frac{1}{\cos x}}{\frac{\sin x}{\cos x}} = \lim_{x \rightarrow \pi/2^-} \frac{1}{\cos x} \times \frac{\cos x}{\sin x}$$

$$= \frac{1}{\sin \pi/2} = \frac{1}{1} = 1$$

$$\textcircled{5} \lim_{x \rightarrow \infty} e^{-x} \sqrt{x} = \lim_{x \rightarrow \infty} \frac{\sqrt{x}}{e^x} \quad (= \frac{\infty}{\infty} = \frac{\infty}{\infty})$$

This is in ID form

Applying L'Hopitals Rule

$$\lim_{x \rightarrow \infty} \frac{\sqrt{x}}{e^x} = \lim_{x \rightarrow \infty} \frac{\frac{1}{2\sqrt{x}}}{e^x} = \lim_{x \rightarrow \infty} \frac{1}{2\sqrt{x}} \times \frac{1}{e^x}$$

$$= \frac{1}{\infty} \frac{1}{\infty} = \frac{1}{\infty} = 0$$

⑥ $\lim_{x \rightarrow 0} \left(\frac{1}{\sin x} - \frac{1}{x} \right) = \lim_{x \rightarrow 0} \frac{x - \sin x}{x \sin x} \quad \left(= \frac{0-0}{0} = \frac{0}{0} \right)$

This is in ID form

apply L'Hopital Rule

$$\lim_{x \rightarrow 0} \frac{x - \sin x}{x \sin x} = \lim_{x \rightarrow 0} \frac{1 - \cos x}{x \cos x + \sin x} = \left(\frac{1-1}{0+0} = \frac{0}{0} \right)$$

again apply L'Hopital's Rule.

$$\begin{aligned} \lim_{x \rightarrow 0} \frac{1 - \cos x}{x \cos x + \sin x} &= \lim_{x \rightarrow 0} \frac{0 + \sin x}{x(-\sin x) + \cos x + \cos x} \\ &= \frac{\sin 0}{0 + \cos 0 + \cos 0} = \frac{0}{2} \\ &= 0 \end{aligned}$$

⑦ $\lim_{x \rightarrow 1^+} \left(\frac{2}{x^2-1} - \frac{x}{x-1} \right)$

$$= \lim_{x \rightarrow 1^+} \left(\frac{2}{(x+1)(x-1)} - \frac{x}{x-1} \right)$$

$$= \lim_{x \rightarrow 1^+} \left(\frac{2 - x(x+1)}{(x+1)(x-1)} \right) = \lim_{x \rightarrow 1^+} \left(\frac{2 - x^2 - x}{x^2 - 1} \right) \quad \left(= \frac{2-1-1}{1-1} = \frac{0}{0} \right)$$

This is in I.D. form.

applying L'Hopital's Rule

$$\begin{aligned} \lim_{x \rightarrow 1^+} \left(\frac{2 - x^2 - x}{x^2 - 1} \right) &= \lim_{x \rightarrow 1^+} \frac{0 - 1 - 2x}{2x} \\ &= \frac{-1-2}{2} = \frac{-3}{2} \end{aligned}$$

8) $\lim_{x \rightarrow 0^+} x^x$

$(= 0^0)$

This is in ID form

composite function Theorem

$$\lim_{x \rightarrow a} \log f(x) = \log \left(\lim_{x \rightarrow a} f(x) \right)$$

Let $y = x^x$

$$\log y = \log x^x$$

$$\log y = x \log x$$

taking limit

$$\lim_{x \rightarrow 0^+} \log y = \lim_{x \rightarrow 0^+} \frac{\log x}{1/x} \quad \left(= \frac{\log 0}{1/0} = \frac{-\infty}{\infty} \right)$$

This is in ID form

applying L'Hopital Rule

$$\begin{aligned} &= \lim_{x \rightarrow 0^+} \frac{\log x}{1/x} = \lim_{x \rightarrow 0^+} \frac{\frac{1}{x}}{-1/x^2} = \lim_{x \rightarrow 0^+} \frac{1}{x} \times \frac{-x^2}{1} \\ &= \lim_{x \rightarrow 0^+} -x = 0 \end{aligned}$$

$$\lim_{x \rightarrow 0^+} \log y = 0$$

$$\log(\lim_{x \rightarrow 0} y) = 0$$

taking antilog

$$(\lim_{x \rightarrow 0} y) = e^0$$

$$\lim_{x \rightarrow 0} x^x = 1$$

9) $\lim_{x \rightarrow \infty} (1 + \frac{1}{x})^x = (1 + \frac{1}{\infty})^\infty = 1^\infty$

Thus is in ID form

Let $y = (1 + \frac{1}{x})^x$

$$\log y = \log(1 + \frac{1}{x})^x$$

$$\log y = x \log(1 + \frac{1}{x})$$

taking limit

$$\lim_{x \rightarrow \infty} \log y = \lim_{x \rightarrow \infty} \frac{\log(1 + \frac{1}{x})}{\frac{1}{x}} \quad (= \frac{\log(1+0)}{\frac{1}{\infty}} = \frac{0}{0})$$

Applying L'Hopital Rule

$$\lim_{x \rightarrow \infty} \frac{\log(1 + \frac{1}{x})}{\frac{1}{x}} = \lim_{x \rightarrow \infty} \frac{\frac{1}{1 + \frac{1}{x}} \cdot (-\frac{1}{x^2})}{(-\frac{1}{x^2})}$$

$$= \frac{1}{1 + \frac{1}{\infty}} = \frac{1}{1+0} = 1$$

$$\lim_{x \rightarrow \infty} \log y = 1$$

Apply composite function Rule $\log(\lim_{x \rightarrow \infty} y) = 1$

taking antilog $\lim_{x \rightarrow \infty} y = e^1$

$$\lim_{x \rightarrow \infty} (1 + \frac{1}{x})^x = e$$

10) $\lim_{x \rightarrow \frac{\pi}{2}} (\sin x)^{\tan x}$

This is in ID form 0^∞

Let $y = \sin x^{\tan x}$

$$\log y = \log \sin x^{\tan x}$$

$$\log y = \tan x \log \sin x$$

$$\log y = \frac{\log \sin x}{\cot x}$$

taking limit $\lim_{x \rightarrow \frac{\pi}{2}} \log y = \lim_{x \rightarrow \frac{\pi}{2}} \frac{\log \sin x}{\cot x} = (\frac{\log 1}{\cot \frac{\pi}{2}} = \frac{0}{0})$

Applying L'Hopital's Rule

$$\lim_{x \rightarrow \frac{\pi}{2}} \frac{\log \sin x}{\cot x} = \lim_{x \rightarrow \frac{\pi}{2}} \frac{\frac{1}{\sin x} \cos x}{-\csc^2 x}$$

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$$= \lim_{x \rightarrow \pi/2} \frac{\cos x}{\sin x} = \lim_{x \rightarrow \pi/2} \cos x \times \frac{1}{\sin x}$$

$$= \cos(\pi/2) (-\sin(\pi/2))$$

$$\lim_{x \rightarrow \pi/2} \log y = 0$$

$$\log(\lim_{x \rightarrow \pi/2} y) = 0$$

$$\text{taking antilog } \lim_{x \rightarrow \pi/2} y = e^0$$

$$\lim_{x \rightarrow \pi/2} (\sin x)^{\tan x} = 1$$

$$11) \lim_{x \rightarrow 0^+} (\cos x)^{1/x^2}$$

$$\text{This is in } 1/0 \text{ form } (\cos 0)^{1/0} = 1^\infty$$

$$\text{Let } y = (\cos x)^{1/x^2}$$

$$\log y = \log \cos x^{1/x^2}$$

$$\log y = \frac{\log \cos x}{x^2}$$

taking limit

$$\lim_{x \rightarrow 0^+} \log y = \lim_{x \rightarrow 0^+} \frac{\log \cos x}{x^2} \left(= \frac{\log 1}{0} = \frac{0}{0} \right)$$

Applying L'Hopital's Rule

$$\lim_{x \rightarrow 0^+} \frac{\log \cos x}{x^2} = \lim_{x \rightarrow 0^+} \frac{\frac{1}{\cos x} \cdot (-\sin x)}{2x}$$

$$= \lim_{x \rightarrow 0^+} \frac{-1}{2 \cos x} \cdot \frac{\sin x}{x}$$

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$$= -\frac{1}{2} \cdot \frac{1}{\cos 0} \times \lim_{x \rightarrow 0^+} \frac{\sin x}{x}$$

$$\lim_{x \rightarrow 0^+} \log y = -\frac{1}{2}$$

$$\log(\lim_{x \rightarrow 0^+} y) = -\frac{1}{2}$$

taking antilog

$$\lim_{x \rightarrow 0^+} y = e^{-1/2} = \frac{1}{e^{1/2}} = \frac{1}{\sqrt{e}}$$

$$\lim_{x \rightarrow 0^+} (\cos x)^{1/x^2} = \frac{1}{\sqrt{e}}$$

⑫ If an initial amount A_0 of money is invested at an interest rate r compounded n times a year, the value of the investment after t years is $A = A_0 \left(1 + \frac{r}{n}\right)^{nt}$. If the interest is compounded continuously (that is $n \rightarrow \infty$) show that the amount after t years is $A = A_0 e^{rt}$.

$$\lim_{n \rightarrow \infty} A = \lim_{n \rightarrow \infty} A_0 \left(1 + \frac{r}{n}\right)^{nt}$$

W.K.T

$$\lim_{n \rightarrow \infty} \left(1 + \frac{r}{n}\right)^n = e^r \quad \left[\because \text{From 1st std.} \right]$$

$$\lim_{n \rightarrow \infty} A = A_0 \lim_{n \rightarrow \infty} \left(1 + \frac{r}{n}\right)^{nt}$$

$$= A_0 \left(\lim_{n \rightarrow \infty} \left(1 + \frac{r}{n}\right)^n \right)^t$$

$$= A_0 (e^r)^t$$

$$\lim_{n \rightarrow \infty} A = A_0 e^{rt} //$$

M-II

$$\lim_{n \rightarrow \infty} A = \lim_{n \rightarrow \infty} A_0 \left(1 + \frac{r}{n}\right)^{nt} = A_0 \left(1 + \frac{r}{\infty}\right)^{\infty} = A_0 (1^{\infty})$$

Let $y = \left(1 + \frac{r}{n}\right)^{nt}$

$$\log y = \log \left(1 + \frac{r}{n}\right)^{nt}$$

$$= nt \log \left(1 + \frac{r}{n}\right)$$

$$\log y = t \log \left(1 + \frac{r}{n}\right)$$

taking limit

$$\lim_{n \rightarrow \infty} \log y = \lim_{n \rightarrow \infty} t \log \left(1 + \frac{r}{n}\right) \quad \left(= \frac{0}{0} \right)$$

Applying L'Hopital's Rule

$$\lim_{n \rightarrow \infty} t \frac{\log \left(1 + \frac{r}{n}\right)}{\frac{1}{n}} = \lim_{n \rightarrow \infty} t \left(\frac{\frac{1}{1 + \frac{r}{n}}}{\left(-\frac{r}{n^2}\right)} \right)$$

$$= rt \lim_{n \rightarrow \infty} \left(\frac{1}{1 + \frac{r}{n}} \right)$$

$$= rt \frac{1}{1+0}$$

$$\lim_{n \rightarrow \infty} \log y = rt$$

$$\log \left(\lim_{n \rightarrow \infty} y \right) = rt$$

taking antilog $\lim_{n \rightarrow \infty} y = e^{rt}$

$$\lim_{n \rightarrow \infty} \left(1 + \frac{r}{n}\right)^{nt} = e^{rt}$$

$$\therefore \lim_{n \rightarrow \infty} A_0 \left(1 + \frac{r}{n}\right)^{nt} = A_0 e^{rt}$$

$$\text{i.e. } \lim_{n \rightarrow \infty} A = A_0 e^{rt} //$$

Monotonicity of functions.

Increasing function

A function $f(x)$ is increasing function in an interval I

$$\text{if } a < b \Rightarrow f(a) \leq f(b) \quad \forall a, b \in I$$

Decreasing function

$f(x)$ is decreasing function in I

$$\text{if } a < b \Rightarrow f(a) \geq f(b) \quad \forall a, b \in I$$

* $f(x)$ is differentiable in an open interval (a, b)

If (i) $f'(x) \geq 0$ then $f(x)$ is increasing in (a, b)

(ii) $f'(x) > 0$ then $f(x)$ is strictly increasing in (a, b)

(iii) $f'(x) \leq 0$ then $f(x)$ is decreasing in (a, b)

(iv) $f'(x) < 0$ then $f(x)$ is strictly decreasing in (a, b)

$$\forall x \in (a, b)$$

Stationary point.

A stationary point $(x_0, f(x_0))$ of a diff. function $f(x)$ is where $f'(x_0) = 0$

critical numbers and critical point

A critical point $(x_0, f(x_0))$ of a function $f(x)$ is where $f'(x_0) = 0$ or does not exist (that x_0 is called critical number)

Absolute maximum & minimum

$f(x_0)$ is called

absolute max of $f(x)$

$$f(x_0) \geq f(x) \quad \forall x \in D$$

absolute min of $f(x)$

$$f(x_0) \leq f(x) \quad \forall x \in D$$

(where x_0 is in the domain D)

A procedure for finding the absolute extrema

- 1) find critical number of $f(x)$
- 2) Evaluate value of $f(x)$ at that critical no, and also find $f(a), f(b)$.
- 3) Largest value in step 2 is absolute max, smallest value in step 2 is absolute min.

Fermat Theorem

If c is a critical number $f(x)$ has relative extremum at $x=c$. Invariably there will be critical numbers of $f(x)$ obtained by $f'(x)=0$ or $f'(x)$ does not exist values.

First Derivative test.

- ① If $f'(x)$ changes its sign +ve to -ve then $f(x)$ has local max $f(c)$
- ② If $f'(x)$ changes from -ve to +ve then $f(x)$ has local min $f(c)$.
- ③ $f'(x)$ is +ve on both sides of c or -ve on both sides of c then $f(c)$ is neither local max nor min.

—x—

Exercise 7.6

① Find the absolute extrema.

(i) $f(x) = x^2 - 12x + 10$; $[1, 2]$

$$f'(x) = 2x - 12$$

$$f'(x) = 0 \Rightarrow 2x - 12 = 0$$

$$2x = 12$$

$$x = \frac{12}{2} = 6$$

critical number is $x = 6 \notin [1, 2]$

$f(6) = 6^2 - 12(6) + 10 = 36 - 72 + 10 = -26$ we cannot take $x = 6$

$$f(1) = 1 - 12 + 10 = -1$$

$$f(2) = 2^2 - 12(2) + 10 = 4 - 24 + 10 = -10$$

absolute maximum = -1, absolute minimum = -26

(ii) $f(x) = 3x^4 - 4x^3$; $[-1, 2]$ [change the interval

$$f'(x) = 12x^3 - 12x^2$$

$$f'(x) = 0 \Rightarrow 12x^2(x-1) = 0$$

$$x^2 = 0 \quad x-1 = 0$$

$$x = 0 \quad x = 1$$

critical numbers $x = 0, 1$

$$f(0) = 3(0) - 4(0) = 0$$

$$f(1) = 3(1) - 4(1) = 3 - 4 = -1$$

$$f(-1) = 3(-1)^4 - 4(-1)^3 = 3 + 4 = 7$$

$$f(2) = 3(2)^4 - 4(2)^3 = 48 - 32 = 16$$

absolute maximum = 16 absolute minimum = -1

(iii) $f(x) = 6x^{4/3} - 3x^{1/3}$ $[-1, 1]$

$$f'(x) = 6 \left(\frac{4}{3}\right) x^{4/3-1} - 3 \frac{1}{3} x^{1/3-1}$$

$$= 8x^{1/3} - x^{-2/3}$$

$$= x^{1/3} \left(8 - \frac{1}{x}\right) = x^{1/3} \frac{(8x-1)}{x} = \frac{8x-1}{x^{2/3}}$$

$$f'(x) = 0 \Rightarrow 8x-1=0$$

$$8x=1$$

$$x = 1/8$$

at $x=0$ $f'(x)$ does not exist critical numbers $x=0, 1/8$

$$f(0) = 0$$

$$f(1/8) = 6 \left(\frac{1}{8}\right)^{4/3} - 3 \left(\frac{1}{8}\right)^{1/3}$$

$$= 6 \left(\frac{1}{2}\right)^4 - 3 \left(\frac{1}{2}\right) = \frac{3 \cdot 3}{16} - \frac{3}{2} = \frac{3-12}{8} = -\frac{9}{8}$$

$$f(-1) = 6(-1)^{4/3} - 3(-1)^{1/3} = 6+3=9$$

$$f(1) = 6-3=3$$

absolute maximum = 9 absolute minimum = $-\frac{9}{8}$

(iv) $f(x) = 2\cos x + \sin 2x$ $[0, \pi/2]$

$$f'(x) = 2(-\sin x) + 2\cos 2x$$

$$f'(x) = 2(-\sin x + \cos 2x)$$

$$f'(x) = 0 \Rightarrow$$

$$-\sin x + 1 - 2\sin^2 x = 0$$

$$2\sin^2 x + \sin x - 1 = 0$$

$$\left(\sin x + \frac{1}{2}\right) \left(\sin x - 1\right) = 0$$

$$\sin x + 1 = 0$$

$$\sin x = 1/2$$

$$\sin x = -1$$

$$\sin x = \sin -\pi/2$$

$$x = n\pi + (-1)^n (-\pi/2)$$

$$\sin x = \sin \pi/6$$

$$x = n\pi + (-1)^n \pi/6$$

take one value

$$\text{critical number } x = 3\pi/2$$

$$x = \pi/6$$

$$f(3\pi/2) = 2\cos 3\pi/2 + \sin 2(3\pi/2) = 0+0=0$$

$$f(\pi/6) = 2\cos \pi/6 + \sin 2\pi/6 = 2 \frac{\sqrt{3}}{2} + \frac{\sqrt{3}}{2} = \frac{3\sqrt{3}}{2}$$

$$f(0) = 2\cos(0) + \sin 0 = 2(1) = 2$$

$$f(\pi/2) = 2\cos \pi/2 + \sin 2\pi/2 = 0+0=0$$

$$\text{absolute maximum} = \frac{3\sqrt{3}}{2}$$

$$\text{absolute minimum} = 0$$

② Find the interval of monotonicities and hence find the local extremum.

(i) $f(x) = 2x^3 + 3x^2 - 12x$

$$f'(x) = 6x^2 + 6x - 12$$

$$f'(x) = 0 \Rightarrow 6x^2 + 6x - 12 = 0$$

$$6(x^2 + x - 2) = 0$$

$$(x+2)(x-1) = 0$$

$$x+2=0 \quad x-1=0$$

$$x = -2, \quad x = 1$$

critical numbers are $-2, 1$



intervals are $(-\infty, -2), (-2, 1), (1, \infty)$

Interval	value x	$f'(x) = 6(x+2)(x-1)$	sign of $f'(x)$	monotonicity
$(-\infty, -2)$	-3	$6(-3+2)(-3-1) = 6(-1)(-4) = 24$	$+$	strictly increasing
$(-2, 1)$	0	$6(-2+2)(-2-1) = 0$	$-$	strictly decreasing
$(1, \infty)$	2	$6(2+2)(2-1) = 6(4)(1) = 24$	$+$	strictly increasing

(i) $f'(x)$ changes from +ve to -ve through $x = -2$

$f(x)$ has local max at $x = -2$

$$\begin{aligned} \text{max value is } f(-2) &= 2(-8) + 3(-2)^2 - 12(-2) \\ &= -16 + 12 + 24 \\ &= 20 \end{aligned}$$

(ii) $f'(x)$ changes from -ve to +ve through $x = 1$

$f(x)$ has local minimum at $x = 1$

$$\begin{aligned} \text{minimum value} &= f(1) \\ &= 2 + 3 - 12 \\ &= -7 \end{aligned}$$

Local maximum = 20, Local minimum = -7

(ii) $f(x) = \frac{x}{x-5}$

$$f'(x) = \frac{(x-5)(1) - x(1)}{(x-5)^2} = \frac{x-5-x}{(x-5)^2} = \frac{-5}{(x-5)^2}$$

$$f'(x) = \frac{-5}{(x-5)^2} \neq 0 \quad \forall x \in \mathbb{R}, \text{ and}$$

$f'(x)$ does not exist at $x=5$

no critical numbers at $x=5$
because at $x=5$ $f(x)$ also does not exist

$\therefore$ domain of $f(x)$ is $\boxed{\mathbb{R} - \{5\}}$

$\therefore x=5$ is not in domain. ($\therefore$ No need to take $x=5$ as a critical no.)

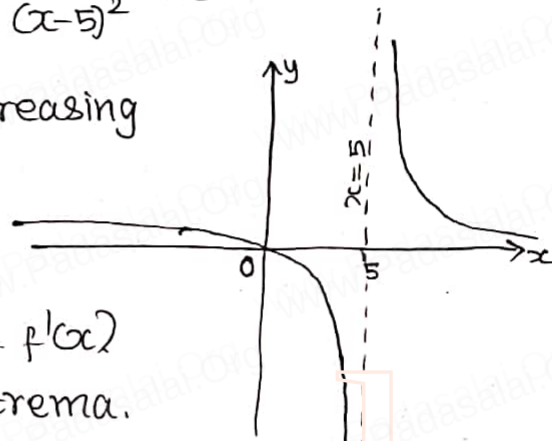
$$\therefore \text{In } \mathbb{R} - \{5\} \quad f'(x) = \frac{-5}{(x-5)^2} < 0$$

$\therefore f(x)$ is strictly decreasing
on $\mathbb{R} - \{5\}$

No critical number

$\therefore$ no change of sign of $f'(x)$

$\therefore$ There is no local extrema.



$$(iii) \quad f(x) = \frac{e^x}{1-e^x}$$

$$f'(x) = \frac{(1-e^x)e^x - e^x(-e^x)}{(1-e^x)^2} \Rightarrow f'(x) = \frac{e^x - e^{-2x} + e^{2x}}{(1-e^x)^2}$$

$$f'(x) = \frac{e^x}{(1-e^x)^2} \neq 0 \quad \forall x \in \mathbb{R} - \{0\}$$

when $x=0$ $f(x)$ does not exist $\therefore$ domain is $\mathbb{R} - \{0\}$

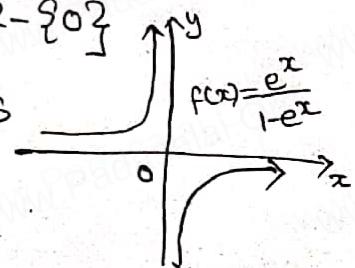
$\therefore x=0$ is not in domain of $f(x)$ ($\therefore$ No need to take $x=0$ as a critical no.)

$$\therefore \text{In } \mathbb{R} - \{0\} \quad f'(x) = \frac{e^x}{(1-e^x)^2} > 0$$

$f(x)$ is strictly increasing on $\mathbb{R} - \{0\}$

$f'(x)$ does not change its sign.

$\therefore$ There is no local extrema,



$$(iv) f(x) = \frac{x^3}{3} - \log x$$

$f(x)$ exist only $x \in (0, \infty)$

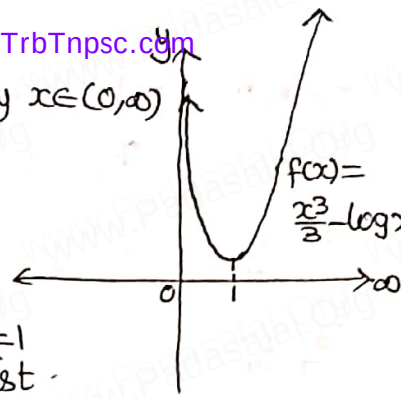
$$f'(x) = \frac{3x^2}{3} - \frac{1}{x} = \frac{x^3 - 1}{x}$$

$$f'(x) = 0 \Rightarrow \frac{x^3 - 1}{x} = 0$$

$$x^3 = 1 \quad x = 1$$

when $x=0$, $f'(x)$ does not exist.

critical numbers are $x=0, 1$



Intervals are $(-\infty, 0)$, $(0, 1)$, $(1, \infty)$

Intervals	value x	$f'(x) = \frac{x^3 - 1}{x}$	sign of $f'(x)$	monotonicity
$(-\infty, 0)$	$-\frac{1}{2}$	$\frac{-1-1}{-\frac{1}{2}} = 2$	$+$	strictly increasing
$(0, 1)$	$\frac{1}{2}$	$\frac{\frac{1}{8} - 1}{\frac{1}{2}}$	$-$	strictly decreasing
$(1, \infty)$	2	$\frac{8 - 1}{2} = \frac{7}{2}$	$+$	strictly increasing

(i) $(-\infty, 0)$ $f(x)$ is not defined.
also at $x=0$ $f(x) \in (-\infty)$ not defined.
 $\therefore f(x)$ lies only on $(0, \infty)$.

from the beginning
we take $x \in (0, \infty)$
is better.
 $(0, 1), (1, \infty)$

(ii) $f'(x)$ changes its sign from -ve to +ve at $x=1$

$\therefore f(x)$ has a local minimum at $x=1$

local minimum value = $f(1)$

$$= \frac{1}{3} - \log 1$$

$$\text{local minimum} = \frac{1}{3}$$

$$(v) f(x) = \sin x \cos x + 5 \quad x \in (0, 2\pi)$$

$$f(x) = \frac{1}{2} 2 \sin x \cos x + 5$$

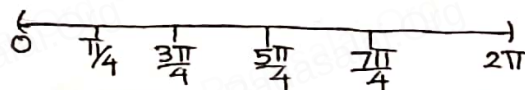
$$f(x) = \frac{1}{2} \sin 2x + 5$$

$$f'(x) = \frac{1}{2} \cos 2x (2) = \cos 2x$$

$$f'(x) = 0 \Rightarrow \cos 2x = 0$$

$$\cos 2x = \frac{\pi}{2}, \frac{3\pi}{2}, \frac{5\pi}{2}, \frac{7\pi}{2}, \dots$$

$$\text{critical numbers } x = \frac{\pi}{4}, \frac{3\pi}{4}, \frac{5\pi}{4}, \frac{7\pi}{4}, \dots$$



intervals are $(0, \frac{\pi}{4})$, $(\frac{\pi}{4}, \frac{3\pi}{4})$, $(\frac{3\pi}{4}, \frac{5\pi}{4})$, $(\frac{5\pi}{4}, \frac{7\pi}{4})$, $(\frac{7\pi}{4}, 2\pi)$

Intervals	value	$f'(x) = \cos 2x$	sign of $f'(x)$	monotonicity
$(0, \pi/4)$	$\pi/6$	$\cos \pi/3 = \frac{1}{2}$	+	strictly increasing
$(\pi/4, 3\pi/4)$	$\pi/2$	$\cos \pi = -1$	-	strictly decreasing
$(3\pi/4, 5\pi/4)$	π	$\cos 2\pi = 1$	+	strictly increasing
$(5\pi/4, 7\pi/4)$	$3\pi/2$	$\cos 3\pi = -1$	-	strictly decreasing
$(7\pi/4, 2\pi)$	$11\pi/6$	$\cos 11\pi/3$ $\cos 660^\circ = \cos -60^\circ = \frac{1}{2}$	+	strictly increasing

$f(x)$ is strictly increasing on $(0, \pi/4), (3\pi/4, 5\pi/4), (7\pi/4, 2\pi)$
and strictly decreasing on $(\pi/4, 3\pi/4), (5\pi/4, 7\pi/4)$

(i) $f'(x)$ changes from +ve to -ve at $x = \pi/4, 5\pi/4$
 $f(x)$ has local maximum at $x = \pi/4, x = 5\pi/4$

$$f(\pi/4) = \frac{1}{2} \sin 2(\pi/4) + 5 = \frac{1}{2} + 5 = 11/2$$

$$f(5\pi/4) = \frac{1}{2} \sin 2(5\pi/4) + 5$$

$$= \frac{1}{2} \sin 5\pi/2 + 5 = \frac{1}{2} + 5 = 11/2$$

$$\text{Local max} = 11/2$$

(ii) $f'(x)$ changes from -ve to +ve through $x = 3\pi/4, 7\pi/4$

$f(x)$ has local minimum at $x = 3\pi/4, 7\pi/4$

Local minimum values are

$$f(3\pi/4) = \frac{1}{2} \sin 2(3\pi/4) + 5 = \frac{1}{2}(-1) + 5 = 9/2$$

$$f(7\pi/4) = \frac{1}{2} \sin 2(7\pi/4) + 5 = \frac{1}{2} \sin(630^\circ) + 5$$

$$= \frac{1}{2} \sin(720 - 90) + 5$$

$$= -\frac{1}{2} + 5 = 9/2$$

$$\text{Local maximum} = 11/2$$

$$\text{Local minimum} = 9/2$$

—x—

Applications of second derivatives

Test of concavity

(i) If $f''(x) > 0$ on an open interval I , then $f(x)$ is concave up on I

If $f''(x) < 0$ on I , then $f(x)$ is concave down on I

point of inflection

(i) If $f''(c)$ exists and $f''(c)$ changes sign. +ve to -ve or -ve to +ve through $x=c$. then $(c, f(c))$ is called point of inflection.

(ii) If $f''(c)$ exists at point of inflection, then $f''(c) = 0$.

The second derivative Test

(i) $f'(x)$ exists and give the critical point $x=c$

(ii) find $f''(x)$. then.

$f''(x) < 0$ at $x=c$. then $f(x)$ has relative maximum

$f''(x) > 0$ at $x=c$ then $f(x)$ has relative minimum.

$f''(x) = 0$ at $x=c$. The test is not informative.

Exercise 7.7

i) Find the intervals of concavity and points of inflection.

(i) $f(x) = x(x-4)^3$

$$f'(x) = x^3(x-4)^2 + (x-4)^3(1)$$

$$= (x-4)^2(3x+x-4)$$

$$= (x-4)^2(4x-4)$$

$$= 4(x-4)^2(x-1)$$

$$f''(x) = 4(x-4)^2(1) + 4(x-1)2(x-4)$$

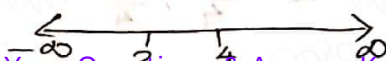
$$= 4(x-4)[x-4 + 2(x-1)]$$

$$= 4(x-4)(x-4+2x-2) = 4(x-4)(3x-6)$$

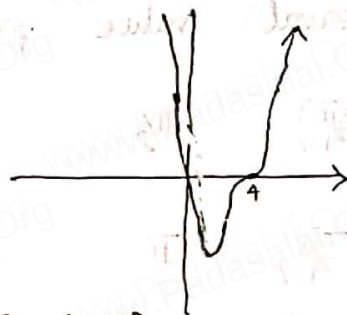
$$f''(x) = 12(x-4)(x-2)$$

$$f''(x) = 0 \Rightarrow 12(x-4)(x-2) = 0$$

$$x = 2, 4$$



intervals are $(-\infty, 2), (2, 4), (4, \infty)$



Interval	value	$f''(x) = 12(-4)(x-2)$	sign of $f''(x)$	concavity
$(-\infty, 2)$	0	$12(-4)(-2) = 96$	+	concave up
$(2, 4)$	3	$12(-1)(1) = -12$	-	concave down
$(4, \infty)$	5	$12(1)(3) = 36$	+	concave up

The curve is concave up on $(-\infty, 2), (4, \infty)$.
concave downward on $(2, 4)$.

$f''(x)$ changes its sign through $x=2$ and $x=4$

$f(2) = 2(-2)^3 = -16$ point of inflection

$f(4) = 4(0) = 0$

$(2, -16), (4, 0)$.

(ii) $f(x) = \sin x + \cos x$, $0 < x < 2\pi$

$f'(x) = \cos x - \sin x$

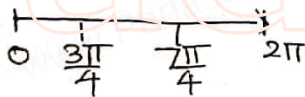
$f''(x) = -\sin x - \cos x$

$f''(x) = 0 \Rightarrow -\sin x - \cos x = 0$

$\sin x = -\cos x$

$\tan x = -1 = \tan -\pi/4$

$x = -\pi/4$



$x = n\pi + \alpha$

$x = n\pi - \pi/4$ $n \in \mathbb{Z}$

$x = \frac{3\pi}{4}, \frac{7\pi}{4} \in (0, 2\pi)$

intervals are $(0, \frac{3\pi}{4}), (\frac{3\pi}{4}, \frac{7\pi}{4}), (\frac{7\pi}{4}, 2\pi)$

Interval	value	$f''(x) = -(\sin x + \cos x)$	$f''(x)$ sign	concavity
$(0, \frac{3\pi}{4})$	$\pi/2$	$-(\sin \pi/2 + \cos \pi/2)$ $-(1+0) = -1$	-	concave down
$(\frac{3\pi}{4}, \frac{7\pi}{4})$	π	$-(0 + (-1)) = 1$	+	concave up
$(\frac{7\pi}{4}, 2\pi)$	$11\pi/6 = 330^\circ$	$-(-\sin 30^\circ + \cos 30^\circ)$ $-(-1/2 + \sqrt{3}/2) = \frac{1-\sqrt{3}}{2}$	-	concave down

$f(x)$ is concave upward on $(\frac{3\pi}{4}, \frac{7\pi}{4})$

concave downward on $(0, \frac{3\pi}{4}), (\frac{7\pi}{4}, 2\pi)$

$f''(x)$ changes its sign through $x = \frac{3\pi}{4}, \frac{7\pi}{4}$

$$f\left(\frac{3\pi}{4}\right) = \sin 135^\circ + \cos 135^\circ = \sin(180-45) + \cos(180-45) \\ = \sin 45 - \cos 45^\circ \\ = \frac{1}{\sqrt{2}} - \frac{1}{\sqrt{2}} = 0$$

$$f\left(\frac{3\pi}{4}\right) = 0$$

$$f\left(\frac{7\pi}{4}\right) = \sin 315^\circ + \cos 315^\circ = -\sin 45 + \cos 45^\circ = -\frac{1}{\sqrt{2}} + \frac{1}{\sqrt{2}} = 0$$

point of inflection: $\left(\frac{3\pi}{4}, 0\right), \left(\frac{7\pi}{4}, 0\right)$

$$(iii) f(x) = \frac{1}{2}(e^x - e^{-x})$$

$$f'(x) = \frac{1}{2}(e^x + e^{-x})$$

$$f''(x) = \frac{1}{2}(e^x - e^{-x})$$

$$f''(x) = 0 \Rightarrow e^x - e^{-x} = 0 \\ e^x = e^{-x}$$

Intervals $(-\infty, 0), (0, \infty)$

$$\boxed{x=0}$$

Interval	value	$f'(x) = \frac{e^x - e^{-x}}{2}$	sign of $f'(x)$	concavity
$(-\infty, 0)$	-1	$\frac{e^{-1} - e^1}{2}$	-	concave downward
$(0, \infty)$	1	$\frac{e^1 - e^{-1}}{2}$	+	concave upward.

$f(x)$ is concave upward on $(0, \infty)$
concave downward on $(-\infty, 0)$

$f'(x)$ changes its sign through $x=0$

$$f(0) = \frac{e^0 - e^0}{2} = \frac{1-1}{2} = 0 \quad \therefore \text{point of inflection} = (0, 0)$$

2) Find the local extrema for the following functions using second derivative test.

$$(i) f(x) = -3x^5 + 5x^3$$

$$f'(x) = -15x^4 + 15x^2$$

$$= 15x^2(x^2 - 1)$$

$$f'(x) = 15x^2(x+1)(x-1)$$

$$f'(x) = 0 \Rightarrow 15x^2(x+1)(x-1) = 0 \Rightarrow x = 0, 1, -1$$

The critical numbers are $x = 0, 1, -1$

$$f''(x) = -60x^3 + 30x$$

$$(i) x = -1 \Rightarrow f''(-1) = +60 - 30 = 30 > 0$$

$f(x)$ has local minimum at $x = -1$.

$$\text{local minimum value } f(-1) = 3 - 5$$

$$(ii) x=1 \quad f''(1) = -60 + 30 = -30 < 0$$

$f(x)$ has local maximum at $x=1$

$$\begin{aligned} \text{local maximum value} &= f(1) \\ &= -3 + 5 \\ &= 2 \end{aligned}$$

(iii) $x=1 \quad f''(1)=0$ second derivative test is not informative of local extremum of $f(x)$.

Local min = -2 Local max = 2.

$$(2) (i) f(x) = x \log x$$

$$f(x) = x \left(\frac{1}{x} \right) + \log x \quad f'(x) = 1 + \log x$$

$$f'(x) = 0 \Rightarrow 1 + \log x = 0$$

$$\log x = -1$$

$$x = e^{-1}$$

$$\boxed{x = \frac{1}{e}}$$

critical number is $x = \frac{1}{e}$

$$f''(x) = 0 + \frac{1}{x}$$

$$f''(x) = \frac{1}{x}$$

$$\text{when } x = \frac{1}{e} \quad f''\left(\frac{1}{e}\right) = \frac{1}{\frac{1}{e}} = e > 0$$

$f(x)$ has local minimum at $x = \frac{1}{e}$.

local minimum value = $f\left(\frac{1}{e}\right)$

$$\text{Local minimum} = -\frac{1}{e}$$

$$= \frac{1}{e} \log \frac{1}{e}$$

$$= \frac{1}{e} (\log 1 - \log e)$$

$$= \frac{1}{e} (0 - 1) = -\frac{1}{e}$$

$$(2) (ii) f(x) = x^2 e^{-2x}$$

$$\begin{aligned} f'(x) &= e^{-2x} (2x) + x^2 e^{-2x} (-2) \\ &= 2e^{-2x} (x - x^2) \end{aligned}$$

$$f'(x) = 0 \Rightarrow 2e^{-2x} (x - x^2) = 0$$

$$e^{-2x} \neq 0, \quad x - x^2 = 0$$

$$-x(x-1) = 0$$

$$x = 0 \quad x = 1$$

critical numbers are $x=0, 1$

$$f''(x) = 2e^{-2x} (1 - 2x) + (x - x^2) 2e^{-2x} (-2)$$

$$= 2e^{-2x} (1 - 2x + (x - x^2)(-2))$$

$$= 2e^{-2x} (1 - 2x - 2x + 2x^2)$$

$$f''(x) = 2e^{-2x} (2x^2 - 4x + 1)$$

i) when $x=0$ $f'(x) = 2e^0(1) = 2 > 0$

$f(x)$ has local minimum at $x=0$

local minimum value $= f(0)$
 $= 0$

(ii) when $x=1$ $f'(1) = 2e^{-2}(2-4+1)$
 $= 2e^{-2}(-1) < 0$

$f(x)$ has local maximum at $x=1$

local maximum value $= f(1)$

$= 1e^{-2} = \frac{1}{e^2}$

Local minimum $= 0$

Local maximum $= \frac{1}{e^2}$

③ For the function $f(x) = 4x^3 + 3x^2 - 6x + 1$ find the intervals of monotonicity, local extrema, intervals of concavity and point of inflection,

$f'(x) = 12x^2 + 6x - 6$

$f'(x) = 6(2x^2 + x - 1) = 6(x+1)(2x-1)$

$f'(x) = 0 \Rightarrow 6(2x^2 + x - 1) = 0$

$(x+\frac{1}{2})(2x-1) = 0$

$x = -1, \frac{1}{2}$

critical numbers are $x = -1, \frac{1}{2}$

$-\infty \leftarrow -1 \quad \frac{1}{2} \rightarrow \infty$

intervals are $(-\infty, -1), (-1, \frac{1}{2}), (\frac{1}{2}, \infty)$

Interval value $f'(x) = 6(x+1)(2x-1)$ sign of $f'(x)$ monotonicity

$(-\infty, -1)$ -2 $6(-1)(-5) = 30$ $+$ strictly increasing

$(-1, \frac{1}{2})$ 0 $6(1)(-1) = -6$ $-$ strictly decreasing

$(\frac{1}{2}, \infty)$ 1 $6(2)(1) = 12$ $+$ strictly increasing

$f(x)$ is strictly increasing on $(-\infty, -1), (\frac{1}{2}, \infty)$

strictly decreasing on $(-1, \frac{1}{2})$

$f'(x)$ changes from +ve to -ve through $x = -1$

$f(x)$ has local maximum at $x = -1$

Local maximum value = $f(-1)$

$$= -4 + 3 + 6 + 1 = 6$$

 $f'(x)$ changes from -ve to +ve through $x = \frac{1}{2}$
 $f(x)$ has local minimum at $x = \frac{1}{2}$
Local minimum value = $f(\frac{1}{2})$

$$= 4\left(\frac{1}{2}\right)^2 + 3\left(\frac{1}{2}\right) - \frac{6}{2} + 1$$

$$= \frac{2+3-12+4}{4} = \frac{9-12}{4} = -\frac{3}{4}$$

Local maximum = 6

Local minimum = $-\frac{3}{4}$

(ii)

$$f'(x) = 6(2x^2 + x - 1)$$

$$f''(x) = 6(4x + 1)$$

$$f''(x) = 0 \Rightarrow 4x + 1 = 0$$

$$4x = -1$$

$$\boxed{x = -\frac{1}{4}}$$

$$\xleftarrow{-\infty} \quad -\frac{1}{4} \quad \xrightarrow{\infty}$$

Interval	value x	$f''(x) = 6(4x+1)$	sign of $f''(x)$	concavity
$(-\infty, -\frac{1}{4})$	-1	$6(-4+1) = -18$	-	concave downward
$(-\frac{1}{4}, \infty)$	0	$6(1) = 6$	+	concave upward

 $f(x)$ is concave downward on $(-\infty, -\frac{1}{4})$
 $f(x)$ is concave upward on $(-\frac{1}{4}, \infty)$
 $f''(x)$ changes its sign through $x = -\frac{1}{4}$

$$f(-\frac{1}{4}) = 4\left(-\frac{1}{4}\right)^2 + 3\left(-\frac{1}{4}\right) - 6\left(-\frac{1}{4}\right) + 1$$

$$= -\frac{1}{16} + \frac{3}{16} + \frac{6}{4} + 1 = \frac{2}{8} + \frac{3}{2} + 1 = \frac{1+12+8}{8} = \frac{21}{8}$$

point of inflection is $(-\frac{1}{4}, \frac{21}{8})$ Applications in optimization,Exercise 7.8

① Find the two +ve numbers whose sum is 12 and their product is maximum.

Let the two numbers be x, y

$$x + y = 12$$

$$y = 12 - x \quad \text{--- ①}$$

(43)

product $P = xy$

$$P = x(12-x)$$

$$P(x) = 12x - x^2$$

$$P'(x) = 12 - 2x$$

$$P'(x) = 0 \Rightarrow 12 - 2x = 0$$

$$+2x = +12$$

$$x = \frac{12}{2}$$

$$\boxed{x=6}$$

$$P''(x) = -2$$

when $x=6$ $P''(x) = -2 < 0$

$P(x)$ has Local maximum at $x=6$

when $x=6$ The product is maximum

$$\Rightarrow y = 12 - 6 = 6$$

The two numbers are 6, 6.

② Find the two +ve numbers whose product is 20 and their sum is minimum.

Let the numbers be x, y

product = 20

$$xy = 20$$

$$y = \frac{20}{x} \quad \text{--- ①}$$

Sum $S = x + y$

$$S = x + \frac{20}{x}$$

$$S'(x) = 1 - \frac{20}{x^2}$$

$$S'(x) = 0 \Rightarrow 1 - \frac{20}{x^2} = 0$$

$$1 = \frac{20}{x^2}$$

$$x^2 = 20$$

$$x = \pm\sqrt{20}$$

$$\boxed{x = 2\sqrt{5}}$$

$$S''(x) = \frac{+20(2)}{x^3}$$

when $x = 2\sqrt{5}$ $S''(x) = \frac{20 \times 2}{(2\sqrt{5})^3} > 0$

when $x = 2\sqrt{5}$ $S(x)$ has Local minimum

$x = 2\sqrt{5}$ sum is minimum

$$\Rightarrow y = \frac{20}{2\sqrt{5}} \quad y = 2\sqrt{5}$$

The numbers are $2\sqrt{5}, 2\sqrt{5}$

3) Find the smallest possible value of $x^2 + y^2$ given that

$$x + y = 10$$

$$f(x) = x^2 + (10-x)^2$$

$$x + y = 10$$

$$y = 10 - x$$

$$f'(x) = 2x + 2(10-x)(-1)$$

$$f'(x) = 2(x+2-10)$$

$$= 2(2x-10)$$

$$f'(x) = 4(x-5)$$

$$f'(x) = 0 \Rightarrow 4(x-5) = 0$$

$$\boxed{x=5}$$

$$f''(x) = 4$$

$$\text{when } x=5, f''(x) = 4 > 0$$

$f(x)$ has local minimum at $x=5$

at $x=5$ $f(x)$ has minimum value

$$f(5) = 5^2 + (10-5)^2$$

$$= 25 + 25$$

Smallest value

$$= 50$$

- ④ A garden is to be laid out in a rectangular area and protected by wire fence, what is the largest possible area of the fenced garden with metres of wire,

Let x, y be the length and breadth of the Rectangular garden

given total fencing = 40

$$2x + 2y = 40$$

$$x + y = 20$$

$$\boxed{y = 20 - x} \quad \text{--- ①}$$

we

$$\text{Area} = xy$$

$$A(x) = x(20-x)$$

$$A(x) = 20x - x^2$$

$$A'(x) = 20 - 2x$$

$$A'(x) = 0 \Rightarrow$$

$$20 - 2x = 0$$

$$-2x = -20$$

$$\boxed{x=10}$$

$$A''(x) = -2$$

$$\text{when } x=10, A''(x) = -2 < 0$$

when $x=10$ $A(x)$ is local maximum

$x=10$ Area is maximum

$$x=10 \Rightarrow y=20-10$$

$$y=10$$

Largest possible area

$$= xy = 10 \times 10$$

$$= 100 \text{ m}^2$$

- ⑤ A rectangular page is contain 24 cm^2 of print. The margins at the top and bottom of the page are 1.5 cm and the margins at other sides of the page is 1 cm . What should be the dimensions of the page, so that the area of the paper is minimum.

Let x, y be the length and breadth of the printed area.

$$\text{Area} = 24$$

$$xy = 24$$

$$y = \frac{24}{x} \quad \text{--- ①}$$

length of the page is $x+2$
breadth = $y+3$ } - ①

Area of paper = $(x+2)(y+3)$

$$A = xy + 2y + 3x + 6$$

$$= 24 + 3x + 2y + 6$$

$$A(x) = 3x + 2\left(\frac{24}{x}\right) + 30$$

$$A'(x) = 3 + 48\left(-\frac{1}{x^2}\right) + 0$$

$$A'(x) = 0 \Rightarrow$$

$$3 - \frac{48}{x^2} = 0$$

$$3 = \frac{48}{x^2}$$

$$x^2 = \frac{48}{3} = 16$$

$$x = 4$$

$$A''(x) = 48 \times \frac{2}{x^3}$$

$$\text{when } x=4 \Rightarrow A''(x) = \frac{96}{4^3} > 0$$

$$= \frac{96}{64} = \frac{3}{2} > 0$$

when $x=4$ $A(x)$ is minimum

$x=4$ Area of the page is minimum

$$\Rightarrow y = \frac{24}{4} = 6$$

length of the page = $x+2 = 4+2 = 6$ cm

breadth of the page = $y+3 = 6+3 = 9$ cm

- 6) A farmer plans to fence a rectangular pasture adjacent to a river. The pasture must contain 1,80,000 sq mtrs in order to provide enough grass for herds. No fencing is needed along the river, what is the length of the minimum needed fencing material?

Let x, y be the length and breadth of the pasture.

given area = 1,80,000

$$xy = 1,80,000$$

$$y = \frac{180000}{x} \text{ --- ①}$$

Length of fencing = $x+2y$

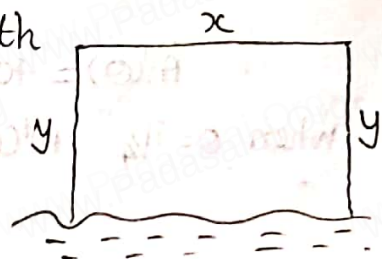
$$L(x) = x + \frac{360000}{x}$$

$$L'(x) = 1 - \frac{360000}{x^2}$$

$$L'(x) = 0 \Rightarrow 1 - \frac{360000}{x^2} = 0$$

$$1 = \frac{360000}{x^2}$$

$$x^2 = 360000 \quad \boxed{x = 600 \text{ m}}$$



$$L''(x) = \frac{720000}{x^3}$$

when $x=600$ $L''(x) = \frac{720000}{(600)^3} > 0$

$x=600$ $L(x)$ is (local) minimum

$x=600$ length of fencing is minimum

① $\Rightarrow y = \frac{180000}{600} = 300 \text{ m}$

minimum Length of fencing $= x+2y$
 $= 600+600 = 1200 \text{ m}$

7) Find the dimensions of the rectangle with maximum area that can be inscribed in a circle of radius 10 cm

Let $2x, 2y$ be the length and breadth of the rectangle. $r=10$

$2x = 2(10 \cos \theta)$ $2y = 2(10 \sin \theta)$

$2x = 20 \cos \theta$, $2y = 20 \sin \theta$

Area $= (2x)(2y)$
 $= (20 \cos \theta)(20 \sin \theta)$
 $= 200 \times 2 \cos \theta \sin \theta$

$A(\theta) = 200 \sin 2\theta$

$A'(\theta) = 200 (\cos 2\theta) 2$

$A'(\theta) = 0 \Rightarrow 400 \cos 2\theta = 0$

$2\theta = \frac{\pi}{2}$

$\theta = \frac{\pi}{4}$

$A''(\theta) = 400 (-\sin 2\theta) 2$

when $\theta = \frac{\pi}{4}$ $A''(\theta) = -800 \sin 2(\frac{\pi}{4})$

$= -800(1)$

$= -800 < 0$

when $\theta = \frac{\pi}{4}$ $A(\theta)$ is maximum (local)

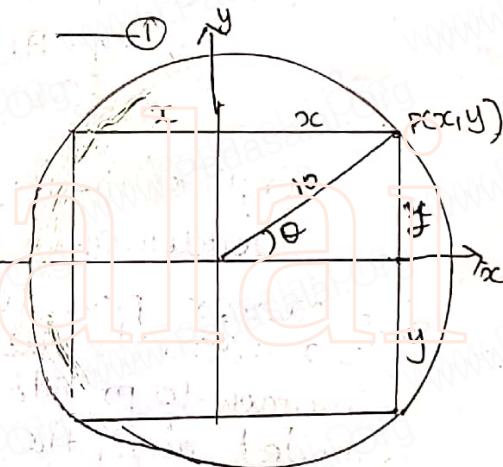
$\theta = \frac{\pi}{4}$, Area of rectangle attains maximum.

① $\Rightarrow 2x = 20 \cos \frac{\pi}{4}$ $2y = 20 \sin \frac{\pi}{4}$

$2x = 20 \frac{1}{\sqrt{2}} = 10\sqrt{2}$ $2y = 10\sqrt{2}$

dimension is $10\sqrt{2}, 10\sqrt{2}$

⑧ prove that among all the rectangles of the given perimeter, the square has the maximum area.



Let x, y be the length and breadth of the rectangle. Let L be the given perimeter

$$2x + 2y = L$$

$$2y = L - 2x$$

$$y = \frac{L}{2} - x \quad \text{--- (1)}$$

rectangle. Area = xy

$$A(x) = x \left(\frac{L}{2} - x \right)$$

$$A(x) = \frac{Lx}{2} - x^2$$

$$A'(x) = \frac{L}{2} - 2x$$

$$A'(x) = 0 \quad \frac{L}{2} - 2x = 0$$

$$\frac{L}{2} = 2x$$

$$A''(x) = -2 < 0$$

when $x = \frac{L}{4}$ $A(x)$ is maximum (local) $\boxed{x = \frac{L}{4}}$

when $x = \frac{L}{4}$ Area of rectangle is maximum

$$\textcircled{1} \Rightarrow y = \frac{L}{2} - \frac{L}{4} \Rightarrow y = \frac{2L - L}{4} = \frac{L}{4}$$

$x = y = \frac{L}{4}$ It is a square.

Square has maximum Area.

9) Find the dimensions of the largest rectangle that can be inscribed in a semicircle of radius r cm.

Let $2x, y$ be the length and breadth of the rectangle.

$$2x = 2r \cos \theta, \quad y = r \sin \theta \quad \text{--- (1)}$$

$$\text{Area} = 2x(y)$$

$$= 2r \cos \theta \cdot r \sin \theta$$

$$A(\theta) = r^2 \sin 2\theta$$

$$A'(\theta) = r^2 \cos 2\theta (2)$$

$$A'(\theta) = 0 \Rightarrow$$

$$2r^2 \cos 2\theta = 0$$

$$\cos 2\theta = 0$$

$$2\theta = \frac{\pi}{2}$$

$$\boxed{\theta = \frac{\pi}{4}}$$

$$A''(\theta) = -4r^2 \sin 2\theta$$

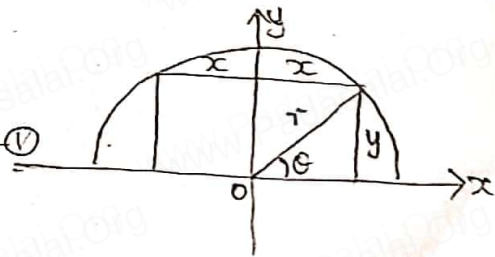
$$\text{When } \theta = \frac{\pi}{4}, \quad A'(\theta) = -4r^2 \sin \frac{\pi}{2}$$

$$A''(\theta) = -4r^2 (1) < 0$$

When $\theta = \frac{\pi}{4}$ $A(\theta)$ is (local) maximum

$\theta = \frac{\pi}{4}$ Area of rectangle is maximum.

$$\textcircled{1} \Rightarrow 2x = 2r \cos \frac{\pi}{4} = \sqrt{2} r \cdot \frac{1}{\sqrt{2}} = \sqrt{2} r \quad y = r \sin \frac{\pi}{4} = r \cdot \frac{1}{\sqrt{2}}$$



$$\text{length} = \sqrt{2}r \quad \text{breadth} = \frac{r}{\sqrt{2}}$$

- 10) A manufacturer wants to design an open box having a square base and a surface area of 108 sq. cm, determine the dimensions of the box for the maximum volume.

Let x, x, y be the dimensions of the box.

$$l=x, b=x, h=y$$

$$\text{base area} = x^2$$

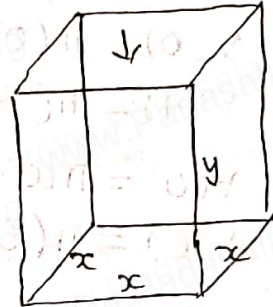
$$\text{sides area} = 4xy$$

$$\text{surface area} = 108$$

$$x^2 + 4xy = 108$$

$$4xy = 108 - x^2$$

$$y = \frac{108 - x^2}{4x} \quad \text{--- (1)}$$



volume of box = lbh

$$V = x^2y$$

(31.07.19)

G. Karthikeyan.
THIRUVARUR DT

$$V(x) = x^2 \left(\frac{108 - x^2}{4x} \right)$$

$$V(x) = \frac{1}{4} (108x - x^3)$$

$$V'(x) = \frac{1}{4} (108 - 3x^2)$$

$$V'(x) = 0 \Rightarrow$$

$$108 - 3x^2 = 0$$

$$3x^2 = 108$$

$$x^2 = \frac{108}{3} = 36$$

$$x^2 = 36$$

$$x = 6$$

$$V''(x) = \frac{1}{4} (-6x) = -\frac{3}{2}x$$

$$\text{when } x=6 \quad V''(x) = -\frac{3}{2}(6) = -9 < 0$$

when $x=6$ $V(x)$ is maximum (local)

when $x=6$ volume of box is maximum

$$\text{--- (1)} \Rightarrow y = \frac{108 - 6^2}{4(6)} = \frac{108 - 36}{24} = \frac{72}{24} = 3$$

dimension of box $x, x, y = 6\text{ cm}, 6\text{ cm}, 3\text{ cm}$

- 11) The volume of a cylinder is given by the formula $V = \pi r^2 h$. Find the greatest and least values of V if $r+h=6$

$$r+h=6$$

$$h = 6 - r \quad \text{--- (1)}$$

$$\text{volume } V = \pi r^2 h$$

$$V(r) = \pi r^2 (6 - r)$$

$$= \pi (6r^2 - r^3)$$

$$V(r) = \pi(6r^2 - r^3)$$

$$V'(r) = \pi(12r - 3r^2)$$

h, r , are +ve. and

maximum value 6

$$0 \leq h, r \leq 6$$

$$\text{and } h+r=6$$

$$V'(r) = 0$$

$$12r - 3r^2 = 0$$

$$-3r(r-4) = 0$$

$$r=0, r-4=0$$

$$r=0, r=4$$

$$V(0) = \pi(0) = 0$$

$$V(4) = \pi(6 \times 16 - 64) = \pi(96 - 64) = 32\pi$$

$$V(6) = \pi(0) = 0$$

$$V(6) = \pi(6(6^2) - 6^3) = \pi(0) = 0$$

absolute maximum = 32π

absolute minimum = 0

- 12) A hollow cone with base radius a cm and height b cm is placed on a table. Show that the volume of the largest cylinder that can be hidden underneath is $\frac{4}{9}$ times volume of the cone.

Let x be the radius of the cylinder and y be the distance from the top of cone to the inscribed cylinder.

given height of cone = b

height of cylinder = $b-y$

volume of cylinder $V = \pi x^2(b-y)$

$$\frac{y}{x} = \frac{b}{a}$$

$$y = \frac{bx}{a}$$

$$V(x) = \pi x^2(b - \frac{bx}{a}) \quad \text{--- ①}$$

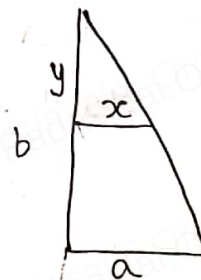
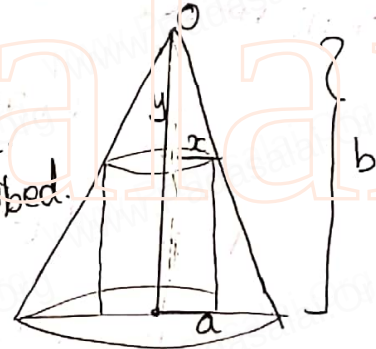
$$V(x) = \pi b(x^2 - \frac{x^3}{a})$$

$$V'(x) = \pi b(2x - \frac{3x^2}{a})$$

$$V'(x) = 0 \Rightarrow$$

$$2x - \frac{3x^2}{a} = 0$$

$$x(2 - \frac{3x}{a}) = 0$$



$$x=0 \text{ or } 2 - \frac{3x}{a} = 0$$

$$\frac{3x}{a} = 2$$

$$x = \frac{2a}{3}$$

$$v'(x) = \pi b \left(2 - \frac{6x}{a} \right)$$

$$\text{when } x = \frac{2a}{3} \Rightarrow v'(x) = \pi b \left(2 - \frac{6}{a} \left(\frac{2a}{3} \right) \right) = \pi b (2 - 4)$$

$$= -2\pi b < 0$$

$$\text{when } x = \frac{2a}{3}, v'(x) < 0$$

$$x = \frac{2a}{3} \quad v(x) \text{ is maximum}$$

$$x = \frac{2a}{3} \quad \text{volume of cylinder is maximum.}$$

$$\textcircled{1} \Rightarrow \text{volume of cylinder } v(x) = \pi x^2 \left(b - \frac{bx}{a} \right)$$

$$= \pi \left(\frac{2a}{3} \right)^2 \left(b - \frac{b}{a} \frac{2a}{3} \right)$$

$$= \pi \frac{4a^2}{9} \left(b \left(1 - \frac{2}{3} \right) \right)$$

$$= \frac{4\pi a^2 b}{9} \left(\frac{1}{3} \right)$$

$$= \frac{4}{9} \left(\frac{1}{3} \pi a^2 b \right)$$

$$\text{max volume of cylinder} = \frac{4}{9} \text{ volume of cone.}$$

Symmetry

$$\textcircled{1} \text{ Symmetric w.r to } y \text{ axis}$$

$$f(-x, y) = f(x, y)$$

(for) all x has x^2 terms

$$\textcircled{2} \text{ Symmetric w.r to } x \text{ axis}$$

$$f(x, -y) = f(x, y)$$

(all y has y^2 terms)

$$\textcircled{3} \text{ Symmetric w.r to origin. if}$$

$$f(-x, -y) = f(x, y)$$

(all x, y has x^2, y^2 terms)

Asymptote

$$\textcircled{1} \text{ Horizontal asymptote}$$

$$\lim_{x \rightarrow \pm\infty} f(x) = L$$

$$\textcircled{ii} \text{ Vertical asymptote}$$

$$\lim_{x \rightarrow \pm\infty} f(x) = \pm\infty$$

(51)

③ Slant asymptote:

numerator degree > denominator degree

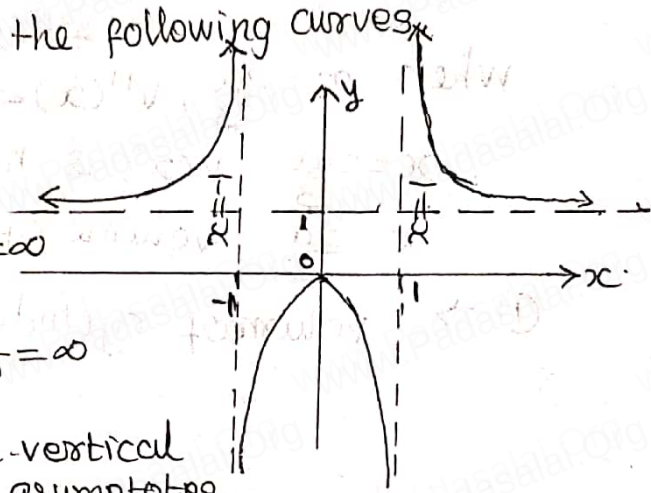
divide the numerator by the denominator
we get the slant asymptote.④ Exercise 7.9

① Find the asymptotes of the following curves

i) $f(x) = \frac{x^2}{x^2-1}$

① $\lim_{x \rightarrow 1} f(x) = \lim_{x \rightarrow 1} \frac{x^2}{x^2-1} = \frac{1}{0} = \infty$

$$\lim_{x \rightarrow -1} f(x) = \lim_{x \rightarrow -1} \frac{x^2}{x^2-1} = \frac{1}{0} = \infty$$

 $\therefore x=1, x=-1$ are vertical asymptotes.

② $\lim_{x \rightarrow \infty} f(x) = \lim_{x \rightarrow \infty} \frac{x^2}{x^2-1} = \lim_{x \rightarrow \infty} \frac{x^2}{(1-\frac{1}{x^2})x^2} = \frac{1}{1-0} = 1$

$$\lim_{x \rightarrow -\infty} f(x) = \lim_{x \rightarrow -\infty} \frac{x^2}{x^2-1} = \frac{1}{1+0} = 1$$

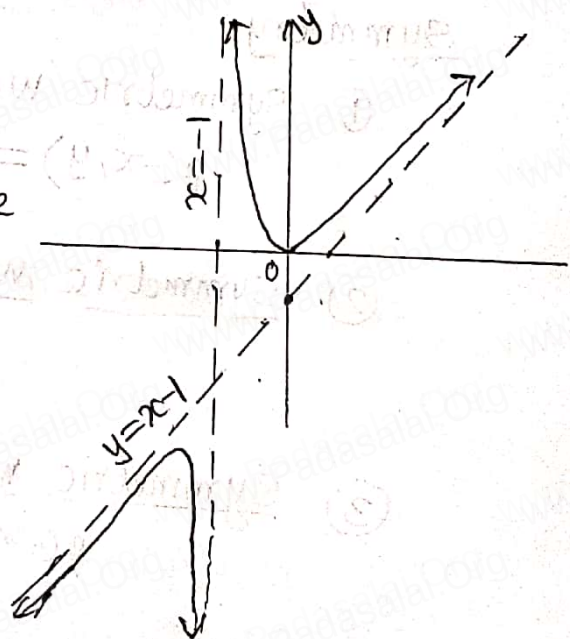
 $\therefore y=1$ is a horizontal asymptote.

ii) $f(x) = \frac{x^2}{x+1}$

$$\lim_{x \rightarrow -1} f(x) = \lim_{x \rightarrow -1} \frac{x^2}{x+1} = \frac{1}{0} = \infty$$

 $x=-1$ is a vertical asymptote

$$\begin{array}{r} x-1 \\ x+1 \overline{) x^2} \\ \underline{x^2+x} \\ -x \\ \underline{-x-1} \\ 1 \end{array}$$

slant asymptote is $y=x-1$ 

iii) $f(x) = \frac{3x}{\sqrt{x^2+2}}$

 $x^2+2 \neq 0 \therefore$ no vertical asymptotes.

$$\lim_{x \rightarrow \infty} f(x) = \lim_{x \rightarrow \infty} \frac{3x}{x\sqrt{1+\frac{2}{x^2}}} = \frac{3}{\sqrt{1+0}} = 3$$

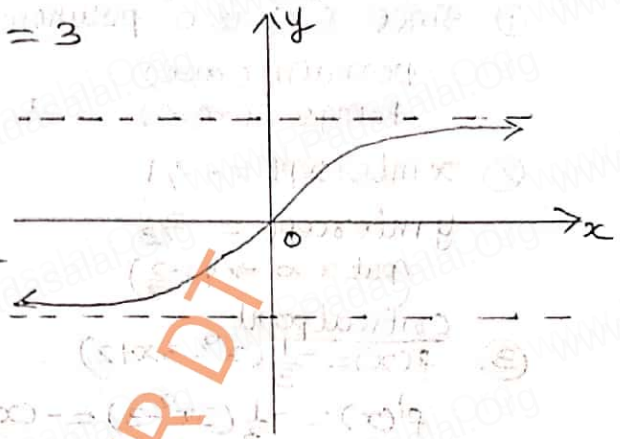
$x \rightarrow \infty \Rightarrow \sqrt{x^2} = x$

$x \rightarrow -\infty \Rightarrow \sqrt{x^2} = -x$

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$$\lim_{x \rightarrow -\infty} f(x) = \lim_{x \rightarrow -\infty} \frac{3x}{x\sqrt{1+\frac{2}{x^2}}} = 3$$

$x=3, x=-3$ are horizontal asymptotes.



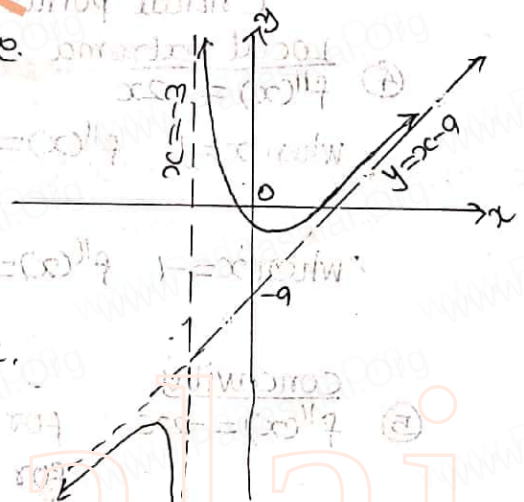
$$(iv) f(x) = \frac{x^2 - 6x - 1}{x + 3}$$

$$\lim_{x \rightarrow -3} f(x) = \lim_{x \rightarrow -3} \frac{9 + 18 - 1}{0} = \infty$$

$x = -3$ is a only vertical asymptote.

$$\begin{array}{r} x-9 \\ x+3 \overline{) x^2 - 6x - 1} \\ \underline{x^2 + 3x} \\ -9x - 1 \\ \underline{-9x - 27} \\ 26 \end{array}$$

$y = x - 9$ is a slant asymptote.



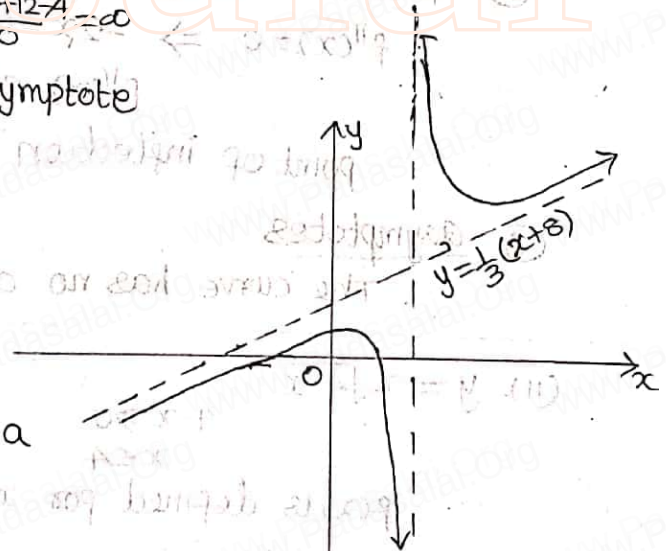
$$(v) f(x) = \frac{x^2 + 6x - 4}{3x - 6}$$

$$\lim_{x \rightarrow 2} f(x) = \lim_{x \rightarrow 2} \frac{x^2 + 6x - 4}{3x - 6} = \frac{4 + 12 - 4}{0} = \infty$$

$x = 2$ is a only vertical asymptote

$$\begin{array}{r} \frac{1}{3}x + \frac{8}{3} \\ 3x-6 \overline{) x^2 + 6x - 4} \\ \underline{x^2 - 2x} \\ 8x - 4 \\ \underline{8x - 16} \\ 12 \end{array}$$

$y = \frac{1}{3}x + \frac{8}{3}$ or $y = \frac{1}{3}(x+8)$ is a slant asymptote.



② Sketch the graphs of the following functions!

$$i) y = -\frac{1}{3}(x^3 - 3x + 2)$$

$$x^3 - 3x + 2 = (x+1)(x-1)(x-1)$$

$$\begin{array}{r|rrrr} 1 & 1 & 0 & -3 & 2 \\ & & 1 & -2 & 0 \\ \hline & 1 & 1 & -2 & 0 \\ & & 2 & 0 & \end{array}$$

when $y = 0$

$$x = -2, 1, 1.$$

① Since $f(x)$ is a polynomial.

Domain $(-\infty, \infty)$

Range $(-\infty, \infty)$

② x-intercept $= -2, 1$

y-intercept $= -2/3$

(put $x=0 \Rightarrow y=-2/3$)

Critical point

③ $f(x) = -\frac{1}{3}(x^3 - 3x + 2)$

$f'(x) = -\frac{1}{3}(3x^2 - 3) = -(x^2 - 1)$

$f'(x) = 0 \Rightarrow x^2 - 1 = 0$
 $x = \pm 1$

Critical points are $1, -1$

Local extrema

④ $f''(x) = -2x$

when $x=1$ $f''(x) = -2 < 0$ $f(x)$ has local maximum
minimum value is $f(1) = 0$

when $x=-1$ $f''(x) = 2 > 0$ $f(x)$ has local minimum
minimum value is $f(-1) = -\frac{1}{3}(-1 + 3 + 2) = -\frac{4}{3}$

Concavity

⑤ $f''(x) = -2x$ for $x > 0$ concave downward
for $x < 0$ concave upward

⑥ Point of Inflection

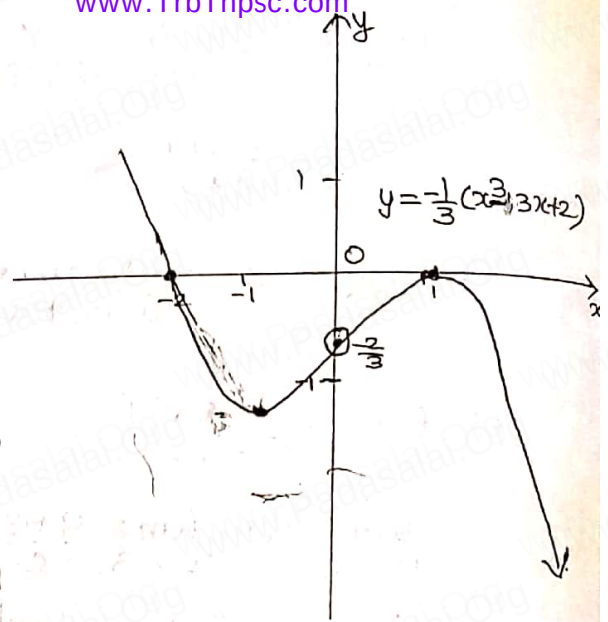
$f''(x) = 0 \Rightarrow -2x = 0, x = 0$

$f''(x)$ changes its sign through $x = 0$

point of inflection is $(0, f(0)) = (0, -2/3)$

⑦ Asymptotes

The curve has no asymptotes



(ii) $y = x\sqrt{4-x}$

$4-x \geq 0$

$x \leq 4$

$f(x)$ is defined for $x \leq 4$

① Domain and Range

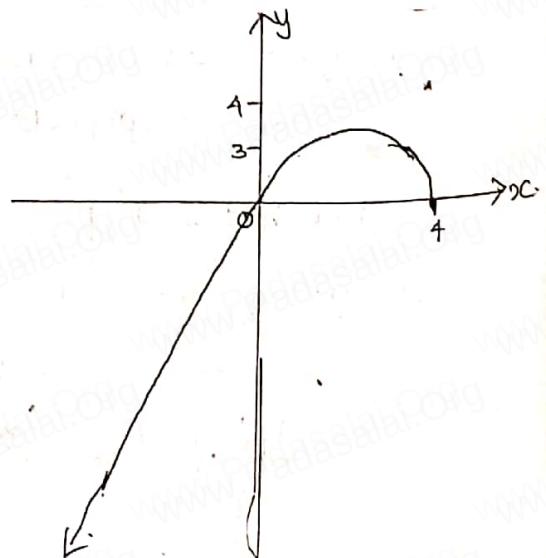
Domain $= (-\infty, 4]$

Range

② Intercepts

x-intercepts $= 0, 4$

y-intercept $= 0$



③ critical numbers

$$f(x) = x(4-x)^{1/2}$$

$$f'(x) = x \cdot \frac{1}{2} (4-x)^{-1/2} (-1) + (4-x)^{1/2} (1)$$

$$= (4-x)^{-1/2} \left[-\frac{x}{2} + (4-x) \right]$$

$$= (4-x)^{-1/2} \left[-\frac{x+8-2x}{2} \right]$$

$$f'(x) = \frac{8-3x}{2(4-x)^{1/2}}$$

$$f'(x) = 0 \Rightarrow 8-3x=0$$

$$x = 8/3$$

when $x=4$, $f'(x)$ does not exist

critical numbers $x = 8/3, 4$

4) Local Extrema

$$f'(x) = \frac{8-3x}{2(4-x)^{1/2}}$$

$$f''(x) = \frac{1}{2} \cdot \frac{(4-x)^{1/2}(-3) - (8-3x) \cdot \frac{1}{2} (4-x)^{-1/2} (-1)}{(4-x)}$$

$$= \frac{(4-x)^{-1/2}}{2(4-x)} \left[(4-x)(-3) + \frac{1}{2} (8-3x) \right]$$

$$= \frac{1}{2(4-x)^{3/2}} \left[-6(4-x) + 8-3x \right]$$

$$= \frac{1}{2(4-x)^{3/2}} \left[-24+6x+8-3x \right] = \frac{1}{4(4-x)^{3/2}} (3x-16)$$

∴ when $x = 8/3$

$$f''(x) = \frac{1}{4(4-8/3)^{3/2}} \left(8 \cdot \frac{8}{3} - 16 \right) = \frac{-8}{4(4/3)^{3/2}} < 0$$

when $x = 8/3$, $f''(x) < 0$, $f(x)$ has local maximum

$$\text{max value} = f(8/3) = \frac{8}{3} \sqrt{4-8/3}$$

$$= \frac{8}{3} \sqrt{\frac{4}{3}} = \frac{16}{3\sqrt{3}}$$

⑤ concavity

$$f''(x) = 0 \Rightarrow \frac{3x-16}{4(4-x)^{3/2}} = 0$$

$$3x-16=0 \quad x = \frac{16}{3} \quad \text{But}$$

domain of $f(x)$ is $(x \leq 4)$ $(-\infty, 4]$

we test only on the domain.

In $(-\infty, 4]$, $f''(x) < 0$ ∴ $f(x)$ is concave downward only.
 $f(x)$ does not change its sign at $x = 16/3$.
 ∴ No point of inflection. (No curve exist at $16/3$)

6) asymptotes

No asymptotes.

(iii) $y = \frac{x^2+1}{x^2-4}$ $f(x) = \frac{x^2+1}{x^2-4}$
 $f(x)$ is not defined for $x = \pm 2$

① Domain ~~can't~~ range.

$x = -2, 2$ $f(x)$ not defined

(Range $(-\infty, -1/4] \cup (1, \infty)$)

domain is $\mathbb{R} - \{-2, 2\}$

② symmetry

$$f(-x, y) = f(x, y)$$

$\therefore f(x)$ is symmetric about y axis

③ intercepts

put $x=0$ $y = -1/4$

$$y=0 \Rightarrow x^2+1=0$$

$$\text{y-intercept} = -1/4$$

no x-intercept

④ asymptotes

$$\lim_{x \rightarrow -2} f(x) = \infty, \lim_{x \rightarrow 2} f(x) = \infty$$

$x = -2, x = 2$ are the vertical asymptotes

$$\lim_{x \rightarrow \infty} f(x) = 1$$

$\therefore y = 1$ is a horizontal asymptotes.

⑤ (monotonicity) critical numbers

$$f(x) = \frac{x^2+1}{x^2-4}$$

$$f'(x) = \frac{(x^2-4)(2x) - (x^2+1)2x}{(x^2-4)^2} = \frac{2x(x^2-4-x^2-1)}{(x^2-4)^2} = \frac{2x(-5)}{(x^2-4)^2}$$

$$f'(x) = \frac{-10x}{(x^2-4)^2}$$

$$f'(x) = 0 \Rightarrow -10x = 0 \quad x = 0$$

$f'(x)$ does not exist at $x = -2, 2$

critical numbers are $x = 0, -2, 2$.

$$(6) f''(x) = \frac{(x^2-4)^2(-10) + 10x(2(x^2-4)2x)}{(x^2-4)^4}$$

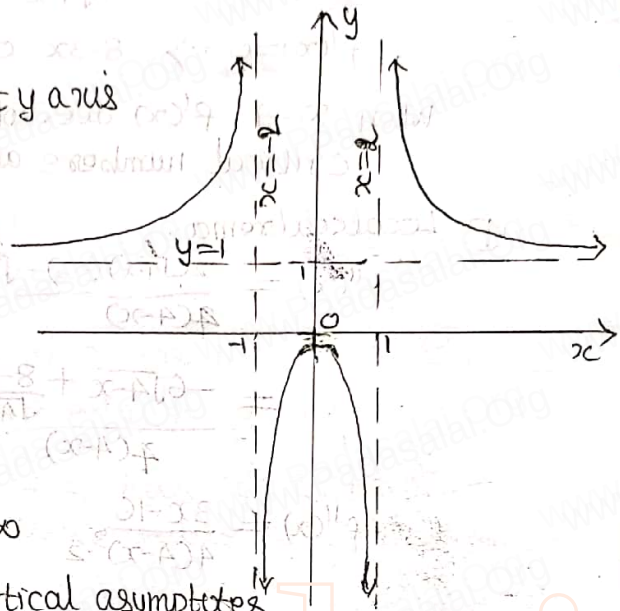
$$= \frac{(x^2-4)(-10x^2+40+40x^2)}{(x^2-4)^3} = \frac{30x^2+40}{(x^2-4)^3}$$

$$\text{w } x=0 \quad f''(x) = \frac{40}{(-4)^3} < 0$$

$f(x)$ has Local maximum at $x=0$

Local maximum value $f(0) = -1/4$.

$x = -2, 2$ is not in the domain and $f'(x) \neq 0 \quad \forall x \in \mathbb{R} - \{-2, 2\}$



• ⑦ concavity

$$f''(x) \neq 0 \text{ but } f''(x) = \frac{30x^2+40}{(x^2-4)^3}$$

does not exist at $x = -2, 2$

Interval	value	$f''(x) = \frac{30x^2+40}{(x^2-4)^3}$	sign of $f''(x)$	Concavity
$(-\infty, -2)$	-3	$\frac{30 \times 9 + 40}{5^3} = \frac{310}{125}$	+	upward
$(-2, 2)$	0	$\frac{40}{-64}$	-	downward
$(2, \infty)$	3	$\frac{310}{125}$	+	upward

$f(x)$ is concave upward on $(-\infty, -2), (2, \infty)$
downward on $(-2, 2)$

⑧ No point of Inflection.

(iv) $y = \frac{1}{1+e^x}$

$y = f(x)$ is defined for all $x \in \mathbb{R}$

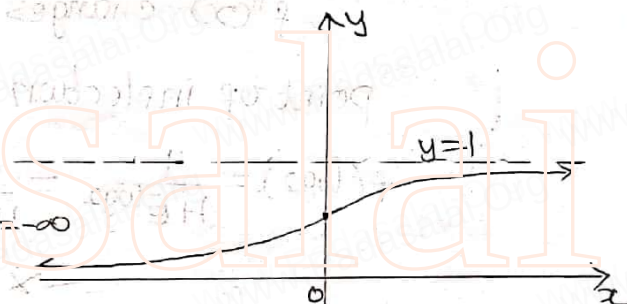
① Domain: $(-\infty, \infty)$

Range: $(0, 1)$

② Intercept

no x intercept (put $y=0$ only at $x=-\infty$)

y intercept = $\frac{1}{2}$



③ asymptotes

$$\lim_{x \rightarrow \infty} f(x) = \frac{1}{1+e^\infty} = \frac{1}{1+0} = 1$$

$$\lim_{x \rightarrow -\infty} f(x) = \frac{1}{1+e^{-\infty}} = \frac{1}{\infty} = 0$$

horizontal asymptotes at $y=1, y=0$

④ critical numbers,

$$f'(x) = \frac{-1 \cdot (-e^{-x})}{(1+e^x)^2} = \frac{e^{-x}}{(1+e^x)^2} > 0 \text{ no critical number}$$

⑤ no critical numbers // no sign changes of $f'(x)$
no local extremum

⑥ concavity

$$f''(x) = \frac{(1+e^x)^2(e^{-x}) - e^{-x}(2(1+e^x)(e^x))}{(1+e^x)^4}$$

$$= \frac{(1+e^x)(-e^{-x}-e^{-x}+2e^x)}{(1+e^x)^3} = \frac{e^{-x}(e^x-1)}{(1+e^x)^3}$$

$$f'(x)=0 \Rightarrow e^{-2x} - e^{-x} = 0$$

$$e^{-x}(e^{-x}-1) = 0$$

$$e^{-x} = 1$$

$$-x = \log 1 \quad \boxed{x=0}$$

Intervals $(-\infty, 0), (0, \infty)$

When $x \in (-\infty, 0)$, $f''(x) > 0$ concave upward
 When $x \in (0, \infty)$, $f''(x) < 0$ concave downward.

(7) point of inflection

$f'(x)$ changes its sign through $x=0$
 point of inflection at $x=0$

$$\therefore \text{i.e. } (0, f(0)) = (0, \frac{1}{1+1}) = (0, \frac{1}{2})$$

point of inflection is $(0, \frac{1}{2})$.

(v) $y = \frac{x^3}{24} - \log x$ (this sum deleted in Tamil medium book)

$f(x)$ exists only for $x > 0$

(1) domain: $(0, \infty)$.

2) intercept

when $x=0$ $\log x \rightarrow -\infty$ no y intercept.

$$y=0 \Rightarrow \frac{x^3}{24} = \log x$$

$$x=1$$

$$x=3$$

3) asymptotes.

$$\lim_{x \rightarrow 0} f(x) = \infty$$

$x=0$ is a vertical asymptotes.

4) critical numbers

$$f'(x) = \frac{3x^2}{24} - \frac{1}{x} = \frac{x^2-8}{8x}$$

$$f'(x)=0 \Rightarrow \frac{x^2-8}{8x} = 0$$

$$x^2-8=0$$

$$x = \pm 2\sqrt{2}$$

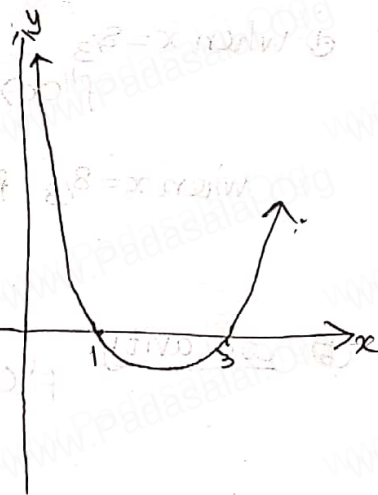
$$x = 2\sqrt{2}, 2\sqrt{2}$$

$$x > 0 \therefore \boxed{x=2\sqrt{2}}$$

when $x=0$, $f'(x)$ does not exist (also $f(x)$ not exist)

$x=0$ is not a critical number

$x=2\sqrt{2}$ is only critical number.



⑤ local extrema

$$f'(x) = \frac{x^2 - 8}{8x}$$

$$f''(x) = \frac{8x(2x) - (x^2 - 8)(8)}{(8x)^2}$$

$$= \frac{16x^2 - 8x^2 + 64}{64x^2} = \frac{8x^2 + 8 \times 8}{64x^2}$$

$$f''(x) = \frac{8(x^2 + 8)}{8 \cdot 64x^2} = \frac{x^2 + 8}{8x^2} > 0$$

when

$$x = 2\sqrt{2} \quad f''(x) = \frac{8+8}{8(8)} = \frac{16}{64} > 0$$

$f(x)$ has a local minimum at $x = 2\sqrt{2}$

$$\text{minimum value} = f(2\sqrt{2})$$

$$= \frac{(2\sqrt{2})^3}{24} - \log 2\sqrt{2}$$

$$= \frac{2 \cdot 16\sqrt{2}}{324} - \log 2\sqrt{2}$$

$$= \frac{2\sqrt{2}}{3} - \log 2\sqrt{2}$$

6) $f''(x) > 0 \therefore f(x)$ is always concave upward on $(0, \infty)$

7) no change of sign of $f''(x)$.

$\therefore$ No point of inflection.

— x —

G. Karthikeyan

9715634957 Need suggestions
(whatsapp only).

8. Differentials and Partial Derivatives.

Linear approximation and differentials

* Linear approximation L of f at x_0

$$L(x) = f(x_0) + f'(x_0)(x - x_0) \quad \forall x \in (a, b)$$

* Absolute error = Actual value - Approximate value.

* Relative error = $\frac{\text{Absolute error}}{\text{Actual value}}$

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Thiruvavur, DT

* percentage error = Relative error $\times 100$

Exercise 8.1

i) Let $f(x) = \sqrt[3]{x}$. Find the linear approximation at $x = 27$, use the linear approximation to approximate $\sqrt[3]{27.2}$.

$$L(x) = f(x_0) + f'(x_0)(x - x_0)$$

$$f(x) = \sqrt[3]{x} = x^{1/3} \quad x_0 = 27 \quad \Delta x = 0.2$$

$$f(x_0) = f(27) = \sqrt[3]{27} = 3$$

$$f'(x) = \frac{1}{3} x^{1/3 - 1}$$

$$f'(27) = \frac{1}{3} \frac{(27)^{1/3}}{27} = \frac{1}{3} \times \frac{1}{27} = \frac{1}{81}$$

$$\therefore L(x) = 3 + \frac{1}{81}(x - 27)$$

$$= 3 + \frac{x}{81} - \frac{27}{81}$$

$L(x) = \frac{x}{81} + 2$ This is the required linear approximation

$$x = 27.2 \quad f(27.2) \approx L(27.2)$$

$$f(27.2) = \sqrt[3]{27.2} \approx \frac{27.2}{81} + 2$$

$$= 1.0074 + 2 = 3.0074$$

$$\begin{array}{r} 1.0074 \\ 27 \overline{) 27.2} \\ \underline{27} \\ 0.200 \\ \underline{189} \\ 110 \end{array}$$

2) Use the linear approximation to find approximate values of $(123)^{2/3}$

$$L(x) = f(x_0) + f'(x_0)(x - x_0)$$

$$f(x) = x^{2/3} \quad x_0 = 125 \quad \Delta x = -2$$

$$f(x_0) = (125)^{2/3} = 5^2 = 25$$

$$f'(x) = \frac{2}{3} x^{2/3-1} = \frac{2}{3} \frac{x^{2/3}}{x}$$

$$f'(x_0) = \frac{2}{3} \frac{(125)^{2/3}}{125} = \frac{2}{3} \frac{25}{125} = \frac{2}{15}$$

$$\therefore L(x) = 25 + \frac{2}{15}(x - 125)$$

$$= 25 + \frac{2x}{15} - \frac{125 \times 2}{15}$$

$$= \frac{2x}{15} + 25 - \frac{50}{3}$$

$$= \frac{2x}{15} + \frac{75-50}{3}$$

$$L(x) = \frac{2x}{15} + \frac{25}{3}$$

$$\begin{array}{r} 24.733 \\ 15 \overline{) 371} \\ \underline{30} \\ 71 \\ \underline{60} \\ 110 \\ \underline{105} \\ 50 \\ \underline{45} \\ 50 \end{array}$$

$$x = 123$$

$$f(123) \approx L(123)$$

$$\begin{aligned} f(123) &= (123)^{2/3} \approx \frac{2(123)}{15} + \frac{25}{3} \\ &= \frac{82}{5} + \frac{25}{3} \\ &= \frac{246+125}{15} = \frac{371}{15} \end{aligned}$$

$$(123)^{2/3} = 24.733$$

(ii) $\sqrt[4]{15}$

$$f(x) = x^{1/4}$$

$$x_0 = 16 \quad \Delta x = -1$$

$$f(x_0) = f(16) = 16^{1/4} = (2^4)^{1/4} = 2$$

$$\begin{aligned} f'(x) &= \frac{1}{4} x^{1/4-1} \\ &= \frac{1}{4} \frac{x^{1/4}}{x} \end{aligned}$$

$$f'(16) = \frac{1}{4} \frac{16^{1/4}}{16} = \frac{2}{64} = \frac{1}{32}$$

③

$$L(x) = f(x_0) + f'(x_0)(x - x_0)$$

$$= 2 + \frac{1}{32}(x - 16)$$

$$= 2 + \frac{x}{32} - \frac{16}{32}$$

$$= \frac{x}{32} + 2 - \frac{1}{2} = \frac{x}{32} + \frac{3}{2}$$

$$L(x) = \frac{x}{32} + \frac{3}{2}$$

$$x = 15$$

$$f(15) = 4\sqrt[4]{15} \approx \frac{15}{32} + \frac{3}{2} = \frac{15+48}{32} = \frac{63}{32}$$

$$4\sqrt[4]{15} = 1.968$$

$$\begin{array}{r} 1.968 \\ 32 \overline{) 63} \\ \underline{32} \\ 310 \\ \underline{288} \\ 220 \\ \underline{192} \\ 280 \\ \underline{256} \end{array}$$

$$(ii) \sqrt[3]{26}$$

$$x_0 = 27$$

$$\Delta x = -1$$

$$f(x) = \sqrt[3]{x} = x^{1/3}$$

$$f(x_0) = \sqrt[3]{27} = 3$$

$$f'(x) = \frac{1}{3} x^{1/3-1} = \frac{1}{3} x^{-2/3}$$

$$f'(x_0) = f'(27) = \frac{1}{3} \frac{(27)^{-2/3}}{27} = \frac{1}{3} \times \frac{2}{27} = -\frac{1}{27}$$

Linear approximation

$$L(x) = f(x_0) + f'(x_0)(x - x_0)$$

$$= 3 + \frac{1}{27}(x - 27)$$

$$= 3 + \frac{x}{27} - 1$$

$$L(x) = \frac{x}{27} + 2$$

$$x = 26$$

$$f(26) = \sqrt[3]{26} \approx L(26)$$

$$\approx \frac{26}{27} + 2$$

$$\approx 0.9629 + 2$$

$$\sqrt[3]{26} \approx 2.9629$$

$$\begin{array}{r} 0.9629 \\ 27 \overline{) 260} \\ \underline{243} \\ 170 \\ \underline{162} \\ 80 \\ \underline{54} \\ 260 \end{array}$$

3) Find the linear approximation for the following functions at the indicated points.

$$i) f(x) = x^3 - 5x + 12, \quad x_0 = 2$$

$$f(x_0) = 2^3 - 5(2) + 12$$

$$= 8 - 10 + 12$$

④

$$f'(x) = 3x^2 - 5$$

$$f'(x_0) = f'(2) = 3(2^2) - 5 = 7$$

$$L(x) = f(x_0) + f'(x_0)(x - x_0)$$

$$L(x) = 10 + 7(x - 2)$$

$$= 10 + 7x - 14$$

$$= 7x - 4$$

$$(ii) g(x) = \sqrt{x^2 + 9}, \quad x_0 = -4$$

$$g(x_0) = \sqrt{16 + 9} = \sqrt{25} = 5$$

$$g'(x) = \frac{1}{\sqrt{x^2 + 9}}$$

$$g'(x_0) = g'(-4) = \frac{-4}{\sqrt{16 + 9}} = -\frac{4}{5}$$

Linear approximation

$$L(x) = g(x_0) + g'(x_0)(x - x_0)$$

$$= 5 + \left(-\frac{4}{5}\right)(x + 4)$$

$$= 5 - \frac{4x}{5} - \frac{16}{5}$$

$$= \frac{25 - 16 - 4x}{5}$$

$$L(x) = \frac{9 - 4x}{5}$$

$$(iii) h(x) = \frac{x}{x+1}, \quad x_0 = 1$$

$$h(x_0) = h(1) = \frac{1}{1+1} = \frac{1}{2}$$

$$h'(x) = \frac{(x+1)(1) - x(1)}{(x+1)^2}$$

$$= \frac{x+1-x}{(x+1)^2}$$

$$h'(x_0) = h'(1) = \frac{1}{(1+1)^2} = \frac{1}{2^2} = \frac{1}{4}$$

Linear approximation

$$L(x) = h(x_0) + h'(x_0)(x - x_0)$$

$$= \frac{1}{2} + \frac{1}{4}(x - 1)$$

$$= \frac{1}{2} + \frac{x}{4} - \frac{1}{4}$$

$$L(x) = \frac{x+1}{4}$$

5)

- 4) The radius of a circular plate is measured as 12.65 cm instead of the actual length 12.5 cm find the following in calculating the area of the circular plate,
 i) Absolute error ii) Relative error iii) percentage error

$$r = 12.65 \quad \Delta r = -0.15$$

$$\text{Area of circle } A = \pi r^2$$

$$\frac{dA}{dr} = 2\pi r \quad dr = \Delta r = -0.15$$

$$dA = 2\pi r dr$$

$$= 2\pi \times 12.65 \times (-0.15)$$

$$= -3.795\pi \text{ cm}^2$$

$$\text{approximate error} = -3.795\pi$$

$$\text{Actual Error} = A(12.5) - A(12.65)$$

$$= \pi(12.5)^2 - \pi(12.65)^2$$

$$= \pi(156.25 - 160.0225)$$

$$= -3.7725\pi \text{ cm}^2$$

$$\text{i) Absolute Error} = \text{Actual error} - \text{Appt. Error}$$

$$= -3.7725\pi - (-3.795\pi)$$

$$= 0.0225\pi \text{ cm}^2$$

$$\text{ii) Relative error} = \frac{\text{absolute error}}{\text{actual error}}$$

$$= \frac{0.0225\pi}{-3.7725\pi}$$

$$= -0.00596$$

$$= -0.006$$

$$\text{iii) Absolute percentage error} = \text{Relative error} \times 100$$

$$= 0.6\%$$

- 5) A sphere is made of ice having radius 10 cm. Its radius decreases from 10 cm to 9.8 cm. Find approximations for the following

i) change in the volume

ii) change in the surface area

$$\text{radius } r = 10$$

$$dr = \Delta r = 9.8 - 10$$

$$\text{volume of sphere } V = \frac{4}{3}\pi r^3$$

$$dr = -0.2$$

$$V'(r) = \frac{4}{3} \cdot 3\pi r^2$$

$$(i) \text{ change in the volume} = V(9.8) - V(10)$$

$$\approx V'(r) dr$$

$$\approx 4\pi(10)^2(-0.2)$$

$$\approx -80\pi \text{ cm}^3$$

change in volume $\approx$ decreased by $80\pi \text{ cm}^3$

ii) change in SA

$$S = 4\pi r^2$$

$$S'(r) = 8\pi r$$

$$\text{change in SA} = S(9.8) - S(10)$$

$$\approx S'(r) dr$$

$$\approx 8\pi(10)(-0.2)$$

$$\approx -16\pi \text{ cm}^2$$

surface area decreased by $16\pi \text{ cm}^2$

6) The time T , taken for a complete oscillation of a single pendulum with length l is given by the equation $T = 2\pi\sqrt{\frac{l}{g}}$, where g is a constant. Find the approximate % error in calculate T corresponding to an error of a number 2% in the value of l .

$$T = 2\pi\sqrt{\frac{l}{g}}$$

taking log on both sides

$$\log T = \log\left(2\pi \left(\frac{l}{g}\right)^{1/2}\right)$$

$$= \log 2\pi + \frac{1}{2} \log l/g$$

$$\log T = \log 2\pi + \frac{1}{2} [\log l - \log g]$$

diff w. r to l

$$\frac{1}{T} \frac{dT}{dl} = 0 + \frac{1}{2} \left(\frac{1}{l} - 0\right)$$

$$\frac{dI}{I} = \frac{20}{100} = \frac{1}{5}$$

$$\frac{dI}{I} = \frac{20}{100}$$

$$\frac{dI}{I} = \frac{1}{100}$$

$$\text{relative error } \frac{\Delta I}{I} \approx \frac{dI}{I} = \frac{1}{100}$$

$$\% \text{ error} = \frac{1}{100} \times 100$$

$$= 1\%$$

7) show that percentage error in the n^{th} root of a number is approximately $\frac{1}{n}$ times the % error in the number.

Let the number be x .
its n^{th} root $x^n = y$

$$y = x^n$$

$$\text{taking log } \log y = \log x^n$$

$$\log y = n \log x$$

diff. w.r to x

$$\frac{1}{y} \frac{dy}{dx} = n \cdot \frac{1}{x}$$

$$\frac{dy}{y} = n \left(\frac{dx}{x} \right)$$

$$\frac{dy}{y} \times 100 = n \left(\frac{dx}{x} \times 100 \right)$$

$$\frac{\Delta y}{y} \times 100 \approx \frac{dy}{y} \times 100 = n \left(\frac{dx}{x} \times 100 \right)$$

$$\% \text{ error of } y \approx \frac{1}{n} (\% \text{ error on } x)$$

DIFFERENTIALS

$$(1) \quad y = f(x) \quad df = f'(x) dx \quad \text{or} \quad dy = f'(x) dx$$

$$(2) \quad \Delta f = f(x + \Delta x) - f(x)$$

Exercise 8.2

1) Find the differential dy

$$(i) \quad y = \frac{(1-2x)^3}{3-4x}$$

$$\frac{dy}{dx} = \frac{(3-4x) 3(1-2x)^2(-2) - (1-2x)^3(0-4)}{(3-4x)^2}$$

$$= (1-2x)^2 \left[\frac{-6(3-4x) + 4(1-2x)}{(3-4x)^2} \right]$$

$$= (1-2x)^2 \left[\frac{-18+24x+4-8x}{(3-4x)^2} \right]$$

$$dy = \frac{(1-2x)^2 (16x-14)}{(3-4x)^2} dx$$

$$dy = 2 \frac{(1-2x)^2 (8x-7)}{(3-4x)^2} dx$$

$$(ii) y = (3 + \sin 2x)^{2/3}$$

$$\frac{dy}{dx} = \frac{2}{3} (3 + \sin 2x)^{\frac{2}{3}-1} (0 + \cos 2x \cdot 2)$$

$$= \frac{4}{3} (3 + \sin 2x)^{-1/3} \cos 2x$$

$$dy = \frac{4 \cos 2x}{3 (3 + \sin 2x)^{1/3}} dx$$

$$(iii) y = e^{x^2-5x+7} \cos(x^2-1)$$

$$\frac{dy}{dx} = e^{x^2-5x+7} (-\sin(x^2-1) (2x)) + \cos(x^2-1) e^{x^2-5x+7} (2x-5)$$

$$\frac{dy}{dx} = e^{x^2-5x+7} [-2x \sin(x^2-1) + (2x-5) \cos(x^2-1)]$$

$$dy = e^{x^2-5x+7} [(2x-5) \cos(x^2-1) - 2x \sin(x^2-1)]$$

2) Find df for $f(x) = x^2 + 3x$ and evaluate it for

(i) $x=2$ and $dx=0.1$ (ii) $x=3$ and $dx=0.02$

$$f(x) = x^2 + 3x$$

$$\frac{df}{dx} = 2x + 3$$

$$df = (2x+3)dx$$

$$= (2(2)+3)(0.1)$$

$$= 0.7$$

$$(ii) x=3 \text{ and } dx=0.02$$

$$df = (2x+3)dx$$

$$df = (2(3)+3) 0.02$$

$$= 0.18$$

3) Find Δf and df for the function f for the indicated values of x , Δx and compare

$$(i) f(x) = x^3 - 2x^2 ; x=2 \quad \Delta x = dx = 0.5$$

$$f'(x) = 3x^2 - 4x$$

$$\frac{df}{dx} = 3x^2 - 4x$$

$$df = (3(2)^2 - 4(2)) 0.5$$

$$= (12 - 8) 0.5$$

$$df = 2.0$$

$$\Delta f = f(x + \Delta x) - f(x)$$

$$= f(2 + 0.5) - f(2)$$

$$= f(2.5) - f(2)$$

$$= \frac{125}{8} - 2\left(\frac{25}{4}\right) - (8 - 8)$$

$$= \frac{25}{8} = 3.125$$

$$\Delta f = 3.125$$

$$(ii) f(x) = x^2 + 2x + 3, x = -0.5, \Delta x = dx = 0.1$$

$$\frac{df}{dx} = 2x + 2$$

$$df = (2x + 2)dx$$

$$= (2(-0.5) + 2) 0.1$$

$$= 1(0.1)$$

$$df = 0.1$$

$$\Delta f = f(x + \Delta x) - f(x)$$

$$= f(-0.4) - f(-0.5)$$

$$= (0.16 - 0.8 + 3) - (0.25 - 1 + 3)$$

$$= 3.16 - 0.8 - 2.25$$

$$\Delta f = 0.11$$

4) Assuming $\log_{10} e = 0.4343$. find an approximate value of $\log_{10} 1003$

$$\text{Let } y = f(x) = \log_{10} x$$

$$x = 1000$$

$$\Delta x = 3$$

$$x + \Delta x = 1003$$

$$\Delta x \approx dx = 3$$

$$f(1000) = \log_{10} 1000 = \log_{10} 10^3 = 3$$

$$y = \log_{10} x$$

$$\frac{dy}{dx} = \frac{1}{x} \log_{10} e$$

$$dy = \frac{0.4343}{x} dx$$

$$= \frac{0.4343}{1000} \times 3$$

$$dy = 0.001303$$

$$f(x + \Delta x) \approx y + dy$$

$$f(1000 + 3) \approx 3 + 0.001303$$

$$\log_{10} 1003 \approx 3.001303$$

5) The trunk of a tree has diameter 30 cm. During the following year, the circumference grew 6 cm.

(i) Approximately, how much did the tree's diameter grew?

(ii) What is the percentage increase in area of the tree's cross-section?

$$\text{diameter } 2r = 30$$

$$r = 15$$

$$\text{Circumference} = 30\pi$$

$$\text{Increase in circumference} = 6$$

$$2\pi r_2 - 2\pi r_1 = 6$$

$$r_2 - r_1 = \frac{6}{2\pi}$$

$$\Delta r = dr = \frac{3}{\pi}$$

$$\text{Area } A = \pi r^2$$

$$\frac{dA}{dr} = 2\pi r$$

$$dA = 2\pi r dr$$

$$dA = 2\pi(15) \frac{3}{\pi}$$

$$= 90$$

$$\% \text{ of Increasing} = \frac{\text{Increasing area}}{\text{actual area}} \times 100$$

$$= \frac{90 \times 2}{\pi \times 15 \times 15} \times 100$$

$$= \frac{40}{\pi} \%$$

- 6) An egg of a particular bird is very nearly spherical. If the radius to the inside of the shell is 5 mm and radius to the outside of the shell is 5.3 mm, find the volume of shell approximately.

$$r = 5$$

$$\Delta r = dr = 5.3 - 5$$

$$dr = 0.3$$

$$\text{volume of sphere } V = \frac{4}{3} \pi r^3$$

$$\frac{dV}{dr} = \frac{4}{3} \pi 3r^2$$

$$dV = 4\pi r^2 dr$$

$$\text{volume of shell} = V(5.3) - V(5)$$

$$\approx dV$$

$$= 4\pi(5 \times 5) 0.3$$

$$\text{volume of shell} \approx 30\pi \text{ mm}^3$$

- 7) Assume that the cross section of the artery of human is circular. A drug is given to a patient to dilate his arteries. If the radius of an artery is increased from 2 mm to 2.1 mm, how much is cross-sectional area increased approximately?

$$r = 2 \text{ mm}$$

$$\Delta r \approx dr = 2.1 - 2$$

$$= 0.1$$

$$\text{Area of circle } A = \pi r^2$$

$$\frac{dA}{dr} = 2\pi r$$

$$dA = 2\pi r dr$$

(12)

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$$\text{Area Increased} = A(2.1) - A(2)$$

$$\approx dA$$

$$= 2\pi(2)(0.1)$$

$$= 0.4\pi$$

$$\text{Area Increased} \approx 0.4\pi \text{ mm}^2$$

8) In a newly developed city, it is estimated that the voting population (in thousands) will increase according to $V(t) = 30 + 12t^2 - t^3$, $0 \leq t \leq 8$ where t is the time in years. Find the approximate change in voters for the time change from 4 to $4\frac{1}{6}$ years.

$$V(t) = 30 + 12t^2 - t^3$$

$$t = 4$$

$$\Delta t \approx dt = 4\frac{1}{6} - 4$$

$$dt = \frac{1}{6}$$

$$\frac{dV}{dt} = 24t - 3t^2$$

$$dV = (24t - 3t^2)dt$$

$$\text{change in } V = V(4\frac{1}{6}) - V(4)$$

$$\approx dV$$

$$= (24(4) - 3(4)^2) \frac{1}{6}$$

$$= (96 - 48) \frac{1}{6}$$

$$= 8 \text{ thousand}$$

$$\text{Change in voters} \approx 8000$$

9) The relation between the number of words y a person learns in x hours is $y = 52\sqrt{x}$, $0 \leq x \leq 9$. What is the approximate number of words learned when x changes from

(i) 1 to 1.1 hour? (ii) 4 to 4.1 hour?

(i)

$$x = 1$$

$$dx = \Delta x = 1.1 - 1 = 0.1$$

$$\frac{dy}{dx} = \frac{26}{\sqrt{x}}$$

$$dy = 26 \frac{1}{\sqrt{x}} dx$$

$$\text{change in word learn} \approx dy$$

$$= 26 \frac{1}{\sqrt{1}} (0.1)$$

$$= 2.6$$

$$\approx 3 \text{ words.}$$

(ii) $x=4$ $dx=\Delta x=4.1-4$
 $=0.1$

$$dy = 26 \frac{1}{\sqrt{x}} dx$$

Change in words learnt $\approx 26 \frac{1}{\sqrt{4}} (0.1) = 1.3$
 ≈ 1 word.

- 10) A circular plate expands uniformly under the influence of heat. If its radius increases from 10.5 cm to 10.75 cm, then find an approximate change in the area and the approximate % change in the area.

$$r=10.5$$

$$\Delta r = dr = 10.75 - 10.5$$

$$dr = 0.25$$

Area of circular plate $A = \pi r^2$
 $dA = 2\pi r dr$

(i) Approximate change in Area $\approx dA$
 $\approx 2\pi (10.5) (0.25)$
 $= 5.25\pi \text{ cm}^2$
 Approximate change in Area $= 5.25\pi \text{ cm}^2$

(ii) approximate % of change $= \frac{dA}{A} \times 100$
 $= \frac{5.25\pi}{\pi \times 10.5 \times 10.5} \times 100$
 $= \frac{525}{10.5 \times 10.5}$
 $= 4.76\%$

- 11) A coat of paint of thickness 0.2 cm is applied to the faces of a cube whose edge is 10 cm. Use the differentials to find approximately how many cubic cm of paint is used to paint this cube. Also calculate the exact amount of paint used to paint this cube.

edge of cube $a = 10 \text{ cm}$

thickness $\Delta a = da = 0.2 \text{ cm}$

volume of cube $V = a^3$

$$\frac{dV}{da} = 3a^2$$

$$dV = 3a^2 da$$

volume of paint $\approx dV$

$$\approx 3(10)^2 \times 0.2$$

$$= 60 \text{ cm}^3$$

volume of paint $\approx 60 \text{ cm}^3$

Exact volume of paint $= V(10.2) - V(10)$

$$= (10.2)^3 - 10^3$$

$$= 1061.208 - 1000$$

$$= 61.208 \text{ cm}^3$$

—x—

Limits and continuity of functions of Two variables.

Limit of a function

$F: A \rightarrow \mathbb{R}$ has a limit L at (u, v)

If for every neighbourhood $(L - \epsilon, L + \epsilon)$, $\epsilon > 0$ of L , there exist a δ -neighbourhood $B_\delta((u, v))$ of (u, v) such that

$$(x, y) \in B_\delta((u, v)) - \{(u, v)\}, \delta > 0,$$

$$\Rightarrow f(x, y) \in (L - \epsilon, L + \epsilon)$$

we denote

$$\lim_{(x, y) \rightarrow (u, v)} F(x, y) = L \quad \text{if such a limit exists.}$$

continuity

$F: A \rightarrow \mathbb{R}$ is continuous at (u, v)

If. 1) F is defined at (u, v)

2) $\lim_{(x, y) \rightarrow (u, v)} F(x, y) = L$ exists

3) $L = F(u, v)$.

Exercise 8.3

1) Evaluate $\lim_{(x,y) \rightarrow (1,2)} g(x,y)$, If the limit exists, where

$$g(x,y) = \frac{3x^2 - xy}{x^2 + y^2 + 3}$$

$$\begin{aligned} \lim_{(x,y) \rightarrow (1,2)} g(x,y) &= \lim_{(x,y) \rightarrow (1,2)} \frac{3x^2 - xy}{x^2 + y^2 + 3} \\ &= \frac{3(1)^2 - 1(2)}{1^2 + 2^2 + 3} \\ &= \frac{3-2}{1+4+3} \\ &= \frac{1}{8} \end{aligned}$$

2) Evaluate $\lim_{(x,y) \rightarrow (0,0)} \cos\left(\frac{x^3 + y^2}{x + y + 2}\right)$, If the limit exists

$$\begin{aligned} \lim_{(x,y) \rightarrow (0,0)} \cos\left(\frac{x^3 + y^2}{x + y + 2}\right) &= \cos\left(\frac{0+0}{0+0+2}\right) \\ &= \cos 0 \\ &= 1 \end{aligned}$$

3) Let $f(x,y) = \frac{y^2 - xy}{\sqrt{x} - \sqrt{y}}$ for $(x,y) \neq (0,0)$, show that

$$\lim_{(x,y) \rightarrow (0,0)} f(x,y) = 0$$

$$\begin{aligned} \lim_{(x,y) \rightarrow (0,0)} f(x,y) &= \lim_{(x,y) \rightarrow (0,0)} \frac{y^2 - xy}{\sqrt{x} - \sqrt{y}} \times \frac{\sqrt{x} + \sqrt{y}}{\sqrt{x} + \sqrt{y}} \\ &= \lim_{(x,y) \rightarrow (0,0)} \frac{y(y-x)(\sqrt{x} + \sqrt{y})}{x - y} \\ &= \lim_{(x,y) \rightarrow (0,0)} -y(\sqrt{x} + \sqrt{y}) \\ &= 0(0+0) \\ &= 0 \end{aligned}$$

4) Evaluate $\lim_{(x,y) \rightarrow (0,0)} \cos\left(\frac{e^x \sin y}{y}\right)$, If the limit exists

$$\lim_{(x,y) \rightarrow (0,0)} \cos\left(\frac{e^x \sin y}{y}\right) = \cos\left(\lim_{(x,y) \rightarrow (0,0)} \frac{e^x \sin y}{y}\right)$$

∴ By composite function theorem

$$= \cos \left(\lim_{(x,y) \rightarrow (0,0)} e^x \cdot \lim_{(x,y) \rightarrow (0,0)} \frac{\sin y}{y} \right)$$

$$= \cos (e^0 \cdot 1)$$

$$= \cos 1$$

5) Let $g(x,y) = \frac{x^2 y}{x^4 + y^2}$ for $(x,y) \neq (0,0)$ and $f(0,0) = 0$

i) show that $\lim_{(x,y) \rightarrow (0,0)} g(x,y) = 0$ along every line

$$y = mx, m \in \mathbb{R}$$

$$\lim_{(x,y) \rightarrow (0,0)} g(x,y) = \lim_{(x,y) \rightarrow (0,0)} \frac{x^2 y}{x^4 + y^2}$$

$$= \lim_{(x,y) \rightarrow (0,0)} \frac{x^2 m x}{x^4 + m^2 x^2}$$

$$= \lim_{(x,y) \rightarrow (0,0)} \frac{x^3 m}{x^2 (x^2 + m^2)}$$

$$= \frac{0}{0 + m^2}$$

$$= 0$$

(ii) show that $\lim_{(x,y) \rightarrow (0,0)} g(x,y) = \frac{k}{1+k^2}$ along every

parabola $y = kx^2, k \in \mathbb{R} - \{0\}$

$$\lim_{(x,y) \rightarrow (0,0)} g(x,y) = \lim_{(x,y) \rightarrow (0,0)} \frac{x^2 y}{x^4 + y^2}$$

$$= \lim_{(x,y) \rightarrow (0,0)} \frac{x^2 k x^2}{x^4 + k^2 x^4}$$

$$= \lim_{(x,y) \rightarrow (0,0)} \frac{x^4 k}{x^4 (1 + k^2)}$$

$$= \frac{k}{1+k^2}$$

6) show that $f(x,y) = \frac{x^2 - y^2}{y^2 + 1}$ is continuous at every $(x,y) \in \mathbb{R}^2$

Let $(a,b) \in \mathbb{R}^2$ be an arbitrary point.

(i) $f(a,b) = \frac{a^2-b^2}{b^2+1}$ is defined. for $\forall (a,b) \in \mathbb{R}^2$

$$(ii) \lim_{(x,y) \rightarrow (a,b)} f(x,y) = \lim_{(x,y) \rightarrow (a,b)} \frac{x^2-y^2}{y^2+1} \\ = \frac{a^2-b^2}{b^2+1} = L$$

i.e Limit exist at $(a,b) \in \mathbb{R}^2$

$$(iii) \lim_{(x,y) \rightarrow (a,b)} f(x,y) = L = f(a,b) = \frac{a^2-b^2}{b^2+1}$$

$\therefore f$ is continuous at every point on $\mathbb{R}^2$.

7) Let $g(x,y) = \frac{e^y \sin x}{x}$, for $x \neq 0$ and $g(0,0) = 1$

show that g is continuous at $(0,0)$.

$$|g(x,y) - g(0,0)| = \left| \frac{e^y \sin x}{x} - 1 \right| = \left| \frac{e^y \sin x - x}{x} \right|$$

(i) $g(0,0) = 1$ g is defined at $(0,0)$

$$(ii) \lim_{(x,y) \rightarrow (0,0)} g(x,y) = \lim_{(x,y) \rightarrow (0,0)} \frac{e^y \sin x}{x} \\ = \lim_{(x,y) \rightarrow (0,0)} e^y \lim_{(x,y) \rightarrow (0,0)} \frac{\sin x}{x} \\ = 1 \cdot 1 = 1 \quad \boxed{L=1}$$

limit exist at $(0,0)$

$$(iii) \lim_{(x,y) \rightarrow (0,0)} g(x,y) = 1 = g(0,0)$$

$\therefore g$ is continuous at $(0,0)$

partial Derivatives

clairaut's Theorem

$f: A \rightarrow \mathbb{R}$ IF f_{xy} and f_{yx} exist in A .

are continuous in A then $f_{xy} = f_{yx}$ in A , where $A \subset \mathbb{R}^2$

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Laplace's Equation

Let $A \subset \mathbb{R}^2$, u is a function $u: A \rightarrow \mathbb{R}^2$ is said to be harmonic in A . If it satisfies $\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0$, $\forall (x, y) \in A$. This equation is called Laplace's Equation.

Exercise 8.4

i). Find the partial derivatives of the following functions at the indicated points.

$$\text{ii) } f(x, y) = 3x^2 - 2xy + y^2 + 5x + 2$$

$$\frac{\partial f}{\partial x} = 6x - 2y + 5$$

$$\frac{\partial f}{\partial x}(2, -5) = 6(2) - 2(-5) + 5 = 12 + 10 + 5 = 27$$

$$\frac{\partial f}{\partial y} = 0 - 2x + 2y$$

$$\frac{\partial f}{\partial y}(2, -5) = -2(2) + 2(-5) = -4 - 10 = -14$$

$$\text{iii) } g(x, y) = 3x^2 + y^2 + 5x + 2, \quad (1, -2)$$

$$\frac{\partial g}{\partial x} = 6x + 5$$

$$\frac{\partial g}{\partial x}(1, -2) = 6(1) + 5 = 11$$

$$\frac{\partial g}{\partial y} = 0 + 2y$$

$$\frac{\partial g}{\partial y}(1, -2) = 2(-2) = -4$$

$$\text{iv) } h(x, y, z) = x \sin(xy) + z^2x, \quad (2, \pi/4, 1)$$

$$\frac{\partial h}{\partial x} = x \cos(xy) y + \sin(xy) (1) + z^2$$

$$\begin{aligned} \frac{\partial h}{\partial x}(2, \pi/4, 1) &= 2 \cos 2\pi/4 \cdot \pi/4 + \sin 2\pi/4 + 1^2 \\ &= 0 + 1 + 1 \\ &= 2 \end{aligned}$$

$$\frac{\partial h}{\partial y} = x \cos xy (x) + 0$$

$$\frac{\partial h}{\partial y}(2, \pi/4, 1) = 2 \cos 2\pi/4 (2) = 0$$

$$\frac{\partial h}{\partial z} = 0 + 2zx$$

$$\frac{\partial h}{\partial z}(2, \pi/4, 1) = 2(2)(1) = 4$$

iv) $G(x, y) = e^{x+3y} \log(x^2+y^2)$, $(-1, 1)$
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$$\frac{\partial G}{\partial x} = e^{x+3y} (\log(x^2+y^2))' + \log(x^2+y^2) (e^{x+3y})'$$

$$= e^{x+3y} \frac{1}{x^2+y^2} (2x) + \log(x^2+y^2) e^{x+3y} (1)$$

$$= e^{x+3y} \left(\frac{2x}{x^2+y^2} + \log(x^2+y^2) \right)$$

$$\frac{\partial G}{\partial x}(-1, 1) = e^{-1+3} \left(\frac{-2}{1+1} + \log(1+1) \right)$$

$$= e^2 (\log 2 - 1)$$

$$\frac{\partial G}{\partial y} = e^{x+3y} \left(\frac{2y}{x^2+y^2} \right) + \log(x^2+y^2) e^{x+3y} (3)$$

$$= e^{x+3y} \left(\frac{2y}{x^2+y^2} + 3 \log(x^2+y^2) \right)$$

$$\frac{\partial G}{\partial y}(-1, 1) = e^{-1+3} \left(\frac{2}{1+1} + 3 \log(1+1) \right)$$

$$= e^2 \left(\frac{12}{7} + 3 \log 2 \right)$$

$$= e^2 (1 + \log 2^3)$$

$$= e^2 (1 + \log 8)$$

2) For each of the following functions find the f_{xy} and show that $f_{xy} = f_{yx}$

i) $f(x, y) = \frac{3x}{y + \sin x}$

$$f_x = \frac{(y + \sin x)(3) - 3x(\cos x)}{(y + \sin x)^2}$$

$$f_x = \frac{3y + 3\sin x - 3x\cos x}{(y + \sin x)^2}$$

$$f_y = \frac{(y + \sin x)(0) - 3x(1)}{(y + \sin x)^2}$$

$$f_y = \frac{-3x}{(y + \sin x)^2}$$

$$f_{xy} = \frac{\partial}{\partial x} \left(\frac{-3x}{(y + \sin x)^2} \right)$$

$$= \frac{(y + \sin x)^2 (-3) + 3x(2(y + \sin x)(\cos x))}{(y + \sin x)^4}$$

$$= \frac{(y + \sin x) (-3(y + \sin x) + 6x\cos x)}{(y + \sin x)^3}$$

$$f_{xy} = \frac{3(2x \cos x - y - \sin x)}{(y + \sin x)^3} \quad \text{--- ①}$$

$$\begin{aligned} f_{yx} &= \frac{\partial}{\partial y} (f_x) = \frac{\partial}{\partial y} \left(\frac{3y + 3\sin x - 3x \cos x}{(y + \sin x)^2} \right) \\ &= \frac{(y + \sin x)^2 (3 + 0 - 0) - (3y + 3\sin x - 3x \cos x) (2(y + \sin x))}{(y + \sin x)^4} \\ &= \frac{(y + \sin x) (3(y + \sin x) - 2 \times 3 (y + \sin x - x \cos x))}{(y + \sin x)^4} \\ &= \frac{3(2x \cos x - y - \sin x)}{(y + \sin x)^3} \end{aligned}$$

$$f_{yx} = \frac{3(2x \cos x - y - \sin x)}{(y + \sin x)^3} \quad \text{--- ②}$$

From ① & ②

$$f_{xy} = f_{yx}$$

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(ii) $f(x, y) = \tan^{-1} \frac{x}{y}$

$$f_x = \frac{1}{1 + \frac{x^2}{y^2}} \cdot \frac{1}{y} = \frac{y^2}{x^2 + y^2} \cdot \frac{1}{y}$$

$$f_x = \frac{y}{x^2 + y^2}$$

$$f_y = \frac{1}{1 + \frac{x^2}{y^2}} \cdot \left(-\frac{x}{y^2}\right) = \frac{y^2}{x^2 + y^2} \left(-\frac{x}{y^2}\right)$$

$$f_y = \frac{-x}{x^2 + y^2}$$

$$\begin{aligned} f_{xy} &= \frac{\partial}{\partial x} (f_y) = \frac{(x^2 + y^2)(-x)' - (-x)(x^2 + y^2)'}{(x^2 + y^2)^2} \\ &= \frac{-x^2 - y^2 + x(2x)}{(x^2 + y^2)^2} \end{aligned}$$

$$f_{xy} = \frac{x^2 - y^2}{(x^2 + y^2)^2} \quad \text{--- ①}$$

$$\begin{aligned} f_{yx} &= \frac{\partial}{\partial y} (f_x) = \frac{(x^2 + y^2)(y)' - y(x^2 + y^2)'}{(x^2 + y^2)^2} \\ &= \frac{x^2 + y^2 - y(2y)}{(x^2 + y^2)^2} \end{aligned}$$

$$f_{yx} = \frac{x^2 - y^2}{(x^2 y)^2} \quad \text{--- (2)}$$

From (1) & (2) $f_{xy} = f_{yx}$.

(iii) $f(x, y) = 3x \cos(x^2 - 3xy)$

$$f_x = -\sin(x^2 - 3xy) (2x - 3y)$$

$$f_y = -\sin(x^2 - 3xy) (0 - 3x) \\ = 3x \sin(x^2 - 3xy)$$

$$f_{xy} = \frac{\partial}{\partial x} (f_y)$$

$$= 3x (\cos(x^2 - 3xy) (2x - 3y) + \sin(x^2 - 3xy) 3)$$

$$f_{xy} = 3 [(2x^2 - 3xy) \cos(x^2 - 3xy) + \sin(x^2 - 3xy)] \quad \text{--- (1)}$$

$$f_{yx} = \frac{\partial}{\partial y} (f_x)$$

$$= -(2x - 3y) (\cos(x^2 - 3xy) (2x - 3y) + \sin(x^2 - 3xy) (-3x))$$

$$f_{yx} = 3 [(2x^2 - 3xy) \cos(x^2 - 3xy) + \sin(x^2 - 3xy)] \quad \text{--- (2)}$$

From (1) & (2)

$$f_{xy} = f_{yx}$$

3) If $U(x, y, z) = \frac{x^2 + y^2}{xy} + 3z^2 y$, find $\frac{\partial U}{\partial x}$, $\frac{\partial U}{\partial y}$ and $\frac{\partial U}{\partial z}$.

$$U = \frac{x^2}{xy} + \frac{y^2}{xy} + 3z^2 y$$

$$U = \frac{x}{y} + \frac{y}{x} + 3z^2 y$$

$$\frac{\partial U}{\partial x} = \frac{1}{y} + y \left(-\frac{1}{x^2}\right) + 0 = \frac{1}{y} - \frac{y}{x^2} = \frac{x^2 - y^2}{yx^2}$$

$$\frac{\partial U}{\partial y} = -\frac{x}{y^2} + \frac{1}{x} + 3z^2 = -\frac{x^2 + y^2}{xy^2} + 3z^2$$

$$\frac{\partial U}{\partial z} = 0 + 0 + 3y(2z) = 6yz.$$

4) If $U(x, y, z) = \log(x^3 + y^3 + z^3)$, find $\frac{\partial U}{\partial x} + \frac{\partial U}{\partial y} + \frac{\partial U}{\partial z}$

$$U = \log(x^3 + y^3 + z^3)$$

$$\frac{\partial U}{\partial x} = \frac{1}{x^3 + y^3 + z^3} (3x^2)$$

$$\frac{\partial U}{\partial y} = \frac{3y^2}{x^3 + y^3 + z^3}, \quad \frac{\partial U}{\partial z} = \frac{3z^2}{x^3 + y^3 + z^3}$$

$$\begin{aligned} \frac{\partial U}{\partial x} + \frac{\partial U}{\partial y} + \frac{\partial U}{\partial z} &= \frac{3x^2 + 3y^2 + 3z^2}{x^3 + y^3 + z^3} \\ &= \frac{3(x^2 + y^2 + z^2)}{x^3 + y^3 + z^3} \end{aligned}$$

5) For each of the following functions find the g_{xy} , g_{xx} , g_{yy} and g_{yx} .

(i) $g(x, y) = xe^y + 3x^2y$

$$g_x = e^y + 3y(2x) = e^y + 6xy$$

$$g_y = xe^y + 3x^2$$

$$g_{xy} = \frac{\partial}{\partial x}(g_y) = e^y + 6x$$

$$g_{xx} = \frac{\partial}{\partial x}(g_x) = 0 + 6y = 6y$$

$$g_{yy} = \frac{\partial}{\partial y}(g_y) = xe^y + 0 = xe^y$$

$$g_{yx} = \frac{\partial}{\partial y}(g_x) = e^y + 6x$$

(ii) $g(x, y) = \log(5x + 3y)$

$$g_x = \frac{1}{5x + 3y} \quad (5)$$

$$g_y = \frac{1}{5x + 3y} \quad (3)$$

$$g_{xx} = 5 \left(\frac{-1}{(5x + 3y)^2} \right) \quad (5) \quad \left| \quad g_{yy} = 3 \left(\frac{-1}{(5x + 3y)^2} \right) \quad (3) \right.$$

$$g_{xx} = \frac{-25}{(5x + 3y)^2} \quad \left| \quad g_{yy} = \frac{-9}{(5x + 3y)^2} \right.$$

$$g_{yx} = \frac{\partial}{\partial y}(g_x)$$

$$= 5 \left(\frac{-1}{(5x + 3y)^2} \right) \quad (5)$$

$$g_{yx} = \frac{-15}{(5x + 3y)^2}$$

$$g_{xy} = \frac{\partial}{\partial x}(g_y)$$

$$= 3 \left(\frac{-1}{(5x + 3y)^2} \right) \quad (3)$$

$$g_{xy} = \frac{-9}{(5x + 3y)^2}$$

$$(iii) g(x, y) = x^2 + 3xy - 7y + \cos 5x.$$

$$g_x = 2x + 3y - \sin 5x (5) \quad g_y = 3x - 7$$

$$g_{xy} = \frac{\partial}{\partial x} (g_y) = 3 (1) = 3 \quad g_{yy} = 0$$

$$g_{xx} = \frac{\partial}{\partial x} (g_x) = 2 - 5 \cos 5x (5) \\ = 2 - 25 \cos 5x$$

$$g_{yx} = \frac{\partial}{\partial y} (g_x) = 3 - 0 = 3$$

6). Let $w(x, y, z) = \frac{1}{\sqrt{x^2 + y^2 + z^2}}$, $(x, y, z) \neq (0, 0, 0)$ show that

$$\frac{\partial^2 w}{\partial x^2} + \frac{\partial^2 w}{\partial y^2} + \frac{\partial^2 w}{\partial z^2} = 0$$

$$w = (x^2 + y^2 + z^2)^{-1/2}$$

$$\frac{\partial w}{\partial x} = -\frac{1}{2} (x^2 + y^2 + z^2)^{-3/2} (2x)$$

$$\frac{\partial w}{\partial x} = -x (x^2 + y^2 + z^2)^{-3/2}$$

$$\frac{\partial^2 w}{\partial x^2} = -x \left(-\frac{3}{2} (x^2 + y^2 + z^2)^{-5/2} (2x) \right) + (x^2 + y^2 + z^2)^{-3/2} (-1)$$

$$= \frac{3x^2}{(x^2 + y^2 + z^2)^{5/2}} - \frac{1}{(x^2 + y^2 + z^2)^{3/2}}$$

$$= \frac{3x^2 - x^2 - y^2 - z^2}{(x^2 + y^2 + z^2)^{5/2}}$$

$$\frac{\partial^2 w}{\partial x^2} = \frac{2x^2 - y^2 - z^2}{(x^2 + y^2 + z^2)^{5/2}} \quad \text{--- ①}$$

similarly $\frac{\partial^2 w}{\partial y^2} = \frac{2y^2 - x^2 - z^2}{(x^2 + y^2 + z^2)^{5/2}} \quad \text{--- ②}$

$$\frac{\partial^2 w}{\partial z^2} = \frac{2z^2 - x^2 - y^2}{(x^2 + y^2 + z^2)^{5/2}} \quad \text{--- ③}$$

adding ①, ② & ③

$$\frac{\partial^2 w}{\partial x^2} + \frac{\partial^2 w}{\partial y^2} + \frac{\partial^2 w}{\partial z^2} = \frac{2x^2 - y^2 - z^2 + 2y^2 - x^2 - z^2 + 2z^2 - x^2 - y^2}{(x^2 + y^2 + z^2)^{5/2}}$$

$$= \frac{0}{(x^2 + y^2 + z^2)^{5/2}}$$

$$= 0$$

7) If $v(x,y) = e^x (x \cos y - y \sin y)$ then prove that $\frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2} = 0$

$$\frac{\partial v}{\partial x} = e^x [\cos y] + (x \cos y - y \sin y) [e^x]$$

$$\frac{\partial v}{\partial x} = e^x [\cos y + x \cos y - y \sin y]$$

$$\begin{aligned} \frac{\partial^2 v}{\partial x^2} &= e^x [\cos y] + (\cos y + x \cos y - y \sin y) [e^x] \\ &= e^x [\cos y + \cos y + x \cos y - y \sin y] \end{aligned}$$

$$\frac{\partial^2 v}{\partial x^2} = e^x [2 \cos y + x \cos y - y \sin y] \text{ --- (1)}$$

$$\frac{\partial v}{\partial y} = e^x [-x \sin y - y \cos y - \sin y]$$

$$\frac{\partial^2 v}{\partial y^2} = e^x [-x \cos y + y \sin y - \cos y - \cos y]$$

$$\frac{\partial^2 v}{\partial y^2} = e^x [-2 \cos y - x \cos y + y \sin y] \text{ --- (2)}$$

$$(1) + (2) \Rightarrow$$

$$\frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2} = [2 \cos y + x \cos y - y \sin y - 2 \cos y - x \cos y + y \sin y] e^x$$

$$\frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2} = 0$$

8) If $w(x,y) = xy + \sin(xy)$, then prove that $\frac{\partial^2 w}{\partial y \partial x} = \frac{\partial^2 w}{\partial x \partial y}$

$$\frac{\partial w}{\partial x} = y + \cos xy (y)$$

$$\frac{\partial w}{\partial x} = y(1 + \cos xy)$$

$$\frac{\partial^2 w}{\partial y \partial x} = y(0 - \sin xy (x) + (1 + \cos xy)(1))$$

$$\frac{\partial^2 w}{\partial y \partial x} = 1 - xy \sin xy + \cos xy \text{ --- (1)}$$

$$\frac{\partial w}{\partial y} = x + \cos xy (x)$$

$$\frac{\partial w}{\partial y} = x(1 + \cos xy)$$

$$\frac{\partial^2 w}{\partial x \partial y} = x(0 - \sin xy (y) + \cos xy (1))$$

$$\frac{\partial^2 w}{\partial x \partial y} = 1 - xy \sin xy + \cos xy \quad \text{--- (2)}$$

From (1) & (2) $\frac{\partial^2 w}{\partial y \partial x} = \frac{\partial^2 w}{\partial x \partial y}$ proved.

9) If $v(x, y, z) = x^3 + y^3 + z^3 + 3xyz$, show that $\frac{\partial^2 v}{\partial y \partial z} = \frac{\partial^2 v}{\partial z \partial y}$

$$\frac{\partial v}{\partial z} = 0 + 0 + 3z^2 + 3xy$$

$$\frac{\partial v}{\partial z} = 3z^2 + 3xy$$

$$\frac{\partial^2 v}{\partial y \partial z} = 3x \quad \text{--- (1)}$$

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$$\frac{\partial v}{\partial y} = 0 + 3y^2 + 0 + 3xz$$

$$\frac{\partial^2 v}{\partial z \partial y} = 3x \quad \text{--- (2)}$$

From (1), (2) $\frac{\partial^2 v}{\partial y \partial z} = \frac{\partial^2 v}{\partial z \partial y}$

10) A firm produces two types of calculators each week, x number of type A, and y number of type B. The weekly revenue and cost functions (in rupees) are $R(x, y) = 80x + 90y + 0.04xy - 0.05y^2$ and $C(x, y) = 8x + 6y + 2000$ respectively

(i) Find the profit function $P(x, y)$

(ii) Find $\frac{\partial P}{\partial x}(1200, 1800)$ and $\frac{\partial P}{\partial y}(1200, 1800)$ and interpret these results.

(i) Profit = Revenue - Cost

$$P(x, y) = R(x, y) - C(x, y)$$

$$= 80x + 90y + 0.04xy - 0.05y^2$$

$$- 8x - 6y - 2000$$

$$P(x, y) = 72x + 84y + 0.04xy - 0.05x^2 - 0.05y^2 - 2000$$

(ii) $\frac{\partial P}{\partial x} = 72 + 0.04y - 0.05(2x)$

$$\frac{\partial P}{\partial x}(1200, 1800) = 72 + (0.04)(1800) - 0.1(1200)$$

$$= 72 + 72 - 120$$

$$= 24$$

$$\frac{\partial P}{\partial y} = 84 + 0.04x - 0.05(2y) \quad \text{www.Padasalai.Net} \quad \text{www.TrbTnpsc.com}$$

$$\begin{aligned} \frac{\partial P}{\partial y}(1200, 1800) &= 84 + 0.04(1200) - 0.1(1800) \\ &= 84 + 48 - 180 \\ &= -48 \end{aligned}$$

$$\text{at } (1200, 1800) \quad \frac{\partial P}{\partial x} = 24, \quad \frac{\partial P}{\partial y} = -48$$

$\therefore$ keeping y constant and increase x values
Then we get profit increasing.

—x—

Linear approximation and Differential of several variables

① The Linear approximation of F at (x_0, y_0) is

$$L(x, y) = F(x_0, y_0) + \frac{\partial F}{\partial x}(x_0, y_0)(x - x_0) + \frac{\partial F}{\partial y}(x_0, y_0)(y - y_0)$$

② The differential of F is

$$dF = \frac{\partial F}{\partial x} dx + \frac{\partial F}{\partial y} dy$$

$$\text{where } \begin{aligned} dx &= \Delta x \\ dy &= \Delta y \end{aligned}$$

③ The linear approximation of F at $(x_0, y_0, z_0) \in A$ is defined to be

$$\begin{aligned} F(x, y, z) &= F(x_0, y_0, z_0) + \frac{\partial F}{\partial x}(x_0, y_0, z_0)(x - x_0) \\ &\quad + \frac{\partial F}{\partial y}(x_0, y_0, z_0)(y - y_0) + \frac{\partial F}{\partial z}(x_0, y_0, z_0)(z - z_0) \end{aligned}$$

④ The differential of F is

$$dF = \frac{\partial F}{\partial x} dx + \frac{\partial F}{\partial y} dy + \frac{\partial F}{\partial z} dz$$

$$\text{where } \begin{aligned} dx &= \Delta x \\ dy &= \Delta y \\ dz &= \Delta z \end{aligned}$$

Exercise 8.5

1) If $w(x, y) = x^3 - 3xy + 2y^2$, $x, y \in \mathbb{R}$, find the linear approximation for w at $(1, -1)$

$$\frac{\partial w}{\partial x} = 3x^2 - 3y$$

$$\frac{\partial w}{\partial y} = -3x + 4y$$

$$\frac{\partial w}{\partial x}(1, -1) = 3 + 3 = 6$$

$$\frac{\partial w}{\partial y}(1, -1) = -3 - 4 = -7$$

$$w(1, -1) = 1 - 3(1)(-1) + 2(-1)^2$$

$$= 1 + 3 + 2$$

$$= 6$$

$$\text{Linear approximation } L(x, y) = f(x_0, y_0) + \frac{\partial f}{\partial x}(x - x_0) + \frac{\partial f}{\partial y}(y - y_0)$$

$$L(x, y) = 6 + 6(x - 1) + (-7)(y + 1)$$

$$= 6 + 6x - 6 - 7y - 7$$

$$= 6x - 7y - 7$$

② Let $z(x, y) = x^2y + 3xy^4$, $x, y \in \mathbb{R}$. Find the linear approximation for z at $(2, -1)$

$$z(2, -1) = 2^2(-1) + 3(2)(-1)^4$$

$$= -4 + 6 = 2$$

$$\frac{\partial z}{\partial x} = 2xy + 3y^4$$

$$\frac{\partial z}{\partial x}(2, -1) = 2(2)(-1) + 3(-1)^4 = -4 + 3$$

$$= -1$$

$$\frac{\partial z}{\partial y} = x^2 + 12xy^3$$

$$\frac{\partial z}{\partial y}(2, -1) = 2^2 + 12(2)(-1)^3 = 4 - 24 = -20$$

Linear approximation

$$L(x, y) = z(2, -1) + \frac{\partial z}{\partial x}(2, -1)(x - 2) + \frac{\partial z}{\partial y}(2, -1)(y + 1)$$

$$= 2 + (-1)(x - 2) + (-20)(y + 1)$$

$$= 2 - x + 2 - 20y - 20$$

$$= -x - 20y + 16$$

$$L(x, y) = -(x + 20y - 16)$$

3) If $v(x, y) = x^2 - xy + \frac{1}{4}y^2 + 7$, $x, y \in \mathbb{R}$, find the differential dv

$$\frac{\partial v}{\partial x} = 2x - y$$

$$\frac{\partial v}{\partial y} = -x + \frac{y}{2}$$

$$dv = \frac{\partial v}{\partial x} dx + \frac{\partial v}{\partial y} dy$$

$$dv = (2x - y)dx + (-x + \frac{y}{2})dy$$

4) Let $w(x, y, z) = x^2 - xy + 3\sin z$, $x, y, z \in \mathbb{R}$. Find the linear approximation at $(2, -1, 0)$

$$w(2, -1, 0) = 2^2 - 2(-1) + 3\sin 0$$

$$= 4 + 2 + 0$$

$$= 6$$

$$\frac{\partial w}{\partial x} = 2x - y$$

$$\frac{\partial w}{\partial x}(2, -1, 0) = 2(2) + 1 = 5$$

$$\frac{\partial w}{\partial y} = 0 - x + 0$$

$$\frac{\partial w}{\partial y}(2, -1, 0) = -2$$

$$\frac{\partial w}{\partial z} = 0 + 0 + 3 \cos z$$

$$\frac{\partial w}{\partial z}(2, -1, 0) = 3 \cos 0 = 3$$

Linear approximation

$$L(x, y, z) = w(2, -1, 0) + \frac{\partial w}{\partial x}(2, -1, 0)(x-2) + \frac{\partial w}{\partial y}(2, -1, 0)(y+1) + \frac{\partial w}{\partial z}(2, -1, 0)(z-0)$$

$$= 6 + 5(x-2) - 2(y+1) + 3(z)$$

$$= 6 + 5x - 10 - 2y - 2 + 3z$$

$$L(x, y, z) = 5x - 2y + 3z - 6$$

5) Let $V(x, y, z) = xy + yz + zx$, $x, y, z \in \mathbb{R}$. Find the differential dv .

$$\frac{\partial v}{\partial x} = y + 0 + z = y + z$$

$$\frac{\partial v}{\partial y} = x + z + 0 = x + z$$

$$\frac{\partial v}{\partial z} = 0 + y + x = y + x$$

$$dv = \frac{\partial v}{\partial x} dx + \frac{\partial v}{\partial y} dy + \frac{\partial v}{\partial z} dz$$

$$dv = (y+z)dx + (x+z)dy + (x+y)dz$$

Function of Function Rule

① $w = f(x, y)$, x, y are function of t .

$$\frac{dw}{dt} = \frac{\partial w}{\partial x} \frac{dx}{dt} + \frac{\partial w}{\partial y} \frac{dy}{dt}$$

② $w = f(x, y, z)$, $x = x(s, t)$, $y = y(s, t)$

$$\frac{\partial w}{\partial s} = \frac{\partial w}{\partial x} \frac{\partial x}{\partial s} + \frac{\partial w}{\partial y} \frac{\partial y}{\partial s}$$

$$\frac{\partial w}{\partial t} = \frac{\partial w}{\partial x} \frac{\partial x}{\partial t} + \frac{\partial w}{\partial y} \frac{\partial y}{\partial t}$$

Exercise 8.6

1) If $u(x, y) = x^2y + 3xy^4$, $x = e^t$ and $y = \sin t$, find $\frac{du}{dt}$ and evaluate it at $t=0$.

$$\frac{\partial u}{\partial x} = 2xy + 3y^4$$

$$\frac{\partial u}{\partial y} = x^2 + 12xy^3$$

$$x = e^t$$

$$y = \sin t$$

$$\frac{dx}{dt} = e^t$$

$$\frac{dy}{dt} = \cos t$$

$$\frac{du}{dt} = \frac{\partial u}{\partial x} \frac{dx}{dt} + \frac{\partial u}{\partial y} \frac{dy}{dt}$$

$$= (2xy + 3y^4) e^t + (x^2 + 12xy^3) \cos t$$

$$\frac{du}{dt} = (2e^t \sin t + 3 \sin^4 t) e^t + (e^{2t} + 12e^t \sin^3 t) \cos t$$

put $t=0$ $\frac{du}{dt} = (2e^0 \sin 0 + 3 \sin^4 0) e^0 + (e^0 + 12e^0 \sin^3 0) \cos 0$

$$= (0+0)1 + (1+0)1$$

$$= 0+1$$

$\frac{du}{dt}$ at $t=0$ is 1

$$\frac{du}{dt} = e^t [2e^t \sin t + 3 \sin^4 t + e^t \cos t + 12 \sin^3 t \cos t]$$

$$\frac{du}{dt} = 1 \text{ at } t=0$$

② If $u(x,y,z) = xy^2z^3$, $x = \sin t$, $y = \cos t$, $z = 1 + e^{2t}$, find $\frac{du}{dt}$

$$u(x,y,z) = xy^2z^3$$

$$\frac{\partial u}{\partial x} = y^2 z^3$$

$$\frac{\partial u}{\partial y} = 2xy z^3$$

$$\frac{\partial u}{\partial z} = 3xy^2 z^2$$

$$x = \sin t$$

$$y = \cos t$$

$$z = 1 + e^{2t}$$

$$\frac{dx}{dt} = \cos t$$

$$\frac{dy}{dt} = -\sin t$$

$$\frac{dz}{dt} = 2e^{2t}$$

$$\frac{du}{dt} = \frac{\partial u}{\partial x} \frac{dx}{dt} + \frac{\partial u}{\partial y} \frac{dy}{dt} + \frac{\partial u}{\partial z} \frac{dz}{dt}$$

$$= (y^2 z^3) \cos t + 2xy z^3 (-\sin t) + 3xy^2 z^2 (2e^{2t})$$

$$\frac{du}{dt} = \cos^2 t (1 + e^{2t})^3 \cos t + 2 \sin t \cos t (1 + e^{2t})^3 (-\sin t) + 3 \sin t \cos^2 t (1 + e^{2t})^2 2e^{2t}$$

$$\frac{du}{dt} = (1 + e^{2t})^2 \left[\cos^3 t (1 + e^{2t}) - 2 \sin^2 t \cos t (1 + e^{2t}) + 6e^{2t} \sin t \cos^2 t \right]$$

$$= (1 + e^{2t})^2 \left[\cos^3 t (1 + e^{2t}) - \sin t \sin 2t (1 + e^{2t}) + 6e^{2t} \sin t \cos^2 t \right]$$

3) If $w(x, y, z) = x^2 + y^2 + z^2$, $x = e^t$, $y = e^t \sin t$ and $z = e^t \cos t$
find $\frac{dw}{dt}$

$$w = x^2 + y^2 + z^2$$

$$\frac{\partial w}{\partial x} = 2x$$

$$\frac{\partial w}{\partial y} = 2y$$

$$\frac{\partial w}{\partial z} = 2z$$

$$x = e^t$$

$$y = e^t \sin t$$

$$z = e^t \cos t$$

$$\frac{dx}{dt} = e^t$$

$$\frac{dy}{dt} = e^t \cos t + \sin t e^t = e^t (\cos t + \sin t)$$

$$\frac{dz}{dt} = e^t (-\sin t) + \cos t e^t = e^t (\cos t - \sin t)$$

$$\frac{dw}{dt} = \frac{\partial w}{\partial x} \frac{dx}{dt} + \frac{\partial w}{\partial y} \frac{dy}{dt} + \frac{\partial w}{\partial z} \frac{dz}{dt}$$

$$= 2x e^t + 2y e^t (\cos t + \sin t) + 2z e^t (\cos t - \sin t)$$

$$= 2e^t [x + y (\cos t + \sin t) + z (\cos t - \sin t)]$$

$$\frac{dw}{dt} = 2e^t [e^t + e^t \sin t (\cos t + \sin t) + e^t \cos t (\cos t - \sin t)]$$

$$= 2e^t e^t [1 + \sin t \cos t + \sin^2 t + \cos^2 t - \sin t \cos t]$$

$$= 2e^{2t} [1 + 1]$$

$$\frac{dw}{dt} = 4e^{2t}$$

4) Let $U(x, y, z) = xyz$, $x = e^{-t}$, $y = e^{-t} \cos t$, $z = \sin t$, t is τ

Find $\frac{dU}{dt}$

$$U = xyz$$

$$\frac{\partial U}{\partial x} = yz, \quad \frac{\partial U}{\partial y} = xz, \quad \frac{\partial U}{\partial z} = xy$$

$$x = e^{-t}$$

$$y = e^{-t} \cos t$$

$$z = \sin t$$

$$\frac{dx}{dt} = -e^{-t}$$

$$\frac{dy}{dt} = e^{-t} (-\sin t) + e^{-t} (-1) \cos t = -e^{-t} (\sin t + \cos t)$$

$$\frac{dz}{dt} = \cos t$$

$$\frac{dU}{dt} = \frac{\partial U}{\partial x} \frac{dx}{dt} + \frac{\partial U}{\partial y} \frac{dy}{dt} + \frac{\partial U}{\partial z} \frac{dz}{dt}$$

$$= yz (-e^{-t}) + xz (-e^{-t} (\sin t + \cos t)) + xy (\cos t)$$

$$= e^{-t} \cos t \sin t (-e^{-t}) + e^{-t} \sin t (-e^{-t} (\sin t + \cos t))$$

$$+ e^{-t} e^{-t} \cos t \cos t$$

(31)

$$= e^{-2t} [-\sin t \cos t - \sin t (\sin t + \cos t) + \cos^2 t]$$

$$= -e^{-2t} [\sin t \cos t + \sin^2 t + \sin t \cos t - \cos^2 t]$$

$$= -e^{-2t} [2 \sin t \cos t - (\cos^2 t - \sin^2 t)]$$

$$\frac{dw}{dt} = -e^{-2t} [\sin 2t - \cos 2t]$$

5) If $w(x, y) = 6x^3 - 3xy + 2y^2$, $x = e^s$, $y = \cos s$ $s \in \mathbb{R}$
find $\frac{dw}{ds}$, and evaluate at $s=0$

$$w = 6x^3 - 3xy + 2y^2$$

$$\frac{\partial w}{\partial x} = 18x^2 - 3y$$

$$\frac{\partial w}{\partial y} = -3x + 4y$$

$$x = e^s$$

$$y = \cos s$$

$$\frac{dx}{ds} = e^s$$

$$\frac{dy}{ds} = -\sin s$$

$$\frac{dw}{ds} = \frac{\partial w}{\partial x} \frac{dx}{ds} + \frac{\partial w}{\partial y} \frac{dy}{ds}$$

$$= (18x^2 - 3y)e^s + (-3x + 4y)(-\sin s)$$

$$= (18e^{2s} - 3\cos s)e^s + (-3e^s + 4\cos s)(-\sin s)$$

$$\frac{dw}{ds} = 18e^{3s} - 3e^s \cos s - 4\sin s \cos s + 3e^s \sin s$$

put $s=0$ $\frac{dw}{ds} = 18e^0 - 3e^0 \cos 0 - 4 \sin 0 \cos 0 + 3e^0 \sin 0$

$$= 18 - 3 - 0 + 0$$

$$\frac{dw}{ds} = 15$$

6) If $z(x, y) = x \tan^{-1}(xy)$, $x = t^2$, $y = se^t$, $s, t \in \mathbb{R}$
Find $\frac{\partial z}{\partial s}$, and $\frac{\partial z}{\partial t}$ at $s=t=1$

$$z = x \tan^{-1}(xy)$$

$$\frac{\partial z}{\partial x} = x \frac{1}{1+x^2y^2} y + \tan^{-1}(xy) (1)$$

$$\frac{\partial z}{\partial y} = x \frac{1}{1+x^2y^2} x$$

$$x = t^2$$

$$y = se^t$$

$$\frac{\partial x}{\partial s} = 0$$

$$\frac{\partial x}{\partial t} = 2t$$

$$\frac{\partial y}{\partial s} = e^t$$

$$\frac{\partial y}{\partial t} = se^t$$

$$\frac{\partial z}{\partial s} = \frac{\partial z}{\partial x} \frac{\partial x}{\partial s} + \frac{\partial z}{\partial y} \frac{\partial y}{\partial s}$$

$$= \left(\frac{xy}{1+x^2y^2} + \tan^{-1} xy \right) 0 + \left(\frac{x^2}{1+x^2y^2} \right) e^t$$

$$= \frac{x^2 e^t}{1+x^2y^2}$$

$$\frac{\partial z}{\partial s} = \frac{t^4 e^t}{1+t^4 e^{2ts^2}}$$

$$\text{put } s=t=1 \quad \frac{\partial z}{\partial s} = \frac{1 \cdot e^1}{1+1 \cdot e^{2(1)}} = \left(\frac{e}{1+e^2} \right)$$

$$\frac{\partial z}{\partial t} = \frac{\partial z}{\partial x} \frac{\partial x}{\partial t} + \frac{\partial z}{\partial y} \frac{\partial y}{\partial t}$$

$$= \left(\frac{xy}{1+x^2y^2} + \tan^{-1} xy \right) 2t + \left(\frac{x^2}{1+x^2y^2} \right) s e^t$$

$$= \left(\frac{t^2 s e^t}{1+t^4 s^2 e^{2t}} + \tan^{-1} (t^2 s e^t) \right) 2t + \left(\frac{t^4}{1+t^4 s^2 e^{2t}} \right) s e^t$$

$$\text{put } s=t=1$$

$$\frac{\partial z}{\partial t} = \left(\frac{1 \cdot e}{1+e^2} + \tan^{-1} e \right) 2 + \left(\frac{1}{1+e^2} \right) e$$

$$= \frac{2e}{1+e^2} + 2 \tan^{-1} e + \frac{e}{1+e^2}$$

$$\frac{\partial z}{\partial t} = \frac{3e}{1+e^2} + 2 \tan^{-1} e$$

7) Let $U(x,y) = e^x \sin y$ where $x = st^2$, $y = s^2 t$, $s, t \in \mathbb{R}$

Find $\frac{\partial U}{\partial s}$, $\frac{\partial U}{\partial t}$ and evaluate them at $s=t=1$

$$U = e^x \sin y$$

$$\frac{\partial U}{\partial x} = e^x \sin y$$

$$\frac{\partial U}{\partial y} = e^x \cos y$$

$$x = st^2$$

$$y = s^2 t$$

$$\frac{\partial x}{\partial s} = t^2$$

$$\frac{\partial x}{\partial t} = 2st$$

$$\frac{\partial y}{\partial s} = 2st$$

$$\frac{\partial y}{\partial t} = s^2$$

$$\frac{\partial U}{\partial s} = \frac{\partial U}{\partial x} \frac{\partial x}{\partial s} + \frac{\partial U}{\partial y} \frac{\partial y}{\partial s}$$

$$= e^x \sin y (t^2) + e^x \cos y (2st)$$

$$= e^x (t^2 \sin y + 2st \cos y)$$

$$\frac{\partial U}{\partial s} = e^{st^2} (t^2 \sin s^2 t + 2st \cos s^2 t)$$

$$= t e^{st^2} (t \sin s^2 t + 2s \cos s^2 t)$$

$$\frac{\partial u}{\partial s} = 1e^{(1)} [\sin 1 + 2(1)\cos 1]$$

$$\frac{\partial u}{\partial s} = e(\sin 1 + \cos 1)$$

$$(ii) \frac{\partial u}{\partial t} = \frac{\partial u}{\partial x} \frac{\partial x}{\partial t} + \frac{\partial u}{\partial y} \frac{\partial y}{\partial t}$$

$$= e^x \sin y (2st) + e^x \cos y (s^2)$$

$$= se^x (2t \sin y + s \cos y)$$

$$\frac{\partial u}{\partial t} = se^{st^2} (2t \sin s^2 t + s \cos s^2 t)$$

$$\text{put } s=t=1 \quad \frac{\partial u}{\partial t} = e^1 (2 \sin 1 + \cos 1)$$

8) Let $z(x, y) = x^3 - 3x^2y^3$, where $x = se^t$, $y = se^{-t}$ $s, t \in \mathbb{R}$

Find $\frac{\partial z}{\partial s}$ and $\frac{\partial z}{\partial t}$

$$z = x^3 - 3x^2y^3$$

$$\frac{\partial z}{\partial x} = 3x^2 - 6xy^3$$

$$\frac{\partial z}{\partial y} = 0 - 9x^2y^2$$

$$x = se^t$$

$$y = se^{-t}$$

$$\frac{\partial x}{\partial s} = e^t \quad \frac{\partial x}{\partial t} = se^t$$

$$\frac{\partial y}{\partial s} = e^{-t} \quad \frac{\partial y}{\partial t} = -se^{-t}$$

$$(i) \frac{\partial z}{\partial s} = \frac{\partial z}{\partial x} \frac{\partial x}{\partial s} + \frac{\partial z}{\partial y} \frac{\partial y}{\partial s}$$

$$= (3x^2 - 6xy^3) e^t + (-9x^2y^2) e^{-t}$$

$$= (3s^2e^{2t} - 6se^t s^3e^{-3t}) e^t + (-9s^2e^{2t} s^2e^{-2t}) e^{-t}$$

$$= 3s^2e^t [e^{2t} - 2e^{-2t} s^2 - 3s^2e^{-2t}]$$

$$\frac{\partial z}{\partial s} = 3s^2e^t [e^{2t} - 5e^{-2t} s^2]$$

$$(ii) \frac{\partial z}{\partial t} = \frac{\partial z}{\partial x} \frac{\partial x}{\partial t} + \frac{\partial z}{\partial y} \frac{\partial y}{\partial t}$$

$$= (3x^2 - 6xy^3) se^t + (-9x^2y^2) (-se^{-t})$$

$$= (3s^2e^{2t} - 6se^t s^3e^{-3t}) se^t + 9s^2e^{2t} s^2e^{-2t} se^{-t}$$

$$= (3s^2e^{2t} - 6s^4e^t e^{-3t}) se^t + 9s^5e^{-t}$$

$$= 3s^3e^{3t} - 6s^5e^{-t} + 9s^5s^{-t}$$

$$= 3s^3e^{3t} + 3s^5e^{-t}$$

9) $w(x, y, z) = xy + yz + zx$ $x = u - v, y = uv, z = u + v, u, v \in \mathbb{R}$
Find $\frac{\partial w}{\partial u}, \frac{\partial w}{\partial v}$, and evaluate them at $(\frac{1}{2}, 1)$

$$w = xy + yz + zx$$

$$\frac{\partial w}{\partial x} = y + z$$

$$\frac{\partial w}{\partial y} = x + z$$

$$\frac{\partial w}{\partial z} = y + x$$

$$x = u - v$$

$$y = uv$$

$$z = u + v$$

$$\frac{\partial x}{\partial u} = 1, \frac{\partial x}{\partial v} = -1$$

$$\frac{\partial y}{\partial u} = v, \frac{\partial y}{\partial v} = u$$

$$\frac{\partial z}{\partial u} = 1, \frac{\partial z}{\partial v} = 1$$

$$(i) \frac{\partial w}{\partial u} = \frac{\partial w}{\partial x} \frac{\partial x}{\partial u} + \frac{\partial w}{\partial y} \frac{\partial y}{\partial u} + \frac{\partial w}{\partial z} \frac{\partial z}{\partial u}$$

$$= (y + z)(1) + (x + z)v + (y + x)1$$

$$= uv + u + v + (u - v + u + v)v + (uv + u - v)$$

$$= 2uv + 2u + 2uv$$

$$= 4uv + 2u$$

$$\frac{\partial w}{\partial u} = 2u(2v + 1)$$

$$\frac{\partial w}{\partial u}(\frac{1}{2}, 1) = 2 \cdot \frac{1}{2} (2 + 1) = 3$$

$$(ii) \frac{\partial w}{\partial v} = \frac{\partial w}{\partial x} \frac{\partial x}{\partial v} + \frac{\partial w}{\partial y} \frac{\partial y}{\partial v} + \frac{\partial w}{\partial z} \frac{\partial z}{\partial v}$$

$$= (y + z)(-1) + (x + z)u + (y + x)(1)$$

$$= -y - z + (x + z)u + y + x$$

$$= -u - v + (u - v + u + v)u + u - v$$

$$= -2v + 2u^2$$

$$\frac{\partial w}{\partial v} = 2(u^2 - v)$$

$$\frac{\partial w}{\partial v}(\frac{1}{2}, 1) = 2(\frac{1}{4} - 1)$$

$$= \frac{1}{2} - 2$$

$$= -\frac{3}{2}$$

Homogeneous Functions and Euler's Theorem

① F is a homogeneous function

If $\psi \quad F(\lambda x, \lambda y) = \lambda^p F(x, y)$

or

(ii) $F(\lambda x, \lambda y, \lambda z) = \lambda^p F(x, y, z) \quad \forall \lambda \in \mathbb{R}$

p is called degree of F

② Euler's Theorem

F is homogeneous on A with degree p then

$$x \frac{\partial f}{\partial x} + y \frac{\partial f}{\partial y} = n f$$

Exercise 8.7

1). In each of the following cases the function homogeneous or not. If it is so, find the degree.

(i) $f(x, y) = x^2 y + 6x^3 + 7$

$$f(\lambda x, \lambda y) = (\lambda x)^2 \lambda y + 6(\lambda x)^3 + 7$$

$$= \lambda^2 x^2 y + \lambda^3 6x^3 + 7$$

we cannot take λ^3 out side as common

$\therefore F$ is not a homogeneous function.

(ii) $h(x, y) = \frac{6x^2 y^3 - \pi y^5 + 9x^4 y}{2020x^2 + 2019y^2}$

$$h(\lambda x, \lambda y) = \frac{6\lambda^2 x^2 \lambda^3 y^3 - \pi \lambda^5 y^5 - 9\lambda^4 x^4 \lambda y}{2020\lambda^2 x^2 + 2019\lambda^2 y^2}$$

$$= \frac{\lambda^5 (6x^2 y^3 - \pi y^5 + 9x^4 y)}{\lambda^2 (2020x^2 + 2019y^2)}$$

$$h(\lambda x, \lambda y) = \lambda^3 h(x, y)$$

$h(x, y)$ is a homogeneous function of degree = 3

(iii) $g(x, y, z) = \frac{\sqrt{3x^2 + 5y^2 + z^2}}{4x + 7y}$

$$g(\lambda x, \lambda y, \lambda z) = \frac{\sqrt{3\lambda^2 x^2 + 5\lambda^2 y^2 + \lambda^2 z^2}}{4\lambda x + 7\lambda y}$$

$$= \frac{\lambda \sqrt{3x^2 + 5y^2 + z^2}}{\lambda (4x + 7y)}$$

$$g(\lambda x, \lambda y, \lambda z) = \lambda^0 g(x, y, z)$$

g is a homogeneous function degree = 0.

$$(iv) \quad u(x, y, z) = xy + \sin\left(\frac{y^2 - 2z^2}{xy}\right)$$

$$u(\lambda x, \lambda y, \lambda z) = \lambda^2 xy + \sin\left(\frac{\lambda^2 y^2 - 2\lambda^2 z^2}{\lambda x \lambda y}\right)$$

$$= \lambda^2 xy + \sin\left(\frac{\lambda^2 (y^2 - 2z^2)}{\lambda^2 xy}\right)$$

$$= \lambda^2 xy + \sin\left(\frac{y^2 - 2z^2}{xy}\right)$$

we cannot take λ^2 outside the function as common.

$\therefore u(x, y, z)$ is not homogeneous.

2) prove that $f(x, y) = x^3 - 2x^2y + 3xy^2 + y^3$ is homogeneous what is the degree? verify Euler's Theorem for f .

$$f(\lambda x, \lambda y) = \lambda^3 x^3 - 2\lambda^2 x^2 \lambda y + 3\lambda x \lambda^2 y^2 + \lambda^3 y^3$$

$$= \lambda^3 (x^3 - 2x^2y + 3xy^2 + y^3)$$

$$f(\lambda x, \lambda y) = \lambda^3 f(x, y)$$

f is a homogeneous function of degree 3.

By Euler's Theorem

$$x \frac{\partial f}{\partial x} + y \frac{\partial f}{\partial y} = 3f$$

verification:-

$$\frac{\partial f}{\partial x} = 3x^2 - 4xy + 3y^2$$

$$x \frac{\partial f}{\partial x} = 3x^3 - 4x^2y + 3xy^2 \quad \text{--- (1)}$$

$$\frac{\partial f}{\partial y} = 0 - 2x^2 + 6xy + 3y^2$$

$$y \frac{\partial f}{\partial y} = -2x^2y + 6xy^2 + 3y^3 \quad \text{--- (2)}$$

$$(1) + (2) \Rightarrow x \frac{\partial f}{\partial x} + y \frac{\partial f}{\partial y} = 3x^3 - 4x^2y + 3xy^2 - 2x^2y + 6xy^2 + 3y^3$$

$$= 3x^3 - 6x^2y + 9xy^2 + 3y^3$$

$$= 3(x^3 - 2x^2y + 3xy^2 + y^3)$$

$$x \frac{\partial f}{\partial x} + y \frac{\partial f}{\partial y} = 3f$$

verified.

3) prove that $g(x,y) = x \log \frac{y}{x}$ is homogeneous, what is the degree? verify Euler's Theorem for g .

$$g(x,y) = x \log \left(\frac{y}{x} \right)$$

$$g(\lambda x, \lambda y) = \lambda x \log \left(\frac{\lambda y}{\lambda x} \right)$$

$$= \lambda \left(x \log \frac{y}{x} \right)$$

$$g(\lambda x, \lambda y) = \lambda^1 g(x,y)$$

g is a homogeneous function of degree $n=1$

$\therefore$ By Euler's Theorem

$$x \frac{\partial g}{\partial x} + y \frac{\partial g}{\partial y} = 1g$$

Verification

$$\frac{\partial g}{\partial x} = x \left(\log \left(\frac{y}{x} \right) \right)' + \log \left(\frac{y}{x} \right) (x)'$$

$$= x \cdot \frac{1}{y/x} \left(-\frac{y}{x^2} \right) + \log \frac{y}{x} (1)$$

$$= -\frac{x^2}{y} \left(\frac{y}{x^2} \right) + \log \frac{y}{x}$$

$$\frac{\partial g}{\partial x} = -1 + \log \frac{y}{x}$$

$$x \frac{\partial g}{\partial x} = -x + x \log \frac{y}{x} \quad (1)$$

$$\frac{\partial g}{\partial y} = x \cdot \frac{1}{y/x} \left(\frac{1}{x} \right)$$

$$= \frac{x^2}{y} \left(\frac{1}{x} \right) = \frac{x}{y}$$

$$y \frac{\partial g}{\partial y} = x \quad (2)$$

$$(1) + (2) \Rightarrow x \frac{\partial g}{\partial x} + y \frac{\partial g}{\partial y} = \frac{-x + x \log \frac{y}{x}}{+x}$$

$$x \frac{\partial g}{\partial x} + y \frac{\partial g}{\partial y} = 1g$$

Euler's Theorem verified.

4) If $u(x,y) = \frac{x^2+y^2}{\sqrt{x+y}}$ prove that $x \frac{\partial u}{\partial x} + y \frac{\partial u}{\partial y} = \frac{3}{2} u$.

$$u(\lambda x, \lambda y) = \frac{\lambda^2 x^2 + \lambda^2 y^2}{\sqrt{\lambda x + \lambda y}}$$

$$= \frac{\lambda^2 (x^2 + y^2)}{\lambda^{1/2} \sqrt{x+y}}$$

(38)

$$u(\lambda x, \lambda y) = \lambda^{3/2} u(x, y)$$

u is a homogeneous function of degree $n = 3/2$

$\therefore$ By Euler's Theorem

$$x \frac{\partial u}{\partial x} + y \frac{\partial u}{\partial y} = \frac{3}{2} u //$$

5) If $v(x, y) = \log \left(\frac{x^2 + y^2}{x + y} \right)$, prove that $x \frac{\partial v}{\partial x} + y \frac{\partial v}{\partial y} = 1$

M-I

v is not a homogeneous function

$$v = \log \left(\frac{x^2 + y^2}{x + y} \right)$$

taking antilog

$$e^v = \frac{x^2 + y^2}{x + y} = u \text{ (say)}$$

$$u(\lambda x, \lambda y) = \frac{\lambda^2 (x^2 + y^2)}{\lambda (x + y)}$$

$$u(\lambda x, \lambda y) = \lambda^1 u(x, y)$$

u is a homogeneous function of degree $n = 1$

$\therefore$ By Euler's Theorem

$$x \frac{\partial u}{\partial x} + y \frac{\partial u}{\partial y} = n u$$

$$x \frac{\partial}{\partial x} (e^v) + y \frac{\partial}{\partial y} (e^v) = 1 e^v$$

$$x e^v \frac{\partial v}{\partial x} + y e^v \frac{\partial v}{\partial y} = e^v$$

divide by e^v $x \frac{\partial v}{\partial x} + y \frac{\partial v}{\partial y} = 1$

M-II

$$v = \log (x^2 + y^2) - \log (x + y)$$

$$\frac{\partial v}{\partial x} = \frac{1}{x^2 + y^2} 2x - \frac{1}{x + y} \quad (1)$$

$$x \frac{\partial v}{\partial x} = \frac{2x^2}{x^2 + y^2} - \frac{x}{x + y}$$

similarly $y \frac{\partial v}{\partial y} = \frac{2y^2}{x^2 + y^2} - \frac{y}{x + y}$

$$x \frac{\partial v}{\partial x} + y \frac{\partial v}{\partial y} = \frac{2x^2 + 2y^2}{x^2 + y^2} - \left(\frac{x + y}{x + y} \right)$$

$$= 2 \left(\frac{x^2 + y^2}{x^2 + y^2} \right) - 1$$

$$= 2 - 1 = 1$$

Hence proved

⑥ If $w(x, y, z) = \log \left(\frac{5x^3y^4 + 7y^2xz^4 - 75y^3z^4}{x^2 + y^2} \right)$

find. $x \frac{\partial w}{\partial x} + y \frac{\partial w}{\partial y} + z \frac{\partial w}{\partial z}$

w is not a homogeneous function

$$w = \log \left(\frac{5x^3y^4 + 7y^2xz^4 - 75y^3z^4}{x^2 + y^2} \right)$$

taking anti log

$$e^w = \frac{5x^3y^4 + 7y^2xz^4 - 75y^3z^4}{x^2 + y^2} = u \text{ (say)}$$

$$u(\lambda x, \lambda y, \lambda z) = \frac{\lambda^7 (5x^3y^4 + 7y^2xz^4 - 75y^3z^4)}{\lambda^2 (x^2 + y^2)}$$

$$u(\lambda x, \lambda y, \lambda z) = \lambda^5 u(x, y, z)$$

u is a homogeneous function of degree 5

$\therefore$ By Euler's Theorem

$$x \frac{\partial u}{\partial x} + y \frac{\partial u}{\partial y} + z \frac{\partial u}{\partial z} = 5u$$

$$x \frac{\partial}{\partial x} (e^w) + y \frac{\partial}{\partial y} (e^w) + z \frac{\partial}{\partial z} (e^w) = 5e^w$$

$$x e^w \frac{\partial w}{\partial x} + y e^w \frac{\partial w}{\partial y} + z e^w \frac{\partial w}{\partial z} = 5e^w$$

divide by e^w

$$x \frac{\partial w}{\partial x} + y \frac{\partial w}{\partial y} + z \frac{\partial w}{\partial z} = 5 //$$

need suggestions

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9. APPLICATIONS OF INTEGRATION

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$$\sum_{i=1}^n f(\xi_i) (x_i - x_{i-1}) = f(\xi_1)(x_1 - x_0) + f(\xi_2)(x_2 - x_1) + \dots + f(\xi_n)(x_n - x_{n-1}) \quad \text{--- ①}$$

is called Riemann sum. of $f(x)$

$$* \int_a^b f(x) dx = \lim_{n \rightarrow \infty \text{ and } \max(x_i - x_{i-1}) \rightarrow 0} \sum_{i=1}^n f(x_{i-1}) (x_i - x_{i-1})$$

is known as left-end Rule

$$* \int_a^b f(x) dx = \lim_{n \rightarrow \infty \text{ and } \max(x_i - x_{i-1}) \rightarrow 0} \sum_{i=1}^n f(x_i) (x_i - x_{i-1})$$

is known as Right-end Rule.

$$\int_a^b f(x) dx = \lim_{n \rightarrow \infty \text{ and } \max(x_i - x_{i-1}) \rightarrow 0} \sum_{i=1}^n f\left(\frac{x_{i-1} + x_i}{2}\right) (x_i - x_{i-1})$$

is known as mid-point Rule.

Exercise 9.1

- 1) Find an approximate value of $\int_1^{1.5} x dx$ by applying the left end Rule with the partition $\{1.1, 1.2, 1.3, 1.4, 1.5\}$

$$n=5$$

$$x_0=1, x_1=1.1, x_2=1.2, x_3=1.3, x_4=1.4, x_5=1.5$$

$$\Delta x = 1.1 - 1 = 0.1$$

$$\text{left end Rule } \int_a^b f(x) dx \approx f(x_0)\Delta x + f(x_1)\Delta x + f(x_2)\Delta x + f(x_3)\Delta x + f(x_4)\Delta x$$

$$\int_1^{1.5} x dx \approx (f(1) + f(1.1) + f(1.2) + f(1.3) + f(1.4)) \times 0.1$$

$$= (1 + 1.1 + 1.2 + 1.3 + 1.4) \times 0.1$$

$$= 6 \times 0.1$$

$$\int_1^{1.5} x dx \approx 0.6$$

$$(1.0) \left[\left(\frac{2.1+1.1}{2} \right) + \left(\frac{1.1+1.1}{2} \right) + \left(\frac{1.1+1.1}{2} \right) + \left(\frac{1.1+1.1}{2} \right) + \left(\frac{1.1+1.1}{2} \right) \right] = 6$$

$$\int_1^{1.5} x dx = \left[\frac{x^2}{2} \right]_1^{1.5} = \frac{2.25 - 1}{2} = 0.625$$

2) Find an approximate value of $\int_1^{1.5} x^2 dx$ by applying the right-end rule with the partitions $\{1.1, 1.2, 1.3, 1.4, 1.5\}$

$$n=5$$

$$x_0=1, x_1=1.1, x_2=1.2, x_3=1.3, x_4=1.4, x_5=1.5$$

$$\Delta x = 1.1 - 1.0 = 0.1$$

Right end Rule

$$\int_a^b f(x) dx = (f(x_1) \Delta x + f(x_2) \Delta x + f(x_3) \Delta x + f(x_4) \Delta x + f(x_5) \Delta x)$$

$$\begin{aligned} \int_1^{1.5} x^2 dx &\approx S = (f(1.1) + f(1.2) + f(1.3) + f(1.4) + f(1.5)) \cdot 0.1 \\ &= (1.1^2 + 1.2^2 + 1.3^2 + 1.4^2 + 1.5^2) \cdot 0.1 \\ &= (1.21 + 1.44 + 1.69 + 1.96 + 2.25) \cdot 0.1 \\ &= 8.55 \times 0.1 \end{aligned}$$

$$\int_1^{1.5} x^2 dx \approx S = 0.855$$

$$\int_1^{1.5} x^2 dx = \left[\frac{x^3}{3} \right]_1^{1.5} = \frac{1.5^3 - 1}{3} = \frac{3.375 - 1}{3} = \frac{2.375}{3} = 0.7916$$

3) Find an approximate value of $\int_1^{1.5} (2-x) dx$ by applying the mid point rule with the partition $\{1.1, 1.2, 1.3, 1.4, 1.5\}$

$$n=5$$

$$x_0=1, x_1=1.1, x_2=1.2, x_3=1.3, x_4=1.4, x_5=1.5$$

$$\Delta x = 1.1 - 1.0 = 0.1$$

$$f(x) = 2 - x$$

mid point rule

$$\int_a^b f(x) dx = \left[f\left(\frac{x_0+x_1}{2}\right) + f\left(\frac{x_1+x_2}{2}\right) + \dots + f\left(\frac{x_{n-1}+x_n}{2}\right) \right] \Delta x$$

$$S = \left[f\left(\frac{1+1.1}{2}\right) + f\left(\frac{1.1+1.2}{2}\right) + f\left(\frac{1.2+1.3}{2}\right) + f\left(\frac{1.3+1.4}{2}\right) + f\left(\frac{1.4+1.5}{2}\right) \right] (0.1)$$

$$S = [f(1.05) + f(1.15) + f(1.25) + f(1.35) + f(1.45)] (0.1)$$

$$S = ((2-1.05) + (2-1.15) + (2-1.25) + (2-1.35) + (2-1.45)) \times 0.1$$

$$= (0.95 + 0.85 + 0.75 + 0.65 + 0.55) \times 0.1$$

$$= (3.75) \times 0.1$$

$$S = 0.375 \quad \int_1^{1.5} (2-x) dx \approx 0.375$$

limit formula to Evaluate $\int_a^b f(x) dx$

$$= \left[-\frac{(2-x)^2}{2} \right]_1^{1.5}$$

$$= -\frac{0.25}{2} + \frac{1}{2} = \frac{0.75}{2} = 0.375$$

$$\int_a^b f(x) dx = \lim_{n \rightarrow \infty} \frac{b-a}{n} \sum_{r=1}^n f\left(a + (b-a) \frac{r}{n}\right)$$

$$a=0, b=1$$

$$\int_0^1 f(x) dx = \lim_{n \rightarrow \infty} \frac{1}{n} \sum_{r=1}^n f\left(\frac{r}{n}\right)$$

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Exercise 9.2

1) Evaluate the following integrals as the limit of sums.

i) $\int_0^1 (5x+4) dx$

$$\int_0^1 f(x) dx = \lim_{n \rightarrow \infty} \frac{1}{n} \sum_{r=1}^n f\left(\frac{r}{n}\right)$$

$$f(x) = 5x+4$$

$$f\left(\frac{r}{n}\right) = \frac{5r}{n} + 4$$

$$\sum_{r=1}^n f\left(\frac{r}{n}\right) = \sum_{r=1}^n \frac{5r}{n} + 4$$

$$= \sum_{r=1}^n \frac{5r}{n} + \sum_{r=1}^n 4$$

$$= \frac{5}{n} \sum_{r=1}^n r + 4 \sum_{r=1}^n (1)$$

$$= \frac{5}{n} (1+2+3+\dots+n) + 4(1+1+\dots+1)$$

$$= \frac{5}{n} \frac{n(n+1)}{2} + 4n$$

$$\sum_{r=1}^n r = \frac{n(n+1)}{2}$$

$$\sum_{r=1}^n f\left(\frac{r}{n}\right) = \frac{5}{2}(n+1) + 4n$$

$$\lim_{n \rightarrow \infty} \frac{1}{n} \sum_{r=1}^n f\left(\frac{r}{n}\right) = \lim_{n \rightarrow \infty} \frac{1}{n} \left(\frac{5}{2}(n+1) + 4n \right)$$

$$= \lim_{n \rightarrow \infty} \frac{5}{2} \left(1 + \frac{1}{n}\right) + 4$$

$$= \frac{5}{2}(1+0) + 4$$

$$= \frac{5}{2} + 4 = \frac{13}{2}$$

$$\therefore \int_0^1 (5x+4) dx = \frac{13}{2}$$

Checking
 $= \left(\frac{5x^2}{2} + 4x \right)_0^1$
 $= \frac{5}{2} + 4 = \frac{13}{2}$

$$(ii) \int_1^2 (4x^2-1) dx.$$

$$a=1, \quad b=2 \quad f(x)=4x^2-1$$

$$\int_a^b f(x) dx = \lim_{n \rightarrow \infty} \frac{b-a}{n} \sum_{r=1}^n f\left(a + (b-a)\frac{r}{n}\right)$$

$$= \lim_{n \rightarrow \infty} \frac{2-1}{n} \sum_{r=1}^n f\left(1 + (2-1)\frac{r}{n}\right)$$

$$\int_a^b f(x) dx = \lim_{n \rightarrow \infty} \frac{1}{n} \sum_{r=1}^n f\left(1 + \frac{r}{n}\right)$$

$$f(x) = 4x^2 - 1$$

$$f\left(1 + \frac{r}{n}\right) = 4\left(1 + \frac{r}{n}\right)^2 - 1$$

$$= 4\left(1 + \frac{r^2}{n^2} + \frac{2r}{n}\right) - 1$$

$$\sum_{r=1}^n f\left(1 + \frac{r}{n}\right) = 4 \sum_{r=1}^n \left(1 + \frac{r^2}{n^2} + \frac{2r}{n}\right) - \sum_{r=1}^n 1$$

$$= 4 \sum_{r=1}^n 1 + \frac{4}{n^2} \sum_{r=1}^n r^2 + \frac{8}{n} \sum_{r=1}^n r - n$$

$$= 4n + \frac{4}{n^2} \frac{n(n+1)(2n+1)}{6} + \frac{8}{n} \frac{n(n+1)}{2} - n$$

$$= 3n + \frac{2}{3} \frac{1}{n} (n+1)(2n+1) + 4(n+1)$$

$$\lim_{n \rightarrow \infty} \frac{1}{n} \sum_{r=1}^n f\left(1 + \frac{r}{n}\right) = \lim_{n \rightarrow \infty} \frac{1}{n} \left(3n + \frac{2}{3} \frac{1}{n} (n+1)(2n+1) + 4(n+1) \right)$$

$$= \lim_{n \rightarrow \infty} \left(3 + \frac{2}{3} \left(1 + \frac{1}{n} \right) \left(2 + \frac{1}{n} \right) + 4 \left(1 + \frac{1}{n} \right) \right)$$

$$= 3 + \frac{2}{3} (1+0)(2+0) + 4(1+0)$$

$$= 3 + \frac{4}{3} + 4 = 7 + \frac{4}{3}$$

$$\int_1^2 (4x^2 - 1) dx = \frac{25}{3}$$

checking

$$\begin{aligned} \int_1^2 (4x^2 - 1) dx &= \left[\frac{4x^3}{3} - x \right]_1^2 \\ &= \left(\frac{4}{3} \cdot 8 - 2 \right) - \left(\frac{4}{3} - 1 \right) \\ &= \frac{28}{3} - 1 = \frac{25}{3} \end{aligned}$$

Fundamental Theorems of Integral calculus and their Applications

* First Fundamental Theorem of Integral calculus

If $f(x)$ be a continuous function defined on a closed interval $[a, b]$ and $F(x) = \int_a^x f(u) du$, $a < x < b$ then $\frac{d}{dx} F(x) = f(x)$.

* Second Fundamental Theorem of Integral calculus

If $f(x)$ be a continuous function defined on a closed interval $[a, b]$ and $F(x)$ is an anti-derivative of $f(x)$, Then $\int_a^b f(x) dx = F(b) - F(a)$.

Properties

$$1) \int_a^b f(x) dx = \int_a^b f(u) du, \quad a < b$$

$$2) \int_a^b f(x) dx = - \int_b^a f(x) dx.$$

$$3) \int_a^b f(x) dx = \int_a^c f(x) dx + \int_c^b f(x) dx, \quad a < c < b$$

$$4) \int_a^b (\alpha f(x) + \beta g(x)) dx = \alpha \int_a^b f(x) dx + \beta \int_a^b g(x) dx.$$

where α, β are constants

$$5) \int_a^b f(x) dx = \int_c^d f(g(u)) \frac{dg(u)}{du} du \quad \text{where } g(c) = a, g(d) = b, x = g(u)$$

6) $\int_a^b f(x) dx = \int_a^b f(a+b-x) dx$

7) $\int_0^{2a} f(x) dx = \int_0^a (f(x) + f(2a-x)) dx$

8) $f(x)$ is an even function $\int_{-a}^a f(x) dx = 2 \int_0^a f(x) dx$

9) $f(x)$ is an odd function $\int_{-a}^a f(x) dx = 0$

10) If $f(2a-x) = f(x)$ then $\int_0^{2a} f(x) dx = 2 \int_0^a f(x) dx$

11) If $f(2a-x) = -f(x)$ then $\int_0^{2a} f(x) dx = 0$

12) $\int_0^a x f(x) dx = \frac{a}{2} \int_0^a f(x) dx$ if $f(a-x) = f(x)$

Exercise 9.3

1) Evaluate the following definite Integrals.

(1) $\int_3^4 \frac{dx}{x^2-4}$

$I = \int_3^4 \frac{dx}{x^2-2^2}$

$= \left[\frac{1}{2(2)} \log \left| \frac{x-2}{x+2} \right| \right]_3^4$

$\int \frac{dx}{x^2-a^2} = \frac{1}{2a} \log \left| \frac{x-a}{x+a} \right| + c$
a=2

$= \frac{1}{4} \left[\log \left| \frac{4-2}{4+2} \right| - \log \left| \frac{3-2}{3+2} \right| \right]$

$= \frac{1}{4} \left[\log \left(\frac{2}{6} \right) - \log \left(\frac{1}{5} \right) \right] = \frac{1}{4} \log \left(\frac{2}{6} \times \frac{5}{1} \right)$

$= \frac{1}{4} \log \frac{5}{3}$

(ii)

$$\int_{-1}^1 \frac{dx}{x^2+2x+5}$$

$$I = \int_{-1}^1 \frac{dx}{x^2+2x+5}$$

$$x^2+2x+5 = x^2+2x+1+4 = (x+1)^2+4 = (x+1)^2+2^2$$

$$I = \int_{-1}^1 \frac{dx}{2^2+(x+1)^2}$$

$$\boxed{\int \frac{dx}{a^2+x^2} = \frac{1}{a} \tan^{-1} \frac{x}{a} + c}$$

$$a=2, \quad x \rightarrow x+1$$

$$I = \left[\frac{1}{2} \tan^{-1} \frac{x+1}{2} \right]_{-1}^1$$

$$= \frac{1}{2} \left[\tan^{-1} \frac{1+1}{2} - \tan^{-1} \frac{-1+1}{2} \right]$$

$$= \frac{1}{2} \left[\tan^{-1} \frac{2}{2} - \tan^{-1} \frac{0}{2} \right] = \frac{1}{2} \left[\frac{\pi}{4} - 0 \right]$$

$$= \frac{\pi}{8}$$

$$(ii) \int_0^1 \sqrt{\frac{1-x}{1+x}} dx$$

$$I = \int_0^1 \sqrt{\frac{1-x}{1+x}} dx$$

$$\text{put } x = \sin t$$

$$\frac{dx}{dt} = \cos t$$

$$dx = \cos t dt$$

$$x=0 \quad t=0$$

$$x=1 \quad t=\frac{\pi}{2}$$

x	0	1
t	0	$\frac{\pi}{2}$

$$I = \int_0^{\pi/2} \sqrt{\frac{1-\sin t}{1+\sin t}} \times \frac{(1-\sin t)}{(1-\sin t)} dt (\cos t)$$

$$= \int_0^{\pi/2} \sqrt{\frac{(1-\sin t)^2}{1-\sin^2 t}} \cos t dt$$

$$= \int_0^{\pi/2} \frac{(1-\sin t)^2}{\cos^2 t} \cos t dt$$

$$= \int_0^{\pi/2} \frac{1-\sin t}{\cos t} \cos t dt$$

$$= (t + \cos t) \Big|_0^{\pi/2}$$

$$= \left(\frac{\pi}{2} + \cos \frac{\pi}{2} \right) - (0 + \cos 0)$$

$$= \frac{\pi}{2} - 1$$

$$(iv) \int_0^{\pi/2} \left(\frac{1+\sin x}{1+\cos x} \right) dx$$

$$I = \int_0^{\pi/2} \left(\frac{1+\sin x}{1+\cos x} \right) dx$$

$$= \int_0^{\pi/2} \frac{1+2\sin x_2 \cos x_2}{2\cos^2 x_2} dx$$

$$= \int_0^{\pi/2} \left(\frac{1}{2\cos^2 x_2} + \frac{2\sin x_2 \cos x_2}{2\cos^2 x_2} \right) dx$$

$$= \int_0^{\pi/2} \left(\frac{1}{2} \sec^2 x_2 + \tan x_2 \right) dx$$

$$= \frac{1}{2} \left[\frac{1}{2} \tan x_2 + \log (\sec x_2) \right]_0^{\pi/2}$$

$$= \frac{2}{2} \left[(\tan \pi/4 + \log \sec \pi/4) - (\tan 0 + \log 1) \right]$$

$$= [1 + \log \sqrt{2}] \quad (\text{Book answer wrong})$$

$$(v) \int_0^{\pi/2} \sqrt{\cos \theta} \sin^3 \theta d\theta$$

$$I = \int_0^{\pi/2} \sqrt{\cos \theta} \sin^3 \theta d\theta$$

$$= \int_0^{\pi/2} \sqrt{\cos \theta} \sin^2 \theta \sin \theta d\theta$$

$$= \int_0^{\pi/2} \sqrt{\cos \theta} (1 - \cos^2 \theta) \sin \theta d\theta$$

put $\cos \theta = t$

$$-\sin \theta = \frac{dt}{d\theta}$$

$$\sin \theta d\theta = -dt$$

$$\theta = 0 \Rightarrow t = \cos 0 = 1$$

$$\theta = \pi/2 \Rightarrow t = \cos \pi/2 = 0$$

θ	0	$\pi/2$
t	1	0

$$I = \int_1^0 \sqrt{t} (1-t^2) (-dt)$$

$$= \int_0^1 (\sqrt{t} - \sqrt{t} t^2) dt = \int_0^1 (t^{1/2} - t^{5/2}) dt$$

$$= \left[\frac{t^{3/2}}{3/2} - \frac{t^{7/2}}{7/2} \right]_0^1 = \left(\frac{2}{3} - \frac{2}{7} \right) - 0 = \frac{14-6}{21} = \frac{8}{21}$$

$$(vi) \int_0^1 \frac{1-x^2}{(1+x^2)^2} dx$$

$$I = \int_0^1 \frac{(1-x^2)}{(1+x^2)^2} dx$$

$$\text{put } x = \tan t$$

$$\frac{dx}{dt} = \sec^2 t$$

$$dx = \sec^2 t dt$$

$$x=0$$

$$x=1$$

$$\tan t = 0 \Rightarrow t = 0$$

$$\tan t = 1 \Rightarrow t = \pi/4$$

x	0	1
t	0	$\pi/4$

$$I = \int_0^{\pi/4} \frac{1-\tan^2 t}{(1+\tan^2 t)^2} \sec^2 t dt$$

$$= \int_0^{\pi/4} \frac{1-\tan^2 t}{(\sec^2 t)^2} \sec^2 t dt$$

$$= \int_0^{\pi/4} \left(\frac{1}{\sec^2 t} - \frac{\sin^2 t}{\cos^2 t} \times \frac{1}{\sec^2 t} \right) dt$$

$$= \int_0^{\pi/4} \left(\cos^2 t - \frac{\sin^2 t}{\cos^2 t} \times \cos^2 t \right) dt$$

$$= \int_0^{\pi/4} \cos^2 t dt = \left[\frac{1}{2} \sin 2t \right]_0^{\pi/4}$$

$$= \frac{1}{2} [\sin 2 \cdot \frac{\pi}{4} - \sin 0]$$

$$= \frac{1}{2} [1 - 0]$$

$$= \frac{1}{2}$$

② Evaluate the following integrals using properties of integration.

$$\psi I = \int_{-\pi}^{\pi} x \cos \left(\frac{e^x - 1}{e^x + 1} \right) dx$$

$$f(x) = x \cos \left(\frac{e^x - 1}{e^x + 1} \right)$$

$$f(-x) = -x \cos \left(\frac{e^{-x} - 1}{e^{-x} + 1} \right)$$

$$= -x \cos \left(\frac{1 - e^x / e^x}{1 + e^x / e^x} \right)$$

$$= -x \cos \left(- \frac{(e^x - 1)}{e^x + 1} \right)$$

$$\boxed{\cos(-\theta) = \cos \theta}$$

$$f(-x) = -x \cos\left(\frac{e^x}{e^x+1}\right)$$

$$f(-x) = -f(x)$$

$f(x)$ is an odd function

$$\therefore \int_{-a}^a f(x) dx = 0$$

$$\therefore I = 0$$

$$(ii) \int_{-\pi/2}^{\pi/2} (x^5 + x \cos x + \tan^3 x + 1) dx$$

$$I = \int_{-\pi/2}^{\pi/2} (x^5 + x \cos x + \tan^3 x + 1) dx \quad \text{--- (1)}$$

$$f(x) = x^5 + x \cos x + \tan^3 x + 1$$

$$a = \pi/2 \quad b = -\pi/2$$

$$a+b-x = \pi/2 - \pi/2 - x = -x$$

$$\int_a^b f(x) dx = \int_a^b f(a+b-x) dx \neq \int$$

$$I = \int_{-\pi/2}^{\pi/2} f(-x) dx$$

$$I = \int_{-\pi/2}^{\pi/2} (-x^5 - x \cos x - \tan^3 x + 1) dx \quad \text{--- (2)}$$

$$(1) + (2) \Rightarrow I + I = \int_{-\pi/2}^{\pi/2} (x^5 + x \cos x + \tan^3 x + 1 - x^5 - x \cos x - \tan^3 x + 1) dx$$

$$2I = 2 \int_{-\pi/2}^{\pi/2} dx$$

$$2I = 2 \left[x \right]_{-\pi/2}^{\pi/2} = 2 \left[\pi/2 + \pi/2 \right]$$

$$2I = 2\pi$$

$$I = \pi$$

$$(iii) \int_{-\pi/4}^{\pi/4} \sin^2 x dx$$

$$I = \int_{-\pi/4}^{\pi/4} \sin^2 x dx$$

$$I = \int_{-\pi/4}^{\pi/4} \left[1 - \frac{\cos 2x}{2} \right] dx$$

$$= \frac{1}{2} \left[x - \frac{1}{2} \sin 2x \right]_{-\pi/4}^{\pi/4}$$

$$= \frac{1}{2} \left[\left(\frac{\pi}{4} - \frac{1}{2} \sin 2 \cdot \frac{\pi}{4} \right) - \left(-\frac{\pi}{4} - \frac{1}{2} \sin 2 \left(-\frac{\pi}{4} \right) \right) \right]$$

$$= \frac{1}{2} \left[\left(\frac{\pi}{4} - \frac{1}{2} \right) - \left(-\frac{\pi}{4} + \frac{1}{2} \right) \right]$$

$$= \frac{1}{2} \left[\frac{\pi}{4} + \frac{\pi}{4} - \frac{1}{2} - \frac{1}{2} \right] = \frac{1}{2} \left[\frac{2\pi}{4} - 1 \right]$$

$$= \frac{\pi}{4} - \frac{1}{2}$$

$$(iv) \int_0^{2\pi} x \log \left(\frac{3+\cos x}{3-\cos x} \right) dx$$

$$I = \int_0^{2\pi} x \log \left(\frac{3+\cos x}{3-\cos x} \right) dx$$

$$f(x) = \log \left(\frac{3+\cos x}{3-\cos x} \right)$$

$$f(2\pi-x) = \log \left(\frac{3+\cos(2\pi-x)}{3-\cos(2\pi-x)} \right) = \log \left(\frac{3+\cos x}{3-\cos x} \right)$$

$$f(2\pi-x) = f(x)$$

$$\boxed{\int_0^a x f(x) dx = \frac{a}{2} \int_0^a f(x) dx \text{ if } f(a-x) = f(x)}$$

$$I = \frac{2\pi}{2} \int_0^{2\pi} \log \left(\frac{3+\cos x}{3-\cos x} \right) dx$$

$$I = \pi \int_0^{2\pi} \log \left(\frac{3+\cos x}{3-\cos x} \right) dx \quad \text{--- (1)}$$

$$f(\pi-x) = \log \left(\frac{3+\cos(\pi-x)}{3-\cos(\pi-x)} \right) = \log \left(\frac{3-\cos x}{3+\cos x} \right)$$

$$= -\log \left(\frac{3+\cos x}{3-\cos x} \right)$$

$$f(\pi-x) = -f(x)$$

(12)

$$I = 2\pi \int_0^{\pi} \log \left(\frac{3-\cos x}{3+\cos x} \right) dx \quad \text{--- (2)}$$

①+② ⇒

$$I+I = 2\pi \int_0^{\pi} \left(\log \left(\frac{3+\cos x}{3-\cos x} \right) + \log \left(\frac{3-\cos x}{3+\cos x} \right) \right) dx$$

$$2I = 2\pi \int_0^{\pi} \log \left(\frac{3+\cos x}{3-\cos x} \times \frac{3-\cos x}{3+\cos x} \right) dx$$

$$= 2\pi \int_0^{\pi} \log 1 dx$$

$$2I = 2\pi (0)$$

$$I = 0$$

$$(v) \int_0^{2\pi} \sin^4 x \cos^3 x dx$$

$$I = \int_0^{2\pi} \sin^4 x \cos^3 x dx$$

$$I = \int_0^{2\pi} (1-\cos^2 x)^2 \cos^3 x dx$$

$$= \int_0^{2\pi} (1 + \cos^4 x - 2\cos^2 x) \cos^3 x dx$$

$$= \int_0^{2\pi} (\cos^3 x + \cos^7 x - 2\cos^5 x) dx$$

$$I = 2 \int_0^{\pi} (\cos^3 x + \cos^7 x - 2\cos^5 x) dx \quad \text{--- (1)}$$

$$a=\pi \quad f(x) = 2(\cos^3 x + \cos^7 x - 2\cos^5 x) \quad \left\{ \int_0^{2\pi} g(\cos x) dx = 2 \int_0^{\pi} g(\cos x) dx \right.$$

$$f(a-x) = f(\pi-x) = 2(-\cos^3 x - \cos^7 x + 2\cos^5 x)$$

$$I = 2 \int_0^{\pi} (-\cos^3 x - \cos^7 x + 2\cos^5 x) dx \quad \text{--- (2)}$$

$$\boxed{\int_0^a f(x) dx = \int_0^a f(a-x) dx}$$

①+② ⇒

$$I+I = 2 \int_0^{\pi} (\cancel{\cos^3 x} + \cancel{\cos^7 x} - 2\cancel{\cos^5 x} - \cancel{\cos^3 x} - \cancel{\cos^7 x} + 2\cancel{\cos^5 x}) dx$$

$$2I = 0$$

(vi) $\int_0^1 |5x-3| dx$

$$|5x-3| = \begin{cases} 5x-3 & 5x-3 > 0 \\ -(5x-3) & 5x-3 < 0 \end{cases}$$

$$= \begin{cases} 5x-3 & x > \frac{3}{5} \\ -(5x-3) & x < \frac{3}{5} \end{cases}$$

$$= \begin{cases} 5x-3 & \frac{3}{5} < x < 1 \\ -(5x-3) & 0 < x < \frac{3}{5} \end{cases}$$

$$I = \int_0^1 |5x-3| dx$$

$$= \int_0^{\frac{3}{5}} -(5x-3) dx + \int_{\frac{3}{5}}^1 (5x-3) dx$$

$$= -\frac{1}{5} \left[\frac{(5x-3)^2}{2} \right]_0^{\frac{3}{5}} + \frac{1}{5} \left[\frac{(5x-3)^2}{2} \right]_{\frac{3}{5}}^1$$

$$= -\frac{1}{10} \left[(0 - (-3)^2) \right] + \frac{1}{10} \left[(5-3)^2 - (0) \right]$$

$$= -\frac{1}{10} (-9) + \frac{1}{10} (4) = \frac{9}{10} + \frac{4}{10}$$

$$= \frac{13}{10}$$

vii $\int_0^{\sin^2 x} \sin^{-1} \sqrt{t} dt + \int_0^{\cos^2 x} \cos^{-1} \sqrt{t} dt$

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$$I = I_1 + I_2$$

$$I_1 = \int_0^{\sin^2 x} \sin^{-1} \sqrt{t} dt$$

$$u = \sin^{-1} \sqrt{t} \quad dv = dt$$

$$du = \frac{1}{\sqrt{1-t}} \cdot \frac{1}{2\sqrt{t}} \quad v = \int dt$$

$$v = t$$

$$\int u dv = uv - \int v du$$

$$I = \int_0^{\sin^2 x} \sin^{-1} \sqrt{t} dt + \int_0^{\cos^2 x} \cos^{-1} \sqrt{t} dt$$

$$u = \sin^{-1} \sqrt{t} \quad dv = dt$$

$$du = \frac{1}{\sqrt{1-t}} \cdot \frac{1}{2\sqrt{t}} \quad v = t$$

$$u = \cos^{-1} \sqrt{t}, \quad dv = dt$$

$$du = \frac{-1}{\sqrt{1-t}} \cdot \frac{1}{2\sqrt{t}}, \quad v = t$$

$$I = \left[\sin^{-1} \sqrt{t} (t) \right]_0^{\sin^2 x} - \int_0^{\sin^2 x} \sqrt{t} \cdot \frac{1}{\sqrt{1-t}} \cdot \frac{1}{2\sqrt{t}} dt + \left[\cos^{-1} \sqrt{t} (t) \right]_0^{\cos^2 x} - \int_0^{\cos^2 x} \sqrt{t} \cdot \left(\frac{-1}{\sqrt{1-t}} \right) \cdot \frac{1}{2\sqrt{t}} dt$$

$$I = \left[\sin^{-1}(\sin x) \sin^2 x - 0 \right] - \frac{1}{2} \int_0^{\sin^2 x} \frac{\sqrt{t}}{\sqrt{1-t}} dt + \left[\cos^{-1}(\cos x) \cos^2 x - 0 \right] + \frac{1}{2} \int_0^{\cos^2 x} \frac{\sqrt{t}}{\sqrt{1-t}} dt$$

$$I = x \sin^2 x + x \cos^2 x - \frac{1}{2} \int_0^{\sin^2 x} \frac{\sqrt{t}}{\sqrt{1-t}} dt + \frac{1}{2} \int_0^{\cos^2 x} \frac{\sqrt{t}}{\sqrt{1-t}} dt$$

$$= x(\sin^2 x + \cos^2 x) + \frac{1}{2} \int_0^{\sin^2 x} \frac{\sqrt{t}}{\sqrt{1-t}} dt + \frac{1}{2} \int_0^{\cos^2 x} \frac{\sqrt{t}}{\sqrt{1-t}} dt$$

$$= x(1) + \frac{1}{2} \int_0^{\sin^2 x} \frac{\sqrt{t}}{\sqrt{1-t}} dt + \frac{1}{2} \int_0^{\cos^2 x} \frac{\sqrt{t}}{\sqrt{1-t}} dt$$

$$= x + \frac{1}{2} \int_x^{\pi/2-x} \frac{\sin \theta \cos \theta}{\sqrt{\cos^2 \theta}} \cdot 2 \sin \theta \cos \theta d\theta$$

put $t = \sin^2 \theta \quad \sqrt{t} = \sin \theta$

$dt = 2 \sin \theta \cos \theta d\theta$

$t = \sin^2 x \Rightarrow \sin^2 \theta = \sin^2 x$

$\theta = x$

$t = \cos^2 x \Rightarrow \sin^2 \theta = \sin^2(\pi/2 - x)$

$\theta = \pi/2 - x$

$$= x + \frac{1}{2} \int_x^{\pi/2-x} \sin^2 \theta d\theta$$

$$= x + \frac{1}{2} \int_x^{\pi/2-x} \frac{(1 - \cos 2\theta)}{2} d\theta$$

t	$\sin^2 x$	$\cos^2 x$
θ	x	$\pi/2 - x$

$$= x + \frac{1}{2} \left(\theta - \frac{1}{2} \sin 2\theta \right)_{\frac{\pi}{2}-x}^x$$

$$= x + \frac{1}{2} \left(\left(\frac{\pi}{2} - x - x \right) - \frac{1}{2} \left(\sin^2 \left(\frac{\pi}{2} - x \right) - \sin 2x \right) \right)$$

$$= x + \frac{1}{2} \left(\frac{\pi}{2} - 2x - \frac{1}{2} \left(\sin(\pi - 2x) - \sin 2x \right) \right)$$

$$= x + \frac{\pi}{4} - x - \frac{1}{4} [\sin 2x - \sin 2x]$$

$$= \cancel{x} + \frac{\pi}{4} - \cancel{x}$$

$$I = \frac{\pi}{4}$$

$$(viii) \int_0^1 \frac{\log(1+x)}{1+x^2} dx$$

$$I = \int_0^1 \frac{\log(1+x)}{1+x^2} dx$$

put $x = \tan t$
 $dx = \sec^2 t dt$

$$I = \int_0^{\pi/4} \frac{\log(1+\tan t)}{1+\tan^2 t} \sec^2 t dt$$

$$x=0 \Rightarrow \tan t=0, t=0$$

$$x=1 \Rightarrow \tan t=1, t=\pi/4$$

x	0	1
t	0	$\pi/4$

$$= \int_0^{\pi/4} \frac{\log(1+\tan t)}{\sec^2 t} \sec^2 t dt$$

$$I = \int_0^{\pi/4} \log(1+\tan t) dt \quad \text{--- (1)}$$

$$f(t) = \log(1+\tan t)$$

$$a = \pi/4$$

$$f\left(\frac{\pi}{4}-t\right) = \log\left(1+\tan\left(\frac{\pi}{4}-t\right)\right)$$

$$= \log\left(1 + \frac{\tan \pi/4 - \tan t}{1 + \tan \pi/4 \tan t}\right)$$

$$= \log\left(1 + \frac{1 - \tan t}{1 + \tan t}\right) = \log\left(\frac{1 + \tan t + 1 - \tan t}{1 + \tan t}\right)$$

$$f\left(\frac{\pi}{4}-t\right) = \log\left(\frac{2}{1+\tan t}\right)$$

$$\int_0^a f(x) dx = \int_0^a f(a-x) dx$$

$$I = \int_0^{\pi/4} \log\left(\frac{2}{1+\tan t}\right) dt \quad \text{--- (2)}$$

$$\textcircled{1} + \textcircled{2} \Rightarrow I + I = \int_0^{\pi/4} \left(\log(1+\tan t) + \log\left(\frac{2}{1+\tan t}\right) \right) dt$$

$$2I = \int_0^{\pi/4} \log\left(\cancel{(1+\tan t)} \times \frac{2}{\cancel{(1+\tan t)}}\right) dt$$

$$2I = \int_0^{\pi/4} \log 2 dt$$

$$2I = \log 2 \left[t \right]_0^{\pi/4}$$

$$2I = \log 2 \left[\frac{\pi}{4} - 0 \right] = \frac{\pi}{4} \log 2$$

$$I = \frac{\pi}{8} \log 2$$

$$(ix) \int_0^{\pi} \frac{x \sin x}{1 + \sin x} dx$$

$$I = \int_0^{\pi} \frac{x \sin x}{1 + \sin x} dx$$

$$f(x) = \frac{\sin x}{1 + \sin x}$$

$$a = \pi$$

$$f(\pi - x) = \frac{\sin(\pi - x)}{1 + \sin(\pi - x)} = \frac{\sin x}{1 + \sin x} = f(x)$$

$$f(x) = f(\pi - x)$$

$$\Rightarrow \int_0^a x f(x) dx = \frac{a}{2} \int_0^a f(x) dx$$

$$I = \frac{\pi}{2} \int_0^{\pi} \frac{\sin x}{1 + \sin x} dx$$

$$= \frac{\pi}{2} \int_0^{\pi} \frac{\sin x}{(1 + \sin x)(1 - \sin x)} dx$$

$$= \frac{\pi}{2} \int_0^{\pi} \frac{\sin x - \sin^2 x}{1 - \sin^2 x} dx = \frac{\pi}{2} \int_0^{\pi} \left(\frac{\sin x}{\cos^2 x} - \frac{\sin^2 x}{\cos^2 x} \right) dx$$

$$= \frac{\pi}{2} \int (\sec x \tan x - \tan^2 x) dx$$

$$= \frac{\pi}{2} \int_0^{\pi} (\sec x \tan x - (\sec^2 x - 1)) dx$$

$$= \frac{\pi}{2} \int_0^{\pi} (\sec x \tan x - \sec^2 x + 1) dx$$

$$= \frac{\pi}{2} \left[\sec x - \tan x + x \right]_0^{\pi}$$

$$= \frac{\pi}{2} \left[(\sec \pi - \tan \pi + \pi) - (\sec 0 - \tan 0 + 0) \right]$$

$$= \frac{\pi}{2} [-1 - 0 + \pi - 1]$$

$$I = \frac{\pi}{2} [\pi - 2]$$

$$(x) \int_{\pi/8}^{3\pi/8} \frac{1}{1+\sqrt{\tan x}} dx \quad I = \int_{\pi/8}^{3\pi/8} \frac{1}{1+\sqrt{\tan x}} dx$$

$$I = \int_{\pi/8}^{3\pi/8} \frac{1}{1+\sqrt{\frac{\sin x}{\cos x}}} dx$$

$$I = \int_{\pi/8}^{3\pi/8} \frac{\sqrt{\cos x} dx}{\sqrt{\cos x} + \sqrt{\sin x}} \quad \text{--- (1)}$$

$$a = \pi/8 \quad b = 3\pi/8 \quad a+b = \frac{\pi}{8} + \frac{3\pi}{8} = \frac{4\pi}{8} = \pi/2$$

$$f(x) = \frac{\sqrt{\cos x}}{\sqrt{\cos x} + \sqrt{\sin x}}$$

$$f(a+b-x) = f(\pi/2-x) = \frac{\sqrt{\cos(\pi/2-x)}}{\sqrt{\cos(\pi/2-x)} + \sqrt{\sin(\pi/2-x)}}$$

$$f(\pi/2-x) = \frac{\sqrt{\sin x}}{\sqrt{\sin x} + \sqrt{\cos x}}$$

$$\int_a^b f(x) dx = \int_a^b f(a+b-x) dx$$

$$I = \int_{\pi/8}^{3\pi/8} \frac{\sqrt{\sin x}}{\sqrt{\sin x} + \sqrt{\cos x}} dx \quad \text{--- (2)}$$

add (1) (2)

$$2I + I = \int_{\pi/8}^{3\pi/8} \frac{\sqrt{\sin x} + \sqrt{\cos x}}{\sqrt{\sin x} + \sqrt{\cos x}} dx$$

$$2I = \int_{\pi/8}^{3\pi/8} dx = \frac{\pi}{4}$$

$$2I = \left[x \right]_{1/8}^{31/8}$$

$$2I = \frac{31\pi}{8} - \frac{\pi}{8}$$

$$2I = \frac{2\pi}{84}$$

$$2I = \frac{\pi}{4}$$

$$I = \frac{\pi}{4 \times 2}$$

$$I = \frac{\pi}{8} //$$

$$(x) I = \int_0^{\pi} x [\sin^2(\sin x) + \cos^2(\cos x)] dx$$

$$f(x) = \sin^2(\sin x) + \cos^2(\cos x)$$

$$a = \pi$$

$$f(a-x) = f(\pi-x)$$

$$= \sin^2(\sin(\pi-x)) + \cos^2(\cos(\pi-x))$$

$$= \sin^2(\sin x) + \cos^2(\cos x)$$

$$= \sin^2(\sin x) + \cos^2(\cos x)$$

$$f(a-x) = f(x)$$

$$\therefore \int_0^a x f(x) dx = \frac{a}{2} \int_0^a f(x) dx$$

$$I = \frac{\pi}{2} \int_0^{\pi} [\sin^2(\sin x) + \cos^2(\cos x)] dx \quad \text{--- (1)}$$

$$\int_0^{\pi} g(\sin x) dx = 2 \int_0^{\pi/2} g(\sin x) dx$$

$$f(x) = \cos^2(\cos x)$$

$$f(\pi+x) = \cos^2(\cos(\pi+x))$$

$$= \cos^2(-\cos x)$$

$$= \cos^2(\cos x)$$

$$f(\pi+x) = f(x)$$

$$\Rightarrow \int_0^{2a} f(x) dx = 2 \int_0^a f(x) dx$$

$$\int_0^{\pi} \cos^2(\cos x) dx = 2 \int_0^{\pi/2} \cos^2(\cos x) dx$$

$$I = \frac{\pi}{2} \int_0^{\pi/2} \sin^2(\sin x) + \cos^2(\cos x) dx \quad \text{--- (1)}$$

$$I = \pi \int_0^{\pi/2} \sin^2(\cos x) + \cos^2(\sin x) dx \quad \text{--- (2)}$$

$$(1) + (2) \Rightarrow$$

$$I + I = \pi \int_0^{\pi/2} (\sin^2(\sin x) + \cos^2(\cos x) + \cos^2(\sin x) + \sin^2(\cos x)) dx$$

$$2I = \pi \int_0^{\pi/2} (1+1) dx$$

$$2I = 2\pi \left[x \right]_0^{\pi/2}$$

$$f(x) = \sin^2(\sin x) + \cos^2(\cos x)$$

$$f(a-x) = f(\pi/2-x) = \sin^2(\cos x) + \cos^2(\sin x)$$

$$\int_0^a f(x) dx = \int_0^a f(a-x) dx$$

$$I = \pi \left[\frac{\pi}{2} - 0 \right]$$

$$I = \frac{\pi^2}{2} - 0$$

$$I = \frac{\pi^2}{2} //$$

$$\int u dv = uv - u'v_1 + u''v_2 - u'''v_3 + \dots$$

Exercise 9.4

Evaluate the following

1) $\int_0^1 x^3 e^{-2x} dx$

$$I = \int_0^1 x^3 e^{-2x} dx$$

$$u = x^3$$

$$dv = e^{-2x}, \quad v = \frac{1}{-2} e^{-2x}$$

$$u' = 3x^2$$

$$v_1 = \frac{1}{4} e^{-2x}$$

$$u'' = 6x$$

$$v_2 = -\frac{1}{8} e^{-2x}$$

$$u''' = 6$$

$$v_3 = +\frac{1}{16} e^{-2x}$$

$$u^{iv} = 0$$

$$I = \int u dv = uv - u'v_1 + u''v_2 - u'''v_3 + \dots$$

$$I = \left[x^3 \left(-\frac{1}{2} e^{-2x} \right) - 3x^2 \left(\frac{1}{4} e^{-2x} \right) + 6x \left(-\frac{1}{8} e^{-2x} \right) - \left(\frac{1}{16} e^{-2x} \right) \right]_0^1$$

$$= \left[\left(-\frac{e^{-2}}{2} - \frac{3}{4} e^{-2} - \frac{3}{8} e^{-2} - \frac{1}{16} e^{-2} \right) - \left(0 + 0 + 0 - \frac{1}{16} e^0 \right) \right]$$

$$= -e^{-2} \left[\frac{1}{2} + \frac{3}{4} + \frac{3}{8} + \frac{1}{16} \right] + \frac{1}{16}$$

$$= -e^{-2} \left[\frac{4+6+6+1}{8} \right] + \frac{1}{16}$$

$$= \frac{1}{16} - e^{-2} \frac{19}{8} = \frac{1}{16} [3 - 19e^{-2}]$$

2) $\int_0^1 \frac{\sin(3 \tan^{-1} x) \tan^{-1} x}{1+x^2} dx$

$$I = \int_0^1 \frac{\sin(3 \tan^{-1} x) \tan^{-1} x}{1+x^2} dx$$

put $\tan^{-1} x = t$
 $\frac{1}{1+x^2} dx = dt$

$x=0 \quad t = \tan^{-1}(0) = 0$
 $x=1 \quad t = \tan^{-1}(1) = \pi/4$

x	0	1
t	0	$\pi/4$

(20)

$$I = \int_0^{\pi/4} \sin 3t \cdot t \, dt$$

$$I = \int_0^{\pi/4} \frac{t}{u} \frac{\sin 3t \, dt}{dv}$$

$$u = t$$

$$dv = \sin 3t \, dt$$

$$v = -\frac{1}{3} \cos 3t$$

$$u' = 1$$

$$v_1 = -\frac{1}{9} \sin 3t$$

$$u'' = 0$$

$$v_2 = +\frac{1}{27} \cos 3t$$

$$\int u \, dv = uv - u'v_1 + u''v_2 - u'''v_3 + \dots$$

$$I = \left[t \left(-\frac{1}{3} \cos 3t \right) - 1 \left(-\frac{1}{9} \sin 3t \right) + 0 \right]_0^{\pi/4}$$

$$\cos \frac{3\pi}{4} = \cos 135$$

$$= \cos(180 - 45)$$

$$= -\cos 45$$

$$= -\frac{1}{\sqrt{2}}$$

$$\sin \frac{3\pi}{4} = \frac{1}{\sqrt{2}}$$

$$= \left(-\frac{\pi}{4} \cdot \frac{1}{3} \cos \frac{3\pi}{4} + \frac{1}{9} \sin \frac{3\pi}{4} \right) - (0 + 0)$$

$$= \frac{\pi}{12} \left(\frac{1}{\sqrt{2}} \right) + \frac{1}{9} \cdot \frac{1}{\sqrt{2}} = \frac{1}{\sqrt{2}} \left(\frac{\pi}{12} + \frac{1}{9} \right)$$

$$I = \frac{1}{\sqrt{2}} \left(\frac{\pi}{12} + \frac{1}{9} \right)$$

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$$3) \quad I = \int_0^{\frac{1}{\sqrt{2}}} \frac{e^{a \sin^{-1} x} \sin^{-1} x}{\sqrt{1-x^2}} \, dx$$

put

$$\sin^{-1} x = t$$

$$x = 0 \Rightarrow t = \sin^{-1}(0) = 0$$

$$x = \frac{1}{\sqrt{2}} \Rightarrow t = \sin^{-1}\left(\frac{1}{\sqrt{2}}\right) = \frac{\pi}{4}$$

$$\frac{1}{\sqrt{1-x^2}} \, dx = dt$$

$$I = \int_0^{\pi/4} e^{at} t \, dt$$

x	0	$\frac{1}{\sqrt{2}}$
t	0	$\frac{\pi}{4}$

$$I = \int_0^{\pi/4} \frac{t}{u} \frac{e^{at} \, dt}{dv}$$

$$u = t$$

$$dv = e^{at} \, dt$$

$$u' = 1$$

$$v = \frac{1}{a} e^{at}$$

$$u'' = 0$$

$$v_1 = \frac{1}{a^2} e^{at}$$

$$\int u dv = uv - \int v du$$

$$I = \left[t \frac{1}{a} e^{at} - \int \frac{1}{a^2} e^{at} dt \right]_{\pi/4}^{\pi/4}$$

$$= \left(\frac{\pi}{4a} e^{a\pi/4} - \frac{1}{a^2} e^{a\pi/4} \right) - \left(0 - \frac{1}{a^2} \right)$$

$$I = e^{a\pi/4} \left(\frac{\pi}{4a} - \frac{1}{a^2} \right) + \frac{1}{a^2}$$

put $a=1$ $I = e^{\pi/4} \left(\frac{\pi}{4} - 1 \right) + 1$ (Book answer)

4)

$$I = \int_0^{\pi/2} \frac{x^2 \cos 2x dx}{u} \frac{dv}{dv}$$

$$u = x^2 \quad dv = \cos 2x dx$$

$$v = \frac{1}{2} \sin 2x$$

$$u' = 2x \quad v_1 = -\frac{1}{4} \cos 2x$$

$$u'' = 2 \quad v_2 = -\frac{1}{8} \sin 2x$$

$$u''' = 0$$

$$\int u dv = uv - u'v_1 + u''v_2 - \dots$$

$$I = \left[x^2 \frac{1}{2} \sin 2x + 2x \left(\frac{1}{4} \cos 2x \right) + 2 \left(-\frac{1}{8} \sin 2x \right) \right]_0^{\pi/2}$$

$$= \left(\frac{\pi^2}{4} \frac{1}{2} \sin 2\frac{\pi}{2} + \frac{\pi}{2} \frac{1}{2} \cos 2\frac{\pi}{2} - \frac{1}{4} \sin 2\frac{\pi}{2} \right)$$

$$= \left(0 + \frac{\pi}{4} - 0 \right)$$

$$= \frac{\pi}{4}$$

$$I = \frac{\pi}{4}$$

Improper Integrals

Exercise 9.5

1). Evaluate

$$(i) \int_0^{\pi/2} \frac{dx}{1+5\cos^2 x}$$

(22)

$$\begin{aligned}
 I &= \int_0^{\pi/2} \frac{dx}{1+5\cos^2 x} \\
 &= \int_0^{\pi/2} \frac{dx}{\cos^2 x \left(\frac{1}{\cos^2 x} + 5 \right)} \\
 &= \int_0^{\pi/2} \frac{\sec^2 x}{\sec^2 x + 5} dx \\
 &= \int_0^{\pi/2} \frac{\sec^2 x}{\sec^2 x - 1 + 6} dx = \int_0^{\pi/2} \frac{\sec^2 x}{\tan^2 x + 6} dx
 \end{aligned}$$

put $\tan x = t$

$$\sec^2 x dx = dt$$

$$x=0$$

$$t = \tan 0 = 0$$

$$x = \pi/2$$

$$t = \tan \pi/2 = \infty$$

$$I = \int_0^{\infty} \frac{dt}{6+t^2}$$

x	0	$\pi/2$
t	0	∞

This is an improper integral

$$a^2 = 6$$

$$a = \sqrt{6}$$

$$= \frac{1}{\sqrt{6}} \left[\tan^{-1} \frac{t}{\sqrt{6}} \right]_0^{\infty}$$

$$\int_0^{\infty} \frac{dx}{a^2+x^2} = \frac{\pi}{2a}$$

$$= \frac{\pi}{2\sqrt{6}}$$

$$= \frac{1}{\sqrt{6}} \left[\tan^{-1} \infty - \tan^{-1}(0) \right] = \frac{1}{\sqrt{6}} \left(\frac{\pi}{2} - 0 \right)$$

$$= \frac{\pi}{2\sqrt{6}}$$

$$(ii) \int_0^{\pi/2} \frac{dx}{5+4\sin^2 x}$$

$$I = \int_0^{\pi/2} \frac{dx}{5+4\sin^2 x}$$

$$\begin{aligned}
 &= \int_0^{\pi/2} \frac{dx}{5+4(1-\cos^2 x)} \\
 &= \int_0^{\pi/2} \frac{dx}{5+4-4\cos^2 x} = \int_0^{\pi/2} \frac{dx}{9-4\cos^2 x} \\
 &= \int_0^{\pi/2} \frac{dx}{\cos^2 x \left(\frac{9}{\cos^2 x} - 4 \right)} \\
 &= \int_0^{\pi/2} \frac{dx \sec^2 x}{9\sec^2 x - 4} \\
 &= \int_0^{\pi/2} \frac{\sec^2 x dx}{9(1+\tan^2 x) - 4} = \int_0^{\pi/2} \frac{\sec^2 x dx}{9+9\tan^2 x - 4} \\
 &= \int_0^{\pi/2} \frac{\sec^2 x dx}{5+9\tan^2 x} = \frac{1}{9} \int_0^{\pi/2} \frac{\sec^2 x dx}{\left(\frac{5}{9}\right) + \tan^2 x}
 \end{aligned}$$

put $\tan x = t$
 $\sec^2 x dx = dt$

$x=0 \quad t=0$
 $x=\pi/2 \quad t=\infty$

$$I = \frac{1}{9} \int_0^{\infty} \frac{dt}{\frac{5}{9} + t^2}$$

$a = \frac{\sqrt{5}}{3}$

$$\int_0^{\infty} \frac{dx}{a^2 + x^2} = \frac{\pi}{2a}$$

$$= \frac{1}{9} \cdot \frac{\pi}{2 \cdot \frac{\sqrt{5}}{3}}$$

$$= \frac{\pi}{6\sqrt{5}}$$

(Book answer wrong)

Reduction Formula.

① n is even

$$\int_0^{\pi/2} \sin^n x dx = \int_0^{\pi/2} \cos^n x dx = \frac{n-1}{n} \times \frac{n-3}{n-2} \times \frac{n-5}{n-4} \dots \frac{1}{2} \frac{\pi}{2}$$

② n is odd

$$\int_0^{\pi/2} \sin^n x dx = \int_0^{\pi/2} \cos^n x dx = \frac{n-1}{n} \times \frac{n-3}{n-2} \times \frac{n-5}{n-4} \dots \frac{2}{3}$$

② If $I_{m,n} = \int_0^1 x^m (1-x)^n dx$, then $I_{m,n} = \frac{n}{m+n+1} I_{m,n-1}$, $n \geq 1$

③ (i) n is even and m is even.

$$\int_0^{\pi/2} \sin^m x \cos^n x dx = \frac{n-1}{m+n} \cdot \frac{n-3}{m+n-2} \cdots \frac{1}{m+2} \cdot \frac{m-1}{m} \cdot \frac{m-3}{m-2} \cdots \frac{1}{2} \cdot \frac{\pi}{2}$$

(ii) n is odd, m is any (even or odd)

$$\int_0^{\pi/2} \sin^m x \cos^n x dx = \frac{n-1}{m+n} \cdot \frac{n-3}{m+n-2} \cdots \frac{2}{m+3} \cdot \frac{1}{m+1}$$

(iii) n is even, m is odd

$$\int_0^{\pi/2} \sin^m x \cos^n x dx = \frac{m-1}{m+n} \cdot \frac{m-3}{m+n-2} \cdots \frac{2}{n+3} \cdot \frac{1}{n+1}$$

Exercise 9.6

i) Evaluate the following

(i) $\int_0^{\pi/2} \sin^{10} x dx$

$n=10$ even

$$I = \int_0^{\pi/2} \sin^{10} x dx$$

$$= \frac{9}{10} \times \frac{7}{8} \times \frac{5}{6} \times \frac{3}{4} \times \frac{1}{2} \times \frac{\pi}{2} = \frac{63\pi}{512}$$

(Book answer π missing)

(ii) $I = \int_0^{\pi/2} \cos^7 x dx$ $n=7$ odd

$$I = \frac{6^2}{7} \times \frac{4}{5} \times \frac{2}{3} = \frac{16}{35}$$

(Book answer wrong)

(iii) $I = \int_0^{\pi/4} \sin^6 2x dx$

put $2x = t$
 $2dx = dt$
 $dx = \frac{1}{2} dt$

$x=0$ $t=0$
 $x=\pi/4$ $t=2\pi/4 = \pi/2$

x	0	$\pi/4$
t	0	$\pi/2$

$$I = \frac{1}{2} \int_0^{\pi/2} \sin^6 t dt$$

$n=6$ even

$$= \frac{1}{2} \cdot \frac{5}{6} \times \frac{3}{4} \times \frac{1}{2} \times \frac{\pi}{2} = \frac{5\pi}{64}$$

$$\int_0^{\pi/2} \sin^n x dx = \frac{n-1}{n} \times \frac{n-3}{n-2} \cdots \frac{1}{2} \cdot \frac{\pi}{2}$$

(iv) $I = \int_0^{\pi/6} \sin^5 3x \, dx$

put $3x = t$

$3dx = dt$

$dx = dt/3$

when $x=0$ $t=0$

$x=\pi/6$ $t=3\pi/6=\pi/2$

x	0	$\pi/6$
t	0	$\pi/2$

$I = \frac{1}{3} \int_0^{\pi/2} \sin^5 t \, dt$

$n=5$

n is odd.

$\int_0^{\pi/2} \sin^n x \, dx = \frac{n-1}{n} \cdot \frac{n-3}{n-2} \dots \frac{2}{3}$

$= \frac{1}{3} \cdot \frac{4}{5} \cdot \frac{2}{3} = \frac{8}{45}$

(v) $I = \int_0^{\pi/2} \sin^2 x \cos^4 x \, dx$

$m=2$ $n=4$

both are even.

$\int_0^{\pi/2} \sin^m x \cos^n x \, dx = \frac{n-1}{m+n} \cdot \frac{n-3}{m+n-2} \dots \frac{1}{m+2} \cdot \frac{m-1}{m} \cdot \frac{m-3}{m-2} \dots$

$I = \frac{4-1}{2+4} \cdot \frac{4-3}{2+4-2} \cdot \frac{2-1}{2} \cdot \frac{\pi}{2}$
 $= \frac{3}{2 \cdot 6} \cdot \frac{1}{4} \cdot \frac{1}{2} \cdot \frac{\pi}{2} = \frac{\pi}{32}$

(vi) Aliter

$I = \int_0^{\pi/2} (1 - \cos^2 x) \cos^4 x \, dx$

$= \int_0^{\pi/2} \cos^4 x \, dx - \int_0^{\pi/2} \cos^6 x \, dx$

$= \frac{3}{4} \cdot \frac{1}{2} \cdot \frac{\pi}{2} - \frac{5}{6} \cdot \frac{3}{4} \cdot \frac{1}{2} \cdot \frac{\pi}{2} = \frac{3\pi}{16} \left[1 - \frac{5}{6} \right]$

$= \frac{3\pi}{16} \times \frac{1}{6} = \frac{\pi}{32}$

(vi) $I = \int_0^{2\pi} \sin^7 \frac{x}{4} \, dx$

x	0	2π
t	0	$\pi/2$

put $\frac{x}{4} = t$

$x = 4t$

$dx = 4dt$

$x=0$ $t=0$

$x=2\pi$ $t=\frac{2\pi}{4}=\pi/2$

x	0	2π
t	0	$\pi/2$

$I = 4 \int_0^{\pi/2} \sin^7 t \, dt$

$$I = 4 \frac{2}{7} \times \frac{4}{5} \frac{2}{3}$$

$$I = \frac{64}{35}$$

$$n=7 \text{ odd}$$

$$\int_0^{\pi/2} \sin^n x dx = \frac{n-1}{n} \cdot \frac{n-3}{n-2} \cdots \frac{2}{3}$$

$$(vii) I = \int_0^{\pi/2} \sin^3 \theta \cos^5 \theta d\theta$$

$$m=3 \quad n=5 \quad (\text{both or odd})$$

$$I = \int_0^{\pi/2} \sin^m x \cos^n x dx = \frac{n-1}{m+n} \cdot \frac{n-3}{m+n-2} \cdots \frac{2}{m+3} \cdot \frac{1}{m+1}$$

$$= \frac{5-1}{3+5} \cdot \frac{5-3}{3+5-2} \cdot \frac{1}{3+1}$$

$$I = \frac{4}{8} \cdot \frac{2}{8} \cdot \frac{1}{4} = \frac{1}{24}$$

$$(viii) I = \int_0^1 x^2 (1-x)^3 dx$$

$$I = \int_0^1 x^m (1-x)^n dx = \frac{m! n!}{(m+n+1)!} = I$$

$$= \frac{2! \times 3!}{(2+3+1)!} = \frac{2 \times 6}{6!} = \frac{2 \times 6}{6 \times 5 \times 4 \times 3 \times 2 \times 1}$$

$$= \frac{1}{60} \times \frac{1}{1} = \frac{1}{60}$$

Gamma Integral

$$\textcircled{1} \int_0^{\infty} e^{-ax} x^n dx = \frac{n!}{a^{n+1}} \quad (A + (1-a)x + 10 = 0)$$

$$a=1$$

$$\textcircled{2} \int_0^{\infty} e^{-x} x^n dx = n! \quad (\text{Gamma } n) \quad \Gamma n = (n-1)!$$

$$\textcircled{3} \int_0^{\infty} e^{-x} x^{n-1} dx = (n-1)! = \Gamma n$$

Exercise 9.7

$$\text{Evaluate } \textcircled{1} \textcircled{ii} \int_0^{\infty} x^5 e^{-3x} dx = I$$

$$n=5 \quad a=3$$

$$\int_0^{\infty} x^n e^{-ax} dx = \frac{n!}{a^{n+1}}$$

$$I = \frac{5!}{3^{5+1}} = \frac{5!}{3^6}$$

$$(ii) \int_0^{\pi/2} \frac{e^{-\tan x}}{\cos^6 x} dx$$

$$I = \int_0^{\pi/2} e^{-\tan x} \sec^6 x dx$$

$$\text{put } \tan x = t$$

$$\sec^2 x dx = dt$$

$$x=0 \quad t = \tan 0 = 0$$

$$x = \pi/2 \quad t = \tan \pi/2 = \infty$$

$$I = \int_0^{\infty} e^{-t} \sec^4 x \sec^2 x dx$$

x	0	$\pi/2$
t	0	∞

$$I = \int_0^{\infty} e^{-t} (\sec^2 x)^2 dt$$

$$= \int_0^{\infty} e^{-t} (1 + \tan^2 x)^2 dt = \int_0^{\infty} e^{-t} (1 + t^2)^2 dt$$

$$= \int_0^{\infty} t^0 e^{-t} dt + \int_0^{\infty} 2t^2 e^{-t} dt + \int_0^{\infty} t^4 e^{-t} dt$$

$$= 0! + 2(2!) + 4!$$

$$= 1 + 4 + 24$$

$$= 29$$

$$\int_0^{\infty} x^n e^{-x} dx = n!$$

$$(2) \text{ If } \int_0^{\infty} e^{-ax^2} x^3 dx = 32, a > 0 \text{ find } a.$$

$$\text{put } ax^2 = t \quad x^2 = \frac{t}{a}$$

$$2x dx = \frac{dt}{a}$$

$$x dx = \frac{dt}{2a}$$

$$x=0 \quad t=0$$

$$x=\infty \quad t=\infty$$

x	0	∞
t	0	∞

$$I = \int_0^{\infty} e^{-ax^2} x^2 x dx$$

(28)

$$I = \int_0^{\infty} e^{-t} \frac{t}{\alpha} \frac{dt}{2\alpha}$$

$$= \frac{1}{2\alpha^2} \int_0^{\infty} t e^{-t} dt$$

$$\int_0^{\infty} e^{-tx} x^n dx = n!$$

$$I = \frac{1}{2\alpha^2} (1)$$

given $I = 32$

$$\frac{1}{2\alpha^2} = 32$$

$$\frac{1}{\alpha^2} = 64$$

$$\alpha^2 = \frac{1}{64} \quad \alpha > 0$$

$$\alpha = \frac{1}{8}$$

Area

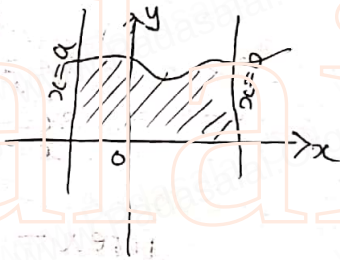
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(i) Area of the region bounded by the curve

$y=f(x)$ and $x=a$, $x=b$

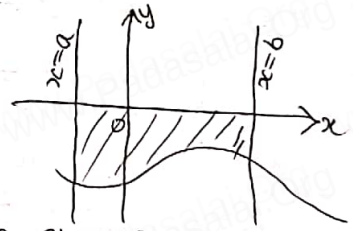
(i) lies above the x-axis

$$\text{Area} = \int_a^b y dx$$



(ii) lies below the x-axis

$$\text{Area} = - \int_a^b y dx$$

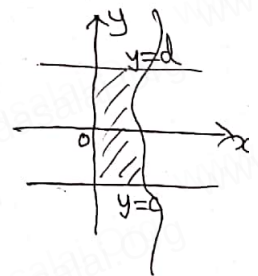


(2) Area of the region bounded by the curve

$x=f(y)$ and $y=c$, $y=d$.

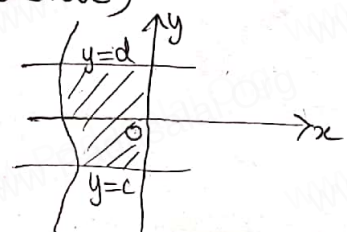
(i) lies +ve direction of x-axis (Right side)

$$\text{Area} = \int_c^d x dy$$



(ii) lies -ve direction of x-axis (Left side)

$$\text{Area} = - \int_c^d x dy$$



③ Area of the region bounded between two curves

i) $y = f(x), y = g(x)$ $f(x) \geq g(x) \forall x \in [a, b]$

Area bounded by two curves
(using vertical strips)

$$A = \int_a^b (y_U - y_L) dx$$

ii) $x = g(y), x = f(y)$ $f(y) \geq g(y) \forall y \in [c, d]$

Area bounded by two curves

$$A = \int_c^d (x_R - x_L) dy$$

(using horizontal strips)

Exercise 9.8

i) Find the area of the region bounded by $3x - 2y + 6 = 0$ to $x = -3, x = 1$ and x axis.

$$3x - 2y + 6 = 0$$

$$-2y = -3x - 6$$

$$y = \frac{3}{2}x + 3$$

$$x = 0 \Rightarrow y = 3 \quad (0, 3)$$

$$y = 0 \Rightarrow 3x = -6 \quad (-2, 0)$$

$$\text{Area} = \int_{-3}^1 y dx$$

$$= \int_{-3}^{-2} -y dx + \int_{-2}^1 y dx$$

$$= \int_{-3}^{-2} \left(-\frac{3}{2}x - 3\right) dx + \int_{-2}^1 \left(\frac{3}{2}x + 3\right) dx$$

$$= \left[-\frac{3x^2}{4} - 3x \right]_{-3}^{-2} + \left[\frac{3x^2}{4} + 3x \right]_{-2}^1$$

$$= \left(-\frac{3(4)}{4} - 3(-2) \right) - \left(-\frac{3(9)}{4} - 3(-3) \right) + \left[\left(\frac{3(1)}{4} + 3 \right) - \left(\frac{3(4)}{4} + 3(-2) \right) \right]$$

$$= (-3 + 6) - \left(-\frac{27}{4} + 9 \right) + \left[\frac{3}{4} + 3 - 3 + 6 \right]$$

$$= -3 + \frac{27}{4} - 9 + 3 + 6$$

$$= \frac{30}{4} = \frac{15}{2}$$

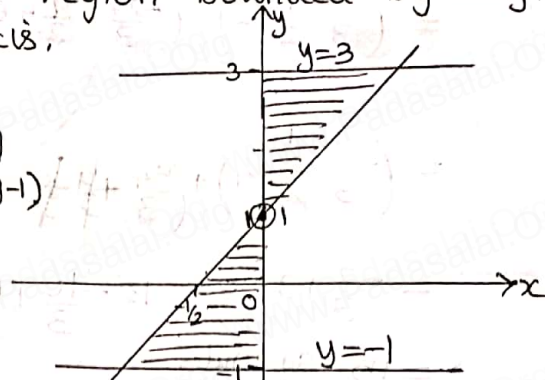
$$= 7.5 \text{ sq units}$$

- 2) Find the area of the region bounded by $2x-y+1=0$, $y=-1$, $y=3$ and y -axis.

$$\begin{aligned} 2x-y+1=0 \\ -y=-2x-1 \\ y=2x+1 \end{aligned} \quad \left| \begin{aligned} x &= y-1 \\ x &= \frac{1}{2}(y-1) \end{aligned} \right.$$

$$x=0 \Rightarrow y=1 \quad (0,1)$$

$$y=0 \Rightarrow x=-\frac{1}{2} \quad (-\frac{1}{2}, 0)$$



$$\text{Area} = \int_{-1}^3 x dy = \int_{-1}^1 -x dy + \int_1^3 x dy$$

$$= \int_{-1}^1 -\frac{1}{2}(y-1) dy + \int_1^3 \frac{1}{2}(y-1) dy$$

$$= -\frac{1}{2} \left[\frac{(y-1)^2}{2} \right]_{-1}^1 + \frac{1}{2} \left[\frac{(y-1)^2}{2} \right]_1^3$$

$$= -\frac{1}{4} [(1-1)^2 - (-1-1)^2] + \frac{1}{4} [(3-1)^2 - (1-1)^2]$$

$$= -\frac{1}{4} (0-4) + \frac{1}{4} (4-0)$$

$$= \frac{1}{4} + \frac{1}{4} = \frac{2}{4} = 1$$

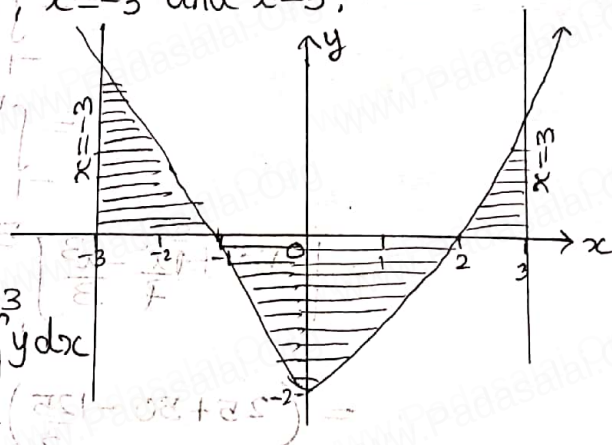
$$= 2 \text{ sq units}$$

- 3) Find the area of the region bounded by the curve $2+x-x^2+y=0$, x axis, $x=-3$ and $x=3$.

$$\begin{aligned} y &= x^2-x-2 \\ \text{put } y=0 & \quad x^2-x-2=0 \\ & \quad (x-2)(x+1)=0 \end{aligned}$$

$$x=2, -1$$

$$(2,0), (-1,0)$$



$$\text{Area} = \int_{-3}^{-1} y dx + \int_{-1}^2 -y dx + \int_2^3 y dx$$

$$= \int_{-3}^{-1} (x^2-x-2) dx - \int_{-1}^2 (x^2-x-2) dx + \int_2^3 (x^2-x-2) dx$$

$$= \left[\left(\frac{x^3}{3} - \frac{x^2}{2} - 2x \right) \right]_{-3}^{-1} - \left[\left(\frac{x^3}{3} - \frac{x^2}{2} - 2x \right) \right]_{-1}^2 + \left[\left(\frac{x^3}{3} - \frac{x^2}{2} - 2x \right) \right]_2^3$$

$$\begin{aligned}
 &= \left(-\frac{1}{3} - \frac{1}{2} + 2\right) - \left(-\frac{27}{3} - \frac{9}{2} + 6\right) - \left(\frac{8}{3} - \frac{4}{2} - 4\right) + \left(-\frac{1}{3} - \frac{1}{2} + 2\right) \\
 &\quad + \left(\frac{27}{3} - \frac{9}{2} - 6\right) - \left(\frac{8}{3} - \frac{4}{2} - 4\right) \\
 &= \left(-\frac{2}{3} - \frac{1}{2} + 4\right) + \left(\frac{54}{3} + \frac{18}{2} - 12\right) - \left(\frac{16}{3} - \frac{4}{2} - 8\right) \\
 &= -\frac{2}{3} + 3 + \frac{54}{3} - 12 - \frac{16}{3} + 12 = \frac{54-18}{3} + 3 = \frac{36}{3} + 3 \\
 &= 15 \text{ sq. units}
 \end{aligned}$$

4) Find the area of the region bounded by the line $y=2x+5$ and the parabola $y=x^2-2x$.

$$y=2x+5 \text{ --- (1)}$$

$$y=x^2-2x \text{ --- (2)}$$

$$x^2-2x=2x+5$$

$$x^2-4x-5=0$$

$$(x-5)(x+1)=0$$

$$x=5, -1$$

$$x=5 \Rightarrow$$

$$y=10+5$$

$$y=15$$

$$(5, 15)$$

$$x=-1 \Rightarrow$$

$$(1) \Rightarrow$$

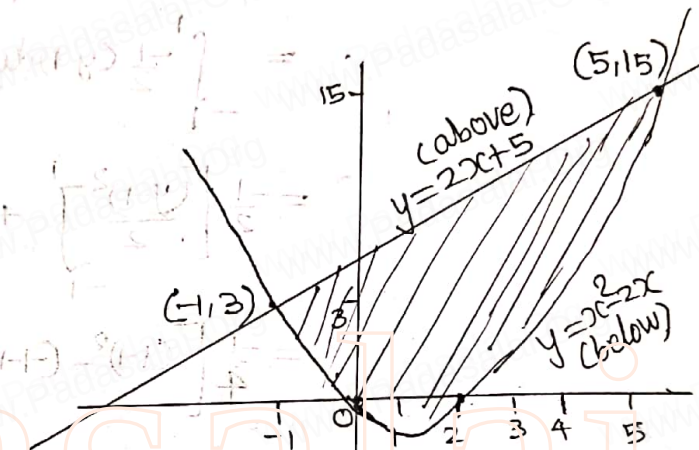
$$y=-2+5$$

$$y=3$$

$$(-1, 3)$$

The point of intersections

$$(-1, 3), (5, 15)$$



$$\text{Area} = \int (y_u - y_l) dx$$

$$= \int_{-1}^5 ((2x+5) - (x^2-2x)) dx$$

$$= \int_{-1}^5 (5+4x-x^2) dx$$

$$= \left[5x + \frac{4x^2}{2} - \frac{x^3}{3} \right]_{-1}^5$$

$$= \left(25 + 50 - \frac{125}{3} \right) - \left(-5 + 2 - \frac{1}{3} \right)$$

$$= 75 - \frac{125}{3} + 3 - \frac{1}{3} = 78 - \frac{126}{3}$$

$$= 36 \text{ sq. units.}$$

- 5) Find the area of the region bounded between the curves $y = \sin x$ and $y = \cos x$ and the lines $x=0$ and $x=\pi$.

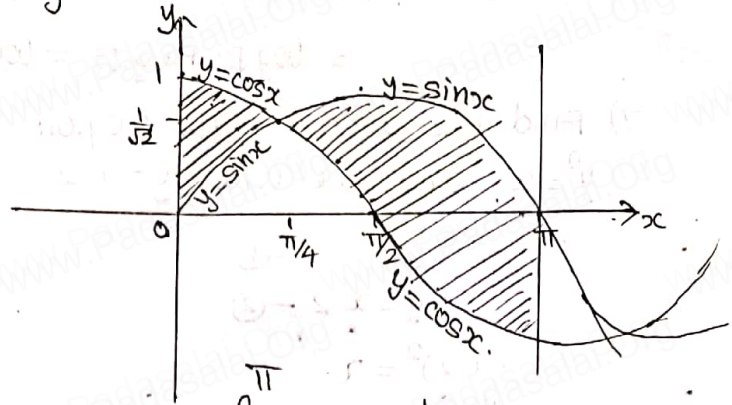
$$y = \sin x$$

$$y = \cos x$$

$$\sin x = \cos x$$

$$x = \pi/4$$

Area of the bounded region



$$= \int_0^{\pi/4} (y_U - y_L) dx + \int_{\pi/4}^{\pi} (y_U - y_L) dx$$

$$= \int_0^{\pi/4} (\cos x - \sin x) dx + \int_{\pi/4}^{\pi} (\sin x - \cos x) dx$$

$$= \left[+\sin x + \cos x \right]_0^{\pi/4} + \left[-\cos x - \sin x \right]_{\pi/4}^{\pi}$$

$$= \left[\left(\sin \frac{\pi}{4} + \cos \frac{\pi}{4} \right) - (\sin 0 + \cos 0) \right] - \left[(\cos \pi + \sin \pi) - \left(\cos \frac{\pi}{4} + \sin \frac{\pi}{4} \right) \right]$$

$$= \left[\frac{1}{\sqrt{2}} + \frac{1}{\sqrt{2}} - 1 \right] - \left[-1 - \frac{1}{\sqrt{2}} - \frac{1}{\sqrt{2}} \right]$$

$$= \frac{2}{\sqrt{2}} - 1 + 1 + \frac{2}{\sqrt{2}} = \frac{4}{\sqrt{2}} = \frac{2\sqrt{2}}{1}$$

$$= 2\sqrt{2} \text{ sq units.}$$

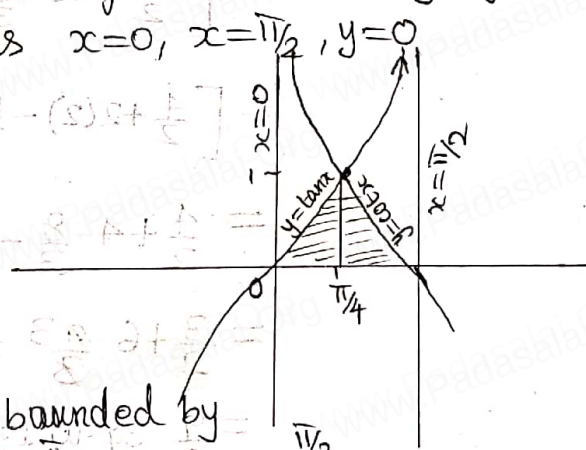
- 6) Find the area of the region bounded by $y = \tan x$, $y = \cot x$ and the lines $x=0$, $x=\pi/2$, $y=0$

$$y = \tan x$$

$$y = \cot x$$

$$\tan x = \cot x$$

$$x = \pi/4$$



Area of the region. bounded by

$$\text{the curves} = \int_0^{\pi/4} \tan x dx + \int_{\pi/4}^{\pi/2} \cot x dx$$

$$= \left[\log \sec x \right]_0^{\pi/4} + \left[\log \sin x \right]_{\pi/4}^{\pi/2}$$

$$= \left[\log \sec \frac{\pi}{4} - \log \sec 0 \right] + \left[\log \sin \frac{\pi}{2} - \log \sin \frac{\pi}{4} \right]$$

$$= \log \sqrt{2} - \log 1 + \log 1 - \log \frac{1}{\sqrt{2}}$$

$$= \log \sqrt{2} + \log \sqrt{2} = \log 2 \text{ sq units.}$$

7) Find the area of the region bounded by the parabola $y^2 = x$ and the line $y = x - 2$

$$y^2 = x \text{ --- (1)}$$

$$y = x - 2 \text{ --- (2)}$$

$$(x-2)^2 = x$$

$$x^2 - 4x + 4 - x = 0$$

$$x^2 - 5x + 4 = 0$$

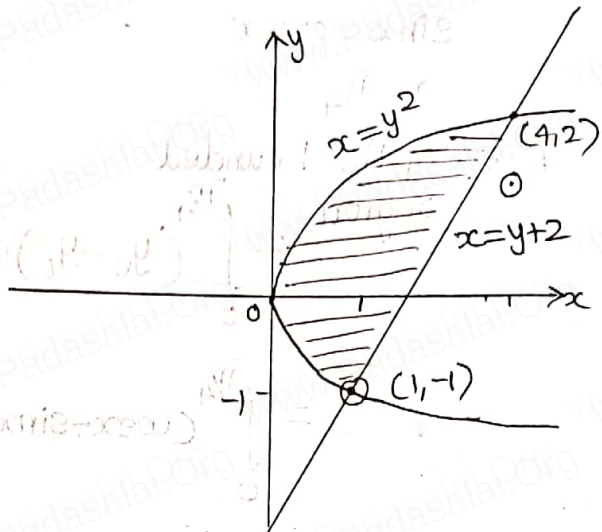
$$(x-1)(x-4) = 0$$

$$x = 1, 4$$

$$\textcircled{2} \Rightarrow x=1 \Rightarrow y=1-2=-1 \quad (1, -1)$$

$$x=4 \Rightarrow y=4-2=2 \quad (4, 2)$$

The point of intersections $(1, -1), (4, 2)$



Area of the bounded region

$$= \int_{-1}^2 (x_R - x_L) dy$$

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$$= \int_{-1}^2 (y+2 - y^2) dy$$

$$= \left[\frac{y^2}{2} + 2y - \frac{y^3}{3} \right]_{-1}^2$$

$$= \left[\frac{4}{2} + 2(2) - \frac{8}{3} \right] - \left[\frac{1}{2} - 2 + \frac{1}{3} \right]$$

$$= \frac{4}{2} + 4 - \frac{8}{3} - \frac{1}{2} + 2 - \frac{1}{3}$$

$$= \frac{3}{2} + 6 - \frac{9}{3} = \frac{3}{2} + 3 = \frac{3+6}{2}$$

$$= \frac{9}{2} \text{ sq. units}$$

8) Father of a family wishes to divide his square field bounded by $x=0$, $x=4$, $y=4$ and $y=0$ along the curve $y^2=4x$ and $x^2=4y$ into three parts for his wife, daughter and son. Is it possible to divide? If so find the area to be divided among them.

$$x^2 = 4y \quad \text{--- (1)}$$

$$y^2 = 4x \quad \text{--- (2)}$$

$$x = \frac{y^2}{4}$$

$$\text{(1)} \Rightarrow \frac{y^4}{16} = 4y$$

$$y^4 = 64y$$

$$y^4 - 64y = 0$$

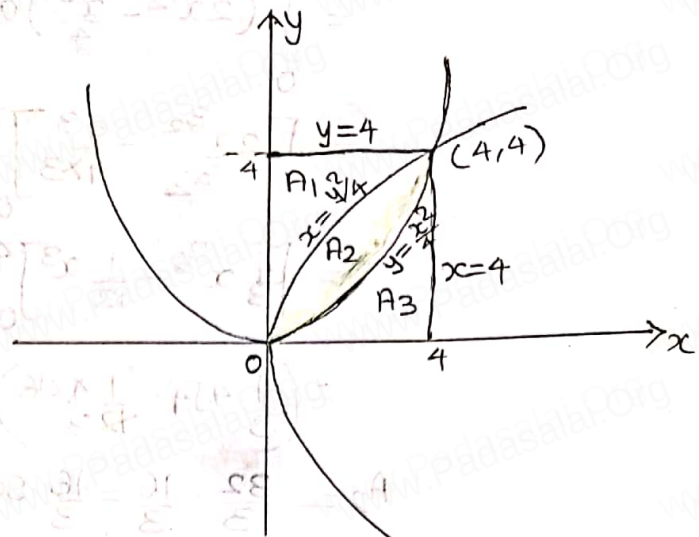
$$y(y^3 - 4^3) = 0$$

$$y = 0, y = 4$$

$$\text{(2)} \Rightarrow y = 0 \Rightarrow x = 0$$

$$y = 4 \Rightarrow 16 = 4x \Rightarrow x = 4$$

$\therefore$ point of intersections
(0,0), (4,4)



Area of the square = 4×4

= 16 sq units

$$\text{Area } A_1 = \int_0^4 x \, dy$$

$$= \int_0^4 \frac{y^2}{4} \, dy$$

$$= \left[\frac{y^3}{4 \times 3} \right]_0^4 = \frac{1}{12} [4^3 - 0]$$

$$= \frac{4 \times 16}{4 \times 3} = \frac{16}{3}$$

$$A_1 = \frac{16}{3} \text{ sq. units}$$

$$\text{Area } A_3 = \int_0^4 y \, dx = \int_0^4 \frac{x^2}{4} \, dx$$

$$= \frac{1}{4} \int_0^4 x^2 \, dx = \frac{1}{4} \left[\frac{x^3}{3} \right]_0^4$$

$$= \frac{1}{4 \times 3} [4^3 - 0] = \frac{1}{12} \times 4 \times 16 = \frac{16}{3} \text{ sq units}$$

$$\text{Area of } A_2 = \text{Total Area} - (A_1 + A_3)$$

$$= 16 - \frac{16}{3} - \frac{16}{3} = \frac{16}{3} \text{ sq. units.}$$

(The total Area is divided into 3 equal parts)

Each area is $\frac{16}{3}$ sq. units

$$[\text{Area of } A_2 = \int_0^4 (y_u - y_l) \, dx]$$

$$= \int_0^4 \left(2x^{1/2} - \frac{x^2}{4} \right) dx$$

$$= \left[2 \frac{x^{3/2}}{3/2} - \frac{x^3}{4 \times 3} \right]_0^4$$

$$= \left[\frac{4}{3} x^{3/2} - \frac{1}{12} x^3 \right]_0^4$$

$$= \left(\frac{4}{3} 4\sqrt{4} - \frac{1}{12} 4 \times 16 \right) - (0-0)$$

$$A_2 = \frac{32}{3} - \frac{16}{3} = \frac{16}{3} \text{ sq. units.}$$

9) The curve $y = (x-2)^2 + 1$ has a minimum point at P. A point Q on the curve is such that the slope of PQ is 2. Find the area bounded by the curve and the chord PQ.

$$y = (x-2)^2 + 1 \quad \text{--- (1)}$$

diff w.r to x

$$y' = \frac{dy}{dx} = 2(x-2)$$

$$\frac{dy}{dx} = 0 \Rightarrow 2(x-2) = 0$$

$$x-2 = 0$$

$$x = 2$$

$$y'' = 2$$

$$\text{When } x=2 \Rightarrow y'' = 2 > 0$$

y has minimum value at $x=2$

$$\text{(1)} \Rightarrow y = (2-2)^2 + 1 = 1$$

minimum point P(2,1)

Slope at Q $m=2$

Equation of PQ is $y = 2x + c$ --- (2)

This passing through P(2,1)

$$1 = 2(2) + c$$

$$1 - 4 = c \quad c = -3$$

Equation of PQ is $y = 2x - 3$ --- (3)

(1), (2) $\Rightarrow$

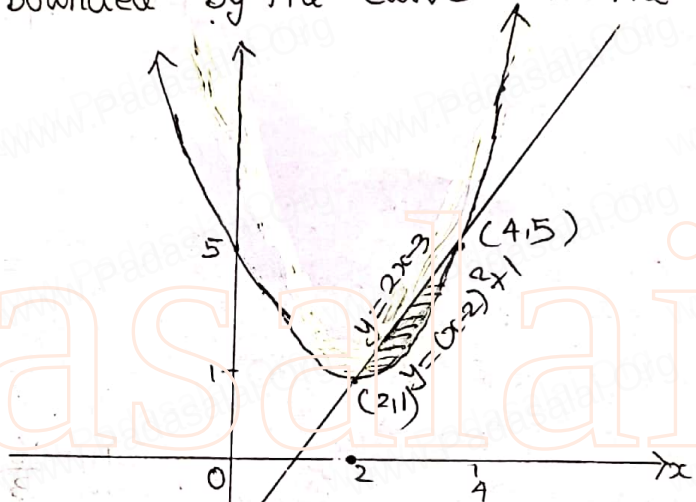
$$(x-2)^2 + 1 = 2x - 3$$

$$x^2 - 4x + 4 + 1 - 2x + 3 = 0$$

$$x^2 - 6x + 8 = 0$$

$$(x-2)(x-4) = 0$$

$$x = 2, 4$$



$$x=2 \quad \text{(1)} \Rightarrow y = 0 + 1 = 1$$

$$(2,1)$$

$$x=4 \quad \text{(1)} \Rightarrow y = 4 + 1 = 5$$

$$(4,5)$$

Area bounded by the curves

$$= \int_2^4 (y_U - y_L) dx$$

$$= \int_2^4 (2x - 3 - (x-2)^2 - 1) dx$$

$$= \int_2^4 (2x - 4 - (x-2)^2) dx$$

$$\begin{aligned}
 &= \left[\frac{2x^2}{2} - 4x - \frac{(x-2)^3}{3} \right]_2^4 \\
 &= \left(16 - 16 - \frac{2^3}{3} \right) - \left(4 - 8 - 0 \right) \\
 &= -\frac{8}{3} + 4 = -\frac{8+12}{3} \\
 &= \frac{4}{3} \text{ sq. units.}
 \end{aligned}$$

10) Find the area of the region common to the circle $x^2 + y^2 = 16$ and the parabola $y^2 = 6x$.

$$x^2 + y^2 = 16 \quad \text{--- (1)}$$

$$y^2 = 6x \quad \text{--- (2)}$$

$$y^2 = 16 - x^2$$

$$y = \sqrt{4^2 - x^2}$$

$$\textcircled{1} \Rightarrow x^2 + 6x = 16$$

$$x^2 + 6x - 16 = 0$$

$$(x+8)(x-2) = 0$$

$$x = -8, 2$$

$$x = -8$$

$$\textcircled{2} \Rightarrow y^2 = 6(-8) \text{ not valid}$$

$$x = 2$$

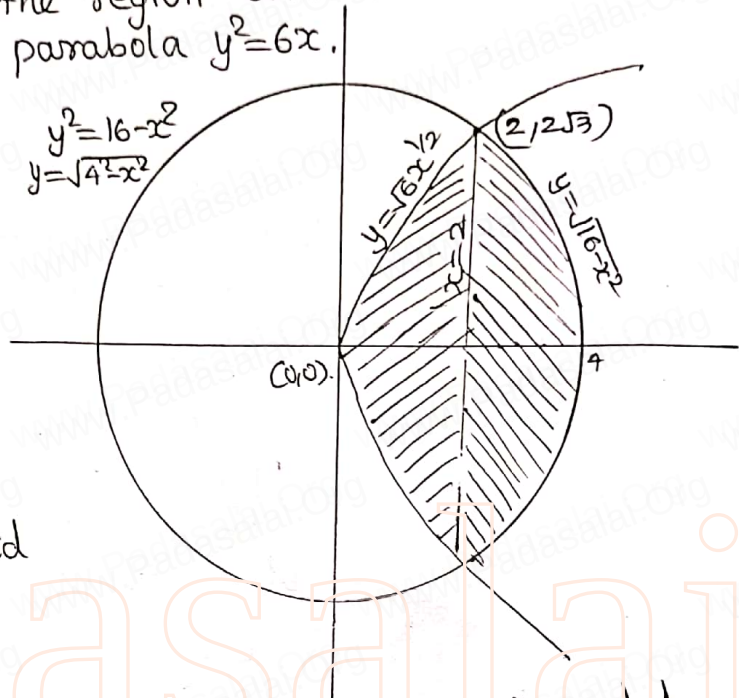
$$\textcircled{2} \Rightarrow y^2 = 6(2)$$

$$y^2 = 4 \times 3$$

$$y = \pm 2\sqrt{3}$$

$\therefore$ point of intersection

$$(2, 2\sqrt{3}), (2, -2\sqrt{3})$$



Area of the region bounded

by the curves

$$= 2 \left[\text{Area lie on the first quadrant} \right]$$

$$= 2 \left[\int_0^2 \sqrt{6x}^{1/2} dx + \int_2^4 \sqrt{4^2 - x^2} dx \right]$$

$$= 2 \left(\left[\sqrt{6} \frac{x^{3/2}}{3/2} \right]_0^2 + \left[\frac{x}{2} \sqrt{16 - x^2} + \frac{16}{2} \sin^{-1} \frac{x}{4} \right]_2^4 \right)$$

$$= 2 \left(\left[\sqrt{6} \frac{2}{3} (2^{3/2} - 0) \right] + \left[\left(\frac{4}{2} (0) + 8 \sin^{-1} \left(\frac{4}{4} \right) \right) - \left(\frac{2}{2} \sqrt{16 - 4} + 8 \sin^{-1} \left(\frac{1}{2} \right) \right) \right] \right)$$

$$= 2 \left(\frac{2\sqrt{6}}{3} 2\sqrt{2} + \frac{4}{2} \pi - \sqrt{12} - 8 \left(\frac{\pi}{6} \right) \right)$$

$$= 2 \left(\frac{8\sqrt{3}}{3} + 4\pi - 2\sqrt{3} - \frac{4\pi}{3} \right)$$

$$= 2 \left(\frac{8\sqrt{3} - 6\sqrt{3}}{3} + \frac{12\pi - 4\pi}{3} \right) = \frac{2}{3} [2\sqrt{3} + 8\pi] = \frac{4}{3} (4\pi + \sqrt{3})$$

sq. units,

Volume

- ① The curve $y=f(x)$ and the line $x=a$ and $x=b$
Then the volume of solid of revolution about x axis
is

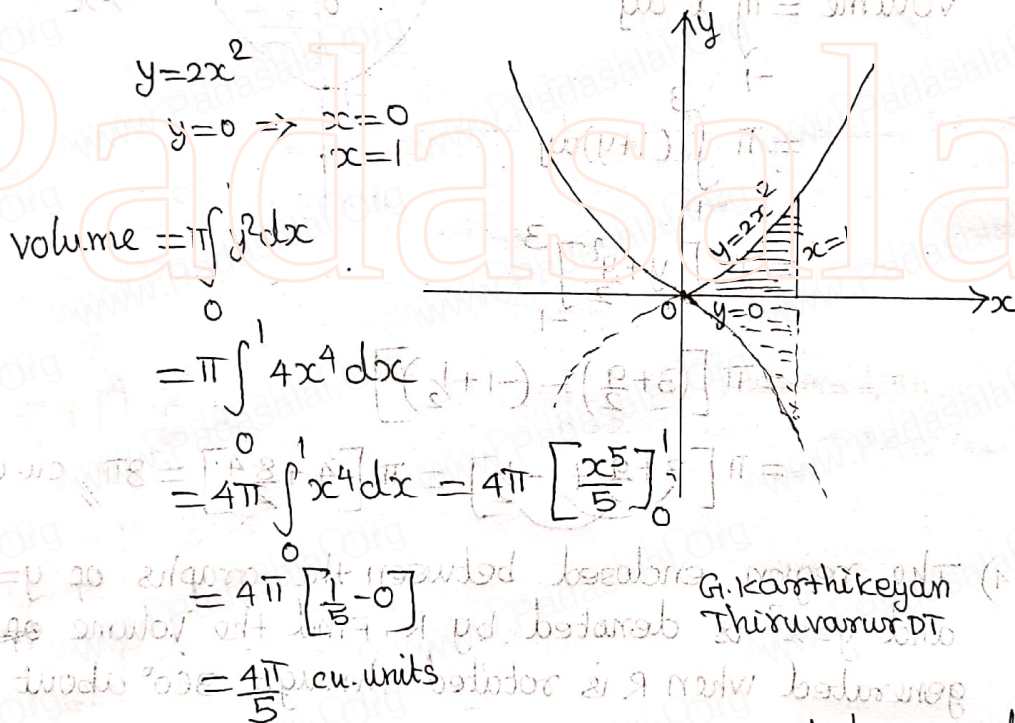
$$V = \pi \int_a^b y^2 dx$$

- ② The curve $x=f(y)$ and the line $y=c$, $y=d$
The volume of solid of revolution about y axis
is

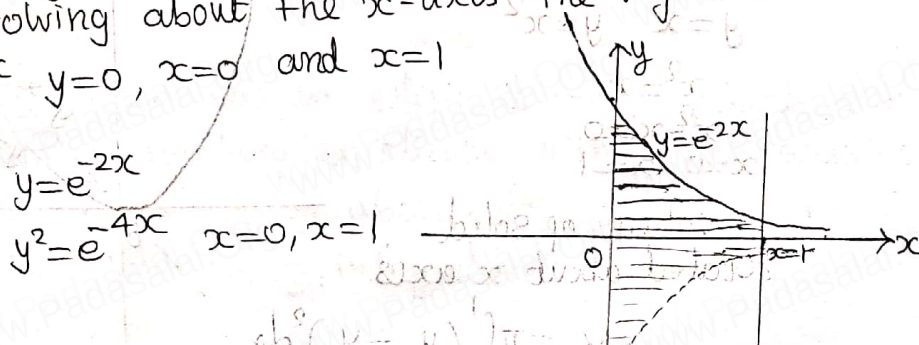
$$V = \pi \int_c^d x^2 dy$$

Exercise 9.9

- i) Find by integration, the volume of the solid generated by revolving about the x -axis, the region enclosed by $y=2x^2$, $y=0$ and $x=1$



- ② Find by integration the volume of solid generated by revolving about the x -axis, the region enclosed by $y=e^{-2x}$, $y=0$, $x=0$ and $x=1$



$$\text{Volume} = \pi \int_0^1 y^2 dx$$

$$= \pi \int_0^1 e^{-4x} dx$$

$$= \pi \left[\frac{e^{-4x}}{-4} \right]_0^1$$

$$= -\frac{\pi}{4} [e^{-4} - e^0] = \frac{\pi}{4} (1 - e^{-4}) \text{ cu. units}$$

- 3) Find, by integration the volume of the solid generated by revolving about the y axis, the region enclosed by $x^2 = 1+y$ and $y=3$

$$x^2 = 1+y$$

$$x=0 \Rightarrow y=-1$$

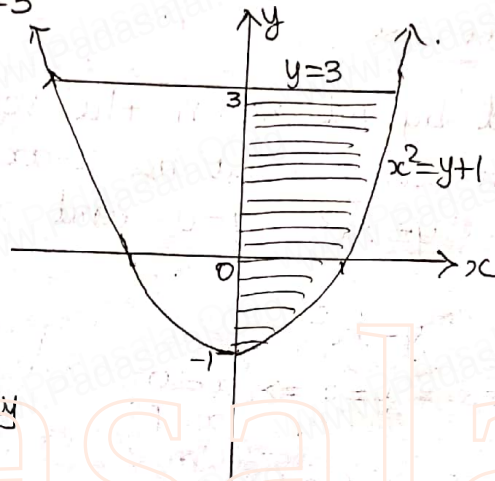
$$\text{Volume} = \pi \int_{-1}^3 x^2 dy$$

$$= \pi \int_{-1}^3 (1+y) dy$$

$$= \pi \left[y + \frac{y^2}{2} \right]_{-1}^3$$

$$= \pi \left[\left(3 + \frac{9}{2} \right) - \left(-1 + \frac{1}{2} \right) \right]$$

$$= \pi \left[3 + \frac{9}{2} + 1 - \frac{1}{2} \right] = \pi \left[4 + \frac{8}{2} \right] = 8\pi \text{ cu. units}$$



- 4) The region enclosed between the graphs of $y=x$ and $y=x^2$ is denoted by R. Find the volume of the solid generated when R is rotated through 360° about x axis.

$$y=x \quad y=x^2$$

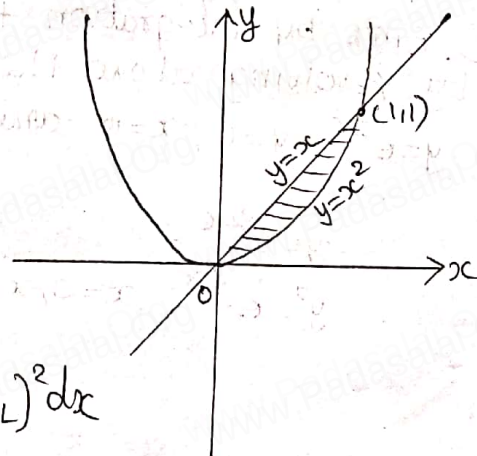
$$x^2=x$$

$$x^2-x=0$$

$$x=0, x=1$$

Volume of solid
rotated about x axis

$$V = \pi \int_0^1 (y_U - y_L)^2 dx$$



$$\begin{aligned}
 V &= \pi \int_0^1 (x-x^2)^2 dx \\
 &= \pi \int_0^1 (x^2+x^4-2x^3) dx \\
 &= \pi \left[\frac{x^3}{3} + \frac{x^5}{5} - 2\frac{x^4}{4} \right]_0^1 \\
 &= \pi \left(\left[\frac{1}{3} + \frac{1}{5} - \frac{2}{2} \right] - 0 \right) = \pi \left[\frac{10+6-15}{30} \right] \\
 V &= \frac{\pi}{30} \quad (\text{Book answer varies})
 \end{aligned}$$

5) Find by Integration the volume of the container which is in the shape of a right circular conical frustum as in the figure.

Equation of line joining $(1,2), (2,4)$ is

$$\frac{y-2}{4-2} = \frac{x-1}{2-1}$$

$$y-2 = 2x-2$$

$$y = 2x$$

$$y^2 = 4x^2$$

volume of frustum

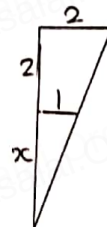
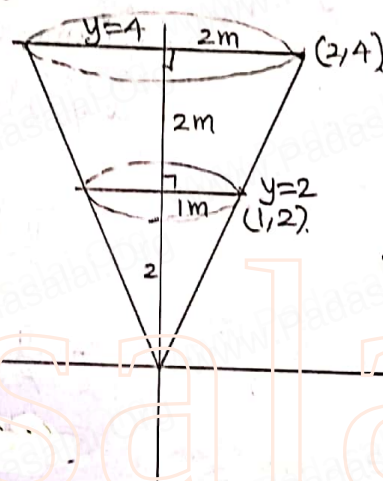
$$= \pi \int_2^4 x^2 dy$$

$$= \pi \int_2^4 \frac{y^2}{4} dy$$

$$= \frac{\pi}{4} \left[\frac{y^3}{3} \right]_2^4$$

$$= \frac{\pi}{3 \times 4} [4^3 - 2^3] = \frac{\pi}{12} [64 - 8]$$

$$= \frac{\pi}{12} 56 = \frac{14\pi}{3} \text{ cu. units}$$



By Similar Δ

$$\frac{x}{1} = \frac{y+2}{2}$$

$$2x = y+2$$

$$x = 2$$

$$\begin{aligned}
 R &= 2 \\
 r &= 1 \\
 h &= 2 \\
 \text{Volume of frustum} &= \frac{1}{3} \pi h (R^2 + r^2 + Rr) \\
 &= \frac{1}{3} \pi (2) (4 + 1 + 2) \\
 &= \frac{14\pi}{3}
 \end{aligned}$$

6) A watermelon has an ellipsoid shape which can be obtained by revolving an ellipse with major-axis 20cm and minor axis 10 cm about its major axis. Find its volume using integration.

$$2a=20$$

$$2b=10$$

$$a=10$$

$$b=5$$

$$a^2=100$$

$$b^2=25$$

Equation of Ellipse

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$$

$$\frac{x^2}{100} + \frac{y^2}{25} = 1$$

$$\frac{y^2}{25} = \frac{100-x^2}{100}$$

$$y^2 = \frac{25}{100} (100-x^2)$$

$$y^2 = \frac{1}{4} (100-x^2)$$

Volume = 2 (Volume generated by area bounded in I quadrant)

$$\text{Volume} = 2 \int_0^{10} \pi y^2 dx$$

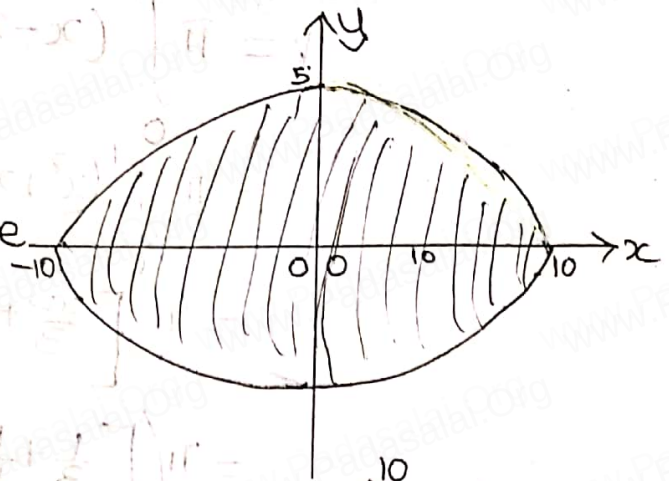
$$= 2\pi \int_0^{10} \frac{1}{4} (100-x^2) dx$$

$$= \frac{\pi}{2} \left[100x - \frac{x^3}{3} \right]_0^{10}$$

$$= \frac{\pi}{2} \left[(1000 - \frac{1000}{3}) - 0 \right]$$

$$= \frac{\pi}{2} \left[\frac{2000}{3} \right]$$

$$= \frac{1000\pi}{3} \text{ cu. cm.}$$



Need suggestions

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①

10. ORDINARY DIFFERENTIAL EQUATIONS

Differential Equation, Order, Degree

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THIRUVARUR DT.Order - highest order derivative present in the D.E.Degree - Integral power of the highest order derivative (D.E. is expressible in a polynomial form)- If not expressible as a highest order derivative as the leading term then the degree of D.E. is not defined.

Exercise 10.1

i) determine the order, degree (if exists) of the D.E.

(i) $\left(\frac{dy}{dx}\right) + xy = \cot x$

order = 1
degree = 1

(ii) $\left(\frac{d^3y}{dx^3}\right)^{2/3} - 3\frac{d^2y}{dx^2} + 5\frac{dy}{dx} + 4 = 0$

$\left(\frac{d^3y}{dx^3}\right)^{2/3} = 3\frac{d^2y}{dx^2} - 5\frac{dy}{dx} - 4$

cubing both sides.

$\left(\frac{d^3y}{dx^3}\right)^2 = \left(3\frac{d^2y}{dx^2} - 5\frac{dy}{dx} - 4\right)^3$

order = 3
degree = 2

(iii) $\left(\frac{d^2y}{dx^2}\right)^2 + \left(\frac{dy}{dx}\right)^2 = x \sin\left(\frac{d^2y}{dx^2}\right)$

order = 2

This cannot be expressed as polynomial with $\frac{d^2y}{dx^2}$ as the leading term.

degree = not defined.

(iv) $\sqrt{\frac{dy}{dx}} - 4\frac{dy}{dx} - 7x = 0$

$\sqrt{\frac{dy}{dx}} = 4\frac{dy}{dx} + 7x$

squaring both sides

$\frac{dy}{dx} = \left(4\frac{dy}{dx} + 7x\right)^2$

$\frac{dy}{dx} = 16\left(\frac{dy}{dx}\right)^2 + 56x\frac{dy}{dx} + 49x^2$

order = 1
degree = 2

$16\left(\frac{dy}{dx}\right)^2 + 56x\frac{dy}{dx} - \frac{dy}{dx} + 49x^2 = 0$

$$(v) y \left(\frac{dy}{dx} \right) = \frac{x}{\left(\frac{dy}{dx} \right) + \left(\frac{dy}{dx} \right)^3}$$

$$y \left(\frac{dy}{dx} \right)^2 + y \left(\frac{dy}{dx} \right)^1 = x$$

order = 1
degree = 4

$$(vi) x^2 \frac{d^2y}{dx^2} + \left[1 + \left(\frac{dy}{dx} \right)^2 \right]^{\frac{1}{2}} = 0$$

$$x^2 \frac{d^2y}{dx^2} = - \left[1 + \left(\frac{dy}{dx} \right)^2 \right]^{\frac{1}{2}}$$

squaring both sides

$$x^4 \left(\frac{d^2y}{dx^2} \right)^2 = 1 + \left(\frac{dy}{dx} \right)^2$$

order = 2
degree = 2

$$(vii) \left(\frac{d^2y}{dx^2} \right)^3 = \sqrt{1 + \left(\frac{dy}{dx} \right)}$$

squaring both sides

$$\left(\frac{d^2y}{dx^2} \right)^6 = 1 + \frac{dy}{dx}$$

order = 2
degree = 6

$$(viii) \frac{d^2y}{dx^2} = xy + \cos \left(\frac{dy}{dx} \right)$$

order = 2
degree = does not exist

$$(ix) \frac{d^2y}{dx^2} + 5 \frac{dy}{dx} + \int y dx = x^3$$

diff. w.r to x

$$\frac{d^3y}{dx^3} + 5 \frac{d^2y}{dx^2} + y = 3x^2$$

order = 3
degree = 1

$$(x) x = e^{xy \frac{dy}{dx}}$$

taking log

$$\log x = xy \frac{dy}{dx}$$

order = 1

degree = 1

(book answer wrong)

classification of D.E.

Ordinary D.Equation (ODE)

contains only ordinary derivatives of one or more functions w.r.to. a single independent variable

Partial D.Equation (PDE)

Involving only partial derivatives of one or more functions of 2 or more independent variable,

1) Express each of the following physical statements in the form of differential equation.

(i) Radium decays at a rate proportional to the amount Q present.

given Q be the amount of Radium at present any time t

$$\frac{dQ}{dt} \propto Q$$

$$\frac{dQ}{dt} = kQ \quad (k \text{ is a constant})$$

(ii) The population P of a city increases at a rate proportional to the product of population and to the difference between 5,00,000 and the population.

P be the population

rate of population $\propto P(500000 - P)$

$$\frac{dP}{dt} \propto P(500000 - P)$$

$$\frac{dP}{dt} = kP(500000 - P)$$

(iii) For a certain substance, the rate of change of vapor pressure P w.r.to temperature T is proportional to the vapor pressure and inversely proportional to the square of the temperature.

rate of change of P w.r.to $T \propto \frac{P}{T^2}$

$$\frac{dP}{dT} \propto \frac{P}{T^2}$$

$$\frac{dP}{dT} = \frac{kP}{T^2}$$

book answer
 $\frac{dP}{dT} = \frac{kP}{T^2}$
wrong

(iv) A saving amount pays 8% interest per year, compounded continuously. In addition, the income from another investment is credited to the amount continuously at the rate of ₹ 400 per year.

Let x be the amount any time t

$$\frac{dx}{dt} \propto x$$

$$\frac{dx}{dt} = kx$$

$$k = 8\% = \frac{8}{100}$$

$$\frac{dx}{dt} = \frac{8}{100}x$$

400 income added

$$\frac{dx}{dt} = \frac{8}{100}x + 400$$

$$\frac{dx}{dt} = \frac{2x}{25} + 400$$

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2) Assume that a spherical rain drop evaporates at a rate proportional to its surface area. Form a D.E. involving the rate of change of the radius of the rain drop.

Let r be the radius V be the volume and A be the surface area of spherical rain drop

$$\frac{dV}{dt} \propto -A$$

$$\frac{dV}{dt} = -kA$$

$$\text{Volume of sphere} = \frac{4}{3}\pi r^3$$

$$SA = 4\pi r^2$$

$$\frac{d}{dt}\left(\frac{4}{3}\pi r^3\right) = -k \cdot 4\pi r^2$$

$$\frac{4\pi}{3} \frac{d}{dt}(r^3) = -4\pi k r^2$$

$$\frac{1}{3} r^2 \frac{dr}{dt} = -k r^2$$

$$\frac{dr}{dt} = -k$$

This is the required D.E.

Formation of D.E.

Exercise 10.3

i) Find the differential equation of the family of

(i) all non-vertical lines in a plane

(ii) all non-horizontal lines in a plane.

(i) The equation of any non-vertical family of line in a plane is

$$ax + by + c = 0 \text{ where } a \in \mathbb{R}, b \neq 0$$

a, b two constants - diff. w. x to x two times

$$a + by' = 0 \therefore y' = -a/b$$

$$y'' = 0$$

$$\text{or } \frac{d^2y}{dx^2} = 0$$

when $b=0 \Rightarrow$
 $x = \text{const vertical line}$

(ii) The equation of all non-horizontal family of line in a plane is

$$ax + by = C \text{ with } a \neq 0$$

diff w. x to y

$$\frac{d}{dy}(ax + b) = 0$$

$$\frac{dx}{dy} = -\frac{b}{a}$$

again diff w. x to y

$$\frac{d^2x}{dy^2} = 0$$

when $a=0 \Rightarrow$
 $y = \text{constant horizontal}$

- 2) Form the differential equation of all straight lines touching the circle $x^2 + y^2 = r^2$

Equation of tangent to the circle $x^2 + y^2 = r^2$ is

to eliminate m only

(m is a arbitrary constant) $y = mx \pm r\sqrt{1+m^2}$ — ①

diff w. r to x

$$\frac{dy}{dx} = m$$

① $\Rightarrow y = \frac{dy}{dx}(x) \pm r\sqrt{1+(\frac{dy}{dx})^2}$

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$$y - x \frac{dy}{dx} = \pm r\sqrt{1+(\frac{dy}{dx})^2}$$

squaring both sides

$$(y - x \frac{dy}{dx})^2 = r^2 (1 + (\frac{dy}{dx})^2)$$

- 3) Find the differential equation of the family of circles passing through the origin and having their centres on the x -axis

Equation of family of circles passing through origin and centre $(a, 0)$ is

$$(x-a)^2 + (y-0)^2 = a^2$$

$$(x-a)^2 + y^2 = a^2 \text{ — ①}$$

a is only arbitrary constant

diff w. r to x

$$2(x-a) + 2y \frac{dy}{dx} = 0$$

$$x-a = -y \frac{dy}{dx}$$

$$a = x + y \frac{dy}{dx}$$

① $\Rightarrow (-y \frac{dy}{dx})^2 + y^2 = (x + y \frac{dy}{dx})^2$

$$y^2 (\frac{dy}{dx})^2 + y^2 = x^2 + y^2 (\frac{dy}{dx})^2 + 2xy \frac{dy}{dx}$$

$$x^2 + 2xy \frac{dy}{dx} - y^2 = 0$$

- 4) Find the differential equation of the family of all the parabolas with the latus rectum $4a$ and whose axes are parallel to the x -axis.

Equation of parabola.

(h, k are arbitrary constants) $(y-k)^2 = 4a(x-h)$ — ①

diff. w. r to x

eliminate h, k only

$$2(y-k) y' = 4a \quad (1)$$

$$(y-k) y' = 2a \quad \text{--- (2)}$$

again diff w.r to x :

$$(y-k) y'' + (y') (y') = 0$$

$$(y-k) y'' = -y'^2$$

$$(y-k) = -\frac{y'^2}{y''}$$

$$\textcircled{2} \Rightarrow -\frac{y'^2}{y''} y' = 2a$$

$$-(y')^3 = 2ay''$$

$$2ay'' + (y')^3 = 0$$

- 5) Find the differential equation of the family of parabolas with vertex at $(0, -1)$ and having axis along the y -axis

Equation of parabola

$$(x-0)^2 = 4a(y+1)$$

$$x^2 = 4a(y+1) \quad \text{--- (1)}$$

eliminate a only

a is arbitrary constant

diff w.r to x

$$2x = 4a \frac{dy}{dx}$$

$$\frac{x}{y'} = 2a$$

$$\textcircled{1} \Rightarrow x^2 = 2\left(\frac{x}{y'}\right)(y+1)$$

$$\frac{y' x^2}{x} = 2y + 2$$

$$xy' - 2y - 2 = 0$$

- 6) Find the differential equations of the family of all the ellipses having foci on the y -axis and centre at the origin.

Equation of Ellipse

$$\frac{x^2}{b^2} + \frac{y^2}{a^2} = 1 \quad \text{--- (1)}$$

eliminate a, b

a, b are arbitrary constant

diff w.r to x

$$\frac{2x}{b^2} + \frac{2yy'}{a^2} = 0$$

$$\frac{x}{b^2} + \frac{yy'}{a^2} = 0 \quad \text{--- (2)}$$

again diff w.r to x

$$\frac{1}{b^2} + \frac{(yy'' + y'y')}{a^2} = 0 \quad \text{--- (3)}$$

From

①②③ Eliminate $\frac{1}{a^2}, \frac{1}{b^2}$

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$$\begin{vmatrix} x^2 & y^2 & 1 \\ x & yy' & 0 \\ 1 & yy'' + y'^2 & 0 \end{vmatrix} = 0$$

$$x^2(0-0) - y^2(0-0) + 1(xy y'' + y'^2 - yy') = 0$$

$$xy y'' + xy'^2 - yy' = 0.$$

7) Find the differential equation corresponding to the family of curves represented by the equation $y = Ae^{8x} + Be^{-8x}$ where A and B are arbitrary constants.

$$y = Ae^{8x} + Be^{-8x}$$

diff w.r to x

$$\frac{dy}{dx} = 8Ae^{8x} + B(-8)e^{-8x}$$

again diff w.r to x

$$\frac{d^2y}{dx^2} = 64Ae^{8x} + 64Be^{-8x}$$

$$= 64(Ae^{8x} + Be^{-8x})$$

$$\frac{d^2y}{dx^2} = 64y$$

8) Find the differential equation of the curve represented by $xy = ae^x + be^{-x} + x^2$

$$xy - x^2 = ae^x + be^{-x} \quad \text{--- ①}$$

a, b are arbitrary constants to eliminate a, b

diff w.r to x

$$xy' + y - 2x = ae^x - be^{-x}$$

again diff w.r to x

$$xy'' + y' + y' - 2 = ae^x + be^{-x}$$

Using ① $\Rightarrow xy'' + 2y' - 2 = xy - x^2$

$$xy'' + 2y' + x^2 - xy - 2 = 0.$$

————— x —————

Example 10.6 Find the D.E of the family of all ellipses having foci on the x-axis and centre at origin.

Equation of ellipse

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1 \quad \text{--- ①}$$

a, b are arbitrary constant

diff w.r to x

$$\frac{2x}{a^2} + \frac{2yy'}{b^2} = 0$$

$$\frac{x}{a^2} + \frac{yy'}{b^2} = 0$$

again diff. w.r to x

$$\frac{1}{a^2} + \frac{(yy'' + y'y')}{b^2} = 0$$

Eliminate $\frac{1}{a^2}, \frac{1}{b^2}$ compare its co-efficients.

$$\begin{vmatrix} x^2 & y^2 & 1 \\ x & yy' & 0 \\ 1 & yy'' + y'y' & 0 \end{vmatrix} = 0$$

$$1(x(yy'' + y'y')) - yy' = 0$$

$$xyy'' + xy'^2 - yy' = 0$$

solution of ODE.

Exercise 10.4

① show that each of the following expressions is a solution of the corresponding given differential equation.

i) $y = 2x^2$; $xy' = 2y$

$y = 2x^2$ — ①
diff w.r to x

$$y' = 4x$$

multiply by x

$$xy' = 4x^2$$

$$xy' = 2(2x^2)$$

$$\text{using ①} \Rightarrow xy' = 2y$$

(ii) $y = ae^x + be^{-x}$; $y'' - y = 0$

$$y = ae^x + be^{-x}$$

a, b are arbitrary constants.

diff w.r to x

$$y' = ae^x - be^{-x}$$

again diff. w.r to x

$$y'' = ae^x + be^{-x}$$

$$y'' = y \Rightarrow y'' - y = 0$$

② Find value of m so that the function $y = e^{mx}$ is a solution of the given differential equation

i) $y' + 2y = 0$ — ①

$$y = e^{mx}$$

diff. w.r to x

$$y' = e^{mx} m$$

$$y' = my$$

$$y' - my = 0 \text{ — ②}$$

compare ① & ② $-m = 2$
 $m = -2$

$$(ii) \quad y'' - 5y' + 6y = 0 \quad \text{--- ②}$$

$$\Rightarrow y' - my = 0$$

$$\boxed{y' = my}$$

again diff w.r to x

$$y'' - my' = 0$$

$$y'' = my'$$

$$= m my'$$

$$\boxed{y'' = m^2 y}$$

$$\Rightarrow m^2 y - 5my + 6y = 0$$

$$y(m^2 - 5m + 6) = 0$$

$$y = 0, \quad m^2 - 5m + 6 = 0$$

$$(m-2)(m-3) = 0$$

$$m = 2, 3$$

③ The slope of the tangent to the curve at any point is the reciprocal of 4 times the ordinate at that point. The curve passes through (2, 5). Find the equation of the curve.

Let the point be P(x, y)

$$\text{slope} = \frac{1}{4y}$$

$$\frac{dy}{dx} = \frac{1}{4y}$$

$$4y dy = dx$$

Integrating

$$4 \int y dy = \int dx$$

$$4 \cdot \frac{y^2}{2} = x + C$$

$$2y^2 = x + C \quad \text{--- ①}$$

This curve passing through (2, 5)

$$\Rightarrow 2(5)^2 = 2 + C$$

$$C = 48$$

$$\Rightarrow \text{Required equation of curve } 2y^2 = x + 48$$

4) Show that $y = e^{-x} + mx + n$ is a solution of the differential equation $e^x \left(\frac{d^2 y}{dx^2} \right) - 1 = 0$

$$y = e^{-x} + mx + n$$

two constants m, n

diff. w.r to x

$$\frac{dy}{dx} = e^{-x}(-1) + m$$

again diff w.r to x.

$$\frac{d^2 y}{dx^2} = e^{-x}$$

$$\Rightarrow \frac{d^2 y}{dx^2} = \frac{1}{e^x}$$

$$e^x \left(\frac{d^2 y}{dx^2} \right) = 1$$

$$e^x \left(\frac{d^2 y}{dx^2} \right) - 1 = 0$$

$\therefore$ given function is a solution of the given D.E.

5) show that $y = ax + \frac{b}{x}$, $x \neq 0$ is a solution of the differential equation $x^2 y'' + xy' - y = 0$

$$y = ax + \frac{b}{x} \quad \text{--- (1)} \quad \text{two constants } a, b$$

diff w.r to x .

$$y' = a(1) + b(-\frac{1}{x^2}) \quad \text{--- (2)}$$

again diff. w.r to x

$$y'' = 0 + \frac{2b}{x^3}$$

$$2b = x^3 y''$$

$$b = \frac{x^3 y''}{2}$$

$$\textcircled{2} \Rightarrow y' = a - \frac{x^3 y''}{2} \times \frac{1}{x^2}$$

$$y' = a - \frac{xy''}{2}$$

$$\frac{y' + \frac{xy''}{2}}{2} = a$$

$$\textcircled{1} \Rightarrow y = \left(\frac{y' + \frac{xy''}{2}}{2} \right) x + \frac{x^2 y''}{2x}$$

$$y = \frac{xy' + \frac{x^2 y''}{2}}{2} + \frac{x^2 y''}{2}$$

$$y = \frac{xy' + 2x^2 y''}{2}$$

$$x^2 y'' + xy' - y = 0$$

$\therefore$ The given function is a solution of given D.E.

6) show that $y = ae^{-3x} + b$, where a and b are arbitrary constants, is a solution of the D.E $\frac{d^2 y}{dx^2} + 3 \frac{dy}{dx} = 0$

$$y = ae^{-3x} + b$$

diff w.r to x

$$\frac{dy}{dx} = a(-3)e^{-3x} + 0 \Rightarrow \frac{dy}{dx} = -3ae^{-3x} \quad \text{--- (1)}$$

again diff w.r to x

$$\frac{d^2 y}{dx^2} = 9ae^{-3x}$$

$$\frac{d^2 y}{dx^2} = -3(-3ae^{-3x})$$

$$\frac{d^2 y}{dx^2} = -3 \frac{dy}{dx}$$

$$\Rightarrow \frac{d^2 y}{dx^2} + 3 \frac{dy}{dx} = 0$$

$\therefore$ given function is a solution of given D.E.

7) show that the D.E. representing the family of curves $y^2 = 2a(x + a^{2/3})$, where a is a +ve parameter, is

$$\left(y^2 - 2xy \frac{dy}{dx} \right)^3 = 8 \left(y \frac{dy}{dx} \right)^5$$

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given curve $y^2 = 2a(x + a^{2/3})$ — (1)

diff w.r to x

$$2y \frac{dy}{dx} = 2a(1+0)$$

$$\boxed{y \frac{dy}{dx} = a}$$

$$\textcircled{1} \Rightarrow y^2 = 2y \frac{dy}{dx} (x + (y \frac{dy}{dx})^{2/3})$$

$$y^2 = 2 \frac{dy}{dx} xy + 2(y \frac{dy}{dx})^{2/3} \frac{dy}{dx} y$$

$$y^2 - 2xy \frac{dy}{dx} = 2 y^{5/3} (\frac{dy}{dx})^{5/3}$$

cubing both sides

$$(y^2 - 2xy \frac{dy}{dx})^3 = 8 (y \frac{dy}{dx})^5$$

∴ The given family of curve gives the D.E.

8) show that $y = a \cos bx$ is a solution of the D.E.

$$\frac{d^2y}{dx^2} + b^2y = 0$$

$$y = a \cos bx$$

diff w.r to x

$$\frac{dy}{dx} = a(-\sin bx) b$$

again diff w.r to x

$$\frac{d^2y}{dx^2} = -a \cos bx (b^2)$$

$$\frac{d^2y}{dx^2} = -b^2y \Rightarrow \frac{d^2y}{dx^2} + b^2y = 0$$

∴ The given function is a solution of the given D.E.

Variable Separable Method

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Exercise 10.5

- 1) If F is the constant force generated by the motor of an automobile of mass m , its velocity V is given by $m \frac{dv}{dt} = F - kV$ where k is a constant, Express V in terms of t given that $V=0$ when $t=0$.

$$m \frac{dv}{dt} = F - kV$$

$$\frac{dv}{F - kV} = \frac{1}{m} dt$$

Integrating

$$\int \frac{dv}{F - kV} = \int \frac{1}{m} dt$$

multiply both sides by $-k$

$$\int \frac{-k dv}{F - kv} = \frac{-k}{M} \int dt$$

$$\log(F - kv) = -\frac{k}{M} t + C \quad \text{--- ①}$$

when $t=0$, $v=0$

$$\text{①} \Rightarrow \log F = 0 + C$$

$$C = \log F$$

$$\text{①} \Rightarrow \log(F - kv) = -\frac{kt}{M} + \log F$$

$$\frac{kt}{M} = \log F - \log(F - kv)$$

$$\frac{kt}{M} = \log \left(\frac{F}{F - kv} \right)$$

taking antilog

$$e^{kt/M} = \frac{F}{F - kv}$$

$$(F - kv) e^{kt/M} = F //$$

- ② The velocity v of a parachute falling vertically satisfies the equation $v \frac{dv}{dx} = g \left(1 - \frac{v^2}{k^2} \right)$ where g and k are constants. If v and x are both initially zero, find v in terms of x .

$$v \frac{dv}{dx} = g \left(1 - \frac{v^2}{k^2} \right)$$

$$\frac{v dv}{\left(1 - \frac{v^2}{k^2} \right)} = g dx$$

multiply both sides by $-\frac{2}{k^2}$

$$\frac{-\frac{2v}{k^2} dv}{\left(1 - \frac{v^2}{k^2} \right)} = -\frac{2g}{k^2} dx$$

integrating

$$\int \frac{-\frac{2v}{k^2} dv}{\left(1 - \frac{v^2}{k^2} \right)} = -\frac{2g}{k^2} \int dx$$

$$\log \left(1 - \frac{v^2}{k^2} \right) = -\frac{2gx}{k^2} + C \quad \text{--- ①}$$

given: $v=0$, $x=0$

$$\text{①} \Rightarrow \log(1 - 0) = 0 + C$$

$$\boxed{C=0}$$

$$\text{①} \Rightarrow \log \left(1 - \frac{v^2}{k^2} \right) = -\frac{2gx}{k^2} + 0$$

taking antilog

$$1 - \frac{v^2}{k^2} = e^{-\frac{2gx}{k^2}}$$

$$1 - e^{-\frac{2gx}{k^2}} = \frac{v^2}{k^2} \quad k^2 \left(1 - e^{-\frac{2gx}{k^2}} \right) = v^2 //$$

- ③ Find the equation of the curve whose slope is $\frac{y-1}{x^2+x}$ and which passes through the point (1,0)

$$\text{slope} = \frac{y-1}{x^2+x}$$

$$\frac{dy}{dx} = \frac{y-1}{x^2+x}$$

$$\frac{dy}{y-1} = \frac{dx}{x^2+x}$$

$$= \frac{1}{x(x+1)} dx$$

(Using partial fraction)

$$\frac{dy}{y-1} = \left(\frac{1}{x} - \frac{1}{x+1} \right) dx$$

$$\text{Integrating } \int \frac{dy}{y-1} = \int \frac{dx}{x} - \int \frac{dx}{x+1}$$

$$\log y - 1 = \log x - \log(x+1) + \log c$$

$$\log(y-1) = \log \frac{xc}{x+1}$$

taking antilog

$$y-1 = \frac{cx}{x+1} \quad \text{--- (1) This passing through (1,0)}$$

$$\therefore 0-1 = \frac{c}{1+1} \quad \boxed{-2=c}$$

$$(1) \Rightarrow y-1 = \frac{-2x}{x+1}$$

$$y = \frac{-2x}{x+1} + 1$$

$$= \frac{-2x + x + 1}{x+1}$$

$$y = \frac{1-x}{1+x}$$

- 4) solve the following differential equations

i) $\frac{dy}{dx} = \sqrt{\frac{1-y^2}{1-x^2}}$

$$\frac{dy}{dx} = \frac{\sqrt{1-y^2}}{\sqrt{1-x^2}}$$

$$\frac{dy}{\sqrt{1-y^2}} = \frac{dx}{\sqrt{1-x^2}}$$

Integrating

$$\int \frac{dy}{\sqrt{1-y^2}} = \int \frac{dx}{\sqrt{1-x^2}}$$

$$\sin^{-1} y = \sin^{-1} x + c$$

(ii) $y dx + (1+x^2) \tan^{-1} x dy = 0$

$$y dx = -(1+x^2) \tan^{-1} x dy$$

$$\frac{dx}{(1+x^2) \tan^{-1} x} = -\frac{dy}{y}$$

take $v = \tan^{-1} x$

$$\frac{dv}{dx} = \frac{1}{1+x^2}$$

$$dv = \frac{dx}{1+x^2}$$

$$\frac{dv}{v} = -\frac{dy}{y}$$

$$\text{Integrating } \int \frac{dv}{v} = -\int \frac{dy}{y}$$

$$\log v = -\log y + \log c$$

$$\log v + \log y = \log c$$

$$\log vy = \log c$$

$$\text{taking antilog } vy = c$$

$$y \tan^{-1} x = c //$$

$$(iii) \sin \frac{dy}{dx} = a, y(0) = 1$$

$$\sin \frac{dy}{dx} = a$$

$$\frac{dy}{dx} = \sin^{-1} a$$

$$dy = \sin^{-1} a \, dx$$

$$\text{Integrating } \int dy = \sin^{-1} a \int dx$$

$$y = (\sin^{-1} a) x + C \quad \text{--- (1)}$$

$$\text{When } x=0, y=1 \Rightarrow 1 = \sin^{-1} a (0) + C$$

$$\boxed{C=1}$$

$$\Rightarrow y = (\sin^{-1} a) x + 1$$

$$\frac{y-1}{x} = \sin^{-1} a$$

$$\sin\left(\frac{y-1}{x}\right) = a //$$

$$(iv) \frac{dy}{dx} = e^{x+y} + x^3 e^y$$

$$\frac{dy}{dx} = e^x \cdot e^y + x^3 e^y$$

$$\frac{dy}{dx} = (x^3 + e^x) e^y$$

$$e^{-y} dy = (x^3 + e^x) dx$$

$$\text{Integrating } \int e^{-y} dy = \int (x^3 + e^x) dx$$

$$-e^{-y} = \frac{x^4}{4} + e^x + C_1$$

$$e^x + e^{-y} + \frac{x^4}{4} = -C_1$$

$$e^x + e^{-y} + \frac{x^4}{4} = C$$

$$(v) (e^y + 1) \cos x \, dx + e^y \sin x \, dy = 0$$

$$(e^y + 1) \cos x \, dx = -e^y \sin x \, dy$$

$$\frac{\cos x \, dx}{\sin x} = -\frac{e^y}{e^y + 1} dy$$

$$\text{Integrating } \int \frac{\cos x \, dx}{\sin x} = -\int \frac{e^y}{e^y + 1} dy$$

$$\log \sin x = -\log(e^y + 1) + \log C$$

$$\log \sin x + \log(e^y + 1) = \log C$$

$$\log \sin x (e^y + 1) = \log C$$

taking anti-log

$$(e^y + 1) \sin x = C$$

$$(vi) (y \, dx - x \, dy) \cot\left(\frac{x}{y}\right) = n y^2 \, dx$$

$$\frac{y \, dx - x \, dy}{y^2} \cot \frac{x}{y} = n \, dx$$

$$\cot\left(\frac{x}{y}\right) d\left(\frac{x}{y}\right) = n dx$$

Integrating $\int \cot\left(\frac{x}{y}\right) d\left(\frac{x}{y}\right) = n \int dx$

$$\log \sin \frac{x}{y} = nx + C$$

taking anti log $\sin\left(\frac{x}{y}\right) = e^{nx+C}$

(vii) $\frac{dy}{dx} - x\sqrt{25-x^2} = 0$

$$\frac{dy}{dx} = x\sqrt{25-x^2}$$

$$dy = x\sqrt{25-x^2} dx$$

put $25-x^2 = t$

$$\therefore dy = \sqrt{t} \left(\frac{dt}{-2}\right)$$

$$-2x = \frac{dt}{dx}$$

$$x dx = \frac{dt}{-2}$$

Integrating $\int dy = \int t^{1/2} dt$

$$+2y = -\frac{t^{3/2}}{3/2} + 2C$$

multiply both sides
by $\frac{3}{2}$

$$\frac{3}{2}(2y) = -t^{3/2} + \frac{3}{2}(2C)$$

$$3y = -(25-x^2)^{3/2} + 3C$$

(viii) $x \cos y dy = e^x (x \log x + 1) dx$

$$\cos y dy = \frac{e^x (x \log x + 1)}{x} dx$$

$$\cos y dy = e^x \left(\log x + \frac{1}{x}\right) dx$$

$$= (e^x \log x + e^x \frac{1}{x}) dx$$

$$= \log x d(e^x) + e^x d(\log x)$$

$$\cos y dy = d(e^x \log x)$$

Integrating

$$\int \cos y dy = \int d(e^x \log x)$$

$$\sin y = e^x \log x + C$$

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$$[d(uv) = u dv + v du]$$

(ix) $\tan y \frac{dy}{dx} = \cos(x+y) + \cos(x-y)$

$$\tan y \frac{dy}{dx} = \cos x \cos y - \sin x \sin y + \cos x \cos y + \sin x \sin y$$

$$\tan y \frac{dy}{dx} = 2 \cos x \cos y$$

$$\frac{\tan y dy}{\cos y} = 2 \cos x dx$$

$$\sec y \tan y dy = 2 \cos x dx$$

Integrating

$$\int \sec y \tan y dy = \int 2 \cos x dx$$

$$\sec y = 2 \sin x + C$$

$$(x) \quad \frac{dy}{dx} = \tan^2(x+y) \quad \text{--- (1)}$$

Let $x+y=z$

diff w. r to x

$$1 + \frac{dy}{dx} = \frac{dz}{dx}$$

$$\frac{dy}{dx} = \frac{dz}{dx} - 1$$

$$\text{(1)} \Rightarrow \frac{dz}{dx} - 1 = \tan^2 z$$

$$\frac{dz}{dx} = 1 + \tan^2 z$$

$$\frac{dz}{dx} = \sec^2 z \Rightarrow \frac{dz}{\sec^2 z} = dx$$

$$\cos^2 z dz = dx$$

$$\left(\frac{1 + \cos 2z}{2} \right) dz = dx$$

$$\text{integrating } \frac{1}{2} \int (1 + \cos 2z) dz = \int dx$$

$$\frac{1}{2} \left[z + \frac{1}{2} \sin 2z \right] = x + C$$

Replace $z = x+y$

$$\frac{1}{2} \left[(x+y) + \frac{1}{2} \sin 2(x+y) \right] = x + C$$

$$\frac{1}{2} \left[(x+y) + \frac{1}{2} (\sin(x+y) \cos(x+y)) \right] = x + C$$

$$\frac{1}{2} \left[(x+y) + \sin(x+y) \cos(x+y) \right] = x + C$$

Homogeneous Differential Equation

Exercise 10.6

(1) solve the following differential equations.

$$1) \quad \left[x + y \cos\left(\frac{y}{x}\right) \right] dx = x \cos\left(\frac{y}{x}\right) dy$$

$$\frac{x + y \cos \frac{y}{x}}{x \cos \frac{y}{x}} = \frac{dy}{dx} \quad \text{--- (2)}$$

This is a homogeneous D.E.

put $y = vx$

$$\frac{dy}{dx} = v + x \frac{dv}{dx}$$

$$\text{(2)} \Rightarrow \frac{x + vx \cos vx}{x \cos vx} = v + x \frac{dv}{dx}$$

$$\frac{x(1 + v \cos v)}{x \cos v} = v + x \frac{dv}{dx}$$

$$\frac{1 + v \cos v}{\cos v} - v = x \frac{dv}{dx}$$

$$\frac{1 + v \cos v - v \cos v}{\cos v} = x \frac{dv}{dx}$$

$$\frac{1}{\cos v} = x \frac{dv}{dx}$$

$$\frac{dx}{x} = \cos v dv$$

Integrating

$$\int \cos v dv = - \int \frac{dx}{x}$$

$$\sin v = \log|x| + \log|C|$$

Replace v by $\frac{y}{x}$

$$\sin \frac{y}{x} = \log|x| + \log|C|$$

When taking
logc. use $\log|C|$
 ~~$\log x$ use $\log|x|$~~

$$2) (x^3 + y^3) dy - x^2 y dx = 0$$

$$(x^3 + y^3) dy = x^2 y dx$$

$$\frac{dy}{dx} = \frac{x^2 y}{x^3 + y^3} \quad \text{--- ①}$$

This is a homogeneous D.E

$$\text{put } y = vx \quad \frac{dy}{dx} = v + x \frac{dv}{dx}$$

$$\text{①} \Rightarrow v + x \frac{dv}{dx} = \frac{x^2 vx}{x^3 + v^3 x^3}$$

$$v + x \frac{dv}{dx} = \frac{x^3 v}{x^3(1+v^3)}$$

$$x \frac{dv}{dx} = \frac{v}{1+v^3} - v$$

$$x \frac{dv}{dx} = \frac{v - v^4}{1+v^3}$$

$$\frac{x dv}{dx} = \frac{-v^4}{1+v^3}$$

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$$\frac{1+v^3}{v^4} dv = - \frac{dx}{x}$$

$$\left(\frac{1}{v^4} + \frac{1}{v} \right) dv = - \frac{dx}{x}$$

$$\text{Integrating } \int v^{-4} dv + \int \frac{dv}{v} = - \int \frac{dx}{x}$$

$$\frac{v^{-3}}{-3} + \log|v| = -\log|x| + \log|C|$$

$$\log|v| + \log|x| - \log|C| = \frac{1}{3v^3}$$

$$\log\left|\frac{vx}{C}\right| = \frac{1}{3v^3}$$

Replace $v = \frac{y}{x}$

$$\log\left|\frac{y}{C}\right| = \frac{1}{3\frac{y}{x^3}} = \frac{x^3}{3y^3}$$

$$\Rightarrow \log\left|\frac{y}{C}\right| = \frac{x^3}{3y^3}$$

taking antilog e

$$y = \pm e^{\frac{x^3}{3y^3}}$$

Thus

$$y = C e^{\frac{x^3}{3y^3}}$$

where $C = \pm C$

$$\text{③ } y e^{\frac{x^3}{y^3}} dx = (x e^{\frac{x^3}{y^3}} + y) dy$$

$$\frac{dx}{dy} = \frac{x e^{\frac{x^3}{y^3}} + y}{y e^{\frac{x^3}{y^3}}} \quad \text{--- ①}$$

This is a Homogeneous D.E.

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 put $x = vy$ $\frac{dx}{dy} = v + y \frac{dv}{dy}$ www.TrbTnpss.com

$$\Rightarrow v + y \frac{dv}{dy} = \frac{vy e^{vy/y+y}}{y e^{vy/y}}$$

$$v + y \frac{dv}{dy} = \frac{y(v e^v + 1)}{y e^v}$$

$$y \frac{dv}{dy} = \frac{v e^v + 1}{e^v} - v$$

$$y \frac{dv}{dy} = \frac{v e^v + 1 - v e^v}{e^v}$$

$$y \frac{dv}{dy} = \frac{1}{e^v}$$

$$e^v dv = \frac{dy}{y}$$

Integrating $\int e^v dv = \int \frac{dy}{y}$

$$e^v = \log|y| + \log|C|$$

$$e^{xy} = \log|cy|$$

4) $2xy dx + (x^2 + 2y^2) dy = 0$

$$2xy dx = -(x^2 + 2y^2) dy$$

$$\frac{dx}{dy} = \frac{-x^2 - 2y^2}{2xy} \quad \text{--- (1)}$$

put $x = vy$ $\frac{dx}{dy} = v + y \frac{dv}{dy}$

$$\Rightarrow v + y \frac{dv}{dy} = \frac{-(v^2 y^2 + 2y^2)}{2vy^2}$$

$$v + y \frac{dv}{dy} = \frac{-y^2(v^2 + 2)}{2vy^2}$$

$$y \frac{dv}{dy} = \frac{-v^2 - 2}{2v} - v \quad y \frac{dv}{dy} = \frac{-v^2 - 2 - 2v^2}{2v}$$

$$y \frac{dv}{dy} = \frac{-(3v^2 + 2)}{2v}$$

$$\frac{2v dv}{(3v^2 + 2)} = -\frac{dy}{y}$$

Integrating $\frac{1}{3} \int \frac{6v dv}{(3v^2 + 2)} = -\int \frac{dy}{y}$

$$\log|3v^2 + 2| = -3 \log|y| + \log|C|$$

$$\log|3v^2 + 2| + 3 \log|y| = \log|C|$$

$$\log|(3v^2 + 2)y^3| = \log|C|$$

put $v = \frac{x}{y}$ and taking anti log $(3 \frac{x^2}{y^2} + 2)y^3 = |C|$ $|3x^2y + 2y^3| = |C|$
 $3x^2y + 2y^3 = \pm C$
 $3x^2y + 2y^3 = C$

$$(5) (y^2 - 2xy) dx = (x^2 - 2xy) dy$$

This is a homogeneous D.E.

$$\frac{dy}{dx} = \frac{y^2 - 2xy}{x^2 - 2xy} \quad \text{--- (1)}$$

put $y = vx$ $\frac{dy}{dx} = v + x \frac{dv}{dx}$

$$(1) \Rightarrow v + x \frac{dv}{dx} = \frac{v^2 x^2 - 2xvx}{x^2 - 2xvx}$$

$$v + x \frac{dv}{dx} = \frac{x^2(v^2 - 2v)}{x^2(1 - 2v)}$$

$$x \frac{dv}{dx} = \frac{v^2 - 2v}{1 - 2v} - v \Rightarrow x \frac{dv}{dx} = \frac{v^2 - 2v - v + 2v^2}{1 - 2v}$$

$$x \frac{dv}{dx} = \frac{3v^2 - 3v}{1 - 2v}$$

$$\frac{1 - 2v}{3(v^2 - v)} dv = \frac{dx}{x}$$

$$\frac{1 - 2v}{v(v - 1)} dv = 3 \frac{dx}{x}$$

Using partial fraction

$$\left(-\frac{1}{v-1} - \frac{1}{v} \right) dv = 3 \frac{dx}{x}$$

Integrating $\int \frac{dv}{v-1} + \int \frac{dv}{v} = -3 \int \frac{dx}{x}$

$$\log|v-1| + \log|v| = -3 \log|x| + \log|c|$$

$$\log|v(v-1)| + 3 \log|x| = \log|c|$$

$$\log|(v^2 - v)| + \log|x^3| = \log|c|$$

$$\log|x^3(v^2 - v)| = \log|c|$$

taking antilog

$$|x^3(v^2 - v)| = |c|$$

putting $v = \frac{y}{x}$

$$|x^3(\frac{y^2}{x^2} - \frac{y}{x})| = |c|$$

$$\frac{x^3 y^2}{x^2} - \frac{x^3 y}{x} = \pm c$$

$$xy^2 - x^2 y = \pm c$$

$$xy^2 - x^2 y = C$$

where $C = \pm c$

$$(6) x \frac{dy}{dx} = y - x \cos^2\left(\frac{y}{x}\right) \quad \text{--- (1)}$$

This is a Homogeneous D.E.

put $y = vx$

$$\frac{dy}{dx} = v + x \frac{dv}{dx}$$

$$(1) \Rightarrow x(v + x \frac{dv}{dx}) = vx - x \cos^2 \frac{vx}{x}$$

$$vx + x^2 \frac{dv}{dx} = vx - x \cos^2 v$$

$$x^2 \frac{dv}{dx} = -x \cos^2 v$$

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$$\frac{dv}{\cos^2 v} = -\frac{dx}{x}$$

$$\text{Integrating } \int \sec^2 v \, dv = -\int \frac{dx}{x}$$

$$\tan v = -\log|x| + \log|C|$$

$$\tan v = \log\left|\frac{C}{x}\right|$$

$$\tan\frac{y}{x} = \log\left|\frac{C}{x}\right|$$

$$\text{taking antilog } e^{\tan\frac{y}{x}} = \left|\frac{C}{x}\right|$$

$$xe^{\tan\frac{y}{x}} = \pm C$$

$$\therefore xe^{\tan\frac{y}{x}} = C$$

$$\textcircled{7} (1+3e^{y/x})dy + 3e^{y/x}(1-y/x)dx = 0, \text{ given that } y=0 \text{ when } x=1$$

$$(1+3e^{y/x})dy = -3e^{y/x}(1-y/x)dx$$

$$\frac{dy}{dx} = \frac{-3e^{y/x}(1-y/x)}{1+3e^{y/x}} \quad \text{--- (1)}$$

This is a Homogeneous D.E.

$$\text{put } y= vx \quad \frac{dy}{dx} = v + x \frac{dv}{dx}$$

$$\textcircled{1} \Rightarrow v + x \frac{dv}{dx} = \frac{-3e^{vx}(1-v)}{1+3e^{vx}}$$

$$v + x \frac{dv}{dx} = \frac{-3e^v(1-v)}{1+3e^v}$$

$$x \frac{dv}{dx} = \frac{-3e^v + 3ve^v}{1+3e^v} - v$$

$$= \frac{-3e^v + 3ve^v - v(1+3e^v)}{1+3e^v}$$

$$x \frac{dv}{dx} = \frac{-(3e^v + v)}{3e^v + 1}$$

$$x dv = \frac{-(3e^v + v)}{(3e^v + 1)} dx$$

$$\left(\frac{3e^v + 1}{3e^v + v}\right) dv = -\frac{dx}{x}$$

$$\text{Integrating } \int \frac{3e^v + 1}{3e^v + v} dv = -\int \frac{dx}{x}$$

$$\log(3e^v + v) = -\log|x| + \log|C|$$

$$\log|x(3e^v + v)| = \log|C|$$

$$\text{taking antilog } |3xe^v + vx| = |C|$$

Replace v by y/x $|3xe^{y/x} + y| = |C| \Rightarrow 3xe^{y/x} + y = \pm C$

$$3xe^{y/x} + y = -C \quad \text{--- (1)}$$

when $x=1 \Rightarrow y=0$

$$\text{--- (1)} \Rightarrow 3e^0 + 0 = C$$

$$C=3$$

$$\text{--- (1)} \Rightarrow 3xe^{y/x} + y = 3 //$$

⑧ $(x^2 + y^2)dy = xy dx$, It is given that $y(0)=1$ and $y(x_0)=e$, Find the value of x_0

$$(x^2 + y^2)dy = xy dx$$

$$\frac{dy}{dx} = \frac{xy}{x^2 + y^2} \quad \text{--- (1)}$$

This is a Homogeneous D.E

put $y=vx$ $\frac{dy}{dx} = v + x \frac{dv}{dx}$

$$\text{--- (1)} \Rightarrow v + x \frac{dv}{dx} = \frac{xvx}{x^2 + v^2x^2} \quad \text{--- (2)}$$

$$x \frac{dv}{dx} = \frac{x^2v}{x^2(1+v^2)} - v$$

$$x \frac{dv}{dx} = \frac{x - x \cdot v^2}{1+v^2}$$

$$dv \frac{1+v^2}{v^3} = -\frac{dx}{x}$$

$$\left(\frac{1}{v^3} + \frac{v}{v^3}\right)dv = -\frac{dx}{x}$$

Integrating

$$\int v^{-3} dv + \int \frac{1}{v} dv = -\int \frac{dx}{x}$$

$$\frac{v^{-2}}{-2} + \log|v| = -\log|x| + \log|C|$$

$$\log\left|\frac{v}{x}\right| = \frac{1}{2v^2}$$

taking antilog $\left|\frac{v}{x}\right| = e^{\frac{1}{2v^2}}$

Replacing $v=y/x$ $\left|\frac{y}{x}\right| = e^{\frac{x^2}{2y^2}}$

$$y = \pm C e^{x^2/2y^2}$$

$$y = C e^{x^2/2y^2} \quad \text{--- (1)}$$

given $x=1, y=1$ $\text{--- (1)} \Rightarrow 1 = C e^{1/2}$

$$C = e^{-1/2}$$

$$\text{--- (1)} \Rightarrow y = e^{-1/2} e^{x^2/2y^2} \quad \text{--- (2)}$$

$y(x_0)=e$ $y=e \Rightarrow e = e^{-1/2} e^{x_0^2/2e^2}$

$$e^1 e^{1/2} = e^{x_0^2/2e^2}$$

$$\frac{3}{2} = \frac{x_0^2}{2e^2}$$

$$x_0^2 = 3e^2$$

$$x_0^2 = 3e^2$$

$$x_0 = \pm \sqrt{3}e //$$

First order Linear Differential equation

I. $\frac{dy}{dx} + Py = Q$ This is Linear in y
where P, Q are function of x

(ii) Integrating factor $IF = e^{\int P dx}$

(iii) solution is $y(IF) = \int Q(IF) dx + C$

II. $\frac{dx}{dy} + Px = Q$ This is linear in x
where P, Q are function of y

(i) $IF = e^{\int P dy}$

(ii) solution is $x(IF) = \int Q(IF) dy + C$

Exercise 10.7

solve the following Linear differential equations

① $\cos x \frac{dy}{dx} + y \sin x = 1$

divide by $\cos x$ $\frac{dy}{dx} + y \frac{\sin x}{\cos x} = \frac{1}{\cos x}$

$P = \frac{\sin x}{\cos x} = \tan x, Q = \frac{1}{\cos x} = \sec x$

$\frac{dy}{dx} + Py = Q$ This is linear in y

$IF = e^{\int P dx}$

$= e^{\int \tan x dx} = e^{\log \sec x} = \sec x$

solution is

$y(IF) = \int Q(IF) dx + C$

$y \sec x = \int \sec x \sec x dx + C$

$y \sec x = \int \sec^2 x dx + C$

$y \sec x = \tan x + C$

$\frac{y}{\cos x} = \frac{\sin x}{\cos x} + C$

multiply by $\cos x$ $y = \sin x + C \cos x$

② $(1-x^2) \frac{dy}{dx} - xy = 1$

divide by $1-x^2$ $\frac{dy}{dx} - \frac{xy}{1-x^2} = \frac{1}{1-x^2}$

$P = \frac{-x}{1-x^2}, Q = \frac{1}{1-x^2}$

$\frac{dy}{dx} + Py = Q$ This is linear in y

$$\begin{aligned} \text{IF} &= e^{\int p dx} \\ &= e^{\int \frac{x}{1-x^2} dx} \\ &= e^{\frac{1}{2} \log(1-x^2)} = e^{\log(1-x^2)^{1/2}} \\ &= e^{\frac{1}{2} \log(1-x^2)} = e^{\log(1-x^2)^{1/2}} \end{aligned}$$

solution is $\text{IF} = \sqrt{1-x^2}$

$$y(\text{IF}) = \int q(\text{IF}) dx + C$$

$$y\sqrt{1-x^2} = \int \frac{1}{1-x^2} \sqrt{1-x^2} dx + C$$

$$y\sqrt{1-x^2} = \int \frac{1}{\sqrt{1-x^2}} dx + C$$

$$y\sqrt{1-x^2} = \sin^{-1} x + C$$

$$\textcircled{3} \quad \frac{dy}{dx} + \frac{y}{x} = \sin x$$

$$P = \frac{1}{x}, \quad Q = \sin x$$

$$\frac{dy}{dx} + Py = Q \quad \text{This is linear in } y$$

$$\begin{aligned} \text{IF} &= e^{\int p dx} \\ &= e^{\int \frac{1}{x} dx} = e^{\log x} \end{aligned}$$

$$\text{IF} = x$$

solution is

$$y(\text{IF}) = \int q(\text{IF}) dx + C$$

$$yx = \int \sin x (x) dx + C$$

$$xy = \int \frac{x}{u} \sin x dx + C$$

$$= x(-\cos x) - \int (-\cos x) dx + C$$

$$xy = -x\cos x + \sin x + C$$

$$xy + x\cos x = \sin x + C$$

$$x(y + \cos x) = \sin x + C$$

$$\textcircled{4} \quad (x^2+1) \frac{dy}{dx} + 2xy = \sqrt{x^2+4}$$

divide by x^2+1

$$\frac{dy}{dx} + \frac{2x}{x^2+1} y = \frac{\sqrt{x^2+4}}{x^2+1}$$

$$P = \frac{2x}{x^2+1} \quad Q = \frac{\sqrt{x^2+4}}{x^2+1}$$

$$\frac{dy}{dx} + Py = Q, \text{ This is linear in } y$$

$$\text{IF} = e^{\int p dx} = e^{\int \frac{2x}{x^2+1} dx} = e^{\log(x^2+1)} = x^2+1$$

solution is $y(\text{IF}) = \int q(\text{IF}) dx + C$

$$y(x^2+1) = \int \frac{\sqrt{x^2+4}}{x^2+1} dx + C$$

$$= \int \sqrt{2+x^2} dx + C$$

$$y(x^2+1) = \frac{x}{2} \sqrt{2^2+x^2} + \frac{2^2}{2} \log|x+\sqrt{x^2+4}| + C$$

$$y(x^2+1) = \frac{x}{2} \sqrt{x^2+4} + 2 \log|x+\sqrt{x^2+4}| + C$$

$$5) (2x-10y^3)dy + ydx = 0$$

$$ydx = -(2x-10y^3)dy$$

$$y \frac{dx}{dy} = -2x + 10y^3$$

$$\frac{dx}{dy} = -\frac{2x}{y} + \frac{10y^3}{y}$$

$$\frac{dx}{dy} + \frac{2x}{y} = 10y^2$$

$$P = \frac{2}{y} \quad Q = 10y^2$$

$$\frac{dx}{dy} + Px = Q$$

Thus is linear in x

$$IF = e^{\int P dy} = e^{\int \frac{2}{y} dy} = e^{2 \log y} = e^{\log y^2} = y^2$$

solution is

$$x(IF) = \int Q(IF) dy + C$$

$$xy^2 = \int 10y^2 (y^2) dy + C$$

$$= 10 \int y^4 dy + C$$

$$= \frac{10y^5}{5} + C$$

$$xy^2 = 2y^5 + C$$

$$6) x \sin x \frac{dy}{dx} + (x \cos x + \sin x) y = \sin x$$

divide by $x \sin x$

$$\frac{dy}{dx} + \frac{x \cos x + \sin x}{x \sin x} y = \frac{\sin x}{x \sin x}$$

$$P = \frac{x \cos x + \sin x}{x \sin x} \quad Q = \frac{1}{x}$$

$$\frac{dy}{dx} + Py = Q \quad \text{This is linear in } y$$

$$IF = e^{\int P dx} = e^{\int \frac{x \cos x + \sin x}{x \sin x} dx}$$

$$= e^{\log(\sin x)} = \sin x$$

$$\text{solution is } y(IF) = \int Q(IF) dx + C$$

$$y(x \sin x) = \int \frac{1}{x} (x \sin x) dx + C$$

$$xysinx = 1 - \cos x + C$$

$$7) (y - e^{\sin^{-1} x}) \frac{dx}{dy} + \sqrt{1-x^2} = 0$$

$$(y - e^{\sin^{-1} x}) \frac{dx}{dy} = -\sqrt{1-x^2}$$

$$y - e^{\sin^{-1}x} = -\frac{dy}{dx} \sqrt{1-x^2}$$

$$\frac{dy}{dx} \sqrt{1-x^2} + y = e^{\sin^{-1}x}$$

divide by $\sqrt{1-x^2}$ $\frac{dy}{dx} + \frac{y}{\sqrt{1-x^2}} = \frac{e^{\sin^{-1}x}}{\sqrt{1-x^2}}$

$$P = \frac{1}{\sqrt{1-x^2}} \quad Q = \frac{e^{\sin^{-1}x}}{\sqrt{1-x^2}} \Rightarrow \frac{dy}{dx} + py = Q$$

This is linear in y

$$IF = e^{\int P dx} = e^{\int \frac{1}{\sqrt{1-x^2}} dx} = e^{\sin^{-1}x}$$

solution is

$$y(IF) = \int Q(IF) dx + C$$

$$y e^{\sin^{-1}x} = \int \frac{e^{\sin^{-1}x}}{\sqrt{1-x^2}} e^{\sin^{-1}x} dx + C$$

$$= \int e^t e^t dt + C$$

$$= \int e^{2t} dt + C$$

$$y e^{\sin^{-1}x} = \frac{e^{2t}}{2} + C$$

$$y e^{\sin^{-1}x} = \frac{e^{2\sin^{-1}x}}{2} + C$$

put $\sin^{-1}x = t$
 $\frac{1}{\sqrt{1-x^2}} dx = dt$

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$$(8) \frac{dy}{dx} + \frac{y}{(1-x)\sqrt{x}} = 1-\sqrt{x}$$

$$P = \frac{1}{(1-x)\sqrt{x}} \quad Q = 1-\sqrt{x} \Rightarrow \frac{dy}{dx} + py = Q$$

This is linear in y

$$IF = e^{\int P dx} = e^{\int \frac{1}{(1-x)\sqrt{x}} dx}$$

$$= e^{\int \frac{2dt}{1-t^2}} = e^{\int \frac{2dt}{(1+t)(1-t)}}$$

$$= e^{\int \left(\frac{1}{1+t} + \frac{1}{1-t} \right) dt} = e^{\log(1+t) - \log(1-t)}$$

$$= e^{\log\left(\frac{1+t}{1-t}\right)} = \frac{1+\sqrt{x}}{1-\sqrt{x}}$$

$$IF = \frac{1+\sqrt{x}}{1-\sqrt{x}}$$

solution is

$$y(IF) = \int Q(IF) dx + C$$

$$y\left(\frac{1+\sqrt{x}}{1-\sqrt{x}}\right) = \int \frac{1+\sqrt{x}}{1-\sqrt{x}} (1-\sqrt{x}) dx + C$$

$$= x + \frac{x^{3/2}}{3/2} + C$$

$$y\left(\frac{1+\sqrt{x}}{1-\sqrt{x}}\right) = x + \frac{2}{3} x\sqrt{x} + C //$$

$$9) (1+x+xy^2) \frac{dy}{dx} + (y+y^3) = 0$$

$$1+x+xy^2 \frac{dy}{dx} = -(y+y^3)$$

$$1+x(1+y^2) = -y(1+y^2) \frac{dx}{dy}$$

$$y(1+y^2) \frac{dx}{dy} + x(1+y^2) = -1$$

$$\text{divide by } (1+y^2)y \quad \frac{dx}{dy} + \frac{x}{y} = \frac{-1}{y(1+y^2)}$$

$$P = \frac{1}{y} \quad Q = \frac{-1}{y(1+y^2)}$$

Thus is linear in x

$$IF = e^{\int P dy} = e^{\int \frac{1}{y} dy} = e^{\log y} = y$$

solution is

$$x(IF) = \int Q(IF) dy + C$$

$$xy = \int \frac{-1}{y(1+y^2)} y dy + C$$

$$xy = -\tan^{-1} y + C$$

$$xy + \tan^{-1} y = C$$

$$10) \frac{dy}{dx} + \frac{y}{x \log x} = \frac{\sin 2x}{\log x}$$

$$P = \frac{1}{x \log x}$$

$$Q = \frac{\sin 2x}{\log x}$$

$$\Rightarrow \frac{dy}{dx} + Py = Q$$

Thus is linear in y

$$IF = e^{\int P dx} = e^{\int \frac{1}{x \log x} dx} = e^{\log \log x}$$

$$IF = \log x$$

solution is

$$y(IF) = \int Q IF dx + C$$

$$y \log x = \int \frac{\sin 2x}{\log x} \log x dx + C$$

$$y \log x = -\frac{\cos 2x}{2} + C$$

$$y \log x + \frac{\cos 2x}{2} = C$$

$$11) (x+a) \frac{dy}{dx} - 2y = (x+a)^4$$

$$\text{divide by } x+a \quad \frac{dy}{dx} - \frac{2y}{x+a} = (x+a)^3$$

$$P = -\frac{2}{x+a} \quad Q = (x+a)^3 \Rightarrow \frac{dy}{dx} + Py = Q$$

Thus is linear in y

$$IF = e^{\int P dx} = e^{-\int \frac{2}{x+a} dx}$$

$$= e^{-2 \log(x+a)} = e^{\log(x+a)^{-2}}$$

$$= (x+a)^{-2}$$

$$IF = \frac{1}{(x+a)^2}$$

solution is

$$y(IF) = \int Q(IF) dx + C$$

$$y \left(\frac{1}{(x+a)^2} \right) = \int \frac{1}{(x+a)^2} (x+a)^2 dx + C$$

$$y \frac{1}{(x+a)^2} = \frac{(x+a)^2}{2} + C$$

$$2y = (x+a)^2 + 2C(x+a)^2$$

$$(2) \frac{dy}{dx} = \frac{\sin^2 x}{1+x^3} - \frac{3x^2}{1+x^3} y$$

$$\frac{dy}{dx} + \frac{3x^2}{1+x^3} y = \frac{\sin^2 x}{1+x^3}$$

$$P = \frac{3x^2}{1+x^3} \quad Q = \frac{\sin^2 x}{1+x^3} \Rightarrow \frac{dy}{dx} + Py = Q$$

This is linear in y

$$IF = e^{\int P dx} = e^{\int \frac{3x^2}{1+x^3} dx} = e^{\log(1+x^3)} = (1+x^3)$$

solution is

$$y(IF) = \int Q(IF) dx + C$$

$$y(1+x^3) = \int \frac{\sin^2 x}{1+x^3} (1+x^3) dx + C$$

$$y(1+x^3) = \int \left(\frac{1}{2} - \frac{\cos 2x}{2} \right) dx + C$$

$$y(1+x^3) = \frac{x}{2} - \frac{1}{4} \sin 2x + C$$

$$13) x \frac{dy}{dx} + y = x \log x$$

divide by x $\frac{dy}{dx} + \frac{y}{x} = \log x$

$$P = \frac{1}{x} \quad Q = \log x \Rightarrow \frac{dy}{dx} + Py = Q$$

This is linear in y

$$IF = e^{\int P dx} = e^{\int \frac{1}{x} dx} = e^{\log x} = x$$

$$y(IF) = \int Q(IF) dx + C$$

$$yx = \int x \log x dx + C$$

$$xy = \int \frac{x \log x}{u} \frac{xdx}{dv} + C$$

$$xy = \frac{x^2}{2} \log x - \int \frac{x^2}{2} \frac{1}{x} dx + C$$

$$xy = \frac{x^2}{2} \log x - \frac{x^2}{4} + C$$

$$4xy = 2x^2 \log x - x^2 + 4C$$

$$u = \log x \quad dv = x dx$$

$$du = \frac{1}{x} dx \quad v = \frac{x^2}{2}$$

$$\int u dv = uv - \int v du$$

$$14) x \frac{dy}{dx} + 2y - x^2 \log x = 0$$

$$x \frac{dy}{dx} + 2y = x^2 \log x$$

$$\text{divide by } x \quad \frac{dy}{dx} + \frac{2y}{x} = x \log x$$

$$P = \frac{2}{x} \quad Q = x \log x \Rightarrow \frac{dy}{dx} + Py = Q$$

This is linear in y

$$IF = e^{\int P dx} = e^{\int \frac{2}{x} dx} = e^{2 \log x} = e^{\log x^2}$$

$$IF = x^2$$

solution is

$$y(IF) = \int Q(IF) dx + C$$

$$yx^2 = \int x \log x \cdot x^2 dx + C$$

$$= \int \log x \cdot \frac{x^3}{dx} dx + C$$

$$= \frac{x^4}{4} \log x - \int \frac{x^3}{4} dx + C$$

$$= \frac{x^4 \log x}{4} - \frac{1}{4} \frac{x^4}{4} + C$$

$$yx^2 = \frac{x^4 \log x}{4} - \frac{x^4}{16} + C$$

$$\begin{aligned} u &= \log x \quad dv = x^3 dx \\ du &= \frac{1}{x} dx \quad v = \frac{x^4}{4} \\ \int u dv &= uv - \int v du \end{aligned}$$

$$15) \frac{dy}{dx} + \frac{3y}{x} = \frac{1}{x^2}, \text{ given that } y=2 \text{ when } x=1$$

$$P = \frac{3}{x} \quad Q = \frac{1}{x^2}$$

$$\frac{dy}{dx} + Py = Q \quad \text{This is linear in } y$$

$$IF = e^{\int P dx} = e^{\int \frac{3}{x} dx} = e^{3 \log x} = e^{\log x^3}$$

$$IF = x^3$$

$$\text{solution is } y(IF) = \int Q(IF) dx + C$$

$$yx^3 = \int \frac{1}{x^2} \cdot x^3 dx + C$$

$$yx^3 = \int x dx + C$$

$$yx^3 = \frac{x^2}{2} + C \quad \text{--- (1)}$$

$$\text{given } y=2 \text{ when } x=1$$

$$(1) \Rightarrow 2(1)^3 = \frac{1^2}{2} + C$$

$$2 - \frac{1}{2} = C$$

$$C = \frac{3}{2}$$

$$(1) \Rightarrow yx^3 = \frac{x^2}{2} + \frac{3}{2}$$

$$2yx^3 = x^2 + 3 //$$

Applications of First order ODE

Exercise 10.8

- ① The rate of increase in the numbers of bacteria in a certain culture is proportional to the number present. Given that the number triples in 5 hours, find how many bacteria will be present after 10 hours.

Let A be the number of bacteria at any time t

$$\frac{dA}{dt} \propto A$$

$$\frac{dA}{dt} = kA$$

$$A = ce^{kt} \quad \text{--- ①}$$

i) when $t=0$ $A=A_0$

$$\text{①} \Rightarrow A_0 = ce^0$$

$$\boxed{c=A_0}$$

$$\text{①} \Rightarrow A = A_0 e^{kt} \quad \text{--- ②}$$

ii) when $t=5$ $A=3A_0$

$$\text{②} \Rightarrow 3A_0 = e^{5kA_0}$$

$$\boxed{e^{5k} = 3}$$

iii) when $t=10$ $A=?$

$$\text{②} \Rightarrow A = A_0 e^{10k}$$

$$= A_0 (e^{5k})^2 = A_0 (3)^2$$

$$A = 9A_0$$

After 10 hours $A = 9$ times the (initial value)
 A_0

- ② Find the population of a city at any time t , given that the rate of increase of population is proportional to the population at that instant and that in a period of 40 years the population increased from 3,00,000 to 4,00,000

Let A be the population of city at any time t

$$\frac{dA}{dt} \propto A$$

$$\frac{dA}{dt} = kA$$

$$A = ce^{kt} \quad \text{--- ①}$$

when $t=0$, $A=3,00,000$

$$3,00,000 = ce^0$$

$$\boxed{c=3,00,000}$$

$$\text{①} \Rightarrow A = 3,00,000 e^{kt} \quad \text{--- ②}$$

when $t=40$ $A=4,00,000$

$$\text{②} \Rightarrow 4,00,000 = 3,00,000 e^{40k}$$

$$\frac{4}{3} = e^{40k}$$

$$e^{40k} = \frac{4}{3}$$

$$(e^k)^{40} = \frac{4}{3} \quad e^k = \left(\frac{4}{3}\right)^{\frac{1}{40}}$$

$$\textcircled{2} \Rightarrow A = 3,00,000 (e^k)^t$$

$$A = 3,00,000 \left(\frac{4}{3}\right)^{\frac{t}{40}}$$

- ③ The equation of electromotive force for an electric circuit containing resistance and self-inductance is $E = Ri + L \frac{di}{dt}$ Where E is the Electromotive force is given to the circuit R is the Resistance and L - coefficient of induction. Find the current i at time t when $E=0$.

given $L \frac{di}{dt} + Ri = E$

divide by $L \quad \frac{di}{dt} + \frac{Ri}{L} = \frac{E}{L}$

$$P = \frac{R}{L} \quad Q = \frac{E}{L} \Rightarrow \frac{di}{dt} + Pi = Q$$

This is Linear in i $\int p dt = \int \frac{R}{L} dt = \frac{Rt}{L}$
 $IF = e^{\int p dt} = e^{\frac{Rt}{L}}$

solution is

$$i(IF) = \int (Q IF) dt + C$$

$$i e^{\frac{Rt}{L}} = \int \frac{E}{L} e^{\frac{Rt}{L}} dt + C$$

$$i e^{\frac{Rt}{L}} = \frac{E}{L} \frac{e^{\frac{Rt}{L}}}{\frac{R}{L}} + C$$

$$i e^{\frac{Rt}{L}} = \frac{E}{R} e^{\frac{Rt}{L}} + C \quad \text{--- (1)}$$

when $E=0 \Rightarrow i e^{\frac{Rt}{L}} = 0 + C$

$$i e^{\frac{Rt}{L}} = C$$

$$i = C e^{-\frac{Rt}{L}}$$

- ④ The engine of a motor boat moving at 10 m/s is shut off. Given that the retardation at any subsequent time (after shutting off the engine) equal to the velocity at that time. Find the velocity after 2 seconds of switching off the engine.

Let v be the velocity at any time t

given retardation = velocity

$$-\frac{dv}{dt} = v$$

$$\frac{dv}{v} = -dt$$

$$\log v = -t + \log c$$

$$\log v/c = -t$$

$$\frac{v}{c} = e^{-t} \quad [v = ce^{-t}] \quad \text{--- ①}$$

when $t=0$, $v=10$

$$\text{①} \Rightarrow$$

$$10 = ce^0 \quad [c=10]$$

$$v = 10e^{-t} \quad \text{--- ②}$$

when $t=2$

$$\text{②} \Rightarrow v = 10e^{-2}$$

$$v = \frac{10}{e^2}$$

⑦ water at temperature 100°C cools in 10 minutes 80°C in a room temperature of 25°C Find

(i) The temperature of water after 20 minutes

(ii) The time when the temperature is 40°C

$$\log\left(\frac{11}{15}\right) = -0.3101, \quad \log 5 = 1.6094$$

→ Let T be temperature of water at any time t
 S be the room temp.

$$S = 25$$

$$\frac{dT}{dt} \propto T - S$$

$$\frac{dT}{dt} = -k(T - S)$$

$$T - S = Ce^{-kt}$$

$$[T - 25 = Ce^{-kt}] \quad \text{--- ①}$$

(i) when $t=0$, $T=100$

$$\text{①} \Rightarrow 100 - 25 = Ce^0$$

$$C = 75$$

$$\text{①} \Rightarrow [T - 25 = 75e^{-kt}] \quad \text{--- ②}$$

(ii). When $t=10$, $T=80$

$$\text{②} \Rightarrow 80 - 25 = 75e^{-10k}$$

$$\left[\frac{55}{75} = e^{-10k}\right] \quad \text{--- ③}$$

(i) when $t=20$

$$\text{②} \Rightarrow T - 25 = 75e^{-20k}$$

$$= 75(e^{-10k})^2$$

$$= 75 \times \frac{55}{75} \times \frac{55}{75} = \frac{121}{3}$$

$$T - 25 = \frac{121}{3}$$

$$T = 25 + 40.33$$

$$[T = 65.33^\circ\text{C}]$$

(ii) $T=40$

$$\text{③} \Rightarrow \frac{11}{15} = e^{-10k}$$

$$\log \frac{11}{15} = -10k$$

$$-0.3101 = -10k$$

$$-\frac{0.3101}{10} = k$$

$$[k = -0.03101]$$

(32)

$$(ii) T=40 \quad ② \Rightarrow 40-25=75 e^{-kt}$$

$$15=75 e^{-kt}$$

$$e^{-kt} = \frac{15}{75}$$

$$-kt = \log 5$$

$$+0.03101 t = 1.6094$$

$$t = \frac{1.6094}{0.03101} \times \frac{10000}{10000} = \frac{16094}{310.1}$$

$$t = 51.89 \text{ minutes.}$$

⑧ At 10.00 A.M a woman took a cup of hot instant coffee from her microwave oven and placed it on nearby kitchen counter to cool. At that instant $T = 180^\circ\text{F}$ and 10 minutes later it was 160°F . Assume that kitchen Temp. 70°F

(i) what was the temperature of the coffee at 10.15 AM

(ii) The woman likes to drink coffee its temperature of the coffee $130^\circ\text{F} - 140^\circ\text{F}$ what time she drunk the coffee.

Let T be the temp of coffee at time t .

$$S=70$$

$$\frac{dT}{dt} \propto T-S$$

$$\frac{dT}{dt} = k(T-S)$$

$$T-S = ce^{kt}$$

$$T-70 = ce^{kt} \quad \text{--- ①}$$

$$\text{when } t=0 \quad T=180 \quad ① \Rightarrow 180-70 = ce^0$$

$$c=110$$

$$T-70 = 110 e^{kt} \quad \text{--- ②}$$

$$\text{when } t=10 \quad T=160 \quad ② \Rightarrow 160-70 = 110 e^{10k}$$

$$\frac{90}{110} = e^{10k} \quad \text{--- ③}$$

not given

$$(i) \text{ when } t=15 \quad T=? \quad ② \Rightarrow T-70 = 110 e^{15k}$$

$$T-70 = 110 e^{3/2 k}$$

$$= 110 (e^{10k})^{3/2}$$

$$= 110 \frac{90}{110} \left(\frac{90}{110}\right)^{1/2}$$

$$= 90 (0.9045)$$

$$T-70 = 81.4$$

$$T = 70 + 81.4 = 151.4^\circ\text{F}$$

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(ii) To find k

$$\textcircled{3} \Rightarrow \frac{9}{11} = e^{10k}$$

$$\log \frac{9}{11} = 10k$$

$$-0.2006 = k$$

$$\boxed{k = -0.02006}$$

$T = 130^\circ$ $t = ?$

$$\textcircled{2} \Rightarrow 130 - 70 = 110 e^{kt}$$

$$\frac{60}{110} = e^{kt}$$

$$\log \frac{6}{11} = e^{-0.02006t}$$

$$-0.606135 = -0.02006t$$

$$\frac{0.606135}{0.02006} = t$$

$$t = 30.216 \text{ min}$$

$T = 140^\circ F$ $t = ?$

$$\textcircled{2} \Rightarrow 140 - 70 = 110 e^{kt}$$

$$\frac{70}{110} = e^{kt}$$

$$\log \frac{7}{11} = kt$$

$$-0.45198 = -0.02006t$$

$$\frac{0.452}{0.02006} = t$$

$$t = 22.53 \text{ min.}$$

tem 130-140 comes between 10.22 - 10.30 AM approx.

- ⑨ A pot of boiling water at $100^\circ C$ is removed from a stove at time $t=0$ and left to cool in the kitchen. After 5 min the water tem $80^\circ C$, and another 5 minutes later it has dropped to $65^\circ C$. Determine the temperature of the kitchen.

Let T be the temperature of boiling water at time t

S be the room (kitchen) Temp.

$$\frac{dT}{dt} \propto T - S$$

$$\frac{dT}{dt} = K(T - S)$$

$$T - S = ce^{kt} \quad \text{--- ①}$$

when $t=0, T=100$

$$100 - S = ce^0$$

$$\boxed{C = 100 - S}$$

$$T - S = (100 - S)e^{kt} \quad \text{--- (1)}$$

when $t = 5$ $T = 80^\circ$

$$(1) \Rightarrow 80 - S = (100 - S)e^{5k}$$

$$e^{5k} = \frac{80 - S}{100 - S}$$

when $t = 10$ $T = 65$

$$(1) \Rightarrow 65 - S = (100 - S)e^{10k}$$

$$= (100 - S)(e^{5k})^2$$

$$65 - S = (100 - S) \frac{(80 - S)(80 - S)}{(100 - S)(100 - S)}$$

$$6500 - 165S + S^2 = 6400 - 160S + S^2$$

$$6500 - 6400 = 165S - 160S$$

$$5S = 100$$

$$S = 20^\circ \text{C}$$

Kitchen Temperature = 20°C .

- (10) A tank initially contains 50 litres of pure water, starting at time $t = 0$ a brine containing with 2 grams of dissolved salt per litre flows into the tank at the rate of 3 litres per minute. The mixture is kept uniform by stirring and the well stirred mixture simultaneously flows out of the tank at the same rate. Find the amount of salt present in the tank at any time $t > 0$.

Let $x(t)$ denote the amount of salt in the tank at time t .

$$\frac{dx}{dt} = \text{inflow rate} - \text{out flow}$$

$$\text{inflow rate} = 2 \times 3 = 6$$

$$\text{out flow} = \frac{3}{50} x$$

$$\frac{dx}{dt} = 6 - \frac{3}{50} x$$

$$= -\frac{3}{50} \left(x - \frac{2 \times 50}{3} \right)$$

$$\frac{dx}{dt} = -\frac{3}{50} (x - 100)$$

$$\frac{dx}{x - 100} = -\frac{3}{50} dt$$

$$x - 100 = ce^{-\frac{3t}{50}} \quad \text{--- (1)}$$

when $t = 0$ $x = 0$

$$(1) \Rightarrow 0 - 100 = ce^0 \quad c = -100$$

$$(1) \Rightarrow x - 100 = -100e^{-\frac{3t}{50}}$$

$$x = 100 - 100e^{-\frac{3t}{50}}$$

$$x = 100(1 - e^{-\frac{3t}{50}})$$

Give your suggestions

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- ⑤ Suppose a person deposits 10,000 Indian rupees in a bank account at the rate of 5% per annum compounded continuously. How much money will be in his bank account 18 months later?

Let A be the Amount at any time t .

$$\frac{dA}{dt} \propto A$$

$$\frac{dA}{dt} = 0.05A$$

$$A = ce^{0.05t} \quad \text{--- (1)}$$

$$t=0, A=10,000 \Rightarrow$$

$$10,000 = ce^0$$

$$c = 10,000$$

$$A = 10,000 e^{0.05t} \quad \text{--- (2)}$$

$$t = 1\frac{1}{2} = \frac{3}{2}$$

$$\Rightarrow A = 10,000 e^{0.05 \times \frac{3}{2}}$$

$$A = 10,000 e^{0.075}$$

- ⑥ Assume that the rate at which radioactive nuclei decay is proportional to the number of such nuclei that are present in a given sample. In a certain sample 10% of the original number of radioactive nuclei have undergone disintegration in a period of 100 years. What percentage of the original radioactive nuclei will remain after 1000 years.

Let A be the Amount of nuclei at any time t

$$\frac{dA}{dt} \propto A$$

$$\frac{dA}{dt} = kA$$

$$A = ce^{kt} \quad \text{--- (1)}$$

$$\text{i) when } t=0, A=A_0 \Rightarrow A_0 = ce^0 \quad \text{--- (2)}$$

$$A = A_0 e^{kt} \quad \text{--- (3)}$$

$$\text{ii) when } t=100, A=90\% A_0$$

$$A = \frac{9}{10} A_0$$

$$\Rightarrow \frac{9}{10} A_0 = A_0 e^{100k}$$

$$e^{100k} = \frac{9}{10}$$

$$\text{iii) when } t=1000, A = A_0 e^{1000k}$$

$$\Rightarrow A = A_0 e^{1000k}$$

$$= A_0 (e^{100k})^{10}$$

$$= A_0 \left(\frac{9}{10}\right)^{10}$$

$$A = A_0 \frac{9^{10}}{10^{10}} \times 100\%$$

$$A = A_0 \left(\frac{9^{10}}{10^8}\right)\%$$

Give your suggestions
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11. PROBABILITY DISTRIBUTIONS

Random Variable

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A random variable x is a function defined on a sample space S into the real numbers R such that the inverse image of points or subset or interval of R is an event in S , for which probability is assigned.

Exercise 11.1

- 1) Suppose x is the number of tails occurred when three fair coins are tossed once simultaneously. Find the values of the random variable x and number of points in its inverse images.

Sample space $S = \{ HHH, HHT, HTH, THH, HTT, THT, TTH, TTT \}$

given that $n(S) = 8$

x be the random variable denotes no of. tails

$$x = 0, 1, 2, 3$$

0 tail.

$$x(HHH) = 0$$

1 tail

$$x(HHT) = 1$$

$$x(HTH) = 1$$

$$x(THH) = 1$$

2 tail

$$x(HTT) = 2$$

$$x(THT) = 2$$

$$x(TTH) = 2$$

3 tail

$$x(TTT) = 3$$

$x(w)$ denotes the number of tails

$$x(w) = \begin{cases} 0 & \text{if } w = HHH \\ 1 & \text{if } w = HHT, THH, HTH \\ 2 & \text{if } w = HTT, THT, TTH \\ 3 & \text{if } w = TTT \end{cases}$$

values of Random Variable	0	1	2	3	Total.
Number of points in inverse image	1	3	3	1	8

Examples 11.1

Suppose two coins are tossed once. If x denote the number of tails, write down the sample space.

(ii) find the inverse image of 1 (iii) the values of the random variable and number of elements in its inverse images.

(i) sample space $S = \{HH, HT, TH, TT\}$ $n(S) = 4$

(ii) x be the number of tails

$$x(TT) = 2 \quad x(TH) = 1 \quad x(HT) = 1$$

$$x(HH) = 0$$

$$x = 0, 1, 2$$

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$$x(w) = \begin{cases} 2 & \text{if } w = TT \\ 1 & \text{if } w = TH, HT \\ 0 & \text{if } w = HH \end{cases}$$

inverse image of 1 is $\{TH, HT\}$

(iii) value of Random variable	0	1	2	Total
Number of elements in inverse image	1	2	1	4

Example 11.2

Suppose a pair of unbiased dice is rolled once. If x denotes the total score of two dice, write down.

(i) the sample space (ii) the values taken by the random variable x (iii) The inverse image of 10, (iv) the number of elements in inverse image of x .

(i) sample space $S = \{ (1,1), (1,2), (1,3), (1,4), (1,5), (1,6) \}$

$(2,1), (2,2), (2,3), (2,4), (2,5), (2,6)$

$(3,1), (3,2), (3,3), (3,4), (3,5), (3,6)$

$(4,1), (4,2), (4,3), (4,4), (4,5), (4,6)$

$(5,1), (5,2), (5,3), (5,4), (5,5), (5,6)$

$(6,1), (6,2), (6,3), (6,4), (6,5), (6,6) \}$

$$n(S) = 36$$

$$x(\alpha, \beta) = \alpha + \beta$$

$$x(1,1) = 2$$

$$x(1,2) = x(2,1) = 3$$

$$x(1,3) = x(2,2) = x(3,1) = 4$$

$$x(1,4) = x(2,3) = x(3,2) = x(4,1) = 5$$

$$x(1,5) = x(2,4) = x(3,3) = x(4,2) = x(5,1) = 6$$

$$x(1,6) = x(2,5) = x(3,4) = x(4,3) = x(5,2) = x(6,1) = 7$$

$$x(2,6) = x(3,5) = x(4,4) = x(5,3) = x(6,2) = 8$$

$$X(3,6) = X(4,5) = X(5,4) = X(6,3) = 9$$

$$X(4,5) = X(5,5) = X(6,4) = 10$$

$$X(5,6) = X(6,5) = 11$$

$$X(6,6) = 12$$

$$(ii) \quad X = 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12$$

$$(iii) \quad \text{Inverse image of 10 is } \{(4,6), (5,5), (6,4)\}$$

Value of random variable	2	3	4	5	6	7	8	9	10	11	12	Σ
Number of elements in inverse image	1	2	3	4	5	6	5	4	3	2	1	36

Example 11.3

An urn contains 2 white and 3 red balls. A sample of 3 balls chosen. If x denotes the number of red balls, find the value of random variable x and its number of inverse images

W	R	Total
2	3	5

$$n(S) = {}^5C_3 = {}^5C_2 = \frac{5 \times 4}{1 \times 2} = 10$$

x denote no. of red balls

$$x = 1, 2, 3 \quad (0 \text{ red not possible})$$

$$X(\text{one red}) = {}^2C_2 \times {}^3C_1 = 1 \times 3 = 3$$

$$X(\text{two red}) = {}^2C_1 \times {}^3C_2 = 2 \times 3 = 6$$

$$X(\text{three red}) = {}^2C_0 \times {}^3C_3 = 1 \times 1 = 1$$

value of random variable x	1	2	3	TOTAL
Number of elements in inverse images	3	6	1	10

Example 11.4

Two balls are chosen randomly from an urn containing 6 white and 4 black balls. Suppose that we win ₹30 for each black ball selected and we lose ₹20 for each white ball selected. If x denotes the winning amount, then find the values of x and number of points in its inverse images

$$n(S) = {}^{10}C_2 = \frac{10 \times 9}{1 \times 2} = 45$$

W	B	Total
6	4	10

x denote winning amount

$$x(2 \text{ black}) = 30 + 30 = 60$$

$$x(1 \text{ black, 1 white}) = 30 - 20 = 10$$

$$x(2 \text{ white}) = 2(-20) = -40$$

3A

$$X = -40, 10, 60$$

$$X(-40) = X(2 \text{ white}) = {}^6C_2 \times {}^4C_0 \\ = \frac{6 \times 5}{1 \times 2} \times 1 = 15$$

$$X(10) = X(1 \text{ white } 1 \text{ black}) = {}^6C_1 \times {}^4C_1 = 6 \times 4 = 24$$

$$X(60) = X(2 \text{ black}) = {}^6C_0 \times {}^4C_2 = 1 \times \frac{4 \times 3}{1 \times 2} = 6$$

Values of the Random Variable	60	10	-40	Total
Number of elements in inverse image	6	24	15	45

- 2) In a pack of 52 playing cards, two cards are drawn at random simultaneously. If the number of black cards drawn is a random variable, find the values of the random variable and number of points in its inverse images.

$$n(S) = {}^{52}C_2$$

$$= \frac{52 \times 51}{1 \times 2}$$

$$n(S) = 1326$$

BLACK	OTHER	TOTAL
26	26	52

Let X be the random variable denote no. of Black cards drawn.

$$X = 0, 1, 2$$

$X(W)$ denotes no. of Black cards

$$X(W) = \begin{cases} 0 & \text{if no black card} \\ 1 & \text{if one is black card} \\ 2 & \text{if two black} \end{cases}$$

$$X(\text{no black}) = X(0) = {}^{26}C_0 \times {}^{26}C_2$$

$$= 1 \times \frac{26 \times 25}{1 \times 2} = 325$$

$$X(\text{one black}) = X(1) = {}^{26}C_1 \times {}^{26}C_1$$

$$= 26 \times 26 = 676$$

$$X(2 \text{ black}) = X(2) = {}^{26}C_2 \times {}^{26}C_0$$

$$= \frac{26 \times 25}{1 \times 2} = 325$$

value of random variable	0	1	2	TOTAL
No. of points in Inverse Image	325	676	325	1326

3) An urn contains 5 mangoes and 4 apples. Three fruits are taken at random. If the number of apples taken is a random variable, Then find the values of the random variable and number of points in the inverse images.

Total Fruits = 5 + 4 = 9

(without replacing) $n(S) = {}^9C_3 = \frac{9 \times 8 \times 7}{1 \times 2 \times 3} = 84$

Let X be the random variable denotes the no. of apples taken.

$$X = 0, 1, 2, 3$$

$X(W)$ denotes no. of apples.

$$X(w) = \begin{cases} 0 & \text{if no apple (3 mangoes) drawn} \\ 1 & \text{if one apple (2 mangoes) taken} \\ 2 & \text{if two apples (1 mango) taken} \\ 3 & \text{if 3 apples (no mango) taken} \end{cases}$$

$$X(\text{no apple}) = X(0) = {}^4C_0 {}^5C_3 = 1 \times \frac{5 \times 4 \times 3}{1 \times 2 \times 1} = 10$$

$$X(0) = 10$$

$$X(\text{one apple}) = X(1) = {}^4C_1 {}^5C_2 = 4 \times \frac{5 \times 4}{1 \times 2} = 40$$

$$X(\text{two apple}) = X(2) = {}^4C_2 {}^5C_1 = \frac{4 \times 3}{1 \times 2} \times 5 = 30$$

$$X(3 \text{ apples}) = X(3) = {}^4C_3 {}^5C_0 = 4C_1 \times 1 = 4$$

value of Random variable X	0	1	2	3	Total
Number of elements in Inverse image	10	40	30	4	84

4) Two balls are chosen randomly from a urn containing 6 red and 8 black balls, suppose that we win ₹15 for each red ball selected and we lose ₹10 for each black ball selected. X denotes the winning amount, then find the values of X and number of points in its inverse images.

$$\text{Total balls} = 6 + 8 = 14$$

$$n(S) = {}^{14}C_2 = \frac{14 \times 13}{1 \times 2} = 91$$

Let X be the amount won

$$X = -10 \times 2, -10 + 15, 15 + 15$$

$$X = -20, 5, 30$$

$$X(w) = \begin{cases} -20 & \text{if two black balls.} \\ 5 & \text{if one black one red.} \\ 30 & \text{if two red balls.} \end{cases}$$

$$X(\text{two black ball.}) = X(-20) = {}^6C_0 {}^8C_2 = 1 \times \frac{8 \times 7}{1 \times 2} = 28$$

$$X(1 \text{ black 1 red}) = X(5) = {}^6C_1 {}^8C_1 = 6 \times 8 = 48$$

$$X(2 \text{ red balls}) = X(30) = {}^6C_2 {}^8C_0 = \frac{6 \times 5}{1 \times 2} \times 1 = 15$$

value of random variable	-20	5	30	Total
No. of points in Inverse Image	28	48	15	91

- 5) A six sided die is marked 2 on one face 3 on two of its faces and 4 on remaining three faces. The die is thrown twice. If X denote the total scores in two throws, find the values of the random variable and number of points in its inverse images.

X	2	3	3	4	4	4
2	4	5	5	6	6	6
3	5	6	6	7	7	7
3	5	6	6	7	7	7
4	6	7	7	8	8	8
4	6	7	7	8	8	8
4	6	7	7	8	8	8

$$n(S) = 36$$

X is assigned to each point (α, β) the sum on the faces

$$X(\alpha, \beta) = \alpha + \beta$$

$$X(2, 2) = 4$$

$$X(2, 3) = X(3, 2) = 5$$

$$X(2, 4) = X(3, 3) = X(4, 2) = 6$$

$$X(4, 3) = X(3, 4) = 7$$

$$X(4, 4) = 8$$

$X(2, 2)$ one time
 $X(2, 3)$ 2 times
 $X(3, 2)$ 2 times

X takes the values 4, 5, 6, 7, 8

values of random variable	4	5	6	7	8	Total
Number of elements in inverse image	1	4	10	12	9	36

A random variable X is said to be a discrete random variable if the range of X is countable.

Probability Mass function (PMF)

If X is a discrete random variable with discrete values $x_1, x_2, \dots, x_n, \dots$ then the function f or P defined by

$$f(x_k) = P(X = x_k), \quad \text{for } k = 1, 2, \dots, n.$$

is called PMF.

$$* f(x_k) \geq 0 \text{ for } k = 1, 2, \dots, n. \text{ and } * \sum_k f(x_k) = 1$$

cumulative distribution Function (or) Distribution Function

The cumulative distribution function $F(x)$ of a discrete random variable X , taking values $x_1, x_2, \dots$ such that $x_1 < x_2 < x_3 < \dots$ with p.m.f $f(x_i)$

$$F(x) = P(X \leq x) = \sum_{x_i \leq x} f(x_i), \quad x \in R$$

$$* f(x_i) = F(x_i) - F(x_{i-1}), \quad i = 1, 2, 3, \dots$$

Exercise 11.2

1) Three fair coins are tossed simultaneously. Find the probability mass function for number of heads occurred.

$$S = \{ HHH, HHT, HTH, THH, HTT, THT, TTH, TTT \}$$

$$n(S) = 8$$

Let X be random variable denotes no. of heads. $X = 0, 1, 2, 3$.

Values of Random variable	0	1	2	3	Total
Number of elements in inverse images	1	3	3	1	8

$$P(X=0) = \frac{1}{8} \quad P(X=1) = \frac{3}{8} \quad P(X=2) = \frac{3}{8} \quad P(X=3) = \frac{1}{8}$$

probability mass function is

x	0	1	2	3
$f(x)$	$\frac{1}{8}$	$\frac{3}{8}$	$\frac{3}{8}$	$\frac{1}{8}$

2) A six sided die is marked '1' on one face, '3' on two of its faces, and '5' on remaining three faces. The die is thrown twice. If X denotes the total score in two throws, find

- (i) the probability mass function, (ii) the cumulative distribution function (iii) $P(4 \leq X < 10)$ (iv) $P(X \geq 6)$

I \ II	1	3	3	5	5	5
1	2	4	4	6	6	6
3	4	6	6	8	8	8
3	4	6	6	8	8	8
5	6	8	8	10	10	10
5	6	8	8	10	10	10
5	6	8	8	10	10	10

X denote the total score in two throws
it takes 2, 4, 6, 8, 10

values of R.V X	2	4	6	8	10	TOTAL
Number of elements in inverse images	1	4	10	12	9	36

$$P(X=2) = \frac{1}{36}, P(X=4) = \frac{4}{36}, P(X=6) = \frac{10}{36}, P(X=8) = \frac{12}{36}$$

$$P(X=10) = \frac{9}{36}$$

(i) The probability mass function is

X	2	4	6	8	10
f(x)	$\frac{1}{36}$	$\frac{4}{36}$	$\frac{10}{36}$	$\frac{12}{36}$	$\frac{9}{36}$

(ii) The cumulative distribution function

x	2	4	6	8	10
F(x)	$\frac{1}{36}$	$\frac{5}{36}$	$\frac{15}{36}$	$\frac{27}{36}$	1

$$\begin{aligned} \text{(iii)} \quad P(4 \leq X \leq 10) &= P(X=4) + P(X=6) + P(X=8) \\ &= \frac{4}{36} + \frac{10}{36} + \frac{12}{36} = \frac{26}{36} \\ &= \frac{13}{18} \end{aligned}$$

$$\begin{aligned} \text{(iv)} \quad P(X \geq 6) &= P(X=6) + P(X=8) + P(X=10) \\ &= \frac{10}{36} + \frac{12}{36} + \frac{9}{36} = \frac{31}{36} \end{aligned}$$

3) Find the probability mass function and cumulative distribution function of number of girls child in families with 4 children, assuming equal probabilities for boys and girls.

$$n(S) = 2^4 = 16$$

Let x be the random variable denotes no. of girls child.

$$x = 0, 1, 2, 3, 4$$

Value of R.V x	0	1	2	3	4
Number of elements in inverse image	$4C_0$	$4C_1$	$4C_2$	$4C_3$	$4C_4$
	1	4	6	4	1

$$P(X=0) = \frac{1}{16} = P(X=4)$$

$$P(X=1) = \frac{4}{16} = P(X=3)$$

$$P(X=2) = \frac{6}{16}$$

probability mass function is $f(x) = \begin{cases} \frac{1}{16} & \text{for } x=0,4 \\ \frac{4}{16} & \text{for } x=1,3 \\ \frac{6}{16} & \text{for } x=2 \end{cases}$

x	0	1	2	3	4
$f(x)$	$\frac{1}{16}$	$\frac{4}{16}$	$\frac{6}{16}$	$\frac{4}{16}$	$\frac{1}{16}$

$$F(x) = P(X \leq x)$$

$$F(0) = P(X \leq 0) = \frac{1}{16}$$

$$F(1) = P(X \leq 1) = \frac{1}{16} + \frac{4}{16} = \frac{5}{16}$$

$$F(2) = P(X \leq 2) = \frac{5}{16} + \frac{6}{16} = \frac{11}{16}$$

$$F(3) = P(X \leq 3) = \frac{11}{16} + \frac{4}{16} = \frac{15}{16}$$

$$F(4) = P(X \leq 4) = \frac{15}{16} + \frac{1}{16} = \frac{16}{16} = 1$$

cumulative distribution function is

x	0	1	2	3	4
$F(x)$	$\frac{1}{16}$	$\frac{5}{16}$	$\frac{11}{16}$	$\frac{15}{16}$	1

4) suppose a discrete random variable can only take the values 0, 1, and 2.

The probability mass function is defined by.

$$f(x) = \begin{cases} \frac{x^2+1}{k}, & \text{for } x=0,1,2 \\ 0 & \text{otherwise} \end{cases}$$

Find the (i) value of k (ii) cumulative distribution function (iii) $P(X \geq 1)$

given $f(x)$ is a P.M.F.

$$\therefore \sum f(x) = 1$$

$$\frac{0^2+1}{k} + \frac{1^2+1}{k} + \frac{2^2+1}{k} = 1$$

$$\frac{1}{K} + \frac{2}{K} + \frac{5}{K} = 1$$

$$\frac{8}{K} = 1$$

$$K = 8$$

$$\therefore f(x) = \begin{cases} \frac{x^2+1}{8} & \text{for } x=0,1,2 \\ 0 & \text{otherwise} \end{cases}$$

ii)

$$F(x) = P(X \leq x)$$

$$F(x) = \begin{cases} 1/8 & \text{for } x \leq 0 \\ 3/8 & \text{for } x \leq 1 \\ 8/8 = 1 & \text{for } x \leq 2 \end{cases}$$

or

x	0	1	2
f(x)	1/8	2/8	5/8

$$F(0) = P(X=0) = 1/8$$

$$F(1) = P(X=0) + P(X=1) = \frac{3}{8}$$

$$F(2) = P(X=0) + P(X=1) + P(X=2) = \frac{8}{8} = 1$$

$$\begin{aligned} \text{iii) } P(X \geq 1) &= P(X=1) + P(X=2) \\ &= \frac{2}{8} + \frac{5}{8} \\ &= \frac{7}{8} \end{aligned}$$

⑤ The cumulative distribution function of a discrete random variable is given by

$$F(x) = \begin{cases} 0 & -\infty < x < -1 \\ 0.15 & -1 \leq x < 0 \\ 0.35 & 0 \leq x < 1 \\ 0.60 & 1 \leq x < 2 \\ 0.85 & 2 \leq x < 3 \\ 1 & 3 \leq x < \infty \end{cases}$$

Find i) Probability mass function

$$\text{ii) } P(X < 1)$$

$$\text{iii) } P(X \geq 2)$$

$$P(X=-1) = F(-1) = 0.15$$

$$P(X=0) = F(0) - F(-1) = 0.35 - 0.15$$

$$P(X=0) = 0.20$$

$$P(X=1) = F(1) - F(0) = 0.60 - 0.35$$

$$P(X=1) = 0.25$$

$$P(X=2) = F(2) - F(1) = 0.85 - 0.60$$

$$= 0.25$$

$$P(X=3) = F(3) - F(2) = 1 - 0.85$$

$$= 0.15$$

$$\text{i) Probability mass function is } x \quad -1 \quad 0 \quad 1 \quad 2 \quad 3$$

$$P(X=x) \quad 0.15 \quad 0.20 \quad 0.25 \quad 0.25 \quad 0.15$$

$$\text{ii) } P(X < 1) = P(X \leq 0) = f(0) + f(-1)$$

$$= P(X=-1) + P(X=0) = 0.15 + 0.20 = 0.35$$

$$\text{iii) } P(X \geq 2) = P(2 \leq X \leq 3) = P(X=2) + P(X=3)$$

$$= 0.25 + 0.15$$

$$= 0.40$$

(9) A random variable x has the following probability mass function

x	1	2	3	4	5
$f(x)$	k^2	$2k^2$	$3k^2$	$2k$	$3k$

Find (i) the value of k (ii) $P(2 \leq x < 5)$ (iii) $P(3 < x)$.

$f(x)$ is a p.m.f.

$$\therefore \sum f(x) = 1$$

$$k^2 + 2k^2 + 3k^2 + 2k + 3k = 1$$

$$6k^2 + 5k - 1 = 0$$

$$\left(\frac{k+8}{6}\right)\left(\frac{k-1}{6}\right) = 0$$

$$k+1=0 \quad k-\frac{1}{6}=0$$

$$k=-1 \quad k=\frac{1}{6}$$

$$\frac{-6}{5} = -1$$

$$\boxed{k = \frac{1}{6}}$$

P.M.F. is

x	1	2	3	4	5
$f(x)$	$\frac{1}{36}$	$\frac{2}{36}$	$\frac{3}{36}$	$\frac{2}{6}$	$\frac{3}{6}$

$$\begin{aligned} \text{(ii)} \quad P(2 \leq x < 5) &= P(x=2) + P(x=3) + P(x=4) \\ &= 2k^2 + 3k^2 + 2k = \frac{2}{36} + \frac{3}{36} + \frac{2}{6} = \frac{2+3+12}{36} \\ &= \frac{17}{36} \end{aligned}$$

$$\text{(iii)} \quad P(3 < x) = P(x=4) + P(x=5) = 2k + 3k = 5k = \frac{5}{6}$$

(7) The cumulative distribution function of a discrete random variable is given by

$$F(x) = \begin{cases} 0 & \text{for } -\infty < x < 0 \\ \frac{1}{2} & \text{for } 0 \leq x < 1 \\ \frac{3}{5} & \text{for } 1 \leq x < 2 \\ \frac{4}{5} & \text{for } 2 \leq x < 3 \\ \frac{9}{10} & \text{for } 3 \leq x < 4 \\ 1 & \text{for } 4 \leq x < \infty \end{cases}$$

Find (i) The probability mass function (ii) $P(x < 3)$ and (iii) $P(x \geq 2)$

$$f(0) = P(x=0) = F(0) = \frac{1}{2}$$

$$f(1) = P(x=1) = F(1) - F(0) = \frac{3}{5} - \frac{1}{2} = \frac{6-5}{10} = \frac{1}{10}$$

$$f(2) = P(x=2) = F(2) - F(1) = \frac{4}{5} - \frac{3}{5} = \frac{1}{5}$$

$$f(3) = P(x=3) = F(3) - F(2) = \frac{9}{10} - \frac{4}{5} = \frac{1}{10}$$

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$$f(4) = P(X=4) = F(4) - F(3) = 1 - \frac{9}{10} = \frac{1}{10}$$

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The probability mass function is

x	0	1	2	3	4
$P(X=x)$	$\frac{1}{5}$	$\frac{1}{10}$	$\frac{1}{5}$	$\frac{1}{10}$	$\frac{1}{10}$

$$(ii) P(X < 3) = (P(X=0) + P(X=1) + P(X=2))$$

$$\text{or}$$

$$= P(X \leq 2) = F(2) = \frac{4}{5}$$

$$(iii) P(X \geq 2) = 1 - P(X < 2) = 1 - P(X \leq 1) = 1 - F(1)$$

$$= 1 - \frac{3}{5} = \frac{2}{5}$$

Continuous Distributions

A random variable takes any value in a set I of R .

Probability Density function (p.d.f)

$f(x)$ is said to be p.d.f. If $x \in [a, b]$ of a continuous random variable x

$$P(a \leq x \leq b) = \int_a^b f(x) dx$$

$$* f(x) \text{ is a p.d.f. } \Leftrightarrow \int_{-\infty}^{\infty} f(x) dx = 1$$

Cumulative distribution function

$$F(x) = P(X \leq x) = \int_{-\infty}^x f(u) du \quad -\infty < u < \infty$$

$$* f(x) = \frac{dF}{dx} \text{ or } F'(x)$$

Exercise 11.3

1) The probability density function of x is given by

$$f(x) = \begin{cases} kx e^{-2x} & \text{for } x > 0 \\ 0 & \text{for } x \leq 0 \end{cases}$$

Find the value of k .

$$f(x) \text{ is a p.d.f. } \therefore \int_{-\infty}^{\infty} f(x) dx = 1$$

$$\int_0^{\infty} kx e^{-2x} dx = 1$$

$$\int_0^{\infty} x^n e^{-ax} dx = \frac{n!}{a^{n+1}} \quad \begin{matrix} a=2 \\ n=1 \end{matrix}$$

$$k \left[\frac{1!}{2^{1+1}} \right] = 1$$

$$\boxed{k = 2^2} \\ \boxed{1 <= 4}$$

② The probability density function of x is

$$f(x) = \begin{cases} x & 0 < x < 1 \\ 2-x & 1 \leq x < 2 \\ 0 & \text{otherwise} \end{cases}$$

Find (i) $P(0.2 \leq x < 0.6)$

(ii) $P(1.2 \leq x < 1.8)$

(iii) $P(0.5 \leq x < 1.5)$

(i) $P(0.2 \leq x < 0.6)$

$$= \int_{0.2}^{0.6} f(x) dx = \int_{0.2}^{0.6} x dx$$

$$= \left[\frac{x^2}{2} \right]_{0.2}^{0.6} = \frac{1}{2} [0.6^2 - 0.2^2]$$

$$= \frac{1}{2} [0.36 - 0.04] = \frac{1}{2} [0.32]$$

$$= 0.16$$

$$(ii) P(1.2 \leq x < 1.8) = \int_{1.2}^{1.8} f(x) dx = \int_{1.2}^{1.8} (2-x) dx$$

$$= \left[\frac{2x - x^2}{2} \right]_{1.2}^{1.8} = \frac{1}{2} [(2 \cdot 1.8 - 1.8^2) - (2 \cdot 1.2 - 1.2^2)]$$

$$= \frac{1}{2} [0.2^2 - 0.8^2] = \frac{1}{2} [0.04 - 0.64]$$

$$= \frac{1}{2} (-0.60) = -0.30$$

$$(iii) P(0.5 \leq x < 1.5) = \int_{0.5}^{1.5} f(x) dx = \int_{0.5}^1 f(x) dx + \int_1^{1.5} f(x) dx$$

$$= \int_{0.5}^1 x dx + \int_1^{1.5} (2-x) dx$$

$$= \left[\frac{x^2}{2} \right]_{0.5}^1 + \left[\frac{2x - x^2}{2} \right]_1^{1.5}$$

$$= \frac{1}{2} [(1^2 - 0.5^2) - ((2 \cdot 1.5 - 1.5^2) - (2 \cdot 1 - 1^2))]$$

$$= \frac{1}{2} [(1 - 0.25) - (0.75 - 1)]$$

$$= \frac{1}{2} [0.75 + 0.25] = 0.5$$

- ③ Suppose the amount of milk sold daily at a milk booth is distributed with a minimum of 200 litres and a maximum of 600 litres with probability density function $f(x) = \begin{cases} k & 200 \leq x \leq 600 \\ 0 & \text{otherwise} \end{cases}$

Find (i) The value of k (ii) The distribution function
(iii) The probability that daily sales will fall between 300 litres and 500 litres.

(i) given $f(x)$ is a P.d.f $\int_{-\infty}^{\infty} f(x) dx = 1$

$$\int_{200}^{600} k dx = 1 \Rightarrow k \left[x \right]_{200}^{600} = 1$$

$$k(600 - 200) = 1$$

$$400k = 1$$

$$\boxed{k = \frac{1}{400}}$$

(ii) distribution function

$$F(x) = \int_{-\infty}^x f(t) dt$$

a) when $x \in (-\infty, 200)$

$$F(x) = \int_{-\infty}^x f(t) dt = \int_{-\infty}^{200} 0 dt = 0$$

b) when $x \in [200, 600]$

$$F(x) = \int_{-\infty}^x f(t) dt = \int_{-\infty}^{200} f(t) dt + \int_{200}^x f(t) dt$$

$$= 0 + \int_{200}^x \frac{1}{400} dt = \frac{1}{400} \left[t \right]_{200}^x$$

$$= \frac{x}{400} - \frac{200}{400}$$

$$= \frac{x}{400} - \frac{1}{2}$$

c) when $x \in (600, \infty)$

$$F(x) = \int_{-\infty}^x f(t) dt = \int_{-\infty}^{200} f(t) dt + \int_{200}^{600} f(t) dt + \int_{600}^x f(t) dt$$

$$= 0 + 1 + 0$$

$$= 1$$

$$F(x) = \begin{cases} 0 & x < 200 \\ \frac{x}{400} - \frac{1}{2} & x \in [200, 600] \\ 1 & x \in (600, \infty) \end{cases}$$

$$\text{iii) } P(300 < x < 500) = F(500) - F(300)$$

$$= \left(\frac{500}{400} - \frac{1}{2} \right) - \left(\frac{300}{400} - \frac{1}{2} \right)$$

$$= \frac{500-300}{400} = \frac{1}{2}$$

$$= \frac{200}{400} = \frac{1}{2}$$

④ The probability density function of x is given

$$\text{by } f(x) = \begin{cases} k e^{-x/3} & \text{for } x > 0 \\ 0 & \text{for } x \leq 0 \end{cases}$$

Find i) the value of k ii) The distribution function
 iii) $P(x < 3)$ iv) $P(5 \leq x)$ v) $P(x \leq 4)$

Given $f(x)$ is a p.d.f $\Rightarrow \int_{-\infty}^{\infty} k e^{-x/3} dx = 1$

$$\therefore \int_{-\infty}^{\infty} f(x) dx = 1$$

$$\frac{k}{-1/3} \left[e^{-x/3} \right]_0^{\infty} = 1$$

$$-3k [e^0 - e^{\infty}] = 1$$

$$-3k [0 - 1] = 1$$

$$3k = 1 \quad k = \frac{1}{3}$$

ii) distribution function

$$F(x) = \int_{-\infty}^x f(t) dt$$

a) $x \in (-\infty, 0]$ $F(x) = \int_{-\infty}^x f(t) dt = 0$

b) when $x \in (0, \infty)$ $F(x) = \int_{-\infty}^x f(t) dt = \int_{-\infty}^0 f(t) dt + \int_0^x f(t) dt$

$$= 0 + \int_0^x \frac{1}{3} e^{-t/3} dt$$

$$= \frac{1}{3} \left[-e^{-t/3} \right]_0^x = -e^{-x/3} + e^0$$

$$= 1 - e^{-x/3}$$

$$F(x) = \begin{cases} 1 - e^{-x/3} & \text{for } x > 0 \\ 0 & \text{for } x \leq 0 \end{cases}$$

$$(ii) P(X < 3) = P(X \leq 3) = F(3) = 1 - e^{-3/3} = 1 - e^{-1}$$

$$(iv) P(5 \leq x) = P(X \geq 5) = 1 - P(X < 5) = 1 - F(5) \\ = 1 - (1 - e^{-5/3}) \\ = e^{-5/3}$$

$$(v) P(X \leq 4) = F(4) = 1 - e^{-4/3}$$

(5) If x is the random variable with probability density function $f(x)$ given by

$$f(x) = \begin{cases} x+1 & -1 \leq x < 0 \\ -x+1 & 0 \leq x < 1 \\ 0 & \text{otherwise} \end{cases}$$

find the distribution function $F(x)$ (ii) $P(-0.5 \leq x \leq 0.5)$

i) distribution function

a) when $x \in (-\infty, -1)$

$$F(x) = \int_{-\infty}^x f(t) dt = 0$$

b) when $x \in [-1, 0)$

$$F(x) = \int_{-\infty}^x f(t) dt = \int_{-\infty}^{-1} f(t) dt + \int_{-1}^x f(t) dt \\ = 0 + \int_{-1}^x (t+1) dt \\ = \left[\frac{t^2}{2} + t \right]_{-1}^x = \frac{x^2}{2} + x - \frac{1}{2} + 1$$

$$F(x) = \frac{x^2}{2} + x + \frac{1}{2}$$

c) when $x \in [0, 1)$

$$F(x) = \int_{-\infty}^x f(t) dt = \int_{-\infty}^{-1} f(t) dt + \int_{-1}^0 f(t) dt + \int_0^x f(t) dt$$

$$= 0 + \int_{-1}^0 (t+1) dt + \int_0^x (-t+1) dt$$

$$= \left[\frac{t^2}{2} + t \right]_{-1}^0 + \left[-\frac{t^2}{2} + t \right]_0^x$$

$$= 0 - \left(\frac{1}{2} - 1 \right) + \left(-\frac{x^2}{2} + x \right) - (0 + 0)$$

$$= +\frac{1}{2} - \frac{x^2}{2} + x$$

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d) when $x \in [1, \infty)$

$$F(x) = \int_{-\infty}^x f(t) dt = \int_{-\infty}^{-1} f(t) dt + \int_{-1}^0 f(t) dt + \int_0^1 f(t) dt + \int_1^x f(t) dt$$

$$= 0 + \frac{1}{2} + \int_0^1 (-t+1) dt + 0$$

$$= \frac{1}{2} + \left(-\frac{t^2}{2} + t\right) \Big|_0^1$$

$$= \frac{1}{2} + \left(-\frac{1}{2} + 1\right) - 0$$

$$= 1$$

$$F(x) = \begin{cases} 0 & x < -1 \\ \frac{x^2}{2} + x + \frac{1}{2} & -1 \leq x < 0 \\ -\frac{x^2}{2} + x + \frac{1}{2} & 0 \leq x < 1 \\ 1 & 1 \leq x \end{cases}$$

(ii) $P(-0.5 \leq x \leq 0.5)$

$$= F(0.5) - F(-0.5)$$

$$= F\left(\frac{1}{2}\right) - F\left(-\frac{1}{2}\right) = \left(\frac{\left(\frac{1}{2}\right)^2}{2} + \frac{1}{2} + \frac{1}{2}\right)$$

$$- \left(\frac{\left(-\frac{1}{2}\right)^2}{2} - \frac{1}{2} + \frac{1}{2}\right)$$

$$= \left(-\frac{1}{8} + 1\right) - \left(\frac{1}{8} - \frac{1}{2} + \frac{1}{2}\right)$$

$$= 1 - \frac{2}{8} = \frac{6}{8} = 0.75$$

6) If x is the random variable with distribution function $F(x)$ given by

$$F(x) = \begin{cases} 0 & x < 0 \\ \frac{1}{2}(x^2 + x) & 0 \leq x < 1 \\ 1 & x \geq 1 \end{cases}$$

then find the (i) The p.d.f, $f(x)$

(ii) $P(0.3 \leq x \leq 0.6)$

$$f(x) = \frac{d}{dx} (F(x))$$

$$f(x) = \begin{cases} 0 & x < 0 \\ \frac{1}{2}(2x+1) & 0 \leq x < 1 \\ 0 & x \geq 1 \end{cases}$$

$$f(x) = \begin{cases} \frac{1}{2}(2x+1) & 0 \leq x < 1 \\ 0 & \text{otherwise} \end{cases}$$

$$\begin{aligned} \text{(ii)} \quad P(0.3 \leq x \leq 0.6) &= F(0.6) - F(0.3) \\ &= \frac{1}{2}(0.6^2 + 0.6) - \frac{1}{2}(0.3^2 + 0.3) \\ &= \frac{1}{2}[0.36 + 0.6 - 0.09 - 0.3] \\ &= \frac{1}{2}[0.96 - 0.39] \\ &= \frac{1}{2}[0.57] \\ &= 0.285 \end{aligned}$$

Alter
(OR)

$$P(0.3 \leq x \leq 0.6) = \int_{0.3}^{0.6} \frac{1}{2}(2x+1) dx$$

$$= \frac{1}{2} \left[\frac{(2x+1)^2}{2} \right]_{0.3}^{0.6}$$

(Book
answer
wrong)

$$= \frac{1}{4} [2.2^2 - 1.6^2]$$

$$= \frac{1}{4} [4.84 - 2.56] = \frac{1}{4} [2.28]$$

$$= 0.285$$

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Mathematical Expectation

Mean

The expected value or mean or mathematical expectation of x , denoted by $E(x)$ or μ is

$$E(x) = \begin{cases} \sum x f(x) & \text{if } x \text{ is discrete} \\ \int_{-\infty}^{\infty} x f(x) dx & \text{if } x \text{ is continuous} \end{cases}$$

$$* \quad E(g(x)) = \begin{cases} \sum g(x) f(x) & \text{if } g(x) \text{ is discrete} \\ \int_{-\infty}^{\infty} g(x) f(x) dx & \text{if } g(x) \text{ is continuous} \end{cases}$$

$$* \quad E(1) = \begin{cases} \sum f(x) = 1 & \text{if } x \text{ is discrete} \\ \int_{-\infty}^{\infty} f(x) dx = 1 & \text{if } x \text{ is continuous} \end{cases}$$

Variance

$$V(X) = E(X^2) - (E(X))^2$$

$$V(X) = E(X - E(X))^2 = E(X - \mu)^2$$

properties

$$(1) E(ax+b) = aE(X) + b$$

$$(2) \text{Var}(ax+b) = a^2 \text{Var}(X)$$

Exercise 11.4

(1) For the random variable x with the given p.m.f. as below, find the mean and variance.

$$(i) f(x) = \begin{cases} \frac{1}{10} & x=2,5 \\ \frac{1}{5} & x=0,1,3,4 \end{cases}$$

x	0	1	2	3	4	5
$f(x)$	$\frac{1}{5}$	$\frac{1}{5}$	$\frac{1}{10}$	$\frac{1}{5}$	$\frac{1}{5}$	$\frac{1}{10}$
$xf(x)$	0	$\frac{1}{5}$	$\frac{2}{10}$	$\frac{3}{5}$	$\frac{4}{5}$	$\frac{5}{10}$
$x^2f(x)$	0	$\frac{1}{5}$	$\frac{4}{10}$	$\frac{9}{5}$	$\frac{16}{5}$	$\frac{25}{10}$

$$\text{mean} = \sum x f(x) = 0 + \frac{1}{5} + \frac{2}{10} + \frac{3}{5} + \frac{4}{5} + \frac{5}{10}$$

$$E(X) = \frac{8}{5} + \frac{7}{10} = \frac{16+7}{10} = \frac{23}{10}$$

$$\boxed{\text{mean} = 2.3}$$

$$E(X^2) = \sum x^2 f(x) = 0 + \frac{1}{5} + \frac{4}{10} + \frac{9}{5} + \frac{16}{5} + \frac{25}{10}$$

$$= \frac{29}{10} + \frac{26}{5} = \frac{29+52}{10} = \frac{81}{10} = 8.1$$

$$\text{Var}(X) = E(X^2) - (E(X))^2$$

$$= 8.1 - 2.3^2$$

$$= 8.1 - 5.29$$

$$= 2.81$$

$$\text{mean} = 2.3$$

$$\text{variance} = 2.81$$

$$(ii) f(x) = \begin{cases} \frac{4-x}{6} & x=1,2,3 \end{cases}$$

x	1	2	3
$f(x)$	$\frac{3}{6}$	$\frac{2}{6}$	$\frac{1}{6}$

$xf(x)$	$\frac{3}{6}$	$\frac{4}{6}$	$\frac{3}{6}$
$x^2f(x)$	$\frac{3}{6}$	$\frac{8}{6}$	$\frac{9}{6}$

$$E(x) = \sum xf(x) = \frac{3}{6} + \frac{4}{6} + \frac{3}{6} = \frac{10}{6} = \frac{5}{3} = 1.667$$

$$E(x^2) = \sum x^2f(x) = \frac{3}{6} + \frac{8}{6} + \frac{9}{6} = \frac{20}{6}$$

$$\text{mean} = 1.67$$

$$\text{var}(x) = E(x^2) - (E(x))^2$$

$$= \frac{20}{6} - \left(\frac{5}{3}\right)^2 = \frac{20}{6} - \frac{25}{9} = \frac{60-50}{18} = \frac{10}{18}$$

$$= 0.56$$

$$\text{Variance} = 0.56$$

$$\text{mean} = 1.67, \text{variance} = 0.56$$

$$(iii) f(x) = \begin{cases} 2(x-1) & 1 < x < 2 \\ 0 & \text{otherwise} \end{cases}$$

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$$E(x) = \int_1^2 xf(x) dx$$

$$= 2 \int_1^2 x(x-1) dx = 2 \int_1^2 (x^2 - x) dx$$

$$= 2 \left[\frac{x^3}{3} - \frac{x^2}{2} \right]_1^2 = 2 \left[\left(\frac{8}{3} - \frac{4}{2} \right) - \left(\frac{1}{3} - \frac{1}{2} \right) \right]$$

$$= 2 \left[\frac{8}{3} - \frac{4}{2} - \frac{1}{3} + \frac{1}{2} \right] = 2 \left[\frac{7}{3} - \frac{3}{2} \right] = 2 \left[\frac{14-9}{6} \right]$$

$$E(x) = \frac{5}{3}$$

$$E(x^2) = \int_1^2 x^2 f(x) dx$$

$$= \int_1^2 x^2 \cdot 2(x-1) dx = 2 \int_1^2 (x^3 - x^2) dx$$

$$= 2 \left[\frac{x^4}{4} - \frac{x^3}{3} \right]_1^2 = 2 \left[\left(\frac{16}{4} - \frac{8}{3} \right) - \left(\frac{1}{4} - \frac{1}{3} \right) \right]$$

$$= 2 \left[\frac{16}{4} - \frac{8}{3} - \frac{1}{4} + \frac{1}{3} \right] = 2 \left[\frac{15}{4} - \frac{7}{3} \right] = 2 \left[\frac{45-28}{12} \right]$$

$$E(x^2) = \frac{17}{6}$$

$$\text{var}(x) = E(x^2) - (E(x))^2$$

$$= \frac{17}{6} - \left(\frac{5}{3}\right)^2 = \frac{17}{6} - \frac{25}{9} = \frac{51-50}{18} = \frac{1}{18}$$

$$\text{mean} = \frac{5}{3} \quad \text{variance} = \frac{1}{18}$$

$$(iv) f(x) = \begin{cases} \frac{1}{2} e^{-x/2} & \text{for } x > 0 \\ 0 & \text{otherwise.} \end{cases}$$

$$\left[e^{+x/2} \rightarrow e^{-x/2} \right] \quad \text{change}$$

$$E(x) = \int_0^{\infty} x f(x) dx$$

$$= \int_0^{\infty} x \cdot \frac{1}{2} e^{-x/2} dx$$

$$= \frac{1}{2} \frac{1!}{(\frac{1}{2})^{1+1}} = \frac{1}{2} \cdot 2^2$$

$$E(x) = 2$$

$$E(x^2) = \int_0^{\infty} x^2 f(x) dx = \int_0^{\infty} x^2 \cdot \frac{1}{2} e^{-x/2} dx$$

$$= \frac{1}{2} \frac{2!}{(\frac{1}{2})^{2+1}} = \frac{1}{2} \cdot 2 \times 2^3$$

$$E(x^2) = 8$$

$$\begin{aligned} \text{Var}(x) &= E(x^2) - (E(x))^2 \\ &= 8 - 2^2 = 8 - 4 \\ &= 4 \end{aligned}$$

$$\begin{aligned} \text{mean} &= 2 \\ \text{variance} &= 4 \end{aligned}$$

2) Two balls are drawn in succession without replacement from an urn containing four red balls and 3 black balls. Let x be the possible outcomes drawing red balls. Find the probability mass function and mean of x .

Let x be the random variable denotes the number of red balls

$$x = 0, 1, 2$$

$$f(0) = P(x=0) = \frac{{}^4C_0 \cdot {}^3C_2}{{}^7C_2} = \frac{1 \times 3}{7 \times 6 / 2} = \frac{3}{21} = \frac{1}{7}$$

$$f(1) = P(x=1) = \frac{{}^4C_1 \times {}^3C_1}{{}^7C_2} = \frac{4 \times 3}{21} = \frac{4}{7}$$

$$f(2) = P(x=2) = \frac{{}^4C_2 \times {}^3C_0}{{}^7C_2} = \frac{6 \times 1}{21} = \frac{2}{7}$$

probability mass function is

x	0	1	2
$P(x=x)$ $f(x)$	$\frac{1}{7}$	$\frac{4}{7}$	$\frac{2}{7}$

$$\begin{aligned}
 E(X) &= \sum x f(x) \\
 &= 0\left(\frac{1}{7}\right) + 1\left(\frac{4}{7}\right) + 2\left(\frac{2}{7}\right) \\
 &= 0 + \frac{4}{7} + \frac{4}{7} \\
 \text{mean} &= \frac{8}{7}
 \end{aligned}$$

- 3) If μ and σ^2 are the mean and variance of the discrete random variable X and $E(X+3)=10$ and $E(X+3)^2=116$ find μ and σ^2

$$E(X+3)=10$$

$$E(X)+3=10$$

$$E(X)=10-3$$

$$E(X)=7$$

$$E(ax+b)=aE(X)+b$$

$$\boxed{\mu=7}$$

$$E(X+3)^2=116$$

$$E(X^2+6X+9)=116$$

$$E(X^2)+6E(X)+9=116$$

$$E(X^2)+6(7)+9=116$$

$$E(X^2)+51=116$$

$$E(X^2)=116-51$$

$$E(X^2)=65$$

$$\text{Var}(X) = E(X^2) - (E(X))^2 = 65 - 7^2 = 65 - 49 = 16$$

$$\boxed{\begin{matrix} \mu=7 \\ \sigma^2=16 \end{matrix}}$$

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- 4) Four fair coins are tossed once, Find the probability mass function, mean and variance for number of heads occurred.

$$n(s) = 2^4 = 16$$

Let X be the random variable denotes number of heads

$$X = 0, 1, 2, 3, 4$$

$$P(X=0) = \frac{{}^4C_0}{16} = \frac{1}{16}, \quad P(X=1) = \frac{{}^4C_1}{16} = \frac{4}{16}$$

$$P(X=2) = \frac{{}^4C_2}{16} = \frac{6}{16}, \quad P(X=3) = \frac{{}^4C_3}{16} = \frac{4}{16}, \quad P(X=4) = \frac{{}^4C_4}{16} = \frac{1}{16}$$

p.m.f vs

X	0	1	2	3	4
$f(x)$ or $P(X=X)$	$\frac{1}{16}$	$\frac{4}{16}$	$\frac{6}{16}$	$\frac{4}{16}$	$\frac{1}{16}$
$xf(x)$	0	$\frac{4}{16}$	$\frac{12}{16}$	$\frac{12}{16}$	$\frac{4}{16}$
$x^2f(x)$	0	$\frac{4}{16}$	$\frac{24}{16}$	$\frac{36}{16}$	$\frac{16}{16}$

$$E(x) = \sum x f(x) = 0 + \frac{4}{16} + \frac{12}{16} + \frac{12}{16} + \frac{4}{16} = \frac{32}{16} = 2$$

$$\boxed{\text{mean} = 2}$$

$$E(x^2) = \sum x^2 f(x) = 0 + \frac{4}{16} + \frac{24}{16} + \frac{36}{16} + \frac{16}{16} = \frac{80}{16} = 5$$

$$\text{var}(x) = E(x^2) - (E(x))^2 = 5 - 2^2 = 5 - 4 = 1$$

$$\boxed{\begin{array}{l} \text{mean} = 2 \\ \text{variance} = 1 \end{array}}$$

- 5) A commuter train punctually at a station every half hour. Each morning, a student leaves his house to the train station. Let x denote the amount of time, in minutes, that the student waits for the train from the time he reaches the train station. It is known that the pdf of x is

$$f(x) = \begin{cases} \frac{1}{30} & 0 < x < 30 \\ 0 & \text{elsewhere} \end{cases}$$

obtain and interpret the expected value of the random variable x ,

given x be the random variable denotes the waiting time. x is continuous on $(0, 30)$.

$$E(x) = \int_{-\infty}^{\infty} x f(x) dx = \int_0^{30} x \frac{1}{30} dx$$

$$= \frac{1}{30} \left[\frac{x^2}{2} \right]_0^{30} = \frac{1}{30 \times 2} [30^2 - 0]$$

$$= \frac{30 \times 30}{30 \times 2} = 15$$

$$E(x) = 15$$

expected value of waiting time = 15 minutes,

- 6) The time to failure in thousands of hours of an electronic equipment used in a manufactured computer has the density function

$$f(x) = \begin{cases} 3e^{-3x} & x > 0 \\ 0 & \text{elsewhere} \end{cases}$$

Find the expected life of this electronic equipment,

$$E(x) = \int_{-\infty}^{\infty} x f(x) dx$$

$$= \int_0^{\infty} x \cdot 3e^{-3x} dx$$

$$= 3 \frac{1!}{3^{1+1}} = \frac{3!}{9 \cdot 3}$$

$$E(x) = \frac{1}{3}$$

$$\boxed{\int_0^{\infty} x^n e^{-ax} dx = \frac{n!}{a^{n+1}}}$$

$$\begin{array}{l} n=1 \\ a=3 \end{array}$$

$$E(x^2) = \int_{-\infty}^{\infty} x^2 f(x) dx$$

$$= \int_0^{\infty} x^2 3e^{-3x} dx$$

$$n=2 \\ a=3$$

$$= 3 \frac{2!}{3^{2+1}} = 3 \times \frac{2}{3^3} = \frac{2}{9} \quad (\text{not asked in question but answer is given})$$

7) The probability density function of the random variable x is given by $f(x) = \begin{cases} 16xe^{-4x} & \text{for } x > 0 \\ 0 & \text{for } x \leq 0 \end{cases}$

find the mean and variance of x .

$$E(x) = \int_{-\infty}^{\infty} xf(x) dx$$

$$= \int_0^{\infty} x 16xe^{-4x} dx = 16 \int_0^{\infty} x^2 e^{-4x} dx$$

$$n=2 \quad a=4$$

$$= 16 \frac{2!}{4^{2+1}} = 16 \times \frac{2}{4^3} = \frac{1}{2}$$

$$\boxed{\text{mean} = \frac{1}{2}}$$

$$E(x^2) = \int_{-\infty}^{\infty} x^2 f(x) dx = 16 \int_0^{\infty} x^3 e^{-4x} dx$$

$$n=3 \\ a=4$$

$$= 16 \frac{3!}{4^4} = 16 \times \frac{6}{256} = \frac{3}{8}$$

$$\text{Var}(x) = E(x^2) - (E(x))^2 = \frac{3}{8} - \left(\frac{1}{2}\right)^2 = \frac{3}{8} - \frac{1}{4} = \frac{3-2}{8} = \frac{1}{8}$$

$$\boxed{\text{variance} = \frac{1}{8}}$$

8) A lottery with 600 tickets gives one prize of ₹200, four prizes of ₹100, and six prizes of ₹50. If the ticket costs is ₹2, find the expected winning amount of ticket.

$$n(s) = 600$$

Let x be the random variable denote the amount of winning

$$x = 200, 100, 50, 0$$

$$P(x=200) = \frac{1}{600}$$

$$P(x=100) = \frac{4}{600}$$

$$P(x=50) = \frac{6}{600}$$

$$P(x=0) = \frac{589}{600}$$

x	200	100	50	0
$P(x=x)$ or $f(x)$	$\frac{1}{600}$	$\frac{4}{600}$	$\frac{6}{600}$	$\frac{589}{600}$

$$E(x) = \sum x f(x)$$

$$= 200 \left(\frac{1}{600} \right) + 100 \times \frac{4}{600} + 50 \times \frac{6}{600} + 0 \times \frac{589}{600}$$

$$= \frac{200 + 400 + 300}{600} = \frac{900}{600} = \frac{3}{2} = 1.50$$

Expected amount winning = ₹ 1.50

one Ticket cost = ₹ 2.00

$$\text{profit (difference)} = 1.50 - 2.00 = -0.50$$

i.e. LOSS = 0.50 Rupees.

The Bernoulli distribution

Let x be a random variable follows Bernoulli's trial. $x(\text{success}) = 1$, $x(\text{failure}) = 0$ such that

$$f(x) = \begin{cases} p & x=1 \\ q=1-p & x=0 \end{cases} \text{ where } 0 < p < 1$$

x is called Bernoulli R.V.

$f(x)$ is called Bernoulli's distribution.

$$* \text{ mean} = p \quad \text{variance} = pq$$

Binomial distribution

The binomial random variable x , equals no. of success with probability p , $q=1-p$ for a failure, p.m.f is

$$f(x) = {}^n C_x p^x (1-p)^{n-x}, \quad x=0, 1, 2, \dots, n.$$

$$\boxed{f(x) = {}^n C_x p^x q^{n-x}}, \quad x=0, 1, 2, \dots, n$$

$$\text{mean} = np \quad \text{variance} = npq$$

$$p+q=1$$

Exercise 11.5

- i) compute $p(x=k)$ for the binomial distribution $B(n, p)$ where i) $n=6$, $p=\frac{1}{3}$, $k=3$

$$p(x=x) = {}^n C_x p^x (1-p)^{n-x}$$

$$p(x=k) = {}^6 C_k \left(\frac{1}{3} \right)^k \left(1 - \frac{1}{3} \right)^{6-k}$$

$$k=3$$

$$p(x=3) = {}^6 C_3 \left(\frac{1}{3} \right)^3 \left(\frac{2}{3} \right)^{6-3}$$

$$= \frac{6 \times 5 \times 4}{1 \times 2 \times 3} \left(\frac{1}{3} \right)^3 \left(\frac{2}{3} \right)^3 = \frac{20 \times 2^3}{3^6}$$

$$P(X=3) = \frac{20 \times 8}{9 \times 9 \times 9} = \frac{160}{729}$$

$$(ii) n=10, p=\frac{1}{5}, k=4$$

$$P(X=k) = {}^nC_k p^k (1-p)^{n-k}$$

$$P(X=4) = {}^{10}C_4 \left(\frac{1}{5}\right)^4 \left(1-\frac{1}{5}\right)^{10-4}$$

$$= \frac{10 \times 9 \times 8 \times 7}{1 \times 2 \times 3 \times 4} \left(\frac{1}{5}\right)^4 \left(\frac{4}{5}\right)^6$$

$$P(X=4) = 210 \left(\frac{1}{5}\right)^4 \left(\frac{4}{5}\right)^6$$

$$(iii) n=9, p=\frac{1}{2}, k=7$$

$$P(X=k) = {}^nC_k p^k (1-p)^{n-k}$$

$$P(X=7) = {}^9C_7 \left(\frac{1}{2}\right)^7 \left(1-\frac{1}{2}\right)^{9-7}$$

$$= {}^9C_2 \left(\frac{1}{2}\right)^7 \left(\frac{1}{2}\right)^2$$

$$= \frac{9 \times 8}{1 \times 2} \left(\frac{1}{2^9}\right) = \frac{9}{2^7} = \frac{9}{128}$$

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2) The probability that Mr. G hits a target at any trial is $\frac{1}{4}$. Suppose he tries at the target 10 times. Find the probability that he hits the target (i) exactly 4 times (ii) atleast one time.

$n=10$ Let x be the random variable denote number of hits $x=0, 1, 2, \dots, 10$.

x follows the Binomial distribution.

$p =$ probability of success $= \frac{1}{4}$

$$p = \frac{1}{4} \quad q = 1-p = 1 - \frac{1}{4} = \frac{3}{4}$$

$$f(x) \text{ or } P(X=x) = {}^nC_x p^x q^{n-x}$$

$$P(X=x) \text{ or } f(x) = {}^{10}C_x \left(\frac{1}{4}\right)^x \left(\frac{3}{4}\right)^{10-x} \quad x=0, 1, \dots, 10$$

(i) exactly 4 times

$$f(4) = {}^{10}C_4 \left(\frac{1}{4}\right)^4 \left(\frac{3}{4}\right)^{10-4}$$

$$= {}^{10}C_4 \left(\frac{1}{4}\right)^4 \left(\frac{3}{4}\right)^6$$

(ii) atleast one time $= P(X \geq 1)$

$$= 1 - P(X < 1)$$

$$= 1 - P(X=0)$$

$$= 1 - {}^{10}C_0 \left(\frac{1}{4}\right)^0 \left(\frac{3}{4}\right)^{10-0}$$

$$= 1 - \left(\frac{3}{4}\right)^{10}$$

3) Using binomial distribution find the mean and variance of x for the following experiments.

(i) A fair coin is tossed 100 times, and x denote the number of heads.

$$n=100 \quad p = \text{probability of getting head} \\ p = \frac{1}{2} \quad q = 1-p = \frac{1}{2}$$

$$\text{mean} = np = 100 \times \frac{1}{2} = 50 \\ \text{variance} = npq = 100 \times \frac{1}{2} \times \frac{1}{2} = 25$$

(ii) A fair die is tossed 240 times, and x denotes the number of times that 4 appeared.

$$n=240 \quad p = \text{probability of getting 4} \\ p = \frac{1}{6} \quad q = 1-p = 1 - \frac{1}{6} = \frac{5}{6}$$

$$\text{mean} = np = 240 \times \frac{1}{6} = 40 \\ \text{variance} = npq = 240 \times \frac{1}{6} \times \frac{5}{6} = \frac{200}{3} = \frac{100}{3}$$

④ The probability that a certain kind of component will survive a electrical test is $\frac{3}{4}$. Find the probability that exactly 3 of the 5 components tested survive.

$$n=5 \\ p=\frac{3}{4}$$

Let x be the random variable denotes no of survive components.

$$x=0,1,2,3,4,5$$

p = probability of a components survive after test

$$X \sim B(5, \frac{3}{4})$$

$$p = \frac{3}{4} \quad q = 1-p \quad q = 1 - \frac{3}{4} = \frac{1}{4}$$

$$P(X=x) = {}^nC_x p^x q^{n-x}, \quad x=0,1,\dots,5$$

$$P(X=x) = {}^5C_x \left(\frac{3}{4}\right)^x \left(\frac{1}{4}\right)^{5-x}$$

$$P(\text{exactly 3 survive}) = P(X=3)$$

$$= {}^5C_3 \left(\frac{3}{4}\right)^3 \left(\frac{1}{4}\right)^{5-3}$$

$$= \frac{5 \times 4}{1 \times 2} \left(\frac{3}{4}\right)^3 \left(\frac{1}{4}\right)^2$$

$$= 10 \times \frac{3^3}{4^3} \times \frac{1}{4^2} = \frac{270}{1024}$$

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⑤ A retailer purchases a certain kind of electronic device from a manufacturer, The manufacturer indicates that the defective device rate is 5%.

The inspector of the retailer randomly picks 10 items from a shipment, what is the probability that there will be

(i) at least one defective item (ii) exactly 2 defective items.

$n=10$ x be the random variable denotes

no. of defective items

$$x \sim B(10, 0.05)$$

p = probability of a defective item

$$p = 5\%$$

$$p = 0.05$$

$$q = 1 - p = 0.95$$

$$P(X=x) = {}^n C_x p^x q^{n-x}$$

$$P(X=x) = {}^{10} C_x (0.05)^x (0.95)^{10-x}$$

(i) at least one defective = $P(X \geq 1)$

$$= 1 - P(X < 1)$$

$$= 1 - P(X = 0)$$

$$= 1 - {}^{10} C_0 (0.05)^0 (0.95)^{10-0}$$

$$= 1 - (0.95)^{10}$$

(ii) exactly 2 defective

$$= P(X=2)$$

$$= {}^{10} C_2 (0.05)^2 (0.95)^{10-2}$$

$$= {}^{10} C_2 (0.05)^2 (0.95)^8$$

6) If the probability that a fluorescent light has a useful life of at least 600 hours is 0.9, find the probability that among 12 such lights

(i) exactly 10 will have a useful life of at least 600 hours,

(ii) at least 11 will not

(iii) at least 11 will not

$$p = 0.9$$

$$q = 1 - p = 1 - 0.9 = 0.1$$

x be the random variable denotes useful life of at least 600 hours of a light,

$$x \sim B(12, 0.9)$$

$$P(X=x) = {}^n C_x p^x q^{n-x}$$

$$P(X=x) = {}^{12} C_x (0.9)^x (0.1)^{12-x}$$

$$x = 0, 1, \dots, 12$$

(i) exactly 10

$$P(X=10) = {}^{12} C_{10} (0.9)^{10} (0.1)^{12-10}$$

$$= {}^{12} C_{10} (0.9)^{10} (0.1)^2$$

(ii) at least 11

$$P(X \geq 11) = P(X=11) + P(X=12)$$

$$= {}^{12} C_{11} (0.9)^{11} (0.1)^{12-11} + {}^{12} C_{12} (0.9)^{12} (0.1)^{12-12}$$

$$= 12 (0.9)^{11} (0.1) + 1 (0.9)^{12} (1)$$

$$= (0.9)^{11} [12 \times 0.1 + 0.9]$$

$$= (0.9)^{11} [1.2 + 0.9]$$

$$P(X \geq 11) = 2.1 (0.9)^{11}$$

(ii) at least 2 will not have a useful life of at least 600 hours.

$$P(\text{at least 2 will not}) = 1 - P(X \geq 11)$$

$$= 1 - (2.1) (0.9)^{11}$$

[each probability having 2 or more defective]

⑦ The mean and standard deviation of a binomial variate. x are respectively 6 and 2.

Find (i) The probability mass function

(ii) $P(X=3)$ (iii) $P(X \geq 2)$

$$\text{mean} = 6 \quad S.D = 2 \quad S.D^2 = 4$$

$$np = 6 \text{ --- (1)} \quad \text{variance} = 4$$

$$npq = 4 \text{ --- (2)}$$

$$\frac{(2)}{(1)} \Rightarrow \frac{npq}{np} = \frac{4}{6}$$

$$q = \frac{2}{3}$$

$$p = 1 - q = 1 - \frac{2}{3}$$

$$p = \frac{1}{3}$$

$$(1) \Rightarrow n \cdot \frac{1}{3} = 6$$

$$n = 18$$

(i) probability mass function

$$P(X=x) = {}^n C_x p^x q^{n-x}$$

$$P(X=x) = {}^{18} C_x \left(\frac{1}{3}\right)^x \left(\frac{2}{3}\right)^{18-x}$$

$$x = 0, 1, 2, \dots, 18$$

$$(ii) P(X=3) = {}^{18} C_3 \left(\frac{1}{3}\right)^3 \left(\frac{2}{3}\right)^{18-3}$$

$$= {}^{18} C_3 \left(\frac{1}{3}\right)^3 \left(\frac{2}{3}\right)^{15}$$

$$(iii) P(X \geq 2) = 1 - P(X < 2)$$

$$= 1 - [P(X=0) + P(X=1)]$$

$$= 1 - \left[{}^{18} C_0 \left(\frac{1}{3}\right)^0 \left(\frac{2}{3}\right)^{18-0} + {}^{18} C_1 \left(\frac{1}{3}\right)^1 \left(\frac{2}{3}\right)^{18-1} \right]$$

$$= 1 - \left[1 \cdot (1) \left(\frac{2}{3}\right)^{18} + {}^{18} C_1 \frac{1}{3} \left(\frac{2}{3}\right)^{17} \right]$$

$$= 1 - \left(\frac{2}{3}\right)^{17} \left[\frac{2}{3} + 6 \right]$$

$$= 1 - \frac{20}{3} \left(\frac{2}{3}\right)^{17}$$

⑧ If $x \sim B(n, p)$ such that $4P(X=4) = P(X=2)$ and $n=6$, Find the distribution, mean and S.D.

$$P(X=x) = {}^n C_x p^x q^{n-x}$$

$$n=6 \quad 4P(X=4) = P(X=2)$$

$$4({}^6 C_4 (p)^4 (q)^{6-4}) = {}^6 C_2 p^2 (q)^{6-2}$$

$$4 \times {}^6 C_4 p^4 q^2 = {}^6 C_2 p^2 q^4$$

$$4 \frac{p^4}{p^2} = \frac{q^4}{q^2}$$

$$4(p^2) = q^2$$

$$4p^2 = (1-p)^2$$

$$4p^2 = 1 + p^2 - 2p$$

$$4p^2 - 1 - p^2 + 2p = 0$$

$$3p^2 + 2p - 1 = 0$$

$$(3p-1)(p+1) = 0$$

$$p = -1 \quad 3p = 1$$

$$p \neq -1$$

$$p = \frac{1}{3}$$

$$q = 1 - p = 1 - \frac{1}{3}$$

$$q = \frac{2}{3}$$

$$\begin{array}{r} 3 \\ \times 3 \\ \hline 9 \end{array}$$

$$(p + \frac{1}{3})(p - \frac{1}{3}) = 0$$

The distribution is

$$P(X=x) = {}^n C_x p^x q^{n-x}$$

$$= {}^6 C_x \left(\frac{1}{3}\right)^x \left(\frac{2}{3}\right)^{6-x}$$

$$x = 0, 1, \dots, 6$$

$$\text{mean} = np = 6 \times \frac{1}{3} = 2$$

$$\text{variance} = npq = 6 \times \frac{1}{3} \times \frac{2}{3} = \frac{4}{3}$$

$$\text{standard deviation} = \sqrt{npq} = \sqrt{\frac{4}{3}} = \frac{2}{\sqrt{3}}$$

9) In a Binomial distribution consisting of 5 independent trials, the probability of 1 and 2 successes are 0.4096 and 0.2048 respectively. Find the mean and variance.

$$n=5$$

$$\text{given } P(X=1) = 0.4096$$

$$P(X=2) = 0.2048$$

$$P(X=x) = {}^n C_x p^x q^{n-x}$$

$$P(X=1) = {}^5 C_1 p^1 q^{5-1} = 0.4096 \Rightarrow 5pq^4 = 0.4096 \quad \text{--- (1)}$$

$$P(X=2) = {}^5 C_2 p^2 q^{5-2} = 0.2048 \Rightarrow 10p^2 q^3 = 0.2048 \quad \text{--- (2)}$$

$$\frac{(1)}{(2)} \Rightarrow \frac{5pq^4}{2 \cdot 10p^2 q^3} = \frac{0.4096}{0.2048} \cdot 2$$

$$\frac{q}{2p} = 2$$

$$1-p = 4p$$

$$1 = 5p$$

$$p = \frac{1}{5}$$

$$q = 1 - p = 1 - \frac{1}{5} = \frac{4}{5}$$

$$\text{mean} = np = 5 \times \frac{1}{5} = 1$$

$$\text{variance} = npq = 1 \times \frac{4}{5} = \frac{4}{5}$$

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(29)

(Book answer wrong)

(checking)

$$n=5 \quad p=\frac{1}{5} \quad q=\frac{4}{5}$$

$$P(X=1) = {}^5C_1 \left(\frac{1}{5}\right)^1 \left(\frac{4}{5}\right)^{5-1}$$

$$= {}^5C_1 \frac{1}{5} \left(\frac{4}{5}\right)^4 = \frac{4^4}{5^4} = \frac{256}{25 \times 25} \times \frac{16}{4 \times 4}$$

$$P(X=1) = \frac{4096}{10000} = 0.4096$$

correct answer

- mean = 1

$$\text{variance} = \frac{4}{5}$$

Need suggestions

$$q = 1 - p = 1 - \frac{1}{5} = \frac{4}{5}$$

$$0 = q + 1 - q = 1 - \frac{1}{5} = \frac{4}{5}$$

$$0 = 1 - q + q = 1 - \frac{1}{5} = \frac{4}{5}$$

$$0 = (1 - q) + q = 1 - \frac{1}{5} = \frac{4}{5}$$

$$\frac{1}{5} \times 1$$

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Binary Operation (or) closure property

$\forall a, b \in S$, $a * b$ is unique and $a * b \in S$

where S is any nonempty set.



Properties

1) commutative property

$$a * b = b * a \quad \forall a, b \in S$$

2) Associative property

$$a * (b * c) = (a * b) * c \quad \forall a, b, c \in S$$

3) Existence of identity property

$e \in S$ is an identity element

$$a * e = e * a = a \quad \forall a \in S$$

4) Existence of Inverse property

$b \in S$ is said to be Inverse element of a

$$a * b = b * a = e \quad \forall a \in S \quad b = a^{-1}$$

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Exercise 12.1

i) Determine whether $*$ is a binary operation on the sets given below

1) $a * b = a + |b|$ on R

Let $a, b \in R$

$$a + |b| \in R \Rightarrow a * b \in R \quad \forall a, b \in R$$

$*$ is the binary operation on R .

(ii) $a * b = \min(a, b)$ on $A = \{1, 2, 3, 4, 5\}$

Let $a, b \in A$

$$\min(a, b) \in A$$

$$a * b \in A$$

$$\forall a, b \in A$$

$\therefore *$ is the binary operation on A .

For example

$$\min(1, 2) = 1 \in A$$

$$\min(1, 5) = 1 \in A$$

$$\min(2, 3) = 2 \in A$$

(iii) $a \times b = a\sqrt{b}$ is binary on $\mathbb{R}$

Let $a, b \in \mathbb{R}$

root of negative numbers not in $\mathbb{R}$

$$\Rightarrow a \times b \notin \mathbb{R} \quad \because \begin{cases} 2, -2 \in \mathbb{R} \\ 2\sqrt{-2} \notin \mathbb{R} \end{cases}$$

$\therefore \times$ is not binary on $\mathbb{R}$.

2) On $\mathbb{Z}$, define $\otimes$ by $(m \otimes n) = m^n + n^m \quad \forall m, n \in \mathbb{Z}$

Is $\otimes$ binary on $\mathbb{Z}$

Let $m, n \in \mathbb{Z}$

take $m=2, n=-2$

$$m^n + n^m = 2^{-2} + (-2)^2 = \frac{1}{4} + 4 = \frac{17}{4} \notin \mathbb{Z}$$

$$m^n + n^m \notin \mathbb{Z}$$

$\therefore m \otimes n \notin \mathbb{Z} \quad \therefore \otimes$ is not binary on $\mathbb{Z}$.

3) Let $*$ be defined on $\mathbb{R}$ by $(a * b) = a + b + ab - 7$.
Is $*$ binary on $\mathbb{R}$? If so find $3 * (-\frac{7}{15})$

Let $a, b \in \mathbb{R}$

clearly $a, b, a+b \in \mathbb{R}$

$$\therefore a + b + ab - 7 \in \mathbb{R}$$

$$\Rightarrow a * b \in \mathbb{R}$$

$\therefore *$ is binary operator on $\mathbb{R}$

$$3 * (-\frac{7}{15}) = 3 - \frac{7}{15} + 3(-\frac{7}{15}) - 7$$

$$= \frac{45 - 7 - 21 - 105}{15} = \frac{45 - 133}{15} = -\frac{88}{15}$$

4) Let $A = \{a + \sqrt{5}b : a, b \in \mathbb{Z}\}$, check whether the usual multiplication is a binary operation on A

$$\text{Let } x = a + \sqrt{5}b, y = c + d\sqrt{5}$$

$$x, y \in A, a, b, c, d \in \mathbb{Z}$$

$$xy = (a + \sqrt{5}b)(c + \sqrt{5}d)$$

$$= ac + 5bd + \sqrt{5}ad + \sqrt{5}bc$$

$$= (ac + 5bd) + \sqrt{5}(ad + bc) \in A$$

$$\therefore xy \in A$$

multiplication is binary on R.

5) (i). Define an operation $*$ on Q as follows

$$a * b = \frac{a+b}{2} ; a, b \in Q. \text{ Examine the closure,}$$

commutative and associative properties satisfied by

$*$ on Q .

① closure property

$$\text{given } a, b \in Q \Rightarrow a+b \in Q \Rightarrow \frac{a+b}{2} \in Q \Rightarrow a * b \in Q. \forall a, b \in Q.$$

② commutative property $*$ is closure on Q .

$$a * b = \frac{a+b}{2}$$

$$= \frac{b+a}{2}$$

$$a * b = b * a$$

$$\forall a, b \in Q$$

commutative property is true. $\therefore *$ is commutative on Q .

③ associative property

$$a * (b * c) = a * \left(\frac{b+c}{2} \right)$$

$$= \frac{a + \left(\frac{b+c}{2} \right)}{2} = \frac{2a+b+c}{2} \times \frac{1}{2}$$

$$a * (b * c) = \frac{2a+b+c}{4} \quad \text{--- (1)}$$

$$(a * b) * c = \left(\frac{a+b}{2} \right) * c$$

$$= \frac{\left(\frac{a+b}{2} \right) + c}{2} = \frac{a+b+2c}{2} \times \frac{1}{2}$$

$$= \frac{a+b+2c}{4} \quad \text{--- (2)}$$

$$\text{From (1) \& (2) } (a * b) * c \neq a * (b * c)$$

$*$ is not associative on Q .

(ii) Define an operation $*$ on Q as follows $a * b = \frac{a+b}{2}$
 $a, b \in Q$. Examine the existence of identity and the existence of inverse for the operation $*$ on Q .

i) Existence of Identity

Let $a \in Q$, e be the identity element on Q .

$$\text{By definition of } * \quad a * e = \frac{a+e}{2}$$

$$\text{By definition of } e \quad a * e = a.$$

$$\frac{a+e}{2} = a$$

$$a+e=2a$$

$$e=2a-a$$

$$e=a \quad \forall a \in G$$

This means every element is a identity element

This is not possible.

* has NO Identity element,

(ii) Existence of Inverse

* has no Identity element

$\therefore$ cannot defined as $axa^{-1} = a^{-1}xa = e$.

$\therefore$ * has no Inverse.

6) Fill in the following table so that the binary operation * on $A = \{a, b, c\}$ is commutative.

*	a	b	c
a	b		
b	c	b	a
c	a		c

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i) From table. $b \times a = c$

* is commutative

$$\Rightarrow a \times b = c$$

ii) $c \times a = a \Rightarrow a \times c = a$

iii) $b \times c = a \Rightarrow c \times b = a$

$\therefore$ given table is

*	a	b	c
a	b	c	a
b	c	b	a
c	a	a	c

7) consider the binary operation * defined on the set $A = \{a, b, c, d\}$ by the following table.

*	a	b	c	d
a	a	c	b	d
b	d	a	b	c
c	c	d	a	a
d	d	b	a	c

Is it commutative and associative?

1) commutative property

$$a * b = c \text{ but } b * a = d \Rightarrow a * b \neq b * a$$

$$a * c = b \text{ but } c * a = e \Rightarrow a * c \neq c * a$$

$\therefore *$ is not commutative.

2) associative Property

$$(a * b) * c = c * c = a$$

$$a * (b * c) = a * b = c$$

$$\therefore (a * b) * c \neq a * (b * c)$$

$*$ is not associative.

8) Let $A = \begin{pmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ 1 & 0 & 0 & 1 \end{pmatrix}$, $B = \begin{pmatrix} 0 & 1 & 0 & 1 \\ 1 & 0 & 1 & 0 \\ 1 & 0 & 0 & 1 \end{pmatrix}$, $C = \begin{pmatrix} 1 & 1 & 0 & 1 \\ 0 & 1 & 1 & 0 \\ 1 & 1 & 1 & 1 \end{pmatrix}$

be any three boolean matrices of the same type

Find i) $A \vee B$ ii) $A \wedge B$ iii) $(A \vee B) \wedge C$ iv) $(A \wedge B) \vee C$.

i) $A \vee B = \begin{pmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ 1 & 0 & 0 & 1 \end{pmatrix} \vee \begin{pmatrix} 0 & 1 & 0 & 1 \\ 1 & 0 & 1 & 0 \\ 1 & 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \\ 1 & 0 & 0 & 1 \end{pmatrix}$

ii) $A \wedge B = \begin{pmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ 1 & 0 & 0 & 1 \end{pmatrix} \wedge \begin{pmatrix} 0 & 1 & 0 & 1 \\ 1 & 0 & 1 & 0 \\ 1 & 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 1 \end{pmatrix}$

iii) $(A \vee B) \wedge C = \begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \\ 1 & 0 & 0 & 1 \end{pmatrix} \wedge \begin{pmatrix} 1 & 1 & 0 & 1 \\ 0 & 1 & 1 & 0 \\ 1 & 1 & 1 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 1 & 0 & 1 \\ 0 & 1 & 1 & 0 \\ 1 & 0 & 0 & 1 \end{pmatrix}$

iv) $(A \wedge B) \vee C = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 1 \end{pmatrix} \vee \begin{pmatrix} 1 & 1 & 0 & 1 \\ 0 & 1 & 1 & 0 \\ 1 & 1 & 1 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 1 & 0 & 1 \\ 0 & 1 & 1 & 0 \\ 1 & 1 & 1 & 1 \end{pmatrix}$

Q (i) Let $M = \left\{ \begin{pmatrix} x & x \\ x & x \end{pmatrix}; x \in \mathbb{R} - \{0\} \right\}$ and let $*$ be the matrix multiplication. Determine whether M is closed under $*$. If so examine the commutative and associative properties satisfied by $*$ on M .

$$\text{Let } A = \begin{pmatrix} x & x \\ x & x \end{pmatrix}, B = \begin{pmatrix} y & y \\ y & y \end{pmatrix} \text{ where } x, y \in \mathbb{R} - \{0\}$$

$$A, B \in M$$

1) closure property

$$A, B \in M$$

$$\Rightarrow A * B = \begin{pmatrix} x & x \\ x & x \end{pmatrix} \begin{pmatrix} y & y \\ y & y \end{pmatrix}$$

$$= \begin{pmatrix} 2xy & 2xy \\ 2xy & 2xy \end{pmatrix} \in M \quad \because \begin{matrix} x, y \in \mathbb{R} - \{0\} \\ xy \in \mathbb{R} - \{0\} \end{matrix}$$

$$A * B \in M$$

$$A, B \in M \Rightarrow A * B \in M$$

$\therefore *$ is closed on M

2) commutative property

$$A, B \in M$$

$$A * B = \begin{pmatrix} x & x \\ x & x \end{pmatrix} \begin{pmatrix} y & y \\ y & y \end{pmatrix}$$

$$= \begin{pmatrix} 2xy & 2xy \\ 2xy & 2xy \end{pmatrix} = \begin{pmatrix} 2yx & 2yx \\ 2yx & 2yx \end{pmatrix}$$

$$= \begin{pmatrix} y & y \\ y & y \end{pmatrix} \begin{pmatrix} x & x \\ x & x \end{pmatrix}$$

$$A * B = B * A$$

$*$ is commutative on M .

3) associative property

Matrix multiplication is always associative.

$$\text{i.e. } A * (B * C) = (A * B) * C \quad \forall A, B, C \in M$$

$*$ is associative on M .

(ii) Let $M = \left\{ \begin{pmatrix} x & x \\ x & x \end{pmatrix}; x \in \mathbb{R} - \{0\} \right\}$ and let $*$ be the matrix multiplication. Determine whether M is closed under $*$. If so examine the existence of identity, inverse property for $*$ on M .

① Closure property

$A, B \in M$ where $A = \begin{pmatrix} x & x \\ x & x \end{pmatrix}$, $B = \begin{pmatrix} y & y \\ y & y \end{pmatrix}$ $x, y \in \mathbb{R} - \{0\}$

$$A * B = \begin{pmatrix} 2xy & 2xy \\ 2xy & 2xy \end{pmatrix} \in M$$

$$A * B \in M$$

$$A, B \in M \Rightarrow A * B \in M$$

$\therefore *$ is closed on M .

② Existence of Identity property

Let $A \in M$, $E = \begin{pmatrix} e & e \\ e & e \end{pmatrix}$ be the Identity element

$$AE = A$$

$$\begin{pmatrix} x & x \\ x & x \end{pmatrix} \begin{pmatrix} e & e \\ e & e \end{pmatrix} = \begin{pmatrix} x & x \\ x & x \end{pmatrix}$$

$$\begin{pmatrix} 2xe & 2xe \\ 2xe & 2xe \end{pmatrix} = \begin{pmatrix} x & x \\ x & x \end{pmatrix}$$

$$2xe = x \quad e = \frac{1}{2} \in \mathbb{R} - \{0\}$$

$$\text{Identity element } E = \begin{pmatrix} \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & \frac{1}{2} \end{pmatrix} \in M$$

satisfies $AE = A$

similarly $EA = A \quad \forall A \in M$

$*$ has identity element on M .

③ Existence of Inverse property.

Let $A \in M$, $A^{-1} = \begin{pmatrix} x^{-1} & x^{-1} \\ x^{-1} & x^{-1} \end{pmatrix}$ be the inverse of A

$$AA^{-1} = E$$

$$\begin{pmatrix} x & x \\ x & x \end{pmatrix} \begin{pmatrix} x^{-1} & x^{-1} \\ x^{-1} & x^{-1} \end{pmatrix} = \begin{pmatrix} \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & \frac{1}{2} \end{pmatrix}$$

$$\begin{pmatrix} 2xx^{-1} & 2xx^{-1} \\ 2xx^{-1} & 2xx^{-1} \end{pmatrix} = \begin{pmatrix} \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & \frac{1}{2} \end{pmatrix}$$

$$2xx^{-1} = \frac{1}{2}$$

$$x^{-1} = \frac{1}{4x} \in \mathbb{R} - \{0\}$$

$$A^{-1} = \begin{pmatrix} \frac{1}{4x} & \frac{1}{4x} \\ \frac{1}{4x} & \frac{1}{4x} \end{pmatrix} \in M$$

is the inverse of A in M

similarly we can find $A^{-1}A = E \quad \forall A \in M$

$*$ is Inverse on M ,

—x—

- 10) i) Let A be $\mathbb{Q} - \{1\}$. Define $*$ on A by $x*y = x+y-xy$. Is $*$ binary on A ? If so examine the commutative and associative properties satisfied by $*$ on A .

$$A = \mathbb{Q} - \{1\} \quad x, y \in A \quad x \neq 1, y \neq 1$$

$$* \text{ is defined by } x*y = x+y-xy$$

① closure property

$$\text{Let } x, y \in A \quad x \neq 1, y \neq 1$$

$$x-1 \neq 0, y-1 \neq 0$$

$$(x-1)(y-1) \neq 0$$

$$xy - x - y + 1 \neq 0$$

$$1 \neq x+y-xy$$

$$x, y \in A \Rightarrow x*y \in A, \quad x*y \neq 1 \Rightarrow x*y \in A$$

$*$ is closed on A .

② commutative property:-

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$$\text{Let } x, y \in A \quad x*y = x+y-xy$$

$$= y+x-yx$$

$$x*y = y*x$$

$*$ is commutative on A .

③ associative property:-

$$(x*y)*z = (x+y-xy)*z$$

$$= (x+y-xy)+z - (x+y-xy)z$$

$$(x*y)*z = x+y+z-xy-xz-yz+xyz \quad \text{--- ①}$$

$$x*(y*z) = x*(y+z-yz)$$

$$= x+(y+z-yz) - x(y+z-yz)$$

$$x*(y*z) = x+y+z-xy-xz-yz+xyz \quad \text{--- ②}$$

From ① & ②

$$(x*y)*z = x*(y*z)$$

$$\forall x, y, z \in A$$

$*$ is associative on A

- (ii) examine the existence of identity, inverse properties for $*$ on A .

① closure property (same as on above)

② existence of identity property.

Let $x \in A$, e be the identity element

By definition of $*$

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By definition of e

$$x * e = x + e - x e$$

$$x * e = x$$

$$x * e - x e = x$$

$$e(1-x) = 0$$

$$e = \frac{0}{1-x}$$

$$e = 0 \in A$$

Identity element

$$e = 0 \in A$$

Existence of $*$ has Identity element on A

③ Inverse property:-

Let $x \in A$, x^{-1} be the inverse of x

By definition of $*$ $x * x^{-1} = x + x^{-1} - x x^{-1}$

By definition of x^{-1} $x * x^{-1} = e$

$$x + x^{-1} - x x^{-1} = 0$$

$$x^{-1}(1-x) = -x$$

$$x^{-1} = \frac{-x}{1-x} \in A$$

Inverse of x is $x^{-1} = \frac{-x}{1-x} \in A \quad \forall x \in A$

$*$ has Inverse element for $\forall x \in A$.

Mathematical Logic

Truth tables

Truth table for NOT

①	p	$\neg p$
	T	F
	F	T

Truth table for AND

②	p	q	$p \wedge q$
	T	T	T
	T	F	F
	F	T	F
	F	F	F

③ Truth table for OR

$p \vee q$

p	q	$p \vee q$
T	T	T
T	F	T
F	T	T
F	F	F

④ Truth table for conditional statement,

$p \rightarrow q$

p	q	$p \rightarrow q$
T	T	T
T	F	F
F	T	T
F	F	T

⑤ Truth table for Bi-conditional statement.

P	q	$P \leftrightarrow q$
T	T	T
T	F	F
F	T	F
F	F	T

⑥ Truth table for Exclusive OR (EOR) ∇

P	q	$P \nabla q$
T	T	F
T	F	T
F	T	T
F	F	F

⑦ Tautology

A statement is said to be tautology if its truth values is always T irrespective of the truth values of its compound statements, (denoted by $\top$)

8) contradiction

A statement is said to be contradiction if its truth value is always F irrespective of the truth values of its compound statements, (denoted by $\perp$)

9) contingency

A statement which is neither a tautology nor a contradiction is called contingency.

10) Duality

The dual of a statement formula is obtained

by replacing $\vee$ by $\wedge$ $\top$ by $\perp$ $\top$ by $\perp$
 $\wedge$ by $\vee$ $\perp$ by $\top$ $\perp$ by $\top$

Exercise 12.2

1) Let p: Jupiter is a planet and q: India is an Island be any two simple statements. Give verbal sentence describing each of the following statements.

p: Jupiter is a planet
 q: India is an Island.

(i) $\neg p$: Jupiter is not a planet

(ii) $p \wedge \neg q$: Jupiter is a planet and India is not an Island.

(iii) $\neg p \vee q$: Jupiter is not a planet or India is an Island.

(iv) $p \rightarrow \neg q$: If Jupiter is a planet then India is not an Island.

(v) $P \leftrightarrow Q$: Jupiter is a planet if and only if India is an Island.

2) Write each of the following sentences in symbolic form using statement variables P and Q .

P : 19 is a prime number

Q : All the angles of a triangle are equal.

(i) 19 is not a prime number and all the angles of a triangle are equal.

$$\neg P \wedge Q$$

(ii) 19 is a prime number or all the angles of a triangle are not equal.

$$P \vee \neg Q$$

(iii) 19 is a prime number and all the angles of a triangle are equal.

$$P \wedge Q$$

(iv) 19 is not a prime number

$$\neg P$$

3) Determine the truth value of each of the following statements.

(i) If $6+2=5$, then the milk is white.

P : $6+2=5$ F

Q : The milk is white T

Symbolic form is

$$P \rightarrow Q$$

$$F \rightarrow T$$

Truth value is T

(ii) China is in Europe or $\sqrt{3}$ is an integer.

P : China is in Europe F

Q : $\sqrt{3}$ is an integer F

Symbolic form is

$$P \vee Q$$

$$F \vee F$$

Truth value is F

(iii) It is not true that $5+5=9$ or Earth is a planet.

P : It is true that $5+5=9$

Q : Earth is a planet.

Symbolic form is $\neg P \vee Q$

Truth value T V T

Truth value is Π

(iv) 11 is a prime number and all the sides of a rectangle are equal.

P : 11 is a prime number

Q : all the sides of a rectangle are equal.

Symbolic form is $P \wedge Q$

Truth value T $\wedge$ F

Truth value is F

4) which one of the following sentences is a proposition?

(i) $4+7=12$ proposition

(ii) what are you doing? (not a proposition)

(iii) $3^n \leq 81, n \in \mathbb{N}$ (proposition)

(iv) Peacock is our national bird. (proposition)

(v) How tall this mountain is! (not a proposition)

$\therefore$ (i) (iii) (iv) are propositions,

5) write the converse, inverse, and contrapositive of each of the following implication

(i) If x and y are numbers such that $x=y$, then $x^2=y^2$

P : x and y are numbers such that $x=y$

Q : $x^2=y^2$

given statement $\Rightarrow$ symbolic form is $P \rightarrow Q$

① converse $Q \rightarrow P$

If x and y are numbers such that $x^2=y^2$
then $x=y$

(Pr) ② Inverse: $\neg p \rightarrow \neg q$

If x and y are numbers such that $x \neq y$
then $x^2 \neq y^2$

③ contrapositive: $\neg q \rightarrow \neg p$

If x and y are numbers such that $x^2 \neq y^2$
then $x \neq y$

ii) If a quadrilateral is a square then it is a rectangle.

p : A quadrilateral is a square.

q : A quadrilateral is a rectangle

given statement is $p \rightarrow q$

① converse: $q \rightarrow p$

If a quadrilateral is a rectangle then
it is a square.

② Inverse: $\neg p \rightarrow \neg q$

If a quadrilateral is not a square, then
It is not a rectangle.

③ Contrapositive:

If a quadrilateral is not a rectangle.
then it is not a square.

6) Construct the truth table for the following

i) $\neg p \wedge \neg q$

p	q	$\neg p$	$\neg q$	$\neg p \wedge \neg q$
T	T	F	F	F
T	F	F	T	F
F	T	T	F	F
F	F	T	T	T

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ii) $\neg(p \wedge \neg q)$

p	q	$\neg q$	$p \wedge \neg q$	$\neg(p \wedge \neg q)$
T	T	F	F	T
T	F	T	T	F
F	T	F	F	T
F	F	T	F	T

(iii) $(p \vee q) \vee \neg q$

P	q	$p \vee q$	$\neg q$	$(p \vee q) \vee (\neg q)$
T	T	T	F	T
T	F	T	T	T
F	T	T	F	T
F	F	F	T	T

(iv) $(\neg p \rightarrow r) \wedge (p \leftrightarrow q)$

P	q	r	$\neg p$	$\neg p \rightarrow r$	$p \leftrightarrow q$	$(\neg p \rightarrow r) \wedge (p \leftrightarrow q)$
T	T	T	F	T	T	T
T	T	F	F	T	T	T
T	F	T	F	T	F	F
T	F	F	F	T	F	F
F	T	T	T	T	F	F
F	T	F	T	F	F	F
F	F	T	T	T	T	T
F	F	F	T	F	T	F

7) Verify whether the following compound propositions are tautologies or contradictions or contingency,

(i) $(p \wedge q) \wedge \neg(p \vee q)$

P	q	$p \wedge q$	$p \vee q$	$\neg(p \vee q)$	$(p \wedge q) \wedge \neg(p \vee q)$
T	T	T	T	F	F
T	F	F	T	F	F
F	T	F	T	F	F
F	F	F	F	T	F

Last column contains only F

This is a contradiction

(ii) $((p \vee q) \wedge \neg p) \rightarrow q$

P	q	$p \vee q$	$\neg p$	$(p \vee q) \wedge \neg p$	$((p \vee q) \wedge \neg p) \rightarrow q$
T	T	T	F	F	T
T	F	T	F	F	T
F	T	T	T	T	T
F	F	F	T	F	T

Last column contains only T

This is a Tautology.

(iii) $(P \rightarrow Q) \leftrightarrow (\neg P \rightarrow Q)$

P	Q	$P \rightarrow Q$	$\neg P$	$\neg P \rightarrow Q$	$(P \rightarrow Q) \leftrightarrow (\neg P \rightarrow Q)$
T	T	T	F	T	T
T	F	F	F	T	F
F	T	T	T	T	T
F	F	T	T	F	F

Last contains T and F

$\therefore$ This is a contingency.

(iv) $((P \rightarrow Q) \wedge (Q \rightarrow R)) \rightarrow (P \rightarrow R)$

P	Q	R	$P \rightarrow Q$	$Q \rightarrow R$	$P \rightarrow R$	$(P \rightarrow Q) \wedge (Q \rightarrow R)$	$((P \rightarrow Q) \wedge (Q \rightarrow R)) \rightarrow (P \rightarrow R)$
T	T	T	T	T	T	T	T
T	T	F	T	F	F	F	T
T	F	T	F	T	T	F	T
T	F	F	F	T	F	F	T
F	T	T	T	T	T	T	T
F	T	F	T	F	T	F	T
F	F	T	T	T	T	T	T
F	F	F	T	T	T	T	T

Last contains only T

$\therefore$ This is a tautology.

8) show that (i) $\neg(P \wedge Q) \equiv \neg P \vee \neg Q$

TABLE ①

P	Q	$P \wedge Q$	$\neg(P \wedge Q)$
T	T	T	F
T	F	F	T
F	T	F	T
F	F	F	T

TABLE ②

P	Q	$\neg P$	$\neg Q$	$\neg P \vee \neg Q$
T	T	F	F	F
T	F	F	T	T
F	T	T	F	T
F	F	T	T	T

FROM Table ① & ② Last columns are Identical.

$$\therefore \neg(P \wedge Q) \equiv \neg P \vee \neg Q.$$

8) (ii) $\neg(P \rightarrow Q) \equiv P \wedge \neg Q$

Table ①

P	Q	$P \rightarrow Q$	$\neg(P \rightarrow Q)$
T	T	T	F
T	F	F	T
F	T	T	F
F	F	T	F

Table ②

P	Q	$\neg Q$	$P \wedge \neg Q$
T	T	F	F
T	F	T	T
F	T	F	F
F	F	T	F

From table ① & ② Last columns are identical $\therefore \neg(P \rightarrow q) \equiv P \wedge (\neg q)$

9) prove that $q \rightarrow p \equiv \neg p \rightarrow \neg q$

TABLE ①			TABLE ②				
P	q	$q \rightarrow p$	P	q	$\neg p$	$\neg q$	$\neg p \rightarrow \neg q$
T	T	T	T	T	F	F	T
T	F	T	T	F	F	T	T
F	T	F	F	T	T	F	F
F	F	T	F	F	T	T	T

From table ① & ② The last columns are identical.

$$\therefore q \rightarrow p \equiv \neg p \rightarrow \neg q$$

10) show that $p \rightarrow q$ and $q \rightarrow p$ are not equivalent.

TABLE ①			TABLE ②		
P	q	$p \rightarrow q$	P	q	$q \rightarrow p$
T	T	T	T	T	T
T	F	F	T	F	T
F	T	T	F	T	F
F	F	T	F	F	T

From table ① & ② Last column s are not identical $\therefore p \rightarrow q \neq q \rightarrow p$

11) Show that $\neg(p \leftrightarrow q) \equiv p \leftrightarrow \neg q$

TABLE ①				TABLE ②			
P	q	$p \leftrightarrow q$	$\neg(p \leftrightarrow q)$	P	q	$\neg q$	$p \leftrightarrow \neg q$
T	T	T	F	T	T	F	F
T	F	F	T	T	F	T	T
F	T	F	T	F	T	F	T
F	F	T	F	F	F	T	F

From table ① & ② Last columns are identical $\therefore \neg(p \leftrightarrow q) \equiv p \leftrightarrow \neg q$

12) check whether the statement $p \rightarrow (q \rightarrow p)$ is a tautology or a contradiction without using the truth table.

$$p \rightarrow (q \rightarrow p) \equiv p \rightarrow (\neg q \vee p)$$

$$\equiv \neg p \vee (\neg q \vee p)$$

$$\equiv \neg p \vee (p \vee \neg q) \quad (\because \text{commutative Law})$$

$$\equiv (\neg p \vee p) \vee \neg q \quad [\because \text{associative Law}]$$

(17)

$$\equiv T \vee \neg q$$

$$\equiv T$$

$\therefore P \rightarrow (q \rightarrow P)$ is a Tautology.

13) Using truth table check whether the statements $\neg(P \vee q) \vee (\neg P \wedge q)$ and $\neg P$ are logically equivalent.

P	q	$P \vee q$	$\neg(P \vee q)$	$\neg P$	$(\neg P \wedge q)$	$\neg(P \vee q) \vee (\neg P \wedge q)$
T	T	T	F	F	F	F
T	F	T	F	F	F	F
F	T	T	F	T	T	T
F	F	F	T	T	F	T

From the table Last column and the Fifth column are identical.

$$\therefore \neg(P \vee q) \vee (\neg P \wedge q) \equiv \neg P$$

14) Prove $P \rightarrow (q \rightarrow r) \equiv (P \wedge q) \rightarrow r$ without using truth table.

$$P \rightarrow (q \rightarrow r) \equiv P \rightarrow (\neg q \vee r)$$

$$\equiv \neg P \vee (\neg q \vee r)$$

$$\equiv (\neg P \vee \neg q) \vee r \quad [\because \text{associative Law}]$$

$$\equiv \neg(P \wedge q) \vee r$$

$$\equiv (P \wedge q) \rightarrow r$$

Hence proved.

15) Prove that $P \rightarrow (\neg q \vee r) \equiv \neg P \vee (\neg q \vee r)$ using truth table.

TABLE ① $P \rightarrow (\neg q \vee r)$

P	q	r	$\neg q$	$(\neg q \vee r)$	$P \rightarrow (\neg q \vee r)$
T	T	T	F	T	T
T	T	F	F	F	F
T	F	T	T	T	T
T	F	F	T	T	T
F	T	T	F	T	T
F	T	F	F	F	T
F	F	T	T	T	T
F	F	F	T	T	T

TABLE ②

P	q	r	$\neg P$	$\neg q$	$(\neg q \vee r)$	$\neg P \vee (\neg q \vee r)$
T	T	T	F	F	T	T
T	T	F	F	F	F	F
T	F	T	F	T	T	T
T	F	F	F	T	T	T
F	T	T	T	F	T	T
F	T	F	T	F	F	T
F	F	T	T	T	T	T
F	F	F	T	T	T	T

From Table ① & ② Last columns are Identical.

$$\therefore P \rightarrow (\neg q \vee r) \equiv \neg P \vee (\neg q \vee r)$$

(Using one table is also good)

—X—

Need suggestions

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