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12. Discrete Mathematics

Example12.1
Examine the binary operation (closure property) of the following operations on the respective sets (if it is not, make it binary):
(i) $a * b = a + 3ab - 5b^2$; $\forall\, a, b \in \mathbb{Z}$
(ii) $a * b = \left(\frac{a-1}{b-1}\right)$; $\forall\, a, b \in \mathbb{Q}$

Solution:

(i) Since $\times$ is binary operation on $\mathbb{Z}$,

$a, b \in \mathbb{Z} \Rightarrow ab \in \mathbb{Z}$ and $b \times b = b^2 \in \mathbb{Z}$ ... (1)

The fact that $+$ is binary operation on $\mathbb{Z}$

and (1) $\Rightarrow 3ab = (ab + ab + ab) \in \mathbb{Z}$ and

$5\,b^2 = (b^2 + b^2 + b^2 + b^2 + b^2) \in \mathbb{Z}$. .... (2)

Also $a \in \mathbb{Z}$ and $3ab \in \mathbb{Z}$ implies $a + 3ab \in \mathbb{Z}$. .....(3)

(2), (3), the closure property of $-$ on $\mathbb{Z}$ yield

$a * b = a + 3ab - 5b^2$; $\forall\, a, b \in \mathbb{Z}$.

Since $a * b$ belongs to $\mathbb{Z}$, $*$ is a binary operation on $\mathbb{Z}$.

(ii) In this problem $a * b$ is in the quotient form. Since the division by 0 is undefined, the denominator $b - 1$must be nonzero.

It is clear that $b - 1 = 0$ if $b = 1$. As $1 \in \mathbb{Q}$, $*$ is not a binary operation on the whole of $\mathbb{Q}$.

However it can be found that by omitting 1 from $\mathbb{Q}$, the output $a * b$ exists in $\mathbb{Q} \setminus \{1\}$.

Hence $*$ is a binary operation on $\mathbb{Q} \setminus \{1\}$.

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Example 12.2
Verify the (i) closure property,
(ii) commutative property,
(iii) associative property
(iv) existence of identity and
(v) existence of inverse for the arithmetic operation $+$ on $\mathbb{Z}$

Solution:

(i) $m + n \in \mathbb{Z}$, $\forall\, m, n \in \mathbb{Z}$.

Hence $+$ is a binary operation on $\mathbb{Z}$.

(ii) Also $m + n = n + m$, $\forall\, m, n \in \mathbb{Z}$.

So the commutative property is satisfied

(iii) $\forall\, m, n, p \in \mathbb{Z}$, $m + (n + p) = (m + n) + p$.

Hence the associative property is satisfied.

(iv) $m + e = e + m = m \Rightarrow e = 0$.

Thus $\exists\, 0 \in \mathbb{Z}, \ni (m + 0) = (0 + m) = m$.

Hence the existence of identity is assured.

(v) $m + m' = m' + m = 0 \Rightarrow m' = -m$.

Thus $\forall\, m \in \mathbb{Z}, \exists -m \in \mathbb{Z}$,

$\ni$ m$+ (-m) = (-m) + m = 0$. Hence, the existence of inverse property is also assured. Thus we see that the usual addition $+$ on $\mathbb{Z}$ satisfies all the above five properties.

Note that the **additive identity** is 0 and the **additive inverse** of any integer m is $-m$.

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Example 12.3
Verify the (i) closure property,
(ii) commutative property,
(iii) associative property
(iv) existence of identity and
(v)existence of inverse for the arithmetic operation $-$ on $\mathbb{Z}$

Solution:

(i) Though $-$ is not binary on $\mathbb{N}$; it is binary on $\mathbb{Z}$. To check the validity of any more properties satisfied by – on $\mathbb{Z}$, it is better to check them for some particular simple values.

(ii) Take $m = 4$, $n = 5$ and

$(m - n) = (4 - 5) = -1$and

$(n - m) = (5 - 4) = 1$.

Hence $(m - n) \neq (n - m)$. So the operation $-$ is not commutative on $\mathbb{Z}$.

(iii) In order to check the associative property, let us put $m = 4$, $n = 5$ and $p = 7$ in both $(m - n) - p$ and $m - (n - p)$.

$(m - n) - p = (4 - 5) - 7 = (-1 - 7) = -8$ ...(1)

$m - (n - p) = 4 - (5 - 7) = (4 + 2) = 6$ . ... (2)

From (1) and (2), it follows that

$(m - n) - p \neq m - (n - p)$.

Hence – is not associative on $\mathbb{Z}$.

(iv) Identity does not exist

(v) So, Inverse does not exist .

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Example 12.4
Verify the (i) closure property,
(ii) commutative property,
(iii) associative property
(iv) existence of identity and
(v) existence of inverse for the arithmetic operation $+$ on $\mathbb{Z}_e$ = the set of all even integers.

Solution : Consider the set of all even integers

$\mathbb{Z}_e = \{2k \mid k \in \mathbb{Z}\} = \{\ldots,-6,-4,-2,0,2,4,6,\ldots\}$.

Let us verify the properties satisfied by $+$ on $\mathbb{Z}_e$

(i) The sum of any two even integers is also an even integer. Because $x, y \in \mathbb{Z}_e \Rightarrow x = 2m$ and $y = 2n$, $m, n \in \mathbb{Z}$.

So $(x + y) = 2m + 2n$

$= 2(m + n) \in \mathbb{Z}_e$

$= 2(n + m)$

$= 2n + 2m$

$= (y + x)$.

Hence $+$ is a binary operation on $\mathbb{Z}_e$.

(ii) $\forall\, x, y \in \mathbb{Z}_e$,

$(x + y) = 2m + 2n$

$= 2(m + n)$

$= 2(n + m)$

$= 2n + 2m$

$= (y + x)$.

So $+$ has commutative property.

(iii) Similarly it can be seen that $\forall\, x, y, z \in \mathbb{Z}_e$,

$(x + y) + z = x + (y + z)$.

Hence the associative property is true.

(iv) Now take $x = 2k$, then

$2k + e = e + 2k = 2k \Rightarrow e = 0$.

Thus $\forall\, x \in \mathbb{Z}_e$, $\exists\, 0\ \mathbb{Z}_e \ni x + 0 = 0 + x = x$.

So, 0 is the identity element.

(v) Taking $x = 2k$ and x' as its inverse,

we have $2k + x' = 0 = x' + 2k$

$\Rightarrow x' = -2k = x' = -x$

Thus $\forall\, x \in \mathbb{Z}_e$, $\exists -x \in \mathbb{Z}_e$,

$\ni x + (-x) = (-x) + x = 0$

Hence $-x$ is the inverse of $x \in \mathbb{Z}_e$.

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Example 12.5
Verify the (i) closure property,
(ii) commutative property,
(iii) associative property
(iv) existence of identity and
(v) existence of inverse for the arithmetic operation $+$ on $\mathbb{Z}_0$ = the set of all odd integers

Solution:

Consider the set $\mathbb{Z}_0$ of all odd integers

$\mathbb{Z}_0 = \{2k+1 : k \in \mathbb{Z}\}$

$= \{\ldots, -5, -3, -1, 1, 3, 5,\}$. $+$ is not a binary operation on $\mathbb{Z}_0$ because when $x = 2m + 1, y = 2n + 1, x + y = 2(m + n) + 2$ is even for all m and n. For instance, consider the two odd numbers $3, 7 \in \mathbb{Z}_0$.

Their sum $3 + 7 = 10$ is an even number. In general, if $x, y \in \mathbb{Z}_0$, $(x + y) \notin \mathbb{Z}_0$. Other properties need not be checked as it is not a binary operation.

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Example 12.6
Verify (i) closure property
(ii) commutative property, and
(iii) associative property of the following operation on the given set.
$(a * b) = a^b$; $\forall\, a, b \in \mathbb{N}$ (exponentiation property)

Solution :

(i) It is true that $a * b = a^b$; $\forall\, a, b \in \mathbb{N}$.

So $*$ is a binary operation on $\mathbb{N}$.

(ii) $a * b = a^b$ and $b * a = b^a$.

Put, $a = 2$ and $b = 3$.

Then $a * b = 2^3 = 8$ but

$b * a = 3^2 = 9$

So a $*$b need not be equal to b $*$ a.

Hence $*$ does not have commutative property.

(iii) Next consider

$a * (b * c) = a * (b^c) = a^{(b^c)}$.

Take Put, $a = 2. b = 3$ and $c = 4$.

$$a * (b * c) = 2 * (3 * 4) = 2 * (3^4) = 2^{(81)}$$

But $(a * b) * c = (a^b) * c = (a^b)^c = a^{bc} = 2^{12}$

Hence $a * (b * c) \neq (a * b) * c$.

So $*$ does not have associative property on $\mathbb{N}$.

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Example 12.7

Verify (i) closure property,
(ii) commutative property,
(iii) associative property,
(iv) existence of identity, and
(v) existence of inverse for following operation on the given set. $m * n = m + n - mn\,; m, n \in \mathbb{Z}$

Solution:

(i) $m + n - mn$ is clearly an integer and hence $*$ is a binary operation on $\mathbb{Z}$.

(ii) $m * n = m + n - mn$ and

$$= n + m - nm$$
$$= n * m$$

So, $*$ has commutative property .

(iii) Consider $(m * n) * p = (m + n - mn) * p$

$$= (m + n - mn) + p - (m + n - mn)p$$
$$= m + n + p - mn - mp - np + mnp \ldots(1)$$
$$m * (n * p) = m * (n + p - np)$$
$$= m + (n + p - np) - m(n + p - np)$$
$$= m + n + p - np - mn - mp + mnp \ldots(2)$$

From (1) and (2) we see that,

$$(m * n) * p = m * (n * p)$$

So, $*$ has associative property .

(iv) An integer e is to be found that

$$m * e = e * m = m, \forall\, m \in \mathbb{Z}$$

So, $m * e = m + e - me = m$

$$e - me = m - m$$
$$e(1 - m) = 0$$

Gives, $e = 0$ *and* $(1 - m) = 0$

But m is an arbitrary integer and hence need not be equal to 1. So the only possibility is $e = 0$. Also $m * 0 = 0 * m = m$. Hence 0 is the identity element and hence the existence of identity is assured.

(v) An integer m' is to be found that

$$m * m' = m' * m = e, \forall\, m \in \mathbb{Z}$$

So, $m * m' = m + m' - mm' = 0$

$$m' - mm' = -m$$
$$m'(1 - m) = -m$$

$m' = -\frac{m}{(1-m)}$ is not defined at $(1 - m) = 0$, that is at $m = 1$

Hence inverse does not exist in $\mathbb{Z}$.

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Example 12.8

Let $A = \begin{bmatrix} 0 & 1 \\ 1 & 1 \end{bmatrix}, B = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$ be any two Boolean matrices of the same type. Find $A \vee B$ and $A \wedge B$

Solution:

Given $A = \begin{bmatrix} 0 & 1 \\ 1 & 1 \end{bmatrix}, B = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$

$$A \vee B = \begin{bmatrix} 0 & 1 \\ 1 & 1 \end{bmatrix} \vee \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$$
$$= \begin{bmatrix} 0 \vee 1 & 1 \vee 1 \\ 1 \vee 0 & 1 \vee 1 \end{bmatrix}$$
$$= \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}$$
$$A \wedge B = \begin{bmatrix} 0 & 1 \\ 1 & 1 \end{bmatrix} \wedge \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$$
$$= \begin{bmatrix} 0 \wedge 1 & 1 \wedge 1 \\ 1 \wedge 0 & 1 \wedge 1 \end{bmatrix}$$
$$= \begin{bmatrix} 0 & 1 \\ 0 & 1 \end{bmatrix}$$

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Example 12.9

Verify (i) closure property,

(ii) commutative property,

(iii) associative property,

(iv) existence of identity, and

(v) existence of inverse for the operation $+_5$ on $\mathbb{Z}_5$ using table corresponding to addition modulo 5.

Solution:

We know that $\mathbb{Z}_5 = \{[0],[1],[2],[3],[4]\}$

$+_5$	0	1	2	3	4
0	0	1	2	3	4
1	1	2	3	4	0
2	2	3	4	0	1
3	3	4	0	1	2
4	4	0	1	2	3

(i) Since each box in the table is filled by exactly one element of $+_5$, the output $a +_5 b$ is unique and hence $+_5$ is a binary operation.

(ii) The entries are symmetrically placed with respect to the main diagonal. So $+_5$ has commutative property.

(iii) The table cannot be used directly for the verification of the associative property. So it is to be verified as usual.

For instance, $(2 +_5 3) +_5 4 = 0 +_5 4 = 4 \pmod 5$

and $2 +_5 (3 +_5 4) = 2 +_5 2 = 4 \pmod 5$

Hence $(2 +_5 3) +_5 4 = 2 +_5 (3 +_5 4)$.

Proceeding like this one can verify this for all possible triples and ultimately it can be shown that $+_5$ is associative.

(iv) The row headed by 0 and the column headed by 0 are identical. Hence the identity element is 0.

(v) The existence of inverse is guaranteed provided the identity 0 exists in each row and each column.

From the table,

Inverse of 0 is 0

Inverse of 1 is 4

Inverse of 2 is 3

Inverse of 3 is 2

Inverse of 4 is 1

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Example 12.10 Verify (i) closure property,

(ii) commutative property,

(iii) associative property,

(iv) existence of identity, and

(v) existence of inverse for the operation $\times_{11}$ on a subset $A = \{1,3,4,5,9\}$ of the set of remainders $\{0,1,2,3,4,5,6,7,8,9,10\}$.

Solution:

Given $A = \{[1],[3],[4],[5],[9]\}$

$\times_{11}$	1	3	4	5	9
1	1	3	4	5	9
3	3	9	1	4	5
4	4	1	5	9	3
5	5	4	9	3	1
9	9	5	3	1	4

(i) Since each box in the table is filled by exactly one element of $\times_{11}$, the output $a \times_{11} b$ is unique and hence $\times_{11}$ is a binary operation.

(ii) The entries are symmetrically placed with respect to the main diagonal. So $\times_{11}$ has commutative property.

(iii) The table cannot be used directly for the verification of the associative property. So it is to be verified as usual.

For instance,

$(3 \times_{11} 4) \times_{11} 5 = 1 \times_{11} 5 = 5 \pmod{11}$

and $3 \times_{11} (4 \times_{11} 5) = 3 \times_{11} 9 = 5 \pmod{11}$

Hence $(3 \times_{11} 4) \times_{11} 5 = 3 \times_{11} (4 \times_{11} 5)$.

Proceeding like this one can verify this for all possible triples and ultimately it can be shown that $\times_{11}$ is associative.

(iv) The row headed by 1 and the column headed by 1 are identical. Hence the identity element is 1.

(v) The existence of inverse is guaranteed provided the identity 1 exists in each row and each column. From the table,

Inverse of 1 is 1

Inverse of 3 is 4

Inverse of 4 is 3

Inverse of 5 is 9

Inverse of 9 is 5

EXERCISE 12.1

1. Determine whether $*$ is a binary operation on the sets given below

(i) $a * b = a.|b|$ on $\mathbb{R}$

(ii) $a * b = \min(a, b)$ on $A = \{1,2,3,4,5\}$

(iii) $(a * b) = a\sqrt{b}$ is binary on $\mathbb{R}$.

Solution:

(i) Let $a, b \in \mathbb{R}$

Then $|b| \in \mathbb{R}$

Hence, $a.|b| \in \mathbb{R}, \forall a, b \in \mathbb{R}$

So, $*$ is the binary operation on $\mathbb{R}$.

(ii) $a * b = \min(a, b)$ on $A = \{1, 2, 3, 4, 5\}$

Let $a, b \in A$

Then $\min(a, b) \in A$

For Example,

$\min(3, 4) = 3 \in A$,

$\min(2, 5) = 2 \in A$

So, $a * b = \min(a, b) \in A, \forall a, b \in A$

So, $*$ is the binary operation on A.

(iii) $(a * b) = a\sqrt{b}$ is binary on $\mathbb{R}$

Let $a, b \in \mathbb{R}$

If b is positive then $\sqrt{b} \in \mathbb{R}$, but even though $-b \in \mathbb{R}$, $\sqrt{-b} \notin \mathbb{R}$.

For example $2, -4 \in \mathbb{R}$

But $2\sqrt{-4} = 2\sqrt{2}i \notin \mathbb{R}$

So, $*$ is not a binary operation on $\mathbb{R}$

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2. On $\mathbb{Z}$, define $\otimes$ by $(m \otimes n) = m^n + n^m : \forall m, n \in \mathbb{Z}$. Is $\otimes$ binary on $\mathbb{Z}$?

Solution:

Let $m, n \in \mathbb{Z}$.

Take $m = 3, n = -2$, then

$$m^n + n^m = 3^{-2} + (-3)^2$$
$$= \frac{1}{3^2} + 9$$
$$= \frac{1}{9} + 9$$
$$= \frac{1+81}{9}$$
$$= \frac{82}{9} \notin \mathbb{Z}$$

So, $(m \otimes n) \neq \mathbb{Z}$

Hence, $\otimes$ is not a binary operation on $\mathbb{Z}$.

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3. Let $*$ be defined on $\mathbb{R}$ by $(a * b) = a + b + ab - 7$. Is $*$ binary on $\mathbb{R}$? If so, find $3 * \left(\frac{-7}{15}\right)$.

Solution:

Let $a, b \in \mathbb{R}$

Then $a + b + ab - 7 \in \mathbb{R}$

So, $*$ is the binary operation on $\mathbb{R}$.

$$3 * \left(\frac{-7}{15}\right) = 3 + \left(-\frac{7}{15}\right) + 3\left(-\frac{7}{15}\right) - 7$$
$$= 3 - \frac{7}{15} - \frac{21}{15} - 7$$
$$= -4 - \frac{28}{15}$$
$$= \frac{-60-28}{15}$$
$$= -\frac{88}{15}$$

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4. Let $A = \{a + \sqrt{5}b : a, b \in \mathbb{Z}\}$. Check whether the usual multiplication is a binary operation on A.

Solution:

Let $A = \{x, y\}$

Such that $x = a + \sqrt{5}b$ and $y = c + \sqrt{5}d$

and $a, b, c, d \in \mathbb{Z}$

Now, $xy = (a + \sqrt{5}b)(c + \sqrt{5}d)$

$$= ac + \sqrt{5}ad + \sqrt{5}bc + 5bd$$
$$= (ac + 5bd) + \sqrt{5}(ad + bc) \in A$$

$\therefore$ $xy \in A$. The usual multiplication is a binary operation on A.

5. (i) Define an operation $*$ on $\mathbb{Q}$ as follows:

$$a \star b = \left(\frac{a+b}{2}\right) : a, b \in \mathbb{Q}.$$

Examine the closure, commutative, and associative properties satisfied by $*$ on $\mathbb{Q}$.

(ii) Define an operation $*$ on as follows:

$$a \star b = \left(\frac{a+b}{2}\right) : a, b \in \mathbb{Q}.$$

Examine the existence of identity and the existence of inverse for the operation $*$ on $\mathbb{Q}$.

Solution: (i)

(1) Given $a, b \in \mathbb{Q}$

It gives $a + b \in \mathbb{Q}$.

Hence $\frac{a+b}{2}$ also $\in \mathbb{Q}$.

So, $*$ is closure on $\mathbb{Q}$.

(2) $a * b = \left(\frac{a+b}{2}\right) : a, b \in \mathbb{Q}$

$$= \left(\frac{b+a}{2}\right)$$

$$= b * a$$

So, $*$ is commutative on $\mathbb{Q}$.

(3) $a * (b * c) = a * \left(\frac{b+c}{2}\right)$

$$= \frac{a + \left(\frac{b+c}{2}\right)}{2}$$

$$= \frac{\frac{2a+b+c}{2}}{2}$$

$$= \frac{2a+b+c}{4}$$

$(a * b) * c = \left(\frac{a+b}{2}\right) * c$

$$= \frac{\left(\frac{a+b}{2}\right) + c}{2}$$

$$= \frac{\frac{a+b+2c}{2}}{2}$$

$$= \frac{a+b+2c}{4}$$

Hence, $a * (b * c) \neq (a * b) * c$

So, $*$ is not associative on $\mathbb{Q}$.

(ii)

(1) Given $a \in \mathbb{Q}$. Let e be the identity.

Then $a * e = e * a = a$

$$\frac{a+e}{2} = a$$

$$a + e = 2a$$

$$e = 2a - a$$

$e = a$, is the identity element.

For every element a belongs to $\mathbb{Q}$, we get a is the identity. Since identity element is unique, $*$ has no identity.

(2) Since there is no identity, inverse of a that is a^{-1} cannot be defined. That is,

$a * a^{-1} = a^{-1} * a = e$, cannot be defined.

Hence $*$ has no inverse.

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6. Fill in the following table so that the binary operation $*$ on $A = \{a, b, c\}$ is commutative.

*	a	b	c
a	b		
b	c	b	a
c	a		c

Solution:

(i) $*$ is commutative

From table $b * a = c$

Hence $a * b = c$

(ii) From table $b * c = a$

Hence $c * b = a$

(iii) From table $c * a = a$

Hence $a * c = a$

So, the table is

*	a	b	c
a	b	c	a
b	c	b	a
c	a	a	c

....

7. Consider the binary operation $*$ defined on the set $A = \{a, b, c, d\}$ by the following table:

$*$	a	b	c	d
a	a	c	b	d
b	d	a	b	c
c	c	d	a	a
d	d	b	a	c

Is it commutative and associative?

Solution:

From table $a * b = c$

and $b * a = d$

So, $a * b \neq b * a$

From table $c * a = c$

and $a * c = b$

So, $c * a \neq a * c$

Hence, $*$ is not commutative

From table $a * (b * c) = a * b = c$ and

$(a * b) * c = c * c = a$

Therefore, $a * (b * c) \neq (a * b) * c$

Hence, $*$ is also not associative.

8. Let $A = \begin{pmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ 1 & 0 & 0 & 1 \end{pmatrix}$, $B = \begin{pmatrix} 0 & 1 & 0 & 1 \\ 1 & 0 & 1 & 0 \\ 1 & 0 & 0 & 1 \end{pmatrix}$,

$C = \begin{pmatrix} 1 & 1 & 0 & 1 \\ 0 & 1 & 1 & 0 \\ 1 & 1 & 1 & 1 \end{pmatrix}$ be any three Boolean matrices of the same type.

Find (i) $A \vee B$ (ii) $A \wedge B$

(iii) $(A \vee B) \wedge C$ (iv) $(A \wedge B) \vee C$.

Solution:

Given $A = \begin{pmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ 1 & 0 & 0 & 1 \end{pmatrix}$, $B = \begin{pmatrix} 0 & 1 & 0 & 1 \\ 1 & 0 & 1 & 0 \\ 1 & 0 & 0 & 1 \end{pmatrix}$

(i) $A \vee B = \begin{pmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ 1 & 0 & 0 & 1 \end{pmatrix} \vee \begin{pmatrix} 0 & 1 & 0 & 1 \\ 1 & 0 & 1 & 0 \\ 1 & 0 & 0 & 1 \end{pmatrix}$

$= \begin{pmatrix} 1 \vee 0 & 0 \vee 1 & 1 \vee 0 & 0 \vee 1 \\ 0 \vee 1 & 1 \vee 0 & 0 \vee 1 & 1 \vee 0 \\ 1 \vee 1 & 0 \vee 0 & 0 \vee 0 & 1 \vee 1 \end{pmatrix}$

$= \begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \\ 1 & 0 & 0 & 1 \end{pmatrix}$

(ii) $A \wedge B = \begin{pmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ 1 & 0 & 0 & 1 \end{pmatrix} \wedge \begin{pmatrix} 0 & 1 & 0 & 1 \\ 1 & 0 & 1 & 0 \\ 1 & 0 & 0 & 1 \end{pmatrix}$

$= \begin{pmatrix} 1 \wedge 0 & 0 \wedge 1 & 1 \wedge 0 & 0 \wedge 1 \\ 0 \wedge 1 & 1 \wedge 0 & 0 \wedge 1 & 1 \wedge 0 \\ 1 \wedge 1 & 0 \wedge 0 & 0 \wedge 0 & 1 \wedge 1 \end{pmatrix}$

$= \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 1 \end{pmatrix}$

(iii) $(A \vee B) \wedge C$

$= \begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \\ 1 & 0 & 0 & 1 \end{pmatrix} \wedge \begin{pmatrix} 1 & 1 & 0 & 1 \\ 0 & 1 & 1 & 0 \\ 1 & 1 & 1 & 1 \end{pmatrix}$

$= \begin{pmatrix} 1 \wedge 1 & 1 \wedge 1 & 1 \wedge 0 & 1 \wedge 1 \\ 1 \wedge 0 & 1 \wedge 1 & 1 \wedge 1 & 1 \wedge 0 \\ 1 \wedge 1 & 0 \wedge 1 & 0 \wedge 1 & 1 \wedge 1 \end{pmatrix}$

$= \begin{pmatrix} 1 & 1 & 0 & 1 \\ 0 & 1 & 1 & 0 \\ 1 & 0 & 0 & 1 \end{pmatrix}$

(iv) $(A \wedge B) \vee C$

$= \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 1 \end{pmatrix} \vee \begin{pmatrix} 1 & 1 & 0 & 1 \\ 0 & 1 & 1 & 0 \\ 1 & 1 & 1 & 1 \end{pmatrix}$

$= \begin{pmatrix} 0 \vee 1 & 0 \vee 1 & 0 \vee 0 & 0 \vee 1 \\ 0 \vee 0 & 0 \vee 1 & 0 \vee 1 & 0 \vee 0 \\ 1 \vee 1 & 0 \vee 1 & 0 \vee 1 & 1 \vee 1 \end{pmatrix}$

$= \begin{pmatrix} 1 & 1 & 0 & 1 \\ 0 & 1 & 1 & 0 \\ 1 & 1 & 1 & 1 \end{pmatrix}$

9. (i) Let $M = \left\{ \begin{pmatrix} x & x \\ x & x \end{pmatrix} : x \in R - (0) \right\}$ and let $*$ be the matrix multiplication. Determine whether M is closed under $*$. If so, examine the commutative and associative properties satisfied by $*$ on M.

(ii) Let $M = \left\{ \begin{pmatrix} x & x \\ x & x \end{pmatrix} : x \in R - (0) \right\}$ and let $*$ be the matrix multiplication. Determine whether M is closed under $*$. If so, examine the existence of identity, existence of inverse properties for the operation $*$ on M.

Solution: (i)

(1) Closure

Let $A = \begin{pmatrix} x & x \\ x & x \end{pmatrix}$ and $B = \begin{pmatrix} y & y \\ y & y \end{pmatrix} : x, y \in R - (0)$

Hence $M = \{A, B\}$

Now, $AB = \begin{pmatrix} x & x \\ x & x \end{pmatrix}\begin{pmatrix} y & y \\ y & y \end{pmatrix}$

$= \begin{pmatrix} xy+xy & xy+xy \\ xy+xy & xy+xy \end{pmatrix}$

$= \begin{pmatrix} 2xy & 2xy \\ 2xy & 2xy \end{pmatrix} \in M$

Since, $x, y \in R-(0)$ gives xy *also* $\in R-(0)$

So, $AB \in M \Rightarrow A * B \in M$

$\therefore *$ is closed on M

(2) Commutative

Let $A = \begin{pmatrix} x & x \\ x & x \end{pmatrix}, B = \begin{pmatrix} y & y \\ y & y \end{pmatrix}$

such that : $x, y \in R-(0)$

Hence $M = \{A, B\}$

Now, $A * B = \begin{pmatrix} x & x \\ x & x \end{pmatrix}\begin{pmatrix} y & y \\ y & y \end{pmatrix}$

$= \begin{pmatrix} xy+xy & xy+xy \\ xy+xy & xy+xy \end{pmatrix}$

$= \begin{pmatrix} 2xy & 2xy \\ 2xy & 2xy \end{pmatrix}$

$= \begin{pmatrix} 2yx & 2yx \\ 2yx & 2yx \end{pmatrix}$

$= \begin{pmatrix} y & y \\ y & y \end{pmatrix}\begin{pmatrix} x & x \\ x & x \end{pmatrix}$

$= B * A$

So, $A * B \in M = B * A \in M$

$\therefore *$ is commutative on M

(3) Associative

Matrix multiplication is always associative.

That is $A*(B*C) = (A*B)*C, \forall A,B,C \in M$.

$\therefore *$ is also associative on M

Solution: (ii)

(1) Closure

Let $A = \begin{pmatrix} x & x \\ x & x \end{pmatrix}$ and $B = \begin{pmatrix} y & y \\ y & y \end{pmatrix} : x, y \in R-(0)$

Hence $M = \{A, B\}$

Now, $AB = \begin{pmatrix} x & x \\ x & x \end{pmatrix}\begin{pmatrix} y & y \\ y & y \end{pmatrix}$

$= \begin{pmatrix} xy+xy & xy+xy \\ xy+xy & xy+xy \end{pmatrix}$

$= \begin{pmatrix} 2xy & 2xy \\ 2xy & 2xy \end{pmatrix} \in M$

Since, $x, y \in R-(0)$ gives xy *also* $\in R-(0)$

So, $AB \in M \Rightarrow A * B \in M$

$\therefore *$ is closed on M

(2) Existence of Identity

Let $A = \begin{pmatrix} x & x \\ x & x \end{pmatrix}$ and

$E = \begin{pmatrix} e & e \\ e & e \end{pmatrix}$ be the identity,

such that : $a, e \in R-(0)$

Hence $M = \{A, E\}$

Now, $A * E = E * A = A$

$\begin{pmatrix} x & x \\ x & x \end{pmatrix}\begin{pmatrix} e & e \\ e & e \end{pmatrix} = \begin{pmatrix} x & x \\ x & x \end{pmatrix}$

$\begin{pmatrix} 2xe & 2xe \\ 2xe & 2xe \end{pmatrix} = \begin{pmatrix} x & x \\ x & x \end{pmatrix}$

$2xe = x$

$2e = 1$

$e = \frac{1}{2} \in R-(0)$

$\therefore E = \begin{pmatrix} \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & \frac{1}{2} \end{pmatrix}$ is the identity $\in M$

$\therefore *$ has identity on M

(3) Existence of Inverse

Let $A = \begin{pmatrix} x & x \\ x & x \end{pmatrix}$ and

$A^{-1} = \begin{pmatrix} x^{-1} & x^{-1} \\ x^{-1} & x^{-1} \end{pmatrix}$ be the inverse of A.

Then $A * A^{-1} = A^{-1} * A = E$

$\begin{pmatrix} x & x \\ x & x \end{pmatrix}\begin{pmatrix} x^{-1} & x^{-1} \\ x^{-1} & x^{-1} \end{pmatrix} = \begin{pmatrix} e & e \\ e & e \end{pmatrix}$

$\begin{pmatrix} 2xx^{-1} & 2xx^{-1} \\ 2xx^{-1} & 2xx^{-1} \end{pmatrix} = \begin{pmatrix} \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & \frac{1}{2} \end{pmatrix}$

$2xx^{-1} = \frac{1}{2}$

$x^{-1} = \frac{1}{4x}, \in R-(0)$

$\therefore A^{-1} = \begin{pmatrix} \frac{1}{4x} & \frac{1}{4x} \\ \frac{1}{4x} & \frac{1}{4x} \end{pmatrix}$ is the inverse of A $\in M$

$\therefore *$ has inverse on M

....

10. (i) Let A *be* $\mathbb{Q} \setminus \{1\}$. Define $*$ on A by $x \star y = x + y - xy$. Is $*$ binary on A? If so, examine the commutative and associative properties satisfied by $*$ on A.

(ii) Let A be $\mathbb{Q} \setminus \{1\}$. Define $*$ on A by $x \star y = x + y - xy$. Is $*$ binary on A? If so, examine the existence of identity, existence of inverse properties for the operation $*$ on A.

Solution: (i)

(1) Closure

Given $A = \{\mathbb{Q} \setminus \{1\}\}$

Let $x, y \in A$. That is $x \neq 1, and\ y \neq 1$

$*$ is defined on A by

$$x \star y = x + y - xy$$

Let us assume that $x \star y = 1$. Then,

$$x + y - xy = 1$$
$$y - xy = 1 - x$$
$$y(1 - x) = 1 - x$$
$$y = \frac{1-x}{(1-x)}$$

It Gives, $y = 1$, which is contradiction to our assumption.

Hence, $x \star y \neq 1 \in A$

$\therefore *$ is closed on A

(2) Commutative

$$x \star y = x + y - xy$$
$$= y + x - yx$$
$$= y \star x$$

$\therefore *$ is commutative on A

(3) Associative

$$x * (y * z) = x * (y + z - yz)$$
$$= x + (y + z - yz) - x(y + z - yz)$$
$$= x + y + z - yz - xy - xz + xyz$$
$$(x * y) * z = (x + y - xy) * z$$
$$= (x + y - xy) + z - (x + y - xy)z$$
$$= x + y - xy + z - xz - yz + xyz$$

From the above results,

$$x * (y * z) = (x * y) * z, \forall\, x, y, z \in A$$

$\therefore *$ is associative on A

Solution: (ii)

(1) Closure

(2) Existence of Identity

Let $x \in A$. That is $x \neq 1$

Let e be the identity.

Then $x \star e = e \star x = x$

$$x + e - xe = x$$
$$e - xe = x - x$$
$$e(1 - x) = 0$$

It gives, $e = 0, and\ (1 - x) = 0$

If $(1 - x) = 0$, gives $x = 1$, which is not applicable as $x \in A$.

Hence, $e = 0 \in A$, is the identity.

$\therefore *$ has identity on A

(3) Existence of Inverse

Let $x \in A$. That is $x \neq 1$

Let x^{-1} be the inverse.

Then $x \star x^{-1} = x^{-1} \star x = e$

$$x + x^{-1} - xx^{-1} = 0$$
$$x^{-1} - xx^{-1} = -x$$
$$x^{-1}(1 - x) = -x$$
$$x^{-1} = -\frac{x}{(1-x)} \in A,$$

is the inverse of $x, \forall\, x \in A$.

$\therefore *$ has inverse on A.

....

Example 12.11

Identify the valid statements from the following sentences.

(1) Mount Everest is the highest mountain of the world.

It is true. So it is a statement.

(2) $3+4=8$.

It is false. So it is a statement.

(3) $7+5>10$.

It is true. So it is a statement.

(4) Give me that book.

It is a command. So it is not a statement.

(5) $(10-x)=7$.

It gives $x=3$. So it is a statement.

(6) How beautiful this flower is!

It is an exclamatory. So it is not a statement.

(7) Where are you going?

It is a question. So it is not a statement.

(8) Wish you all success.

It is expressing wishes. So it is not a statement.

(9) This is the beginning of the end.

It is a paradox. So it is not a statement.

....

Example 12.12

Write the statements in words corresponding to $\neg p$, $p \wedge q$, $p \vee q$ and $q \vee \neg p$, where p is 'It is cold' and q is 'It is raining.'

Solution:

Sentence p : 'It is cold'

Sentence q : 'It is raining.'

So, $\neg p$ = It is not cold.

$p \wedge q$ = It is cold and raining.

$p \vee q$ = It is cold or raining.

$q \vee \neg p$ = It is raining or it is not cold.

....

Example 12.13

How many rows are needed for following statement formulae?

(i) $p \vee \neg t \wedge (p \vee \neg s)$

(ii) $((p \wedge q) \vee (\neg r \vee \neg s)) \wedge (\neg t \wedge v)$

Solution:

(i) It contains 3 variables p, t and s

So, number of rows needed $= 2^3 = 8$

(ii) It contains 6 variables p, q, r, s t and v

So, number of rows needed $= 2^6 = 64$

....

Example 12.15

Write down the (i) conditional statement (ii) converse statement (iii) inverse statement, and(iv) contra positive statement for the two statements p and q given below.

p : The number of primes is infinite.

q: Ooty is in Kerala.

Solution:

Then the four types of conditional statements corresponding to p and q are respectively listed below.

(i) $p \to q$: (conditional statement)

"If the number of primes is infinite then Ooty is in Kerala".

(ii) $q \to p$: (converse statement)

"If Ooty is in Kerala then the number of primes is infinite"

(iii) $\neg p \to \neg q$ (inverse statement)

"If the number of primes is not infinite then Ooty is not in Kerala".

(iv) $\neg q \to \neg p$ (contra positive statement)

"If Ooty is not in Kerala then the number of primes is not infinite

LOGIC TRUTH FORMULAE:

(1) For NOT ($\neg$)

p	$\neg p$
T	F
F	T

(2) For AND ($\wedge$)

p	q	$p \wedge q$
T	T	T
T	F	F
F	T	F
F	F	F

(3) For OR ($\vee$)

p	q	$p \vee q$
T	T	T
T	F	T
F	T	T
F	F	F

(4) For CONDITIONAL ($\rightarrow$)

p	q	$p \rightarrow q$
T	T	T
T	F	F
F	T	T
F	F	T

(5) For BI - CONDITIONAL ($\leftrightarrow$)

p	q	$p \leftrightarrow q$
T	T	T
T	F	F
F	T	F
F	F	T

(6) For EXCLUSIVE $p\ EOR\ q$ ($\veebar$)

p	q	$p \veebar q$
T	T	F
T	F	T
F	T	T
F	F	F

(7) If the last column contains only T, then the statement called as Tautology.

(8) If the last column contains only F, then the statement called as Contradiction.

(9) The statement which is neither Tautology nor Contradiction is called Contingency.

Example 12.16

Construct the truth table for $(p \veebar q) \wedge (p \veebar \neg q)$

Let r: $(p \veebar q)$ and s: $(p \veebar \neg q)$

So, $(p \veebar q) \wedge (p \veebar \neg q) \equiv r \wedge s$

p	q	$p \vee q$	$r{:}p \veebar q$	$\neg q$	$s{:}p \veebar \neg q$	$r \wedge s$
T	T	T	F	F	T	F
T	F	T	T	T	F	F
F	T	T	T	F	F	F
F	F	F	F	T	T	F

Example 12.17

Establish the equivalence property: $p \rightarrow q \equiv \neg p \vee q$

p	q	$\neg p$	$p \rightarrow q$	$\neg p \vee q$
T	T	F	T	T
T	F	F	F	F
F	T	T	T	T
F	F	T	T	T

The entries in the columns corresponding to $p \rightarrow q$ and $\neg p \vee q$ are identical and hence they are equivalent.

Example 12.18 Establish the equivalence property connecting the bi-conditional with conditional:

$p \leftrightarrow q \equiv (p \rightarrow q) \wedge (q \rightarrow p)$

Solution:

p	q	$p \leftrightarrow q$	$r: p \rightarrow q$	$s: q \rightarrow p$	$r \wedge s$
T	T	T	T	T	T
T	F	F	F	T	F
F	T	F	T	F	F
F	F	T	T	T	T

The entries in the columns corresponding to $p \leftrightarrow q$ and $(p \rightarrow q) \wedge (q \rightarrow p)$ are identical and hence they are equivalent.

Example 12.19

Using the equivalence property, show that: $p \leftrightarrow q \equiv (p \wedge q) \vee (\neg p \wedge \neg q)$.

Solution:

LHS

p	q	$p \leftrightarrow q$
T	T	T
T	F	F
F	T	F
F	F	T

RHS

$r: p \wedge q$	$\neg p$	$\neg q$	$s: \neg p \wedge \neg q$	$r \vee s$
T	F	F	F	T
F	F	T	F	F
F	T	F	F	F
F	T	T	T	T

The entries in the columns corresponding to $p \leftrightarrow q$ and $(p \wedge q) \vee (\neg p \wedge \neg q)$ are identical and hence they are equivalent.

....

EXERCISE 12.2

1. Let p: Jupiter is a planet and q: India is an island be any two simple statements.

Give verbal sentence describing each of the following statements.

(i) $\neg p$ (ii) $p \wedge \neg q$ (iii) $\neg p \vee q$

(iv) $p \rightarrow \neg q$ (v) $p \leftrightarrow q$

Solution:

p: Jupiter is a planet

q: India is an island

(i) $\neg p$: Jupiter is not a planet

(ii) $p \wedge \neg q$: Jupiter is a planet and India is not an island

(iii) $\neg p \vee q$: Jupiter is not a planet or India is an island

(iv) $p \rightarrow \neg q$: Jupiter is a planet then India is not an island

(v) $p \leftrightarrow q$: Jupiter is a planet if and only if India is an island

....

2. Write each of the following sentences in symbolic form using statement variables p and q.

p: 19 is a prime number

q: all the angles of a triangle are equal.

(i) 19 is not a prime number and all the angles of a triangle are equal.

(ii) 19 is a prime number or all the angles of a triangle are not equal

(iii) 19 is a prime number and all the angles of a triangle are equal

(iv) 19 is not a prime number

Solution:

(i) 19 is not a prime number and all the angles of a triangle are equal : $\neg p \wedge q$

(ii) 19 is a prime number or all the angles of a triangle are not equal : $p \vee \neg q$

(iii) 19 is a prime number and all the angles of a triangle are equal : $p \wedge q$

(iv) 19 is not a prime number : $\neg p$

....

3. Determine the truth value of each of the following statements

(i) If $6 + 2 = 5$, then the milk is white.

(ii) China is in Europe or 3 is an integer

(iii) It is not true that $5 + 5 = 9$ or Earth is a planet

(iv) 11 is a prime number and all the sides of a rectangle are equal

Solution:

(i) If $6 + 2 = 5$, then the milk is white.

p: $6 + 2 = 5$ F

q: The milk is white. T

Symbolic Form: $p \rightarrow q$ Truth value is T

(ii) China is in Europe or $\sqrt{3}$ is an integer.

p: China is in Europe F

q: $\sqrt{3}$ is an integer. F

Symbolic Form: $p \vee q$ Truth value is F

(iii) It is not true that $5 + 5 = 9$ or Earth is a planet.

$p: 5 + 5 = 9$ F

q: Earth is a planet. T

Symbolic Form: $\neg p \vee q$

Truth value is T ∨ T = T

(iv) 11 is a prime number and all the sides of a rectangle are equal

p: 11 is a prime number T

q: all the sides of a rectangle are equal. F

Symbolic Form: $p \wedge q$

Truth value is T ∧ F = F

---

4. Which one of the following sentences is a proposition?

(i) 4 + 7 =12

(ii) What are you doing?

(iii) $3^n \leq 81, n \in N$

(iv) Peacock is our national bird

(v) How tall this mountain is!

Solution:

(i) 4 + 7 =12 Proposition

(ii) What are you doing? Not a Proposition

(iii) $3^n \leq 81, n \in N$ Proposition

(iv) Peacock is our national bird. Proposition

(v) How tall this mountain is! Not Proposition

---

5. Write the converse, inverse, and contra positive of each of the following implication.

(i) If x and y are numbers such that $x = y$, then $x^2 = y^2$

(ii) If a quadrilateral is a square then it is a rectangle

Solution:

(i) If x and y are numbers such that $x = y$, then $x^2 = y^2$

p: x and y are numbers such that $x = y$

q: $x^2 = y^2$

Symbolic form of the given statement : $p \rightarrow q$

(a) Converse: $q \rightarrow p$

If x and y are numbers such that $x^2 = y^2$, then $x = y$

(b) Inverse: $\neg p \rightarrow \neg q$

If x and y are numbers such that $x \neq y$, then $x^2 \neq y^2$

(c) Contra positive: $\neg q \rightarrow \neg p$

If x and y are numbers such that $x^2 \neq y^2$, then $x \neq y$

(ii) If a quadrilateral is a square then it is a rectangle

p: A quadrilateral is a square.

q: A quadrilateral is a rectangle.

Symbolic form of the given statement : $p \rightarrow q$

(a) Converse: $q \rightarrow p$

If a quadrilateral is a rectangle then it is a square.

(b) Inverse: $\neg p \rightarrow \neg q$

If a quadrilateral is not a square then it is not a rectangle.

(c) Contra positive: $\neg q \rightarrow \neg p$

If a quadrilateral is not a rectangle then it is not a square.

---

6. Construct the truth table for the following statements.

(i) $\neg p \wedge \neg q$

p	q	$\neg p$	$\neg q$	$\neg p \wedge \neg q$
T	T	F	F	F
T	F	F	T	F
F	T	T	F	F
F	F	T	T	T

(ii) $\neg(p \wedge \neg q)$

p	q	$\neg q$	$p \wedge \neg q$	$\neg(p \wedge \neg q)$
T	T	F	F	T
T	F	T	T	F
F	T	F	F	T
F	F	T	F	T

(iii) $(p \vee q) \vee \neg q$

p	q	$p \vee q$	$\neg q$	$(p \vee q) \vee \neg q$
T	T	T	F	T
T	F	T	T	T
F	T	T	F	T
F	F	F	T	T

(iv) $(\neg p \rightarrow r) \wedge (p \leftrightarrow q)$

p	q	r	$\neg p$	$(\neg p \rightarrow r)$	$p \leftrightarrow q$	$(\neg p \rightarrow r) \wedge (p \leftrightarrow q$
T	T	T	F	T	T	T
T	T	F	F	T	T	T
T	F	T	F	T	F	F
T	F	F	F	T	F	F
F	T	T	T	T	F	F
F	T	F	T	F	F	F
F	F	T	T	T	T	T
F	F	F	T	F	T	F

7. Verify whether the following compound propositions are tautologies or contradictions or contingency

(i) $(p \wedge q) \wedge \neg(p \vee q)$

p	q	$p \wedge q$	$p \vee q$	$\neg(p \vee q)$	$(p \wedge q) \wedge \neg(p \vee q)$
T	T	T	T	F	F
T	F	F	T	F	F
F	T	F	T	F	F
F	F	F	F	T	F

The last column contains only F. Hence it is contradiction.

(ii) $((p \vee q) \wedge \neg p) \rightarrow q$

p	q	$p \vee q$	$\neg p$	$(p \vee q) \wedge \neg p$	$((p \vee q) \wedge \neg p) \rightarrow q$
T	T	T	F	F	T
T	F	T	F	F	T
F	T	T	T	T	T
F	F	F	T	F	T

The last column contains only T. Hence it is tautology.

(iii) $(p \rightarrow q) \leftrightarrow (\neg p \rightarrow q)$

p	q	$p \rightarrow q$	$\neg p$	$(\neg p \rightarrow q)$	$(p \rightarrow q) \leftrightarrow (\neg p \rightarrow q)$
T	T	T	F	T	T
T	F	F	F	T	F
F	T	T	T	T	T
F	F	T	T	F	F

The last column contains T and F. Hence it is contingency.

(iv) $((p \rightarrow q) \wedge (q \rightarrow r)) \rightarrow (p \rightarrow r)$

p	q	r	$(p \rightarrow q)$	$(q \rightarrow r)$	$((p \rightarrow q) \wedge (q \rightarrow r))$
T	T	T	T	T	T
T	T	F	T	F	F
T	F	T	F	T	F
T	F	F	F	T	F
F	T	T	T	T	T
F	T	F	T	F	F
F	F	T	T	T	T
F	F	F	T	T	T

$(p \rightarrow r)$	$((p \rightarrow q) \wedge (q \rightarrow r)) \rightarrow (p \rightarrow r)$
T	T
F	T
T	T
F	T
T	T
T	T
T	T
T	T

The last column contains only T. Hence it is tautology.

8. Show that

(i) $\neg(p \wedge q) \equiv \neg p \vee \neg q$

p	q	$p \wedge q$	$\neg(p \wedge q)$	$\neg p$	$\neg q$	$\neg p \vee \neg q$
T	T	T	F	F	F	F
T	F	F	T	F	T	T
F	T	F	T	T	F	T
F	F	F	T	T	T	T

From column (4) and (7), it is proved that

$\neg(p \wedge q) \equiv \neg p \vee \neg q$

(ii) $\neg(p\rightarrow q)\equiv p\wedge\neg q$.

p	q	$p\rightarrow q$	$\neg(p\rightarrow q)$	$\neg q$	$p\wedge\neg q$
T	T	T	F	F	F
T	F	F	T	T	T
F	T	T	F	F	F
F	F	T	F	T	F

From column (4) and (6), it is proved that

$\neg(p\rightarrow q)\equiv p\wedge\neg q$.

....

9. Prove that $q\rightarrow p\equiv\neg p\rightarrow\neg q$

p	q	$q\rightarrow p$	$\neg p$	$\neg q$	$\neg p\rightarrow\neg q$
T	T	T	F	F	T
T	F	T	F	T	T
F	T	F	T	F	F
F	F	T	T	T	T

From column (3) and (6), it is proved that

$q\rightarrow p\equiv\neg p\rightarrow\neg q$

....

10. Show that $p\rightarrow q$ and $q\rightarrow p$ are not equivalent

p	q	$p\rightarrow q$	$q\rightarrow p$
T	T	T	T
T	F	F	T
F	T	T	F
F	F	T	T

$p\rightarrow q$ and $q\rightarrow p$ are not equivalent.

....

11. Show that $\neg(p\leftrightarrow q)\equiv p\leftrightarrow\neg q$

p	q	$p\leftrightarrow q$	$\neg(p\leftrightarrow q)$	$\neg q$	$p\leftrightarrow\neg q$
T	T	T	F	F	F
T	F	F	T	T	T
F	T	F	T	F	T
F	F	T	F	T	F

From column (4) and (6), it is proved that

$\neg(p\leftrightarrow q)\equiv p\leftrightarrow\neg q$.

....

12. Check whether the statement $p\rightarrow(q\rightarrow p)$ is a tautology or a contradiction without using the truth table.

Solution:

$$p\rightarrow(q\rightarrow p)\equiv p\rightarrow(\neg q\vee p)$$
$$\equiv\neg p\vee(\neg q\vee p)$$
$$\equiv\neg p\vee(p\vee\neg q)$$
$$\equiv(\neg p\vee p)\vee\neg q$$
$$\equiv T\vee\neg q$$
$$\equiv T$$

Hence, $p\rightarrow(q\rightarrow p)$ is a tautology.

....

13. Using truth table check whether the statements $\neg(p\vee q)\vee(\neg p\wedge q)$ and $\neg p$ are logically equivalent.

p	q	$p\vee q$	$\neg(p\vee q)$	$\neg p$	$\neg p\wedge q$	$\neg(p\vee q)\vee(\neg p\wedge q)$
T	T	T	F	F	F	F
T	F	T	F	F	F	F
F	T	T	F	T	T	T
F	F	F	T	T	F	T

From column (5) and (7), it is proved that

$\neg p\equiv\neg(p\vee q)\vee(\neg p\wedge q)$.

....

14. Prove $p\rightarrow(q\rightarrow r)\equiv(p\wedge q)\rightarrow r$ without using truth table.

Solution:

$$p\rightarrow(q\rightarrow r)\equiv p\rightarrow(\neg q\vee r)$$
$$\equiv\neg p\vee(\neg q\vee r)$$
$$\equiv(\neg p\vee\neg q)\vee r$$
$$\equiv\neg(p\wedge q)\vee r$$
$$\equiv(p\wedge q)\rightarrow r$$ Hence proved.

....

15. Prove that $p \to (\neg q \vee r) \equiv \neg p \vee (\neg q \vee r)$ using truth table.

LHS : $p \to (\neg q \vee r)$

p	q	r	$\neg q$	$(\neg q \vee r)$	$p \to (\neg q \vee r)$
T	T	T	F	T	T
T	T	F	F	F	F
T	F	T	T	T	T
T	F	F	T	T	T
F	T	T	F	T	T
F	T	F	F	F	T
F	F	T	T	T	T
F	F	F	T	T	T

RHS : $\neg p \vee (\neg q \vee r)$

p	q	r	$\neg q$	$(\neg q \vee r)$	$\neg p$	$\neg p \vee (\neg q \vee r)$
T	T	T	F	T	F	T
T	T	F	F	F	F	F
T	F	T	T	T	F	T
T	F	F	T	T	F	T
F	T	T	F	T	T	T
F	T	F	F	F	T	T
F	F	T	T	T	T	T
F	F	F	T	T	T	T

From the above tables, the last columns are identical. Hence $p \to (\neg q \vee r) \equiv \neg p \vee (\neg q \vee r)$.

....

EXERCISE 12.3

Choose the correct or the most suitable answer from the given four alternatives.

1. A binary operation on a set S is a function from
   (1) $S \to S$ **(2) $(S \times S) \to S$**
   (3) $S \to (S \times S)$ (4) $(S \times S) \to (S \times S)$

2. Subtraction is not a binary operation in
   (1) $\mathbb{R}$ (2) $\mathbb{Z}$ **(3) $\mathbb{N}$** (4) $\mathbb{Q}$

3. Which one of the following is a binary operation on $\mathbb{N}$?
   (1) Subtraction **(2) Multiplication**
   (3) Division (4) All the above

4. In the set $\mathbb{R}$ of real numbers '*' is defined as follows. Which one of the following is not a binary operation on $\mathbb{R}$?
   (1) $a * b = \min(a, b)$ (2) $a * b = \max(a, b)$
   (3) $a * b = a$ **(4) $a * b = a^b$**

5. The operation * defined by $a * b = \frac{ab}{7}$ is not a binary operation on
   (1) $\mathbb{Q}^+$ **(2) $\mathbb{Z}$** (3) $\mathbb{R}$ (4) $\mathbb{C}$

6. In the set $\mathbb{Q}$ define $\odot b = a + b + ab$. For what value of y, $3 \odot (y \odot 5) = 7$?
   (1) $y = \frac{2}{3}$ **(2) $y = \frac{-2}{3}$**
   (3) $y = \frac{-3}{2}$ (4) $y = 4$

7. If $a * b = \sqrt{a^2 + b^2}$ on the real numbers then * is
   (1) commutative but not associative
   (2) associative but not commutative
   (3) both commutative and associative
   (4) neither commutative nor associative

8. Which one of the following statements has the truth value T ?
   (1) $\sin x$ is an even function.
   (2) Every square matrix is non-singular
   (3) The product of complex number and its conjugate is purely imaginary
   (4) $\sqrt{5}$ is an irrational number

9. Which one of the following statements has truth value F ?
   (1) Chennai is in India or $\sqrt{2}$ is an integer
   (2) Chennai is in India or $\sqrt{2}$ is an irrational number
   (3) Chennai is in China or $\sqrt{2}$ is an integer
   (4) Chennai is in China or $\sqrt{2}$ is an irrational number

10. If a compound statement involves 3 simple statements, then the number of rows in the truth table is
(1) 9 **(2) 8** (3) 6 (4) 3

11. Which one is the inverse of the statement $(p \vee q) \rightarrow (p \wedge q)$?
(1) $(p \wedge q) \rightarrow (p \vee q)$
(2) $\neg (p \vee q) \rightarrow (p \wedge q)$
(3) $(\neg p \vee \neg q) \rightarrow (\neg p \wedge \neg q)$
(4) $(\neg p \wedge \neg q) \rightarrow (\neg p \vee \neg q)$

12. Which one is the contra positive of the statement $(p \vee q) \rightarrow r$?
(1) $\neg r \rightarrow (\neg p \wedge \neg q)$ (2) $\neg r \rightarrow (p \vee q)$
(3) $r \rightarrow (p \wedge q)$ (4) $p \rightarrow (q \vee r)$

13. The truth table for $(p \wedge q) \vee \neg q$ is given below

p	q	$(p \wedge q) \vee \neg q$
T	T	(a)
T	F	(b)
F	T	(c)
F	F	(d)

Which one of the following is true?

	(a)	(b)	(c)	(d)
(1)	T	T	T	T
(2)	T	F	T	T
(3)	**T**	**T**	**F**	**T**
(4)	T	F	F	F

14. In the last column of the truth table for $\neg (p \vee \neg q)$ the number of final outcomes of the truth value 'F' are
(1) 1 (2) 2 **(3) 3** (4) 4

15. Which one of the following is incorrect? For any two propositions p and , we have
(1) $\neg (p \vee q) \equiv \neg p \wedge \neg q$
(2) $\neg (p \wedge q) \equiv \neg p \vee \neg q$
(3) $\neg (p \vee q) \equiv \neg p \vee \neg q$
(4) $\neg (\neg p) \equiv p$

16. Which one of the following is correct for the truth value of $(p \wedge q) \rightarrow \neg p$?

	(a)	(b)	(c)	(d)
(1)	T	T	T	T
(2)	**F**	**T**	**T**	**T**
(3)	F	F	T	T
(4)	T	T	T	F

17. The dual of $\neg (p \vee q) \vee [p \vee (p \wedge \neg r)]$ is
(1) $\neg (p \wedge q) \wedge [p \vee (p \wedge \neg r)]$
(2) $(p \wedge q) \wedge [p \wedge (p \vee \neg r)]$
(3) $\neg (p \wedge q) \wedge [p \wedge (p \wedge r)]$
(4) $\neg (p \wedge q) \wedge [p \wedge (p \vee \neg r)]$

18. The proposition $p \wedge (\neg p \vee q)$ is
(1) a tautology
(2) a contradiction
(3) logically equivalent to $p \wedge q$
(4) logically equivalent to $p \vee q$

19. Determine the truth value of each of the following statements:
(a) 4+2=5 and 6+3=9
(b) 3+2=5 and 6+1=7
(c) 4+5=9 and1+2= 4
(d) 3+2=5 and 4+7=11

	(a)	(b)	(c)	(d)
(1)	**F**	**T**	**F**	**T**
(2)	T	F	T	F
(3)	T	T	F	F
(4)	F	F	T	T

20. Which one of the following is not true?
(1) Negation of a negation of a statement is the statement itself.
(2) If the last column of the truth table contains only T then it is a tautology.
(3) If the last column of its truth table contains only F then it is a contradiction
(4) If p and q are any two statements then $p \leftrightarrow q$ is a tautology.

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