Padasalai's Telegram Groups!

(தலைப்பிற்கு கீழே உள்ள லிங்கை கிளிக் செய்து குழுவில் இணையவும்!)

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- Lesson Plan - Group
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1.APPLICATIONS OF MATRICES AND DETERMINANTS

- If $|A| \neq 0$, then the square matrix A is called a **non-singular matrix.**
- If $|A| = 0$, then the square matrix A is called a **singular matrix.**
- **ADJOINT OF A SQUARE MATRIX:** Let A be the square matrix of order n, then the adjoint matrix of A is defined as the transpose of the matrix of cofactors of A. It is denoted by adj A. $\boldsymbol{adj\ A = [A_{ij}]^T = [(-1)^{i+j}M_{ij}]^T}$
- $A(adjA) = (adjA)A = |A|I_n$
- **INVERSE MATRIX:** Let A be a square matrix of order n. If there exists a square matrix B of order n such that $AB = BA = I_n$, then the matrix B is called an inverse of A.
- $AA^{-1} = A^{-1}A = I_n$
- A singular matrix has no inverse.
- If A is non-singular, then
  $(i)|A^{-1}| = \frac{1}{|A|}$ $(ii)\ (A^T)^{-1} = (A^{-1})^T$ $(iii)(\lambda A)^{-1} = \frac{1}{\lambda}A^{-1}$
- If A is non-singular, then $(\lambda A)^{-1} = \frac{1}{\lambda}A^{-1}$, where $\boldsymbol{\lambda}$ is a non-zero scalar.
- $(AB)^{-1} = B^{-1}A^{-1}$
- If A is non-singular, then A^{-1} is also non-singular and $(A^{-1})^{-1} = A$
- If A is non-singular square matrix of order n, then
  $(i)(adjA)^{-1} = adj(A^{-1}) = \frac{1}{|A|}A \implies adj\ A = |A|A^{-1}$
  $(ii)|adjA| = |A|^{n-1}$ $(iii)adj(adjA) = |A|^{n-2}A$
  $(iv)adj(\lambda A) = \lambda^{n-1}adj(A)$ $(v)|adj(adjA)| = |A|^{(n-1)^2}$
  $(vi)(adjA)^T = adj(A^T)$ $(vii)adj(AB) = (adjB)(adjA)$
- $|adj\ A| = |A|^2 \implies |A| = \pm\sqrt{|adj\ A|}$
- $A^{-1} = \pm\frac{1}{\sqrt{|adjA|}}adjA$
- $A = \pm\frac{1}{\sqrt{|adjA|}}adj(adjA)$
- If A and B are any two non-singular square matrices of order n, then $adj(AB) = (adj\ B)(adj\ A)$
- A square matrix A is called orthogonal if $AA^T = A^TA = I$
- A is orthogonal if and only if A is non-singular and $A^{-1} = A^T$
- Application of matrices is in computer graphic and cryptography.

THEOREM 1: For every square matrix A of order n, $A(adj\ A) = (adj\ A)A = |A|I_n$

Proof:

For simplicity, we prove the theorem for n=3 only.

Consider $A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$ and $adj\ A = \begin{bmatrix} A_{11} & A_{12} & A_{13} \\ A_{21} & A_{22} & A_{23} \\ A_{31} & A_{32} & A_{33} \end{bmatrix}$

$$A(adj\ A) = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}\begin{bmatrix} A_{11} & A_{12} & A_{13} \\ A_{21} & A_{22} & A_{23} \\ A_{31} & A_{32} & A_{33} \end{bmatrix} = \begin{bmatrix} |A| & 0 & 0 \\ 0 & |A| & 0 \\ 0 & 0 & |A| \end{bmatrix}$$

$$= |A|\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = |A|I_3 \rightarrow (1)$$

$$(adj\ A)A = \begin{bmatrix} A_{11} & A_{12} & A_{13} \\ A_{21} & A_{22} & A_{23} \\ A_{31} & A_{32} & A_{33} \end{bmatrix}\begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix} = \begin{bmatrix} |A| & 0 & 0 \\ 0 & |A| & 0 \\ 0 & 0 & |A| \end{bmatrix}$$

$$= |A|\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = |A|I_3 \rightarrow (2)$$

From (1) & (2), we get $A(adj\ A) = (adj\ A)A = |A|I_n$

THEOREM 2: If a square matrix has an inverse, then it is unique.

Proof:

Let A be a square matrix of order n such that an inverse of A exists. If possible, let there be two inverse B and C of A.

By definition, $AB = BA = I_n\ and\ AC = CA = I_n$

$C = CI_n = C(AB) = (CA)B = I_nB = B \implies B = C$

Hence it is proved.

THEOREM 3: Let A be a square matrix of order n. Then A^{-1} exists if and only if A is non-singular.

Proof:

Case (i) : Suppose that A^{-1} exists, then $AA^{-1} = A^{-1}A = I_n$.

By the product rule for determinants,

$det(AA^{-1}) = det(A^{-1})det(A) = det(I_n) = 1 \implies |A| \neq 0$.

Hence A is non-singular.

Case (ii) : Conversely, suppose A is non-singular, then $|A| \neq 0$.

W.k.T. $A(\text{adj } A) = (\text{adj } A)A = |A|I_n$

$\div$ by $|A| \Rightarrow A\left(\frac{1}{|A|} adj\ A\right) = \left(\frac{1}{|A|} adj\ A\right)A = I_n$

$\Rightarrow AB = BA = I_n$; $B = \frac{1}{|A|} adj\ A$

Hence the inverse of A exists and the inverse is $A^{-1} = \frac{1}{|A|} adj\ A$

PROPERTIES OF INVERSES OF MATRICES

THEOREM 4: If A is non-singular, then $|A^{-1}| = \frac{1}{|A|}$

Proof:

Let A be non-singular then $|A| \neq 0$ and A^{-1} exists.

By definition, $AA^{-1} = A^{-1}A = I_n \Rightarrow |AA^{-1}| = |A^{-1}A| = |I_n|$

By the product rule for determinants,

$|AA^{-1}| = |I_n| \Rightarrow |A||A^{-1}| = 1 \Rightarrow |A^{-1}| = \frac{1}{|A|}$

THEOREM 5: If A is non-singular, then $(A^T)^{-1} = (A^{-1})^T$

Proof:

By definition, $AA^{-1} = A^{-1}A = I_n$

Taking transpose ,$(AA^{-1})^T = (A^{-1}A)^T = (I_n)^T$

By the reversal law of transpose, $(A^{-1})^T A^T = A^T(A^{-1})^T = |I_n|$

Hence $(A^T)^{-1} = (A^{-1})^T$

THEOREM 6: If A is non-singular, then $(\lambda A)^{-1} = \frac{1}{\lambda}A^{-1}$, where λ is a non-zero scalar.

Proof:

By definition, $AA^{-1} = A^{-1}A = I_n$

Mulipl y & divide by λ, $(\lambda A)\left(\frac{1}{\lambda}A^{-1}\right) = \left(\frac{1}{\lambda}A^{-1}\right)(\lambda A) = I_n \Rightarrow (\lambda A)^{-1} = \frac{1}{\lambda}A^{-1}$

THEOREM 7: LEFT CANCELLATION LAW

Let A,B and C be square matrices of order n. If A is non-singular and AB=AC then B=C

Proof:

Since A is non-singular, A^{-1} exists and $AA^{-1} = A^{-1}A = I_n$.

Given AB=AC. Pre multiply by A^{-1} on both sides,

$A^{-1}(AB) = A^{-1}(AC) \Rightarrow (A^{-1}A)B = (A^{-1}A)C \Rightarrow I_nB = I_nC \Rightarrow B = C$

THEOREM 8: RIGHT CANCELLATION LAW

Let A,B and C be square matrices of order n. If A is non-singular and BA=CA then B=C

Proof:

Since A is non-singular, A^{-1} exists and $AA^{-1} = A^{-1}A = I_n$.

Given BA=CA. Post multiply by A^{-1} on both sides,

$(BA)A^{-1} = (CA)A^{-1} \Rightarrow B(AA^{-1}) = C(AA^{-1}) \Rightarrow BI_n = CI_n \Rightarrow B = C$

THEOREM 9: REVERSAL LAW FOR INVERSES

If A and B are non-singular matrices of the same order, then the product AB is also non-singular and $(AB)^{-1} = B^{-1}A^{-1}$

Proof:

Since A and B are non-singular matrices of the same order n, then $|A| \neq 0$, $|B| \neq 0 \Rightarrow A^{-1}$ *and* B^{-1} of order n exists. The product of AB and $B^{-1}A^{-1}$ can also found.

$|AB| = |A||B| \neq 0 \Rightarrow AB$ is also non-singular and $(AB)^{-1}$ exists.

$(AB)(B^{-1}A^{-1}) = A(BB^{-1})A^{-1} = A(I_n)A^{-1} = AA^{-1} = I_n$

$(B^{-1}A^{-1})(AB) = B^{-1}(A^{-1}A)B = B^{-1}(I_n)B = B^{-1}B = I_n$

Hence $(AB)^{-1} = B^{-1}A^{-1}$

THEOREM 10: LAW OF DOUBLE INVERSE

If A is non-singular, then A^{-1} is also non-singular and $(A^{-1})^{-1} = A$

Proof:

Since A is non-singular,then $|A| \neq 0$ and A^{-1} exists.

Now $|A^{-1}| = \frac{1}{|A|} \neq 0 \Rightarrow A^{-1}$ is also non-singular and $AA^{-1} = A^{-1}A = I_n$.

Now $AA^{-1} = I_n \Rightarrow (AA^{-1})^{-1} = I \Rightarrow (A^{-1})^{-1}A^{-1} = I$

Post- multiply by A on both sides, $(A^{-1})^{-1} = A$

THEOREM 11: If A is non-singular square matrix of order n, then $(\mathbf{adjA})^{-1} = \mathbf{adj}(\mathbf{A}^{-1}) = \frac{1}{|\mathbf{A}|}\mathbf{A}$

Proof:

$A^{-1} = \frac{1}{|A|}(adj\ A) \Rightarrow adj\ A = |A|A^{-1}$

$\Rightarrow (adj\ A)^{-1} = (|A|A^{-1})^{-1} = (A^{-1})^{-1}|A|^{-1} = \frac{1}{|A|}A \rightarrow (1)$

$A^{-1} = \frac{1}{|A|}(adj\ A) \Rightarrow adj\ A = |A|A^{-1}$

Replacing A by A^{-1}, $adj\ (A^{-1}) = |A^{-1}|(A^{-1})^{-1} = \frac{1}{|A|}A \rightarrow (2)$

From (1) and (2), $(\mathrm{adj}A)^{-1} = \mathrm{adj}(A^{-1}) = \frac{1}{|A|}A$

THEOREM 12: If A is non-singular square matrix of order n, then $|\mathbf{adjA}| = |\mathbf{A}|^{\mathbf{n-1}}$

Proof:

$A(\mathrm{adj}\ A) = (\mathrm{adj}\ A)A = |A|I_n \Rightarrow det[A(\mathrm{adj}\ A)] = \det(A)\det(adj A) = \det[|A|I_n]$

$\Rightarrow |A||adj\ A| = |A|^n \Rightarrow |\mathrm{adj}A| = |A|^{n-1}$

THEOREM 13: If A is non-singular square matrix of order n, then $\mathbf{adj}(\mathbf{adjA}) = |\mathbf{A}|^{\mathbf{n-2}}\mathbf{A}$

Proof:

For any non-singular matrix B of order n, $B(\mathrm{adj}\ B) = (\mathrm{adj}\ B)B = |B|I_n$

Put $B = adjA \Rightarrow adjA(\mathrm{adj}\ (adjA)) = |adjA|I_n$

$\Rightarrow adjA(\mathrm{adj}\ (adjA)) = |A|^{n-1}I_n$

Pre-multiply by A on both sides,

$\Rightarrow [A(adjA)](\mathrm{adj}\ (adjA)) = A|A|^{n-1}I_n \Rightarrow |A|I_n(\mathrm{adj}\ (adjA)) = A|A|^{n-1}I_n$

$\Rightarrow \mathrm{adj}\ (adjA) = |A|^{n-2}A$

THEOREM 14: If A is non-singular square matrix of order n, then $\mathbf{adj}(\boldsymbol{\lambda}\mathbf{A}) = \boldsymbol{\lambda}^{\mathbf{n-1}}\mathbf{adj}(\mathbf{A})$

Proof:

W.K.T, $(\mathrm{adj}A)^{-1} = \frac{1}{|A|}A \Rightarrow adj\ A = |A|A^{-1}$

Replace A by λA $\Rightarrow adj\ (\lambda A) = |\lambda A|(\lambda A)^{-1} = \lambda^n|A|\frac{1}{\lambda}A^{-1} = \lambda^{n-1}|A|A^{-1} = \lambda^{n-1}\mathrm{adj}(A)$

THEOREM 15: If A is non-singular square matrix of order n, then $|\mathbf{adj}(\mathbf{adjA})| = |\mathbf{A}|^{(\mathbf{n-1})^2}$

Proof:

W.K.T, $\mathrm{adj}\ (adjA) = |A|^{n-2}A$

$\Rightarrow |\mathrm{adj}\ (adjA)| = ||A|^{n-2}A| = (|A|^{n-2})^n|A| = |A|^{n^2-2n+1} = |A|^{(n-1)^2}$

THEOREM 16: If A is non-singular square matrix of order n, then $(\mathbf{adjA})^{\mathbf{T}} = \mathbf{adj}(\mathbf{A}^{\mathbf{T}})$

Proof:

W.K.T, $A^{-1} = \frac{1}{|A|}(adj\ A)$

Replace A by $A^T \Rightarrow (A^T)^{-1} = \frac{1}{|A^T|}(adj\ (A^T))$

$\Rightarrow adj\ (A^T) = |A^T|(A^T)^{-1} = (|A|A^{-1})^T = (\mathrm{adj}A)^T$

THEOREM 17: If A and B are any two non-singular matrices of order n, then $\mathbf{adj}(\mathbf{AB}) = (\mathbf{adjB})(\mathbf{adjA})$

Proof:

W.K.T, $adj\ A = |A|A^{-1}$

Replace A by AB $\Rightarrow adj\ (AB) = |AB|(AB)^{-1} = |A||B|A^{-1}B^{-1}$

$= (|B|B^{-1})(|A|A^{-1}) = (\mathrm{adj}B)(\mathrm{adj}A)$

EXERCISE 1.1

1.Find the adjoint of the following matrices.

(i) $\begin{bmatrix} -3 & 4 \\ 6 & 2 \end{bmatrix}$

Let $A = \begin{bmatrix} -3 & 4 \\ 6 & 2 \end{bmatrix} \Rightarrow adj\ A = \begin{bmatrix} 2 & -4 \\ -6 & -3 \end{bmatrix}$

(ii) $\begin{bmatrix} 2 & 3 & 1 \\ 3 & 4 & 1 \\ 3 & 7 & 2 \end{bmatrix}$

Let $A = \begin{bmatrix} 2 & 3 & 1 \\ 3 & 4 & 1 \\ 3 & 7 & 2 \end{bmatrix}$

$$\begin{matrix} 4 & 7 & 3 & 4 \\ 1 & 2 & 1 & 1 \\ 3 & 3 & 2 & 3 \\ 4 & 7 & 3 & 4 \end{matrix} \Rightarrow adj\ A = \begin{bmatrix} 8-7 & 7-6 & 3-4 \\ 3-6 & 4-3 & 3-2 \\ 21-12 & 9-14 & 8-9 \end{bmatrix} = \begin{bmatrix} 1 & 1 & -1 \\ -3 & 1 & 1 \\ 9 & -5 & -1 \end{bmatrix}$$

(iii) $\frac{1}{3}\begin{bmatrix} 2 & 2 & 1 \\ -2 & 1 & 2 \\ 1 & -2 & 2 \end{bmatrix}$

$adj(\lambda A) = \lambda^{n-1} adj(A)$

$$\begin{matrix} 1 & -2 & 2 & 1 \\ 2 & 2 & 1 & 2 \\ -2 & 1 & 2 & -2 \\ 1 & -2 & 2 & 1 \end{matrix} \Rightarrow adj\ A = \left(\frac{1}{3}\right)^{3-1} \begin{bmatrix} 2+4 & -2-4 & 4-1 \\ 2+4 & 4-1 & -2-4 \\ 4-1 & 2+4 & 2+4 \end{bmatrix}$$

$$= \frac{1}{9}\begin{bmatrix} 6 & -6 & 3 \\ 6 & 3 & -6 \\ 3 & 6 & 6 \end{bmatrix} = \frac{1}{3}\begin{bmatrix} 2 & -2 & 1 \\ 2 & 1 & -2 \\ 1 & 2 & 2 \end{bmatrix}$$

2. Find the inverse of the following

(i) $\begin{bmatrix} -2 & 4 \\ 1 & -3 \end{bmatrix}$

Let $A = \begin{bmatrix} -2 & 4 \\ 1 & -3 \end{bmatrix}$; $|A| = \begin{vmatrix} -2 & 4 \\ 1 & -3 \end{vmatrix} = 6 - 4 = 2 \neq 0$; $adj\ A = \begin{bmatrix} -3 & -4 \\ -1 & -2 \end{bmatrix}$

$A^{-1} = \frac{1}{|A|} adj\ A \Rightarrow A^{-1} = \frac{1}{2}\begin{bmatrix} -3 & -4 \\ -1 & -2 \end{bmatrix}$

(ii) $\begin{bmatrix} 5 & 1 & 1 \\ 1 & 5 & 1 \\ 1 & 1 & 5 \end{bmatrix}$

Let $A = \begin{bmatrix} 5 & 1 & 1 \\ 1 & 5 & 1 \\ 1 & 1 & 5 \end{bmatrix}$

$$|A| = \begin{vmatrix} 5 & 1 & 1 \\ 1 & 5 & 1 \\ 1 & 1 & 5 \end{vmatrix} = 5(25-1) - 1(5-1) + 1(1-5)$$
$$= 120 - 4 - 4$$
$$= 112 \neq 0$$

$$\begin{matrix} 5 & 1 & 1 & 5 \\ 1 & 5 & 1 & 1 \\ 1 & 1 & 5 & 1 \\ 5 & 1 & 1 & 5 \end{matrix} \Rightarrow adj\ A = \begin{bmatrix} 25-1 & 1-5 & 1-5 \\ 1-5 & 25-1 & 1-5 \\ 1-5 & 1-5 & 25-1 \end{bmatrix} = \begin{bmatrix} 24 & -4 & -4 \\ -4 & 24 & -4 \\ -4 & -4 & 24 \end{bmatrix}$$

$$A^{-1} = \frac{1}{|A|} adj\ A \Rightarrow A^{-1} = \frac{1}{112}\begin{bmatrix} 24 & -4 & -4 \\ -4 & 24 & -4 \\ -4 & -4 & 24 \end{bmatrix} = \frac{1}{28}\begin{bmatrix} 6 & -1 & -1 \\ -1 & 6 & -1 \\ -1 & -1 & 6 \end{bmatrix}$$

(iii) $\begin{bmatrix} 2 & 3 & 1 \\ 3 & 4 & 1 \\ 3 & 7 & 2 \end{bmatrix}$

Let $A = \begin{bmatrix} 2 & 3 & 1 \\ 3 & 4 & 1 \\ 3 & 7 & 2 \end{bmatrix}$

$$|A| = \begin{vmatrix} 2 & 3 & 1 \\ 3 & 4 & 1 \\ 3 & 7 & 2 \end{vmatrix} = 2(8-7) - 3(6-3) + 1(21-12) = 2 - 9 + 9 = 2 \neq 0$$

$$\begin{matrix} 4 & 7 & 3 & 4 \\ 1 & 2 & 1 & 1 \\ 3 & 3 & 2 & 3 \\ 4 & 7 & 3 & 4 \end{matrix} \Rightarrow adj\ A = \begin{bmatrix} 8-7 & 7-6 & 3-4 \\ 3-6 & 4-3 & 3-2 \\ 21-12 & 9-14 & 8-9 \end{bmatrix} = \begin{bmatrix} 1 & 1 & -1 \\ -3 & 1 & 1 \\ 9 & -5 & -1 \end{bmatrix}$$

$$A^{-1} = \frac{1}{|A|} adj\ A \Rightarrow A^{-1} = \frac{1}{2}\begin{bmatrix} 1 & 1 & -1 \\ -3 & 1 & 1 \\ 9 & -5 & -1 \end{bmatrix}$$

EXAMPLE 1.2 Find the inverse of $\begin{bmatrix} a & b \\ c & d \end{bmatrix}$

EXAMPLE 1.3 Find the inverse of $\begin{bmatrix} 2 & -1 & 3 \\ -5 & 3 & 1 \\ -3 & 2 & 3 \end{bmatrix}$

3. *If* $F(\alpha) = \begin{bmatrix} \cos\alpha & 0 & \sin\alpha \\ 0 & 1 & 0 \\ -\sin\alpha & 0 & \cos\alpha \end{bmatrix}$, *show that* $[F(\alpha)]^{-1} = F(-\alpha)$

$$F(-\alpha) = \begin{bmatrix} \cos(-\alpha) & 0 & \sin(-\alpha) \\ 0 & 1 & 0 \\ -\sin(-\alpha) & 0 & \cos(-\alpha) \end{bmatrix} = \begin{bmatrix} \cos\alpha & 0 & -\sin\alpha \\ 0 & 1 & 0 \\ \sin\alpha & 0 & \cos\alpha \end{bmatrix} \rightarrow (1)$$

$$\left(\because \begin{matrix} \cos(-\alpha) = \cos\alpha \\ \sin(-\alpha) = -\sin\alpha \end{matrix}\right)$$

$$|F(\alpha)| = \begin{vmatrix} \cos\alpha & 0 & \sin\alpha \\ 0 & 1 & 0 \\ -\sin\alpha & 0 & \cos\alpha \end{vmatrix} = \cos\alpha(\cos\alpha) - 0 + \sin\alpha(\sin\alpha)$$

$$= cos^2\alpha + sin^2\alpha = 1$$

$$\begin{array}{cccc} 1 & 0 & 0 & 1 \\ 0 & \cos\alpha & \sin\alpha & 0 \\ 0 & -\sin\alpha & \cos\alpha & 0 \\ 1 & 0 & 0 & 1 \end{array} \Rightarrow$$

$$adjF(\alpha) = \begin{bmatrix} \cos\alpha - 0 & 0-0 & 0-\sin\alpha \\ 0-0 & cos^2\alpha + sin^2\alpha & 0-0 \\ 0+\sin\alpha & 0-0 & \cos\alpha - 0 \end{bmatrix} = \begin{bmatrix} \cos\alpha & 0 & -\sin\alpha \\ 0 & 1 & 0 \\ \sin\alpha & 0 & \cos\alpha \end{bmatrix}$$

$$[F(\alpha)]^{-1} = \frac{1}{|F(\alpha)|} adj\, F(\alpha) \Rightarrow [F(\alpha)]^{-1} = \frac{1}{1}\begin{bmatrix} \cos\alpha & 0 & -\sin\alpha \\ 0 & 1 & 0 \\ \sin\alpha & 0 & \cos\alpha \end{bmatrix}$$

$$[F(\alpha)]^{-1} = \begin{bmatrix} \cos\alpha & 0 & -\sin\alpha \\ 0 & 1 & 0 \\ \sin\alpha & 0 & \cos\alpha \end{bmatrix} \rightarrow (2)$$

From (1) & (2), $[F(\alpha)]^{-1} = F(-\alpha)$

4. If $A = \begin{bmatrix} 5 & 3 \\ -1 & -2 \end{bmatrix}$, show that $A^2 - 3A - 7I_2 = 0_2$. Hence find A^{-1}

$$A^2 = \begin{bmatrix} 5 & 3 \\ -1 & -2 \end{bmatrix}\begin{bmatrix} 5 & 3 \\ -1 & -2 \end{bmatrix} = \begin{bmatrix} 25-3 & 15-6 \\ -5+2 & -3+4 \end{bmatrix} = \begin{bmatrix} 22 & 9 \\ -3 & 1 \end{bmatrix}$$

$$-3A = -3\begin{bmatrix} 5 & 3 \\ -1 & -2 \end{bmatrix} = \begin{bmatrix} -15 & -9 \\ 3 & 6 \end{bmatrix}$$

$$-7I_2 = -7\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} -7 & 0 \\ 0 & -7 \end{bmatrix}$$

$$A^2 - 3A - 7I_2 = \begin{bmatrix} 22 & 9 \\ -3 & 1 \end{bmatrix} + \begin{bmatrix} -15 & -9 \\ 3 & 6 \end{bmatrix} + \begin{bmatrix} -7 & 0 \\ 0 & -7 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} = 0_2$$

$A^2 - 3A - 7I_2 = 0_2$ pre-multiply by A^{-1} on both sides, we get

$$A - 3I_2 - 7A^{-1} = 0_2 \Rightarrow 7A^{-1} = A - 3I_2 \Rightarrow A^{-1} = \frac{1}{7}[A - 3I_2]$$

$$A^{-1} = \frac{1}{7}\left\{\begin{bmatrix} 5 & 3 \\ -1 & -2 \end{bmatrix} - 3\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\right\} = \frac{1}{7}\left\{\begin{bmatrix} 5 & 3 \\ -1 & -2 \end{bmatrix} + \begin{bmatrix} -3 & 0 \\ 0 & -3 \end{bmatrix}\right\} = \frac{1}{7}\begin{bmatrix} 2 & 3 \\ -1 & -5 \end{bmatrix}$$

EXAMPLE 1.10 If $A = \begin{bmatrix} 4 & 3 \\ 2 & 5 \end{bmatrix}$, find x and y such that $A^2 + xA + yI_2 = 0_2$.

Hence find A^{-1}

$$A^2 = \begin{bmatrix} 4 & 3 \\ 2 & 5 \end{bmatrix}\begin{bmatrix} 4 & 3 \\ 2 & 5 \end{bmatrix} = \begin{bmatrix} 16+6 & 12+15 \\ 8+10 & 6+25 \end{bmatrix} = \begin{bmatrix} 22 & 27 \\ 18 & 31 \end{bmatrix}$$

$$xA = x\begin{bmatrix} 4 & 3 \\ 2 & 5 \end{bmatrix} = \begin{bmatrix} 4x & 3x \\ 2x & 5x \end{bmatrix} \quad ; \quad yI_2 = y\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} y & 0 \\ 0 & y \end{bmatrix}$$

$$A^2 + xA + yI_2 = 0_2 \Rightarrow \begin{bmatrix} 22 & 27 \\ 18 & 31 \end{bmatrix} + \begin{bmatrix} 4x & 3x \\ 2x & 5x \end{bmatrix} + \begin{bmatrix} y & 0 \\ 0 & y \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} 22+4x+y & 27+3x \\ 18+2x & 31+5x+y \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$$

$$\Rightarrow \begin{cases} 27 + 3x = 0 \Rightarrow 3x = -27 \Rightarrow \boxed{x = -9} \\ 22 + 4x + y = 0 \Rightarrow 22 + 4(-9) + y = 0 \Rightarrow 22 - 36 + y = 0 \Rightarrow \boxed{y = 14} \end{cases}$$

$$A^2 + xA + yI_2 = 0_2 \Rightarrow A^2 - 9A + 14I_2 = 0_2$$

pre-multiply by A^{-1} on both sides, we get

$$A - 9I_2 + 14A^{-1} = 0_2 \Rightarrow 14A^{-1} = 9I_2 - A \Rightarrow A^{-1} = \frac{1}{14}[9I_2 - A]$$

$$A^{-1} = \frac{1}{14}\left\{9\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} - \begin{bmatrix} 4 & 3 \\ 2 & 5 \end{bmatrix}\right\} = \frac{1}{14}\left\{\begin{bmatrix} 9 & 0 \\ 0 & 9 \end{bmatrix} - \begin{bmatrix} 4 & 3 \\ 2 & 5 \end{bmatrix}\right\} = \frac{1}{14}\begin{bmatrix} 5 & -3 \\ -2 & 4 \end{bmatrix}$$

5. If $A = \frac{1}{9}\begin{bmatrix} -8 & 1 & 4 \\ 4 & 4 & 7 \\ 1 & -8 & 4 \end{bmatrix}$, prove that $A^{-1} = A^T$

$$A^T = \frac{1}{9}\begin{bmatrix} -8 & 4 & 1 \\ 1 & 4 & 8 \\ 4 & 7 & 4 \end{bmatrix} \rightarrow (1)$$

$$|A| = \left(\frac{1}{9}\right)^3 [-8(16-56) - 1(16-7) + 4(7-16)] \qquad \boxed{(\because |KA| = K^n|A|)}$$

$$= \frac{1}{729}[-576 - 9 - 144] = \frac{1}{729}[-729] = -1$$

$$\begin{array}{cccc} 4 & -8 & 1 & 4 \\ 7 & 4 & 4 & 7 \\ 4 & 1 & -8 & 4 \\ 4 & -8 & 1 & 4 \end{array} \Rightarrow \qquad \boxed{adj(\lambda A) = \lambda^{n-1} adj(A)}$$

$$adj\, A = \left(\frac{1}{9}\right)^{3-1}\begin{bmatrix} 16+56 & -32-4 & 7-16 \\ 7-16 & -32-4 & 16+56 \\ -32-4 & 1-64 & -32-4 \end{bmatrix} = \frac{1}{81}\begin{bmatrix} 72 & -36 & -9 \\ -9 & -36 & 72 \\ -36 & -63 & -36 \end{bmatrix}$$

$$= \frac{1}{9}\begin{bmatrix} 8 & -4 & -1 \\ -1 & -4 & 8 \\ -4 & -7 & -4 \end{bmatrix}$$

$$A^{-1} = \frac{1}{|A|} adj\, A \Rightarrow A^{-1} = \frac{1}{-1}\cdot\frac{1}{9}\begin{bmatrix} 8 & -4 & -1 \\ -1 & -4 & 8 \\ -4 & -7 & -4 \end{bmatrix} = \frac{1}{9}\begin{bmatrix} -8 & 4 & 1 \\ 1 & 4 & 8 \\ 4 & 7 & 4 \end{bmatrix} \rightarrow (2)$$

From (1) & (2), $A^{-1} = A^T$

6. If $A = \begin{bmatrix} 8 & -4 \\ -5 & 3 \end{bmatrix}$, verify that $A(adjA) = (adjA)A = |A|I_2$

$|A| = 24 - 20 = 4$

$|A|I_2 = 4\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 4 & 0 \\ 0 & 4 \end{bmatrix} \to (1)$

$adjA = \begin{bmatrix} 3 & 4 \\ 5 & 8 \end{bmatrix}$

$A(adjA) = \begin{bmatrix} 8 & -4 \\ -5 & 3 \end{bmatrix}\begin{bmatrix} 3 & 4 \\ 5 & 8 \end{bmatrix} = \begin{bmatrix} 4 & 0 \\ 0 & 4 \end{bmatrix} \to (2)$

$(adjA)A = \begin{bmatrix} 3 & 4 \\ 5 & 8 \end{bmatrix}\begin{bmatrix} 8 & -4 \\ -5 & 3 \end{bmatrix} = \begin{bmatrix} 4 & 0 \\ 0 & 4 \end{bmatrix} \to (3)$

From (1),(2) & (3), $A(adjA) = (adjA)A = |A|I_2$

EXAMPLE 1.1 If $A = \begin{bmatrix} 8 & -6 & 2 \\ -6 & 7 & -4 \\ 2 & -4 & 3 \end{bmatrix}$, verify that $A(adjA) = (adjA)A = |A|I_3$

$|A| = 8(21-16) - (-6)(-18+8) + 2(24-14) = 40 - 60 + 20 = 0$

$|A|I_3 = 0\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \to (1)$

$$\begin{array}{cccc} 7 & -4 & -6 & 7 \\ -4 & 3 & 2 & -4 \\ -6 & 2 & 8 & -6 \\ 7 & -4 & -6 & 7 \end{array} \Rightarrow$$

$adj\ A = \begin{bmatrix} 21-16 & -8+18 & 24-14 \\ -8+18 & 24-4 & -12+32 \\ 24-14 & -12+32 & 56-36 \end{bmatrix} = \begin{bmatrix} 5 & 10 & 10 \\ 10 & 20 & 20 \\ 10 & 20 & 20 \end{bmatrix}$

$A(adjA) = \begin{bmatrix} 8 & -6 & 2 \\ -6 & 7 & -4 \\ 2 & -4 & 3 \end{bmatrix}\begin{bmatrix} 5 & 10 & 10 \\ 10 & 20 & 20 \\ 10 & 20 & 20 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \to (2)$

$(adjA)A = \begin{bmatrix} 5 & 10 & 10 \\ 10 & 20 & 20 \\ 10 & 20 & 20 \end{bmatrix}\begin{bmatrix} 8 & -6 & 2 \\ -6 & 7 & -4 \\ 2 & -4 & 3 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \to (3)$

From (1),(2) & (3), $A(adjA) = (adjA)A = |A|I_3$

7. If $A = \begin{bmatrix} 3 & 2 \\ 7 & 5 \end{bmatrix}$ and $B = \begin{bmatrix} -1 & -3 \\ 5 & 2 \end{bmatrix}$ verify that $(AB)^{-1} = B^{-1}A^{-1}$

$AB = \begin{bmatrix} 3 & 2 \\ 7 & 5 \end{bmatrix}\begin{bmatrix} -1 & -3 \\ 5 & 2 \end{bmatrix} = \begin{bmatrix} -3+10 & -9+4 \\ -7+25 & -21+10 \end{bmatrix} = \begin{bmatrix} 7 & -5 \\ 18 & -11 \end{bmatrix}$

$|AB| = -77 + 90 = 13$; $adj(AB) = \begin{bmatrix} -11 & 5 \\ -18 & 7 \end{bmatrix}$

$(AB)^{-1} = \frac{1}{|AB|} adj(AB) = \frac{1}{13}\begin{bmatrix} -11 & 5 \\ -18 & 7 \end{bmatrix} \to (1)$

$|B| = -2 + 15 = 13$; $adj\ B = \begin{bmatrix} 2 & -2 \\ -7 & -1 \end{bmatrix}$; $B^{-1} = \frac{1}{|B|} adj\ B = \frac{1}{13}\begin{bmatrix} 2 & 3 \\ -5 & -1 \end{bmatrix}$

$|A| = 15 - 14 = 1$; $adj\ A = \begin{bmatrix} 5 & -2 \\ -7 & 3 \end{bmatrix}$

$A^{-1} = \frac{1}{|A|} adj\ A = \frac{1}{1}\begin{bmatrix} 5 & -2 \\ -7 & 3 \end{bmatrix} = \begin{bmatrix} 5 & -2 \\ -7 & 3 \end{bmatrix}$

$B^{-1}A^{-1} = \frac{1}{13}\begin{bmatrix} 2 & 3 \\ -5 & -1 \end{bmatrix}\begin{bmatrix} 5 & -2 \\ -7 & 3 \end{bmatrix} = \frac{1}{13}\begin{bmatrix} 10-21 & -4+9 \\ -25+7 & 10-3 \end{bmatrix}$

$= \frac{1}{13}\begin{bmatrix} -11 & 5 \\ -18 & 7 \end{bmatrix} \to (2)$

From (1) & (2), $(AB)^{-1} = B^{-1}A^{-1}$

EXAMPLE 1.9 If $A = \begin{bmatrix} 0 & -3 \\ 1 & 4 \end{bmatrix}$ and $B = \begin{bmatrix} -2 & -3 \\ 0 & -1 \end{bmatrix}$ verify that $(AB)^{-1} = B^{-1}A^{-1}$

8. If $adj\ A = \begin{bmatrix} 2 & -4 & 2 \\ -3 & 12 & -7 \\ -2 & 0 & 2 \end{bmatrix}$, find A.

$$A = \pm \frac{1}{\sqrt{|adjA|}} adj(adjA)$$

$|adjA| = 2(24-0) - (-4)(-6-14) + 2(0+24) = 48 - 80 + 48 = 16$

$\sqrt{|adjA|} = \sqrt{16} = 4$

$$\begin{array}{cccc} 12 & 0 & -4 & 12 \\ -7 & 2 & 2 & -7 \\ -3 & -2 & 2 & -3 \\ 12 & 0 & -4 & 12 \end{array} \Rightarrow adj(adjA) = \begin{bmatrix} 24-0 & 0+8 & 28-24 \\ 14+6 & 4+4 & -6+14 \\ 0+24 & 8-0 & 24-12 \end{bmatrix}$$

$$= \begin{bmatrix} 24 & 8 & 4 \\ 20 & 8 & 8 \\ 24 & 8 & 12 \end{bmatrix}$$

$A = \pm \frac{1}{\sqrt{|adjA|}} adj(adjA) = \pm\frac{1}{4}\begin{bmatrix} 24 & 8 & 4 \\ 20 & 8 & 8 \\ 24 & 8 & 12 \end{bmatrix} = \pm\begin{bmatrix} 6 & 2 & 1 \\ 5 & 2 & 2 \\ 6 & 2 & 3 \end{bmatrix}$

EXAMPLE1.5 Find a matrix A if $adj\ A = \begin{bmatrix} 7 & 7 & -7 \\ -1 & 11 & 7 \\ 11 & 5 & 7 \end{bmatrix}$

9. If $adj\ A = \begin{bmatrix} 0 & -2 & 0 \\ 6 & 2 & -6 \\ -3 & 0 & 6 \end{bmatrix}$, find A^{-1}.

$$\boxed{A^{-1} = \pm \frac{1}{\sqrt{|adjA|}} adjA}$$

$|adjA| = 0(12-0) - (-2)(36-18) + 0(0+6) = 0 + 36 + 0 = 36$

$\sqrt{|adjA|} = \sqrt{36} = 6$

$A^{-1} = \pm \frac{1}{\sqrt{|adjA|}} adjA = \pm \frac{1}{6} \begin{bmatrix} 0 & -2 & 0 \\ 6 & 2 & -6 \\ -3 & 0 & 6 \end{bmatrix}$

EXAMPLE 1.6 If $adj\ A = \begin{bmatrix} -1 & 2 & 2 \\ 1 & 1 & 2 \\ 2 & 2 & 1 \end{bmatrix}$, find A^{-1}.

10.Find adj(adjA) if $adj\ A = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 2 & 0 \\ -1 & 0 & 1 \end{bmatrix}$

$\begin{matrix} 2 & 0 & 0 & 2 \\ 0 & 1 & 1 & 0 \\ 0 & -1 & 1 & 0 \\ 2 & 0 & 0 & 2 \end{matrix} \Rightarrow \quad adj(adjA) = \begin{bmatrix} 2-0 & 0-0 & 0-2 \\ 0-0 & 1+1 & 0-0 \\ 0+2 & 0-0 & 2-0 \end{bmatrix} = \begin{bmatrix} 2 & 0 & -2 \\ 0 & 2 & 0 \\ 2 & 0 & 2 \end{bmatrix}$

11.$A = \begin{bmatrix} 1 & \tan x \\ -\tan x & 1 \end{bmatrix}$, show that $A^T A^{-1} = \begin{bmatrix} \cos 2x & -\sin 2x \\ \sin 2x & \cos 2x \end{bmatrix}$

$A^T = \begin{bmatrix} 1 & -\tan x \\ \tan x & 1 \end{bmatrix}$

$|A| = 1 + tan^2 x = sec^2 x = \frac{1}{cos^2 x}$

$adjA = \begin{bmatrix} 1 & -\tan x \\ \tan x & 1 \end{bmatrix}$

$A^{-1} = \frac{1}{|A|} adj\ A = \frac{1}{\frac{1}{cos^2 x}} \begin{bmatrix} 1 & -\tan x \\ \tan x & 1 \end{bmatrix} = cos^2 x \begin{bmatrix} 1 & -\tan x \\ \tan x & 1 \end{bmatrix}$

$= \begin{bmatrix} cos^2 x & -sinxcosx \\ sinxcosx & cos^2 x \end{bmatrix}$

$A^T A^{-1} = \begin{bmatrix} 1 & -\tan x \\ \tan x & 1 \end{bmatrix} \begin{bmatrix} cos^2 x & -sinxcosx \\ sinxcosx & cos^2 x \end{bmatrix}$

$= \begin{bmatrix} cos^2 x - sin^2 x & -sinxcosx - sinxcosx \\ sinxcosx + sinxcosx & cos^2 x - sin^2 x \end{bmatrix} = \begin{bmatrix} \cos 2x & -\sin 2x \\ \sin 2x & \cos 2x \end{bmatrix}$

12.Find the matrix A for which $A \begin{bmatrix} 5 & 3 \\ -1 & -2 \end{bmatrix} = \begin{bmatrix} 14 & 7 \\ 7 & 7 \end{bmatrix}$

$A \begin{bmatrix} 5 & 3 \\ -1 & -2 \end{bmatrix} = \begin{bmatrix} 14 & 7 \\ 7 & 7 \end{bmatrix} \Rightarrow AB = \begin{bmatrix} 14 & 7 \\ 7 & 7 \end{bmatrix} \Rightarrow A = \begin{bmatrix} 14 & 7 \\ 7 & 7 \end{bmatrix} B^{-1} \rightarrow (1)$

$B = \begin{bmatrix} 5 & 3 \\ -1 & -2 \end{bmatrix} \Rightarrow |B| = -10 + 3 = -7\ ; adj\ B = \begin{bmatrix} -2 & -3 \\ 1 & 5 \end{bmatrix}$

$B^{-1} = \frac{1}{|B|} adjB \Rightarrow B^{-1} = \frac{1}{-7} \begin{bmatrix} -2 & -3 \\ 1 & 5 \end{bmatrix}$

$(1) \Rightarrow A = \begin{bmatrix} 14 & 7 \\ 7 & 7 \end{bmatrix} \frac{1}{-7} \begin{bmatrix} -2 & -3 \\ 1 & 5 \end{bmatrix} = -\frac{1}{7} \begin{bmatrix} -28+7 & -42+35 \\ -14+7 & -21+35 \end{bmatrix}$

$= -\frac{1}{7} \begin{bmatrix} -21 & -7 \\ -7 & 14 \end{bmatrix} = \begin{bmatrix} 3 & 1 \\ 1 & -2 \end{bmatrix} \qquad \therefore A = \begin{bmatrix} 3 & 1 \\ 1 & -2 \end{bmatrix}$

13. Given $A = \begin{bmatrix} 1 & -1 \\ 2 & 0 \end{bmatrix}, B = \begin{bmatrix} 3 & -2 \\ 1 & 1 \end{bmatrix}$ and $C = \begin{bmatrix} 1 & 1 \\ 2 & 2 \end{bmatrix}$, find a matrix X such that $AXB = C$.

$$AXB = C$$

Pre multipiying by A^{-1} and post multiplying by $B^{-1} \Rightarrow X = A^{-1} C B^{-1} \rightarrow (1)$

$|A| = 0 + 2 = 2\ ; adj\ A = \begin{bmatrix} 0 & 1 \\ -2 & 1 \end{bmatrix};\ A^{-1} = \frac{1}{|A|} adj\ A = \frac{1}{2} \begin{bmatrix} 0 & 1 \\ -2 & 1 \end{bmatrix}$

$|B| = 3 + 2 = 5\ ; adj\ B = \begin{bmatrix} 1 & 2 \\ -1 & 3 \end{bmatrix} \quad A^{-1} = \frac{1}{|A|} adj\ A = \frac{1}{5} \begin{bmatrix} 1 & 2 \\ -1 & 3 \end{bmatrix}$

$(1) \Rightarrow X = \frac{1}{2} \begin{bmatrix} 0 & 1 \\ -2 & 1 \end{bmatrix} \begin{bmatrix} 1 & 1 \\ 2 & 2 \end{bmatrix} \frac{1}{5} \begin{bmatrix} 1 & 2 \\ -1 & 3 \end{bmatrix} = \frac{1}{10} \begin{bmatrix} 0+2 & 0+2 \\ -2+2 & -2+2 \end{bmatrix} \begin{bmatrix} 1 & 2 \\ -1 & 3 \end{bmatrix}$

$= \frac{1}{10} \begin{bmatrix} 2 & 2 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} 1 & 2 \\ -1 & 3 \end{bmatrix} = \frac{1}{10} \begin{bmatrix} 2-2 & 4+6 \\ 0 & 0 \end{bmatrix} = \frac{1}{10} \begin{bmatrix} 0 & 10 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$

14.If $A = \begin{bmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{bmatrix}$, show that $A^{-1} = \frac{1}{2}(A^2 - 3I)$

$|A| = 0 - 1(0-1) + 1(1-0) = 1 + 1 = 2$

$\begin{matrix} 0 & 1 & 1 & 0 \\ 1 & 0 & 1 & 1 \\ 1 & 1 & 0 & 1 \\ 0 & 1 & 1 & 0 \end{matrix} \Rightarrow adj\ A = \begin{bmatrix} 0-1 & 1-0 & 1-0 \\ 1-0 & 0-1 & 1-0 \\ 1-0 & 1-0 & 0-1 \end{bmatrix} = \begin{bmatrix} -1 & 1 & 1 \\ 1 & -1 & 1 \\ 1 & 1 & -1 \end{bmatrix}$

$A^{-1} = \frac{1}{|A|} adj\ A = \frac{1}{2} \begin{bmatrix} -1 & 1 & 1 \\ 1 & -1 & 1 \\ 1 & 1 & -1 \end{bmatrix} \rightarrow (1)$

$$A^2 = \begin{bmatrix}0 & 1 & 1\\1 & 0 & 1\\1 & 1 & 0\end{bmatrix}\begin{bmatrix}0 & 1 & 1\\1 & 0 & 1\\1 & 1 & 0\end{bmatrix} = \begin{bmatrix}0+1+1 & 0+0+1 & 0+1+0\\0+0+1 & 1+0+1 & 1+0+0\\0+1+0 & 1+0+0 & 1+1+0\end{bmatrix} = \begin{bmatrix}2 & 1 & 1\\1 & 2 & 1\\1 & 1 & 2\end{bmatrix}$$

$$\frac{1}{2}(A^2 - 3I) = \frac{1}{2}\left(\begin{bmatrix}2 & 1 & 1\\1 & 2 & 1\\1 & 1 & 2\end{bmatrix} - 3\begin{bmatrix}1 & 0 & 0\\0 & 1 & 0\\0 & 0 & 1\end{bmatrix}\right)$$

$$= \frac{1}{2}\left(\begin{bmatrix}2 & 1 & 1\\1 & 2 & 1\\1 & 1 & 2\end{bmatrix} + \begin{bmatrix}-3 & 0 & 0\\0 & -3 & 0\\0 & 0 & -3\end{bmatrix}\right) = \frac{1}{2}\begin{bmatrix}-1 & 1 & 1\\1 & -1 & 1\\1 & 1 & -1\end{bmatrix} \to (2)$$

From (1) & (2), $A^{-1} = \frac{1}{2}(A^2 - 3I)$

15. Decrypt the received encoded message $[2 \quad -3], [20 \quad 4]$ with the encryption matrix $\begin{bmatrix}-1 & -1\\2 & 1\end{bmatrix}$ and the decryption matrix as its inverse, where the system of code are described by the numbers 1-26 to the letters A-Z respectively, and the number 0 to a blank space.

$A = [2 \quad -3], B = [20 \quad 4]$

Encoded matrix C= $\begin{bmatrix}-1 & -1\\2 & 1\end{bmatrix} \Longrightarrow$ Decoded matrix = C^{-1}

$|C| = -1 + 2 = 1$; $adj\ C = \begin{bmatrix}1 & 1\\-2 & -1\end{bmatrix}$; $C^{-1} = \frac{1}{|C|} adj\ C = \begin{bmatrix}1 & 1\\-2 & -1\end{bmatrix}$

Received message= $(AC^{-1})(BC^{-1})$

$= \left\{[2 \quad -3]\begin{bmatrix}1 & 1\\-2 & -1\end{bmatrix}\right\}\left\{[20 \quad 4]\begin{bmatrix}1 & 1\\-2 & -1\end{bmatrix}\right\}$

$= [2+6 \quad 2+3][20-8 \quad 20-4] = [8 \quad 5][12 \quad 16] = HELP$

(∵ In alphabetical order, 8=H, 5=E, 12=L, 16=P)

EXAMPLE 1.8 Verify the property $(A^T)^{-1} = (A^{-1})^T$ with $A = \begin{bmatrix}2 & 9\\1 & 7\end{bmatrix}$

$A^T = \begin{bmatrix}2 & 1\\9 & 7\end{bmatrix}$; $|A^T| = 14 - 9 = 5$; $adj(A^T) = \begin{bmatrix}7 & -1\\-9 & 2\end{bmatrix}$

$(A^T)^{-1} = \frac{1}{|A^T|} adj(A^T) = \frac{1}{5}\begin{bmatrix}7 & -1\\-9 & 2\end{bmatrix} \to (1)$

$|A| = 14 - 9 = 5$; $adj\ A = \begin{bmatrix}7 & -9\\-1 & 2\end{bmatrix}$; $A^{-1} = \frac{1}{|A|} adj\ A = \frac{1}{5}\begin{bmatrix}7 & -9\\-1 & 2\end{bmatrix}$

$(A^{-1})^T = \frac{1}{5}\begin{bmatrix}7 & -1\\-9 & 2\end{bmatrix} \to (2)$

From (1) & (2), $(A^T)^{-1} = (A^{-1})^T$

EXAMPLE 1.11 Prove that $\begin{bmatrix}\cos\theta & -\sin\theta\\\sin\theta & \cos\theta\end{bmatrix}$ is orthogonal.

A square matrix A is called orthogonal if $\boxed{AA^T = A^TA = I_n}$

$A = \begin{bmatrix}\cos\theta & -\sin\theta\\\sin\theta & \cos\theta\end{bmatrix} \Rightarrow A^T = \begin{bmatrix}\cos\theta & \sin\theta\\-\sin\theta & \cos\theta\end{bmatrix}$

$AA^T = \begin{bmatrix}\cos\theta & -\sin\theta\\\sin\theta & \cos\theta\end{bmatrix}\begin{bmatrix}\cos\theta & \sin\theta\\-\sin\theta & \cos\theta\end{bmatrix}$

$= \begin{bmatrix}cos^2\theta + sin^2\theta & \sin\theta\cos\theta - \sin\theta\cos\theta\\\sin\theta\cos\theta - \sin\theta\cos\theta & sin^2\theta + cos^2\theta\end{bmatrix} = \begin{bmatrix}1 & 0\\0 & 1\end{bmatrix} = I_2 \to (1)$

$A^TA = \begin{bmatrix}\cos\theta & \sin\theta\\-\sin\theta & \cos\theta\end{bmatrix}\begin{bmatrix}\cos\theta & -\sin\theta\\\sin\theta & \cos\theta\end{bmatrix}$

$= \begin{bmatrix}cos^2\theta + sin^2\theta & \sin\theta\cos\theta - \sin\theta\cos\theta\\\sin\theta\cos\theta - \sin\theta\cos\theta & sin^2\theta + cos^2\theta\end{bmatrix} = \begin{bmatrix}1 & 0\\0 & 1\end{bmatrix} = I_2 \to (2)$

From (1) & (2), $AA^T = A^TA = I_2$, hence A is orthogonal

EXAMPLE 1.12 If $A = \frac{1}{7}\begin{bmatrix}6 & -3 & a\\b & -2 & 6\\2 & c & 3\end{bmatrix}$ is orthogonal, find a, b and c and hence A^{-1}

A square matrix A is called orthogonal if $\boxed{AA^T = A^TA = I_n}$

$A = \frac{1}{7}\begin{bmatrix}6 & -3 & a\\b & -2 & 6\\2 & c & 3\end{bmatrix} \Rightarrow A^T = \frac{1}{7}\begin{bmatrix}6 & b & 2\\-3 & -2 & c\\a & 6 & 3\end{bmatrix}$

$AA^T = I_3 \Rightarrow \frac{1}{7}\begin{bmatrix}6 & -3 & a\\b & -2 & 6\\2 & c & 3\end{bmatrix}\frac{1}{7}\begin{bmatrix}6 & b & 2\\-3 & -2 & c\\a & 6 & 3\end{bmatrix} = \begin{bmatrix}1 & 0 & 0\\0 & 1 & 0\\0 & 0 & 1\end{bmatrix}$

$\Rightarrow \frac{1}{49}\begin{bmatrix}36+9+a^2 & 6b+6+6a & 12-3c+3a\\6b+6+6a & b^2+4+36 & 2b-2c+18\\12-3c+3a & 2b-2c+18 & 4+c^2+9\end{bmatrix} = \begin{bmatrix}1 & 0 & 0\\0 & 1 & 0\\0 & 0 & 1\end{bmatrix}$

$\Rightarrow \begin{bmatrix}a^2+45 & 6a+6b+6 & 3a-3c+12\\6a+6b+6 & b^2+40 & 2b-2c+18\\3a-3c+12 & 2b-2c+18 & c^2+13\end{bmatrix} = \begin{bmatrix}49 & 0 & 0\\0 & 49 & 0\\0 & 0 & 49\end{bmatrix}$

$$\Rightarrow \begin{cases} a^2+45=49 \Rightarrow a^2=4 \Rightarrow a=\pm 2 \\ b^2+40=49 \Rightarrow b^2=9 \Rightarrow b=\pm 3 \\ c^2+13=49 \Rightarrow c^2=36 \Rightarrow c=\pm 6 \\ 6a+6b+6=0 \Rightarrow a+b=-1 \\ 3a-3c+12=0 \Rightarrow a-c=-4 \\ 2b-2c+18=0 \Rightarrow b-c=-9 \end{cases} \Rightarrow \boxed{a=2, b=-3, c=6}$$

Since A is orthogonal, $A^{-1}=A^T \Rightarrow A^{-1}=\frac{1}{7}\begin{bmatrix} 6 & -3 & 2 \\ -3 & -2 & 6 \\ 2 & 6 & 3 \end{bmatrix}$

RANK OF A MATRIX

- **Rank of a matrix in minor method:** The rank of a matrix A is defined as the order of a highest order non-vanishing minor of the matrix A. It is denoted by $\rho(A)$.
- The rank of zero matrix is 0.
- The rank of the identity matrix I_n is n.
- If a matrix contains at-least one non-zero element then $\rho(A) \geq 1$.
- If A is an $m \times n$ matrix then $\rho(A) \leq min\{m,n\}$
- A square matrix A order n is invertible $\Leftrightarrow \rho(A)=n$
- If all the entries below the leading diagonal are zero, use minor method for finding the rank.
- **Rank of a matrix in row echelon form:** The rank of a matrix in row echelon form is the number of non-zero rows in it.
- The rank of the non-zero matrix is equal to the number of non-zero rows in a row-echelon form of the matrix.
- **Elementary matrix:** An elementary matrix is defined as a matrix which is obtained from an identity matrix by applying only one elementary transformation.
- **Gauss-Jordan Method:** Transforming a non-singular matrix A to the form I_n by applying elementary row operations is called Gauss-Jordan method.
- **Steps in finding A^{-1} by Gauss-Jordan method:**
  - Write matrix of the form $[A|I_n]$
  - Apply elementary row operations and convert $[A|I_n]$ *to* $[I_n|A^{-1}]$ and find A^{-1}.

RANK OF A MATRIX IN MINOR METHOD

EXERCISE 1.2

1.Find the rank of the following matrices by minor method:

(i) $\begin{bmatrix} 2 & -4 \\ -1 & 2 \end{bmatrix}$

Let $A=\begin{bmatrix} 2 & -4 \\ -1 & 2 \end{bmatrix}$. Order of A is 2×2. So $\rho(A) \leq min\{2,2\}=2$

$|A|=4-4=0 \Rightarrow \rho(A)<2$

Since A contains at-least one non-Zero element, $\rho(A)=1$.

(ii) $\begin{bmatrix} -1 & 3 \\ 4 & -7 \\ 3 & -4 \end{bmatrix}$

Let $A=\begin{bmatrix} -1 & 3 \\ 4 & -7 \\ 3 & -4 \end{bmatrix}$. Order of A is 3×2. So $\rho(A) \leq min\{3,2\}=2$

2 × 2 Minor:

$\begin{vmatrix} -1 & 3 \\ 4 & -7 \end{vmatrix} = 7-12=5 \neq 0 \Rightarrow \rho(A)=2$

(iii) $\begin{bmatrix} 1 & -2 & -1 & 0 \\ 3 & -6 & -3 & 1 \end{bmatrix}$

Let $A=\begin{bmatrix} 1 & -2 & -1 & 0 \\ 3 & -6 & -3 & 1 \end{bmatrix}$. Order of A is 2×4. So $\rho(A) \leq min\{2,4\}=2$

2 × 2 Minors:

$\begin{vmatrix} 1 & -2 \\ 3 & -6 \end{vmatrix} = 6-6=0$; $\begin{vmatrix} 1 & -1 \\ 3 & -3 \end{vmatrix} = -3+3=0$; $\begin{vmatrix} 1 & 0 \\ 3 & 1 \end{vmatrix} = 1-0=1 \neq 0$

$\therefore \rho(A)=2$

(iv) $\begin{bmatrix} 1 & -2 & 3 \\ 2 & 4 & -6 \\ 5 & 1 & -1 \end{bmatrix}$

Let $A=\begin{bmatrix} 1 & -2 & 3 \\ 2 & 4 & -6 \\ 5 & 1 & -1 \end{bmatrix}$. Order of A is 3×3. So $\rho(A) \leq min\{3,3\}=3$

$|A|=1(-4+6)-(-2)(-2+30)+3(2-20)=2+56-54=4 \neq 0$

$\therefore \rho(A)=3$

(v) $\begin{bmatrix} 0 & 1 & 2 & 1 \\ 0 & 2 & 4 & 3 \\ 8 & 1 & 0 & 2 \end{bmatrix}$

Let $A=\begin{bmatrix}0&1&2&1\\0&2&4&3\\8&1&0&2\end{bmatrix}$. Order of A is$3\times4$. So $\rho(A)\le min\{3,4\}=3$

3 × 3 Minors:

$\begin{vmatrix}0&1&2\\0&2&4\\8&1&0\end{vmatrix}=0-0+8(4-4)=0$; $\begin{vmatrix}0&1&1\\0&2&3\\8&1&2\end{vmatrix}0-0+8(3-2)=8\ne0$

$\therefore\rho(A)=3$

EXAMPLE 1.15 $\begin{bmatrix}3&2&5\\1&1&2\\3&3&6\end{bmatrix}$

Let $A=\begin{bmatrix}3&2&5\\1&1&2\\3&3&6\end{bmatrix}$. Order of A is$3\times3$. So $\rho(A)\le min\{3,3\}=3$

$|A|=0\quad(\because R_2:R_3)$

$\therefore\rho(A)<3$

2×2 *Minor:*

$\begin{vmatrix}3&2\\1&1\end{vmatrix}=3-2=1\ne0$. So $\rho(A)=2$

EXAMPLE1.15 $\begin{bmatrix}4&3&1&-2\\-3&-1&-2&4\\6&7&-1&2\end{bmatrix}$

Let $A=\begin{bmatrix}4&3&1&-2\\-3&-1&-2&4\\6&7&-1&2\end{bmatrix}$. Order of A is 3×4. So $\rho(A)\le min\{3,4\}=3$

3 × 3 Minors:

$\begin{vmatrix}4&3&1\\-3&-1&-2\\6&7&-1\end{vmatrix}=0$; $\begin{vmatrix}4&3&-2\\-3&-1&4\\6&7&2\end{vmatrix}=0$;

$\begin{vmatrix}4&1&-2\\-3&-2&4\\6&-1&2\end{vmatrix}=0$; $\begin{vmatrix}3&1&-2\\-1&-2&4\\6&-1&2\end{vmatrix}=0$

So $\rho(A)<3$

2 × 2 Minor:

$\begin{vmatrix}4&3\\-3&-1\end{vmatrix}=-4+9=5\ne0$. So $\rho(A)=2$

EXAMPLE 1.16 Find the rank of the following matrices which are in row-echelon form:

(i) $\begin{bmatrix}2&0&-7\\0&3&1\\0&0&1\end{bmatrix}$

Let $A=\begin{bmatrix}2&0&-7\\0&3&1\\0&0&1\end{bmatrix}$. Order of A is$3\times3$. So $\rho(A)\le min\{3,3\}=3$

3 × 3 Minor:

$\begin{vmatrix}2&0&-7\\0&3&1\\0&0&1\end{vmatrix}=2(3-0)-0+0=6\ne0$. So $\rho(A)=3$

(ii) $\begin{bmatrix}-2&2&-1\\0&5&1\\0&0&0\end{bmatrix}$

Let $A=\begin{bmatrix}-2&2&-1\\0&5&1\\0&0&0\end{bmatrix}$. Order of A is$3\times3$. So $\rho(A)\le min\{3,3\}=3$

3 × 3 Minor:

$\begin{vmatrix}-2&2&-1\\0&5&1\\0&0&0\end{vmatrix}=0$ So $\rho(A)<3$

2 × 2 Minor:

$\begin{vmatrix}-2&2\\0&5\end{vmatrix}=-10-0=-10\ne0$. So $\rho(A)=2$

(iii) $\begin{bmatrix}6&0&-9\\0&2&0\\0&0&0\\0&0&0\end{bmatrix}$

Let $A=\begin{bmatrix}6&0&-9\\0&2&0\\0&0&0\\0&0&0\end{bmatrix}$. Order of A is$4\times3$. So $\rho(A)\le min\{4,3\}=3$

3 × 3 Minor:

All the 3×3 minors are equal to zero. So $\rho(A)<3$

2 × 2 Minor:

$\begin{vmatrix} 6 & 0 \\ 0 & 2 \end{vmatrix} = 12 - 0 = 12 \neq 0.$

So $\rho(A) = 2$

RANK OF A MATRIX IN ROW REDUCTION METHOD

2.Find the rank of the following matrices by row reduction (row-echelon) method:

(i) $\begin{bmatrix} 1 & 1 & 1 & 3 \\ 2 & -1 & 3 & 4 \\ 5 & -1 & 7 & 11 \end{bmatrix}$

Let $A = \begin{bmatrix} 1 & 1 & 1 & 3 \\ 2 & -1 & 3 & 4 \\ 5 & -1 & 7 & 11 \end{bmatrix}$

$\sim \begin{bmatrix} 1 & 1 & 1 & 3 \\ 0 & -3 & 1 & -2 \\ 0 & -6 & 2 & -4 \end{bmatrix} \begin{matrix} R_2 \to R_2 - 2R_1 \\ R_3 \to R_3 - 5R_1 \end{matrix}$

$\sim \begin{bmatrix} 1 & 1 & 1 & 3 \\ 0 & -3 & 1 & -2 \\ 0 & 0 & 0 & 0 \end{bmatrix} R_3 \to R_3 - 2R_2$

The equivalent matrix is in row-echelon form and the number of non-zero rows is 2. So $\rho(A) = 2$

(ii) $\begin{bmatrix} 1 & 2 & -1 \\ 3 & -1 & 2 \\ 1 & -2 & 3 \\ 1 & -1 & 1 \end{bmatrix}$

Let $A = \begin{bmatrix} 1 & 2 & -1 \\ 3 & -1 & 2 \\ 1 & -2 & 3 \\ 1 & -1 & 1 \end{bmatrix}$

$\sim \begin{bmatrix} 1 & 2 & -1 \\ 0 & -7 & 5 \\ 0 & -4 & 4 \\ 0 & -3 & 2 \end{bmatrix} \begin{matrix} R_2 \to R_2 - 3R_1 \\ R_3 \to R_3 - R_1 \\ R_4 \to R_4 - R_1 \end{matrix}$

$\sim \begin{bmatrix} 1 & 2 & -1 \\ 0 & -7 & 5 \\ 0 & 0 & 8 \\ 0 & 0 & -4 \end{bmatrix} \begin{matrix} R_3 \to 7R_3 - 4R_2 \\ R_4 \to 4R_4 - 3R_3 \end{matrix}$

$\sim \begin{bmatrix} 1 & 2 & -1 \\ 0 & -7 & 5 \\ 0 & 0 & 8 \\ 0 & 0 & 0 \end{bmatrix} R_4 \to R_4 + 2R_3$

The equivalent matrix is in row-echelon form and the number of non-zero rows is 3. So $\rho(A) = 3$

(iii) $\begin{bmatrix} 3 & -8 & 5 & 2 \\ 2 & -5 & 1 & 4 \\ -1 & 2 & 3 & -2 \end{bmatrix}$

Let $A = \begin{bmatrix} 3 & -8 & 5 & 2 \\ 2 & -5 & 1 & 4 \\ -1 & 2 & 3 & -2 \end{bmatrix}$

$\sim \begin{bmatrix} -1 & 2 & 3 & -2 \\ 2 & -5 & 1 & 4 \\ 3 & -8 & 5 & 2 \end{bmatrix} R_1 \to R_3$

$\sim \begin{bmatrix} -1 & 2 & 3 & -2 \\ 0 & -1 & 7 & 0 \\ 0 & -2 & 14 & -4 \end{bmatrix} \begin{matrix} R_2 \to R_2 + 2R_1 \\ R_3 \to R_3 + 3R_1 \end{matrix}$

$\sim \begin{bmatrix} -1 & 2 & 3 & -2 \\ 0 & -1 & 7 & 0 \\ 0 & 0 & 0 & -4 \end{bmatrix} R_3 \to R_3 - 2R_2$

The equivalent matrix is in row-echelon form and the number of non-zero rows is 3. So $\rho(A) = 3$

EXAMPLE 1.17 Find the rank of the matrix $\begin{bmatrix} 1 & 2 & 3 \\ 2 & 1 & 4 \\ 3 & 0 & 5 \end{bmatrix}$ by reducing it to a row - echelon form.

Let $A = \begin{bmatrix} 1 & 2 & 3 \\ 2 & 1 & 4 \\ 3 & 0 & 5 \end{bmatrix}$

$\sim \begin{bmatrix} 1 & 2 & 3 \\ 0 & -3 & -2 \\ 0 & -6 & -4 \end{bmatrix} \begin{matrix} R_2 \to R_2 - 2R_1 \\ R_3 \to R_3 - 3R_1 \end{matrix}$

$\sim \begin{bmatrix} 1 & 2 & 3 \\ 0 & -3 & -2 \\ 0 & 0 & 0 \end{bmatrix} R_3 \to R_3 - 2R_2$

The equivalent matrix is in row-echelon form and the number of non-zero rows is 2. So $\rho(A) = 2$

EXAMPLE 1.18 Find the rank of the matrix $\begin{bmatrix} 2 & -2 & 4 & 3 \\ -3 & 4 & -2 & -1 \\ 6 & 2 & -1 & 7 \end{bmatrix}$ by reducing it to a row -echelon form.

$$A = \begin{bmatrix} 2 & -2 & 4 & 3 \\ -3 & 4 & -2 & -1 \\ 6 & 2 & -1 & 7 \end{bmatrix}$$

$$\sim \begin{bmatrix} 2 & -2 & 4 & 3 \\ -6 & 8 & -4 & -2 \\ 6 & 2 & -1 & 7 \end{bmatrix} R_2 \to 2R_2$$

$$\sim \begin{bmatrix} 2 & -2 & 4 & 3 \\ 0 & 2 & 8 & 7 \\ 0 & 8 & -13 & -2 \end{bmatrix} \begin{matrix} R_2 \to R_2 + 3R_1 \\ R_3 \to R_3 - 3R_1 \end{matrix}$$

$$\sim \begin{bmatrix} 2 & -2 & 4 & 3 \\ 0 & 2 & 8 & 7 \\ 0 & 0 & -45 & -30 \end{bmatrix} R_3 \to R_3 - 4R_2$$

The equivalent matrix is in row-echelon form and the number of non-zero rows is 3. So $\rho(A) = 3$

GAUSS-JORDAN METHOD FOR FINDING INVERSE OF THE MATRIX

EXAMPLE 1.19 Show that the matrix $\begin{bmatrix} 3 & 1 & 4 \\ 2 & 0 & -1 \\ 5 & 2 & 1 \end{bmatrix}$ is non-singular and reduce it to the identity matrix by elementary row transformation.

Let $A = \begin{bmatrix} 3 & 1 & 4 \\ 2 & 0 & -1 \\ 5 & 2 & 1 \end{bmatrix}$

$|A| = 3(0+2) - 1(2+5) + 4(4-0) = 6 - 7 + 16 = 15 \neq 0$

∴ A is non-singular.

$$\begin{bmatrix} 3 & 1 & 4 \\ 2 & 0 & -1 \\ 5 & 2 & 1 \end{bmatrix} \sim \begin{bmatrix} 1 & \frac{1}{3} & \frac{4}{3} \\ 2 & 0 & -1 \\ 5 & 2 & 1 \end{bmatrix} R_1 \to \frac{1}{3}R_1$$

$$\sim \begin{bmatrix} 1 & \frac{1}{3} & \frac{4}{3} \\ 0 & -\frac{2}{3} & -\frac{11}{3} \\ 0 & \frac{1}{3} & -\frac{17}{3} \end{bmatrix} \begin{matrix} R_2 \to R_2 - 2R_1 \\ R_3 \to R_3 - 5R_1 \end{matrix}$$

$$\sim \begin{bmatrix} 1 & \frac{1}{3} & \frac{4}{3} \\ 0 & 1 & \frac{11}{2} \\ 0 & \frac{1}{3} & -\frac{17}{3} \end{bmatrix} R_2 \to \left(-\frac{3}{2}\right) R_2$$

$$\sim \begin{bmatrix} 1 & 0 & -\frac{1}{2} \\ 0 & 1 & \frac{11}{2} \\ 0 & 0 & -\frac{15}{2} \end{bmatrix} \begin{matrix} R_1 \to R_1 - \frac{1}{3}R_2 \\ R_3 \to R_3 - \frac{1}{3}R_2 \end{matrix}$$

$$\sim \begin{bmatrix} 1 & 0 & -\frac{1}{2} \\ 0 & 1 & \frac{11}{2} \\ 0 & 0 & 1 \end{bmatrix} R_3 \to \left(-\frac{2}{15}\right) R_3$$

$$\sim \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{matrix} R_1 \to R_1 - \frac{1}{2}R_3 \\ R_2 \to R_2 - \frac{11}{2}R_3 \end{matrix}$$

3.Find the inverse of each of the following by Gauss-Jordan method:

(i) $\begin{bmatrix} 2 & -1 \\ 5 & -2 \end{bmatrix}$

$$[A|I_2] = \left(\begin{array}{cc|cc} 2 & -1 & 1 & 0 \\ 5 & -2 & 0 & 1 \end{array}\right)$$

$$\sim \left(\begin{array}{cc|cc} 1 & -\frac{1}{2} & \frac{1}{2} & 0 \\ 5 & -2 & 0 & 1 \end{array}\right) R_2 \to \frac{1}{2}R_2$$

$$\sim \left(\begin{array}{cc|cc} 1 & -\frac{1}{2} & \frac{1}{2} & 0 \\ 0 & \frac{1}{2} & -\frac{5}{2} & 1 \end{array}\right) R_2 \to R_2 - 5R_1$$

$$\sim \left(\begin{array}{cc|cc} 1 & -\frac{1}{2} & \frac{1}{2} & 0 \\ 0 & 1 & -5 & 2 \end{array}\right) R_2 \to 2R_2$$

$$\sim\left(\begin{array}{cc|cc}1 & 0 & -2 & 1\\ 0 & 1 & -5 & 2\end{array}\right) R_1 \to R_1 + \frac{1}{2}R_2$$

$$= [I_2|A^{-1}]$$

$$\Rightarrow A^{-1} = \begin{bmatrix}-2 & 1\\ -5 & 2\end{bmatrix}$$

(ii) $\begin{bmatrix}1 & -1 & 0\\ 1 & 0 & -1\\ 6 & -2 & -3\end{bmatrix}$

$$[A|I_3] = \left(\begin{array}{ccc|ccc}1 & -1 & 0 & 1 & 0 & 0\\ 1 & 0 & -1 & 0 & 1 & 0\\ 6 & -2 & -3 & 0 & 0 & 1\end{array}\right)$$

$$\sim\left(\begin{array}{ccc|ccc}1 & -1 & 0 & 1 & 0 & 0\\ 0 & 1 & -1 & -1 & 1 & 0\\ 0 & 4 & -3 & -6 & 0 & 1\end{array}\right) \begin{array}{l}R_2 \to R_2 - R_1\\ R_3 \to R_3 - 6R_1\end{array}$$

$$\sim\left(\begin{array}{ccc|ccc}1 & -1 & 0 & 1 & 0 & 0\\ 0 & 1 & -1 & -1 & 1 & 0\\ 0 & 0 & 1 & -2 & -4 & 1\end{array}\right) R_3 \to R_3 - 4R_2$$

$$\sim\left(\begin{array}{ccc|ccc}1 & -1 & 0 & 1 & 0 & 0\\ 0 & 1 & 0 & -3 & -3 & 1\\ 0 & 0 & 1 & -2 & -4 & 1\end{array}\right) R_2 \to R_2 + R_3$$

$$\sim\left(\begin{array}{ccc|ccc}1 & 0 & 0 & -2 & -3 & 1\\ 0 & 1 & 0 & -3 & -3 & 1\\ 0 & 0 & 1 & -2 & -4 & 1\end{array}\right) R_1 \to R_1 + R_2$$

$$= [I_3|A^{-1}]$$

$$\Rightarrow A^{-1} = \begin{bmatrix}-2 & -3 & 1\\ -3 & -3 & 1\\ -2 & -4 & 1\end{bmatrix}$$

(iii) $\begin{bmatrix}1 & 2 & 3\\ 2 & 5 & 3\\ 1 & 0 & 8\end{bmatrix}$

$$[A|I_3] = \left(\begin{array}{ccc|ccc}1 & 2 & 3 & 1 & 0 & 0\\ 2 & 5 & 3 & 0 & 1 & 0\\ 1 & 0 & 8 & 0 & 0 & 1\end{array}\right)$$

$$\sim\left(\begin{array}{ccc|ccc}1 & 2 & 3 & 1 & 0 & 0\\ 0 & 1 & -3 & -2 & 1 & 0\\ 0 & -2 & 5 & -1 & 0 & 1\end{array}\right) \begin{array}{l}R_2 \to R_2 - 2R_1\\ R_3 \to R_3 - R_1\end{array}$$

$$\sim\left(\begin{array}{ccc|ccc}1 & 0 & 9 & 5 & -2 & 0\\ 0 & 1 & -3 & -2 & 1 & 0\\ 0 & 0 & -1 & -5 & 2 & 1\end{array}\right) \begin{array}{l}R_1 \to R_1 - 2R_2\\ R_3 \to R_3 + 2R_2\end{array}$$

$$\sim\left(\begin{array}{ccc|ccc}1 & 0 & 0 & -40 & 16 & 9\\ 0 & 1 & 0 & 14 & -5 & -3\\ 0 & 0 & -1 & -5 & 2 & 1\end{array}\right) \begin{array}{l}R_1 \to R_1 + 9R_3\\ R_2 \to R_2 - 3R_3\end{array}$$

$$\sim\left(\begin{array}{ccc|ccc}1 & 0 & 0 & -44 & 16 & 9\\ 0 & 1 & 0 & 13 & -5 & -3\\ 0 & 0 & 1 & 5 & -2 & -1\end{array}\right) R_3 \to (-)R_3$$

$$= [I_3|A^{-1}]$$

$$\Rightarrow A^{-1} = \begin{bmatrix}-40 & 16 & 9\\ 13 & -5 & -3\\ 5 & -2 & -1\end{bmatrix}$$

EXAMPLE 1.20 Find the inverse of the non-singular matrix $A = \begin{bmatrix}0 & 5\\ -1 & 6\end{bmatrix}$, by Gauss-Jordan method.

$$[A|I_2] = \left(\begin{array}{cc|cc}0 & 5 & 1 & 0\\ -1 & 6 & 0 & 1\end{array}\right)$$

$$\sim\left(\begin{array}{cc|cc}-1 & 6 & 0 & 1\\ 0 & 5 & 1 & 0\end{array}\right) R_1 \leftrightarrow R_2$$

$$\sim\left(\begin{array}{cc|cc}1 & -6 & 0 & -1\\ 0 & 5 & 1 & 0\end{array}\right) R_1 \to (-)R_1$$

$$\sim\left(\begin{array}{cc|cc}1 & -6 & 0 & -1\\ 0 & 1 & \frac{1}{5} & 0\end{array}\right) R_2 \to \frac{1}{5}R_2$$

$$\sim\left(\begin{array}{cc|cc}1 & 0 & \frac{6}{5} & -1\\ 0 & 1 & \frac{1}{5} & 0\end{array}\right) R_1 \to R_1 + 6R_2$$

$$= [I_2|A^{-1}] \qquad \Rightarrow A^{-1} = \begin{bmatrix}\frac{6}{5} & -1\\ \frac{1}{5} & 0\end{bmatrix} = \frac{1}{5}\begin{bmatrix}6 & -5\\ 1 & 0\end{bmatrix}$$

EXAMPLE 1.21 Find the inverse of $A = \begin{bmatrix}2 & 1 & 1\\ 3 & 2 & 1\\ 2 & 1 & 2\end{bmatrix}$, by Gauss-Jordan method.

$$[A|I_3] = \left(\begin{array}{ccc|ccc}2 & 1 & 1 & 1 & 0 & 0\\ 3 & 2 & 1 & 0 & 1 & 0\\ 2 & 1 & 2 & 0 & 0 & 1\end{array}\right)$$

$$\sim\left(\begin{array}{ccc|ccc}1 & \frac{1}{2} & \frac{1}{2} & \frac{1}{2} & 0 & 0\\ 3 & 2 & 1 & 0 & 1 & 0\\ 2 & 1 & 2 & 0 & 0 & 1\end{array}\right) R_1 \to \frac{1}{2}R_1$$

$$\sim\left(\begin{array}{ccc|ccc}1 & \frac{1}{2} & \frac{1}{2} & \frac{1}{2} & 0 & 0\\ 0 & \frac{1}{2} & -\frac{1}{2} & -\frac{3}{2} & 1 & 0\\ 0 & 0 & 1 & -1 & 0 & 1\end{array}\right) \begin{array}{l}R_2 \to R_2 - 3R_1\\ R_3 \to R_3 - 2R_1\end{array}$$

$$\sim\left(\begin{array}{ccc|ccc}1 & \frac{1}{2} & \frac{1}{2} & \frac{1}{2} & 0 & 0\\ 0 & 1 & -1 & -3 & 2 & 0\\ 0 & 0 & 1 & -1 & 0 & 1\end{array}\right) R_2 \to 2R_2$$

$$\sim\left(\begin{array}{ccc|ccc}1 & 0 & 1 & 2 & -1 & 0\\ 0 & 1 & -1 & -3 & 2 & 0\\ 0 & 0 & 1 & -1 & 0 & 1\end{array}\right) R_1 \to R_1 - \frac{1}{2}R_2$$

$$\sim\left(\begin{array}{ccc|ccc}1 & 0 & 0 & 3 & -1 & -1\\ 0 & 1 & 0 & -4 & 2 & 1\\ 0 & 0 & 1 & -1 & 0 & 1\end{array}\right) \begin{array}{l}R_1 \to R_1 - R_3\\ R_2 \to R_2 + R_3\end{array}$$

$$= [I_3|A^{-1}] \implies A^{-1} = \begin{bmatrix}3 & -1 & -1\\ -4 & 2 & 1\\ -1 & 0 & 1\end{bmatrix}$$

MATRIX INVERSION METHOD

EXERCISE 1.3

1.Solve the following system of linear equations by matrix inversion method.

$(i) 2x + 5y = -2, x + 2y = -3$

$$\begin{bmatrix}2 & 5\\ 1 & 2\end{bmatrix}\begin{bmatrix}x\\ y\end{bmatrix} = \begin{bmatrix}-2\\ -3\end{bmatrix} \implies AX = B \implies X = A^{-1}B$$

$$|A| = 4 - 5 = -1\ ; adj\ A = \begin{bmatrix}2 & -5\\ -1 & 2\end{bmatrix}$$

$$A^{-1} = \frac{1}{|A|}adjA = \frac{1}{-1}\begin{bmatrix}2 & -5\\ -1 & 2\end{bmatrix} = \begin{bmatrix}-2 & 5\\ 1 & -2\end{bmatrix}$$

$$X = A^{-1}B \implies \begin{bmatrix}x\\ y\end{bmatrix} = \begin{bmatrix}-2 & 5\\ 1 & -2\end{bmatrix}\begin{bmatrix}-2\\ -3\end{bmatrix} = \begin{bmatrix}4-15\\ -2+6\end{bmatrix} = \begin{bmatrix}-11\\ 4\end{bmatrix}$$

Solution: $(x, y) = (-11, 4)$

$(ii) 2x - y = 8, 3x + 2y = -2$

$$\begin{bmatrix}2 & -1\\ 3 & 2\end{bmatrix}\begin{bmatrix}x\\ y\end{bmatrix} = \begin{bmatrix}8\\ -2\end{bmatrix} \implies AX = B \implies X = A^{-1}B$$

$$|A| = 4 + 3 = 7\ ; adj\ A = \begin{bmatrix}2 & 1\\ -3 & 2\end{bmatrix}$$

$$A^{-1} = \frac{1}{|A|}adjA = \frac{1}{7}\begin{bmatrix}2 & 1\\ -3 & 2\end{bmatrix}$$

$$X = A^{-1}B \implies \begin{bmatrix}x\\ y\end{bmatrix} = \frac{1}{7}\begin{bmatrix}2 & 1\\ -3 & 2\end{bmatrix}\begin{bmatrix}8\\ -2\end{bmatrix} = \frac{1}{7}\begin{bmatrix}16-2\\ -24-4\end{bmatrix} = \frac{1}{7}\begin{bmatrix}14\\ -28\end{bmatrix} = \begin{bmatrix}2\\ -4\end{bmatrix}$$

Solution: $(x, y) = (2, -4)$

EXAMPLE 1.22 Solve the following system of linear equations by matrix inversion method: $5x + 2y = 3, 3x + 2y = 5$

$(iii) 2x + 3y - z = 9, x + y + z = 9, 3x - y - z = -1$

$$\begin{bmatrix}2 & 3 & -1\\ 1 & 1 & 1\\ 3 & -1 & -1\end{bmatrix}\begin{bmatrix}x\\ y\\ z\end{bmatrix} = \begin{bmatrix}9\\ 9\\ -1\end{bmatrix} \implies AX = B \implies X = A^{-1}B$$

$$|A| = 2(-1+1) - 3(-1-3) - 1(-1-3) = 12 + 4 = 16$$

$$\begin{array}{cccc}1 & -1 & 3 & 1\\ 1 & -1 & -1 & 1\\ 1 & 3 & 2 & 1\\ 1 & -1 & 3 & 1\end{array} \implies \text{adjA} = \begin{bmatrix}-1+1 & 1+3 & 3+1\\ 3+1 & -2+3 & -1-2\\ -1-3 & 9+2 & 2-3\end{bmatrix} = \begin{bmatrix}0 & 4 & 4\\ 4 & 1 & -3\\ -4 & 11 & -1\end{bmatrix}$$

$$A^{-1} = \frac{1}{|A|}adjA = \frac{1}{16}\begin{bmatrix}0 & 4 & 4\\ 4 & 1 & -3\\ -4 & 11 & -1\end{bmatrix}$$

$$X = A^{-1}B \implies \begin{bmatrix}x\\ y\\ z\end{bmatrix} = \frac{1}{16}\begin{bmatrix}0 & 4 & 4\\ 4 & 1 & -3\\ -4 & 11 & -1\end{bmatrix}\begin{bmatrix}9\\ 9\\ -1\end{bmatrix} = \frac{1}{16}\begin{bmatrix}0+36-4\\ 36+9+3\\ -36+99+1\end{bmatrix} = \frac{1}{16}\begin{bmatrix}32\\ 48\\ 64\end{bmatrix}$$

$$\begin{bmatrix}x\\ y\\ z\end{bmatrix} = \begin{bmatrix}2\\ 3\\ 4\end{bmatrix}$$ Solution : $(x, y, z) = (2,3,4)$

$(iv) x + y + z - 2 = 0, 6x - 4y + 5z - 31 = 0, 5x + 2y + 2z = 13$

$$\begin{bmatrix}1 & 1 & 1\\ 6 & -4 & 5\\ 5 & 2 & 2\end{bmatrix}\begin{bmatrix}x\\ y\\ z\end{bmatrix} = \begin{bmatrix}2\\ 31\\ 13\end{bmatrix} \implies AX = B \implies X = A^{-1}B$$

$$|A| = 1(-8-10) - 1(12-25) + 1(12+20) = -18 + 13 + 32 = 27$$

$$\begin{array}{cccc}-4 & 2 & 1 & -4\\ 5 & 2 & 1 & 5\\ 6 & 5 & 1 & 6\\ -4 & 2 & 1 & -4\end{array} \implies \text{adjA} = \begin{bmatrix}-8-10 & 2-2 & 5+4\\ 25-12 & 2-5 & 6-5\\ 12+20 & 5-2 & -4-6\end{bmatrix} = \begin{bmatrix}-18 & 0 & 9\\ 13 & -3 & 1\\ 32 & 3 & -10\end{bmatrix}$$

$$A^{-1} = \frac{1}{|A|} adjA = \frac{1}{27}\begin{bmatrix} -18 & 0 & 9 \\ 13 & -3 & 1 \\ 32 & 3 & -10 \end{bmatrix}$$

$$X = A^{-1}B \Rightarrow \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \frac{1}{27}\begin{bmatrix} -18 & 0 & 9 \\ 13 & -3 & 1 \\ 32 & 3 & -10 \end{bmatrix}\begin{bmatrix} 2 \\ 31 \\ 13 \end{bmatrix} = \frac{1}{27}\begin{bmatrix} -36+0+117 \\ 26-93+13 \\ 64+93-130 \end{bmatrix}$$

$$= \frac{1}{27}\begin{bmatrix} 81 \\ -54 \\ 27 \end{bmatrix} \Rightarrow \quad \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 3 \\ -2 \\ 1 \end{bmatrix}$$

Solution : $(x, y, z) = (3, -2, 1)$

EXAMPLE 1.23 Solve the following system of linear equations by matrix inversion method: $2x_1 + 3x_2 + 3x_3 = 5, x_1 - 2x_2 + x_3 = -4, 3x_1 - x_2 - 2x_3 = 3$

$$\begin{bmatrix} 2 & 3 & 3 \\ 1 & -2 & 1 \\ 3 & -1 & -2 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 5 \\ -4 \\ 3 \end{bmatrix} \Rightarrow AX = B \Rightarrow X = A^{-1}B$$

$$|A| = 2(4+1) - 3(-2-3) + 3(-1+6) = 10 + 15 + 15 = 40$$

$$\begin{array}{cccc} -2 & -1 & 3 & -2 \\ 1 & -2 & 3 & 1 \\ 1 & 3 & 2 & 1 \\ -2 & -1 & 3 & -2 \end{array} \Rightarrow \text{adjA} = \begin{bmatrix} 4+1 & -3+6 & 3+6 \\ 3+2 & -4-9 & 3-2 \\ -1+6 & 9+2 & -4-3 \end{bmatrix} = \begin{bmatrix} 5 & 3 & 9 \\ 5 & -13 & 1 \\ 5 & 11 & -7 \end{bmatrix}$$

$$A^{-1} = \frac{1}{|A|} adjA = \frac{1}{40}\begin{bmatrix} 5 & 3 & 9 \\ 5 & -13 & 1 \\ 5 & 11 & -7 \end{bmatrix}$$

$$X = A^{-1}B \Rightarrow \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \frac{1}{40}\begin{bmatrix} 5 & 3 & 9 \\ 5 & -13 & 1 \\ 5 & 11 & -7 \end{bmatrix}\begin{bmatrix} 5 \\ -4 \\ 3 \end{bmatrix} = \frac{1}{40}\begin{bmatrix} 25-12+27 \\ 25+52+3 \\ 25-44-21 \end{bmatrix} = \frac{1}{40}\begin{bmatrix} 40 \\ 80 \\ -40 \end{bmatrix}$$

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \\ -1 \end{bmatrix}$$

Solution : $(x, y, z) = (1, 2, -1)$

2.If $A = \begin{bmatrix} -5 & 1 & 3 \\ 7 & 1 & -5 \\ 1 & -1 & 1 \end{bmatrix}$ and $B = \begin{bmatrix} 1 & 1 & 2 \\ 3 & 2 & 1 \\ 2 & 1 & 3 \end{bmatrix}$, find the products AB and BA and hence solve the system of equations $x + y + 2z = 1, 3x + 2y + z = 7, 2x + y + 3z = 2$

$$AB = \begin{bmatrix} -5 & 1 & 3 \\ 7 & 1 & -5 \\ 1 & -1 & 1 \end{bmatrix}\begin{bmatrix} 1 & 1 & 2 \\ 3 & 2 & 1 \\ 2 & 1 & 3 \end{bmatrix} = \begin{bmatrix} -5+3+6 & -5+2+3 & -10+1+9 \\ 7+3-10 & 7+2-5 & 14+1-15 \\ 1-3+2 & 1-2+1 & 2-1+3 \end{bmatrix}$$

$$= \begin{bmatrix} 4 & 0 & 0 \\ 0 & 4 & 0 \\ 0 & 0 & 4 \end{bmatrix} = 4I_3$$

$$BA = \begin{bmatrix} 1 & 1 & 2 \\ 3 & 2 & 1 \\ 2 & 1 & 3 \end{bmatrix}\begin{bmatrix} -5 & 1 & 3 \\ 7 & 1 & -5 \\ 1 & -1 & 1 \end{bmatrix} = \begin{bmatrix} 4 & 0 & 0 \\ 0 & 4 & 0 \\ 0 & 0 & 4 \end{bmatrix} = 4I_3$$

$$AB = BA = 4I_3 \Rightarrow B^{-1} = \frac{1}{4}A \rightarrow (1)$$

$$\begin{bmatrix} 1 & 1 & 2 \\ 3 & 2 & 1 \\ 2 & 1 & 3 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 1 \\ 7 \\ 2 \end{bmatrix} \Rightarrow BX = C \Rightarrow X = B^{-1}C \rightarrow (2)$$

From (1) $\Rightarrow B^{-1} = \frac{1}{4}A = \frac{1}{4}\begin{bmatrix} -5 & 1 & 3 \\ 7 & 1 & -5 \\ 1 & -1 & 1 \end{bmatrix}$

From (2) $\Rightarrow X = B^{-1}C \Rightarrow \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \frac{1}{4}\begin{bmatrix} -5 & 1 & 3 \\ 7 & 1 & -5 \\ 1 & -1 & 1 \end{bmatrix}\begin{bmatrix} 1 \\ 7 \\ 2 \end{bmatrix} = \frac{1}{4}\begin{bmatrix} -5+7+6 \\ 7+7-10 \\ 1-7+2 \end{bmatrix}$

$$= \frac{1}{4}\begin{bmatrix} 8 \\ 4 \\ -4 \end{bmatrix} = \begin{bmatrix} 2 \\ 1 \\ -1 \end{bmatrix}$$

Solution : $(x, y, z) = (2, 1, -1)$

EXAMPLE 1.24 If $A = \begin{bmatrix} -4 & 4 & 4 \\ -7 & 1 & 3 \\ 5 & -3 & -1 \end{bmatrix}$ and $B = \begin{bmatrix} 1 & -1 & 1 \\ 1 & -2 & -2 \\ 2 & 1 & 3 \end{bmatrix}$, find the products AB and BA and hence solve the system of equations $x - y + z = 4, x - 2y - 2z = 9, 2x + y + 3z = 1$

3. A man is appointed in a job with a monthly salary of certain amount and a fixed amount of annual increment. If his salary was Rs.19800 per month at the end of the first month after 3 years of service and Rs.23400 per month at the end of the first month after 9 years of service, find his starting salary and his annual increment. (Use matrix inversion method)

Let his starting salary be x and the annual increment be y.

From the given data, $x + 3y = 19800 \rightarrow (1)\ ;\ x + 9y = 23400 \rightarrow (2)$

$$\begin{bmatrix} 1 & 3 \\ 1 & 9 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 19800 \\ 23400 \end{bmatrix} \Rightarrow AX = B \Rightarrow X = A^{-1}B$$

$$|A| = 9 - 3 = 6\ ;\ adj\ A = \begin{bmatrix} 9 & -3 \\ -1 & 1 \end{bmatrix}$$

$A^{-1} = \frac{1}{|A|} adjA = \frac{1}{6}\begin{bmatrix} 9 & -3 \\ -1 & 1 \end{bmatrix}$

$X = A^{-1}B \Longrightarrow \begin{bmatrix} x \\ y \end{bmatrix} = \frac{1}{6}\begin{bmatrix} 9 & -3 \\ -1 & 1 \end{bmatrix}\begin{bmatrix} 19800 \\ 23400 \end{bmatrix} = \frac{1}{6}\begin{bmatrix} 178200 - 70200 \\ -19800 + 23400 \end{bmatrix} = \frac{1}{6}\begin{bmatrix} 108000 \\ 3600 \end{bmatrix}$

$\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 18000 \\ 600 \end{bmatrix}$

Solution: His starting salary = Rs.18000 ; Annual increment = Rs.600

4. 4 men and 4 women can finish a piece of work jointly in 3 days while 2 men and 5 women can finish the same work jointly in 4 days. Find the time taken by one man alone and that of one woman alone to finish the same work by using matrix inversion method.

Let the number of days taken by a man and woman be a and y respectively.

Work finished by a man in one day = $\frac{1}{x}$

Work finished by a woman in one day = $\frac{1}{y}$

From the given data, $\frac{4}{x} + \frac{4}{y} = \frac{1}{3}$; $\frac{2}{x} + \frac{5}{y} = \frac{1}{4}$

Let $\frac{1}{x} = a$, $\frac{1}{y} = b$

$\frac{4}{x} + \frac{4}{y} = \frac{1}{3} \Longrightarrow 4a + 4b = \frac{1}{3} \Longrightarrow 12a + 12b = 1 \rightarrow (1)$

$\frac{2}{x} + \frac{5}{y} = \frac{1}{4} \Longrightarrow 2a + 5b = \frac{1}{4} \Longrightarrow 8a + 20b = 1 \rightarrow (2)$

$\begin{bmatrix} 12 & 12 \\ 8 & 20 \end{bmatrix}\begin{bmatrix} a \\ b \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \end{bmatrix} \Longrightarrow AX = B \Longrightarrow X = A^{-1}B$

$|A| = 240 - 96 = 144$; $adj\ A = \begin{bmatrix} 20 & -12 \\ -8 & 12 \end{bmatrix}$

$A^{-1} = \frac{1}{|A|} adjA = \frac{1}{144}\begin{bmatrix} 20 & -12 \\ -8 & 12 \end{bmatrix}$

$X = A^{-1}B \Longrightarrow \begin{bmatrix} a \\ b \end{bmatrix} = \frac{1}{144}\begin{bmatrix} 20 & -12 \\ -8 & 12 \end{bmatrix}\begin{bmatrix} 1 \\ 1 \end{bmatrix} = \frac{1}{144}\begin{bmatrix} 20 - 12 \\ -8 + 12 \end{bmatrix} = \frac{1}{144}\begin{bmatrix} 8 \\ 4 \end{bmatrix} = \begin{bmatrix} \frac{1}{18} \\ \frac{1}{36} \end{bmatrix}$

$\begin{bmatrix} a \\ b \end{bmatrix} = \begin{bmatrix} \frac{1}{18} \\ \frac{1}{36} \end{bmatrix} \Longrightarrow x = \frac{1}{a} = 18 ; y = \frac{1}{b} = 36$

Solution: A man can finish the work in 18 days and a woman can finish the work in 36 days.

5. The prices of three commodities A, B and C are Rs.x, y and z per units respectively. A person P purchases 4 units of B and sells 2 units of A and 5 units of C. Person Q purchase 2 unit of C and sells 3 units of A and one unit of B. Person R purchases one unit of A and sells 3 unit of B and one unit of C. In the process P, Q and R earn Rs.15000, Rs.1000 and Rs.4000 respectively. Find the prices per unit of A, B and C. (Use matrix inversion method)

Let the prices of one unit of A, B and C are Rs. x, y and z respectively.

From the given data,

$2x - 4y + 5z = 15000 \rightarrow (1)$

$3x + y - 2z = 1000 \rightarrow (1)$

$-x + 3y + z = 4000 \rightarrow (1)$

$\begin{bmatrix} 2 & -4 & 5 \\ 3 & 1 & -2 \\ -1 & 3 & 1 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 15000 \\ 1000 \\ 4000 \end{bmatrix} \Longrightarrow AX = B \Longrightarrow X = A^{-1}B$

$|A| = 2(1 + 6) - (-4)(3 - 2) + 5(9 + 1) = 14 + 4 + 50 = 68$

$\begin{array}{cccc} 1 & 3 & -4 & 1 \\ -2 & 1 & 5 & -2 \\ 3 & -1 & 2 & 3 \\ 1 & 3 & -4 & 1 \end{array} \Longrightarrow \text{adjA} = \begin{bmatrix} 1 + 6 & 15 + 4 & 8 - 5 \\ 2 - 3 & 2 + 5 & 15 + 4 \\ 9 + 1 & 4 - 6 & 2 + 12 \end{bmatrix} = \begin{bmatrix} 7 & 19 & 3 \\ -1 & 7 & 19 \\ 10 & -2 & 14 \end{bmatrix}$

$A^{-1} = \frac{1}{|A|} adjA = \frac{1}{68}\begin{bmatrix} 7 & 19 & 3 \\ -1 & 7 & 19 \\ 10 & -2 & 14 \end{bmatrix}$

$X = A^{-1}B \Longrightarrow \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \frac{1}{68}\begin{bmatrix} 7 & 19 & 3 \\ -1 & 7 & 19 \\ 10 & -2 & 14 \end{bmatrix}\begin{bmatrix} 15000 \\ 1000 \\ 4000 \end{bmatrix}$

$= \frac{1000}{68}\begin{bmatrix} 7 & 19 & 3 \\ -1 & 7 & 19 \\ 10 & -2 & 14 \end{bmatrix}\begin{bmatrix} 15 \\ 1 \\ 4 \end{bmatrix}$

$= \frac{1000}{68}\begin{bmatrix} 105 + 19 + 12 \\ -15 + 7 + 76 \\ 150 - 2 + 56 \end{bmatrix}$

$= \frac{1000}{68}\begin{bmatrix} 136 \\ 68 \\ 204 \end{bmatrix}$

$$\Rightarrow \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 2000 \\ 1000 \\ 3000 \end{bmatrix}$$

Solution: The price of one unit of A, B and C are Rs.2000, 1000 and 3000 respectively.

CRAMER'S RULE
EXERCISE 1.4

1. Solve the following systems of linear equations by Cramer's rule:

$(i) 5x - 2y + 16 = 0, x + 3y - 7 = 0$

$\Delta = \begin{vmatrix} 5 & -2 \\ 1 & 3 \end{vmatrix} = 15 + 2 = 17$ $\quad \Delta_x = \begin{vmatrix} -16 & -2 \\ 7 & 3 \end{vmatrix} = -48 + 14 = -34$

$\Delta_y = \begin{vmatrix} 5 & -16 \\ 1 & 7 \end{vmatrix} = 35 + 16 = 51$ $\quad x = \frac{\Delta_x}{\Delta} = \frac{-34}{17} = -2 \quad ; y = \frac{\Delta_y}{\Delta} = \frac{51}{17} = 3$

Solution: $(x, y) = (-2, 3)$

$(ii) \frac{3}{x} + 2y = 12, \frac{2}{x} + 3y = 13$

Let $\frac{1}{x} = a, y = b \Rightarrow 3a + 2b = 12 ; 2a + 3b = 13$

$\Delta = \begin{vmatrix} 3 & 2 \\ 2 & 3 \end{vmatrix} = 9 - 4 = 5$ $\quad \Delta_a = \begin{vmatrix} 12 & 2 \\ 13 & 3 \end{vmatrix} = 36 - 26 = 10$

$\Delta_y = \begin{vmatrix} 3 & 12 \\ 2 & 13 \end{vmatrix} = 39 - 24 = 15$

$a = \frac{\Delta_a}{\Delta} = \frac{10}{5} = 2 \Rightarrow x = \frac{1}{a} \Rightarrow x = \frac{1}{2}$ $\quad b = \frac{\Delta_b}{\Delta} = \frac{15}{5} = 3 \Rightarrow y = 3$

Solution: $(x, y) = \left(\frac{1}{2}, 3\right)$

$(iii) 3x + 3y - z = 11, 2x - y + 2z = 9, 4x + 3y + 2z = 25$

$\Delta = \begin{vmatrix} 3 & 3 & -1 \\ 2 & -1 & 2 \\ 4 & 3 & 2 \end{vmatrix} = 3(-2 - 6) - 3(4 - 8) - 1(6 + 4) = -24 + 12 - 10 = -22$

$\Delta_x = \begin{vmatrix} 11 & 3 & -1 \\ 9 & -1 & 2 \\ 25 & 3 & 2 \end{vmatrix} = 11(-2 - 6) - 3(18 - 50) - 1(27 + 25) = -88 + 96 - 52 = -44$

$\Delta_y = \begin{vmatrix} 3 & 11 & -1 \\ 2 & 9 & 2 \\ 4 & 25 & 2 \end{vmatrix} = 3(18 - 50) - 11(4 - 8) - 1(50 - 36) = -96 + 44 - 14 = -66$

$\Delta_z = \begin{vmatrix} 3 & 3 & 11 \\ 2 & -1 & 9 \\ 4 & 3 & 25 \end{vmatrix} = 3(-25 - 27) - 3(50 - 36) + 11(6 + 4) = -156 - 42 + 110 = -88$

$x = \frac{\Delta_x}{\Delta} = \frac{-44}{-22} = 2 \; ; y = \frac{\Delta_y}{\Delta} = \frac{-66}{-22} = 3 \; ; z = \frac{\Delta_z}{\Delta} = \frac{-88}{-22} = 4$

Solution : $(x, y, z) = (2, 3, 4)$

$(iv) \frac{3}{x} - \frac{4}{y} - \frac{2}{z} - 1 = 0, \frac{1}{x} + \frac{2}{y} + \frac{1}{z} - 2 = 0, \frac{2}{x} - \frac{5}{y} - \frac{4}{z} + 1 = 0$

Let $\frac{1}{x} = a, \frac{1}{y} = b, \frac{1}{z} = c$

$\Rightarrow 3a - 4b - 2c = 1 \rightarrow (1), a + 2b + c = 2 \rightarrow (2), 2a - 5b - 4c = -1 \rightarrow (3)$

$\Delta = \begin{vmatrix} 3 & -4 & -2 \\ 1 & 2 & 1 \\ 2 & -5 & -4 \end{vmatrix} = 3(-8 + 5) - (-4)(-4 - 2) - 2(-5 - 4) = -9 - 24 + 18 = -15$

$\Delta_a = \begin{vmatrix} 1 & -4 & -2 \\ 2 & 2 & 1 \\ -1 & -5 & -4 \end{vmatrix} = 1(-8 + 5) - (-4)(-8 + 1) - 2(-10 + 2) = -3 - 28 + 16 = -15$

$\Delta_b = \begin{vmatrix} 3 & 1 & -2 \\ 1 & 2 & 1 \\ 2 & -1 & -4 \end{vmatrix} = 3(-8 + 1) - 1(-4 - 2) - 2(-1 - 4) = -21 + 6 + 10 = -5$

$\Delta_c = \begin{vmatrix} 3 & -4 & 1 \\ 1 & 2 & 2 \\ 2 & -5 & -1 \end{vmatrix} = 3(-2 + 10) - (-4)(-1 - 4) + 1(-5 - 4) = 24 - 20 - 9 = -5$

$a = \frac{\Delta_a}{\Delta} = \frac{-15}{-15} = 1 \Rightarrow x = \frac{1}{a} = 1$

$b = \frac{\Delta_b}{\Delta} = \frac{-5}{-15} = \frac{1}{3} \Rightarrow y = \frac{1}{b} = 3$

$c = \frac{\Delta_c}{\Delta} = \frac{-5}{-15} = \frac{1}{3} \Rightarrow z = \frac{1}{c} = 3$

Solution : $(x, y, z) = (1, 3, 3)$

EXAMPLE 1.25 Solve by Cramer's rule, the system of equations $x_1 - x_2 = 3, 2x_1 + 3x_2 + 4x_3 = 17, x_2 + 2x_3 = 7$

$\Delta = \begin{vmatrix} 1 & -1 & 0 \\ 2 & 3 & 4 \\ 0 & 1 & 2 \end{vmatrix} = 1(6 - 4) - (-1)(4 - 0) + 0 = 2 + 4 = 6$

$\Delta_1 = \begin{vmatrix} 3 & -1 & 0 \\ 17 & 3 & 4 \\ 7 & 1 & 2 \end{vmatrix} = 3(6 - 4) - (-1)(34 - 28) + 0 = 6 + 6 = 12$

$\Delta_2 = \begin{vmatrix} 1 & 3 & 0 \\ 2 & 17 & 4 \\ 0 & 7 & 2 \end{vmatrix} = 1(34 - 28) - 3(4 - 0) + 0 = 6 - 12 = -6$

$\Delta_3 = \begin{vmatrix} 1 & -1 & 3 \\ 2 & 3 & 17 \\ 0 & 1 & 7 \end{vmatrix} = 1(21 - 17) - (-1)(14 - 0) + 3(2 - 0) = 4 + 14 + 6 = 24$

$x_1=\frac{\Delta_1}{\Delta}=\frac{12}{6}=2$; $x_2=\frac{\Delta_2}{\Delta}=\frac{-6}{6}=-1$; $x_3=\frac{\Delta_3}{\Delta}=\frac{24}{6}=4$

Solution : $(x_1,x_2,x_3)=(2,-1,4)$

2. In a competitive examination, one mark is awarded for every correct answer while $\frac{1}{4}$ mark is deducted for every wrong answer. A student answered 100 questions and get 80 marks. How many questions did he answer correctly? (Use Cramer's rule)

Let the number questions answered correctly be x and y be the number of questions answered wrong.

From the given data, $x+y=100\rightarrow(1)$; $x-\frac{1}{4}y=80\Longrightarrow 4x-y=320\rightarrow(2)$

$\Delta=\begin{vmatrix}1 & 1\\4 & -1\end{vmatrix}=-1-4=-5$

$\Delta_x=\begin{vmatrix}100 & 1\\320 & -1\end{vmatrix}=-100-320=-420$

$\Delta_y=\begin{vmatrix}1 & 100\\4 & 320\end{vmatrix}=320-400=-80$

$x=\frac{\Delta_x}{\Delta}=\frac{-420}{-5}=84$; $y=\frac{\Delta_y}{\Delta}=\frac{-80}{-5}=16$

Solution:

The number questions answered correctly= 84

The number questions answered wrong= 16

3. A chemist has one solution which is 50% acid and another solution which is 25% acid. How much each should be mixed to make 10 litres of 40% acid solution? (Use Cramer's rule)

Let x and y be the amount of solution containing 50% and 25% acid respectively.

From the given data,

$x+y=10\rightarrow(1)$

$50\%\ of\ x\ +25\%\ of\ y=40\%\ of\ 10\Longrightarrow\frac{50}{100}x+\frac{25}{100}y=\frac{40}{100}.10$

$\Longrightarrow 50x+25y=400\rightarrow(2)$

$\Delta=\begin{vmatrix}1 & 1\\50 & 25\end{vmatrix}=25-50=-25$

$\Delta_x=\begin{vmatrix}10 & 1\\400 & 25\end{vmatrix}=250-400=-150$

$\Delta_y=\begin{vmatrix}1 & 100\\50 & 400\end{vmatrix}=400-500=-100$

$x=\frac{\Delta_x}{\Delta}=\frac{-150}{-25}=6$; $y=\frac{\Delta_y}{\Delta}=\frac{-100}{-25}=4$

Solution:

6 litres of solution containing 50% acid and 4 litres of solution containing 25% acid must be mixed to make 40% acid solution.

4. A fish tank can be filled in 10 minutes using both pumps A and B simultaneously. However, pump B can pump water in or out at the same rate. If pump B is inadvertently run in reverse, then the tank will be filled in 30 minutes. How long would it take each pump to fill the tank by itself? (Use Cramer's rule)

Let x and y be the time taken by pumps A and B respectively.

Amount of water filled by pump A in one minute= $\frac{1}{x}$

Amount of water filled by pump B in one minute= $\frac{1}{y}$

From the given data, $\frac{1}{x}+\frac{1}{y}=\frac{1}{10}\rightarrow(1)$; $\frac{1}{x}-\frac{1}{y}=\frac{1}{30}\rightarrow(2)$

Let $\frac{1}{x}=a$, $\frac{1}{y}=b$

$(1)\Longrightarrow a+b=\frac{1}{10}\Longrightarrow 10a+10b=1\rightarrow(3)$

$(2)\Longrightarrow a-b=\frac{1}{30}\Longrightarrow 30a-30b=1\rightarrow(4)$

Solving (3) & (4) ,

$\Delta=\begin{vmatrix}10 & 10\\30 & -30\end{vmatrix}=-300-300=-600$

$\Delta_a=\begin{vmatrix}1 & 10\\1 & -30\end{vmatrix}=-30-10=-40$

$\Delta_b=\begin{vmatrix}10 & 1\\30 & 1\end{vmatrix}=10-30=-20$

$a=\frac{\Delta_a}{\Delta}=\frac{-40}{-600}=\frac{1}{15}$; $b=\frac{\Delta_b}{\Delta}=\frac{-20}{-600}=\frac{1}{30}\Longrightarrow x=\frac{1}{a}=15$, $y=\frac{1}{b}=30$

Solution:

Time taken by pump A= 15 minutes ; Time taken by pump B= 30 minutes

5.A family of 3 people went out for dinner in a restaurant. The cost of two dosai, three idlies and two vadais is Rs. 150. The cost of two dosai, two idlies and four vadais is Rs. 200. The cost of five dosai, four idlies and two vadais is Rs. 250. The family has Rs. 350 in hand and they ate 3 dosai and six idlies and six vadais. Will they be able to manage to pay the bill within the amount they had? (Use Cramer's rule)

Let Rs.x, y and z be the cost of I dosai, 1 idly and I vadai respectively.

From the given data,

$2x+3y+2z=150 \rightarrow (1)$

$2x+2y+4z=200, \div 2 \Longrightarrow x+y+2z=100 \rightarrow (2)$

$5x+4y+2z=250 \rightarrow (3)$

$$\Delta=\begin{vmatrix}2&3&2\\1&1&2\\5&4&2\end{vmatrix}=2(2-8)-3(2-10)+2(4-5)=-12+24-2=10$$

$$\Delta_x=\begin{vmatrix}150&3&2\\100&1&2\\250&4&2\end{vmatrix}=150(2-8)-3(200-500)+2(400-250)$$

$=-900+900+300=300$

$$\Delta_y=\begin{vmatrix}2&150&2\\1&100&2\\5&250&2\end{vmatrix}=2(200-500)-150(2-10)+2(250-500)$$

$=-600+1200-500=100$

$$\Delta_z=\begin{vmatrix}2&3&150\\1&1&100\\5&4&250\end{vmatrix}=2(250-400)-3(250-500)+150(4-5)$$

$-300+750-150=300$

$x=\frac{\Delta_x}{\Delta}=\frac{300}{10}=30$; $y=\frac{\Delta_y}{\Delta}=\frac{100}{10}=10$; $z=\frac{\Delta_z}{\Delta}=\frac{300}{10}=30$

The cost of 3 dosai , 6 idlies and 6 vadais $=3x+6y+6z$

$=3(30)+6(10)+6(30)=90+60+180=330<350$

Hence they can manage to pay the bill.

EXAMPLE 1.26 In a T20 match, Chennai Super Kings needed just 6 runs to win with 1 ball left to go in the last over. The last ball was bowled and the batsman at the crease hit it high up. The ball travelled along the path in a vertical plane and the equation of the path is $y=ax^2+bx+c$ with respect to a xy-coordinate ystem in the vertical plane and the ball travelled through the points (10,8), (20,16), (40,22), can you conclude that , Chennai Super Kings won the match ? Justify your answer. (All the distances are measured in metres and the meeting point of the plane of the path with the farthest boundary line is (70,0).)

$$y=ax^2+bx+c$$

$(10,8) \Longrightarrow 100a+10b+c=8 \rightarrow (1)$

$(20,16) \Longrightarrow 400a+20b+c=16 \rightarrow (2)$

$(40,22) \Longrightarrow 1600a+40b+c=22 \rightarrow (3)$

$$\Delta=\begin{vmatrix}100&10&1\\400&20&1\\1600&40&1\end{vmatrix}=100\times 10\begin{vmatrix}1&1&1\\4&2&1\\16&4&1\end{vmatrix}$$

$=1000[1(2-4)-1(4-16)+1(16-32)]=1000[-2+12-16]=-6000$

$$\Delta_a=\begin{vmatrix}8&10&1\\16&20&1\\22&40&1\end{vmatrix}=2\times 10\begin{vmatrix}4&1&1\\8&2&1\\11&4&1\end{vmatrix}=20[4(2-4)-1(8-11)+1(32-22)]$$

$=20[-8+3+10]=100$

$$\Delta_b=\begin{vmatrix}100&8&1\\400&16&1\\1600&22&1\end{vmatrix}=100\times 2\begin{vmatrix}1&4&1\\4&8&1\\16&11&1\end{vmatrix}$$

$=200[1(8-11)-4(4-16)+1(44-128)]=200[-3+48-84]=-7800$

$$\Delta_c=\begin{vmatrix}100&10&8\\400&20&16\\1600&40&22\end{vmatrix}=100\times 10\times 2\begin{vmatrix}1&1&4\\4&2&8\\16&4&11\end{vmatrix}$$

$=2000[1(22-32)-1(44-128)+4(16-32)]=2000[-10+84-64]=20000$

$a=\frac{\Delta_a}{\Delta}=\frac{100}{-6000}=-\frac{1}{60}$; $b=\frac{\Delta_b}{\Delta}=\frac{-7800}{-6000}=\frac{13}{10}$; $c=\frac{\Delta_c}{\Delta}=\frac{20000}{-6000}=-\frac{10}{3}$

$\therefore$ The equation of the path is $y=-\frac{1}{60}x^2+\frac{13}{10}x-\frac{10}{3}$

Sub. $x=70$, we get $y=-\frac{1}{60}(70)^2+\frac{13}{10}(70)-\frac{10}{3}=-\frac{4900}{60}+\frac{910}{10}-\frac{10}{3}=6$

$\Longrightarrow$The ball went 6 m high over the boundary line and it is impossible for a fielder to catch the ball. Hence the ball went for six and the Chennai Super Kings won the match.

GAUSSIAN ELIMINATION METHOD

EXERCISE 1.5

1.Solve the following systems of linear equations by Gaussian elimination method:

$(i)\, 2x-2y+3z=2, x+2y-z=3, 3x-y+2z=1$

$$[A|B]=\left(\begin{array}{ccc|c}2&-2&3&2\\1&2&-1&3\\3&-1&2&1\end{array}\right)$$

$$\sim\left(\begin{array}{ccc|c}1&2&-1&3\\2&-2&3&2\\3&-1&2&1\end{array}\right) R_1 \leftrightarrow R_2$$

$$\sim\left(\begin{array}{ccc|c}1&2&-1&3\\0&-6&5&-4\\0&-7&5&-8\end{array}\right)\begin{array}{l}R_2\to R_2-2R_1\\R_3\to R_3-3R_1\end{array}$$

$$\sim\left(\begin{array}{ccc|c}1&2&-1&3\\0&-6&5&-4\\0&0&-5&-20\end{array}\right)R_3\to 6R_3-7R_2$$

$\Longrightarrow x+2y-z=3\to(1)\,;\,-6y+5z=-4\to(2);\ -5z=-20\Longrightarrow z=4$

Put $z=4$ in $(2)\Longrightarrow -6y+5(4)=-4\Longrightarrow -6y+20=-4$

$\Longrightarrow -6y=-24\Longrightarrow y=4$

Put $y=16, z=4$ in $(1)\Longrightarrow x+2(4)-4=3\Longrightarrow x+8-4=3\Longrightarrow x=-1$

Solution: $(x,y,z)=(-1,4,4)$

$(ii) 2x+4y+6z=22, 3x+8y+5z=27, -x+y+2z=2$

$2x+4y+6z=22\,,\div 2\Longrightarrow x+2y+3z=11$

$$[A|B]=\left(\begin{array}{ccc|c}1&2&3&11\\3&8&5&27\\-1&1&2&2\end{array}\right)$$

$$\sim\left(\begin{array}{ccc|c}1&2&3&11\\0&2&-4&-6\\0&3&5&13\end{array}\right)\begin{array}{l}R_2\to R_2-3R_1\\R_3\to R_3+R_1\end{array}$$

$$\sim\left(\begin{array}{ccc|c}1&2&3&11\\0&2&-4&-6\\0&0&22&44\end{array}\right)R_3\to 2R_3-3R_2$$

$\Longrightarrow x+2y+3z=11\to(1)\,;\,2y-4z=-6\to(2)\,;\,22z=44\Longrightarrow z=2$

Put $z=2\ in\ (2)\Longrightarrow 2y-4(2)=-6\ \Longrightarrow 2y=-6+8\Longrightarrow 2y=2\Longrightarrow y=1$

Put $y=1, z=2\ in\ (1)\Longrightarrow x+2(1)+3(2)=11\Longrightarrow x+8=11\Longrightarrow x=3$

Solution: $(x,y,z)=(3,1,2)$

EXAMPLE 1.27 Solve the following systems of linear equations by Gaussian elimination method: $4x+3y+6z=25, x+5+7z=13, 2x+9y+z=1$

2. If ax^2+bx+c is divided by $x+3, x-5$ and $x-1$ the remainders are 21, 61 and 9 respectively. Find a, b and c. (Use Gaussian elimination method)

Let $p(x)=ax^2+bx+c$

$p(-3)=21\Longrightarrow= a(-3)^2+b(-3)+c=21\Longrightarrow 9a-3b+c=21\to(1)$

$p(5)=61\Longrightarrow= a(5)^2+b(5)+c=61\Longrightarrow 25a+5b+c=61\to(2)$

$p(1)=9\Longrightarrow= a(1)^2+b(1)+c=9\Longrightarrow a+b+c=9\to(3)$

$$[A|B]=\left(\begin{array}{ccc|c}9&-3&1&21\\25&5&1&61\\1&1&1&9\end{array}\right)$$

$$\sim\left(\begin{array}{ccc|c}1&1&1&9\\25&5&1&61\\9&-3&1&21\end{array}\right)R_1\leftrightarrow R_3$$

$$\sim\left(\begin{array}{ccc|c}1&1&1&9\\0&-20&-24&-164\\0&-12&-8&-60\end{array}\right)\begin{array}{l}R_2\to R_2-25R_1\\R_3\to R_3-9R_1\end{array}$$

$$\sim\left(\begin{array}{ccc|c}1&1&1&9\\0&5&6&41\\0&3&2&-15\end{array}\right)\begin{array}{l}R_2\to\left(-\frac{1}{4}\right)R_2\\R_3\to\left(-\frac{1}{4}\right)R_3\end{array}$$

$$\sim\left(\begin{array}{ccc|c}1&1&1&9\\0&5&6&41\\0&0&-8&-48\end{array}\right)R_3\to 5R_3-3R_2$$

$\Longrightarrow a+b+c=9\to(1)\,;5b+6c=41\to(2)\,;-8c=-48\Longrightarrow c=6$

Put $c=6\ in\ (2)\Longrightarrow 5b+6(6)=41\ \Longrightarrow 5b=41-36\Longrightarrow 5b=5\Longrightarrow b=1$

Put $b=1, c=6\ in\ (1)\Longrightarrow a+1+6=9\Longrightarrow a+7=9\Longrightarrow a=2$

Solution: $(a,b,c)=(2,1,6)$

3. An amount of Rs.65,000 is invested in three bonds at the rates of 6%, 8% and 10% per annum respectively. The total annual income is Rs.5,000. The income from the third bond is Rs.800 more than that from the second bond. Determine the price of each bond. (Use Gaussian elimination method)

Let the price of three bond be x, y and z respectively.

From the given data,

$x+y+z=65000\to(1)$

Total annual income = Rs.5,000$\Longrightarrow 6\%\ of\ x+8\%\ of\ y+10\%\ of\ z=5000$

$\Longrightarrow \frac{6}{100}x+\frac{8}{100}y+\frac{10}{100}z=5000\ \Longrightarrow 6x+8y+10z=500000$

$\Longrightarrow\div 2, 3x+4y+5z=250000\to(2)$

Income from third bond = income from second bond + Rs.800

$\Longrightarrow 10\%\ of\ z=8\%\ of\ y+800\Longrightarrow \frac{10}{100}z=\frac{8}{100}y+800$

$\Longrightarrow \frac{10}{100}z=\frac{8y+80000}{100}\Longrightarrow 10z=8y+80000\Longrightarrow -8y+10z=80000$

$\Longrightarrow\div 2, -4y+5z=40000\to(3)$

$$[A|B]=\left(\begin{array}{ccc|c}1&1&1&65000\\3&4&5&250000\\0&-4&5&40000\end{array}\right)$$

$$\sim\left(\begin{array}{ccc|c}1&1&1&65000\\0&1&2&55000\\0&-4&5&40000\end{array}\right)R_2\to R_2-3R_1$$

$$\sim\left(\begin{array}{ccc|c}1&1&1&65000\\0&1&2&55000\\0&0&13&260000\end{array}\right)R_3\to R_3+4R_2$$

$\Rightarrow x+y+z=65000\to(1)\,;y+2z=55000\to(2)\,;$

$13z=260000\Rightarrow z=20000$

Put $z=20000\ in\ (2)\Rightarrow y+2(20000)=55000\ \Rightarrow y=55000-40000\Rightarrow$ $y=15000$

Put $y=15000, z=20000\ in\ (1)\Rightarrow x+15000+20000=65000$

$\Rightarrow x+35000=65000\Rightarrow x=30000$

Solution:

The price of 6%, 8% and 10% are Rs.30000. Rs.15000 and Rs.20000 respectively.

4. A boy is walking along the path $y=ax^2+bx+c$ through the points$(-6,8),(-2,-12),(3,8)$. He wants to meet his friend at$P(7,60)$. Will he meet his friend? (Use Gaussian elimination method)

$$y=ax^2+bx+c$$

$(-6,8)\Rightarrow 8=a(-6)^2+b(-6)+c\Rightarrow 36a-6b+c=8\to(1)$

$(-2,-12)\Rightarrow -12=a(-2)^2+b(-2)+c\Rightarrow 4a-2b+c=-12\to(2)$

$(3,8)\Rightarrow 8=a(3)^2+b(3)+c\Rightarrow 9a+3b+c=8\to(3)$

$$[A|B]=\left(\begin{array}{ccc|c}36&-6&1&8\\4&-2&1&-12\\9&3&1&8\end{array}\right)$$

$$\sim\left(\begin{array}{ccc|c}36&-6&1&8\\0&-12&8&-116\\0&18&3&24\end{array}\right)\begin{array}{l}R_2\to 9R_2-R_1\\R_3\to 4R_3-R_1\end{array}$$

$$\sim\left(\begin{array}{ccc|c}36&-6&1&8\\0&-6&4&-58\\0&6&1&8\end{array}\right)\begin{array}{l}R_2\to\left(\frac{1}{2}\right)R_2\\R_3\to\left(\frac{1}{3}\right)R_3\end{array}$$

$$\sim\left(\begin{array}{ccc|c}36&-6&1&8\\0&-6&4&-58\\0&0&5&-50\end{array}\right)R_3\to R_3+R_2$$

$\Rightarrow 36a-6b+c=8\to(1)\,;-6b+4c=-58\to(2)\,;5c=-50\Rightarrow c=-10$

Put $c=-10\ in\ (2)\Rightarrow -6b+4(-10)=-58\ \Rightarrow -6b-40=-58$

$\Rightarrow -6b=-58+40\Rightarrow -6b=-18\Rightarrow b=3$

Put $b=3, c=-10\ in\ (1)\Rightarrow 36a-6(3)-10=8\Rightarrow 36a-18-10=8$

$\Rightarrow 36a=8+28\Rightarrow 36a=36\Rightarrow a=1$

Solution: $(a,b,c)=(1,3,-10)\Rightarrow \boxed{y=x^2+3x-10}$

Put $x=7\Rightarrow y=(7)^2+3(7)-10=49+21-10=60\Rightarrow y=60$

The point $P(7,60)$ satisfies the equation $y=x^2+3x-10$, hence the boy will meet friend at $P(7,60)$.

EXAMPLE 1.28 The upward speed $v(t)$ a rocket at time t is approximated by $v(t)=at^2+bt+c, 0\le t\le 100$ where a, b and c are constant. It has been found that the speed at times $t=3$, $t=6$ and $t=9$ seconds are respectively 64, 133 and 208 miles per second respectively. Find the speed at time t=15 seconds. (Use Gaussian elimination method)

RANK METHOD – NON-HOMOGENEOUS LINEAR EQUATIONS

- If $\rho(A)=\rho([A|B])=3$, then the system of equations is consistent and has unique solution.
- If $\rho(A)=\rho([A|B])=2$, then the system of equations is consistent and has infinitely many solutions.(Put z=k and find x, y)
- If $\rho(A)=\rho([A|B])=1$, then the system of equations is consistent and has infinitely many solutions.(Put y=s and z=t and find x)
- If $\rho(A)\neq\rho([A|B])$, then the system of equations is inconsistent and has no solution.

EXERCISE 1.6

1.Test for consistency and if possible, solve the following systems of equations by rank method:

$(i) x-y+2z=2, 2x+y+4z=7, 4x-y+z=4$

$$AX=B\Rightarrow\begin{bmatrix}1&-1&2\\2&1&4\\4&-1&1\end{bmatrix}\begin{bmatrix}x\\y\\z\end{bmatrix}=\begin{bmatrix}2\\7\\4\end{bmatrix}$$

Augmented matrix $[A|B] = \left(\begin{array}{ccc|c} 1 & -1 & 2 & 2 \\ 2 & 1 & 4 & 7 \\ 4 & -1 & 1 & 4 \end{array}\right)$

$$\sim \left(\begin{array}{ccc|c} 1 & -1 & 2 & 2 \\ 0 & 3 & 0 & 3 \\ 0 & 3 & -7 & -4 \end{array}\right) \begin{array}{l} R_2 \to R_2 - 2R_1 \\ R_3 \to R_3 - 4R_1 \end{array}$$

$$\sim \left(\begin{array}{ccc|c} 1 & -1 & 2 & 2 \\ 0 & 3 & 0 & 3 \\ 0 & 0 & -7 & -7 \end{array}\right) R_3 \to R_3 - R_2$$

The last equivalent matrix is in echelon form.

$\rho([A|B]) = 3 = \rho(A) \Longrightarrow$ The system of equations is consistent and has unique solution

$$AX = B \Longrightarrow \begin{pmatrix} 1 & -1 & 2 \\ 0 & 3 & 0 \\ 0 & 0 & -7 \end{pmatrix}\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 2 \\ 3 \\ -7 \end{pmatrix}$$

$$\Longrightarrow \begin{array}{l} x - y + 2z = 2 \to (1) \\ 3y = 3 \Longrightarrow y = 1 \\ -7z = -7 \Longrightarrow z = 1 \end{array}$$

Put $y = 1, z = 1$ in $(1) \Longrightarrow x - 1 + 2 = 2 \Longrightarrow x = 1$

Solution: $(x, y, z) = (1,1,1)$

EXAMPLE 1.29 Test for consistency of the following systems of equations and if possible solve: $x + 2y - z = 3, 3x - y + 2z = 1, x - 2y + 3z = 3, x - y + z + 1 = 0$

$(ii) 3x + y + z = 2, x - 3y + 2z = 1, 7x - y + 4z = 5$

$$AX = B \Longrightarrow \begin{bmatrix} 3 & 1 & 1 \\ 1 & -3 & 2 \\ 7 & -1 & 4 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 2 \\ 1 \\ 5 \end{bmatrix}$$

Augmented matrix $[A|B] = \left(\begin{array}{ccc|c} 3 & 1 & 1 & 2 \\ 1 & -3 & 2 & 1 \\ 7 & -1 & 4 & 5 \end{array}\right)$

$$\sim \left(\begin{array}{ccc|c} 1 & -3 & 2 & 1 \\ 3 & 1 & 1 & 2 \\ 7 & -1 & 4 & 5 \end{array}\right) R_1 \leftrightarrow R_2$$

$$\sim \left(\begin{array}{ccc|c} 1 & -3 & 2 & 1 \\ 0 & 10 & -5 & -1 \\ 0 & 20 & -10 & -2 \end{array}\right) \begin{array}{l} R_2 \to R_2 - 3R_1 \\ R_3 \to R_3 - 7R_1 \end{array}$$

$$\sim \left(\begin{array}{ccc|c} 1 & -3 & 2 & 1 \\ 0 & 10 & -5 & -1 \\ 0 & 0 & 0 & 0 \end{array}\right) R_3 \to R_3 - 2R_2$$

The last equivalent matrix is in echelon form.

$\rho([A|B]) = 2 = \rho(A) \Longrightarrow$ The system of equations is consistent and has unique solution

$$AX = B \Longrightarrow \begin{pmatrix} 1 & -3 & 2 \\ 0 & 10 & -5 \\ 0 & 0 & 0 \end{pmatrix}\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 1 \\ -1 \\ 0 \end{pmatrix}$$

$$\Longrightarrow \begin{array}{l} x - 3y + 2z = 1 \to (1) \\ 10y - 5z = -1 \to (2) \end{array}$$

Put $z = t$ in $(2) \Longrightarrow 10y - 5t = -1 \Longrightarrow 10y = 5t - 1 \Longrightarrow y = \frac{1}{10}(5t - 1)$

Put $y = \frac{1}{10}(5t - 1), z = t$ in $(1) \Longrightarrow x - 3\frac{1}{10}(5t - 1) + 2t = 1$

$\Longrightarrow x = 1 - 2t + 3\frac{1}{10}(5t - 1) = \frac{10 - 20t + 15t - 3}{10} = \frac{1}{10}(7 - 5t)$

Solution: $(x, y, z) = \left(\frac{1}{10}(7 - 5t), \frac{1}{10}(5t - 1), t\right); t \in R$

EXAMPLE 1.30 Test for consistency of the following systems of equations and if possible solve: $4x - 2y + 6z = 8, x + y - 3z = -1, 15x - 3y + 9z = 21$

$(iii) 2x + 2y + z = 5, x - y + z = 1, 3x + y + 2z = 4$

$$AX = B \Longrightarrow \begin{bmatrix} 2 & 2 & 1 \\ 1 & -1 & 1 \\ 3 & 1 & 2 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 5 \\ 1 \\ 4 \end{bmatrix}$$

Augmented matrix $[A|B] = \left(\begin{array}{ccc|c} 2 & 2 & 1 & 5 \\ 1 & -1 & 1 & 1 \\ 3 & 1 & 2 & 4 \end{array}\right)$

$$\sim \left(\begin{array}{ccc|c} 1 & -1 & 1 & 1 \\ 2 & 2 & 1 & 5 \\ 3 & 1 & 2 & 4 \end{array}\right) R_1 \leftrightarrow R_2$$

$$\sim \left(\begin{array}{ccc|c} 1 & -1 & 1 & 1 \\ 0 & 4 & -1 & 3 \\ 0 & 4 & -1 & 1 \end{array}\right) \begin{array}{l} R_2 \to R_2 - 2R_1 \\ R_3 \to R_3 - 3R_1 \end{array}$$

$$\sim \left(\begin{array}{ccc|c} 1 & -1 & 1 & 1 \\ 0 & 4 & -1 & 3 \\ 0 & 0 & 0 & -2 \end{array}\right) R_3 \to R_3 - R_2$$

The last equivalent matrix is in echelon form.

$\rho([A|B]) = 3, \rho(A) = 2 \Longrightarrow \rho([A|B]) \neq \rho(A)$, The system of equations is inconsistent and has no solution

EXAMPLE 1.32 Test for consistency of the following systems of equations and if possible solve: $x - y + z = -9, 2x - y + z = 4, 3x - y + z = 6, 4x - y + 2z = 7$

$(iv) 2x - y + z = 2, 6x - 3y + 3z = 6, 4x - 2y + 2z = 4$

$$AX = B \Longrightarrow \begin{bmatrix} 2 & -1 & 1 \\ 6 & -3 & 3 \\ 4 & -2 & 2 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 2 \\ 6 \\ 4 \end{bmatrix}$$

Augmented matrix $[A|B] = \left(\begin{array}{ccc|c} 2 & -1 & 1 & 2 \\ 6 & -3 & 3 & 6 \\ 4 & -2 & 2 & 4 \end{array}\right)$

$$\sim \left(\begin{array}{ccc|c} 2 & -1 & 1 & 2 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right) \begin{matrix} R_2 \to R_2 - 3R_1 \\ R_3 \to R_3 - 2R_1 \end{matrix}$$

The last equivalent matrix is in echelon form.

$\rho([A|B]) = 1, \rho(A) = 1 \Longrightarrow$ The system of equations is consistent and has infinitely many solutions.

$$AX = B \Longrightarrow \begin{pmatrix} 2 & -1 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 2 \\ 0 \\ 0 \end{pmatrix}$$

$\Longrightarrow 2x - y + z = 2$

Put y= and z=t in the above equation, $2x - s + t = 2 \Longrightarrow 2x = s - t + 2$

$\Longrightarrow x = \frac{1}{2}(s - t + 2)$

Solution: $(x, y, z) = \left(\frac{1}{2}(s - t + 2), s, t\right); s, t \in R$

EXAMPLE 1.31 Test for consistency of the following systems of equation and if possible solve: $x - y + z = -9, 2x - 2y + 2z = -18, 3x - 3y + 3z + 27 = 0$

2.Find the value of k for which the equations $kx - 2y + z = 1, x - 2ky + z = -2, x - 2y + kz = 1$ have (i) no solution (ii) unique solution (iii) infinitely many solution

$$AX = B \Longrightarrow \begin{bmatrix} k & -2 & 1 \\ 1 & -2k & 1 \\ 1 & -2 & k \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 1 \\ -2 \\ 1 \end{bmatrix}$$

Augmented matrix $[A|B] = \left(\begin{array}{ccc|c} k & -2 & 1 & 1 \\ 1 & -2k & 1 & -2 \\ 1 & -2 & k & 1 \end{array}\right)$

$$\sim \left(\begin{array}{ccc|c} 1 & -2 & k & 1 \\ 1 & -2k & 1 & -2 \\ k & -2 & 1 & 1 \end{array}\right) R_1 \leftrightarrow R_3$$

$$\sim \left(\begin{array}{ccc|c} 1 & -2 & k & 1 \\ 0 & 2-2k & 1-k & -3 \\ 0 & 2k-2 & 1-k^2 & 1-k \end{array}\right) \begin{matrix} R_2 \to R_2 - R_1 \\ R_3 \to R_3 - kR_1 \end{matrix}$$

$$\sim \left(\begin{array}{ccc|c} 1 & -2 & k & 1 \\ 0 & 2(1-k) & 1-k & -3 \\ 0 & 0 & 2-k-k^2 & -k-2 \end{array}\right) R_3 \to R_3 + R_2$$

$$\sim \left(\begin{array}{ccc|c} 1 & -2 & k & 1 \\ 0 & 2(1-k) & 1-k & -3 \\ 0 & 0 & (k+2)(1-k) & -(k+2) \end{array}\right)$$

(i) When $k = 1, k \neq -2, \rho([A|B]) = 3, \rho(A) = 1 \Longrightarrow \rho([A|B]) \neq \rho(A)$, the system of equations is inconsistent and has no solution.

(ii) When $k \neq 1,\ k \neq -2, \rho([A|B]) = 3 = \rho(A)$, the system of equations is consistent and has unique solution.

(iii) When $k = -2, k \neq 1, \rho([A|B]) = 2 = \rho(A)$, the system of equations is consistent and has infinitely many solutions.

3.Investigate the values of λ and μ the system of linear equations $2x + 3y + 5z = 9, 7x + 3y - 5z = 8, 2x + 3y + \lambda z = \mu$, have (i) no solution (ii) unique solution (iii) infinitely many solution

$$AX = B \Longrightarrow \begin{bmatrix} 2 & 3 & 5 \\ 7 & 3 & -5 \\ 2 & 3 & \lambda \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 9 \\ 8 \\ \mu \end{bmatrix}$$

Augmented matrix $[A|B] = \left(\begin{array}{ccc|c} 2 & 3 & 5 & 9 \\ 7 & 3 & -5 & 8 \\ 2 & 3 & \lambda & \mu \end{array}\right)$

$$\sim \left(\begin{array}{ccc|c} 2 & 3 & 5 & 9 \\ 0 & -15 & -25 & --9 \\ 0 & 0 & \lambda-5 & \mu-9 \end{array}\right) \begin{matrix} R_2 \to 2R_2 - 7R_1 \\ R_3 \to R_3 - R_1 \end{matrix}$$

(i) When $\lambda = 5, \mu \neq 9, \rho([A|B]) = 3, \rho(A) = 2 \Longrightarrow \rho([A|B]) \neq \rho(A)$, the system of equations is inconsistent and has no solution.

(ii) When $\lambda \neq 5, \mu \neq 9, \rho([A|B]) = 3 = \rho(A)$, the system of equations is consistent and has unique solution.

(iii) When $\lambda = 5, \mu = 9, \rho([A|B]) = 2 = \rho(A)$, the system of equations is consistent and has infinitely many solutions.

EXAMPLE 1.34 Investigate for what values of λ and μ the system of linear equations $x + 2y + z = 7, x + y + \lambda z = \mu, x + 3y - 5z = 5$, have (i) no solution (ii) unique solution (iii) infinitely many solution

$$AX = B \Rightarrow \begin{bmatrix}1 & 2 & 1\\1 & 1 & \lambda\\1 & 3 & -5\end{bmatrix}\begin{bmatrix}x\\y\\z\end{bmatrix} = \begin{bmatrix}7\\\mu\\5\end{bmatrix}$$

Augmented matrix $[A|B] = \left(\begin{array}{ccc|c}1 & 2 & 1 & 7\\1 & 1 & \lambda & \mu\\1 & 3 & -5 & 5\end{array}\right)$

$$\sim \left(\begin{array}{ccc|c}1 & 2 & 1 & 7\\1 & 3 & -5 & 5\\1 & 1 & \lambda & \mu\end{array}\right) R_2 \leftrightarrow R_3$$

$$\sim \left(\begin{array}{ccc|c}1 & 2 & 1 & 7\\0 & 1 & -6 & -2\\0 & -1 & \lambda - 1 & \mu - 7\end{array}\right) \begin{array}{l}R_2 \to R_2 - R_1\\R_3 \to R_3 - R_1\end{array}$$

$$\sim \left(\begin{array}{ccc|c}1 & 2 & 1 & 7\\0 & 1 & -6 & -2\\0 & 0 & \lambda - 7 & \mu - 9\end{array}\right) R_3 \to R_3 + R_2$$

(i) When $\lambda = 7, \mu \neq 9, \rho([A|B]) = 3, \rho(A) = 2 \Rightarrow \rho([A|B]) \neq \rho(A)$, the system of equations is inconsistent and has no solution.

(ii) When $\lambda \neq 7, \mu \neq 9, \rho([A|B]) = 3 = \rho(A)$, the system of equations is consistent and has unique solution.

(iii) When $\lambda = 7, \mu = 9, \rho([A|B]) = 2 = \rho(A)$, the system of equations is consistent and has infinitely many solutions.

EXAMPLE 1.33 Find the condition on a, b and c so that the following system of linear equations has one parameter family of solutions: $x + y + z = a, x + 2y + 3z = b, 3x + 5y + 7z = c$

$$AX = B \Rightarrow \begin{bmatrix}1 & 1 & 1\\1 & 2 & 3\\3 & 5 & 7\end{bmatrix}\begin{bmatrix}x\\y\\z\end{bmatrix} = \begin{bmatrix}a\\b\\c\end{bmatrix}$$

Augmented matrix $[A|B] = \left(\begin{array}{ccc|c}1 & 1 & 1 & a\\1 & 2 & 3 & b\\3 & 5 & 7 & c\end{array}\right)$

$$\sim \left(\begin{array}{ccc|c}1 & 1 & 1 & a\\0 & 1 & 2 & b - a\\0 & 2 & 4 & c - 3a\end{array}\right) \begin{array}{l}R_2 \to R_2 - R_1\\R_3 \to R_3 - 3R_1\end{array}$$

$$\sim \left(\begin{array}{ccc|c}1 & 1 & 1 & a\\0 & 1 & 2 & b - a\\0 & 0 & 0 & (c - 3a) - 2(b - a)\end{array}\right) R_3 \to R_3 - 2R_2$$

$$\sim \left(\begin{array}{ccc|c}1 & 1 & 1 & a\\0 & 1 & 2 & b - a\\0 & 0 & 0 & c - 2b - a\end{array}\right)$$

Since given that the system have one parameter family of solutions, $\rho([A|B]) = \rho(A) = 2$. So the third row must be a zero row. So $c - 2b - a = 0$ $\Rightarrow c = a + 2b$

RANK METHOD – HOMOGENEOUS LINEAR EQUATIONS

- If $\rho(A) = \rho([A|O]) = 3$, then the system of equations is consistent and has trivial solution.ie., $(x, y, z) = (0,0,0)$
- If $\rho(A) = \rho([A|O]) = 2$ or $1 < n$, then the system of equations is consistent and has non-trivial solutions.
- If the system of equations has non-trivial solution, $\rho([A|O]) < n$, so the determinant of the coefficient matrix is 0.

EXERCISE 1.7

1. Solve the following system of homogenous equations:

$(i) 3x + 2y + 7z = 0, 4x - 3y - 2z = 0, 5x + 9y + 23z = 0$

$$AX = O \Rightarrow \begin{bmatrix}3 & 2 & 7\\4 & -3 & -2\\5 & 9 & 23\end{bmatrix}\begin{bmatrix}x\\y\\z\end{bmatrix} = \begin{bmatrix}0\\0\\0\end{bmatrix}$$

Augmented matrix $[A|O] = \left(\begin{array}{ccc|c}3 & 2 & 7 & 0\\4 & -3 & -2 & 0\\5 & 9 & 23 & 0\end{array}\right)$

$$\sim \left(\begin{array}{ccc|c}3 & 2 & 7 & 0\\0 & -17 & -34 & 0\\0 & 17 & 34 & 0\end{array}\right) \begin{array}{l}R_2 \to 3R_2 - 4R_1\\R_3 \to 3R_3 - 5R_1\end{array}$$

$$\sim\begin{pmatrix}3 & 2 & 7 & 0\\ 0 & -17 & -34 & 0\\ 0 & 0 & 0 & 0\end{pmatrix} R_3 \to R_3 + R_2$$

$\Rightarrow \rho([A|B]) = 2 = \rho(A)$, the system of equations is consistent and has non-trivial solution.

$$AX = O \Rightarrow \begin{pmatrix}3 & 2 & 7\\ 0 & -17 & -34\\ 0 & 0 & 0\end{pmatrix}\begin{pmatrix}x\\ y\\ z\end{pmatrix} = \begin{pmatrix}0\\ 0\\ 0\end{pmatrix} \Rightarrow \begin{matrix}3x + 2y + 7z = 0 \to (1)\\ -17y - 34z = 0 \to (2)\end{matrix}$$

Put z=t in (2) $\Rightarrow -17y - 34t = 0 \Rightarrow y = -2t$

Put $y = -2t, z = t \; in \; (1) \Rightarrow 3x + 2(-2t) + 7t = 0 \Rightarrow 3x = -3t \Rightarrow x = -t$

Solution : $(x, y, z) = (-t, -2t, t); t \in R$

EXAMPLE 1.36 Solve the system : $x + 3y - 2z = 0, 2x - y + 4z = 0, x - 11y + 14z = 0$

1(ii) $2x + 3y - z = 0, x - y - 2z = 0, 3x + y + 3z = 0$

$$AX = O \Rightarrow \begin{bmatrix}2 & 3 & -1\\ 1 & -1 & -2\\ 3 & 1 & 3\end{bmatrix}\begin{bmatrix}x\\ y\\ z\end{bmatrix} = \begin{bmatrix}0\\ 0\\ 0\end{bmatrix}$$

Augmented matrix $[A|O] = \begin{pmatrix}2 & 3 & -1 & 0\\ 1 & -1 & -2 & 0\\ 3 & 1 & 3 & 0\end{pmatrix}$

$$\sim\begin{pmatrix}1 & -1 & -2 & 0\\ 2 & 3 & -1 & 0\\ 3 & 1 & 3 & 0\end{pmatrix} R_1 \leftrightarrow R_2$$

$$\sim\begin{pmatrix}1 & -1 & -2 & 0\\ 0 & 5 & 3 & 0\\ 0 & 4 & 9 & 0\end{pmatrix} \begin{matrix}R_2 \to R_2 - 2R_1\\ R_3 \to R_3 - 3R_1\end{matrix}$$

$$\sim\begin{pmatrix}1 & -1 & -2 & 0\\ 0 & 5 & 3 & 0\\ 0 & 0 & 33 & 0\end{pmatrix} R_3 \to 5R_3 - 4R_2$$

$\Rightarrow \rho([A|B]) = 3 = \rho(A)$, the system of equations is consistent and has trivial solution. $(x, y, z) = (0,0,0)$

EXAMPLE 1.35 Solve the system : $x + 2y + 3z = 0, 3x + 4y + 4z = 0, 7x + 10y + 12z = 0$

EXAMPLE 1.37 Solve the system : $x + y - 2z = 0, 2x - 3y + z = 0, 3x - 7y + 10z = 0, 6x - 9y + 10z = 0$

2.Determine the values of **λ** for which the following system of equations $x + y + 3z = 0, 4x + 3y + \lambda z = 0, 2x + y + 2z = 0$ has (i) a unique solution (ii) a non- trivial solution

$$AX = O \Rightarrow \begin{bmatrix}1 & 1 & 3\\ 4 & 3 & \lambda\\ 2 & 1 & 2\end{bmatrix}\begin{bmatrix}x\\ y\\ z\end{bmatrix} = \begin{bmatrix}0\\ 0\\ 0\end{bmatrix}$$

Augmented matrix $[A|O] = \begin{pmatrix}1 & 1 & 3 & 0\\ 4 & 3 & \lambda & 0\\ 2 & 1 & 2 & 0\end{pmatrix}$

$$\sim\begin{pmatrix}1 & 1 & 3 & 0\\ 2 & 1 & 2 & 0\\ 4 & 3 & \lambda & 0\end{pmatrix} R_2 \leftrightarrow R_3$$

$$\sim\begin{pmatrix}1 & 1 & 3 & 0\\ 0 & -1 & -4 & 0\\ 0 & -1 & \lambda - 12 & 0\end{pmatrix} \begin{matrix}R_2 \to R_2 - 2R_1\\ R_3 \to R_3 - 4R_1\end{matrix}$$

$$\sim\begin{pmatrix}1 & 1 & 3 & 0\\ 0 & -1 & -4 & 0\\ 0 & 0 & \lambda - 8 & 0\end{pmatrix} R_3 \to R_3 - R_2$$

(i)When $\lambda \neq 8 \Rightarrow \rho(A) = \rho([A|O)]) = 3$, then the system of equations is consistent and has trivial solution. $(x, y, z) = (0,0,0)$

(ii)When $\lambda = 8 \Rightarrow \rho(A) = \rho([A|O)]) = 2$, then the system of equations is consistent and has non- trivial solution.

EXAMPLE 1.38 Determine the values of **λ** for which the following system of equations $(3\lambda - 8)x + 3y + 3z = 0, 3x + (3\lambda - 8)y + 3z = 0, 3x + 3y + (3\lambda - 8)z = 0$ has a non- trivial solution.

Here the number of unknowns is 3.

Since the system of equations has non-trivial solution, $\rho([A|O)]) < 3$, so the determinant of the coefficient matrix is 0.

$$\begin{vmatrix}3\lambda - 8 & 3 & 3\\ 3 & 3\lambda - 8 & 3\\ 3 & 3 & 3\lambda - 8\end{vmatrix} = 0$$

$$\Rightarrow \begin{vmatrix}3\lambda - 2 & 3\lambda - 2 & 3\lambda - 2\\ 3 & 3\lambda - 8 & 3\\ 3 & 3 & 3\lambda - 8\end{vmatrix} = 0 \quad (R_1 \to R_1 + R_2 + R_3)$$

$\Rightarrow (3\lambda-2)\begin{vmatrix}1 & 1 & 1\\ 3 & 3\lambda-8 & 3\\ 3 & 3 & 3\lambda-8\end{vmatrix} = 0$

$\Rightarrow (3\lambda-2)\begin{vmatrix}1 & 1 & 1\\ 0 & 3\lambda-11 & 0\\ 0 & 0 & 3\lambda-11\end{vmatrix} = 0 \quad \begin{pmatrix}R_2 \to R_2 - 3R_1\\ R_3 \to R_3 - 3R_1\end{pmatrix}$

Expanding through C_1,

$\Rightarrow (3\lambda-2)(3\lambda-11)^2 = 0 \Rightarrow \lambda = \frac{2}{3}, \lambda = \frac{11}{3}$

EXAMPLE 1.40 If the system of equations $px + by + cz = 0, ax + qy + cz = 0, ax + by + rz = 0$ has a non-trivial solution and $p \neq a, q \neq b, r \neq c$, prove that $\frac{p}{p-a} + \frac{q}{q-b} + \frac{r}{r-c} = 2$

Since the system of equations has non-trivial solution, $\rho([A|O])) < 3$, so the determinant of the coefficient matrix is 0.

$\begin{vmatrix}p & b & c\\ a & q & c\\ a & b & r\end{vmatrix} = 0$

$\Rightarrow \begin{vmatrix}p & b & c\\ a-p & q-b & 0\\ a-p & 0 & r-c\end{vmatrix} = 0 \quad \begin{pmatrix}R_2 \to R_2 - R_1\\ R_3 \to R_3 - R_1\end{pmatrix}$

$\Rightarrow \begin{vmatrix}p & b & c\\ -(p-a) & q-b & 0\\ -(p-a) & 0 & r-c\end{vmatrix} = 0$

$\Rightarrow (p-a)(q-b)(r-c)\begin{vmatrix}\frac{p}{p-a} & \frac{q}{q-b} & \frac{r}{r-c}\\ -1 & 1 & 0\\ -1 & 0 & 1\end{vmatrix} = 0 \quad \Rightarrow \begin{vmatrix}\frac{p}{p-a} & \frac{q}{q-b} & \frac{r}{r-c}\\ -1 & 1 & 0\\ -1 & 0 & 1\end{vmatrix} = 0$

$\Rightarrow \frac{p}{p-a} + \frac{q}{q-b} + \frac{r}{r-c} = 0$

$\Rightarrow \frac{p}{p-a} + \frac{q-(q-b)}{q-b} + \frac{r-(r-c)}{r-c} = 0$

$\Rightarrow \frac{p}{p-a} + \frac{q}{q-b} + \frac{r}{r-c} - 1 - 1 = 0 \quad \Rightarrow \frac{p}{p-a} + \frac{q}{q-b} + \frac{r}{r-c} = 2$

3.By using Gaussian elimination method, balance the chemical reaction equation :

$C_2H_6 + O_2 \to H_2O + CO_2$

$C_2H_6 + O_2 \to H_2O + CO_2 \Rightarrow x_1C_2H_6 + x_2O_2 \to x_3H_2O + x_4CO_2 \to (1)$

x_1			x_2			→	x_3			x_4		
C	H	O	C	H	O		C	H	O	C	H	O
2	6	0	0	0	2		0	2	1	1	0	2

Equating the number of atom of C on both sides with x_1, x_2, x_3, x_4 , we get

$2x_1 + 0x_2 = 0x_3 + 1x_4 \Rightarrow 2x_1 + 0x_2 - 0x_3 - 1x_4 = 0 \to (2)$

Equating the number of atom of H on both sides with x_1, x_2, x_3, x_4 , we get

$6x_1 + 0x_2 = 2x_3 + 0x_4 \Rightarrow 6x_1 + 0x_2 - 2x_3 - 0x_4 = 0$

$\div 2 \Rightarrow 3x_1 + 0x_2 - x_3 - 0x_4 = 0 \to (3)$

Equating the number of atom of O on both sides with x_1, x_2, x_3, x_4 , we get

$0x_1 + 2x_2 = 1x_3 + 2x_4 \Rightarrow 0x_1 + 2x_2 - 1x_3 - 2x_4 = 0 \to (4)$

From $(2), (3), (4) \Rightarrow$ Augmented matrix $[A|O] = \left(\begin{array}{cccc|c}2 & 0 & 0 & -1 & 0\\ 3 & 0 & -1 & 0 & 0\\ 0 & 2 & -1 & -2 & 0\end{array}\right)$

$\sim \left(\begin{array}{cccc|c}2 & 0 & 0 & -1 & 0\\ 0 & 0 & -2 & 3 & 0\\ 0 & 2 & -1 & -2 & 0\end{array}\right) \sim R_2 \to 2R_2 - 3R_1$

$\sim \left(\begin{array}{cccc|c}2 & 0 & 0 & -1 & 0\\ 0 & 2 & -1 & -2 & 0\\ 0 & 0 & -2 & 3 & 0\end{array}\right) R_2 \leftrightarrow R_3$

$\rho(A) = \rho([A|O]) = 3 < 4$, the systems of equations is consistent and has infinite solution.

$2x_1 - 1x_4 = 0 \to (5)$; $2x_2 - x_3 - 2x_4 = 0 \to (6)$; $-2x_3 + 3x_4 = 0 \to (7)$

Put $x_4 = t$ in (5) $\Rightarrow 2x_1 - t = 0 \Rightarrow x_1 = \frac{t}{2}$

Put $x_4 = t$ in (7) $\Rightarrow -2x_3 + 3x_4 = 0 \Rightarrow -2x_3 + 3t = 0 \Rightarrow x_3 = \frac{3t}{2}$

Put $x_3 = \frac{3t}{2}, x_4 = t$ in (6) $\Rightarrow 2x_2 - \frac{3t}{2} - 2t = 0 \Rightarrow x_2 = \frac{7t}{4}$

$(x_1, x_2, x_3, x_4) = \left(\frac{t}{2}, \frac{7t}{4}, \frac{3t}{2}, t\right)$

Put $t = 4 \Rightarrow (x_1, x_2, x_3, x_4) = (2,7,6,4)$

Sub. The above values in (1) $\Rightarrow 2C_2H_6 + 7O_2 \to 6H_2O + 4CO_2$

EXAMPLE 1.38 By using Gaussian elimination method, balance the chemical reaction equation : $C_5H_8 + O_2 \to CO_2 + H_2O$

MARUSHIKAA MATHS ACADEMY

Shevapet, Salem -2.

XII - Mathematics

P.C. Senthil Kumar

Study Material

2. Complex Numbers

Exercise 2.1

1. $i^{1947}+i^{1950}$

$=i^{4(486)+3}+i^{4(487)+2}$

$=i^3+i^2=-i-1=-1-i$

2. $i^{1948}-i^{-1869}$

$=i^{4(487)}-i^{4(-468)+3}$

$=1+i^3=1-i$

3. $\sum_{n=1}^{12} i^n = i+i^2+i^3+i^4+\cdots+i^{12}$

$=\left(i+i^2+i^3+i^4\right)+\left(i^5+i^6+i^7+i^8\right)+$

$+\left(i^9+i^{10}+i^{11}+i^{12}\right)$

$=0+0+0=0$

4. $i^{59}+\dfrac{1}{i^{59}}$

$=i^{4(14)+3}+\dfrac{1}{i^{4(14)+3}}$

$=i^3+\dfrac{1}{i^3}=-i-\dfrac{1}{i}$

$=-i-\dfrac{-i^2}{i}=-i+i=0$

5. $i\cdot i^2\cdot i^3\cdots\cdots i^{2000}$

$=i^{\frac{2000(2000+1)}{2}}$

$=i^{1000*2001}=i^{2001000}$

$=i^{2001000}=i^{4(500250)}=1$

6. $\sum_{n=1}^{10} i^{n+50}=i^{51}+i^{52}+i^{53}+i^{54}+\cdots+i^{60}$

$=\left(i^{51}+i^{52}+i^{53}+i^{54}\right)+\left(i^{55}+i^{56}+i^{57}+i^{58}\right)$

$+i^{59}+i^{60}$

$=0+0+i^{4(14)+3}+i^{4(15)}$

$=i^3+1=-i+1=1-i$

Exercise 2.2

1. (i) $z+w=5-2i+(-1+3i)$

$=5-2i-1+3i=4+i$

1. (ii) $z-i\,w=5-2i-i\,(-1+3i)$

$=5-2i+i-3i^2=5-i-3(-1)$

$=5-i+3=8-i$

1. (iii) $2z+3w=2(5-2i)+3\,(-1+3i)$

$=10-4i-3+9i=7+5i$

1. (iv) $z\,w=(5-2i)(-1+3i)$

$=-5+15i+2i-6i^2$

$=-5+17i-6(-1)$

$=-5+17i+6$

$=1+17i$

1. (v) $z^2+2z\,w+w^2$

$=(5-2i)^2+2(5-2i)(-1+3i)+(-1+3i)^2$

$=25-20i+4i^2+2\,(1+17i)+1-6i+9i^2$

$=25-20i+4(-1)+2+34i$

$+1-6i+9(-1)$

$=25-4+2+1-20i+34i-6i$

$=15+8i$

1. (vi) $(z+w)^2=(4+i)^2$

$=16+8i+i^2=16+8i+(-1)$

$=16-1+8i=15+8i$

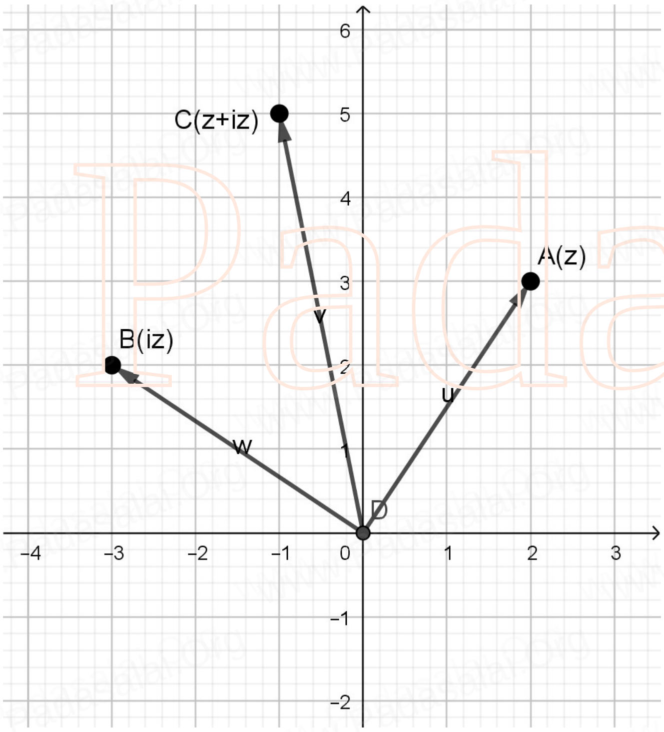
2. (i) $z=2+3i$

$iz=i(2+3i)=2i+3i^2$

$=2i+3(-1)=-3+2i$

$z+iz=2+3i+(-3+2i)$

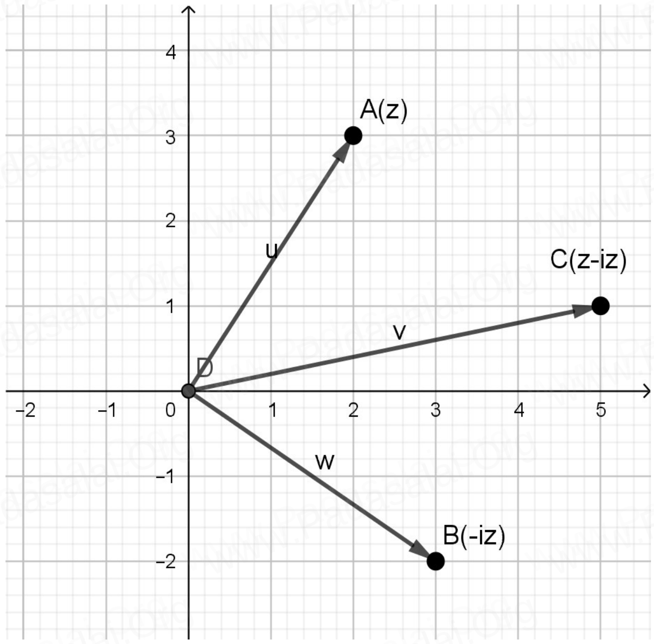
$=2+3i-3+2i=-1+5i$



(ii) $-iz=-(-3+2i)=3-2i$

$z-iz=2+3i-(-3+2i)$

$=2+3i+3-2i=5+i$



3. $(3-i)x-(2-i)y+2i+5$

$=2x+(-1+2i)y+3+2i$

$3x-ix-2y+iy+2i+5$

$=2x-y+2iy+3+2i$

$(3x-2y+5)+i(-x+y+2)$

$=2x-y+3+i(2y+2)$

Equating the real and imaginary parts, we get,

$3x-2y+5=2x-y+3$ and $-x+y+2=2y+2$

$3x-2x-2y+y=3-5$ and $-x+y-2y=2-2$

$x-y=-2$ and $-x-y=0$

$x-y=-2$ and $x+y=0$

$x-y=-2$

$x+y=0$

$2x=-2$

$x=-1$

Substituting $x=-1$ in $x+y=0$, we get

$-1+y=0 \Rightarrow y=1$

Hence $x=-1;\ y=1$.

Exercise 2.3

1. (i) $z_1+z_2=1-3i+(-4i)$

$$=1-3i-4i=1-7i$$

$$(z_1+z_2)+z_3=1-7i+5=6-7i \text{ .....(1)}$$

$$z_2+z_3=-4i+5=5-4i$$

$$z_1+(z_2+z_3)=1-3i+5-4i=6-7i \text{ ....(2)}$$

From (1) and (2),

$$(z_1+z_2)+z_3=z_1+(z_2+z_3)$$

1. (ii) $z_1\cdot z_2=(1-3i)(-4i)=-4i+12i^2$

$$=-4i+12(-1)=-12-4i$$

$$(z_1\cdot z_2)\cdot z_3=(-12-4i)(5)=-60-20i \text{ ....(3)}$$

$$z_2\cdot z_3=(-4i)(5)=-20i$$

$$z_1\cdot(z_2\cdot z_3)=(1-3i)(-20i)$$

$$=-20i+60i^2=-20+60(-1)$$

$$=-20i-60=-60-20i \text{ ....(3)}$$

From (3) and (4)

$$(z_1\cdot z_2)\cdot z_3=z_1\cdot(z_2\cdot z_3)$$

2. (i) $z_2+z_3=-7i+5+4i=5-3i$

$$z_1\cdot(z_2+z_3)=3(5-3i)=15-9i \text{ ....(1)}$$

Now $z_1\cdot z_2=3(-7i)=-21i$

$$z_1\cdot z_3=3(5+4i)=15+12i$$

$$z_1\cdot z_2+z_1\cdot z_3=-21i+15+12i=15-9i \text{ .....(2)}$$

From (1) and (2), we get,

$$z_1\cdot(z_2+z_3)=z_1\cdot z_2+z_1\cdot z_3$$

2. (ii) $z_1+z_2=3+(-7i)=3-7i$

$$(z_1+z_2)\cdot z_3=(3-7i)(5+4i)$$

$$=15+12i-35i-28i^2$$

$$=15-23i-28(-1)$$

$$=15-23i+28=43-23i \text{ .....(3)}$$

$$z_1\cdot z_3=3(5+4i)=15+12i$$

$$z_2\cdot z_3=(-7i)(5+4i)=-35i-28i^2$$

$$=-35i-28(-1)=28-35i$$

$$z_1\cdot z_3+z_2\cdot z_3=15+12i+28-35i$$

$$=43-23i \text{ ......(4)}$$

From (3) and (4), we get

$$(z_1+z_2)\cdot z_3=z_1\cdot z_3+z_2\cdot z_3$$

3. (i) $z_1=2+5i$

Additive Inverse $=-z_1=-(2+5i)=-2-5i$

Multiplicative inverse :

Here $x=2;\ y=5$.

$$\frac{1}{z_1}=\frac{x}{x^2+y^2}+i\frac{-y}{x^2+y^2}$$

$$=\frac{2}{2^2+5^2}+i\frac{-5}{2^2+5^2}$$

$$=\frac{2}{4+25}+i\frac{-5}{4+25}=\frac{2}{29}-\frac{5}{29}i$$

Aliter :

$$\frac{1}{z_1}=\frac{1}{2+5i}=\frac{1}{2+5i}\times\frac{2-5i}{2-5i}$$

$$=\frac{2-5i}{2^2+5^2}=\frac{2-5i}{4+25}=\frac{2-5i}{29}$$

$$=\frac{2}{29}-\frac{5}{29}i$$

3. (ii) $z_2=-3-4i$

Additive Inverse $=-z_2=-(-3-4i)=3+4i$

Multiplicative inverse :

Here $x=-3;\ y=-4$.

$$\frac{1}{z_2}=\frac{x}{x^2+y^2}+i\frac{-y}{x^2+y^2}$$

$$=\frac{-3}{(-3)^2+(-4)^2}+i\frac{-(-4)}{2^2+5^2}$$

$$=\frac{-3}{9+16}+i\frac{4}{9+16}=\frac{-3}{25}+\frac{4}{25}i$$

Aliter :

$\frac{1}{z_2}=\frac{1}{-3-4i}=-\frac{1}{3+4i}$

$=-\frac{1}{3+4i}\times\frac{3-4i}{3-4i}$

$=\frac{-(3+4i)}{3^2+4^2}=\frac{-3+4i}{9+16}\quad=\frac{-3}{25}+\frac{4}{25}i$

3. (iii) $z_3=1+i$

Additive Inverse $=-z_3=-(1+i)=-1-i$

Multiplicative inverse :

Here $x=1;\ y=1$.

$\frac{1}{z_3}=\frac{x}{x^2+y^2}+i\frac{-y}{x^2+y^2}$

$=\frac{1}{1^2+1^2}+i\frac{-1}{1^2+1^2}$

$=\frac{1}{1+1}+i\frac{-1}{1+1}=\frac{1}{2}-\frac{1}{2}i$

Aliter :

$\frac{1}{z_3}=\frac{1}{1+i}=\frac{1}{1+i}\times\frac{1-i}{1-i}$

$=\frac{1-i}{1^2+1^2}=\frac{1-i}{1+1}=\frac{1-i}{2}=\frac{1}{2}-\frac{1}{2}i$

Exercise 2.4

1. (i) $\overline{(5+9i)+(2-4i)}=\overline{7+5i}=7-5i$

Aliter :

$\overline{(5+9i)+(2-4i)}=\overline{5+9i}+\overline{2-4i}$

$=5-9i+2+4i=7-5i$

1. (ii) $\frac{10-5i}{6+2i}=\frac{10-5i}{6+2i}\times\frac{6-2i}{6-2i}$

$=\frac{60-20i-30i+10i^2}{6^2+2^2}$

$=\frac{60-50i+10(-1)}{36+4}$

$=\frac{60-50i-10}{40}=\frac{50-50i}{40}$

$=\frac{50(1-i)}{40}=\frac{5}{4}(1-i)$

1. (iii) $\overline{3i}+\frac{1}{2-i}$

$=-3i+\frac{1}{2-i}\times\frac{2+i}{2+i}$

$=-3i+\frac{2+i}{2^2+(-1)^2}$

$=-3i+\frac{2+i}{4+1}=-3i+\frac{2+i}{5}$

$=\frac{-15i+2+i}{5}=\frac{2-14i}{5}$

$=\frac{2}{5}+i\left(-\frac{14}{5}\right)$

2. (i) $z=x+iy$

$\frac{1}{z}=\frac{1}{x+iy}=\frac{1}{x+iy}\times\frac{x-iy}{x-iy}$

$=\frac{x-iy}{x^2+y^2}=\frac{x}{x^2+y^2}+i\frac{-y}{x^2+y^2}$

Hence $\text{Re}\left(\frac{1}{z}\right)=\frac{x}{x^2+y^2}$

2. (ii) $z=x+iy$

$iz=i(x+iy)=ix+i^2y=ix+(-1)y$

$=-y+ix$

Hence $\text{Re}(iz)=-y$

2. (iii) $z=x+iy$

$3z+4\bar{z}-4i$

$=3(x+iy)+4\overline{(x+iy)}-4i$

$= 3x + i3y + 4(x - iy) - 4i$

$= 3x + i3y + 4x - i4y - 4i$

$= 7x - iy - 4i$

$= 7x + i(-y - 4)$

Hence $\text{Im}(3z + 4\bar{z} - 4i) = -y - 4$

3. Here $z_1 = 2 - i;\ z_2 = -4 + 3i$

$z_1 \cdot z_2 = (2 - i)(-4 + 3i)$

$= -8 + 6i + 4i - 3i^2$

$= -8 + 10i - 3(-1)$

$= -8 + 10i + 3 = -5 + 10i$

$$\frac{1}{z_1 \cdot z_2} = \frac{1}{-5 + 10i} = -\frac{1}{5 - 10i}$$

$$= -\frac{1}{5 - 10i} \times \frac{5 + 10i}{5 + 10i}$$

$$= -\frac{5 + 10i}{5^2 + 10^2} = -\frac{5(1 + 2i)}{25 + 100}$$

$$= -\frac{5(1 + 2i)}{125} = -\frac{1}{25}(1 + 2i)$$

$$\left(\frac{z_1}{z_2}\right)^{-1} = \frac{1}{z_1/z_2} = \frac{z_2}{z_1}$$

$$= \frac{-4 + 3i}{2 - i} = \frac{-4 + 3i}{2 - i} \times \frac{2 + i}{2 + i}$$

$$= \frac{-8 - 4i + 6i + 3i^2}{2^2 + 1^2}$$

$$= \frac{-8 + 2i + 3(-1)}{4 + 1}$$

$$= \frac{-8 + 2i - 3}{5} = \frac{-11 + 2i}{5}$$

4. Here $v = 3 - 4i;\ w = 4 + 3i$

$$\frac{1}{v} = \frac{1}{3 - 4i} = \frac{1}{3 - 4i} \times \frac{3 + 4i}{3 + 4i}$$

$$= \frac{3 + 4i}{3^2 + 4^2} = \frac{3 + 4i}{9 + 16} = \frac{3 + 4i}{25}$$

$$\frac{1}{w} = \frac{1}{4 + 3i} = \frac{1}{4 + 3i} \times \frac{4 - 3i}{4 - 3i}$$

$$= \frac{4 - 3i}{4^2 + 3^2} = \frac{4 - 3i}{16 + 9} = \frac{4 - 3i}{25}$$

$$\frac{1}{v} + \frac{1}{w} = \frac{3 + 4i}{25} + \frac{4 - 3i}{25} = \frac{7 + i}{25}$$

Now $\frac{1}{u} = \frac{1}{v} + \frac{1}{w} = \frac{7 + i}{25}$

$$u = \frac{25}{7 + i} = \frac{25}{7 + i} \times \frac{7 - i}{7 - i}$$

$$= \frac{25(7 - i)}{7^2 + 1^2} = \frac{25(7 - i)}{49 + 1}$$

$$= \frac{25(7 - i)}{50} = \frac{1}{2}(7 - i)$$

Aliter :

Given that $\frac{1}{u} = \frac{1}{v} + \frac{1}{w} \Rightarrow \frac{1}{u} = \frac{w + v}{vw}$

$$u = \frac{vw}{v + w} = \frac{(3 - 4i)(4 + 3i)}{3 - 4i + 4 + 3i}$$

$$= \frac{12 + 9i - 16i - 12i^2}{7 - i}$$

$$= \frac{12 - 7i - 12(-1)}{7 - i} = \frac{12 - 7i + 12}{7 - i}$$

$$= \frac{24 - 7i}{7 - i} = \frac{24 - 7i}{7 - i} \times \frac{7 + i}{7 + i}$$

$$= \frac{168 + 24i - 49i - 7i^2}{7^2 + 1^2}$$

$$= \frac{168 - 25i - 7(-1)}{49 + 1} = \frac{168 - 25i + 7}{50}$$

$$= \frac{175 - 25i}{50} = \frac{25(7 - i)}{50} = \frac{1}{2}(7 - i)$$

5. Let $z = x + iy$. Then $\text{Re}(z) = x$; $\text{Im}(z) = y$ and $\overline{z} = x - iy$.

(i) If z is purely real, then $y = 0$.

$\Rightarrow z = x$ and also $\overline{z} = x$ and hence $z = \overline{z}$.

Conversely, if $z = \overline{z}$, then

$$x + iy = x - iy$$
$$iy = -iy$$
$$iy + iy = 0$$
$$2iy = 0 \Rightarrow y = 0$$

$\Rightarrow z$ is purely real.

Hence z is real if and only if $z = \overline{z}$.

(ii) $z + \overline{z} = x + iy + x - iy$

$$z + \overline{z} = 2x \Rightarrow x = \frac{z + \overline{z}}{2} \Rightarrow \text{Re}(z) = \frac{z + \overline{z}}{2}$$

(iii) $z - \overline{z} = x + iy - (x - iy)$

$$z - \overline{z} = x + iy - x + iy$$

$$z - \overline{z} = 2iy \Rightarrow y = \frac{z + \overline{z}}{2i} \Rightarrow \text{Im}(z) = \frac{z - \overline{z}}{2i}$$

6. $\left(\sqrt{3} + i\right)^1 = \sqrt{3} + i$

$$\left(\sqrt{3} + i\right)^2 = \left(\sqrt{3}\right)^2 + 2\left(\sqrt{3}\right)i + i^2$$
$$= 3 + 2\sqrt{3}\,i + (-1) = 3 + 2\sqrt{3}\,i - 1$$
$$= 2 + 2\sqrt{3}\,i$$

$$\left(\sqrt{3} + i\right)^3 = \left(\sqrt{3} + i\right)^2\left(\sqrt{3} + i\right)$$
$$= \left(2 + 2\sqrt{3}\,i\right)\left(\sqrt{3} + i\right)$$
$$= 2\sqrt{3} + 2i + 6i + 2\sqrt{3}\,i^2$$
$$= 2\sqrt{3} + 8i + 2\sqrt{3}(-1)$$
$$= 2\sqrt{3} + 8i - 2\sqrt{3}$$

$= 8i$ Which is purely imaginary

$$\left(\sqrt{3} + i\right)^4 = \left(\sqrt{3} + i\right)^3\left(\sqrt{3} + i\right)$$
$$= 8i\left(\sqrt{3} + i\right) = 8\sqrt{3}\,i + 8i^2$$
$$= 8\sqrt{3}\,i + 8(-1) = 8\sqrt{3}\,i - 8$$
$$= -8 + 8\sqrt{3}\,i$$

$$\left(\sqrt{3} + i\right)^5 = \left(\sqrt{3} + i\right)^3\left(\sqrt{3} + i\right)^2$$
$$= 8i\left(2 + 2\sqrt{3}\,i\right) = 16i + 16\sqrt{3}\,i^2$$
$$= 16i + 16\sqrt{3}(-1) = 16i - 16\sqrt{3}$$
$$= -16\sqrt{3} + 16i$$

$$\left(\sqrt{3} + i\right)^6 = \left(\sqrt{3} + i\right)^3\left(\sqrt{3} + i\right)^3$$
$$= 8i \times 8i = 64i^2 = 64(-1)$$

$= -64$ which is purely real.

(i) $n = 6$ is the least positive integer for which $\left(\sqrt{3} + i\right)^n$ is purely real.

(ii) $n = 3$ is the least positive integer for which $\left(\sqrt{3} + i\right)^n$ is purely imaginary.

7. (i) Let $z = \left(2 + i\sqrt{3}\right)^{10} - \left(2 - i\sqrt{3}\right)^{10}$.

$$\overline{z} = \overline{\left(2 + i\sqrt{3}\right)^{10} - \left(2 - i\sqrt{3}\right)^{10}}$$
$$= \overline{\left(2 + i\sqrt{3}\right)^{10}} - \overline{\left(2 - i\sqrt{3}\right)^{10}}$$
$$= \left(\overline{2 + i\sqrt{3}}\right)^{10} - \left(\overline{2 - i\sqrt{3}}\right)^{10}$$
$$= \left(2 - i\sqrt{3}\right)^{10} - \left(2 + i\sqrt{3}\right)^{10}$$
$$= -\left[\left(2 + i\sqrt{3}\right)^{10} - \left(2 - i\sqrt{3}\right)^{10}\right]$$
$$\overline{z} = -z$$

Hence z is purely imaginary.

7. (ii) Let $z = \left(\dfrac{19 - 7i}{9 + i}\right)^{12} + \left(\dfrac{20 - 5i}{7 - 6i}\right)^{12}$

$$\frac{19 - 7i}{9 + i} = \frac{19 - 7i}{9 + i} \times \frac{9 - i}{9 - i}$$

$$=\frac{171-19i-63i+7i^2}{9^2+1^2}$$

$$=\frac{171-82i+7(-1)}{81+1}$$

$$=\frac{171-82i-7}{82}$$

$$=\frac{164-82i}{82}$$

$$=\frac{82(2-i)}{82}=2-i$$

$$\frac{20-5i}{7-6i}=\frac{20-5i}{7-6i}\times\frac{7+6i}{7+6i}$$

$$=\frac{140+120i-35i-30i^2}{7^2+6^2}$$

$$=\frac{140+85i-30(-1)}{49+36}$$

$$=\frac{140+85i+30}{85}$$

$$=\frac{170+85i}{85}$$

$$=\frac{85(2+i)}{85}=2+i$$

Now $z=\left(\frac{19-7i}{9+i}\right)^{12}+\left(\frac{20-5i}{7-6i}\right)^{12}$

$$z=(2-i)^{12}+(2+i)^{12}$$

$$\bar{z}=\overline{(2-i)^{12}+(2+i)^{12}}$$

$$=\overline{(2-i)^{12}}+\overline{(2+i)^{12}}$$

$$=\left(\overline{2-i}\right)^{12}+\left(\overline{2+i}\right)^{12}$$

$$=(2+i)^{12}+(2-i)^{12}$$

$$=(2-i)^{12}+(2+i)^{12}$$

$$\bar{z}=z$$

Hence z is purely real.

Exercise 2.5

1. (i) $\left|\frac{2i}{3+4i}\right|=\frac{|2i|}{|3+4i|}=\frac{\sqrt{0^2+2^2}}{\sqrt{3^2+4^2}}$

$$=\frac{\sqrt{4}}{\sqrt{9+16}}=\frac{2}{\sqrt{25}}=\frac{2}{5}$$

1. (ii) $\frac{2-i}{1+i}+\frac{1-2i}{1-i}$

$$=\frac{(2-i)(1-i)+(1-2i)(1+i)}{(1+i)(1-i)}$$

$$=\frac{2-2i-i+i^2+1+i-2i-2i^2}{1^2+1^2}$$

$$=\frac{2-3i+(-1)+1-i-2(-1)}{1+1}$$

$$=\frac{2-4i-1+1+2}{2}$$

$$=\frac{4-4i}{2}=\frac{2(2-2i)}{2}$$

$$=2-2i$$

$$\left|\frac{2-i}{1+i}+\frac{1-2i}{1-i}\right|=|2-2i|$$

$$=\sqrt{2^2+(-2)^2}=\sqrt{4+4}$$

$$=\sqrt{8}=2\sqrt{2}$$

1. (iii) $\left|(1-i)^{10}\right|=|(1-i)|^{10}$

$$=\left(\sqrt{1^2+(-1)^2}\right)^{10}=\left(\sqrt{1+1}\right)^{10}$$

$$=\left(\sqrt{2}\right)^{10}=32$$

1. (iv) $|2i(3-4i)(4-3i)|$

$$=|2i||3-4i||4-3i|$$

$= \sqrt{0^2+2^2}\sqrt{3^2+(-4)^2}\sqrt{4^2+(-3)^2}$

$= \sqrt{0+4}\sqrt{9+16}\sqrt{16+9}$

$= \sqrt{4}\sqrt{25}\sqrt{25}$

$= 2\times 5\times 5 = 50$

2. (i) Given that $|z_1| = |z_2| = 1$.

$|z_1| = 1 \Rightarrow |z_1|^2 = 1 \Rightarrow z_1\overline{z_1} = 1 \Rightarrow \overline{z_1} = \frac{1}{z_1}$

Similarly, $\overline{z_2} = \frac{1}{z_2}$

Let $w = \frac{z_1+z_2}{1+z_1z_2}$

Now $\overline{w} = \overline{\left(\frac{z_1+z_2}{1+z_1z_2}\right)}$

$= \frac{\overline{z_1+z_2}}{\overline{1+z_1z_2}} = \frac{\overline{z_1}+\overline{z_2}}{1+\overline{z_1z_2}} = \frac{\overline{z_1}+\overline{z_2}}{1+\overline{z_1}\,\overline{z_2}}$

$= \frac{\frac{1}{z_1}+\frac{1}{z_2}}{1+\frac{1}{z_1}\times\frac{1}{z_2}} = \frac{\frac{1}{z_1}+\frac{1}{z_2}}{1+\frac{1}{z_1z_2}} = \frac{\frac{z_2+z_1}{z_1z_2}}{\frac{z_1z_2+1}{z_1z_2}}$

$= \frac{z_2+z_1}{z_1z_2+1} = \frac{z_1+z_2}{1+z_1z_2}$

$\overline{w} = w$

Hence w is purely real.

3. Let A, B and C be the points representing the complex numbers $z_1 = 10-8i$, $z_2 = 11+6i$ and $z_3 = 1+i$.

$AC = |10-8i-(1+i)|$

$= |10-8i-1-i| = |9-9i|$

$= \sqrt{9^2+(-9)^2} = \sqrt{81+81}$

$= \sqrt{162} = 9\sqrt{2}$

$BC = |11+6i-(1+i)|$

$= |11+6i-1-i| = |10+5i|$

$= \sqrt{10^2+5^2} = \sqrt{100+25}$

$= \sqrt{125} = 5\sqrt{5}$

Since $5\sqrt{5} < 9\sqrt{2}$, B is closer to C than A. Hence $11+6i$ is closer to $1+i$ than $10-8i$.

4. Let $z_1 = z$ and $z_2 = 6-8i$.

Then $|z_1| = |z| = 3$ and

$|z_2| = \sqrt{6^2+(-8)^2} = \sqrt{36+64} = \sqrt{100} = 10$

We konw that

$||z_1|-|z_2|| \le |z_1+z_2| \le |z_1|+|z_2|$

$|3-10| \le |z_1+6-8i| \le 3+10$

$7 \le |z_1+6-8i| \le 13$

5. Let $z_1 = z^2$ and $z_2 = -3 = -3+i0$

Then $|z_1| = |z^2| = |z|^2 = 1^2 = 1$ and

$|z_2| = \sqrt{(-3)^2+0^2} = \sqrt{9+0} = \sqrt{9} = 3$

We konw that

$||z_1|-|z_2|| \le |z_1+z_2| \le |z_1|+|z_2|$

$|1-3| \le |z^2+(-3)| \le 1+3$

$2 \le |z^2-3| \le 4$

6. Given that $\left|z-\frac{2}{z}\right| = 2$

$2 = \left|z-\frac{2}{z}\right| \ge \left||z|-\left|\frac{2}{z}\right|\right|$

$\left||z|-\left|\frac{2}{z}\right|\right| \le 2$

$$-2 \le |z| - \left|\frac{2}{z}\right| \le 2$$

$$-2 \le |z| - \frac{|2|}{|z|} \le 2$$

$$-2 \le |z| - \frac{2}{|z|} \le 2$$

$$-2|z| \le |z|^2 - 2 \le 2|z|$$

From $-2|z| \le |z|^2 - 2$, we get,

$$0 \le |z|^2 + 2|z| - 2$$

$$|z|^2 + 2|z| - 2 \ge 0$$

$$\left(|z| - \left(\sqrt{3}-1\right)\right)\left(|z| - \left(-\sqrt{3}-1\right)\right) \ge 0$$

$$|z| \ge \sqrt{3}-1 \text{ (or) } |z| \le -1-\sqrt{3}$$

As $|z|$ is a non-negative real number, $|z|$ cannot be less than $-1-\sqrt{3}$. Hence we must have,

$$|z| \ge \sqrt{3}-1 \quad \text{....(1)}$$

From $|z|^2 - 2 \le 2|z|$, we get,

$$|z|^2 - 2|z| - 2 \le 0$$

$$\left(|z| - \left(1+\sqrt{3}\right)\right)\left(|z| - \left(1-\sqrt{3}\right)\right) \le 0$$

$$1-\sqrt{3} \le |z| \le 1+\sqrt{3}$$

Since $|z| \ge 0$, we must have to omit the negative interval. But

$$|z| \le 1+\sqrt{3} \quad \text{....(2)}$$

Combining (1) and (2), we get,

$$\sqrt{3}-1 \le |z| \le \sqrt{3}+1$$

Hence the greatest value and least value of $|z|$ are $\sqrt{3}+1$ and $\sqrt{3}-1$ respectively.

Note : If $|z|^2 + 2|z| - 2 \ge 0$, then

$$|z| = \frac{-2 \pm \sqrt{2^2 - 4(1)(-2)}}{2(1)}$$

$$= \frac{-2 \pm \sqrt{4+8}}{2} = \frac{-2 \pm \sqrt{12}}{2}$$

$$= \frac{-2 \pm 2\sqrt{3}}{2} = \frac{2\left(-1 \pm \sqrt{3}\right)}{2}$$

$$= -1 \pm \sqrt{3}$$

If $|z|^2 - 2|z| - 2 = 0$, then

$$|z| = \frac{-(-2) \pm \sqrt{(-2)^2 - 4(1)(-2)}}{2(1)}$$

$$= \frac{2 \pm \sqrt{4+8}}{2} = \frac{2 \pm \sqrt{12}}{2}$$

$$= \frac{2 \pm 2\sqrt{3}}{2} = \frac{2\left(1 \pm \sqrt{3}\right)}{2}$$

$$= 1 \pm \sqrt{3}$$

7. Given $|z_1| = 1; |z_2| = 2; |z_3| = 3$. Hence

$$|z_1|^2 = 1; |z_2|^2 = 4; |z_3|^2 = 9$$

$$z_1 \overline{z_1} = 1; \; z_2 \overline{z_2} = 4; \; z_3 \overline{z_3} = 9$$

$$z_1 = \frac{1}{\overline{z_1}}; \; z_2 = \frac{4}{\overline{z_2}}; \; z_3 = \frac{9}{\overline{z_3}}$$

Then

$$z_1 + z_2 + z_3 = \frac{1}{\overline{z_1}} + \frac{4}{\overline{z_2}} + \frac{9}{\overline{z_3}}$$

$$z_1 + z_2 + z_3 = \frac{\overline{z_2}\,\overline{z_3} + 4\overline{z_1}\,\overline{z_3} + 9\overline{z_1}\,\overline{z_2}}{\overline{z_1}\,\overline{z_2}\,\overline{z_3}}$$

$$= \frac{9\overline{z_1}\,\overline{z_2} + 4\overline{z_1}\,\overline{z_3} + \overline{z_2}\,\overline{z_3}}{\overline{z_1}\,\overline{z_2}\,\overline{z_3}}$$

$$= \frac{\overline{9z_1z_2} + \overline{4z_1z_3} + \overline{z_2z_3}}{\overline{z_1z_2z_3}}$$

$$= \frac{\overline{9z_1z_2 + 4z_1z_3 + z_2z_3}}{\overline{z_1z_2z_3}}$$

$$= \overline{\left(\frac{9z_1z_2 + 4z_1z_3 + z_2z_3}{z_1z_2z_3}\right)}$$

Now

$$|z_1 + z_2 + z_3| = \left|\overline{\left(\frac{9z_1z_2 + 4z_1z_3 + z_2z_3}{z_1z_2z_3}\right)}\right|$$

$$1 = \left|\frac{9z_1z_2 + 4z_1z_3 + z_2z_3}{z_1z_2z_3}\right|$$

$$|9z_1z_2 + 4z_1z_3 + z_2z_3| = |z_1z_2z_3|$$

$$|9z_1z_2 + 4z_1z_3 + z_2z_3| = |z_1||z_2||z_3|$$

$$|9z_1z_2 + 4z_1z_3 + z_2z_3| = 1 \times 2 \times 3$$

$$|9z_1z_2 + 4z_1z_3 + z_2z_3| = 6$$

8. Let $z = a + ib$. Then $|z| = \sqrt{a^2 + b^2}$.

Also

$iz = i(a+ib) = ia + i^2b = ia + (-1)b = -b + ia$

Then the area of the traiangle is formed by the vertices (a, b), $(-b, a)$, and $(a-b, a+b)$.

From the given information, we have

Area of the triangle = 50 square units

$$\frac{1}{2}\begin{vmatrix} a & -b & a-b & a \\ b & a & a+b & b \end{vmatrix} = \pm 50$$

$$a^2 - b(a+b) + b(a-b)$$
$$-\{-b^2 + a(a-b) + a(a+b)\} = \pm 100$$

$$a^2 - ab - b^2 + ab - b^2$$
$$-\{-b^2 + a^2 - ab + a^2 + ab\} = \pm 100$$

$$a^2 - 2b^2 - \{-b^2 + 2a^2\} = \pm 100$$

$$a^2 - 2b^2 + b^2 - 2a^2 = \pm 100$$

$$-a^2 - b^2 = \pm 100$$

$$-(a^2 + b^2) = \pm 100$$

$$a^2 + b^2 = \mp 100$$

As we $a^2 + b^2$ cannot be negative, we must have

$$a^2 + b^2 = 100$$

$$\sqrt{a^2 + b^2} = 10$$

$$|z| = 10$$

9. Given that

$$z^3 + 2\bar{z} = 0 \;....(1)$$

$$z^3 = -2\bar{z}$$

Taking modulus on both sides, we get,

$$|z^3| = |-2\bar{z}|$$

$$|z|^3 = |-2||\bar{z}|$$

$$|z|^3 = 2|z|$$

$$|z|^3 - 2|z| = 0$$

$$|z|\left(|z|^2 - 2\right) = 0$$

$$|z| = 0 \text{ (or) } |z|^2 - 2 = 0$$

If $|z| = 0 \Rightarrow z = 0$ is a solution.

If $|z|^2 - 2 = 0 \Rightarrow |z|^2 = 2$

$$z\bar{z} = 2 \Rightarrow \bar{z} = \frac{2}{z}$$

Substituting $\bar{z} = \frac{2}{z}$, in (1), we get,

$$z^3 + 2\left(\frac{2}{z}\right) = 0$$

$$z^4 + 4 = 0$$

which has four solution. Hence including zero solution, there are five solutions.

10. (i) Let $z=4+3i$. Here $a=4;\ b=3$

$$|z|=\sqrt{4^2+3^2}=\sqrt{16+9}=\sqrt{25}=5$$

Here b is positive. Hence

$$\sqrt{z}=\pm\left(\sqrt{\frac{|z|+a}{2}}+i\sqrt{\frac{|z|-a}{2}}\right)$$

$$\sqrt{4+3i}=\pm\left(\sqrt{\frac{5+4}{2}}+i\sqrt{\frac{5-4}{2}}\right)$$

$$=\pm\left(\sqrt{\frac{9}{2}}+i\sqrt{\frac{1}{2}}\right)$$

$$=\pm\left(\frac{3}{\sqrt{2}}+i\frac{1}{\sqrt{2}}\right)=\pm\left(\frac{3+i}{\sqrt{2}}\right)$$

10. (ii) Let $z=-6+8i$. Here $a=-6;\ b=8$

$$|z|=\sqrt{(-6)^2+8^2}=\sqrt{36+64}=\sqrt{100}=10$$

Here b is positive. Hence

$$\sqrt{z}=\pm\left(\sqrt{\frac{|z|+a}{2}}+i\sqrt{\frac{|z|-a}{2}}\right)$$

$$\sqrt{-6+8i}=\pm\left(\sqrt{\frac{10+(-6)}{2}}+i\sqrt{\frac{10-(-6)}{2}}\right)$$

$$=\pm\left(\sqrt{\frac{10-6}{2}}+i\sqrt{\frac{10+6}{2}}\right)$$

$$=\pm\left(\sqrt{\frac{4}{2}}+i\sqrt{\frac{16}{2}}\right)$$

$$=\pm\left(\sqrt{2}+i\sqrt{8}\right)=\pm\left(\sqrt{2}+i2\sqrt{2}\right)$$

$$=\pm\sqrt{2}\,(1+i2)$$

10. (iii) Let $z=-5-12i$. Here $a=-5;\ b=-12$

$$|z|=\sqrt{(-5)^2+(-12)^2}=\sqrt{25+144}=\sqrt{169}=13$$

Here b is negative. Hence

$$\sqrt{z}=\pm\left(\sqrt{\frac{|z|+a}{2}}-i\sqrt{\frac{|z|-a}{2}}\right)$$

$$\sqrt{-5-12i}=\pm\left(\sqrt{\frac{13+(-5)}{2}}-i\sqrt{\frac{13-(-5)}{2}}\right)$$

$$=\pm\left(\sqrt{\frac{13-5}{2}}-i\sqrt{\frac{13+5}{2}}\right)$$

$$=\pm\left(\sqrt{\frac{8}{2}}+i\sqrt{\frac{18}{2}}\right)$$

$$=\pm\left(\sqrt{\frac{8}{2}}-i\sqrt{\frac{18}{2}}\right)$$

$$=\pm\left(\sqrt{4}-i\sqrt{9}\right)=\pm(2-3i)$$

Exercise 2.6

1. Given $\left|\frac{z-4i}{z+4i}\right|=1\Rightarrow\frac{|z-4i|}{|z+4i|}=1$

$$\Rightarrow|z-4i|=|z+4i|$$

$$\Rightarrow|x+iy-4i|=|x+iy+4i|$$

$$|x+i(y-4)|=|x+i(y+4)|$$

$$\sqrt{x^2+(y-4)^2}=\sqrt{x^2+(y+4)^2}$$

Squaring on both sides,

$$x^2+(y-4)^2=x^2+(y+4)^2$$

$$(y-4)^2=(y+4)^2$$

$$y^2-8y+16=y^2+8y+16$$

$$-8y=8y$$

$$-8y-8y=0$$

$$-16y=0$$

$$y=0$$

which is the required locus and it is the real axis.

2. Given $z = x + iy$

$$\frac{2z+1}{iz+1} = \frac{2(x+iy)+1}{i(x+iy)+1}$$

$$= \frac{2x+i2y+1}{ix+i^2y+1}$$

$$= \frac{2x+1+i2y}{ix+(-1)y+1}$$

$$= \frac{2x+1+i2y}{ix-y+1}$$

$$= \frac{(2x+1)+i2y}{(1-y)+ix}$$

$$= \frac{(2x+1)+i2y}{(1-y)+ix} \times \frac{(1-y)-ix}{(1-y)-ix}$$

$$= \frac{(2x+1)(1-y)-ix(2x+1)+i2y(1-y)-i^2 2xy}{(1-y)^2+x^2}$$

$$= \frac{(2x+1)(1-y)-(-1)2xy+i[2y(1-y)-x(2x+1)]}{(1-y)^2+x^2}$$

$$= \frac{(2x+1)(1-y)+2xy+i[2y(1-y)-x(2x+1)]}{(1-y)^2+x^2}$$

$$\frac{2z+1}{iz+1} = \frac{(2x+1)(1-y)+2xy}{(1-y)^2+x^2} + i\frac{[2y(1-y)-x(2x+1)]}{(1-y)^2+x^2}$$

$$\text{Im}\left(\frac{2z+1}{iz+1}\right) = \frac{2y(1-y)-x(2x+1)}{(1-y)^2+x^2}$$

Given that $\text{Im}\left(\frac{2z+1}{iz+1}\right) = 0$

$$\Rightarrow \frac{2y(1-y)-x(2x+1)}{(1-y)^2+x^2} = 0$$

$$\Rightarrow 2y(1-y)-x(2x+1) = 0$$

$$2y-2y^2-2x^2-x = 0$$

$$-2x^2-2y^2-x+2y = 0$$

$$2x^2+2y^2+x-2y = 0$$

which is the required locus

3. (i) Given $z = x + iy$

$$iz = i(x+iy) = ix+i^2y = ix+(-1)y = ix-y$$

$$iz = -y+ix$$

Now $\text{Re}(iz) = -y$

$$(\text{Re}(iz))^2 = (-y)^2$$

$$(\text{Re}(iz))^2 = y^2$$

Given that $(\text{Re}(iz))^2 = 3$

and hence $y^2 = 3$

3. (ii) $(1-i)z+1 = (1-i)(x+iy)+1$

$$= x+iy-ix-i^2y+1$$

$$= x+i(y-x)-(-1)y+1$$

$$= x+i(y-x)+y+1$$

$$= x+y+1+i(y-x)$$

Here $\text{Im}[(1-i)z+1] = y-x$

Given that $\text{Im}[(1-i)z+1] = 0$

Hence $y - x = 0$

(or) $x - y = 0$

3. (iii) $|z+i| = |z-1|$

$$|x+iy+i| = |x+iy-1|$$

$$|x+i(y+1)| = |(x-1)+iy|$$

$$\sqrt{x^2+(y+1)^2} = \sqrt{(x-1)^2+y^2}$$

Squaring on both sides, we get,

$$x^2+(y+1)^2=(x-1)^2+y^2$$
$$x^2+y^2+2y+1=x^2-2x+1+y^2$$
$$2x+2y=0$$
$$x+y=0$$

3. (iv) Given that $\overline{z}=z^{-1}$

$$\overline{z}=\frac{1}{z}\Rightarrow z\overline{z}=1\Rightarrow |z|^2=1$$
$$\Rightarrow\left(\sqrt{x^2+y^2}\right)^2=1$$
$$x^2+y^2=1$$

4. (i) Given that $|z-2-i|=3$

$$|z-(2+i)|=3$$

It is of the form $|z-z_0|=r$ and so it represents a circle, whose centre and radius is $(2, 1)$ and 3 respectively.

4. (ii) Given that $|2z+2-4i|=2$

$$|2(z+1-2i)|=2$$
$$|2||z+1-2i|=2$$
$$2|z-(-1+2i)|=2$$
$$|z-(-1+2i)|=1$$

It is of the form $|z-z_0|=r$ and so it represents a circle, whose centre and radius is $(-1, 2)$ and 1 respectively.

4. (iii) Given that $|3z-6+12i|=8$

$$|3(z-2+4i)|=8$$
$$|3||z-2+4i|=8$$
$$3|z-(2-4i)|=8$$
$$|z-(2-4i)|=\frac{8}{3}$$

It is of the form $|z-z_0|=r$ and so it represents a circle, whose centre and radius is $(2, -4)$ and $\frac{8}{3}$ respectively.

5. (i) Given $z=x+iy$

$$|z-4|=16$$
$$|x+iy-4|=16$$
$$|(x-4)+iy|=16$$
$$\sqrt{(x-4)^2+y^2}=16$$

Squaring on both sides,

$$(x-4)^2+y^2=16^2$$
$$x^2-8x+16+y^2=256$$
$$x^2+y^2-8x+16-256=0$$
$$x^2+y^2-8x-240=0$$

5. (ii) Given $z=x+iy$

$$|z-4|^2-|z-1|^2=16$$
$$|x+iy-4|^2-|x+iy-1|^2=16$$
$$|(x-4)+iy|^2-|x-1+iy|^2=16$$
$$\left(\sqrt{(x-4)^2+y^2}\right)^2-\left(\sqrt{(x-1)^2+y^2}\right)^2=16$$
$$(x-4)^2+y^2-\left((x-1)^2+y^2\right)=16$$
$$(x-4)^2+y^2-(x-1)^2-y^2=16$$
$$x^2-8x+16-\left(x^2-2x+1\right)=16$$
$$x^2-8x-x^2+2x-1=0$$
$$-6x-1=0$$
$$6x+1=0$$

Exercise 2.7

1. (i) Let $z=2+2\sqrt{3}i$.

$$r = |z| = \sqrt{2^2 + (2\sqrt{3})^2} = \sqrt{4+12} = \sqrt{16} = 4$$

$$\alpha = \tan^{-1}\left|\frac{y}{x}\right| = \tan^{-1}\left|\frac{2\sqrt{3}}{2}\right| = \tan^{-1}(\sqrt{3}) = \frac{\pi}{3}$$

The complex number z lies on the first quadrant. Hence we have,

$$\theta = Arg\, z = \alpha = \frac{\pi}{3}$$

Now $z = r(\cos\theta + i\sin\theta)$

$$2 + 2\sqrt{3}i = 4\left(\cos\left(\frac{\pi}{3}\right) + i\sin\left(\frac{\pi}{3}\right)\right)$$

$$= 4\left(\cos\left(2k\pi + \frac{\pi}{3}\right) + i\sin\left(2k\pi + \frac{\pi}{3}\right)\right),\ k \in Z$$

1. (ii) Let $z = 3 - \sqrt{3}\, i$.

$$r = |z| = \sqrt{3^2 + (-\sqrt{3})^2} = \sqrt{9+3} = \sqrt{12} = 2\sqrt{3}$$

$$\alpha = \tan^{-1}\left|\frac{y}{x}\right| = \tan^{-1}\left|\frac{-\sqrt{3}}{3}\right| = \tan^{-1}\left(\frac{1}{\sqrt{3}}\right) = \frac{\pi}{6}$$

The complex number z lies on the fourth quadrant. Hence we have,

$$\theta = Arg\, z = -\alpha = -\frac{\pi}{6}$$

Now $z = r(\cos\theta + i\sin\theta)$

$$3 - \sqrt{3}\, i = 2\sqrt{3}\left(\cos\left(-\frac{\pi}{6}\right) + i\sin\left(-\frac{\pi}{6}\right)\right)$$

$$= 2\sqrt{3}\left(\cos\left(2k\pi - \frac{\pi}{6}\right) + i\sin\left(2k\pi - \frac{\pi}{6}\right)\right),\ k \in Z$$

1. (iii) Let $z = -2 - 2i$

$$r = |z| = \sqrt{(-2)^2 + (-2)^2} = \sqrt{4+4} = \sqrt{8} = 2\sqrt{2}$$

$$\alpha = \tan^{-1}\left|\frac{y}{x}\right| = \tan^{-1}\left|\frac{-2}{2}\right| = \tan^{-1}(1) = \frac{\pi}{4}$$

The complex number z lies on the third quadrant. Hence we have,

$$\theta = Arg\, z = \alpha - \pi = \frac{\pi}{4} - \pi = \frac{\pi - 4\pi}{4} = \frac{-3\pi}{4}$$

Now $z = r(\cos\theta + i\sin\theta)$

$$-2 - 2i = 2\sqrt{2}\left(\cos\left(-\frac{3\pi}{4}\right) + i\sin\left(-\frac{3\pi}{4}\right)\right)$$

$$= 2\sqrt{2}\left(\cos\left(2k\pi - \frac{3\pi}{4}\right) + i\sin\left(2k\pi - \frac{3\pi}{4}\right)\right),$$

$k \in Z$

1. (iv) Let $z = i - 1 = -1 + i$

$$r = |z| = \sqrt{(-1)^2 + 1^2} = \sqrt{1+1} = \sqrt{2}$$

$$\alpha = \tan^{-1}\left|\frac{y}{x}\right| = \tan^{-1}\left|\frac{1}{-1}\right| = \tan^{-1}(1) = \frac{\pi}{4}$$

The complex number z lies on the second quadrant. Hence we have,

$$\theta = Arg\, z = \pi - \alpha = \pi - \frac{\pi}{4} = \frac{4\pi - \pi}{4} = \frac{3\pi}{4}$$

Now $z = r(\cos\theta + i\sin\theta)$

$$-1 + i = \sqrt{2}\left(\cos\left(\frac{3\pi}{4}\right) + i\sin\left(\frac{3\pi}{4}\right)\right)$$

$$\frac{i-1}{\cos\frac{\pi}{3} + i\sin\frac{\pi}{3}}$$

$$= \frac{\sqrt{2}\left(\cos\left(\frac{3\pi}{4}\right) + i\sin\left(\frac{3\pi}{4}\right)\right)}{\cos\frac{\pi}{3} + i\sin\frac{\pi}{3}}$$

$$= \sqrt{2}\left(\cos\left(\frac{3\pi}{4} - \frac{\pi}{3}\right) + i\sin\left(\frac{3\pi}{4} - \frac{\pi}{3}\right)\right)$$

$$= \sqrt{2}\left(\cos\left(\frac{9\pi - 4\pi}{12}\right) + i\sin\left(\frac{9\pi - 4\pi}{12}\right)\right)$$

$$= \sqrt{2}\left(\cos\left(\frac{5\pi}{12}\right) + i\sin\left(\frac{5\pi}{12}\right)\right)$$

$$= \sqrt{2}\left(\cos\left(2k\pi + \frac{5\pi}{12}\right) + i\sin\left(2k\pi + \frac{5\pi}{12}\right)\right),\quad k \in Z$$

2. (i) $\left(\cos\left(\frac{\pi}{6}\right) + i\sin\left(\frac{\pi}{6}\right)\right)\left(\cos\left(\frac{\pi}{12}\right) + i\sin\left(\frac{\pi}{12}\right)\right)$

$$= \cos\left(\frac{\pi}{6} + \frac{\pi}{12}\right) + i\sin\left(\frac{\pi}{6} + \frac{\pi}{12}\right)$$

$$= \cos\left(\frac{2\pi}{12} + \frac{\pi}{12}\right) + i\sin\left(\frac{2\pi}{12} + \frac{\pi}{12}\right)$$

$$= \cos\left(\frac{3\pi}{12}\right) + i\sin\left(\frac{3\pi}{12}\right)$$

$$= \cos\left(\frac{\pi}{4}\right) + i\sin\left(\frac{\pi}{4}\right)$$

$$= \frac{1}{\sqrt{2}} + \frac{1}{\sqrt{2}}i$$

2. (ii) $\dfrac{\cos\left(\frac{\pi}{6}\right) - i\sin\left(\frac{\pi}{6}\right)}{2\left(\cos\left(\frac{\pi}{3}\right) + i\sin\left(\frac{\pi}{3}\right)\right)}$

$$= \frac{\cos\left(-\frac{\pi}{6}\right) + i\sin\left(-\frac{\pi}{6}\right)}{2\left(\cos\left(\frac{\pi}{3}\right) + i\sin\left(\frac{\pi}{3}\right)\right)}$$

$$= \frac{1}{2}\left(\cos\left(-\frac{\pi}{6} - \frac{\pi}{3}\right) + i\sin\left(-\frac{\pi}{6} - \frac{\pi}{3}\right)\right)$$

$$= \frac{1}{2}\left(\cos\left(-\frac{\pi}{6} - \frac{2\pi}{6}\right) + i\sin\left(-\frac{\pi}{6} - \frac{2\pi}{6}\right)\right)$$

$$= \frac{1}{2}\left(\cos\left(-\frac{3\pi}{6}\right) + i\sin\left(-\frac{3\pi}{6}\right)\right)$$

$$= \frac{1}{2}\left(\cos\left(-\frac{\pi}{2}\right) + i\sin\left(-\frac{\pi}{2}\right)\right)$$

$$= \frac{1}{2}\left(\cos\left(\frac{\pi}{2}\right) - i\sin\left(\frac{\pi}{2}\right)\right)$$

$$= \frac{1}{2}(0 - i(1)) = -\frac{1}{2}i$$

3. (i) Given that

$$(x_1 + iy_1)(x_2 + iy_2)(x_3 + iy_3)\cdots(x_n + iy_n) = a + ib$$

Taking modulus on both sides, we get,

$$|(x_1 + iy_1)(x_2 + iy_2)(x_3 + iy_3)\cdots(x_n + iy_n)| = |a + ib|$$

$$|x_1 + iy_1||x_2 + iy_2||x_3 + iy_3|\cdots|x_n + iy_n| = |a + ib|$$

$$\sqrt{x_1^2 + iy_1^2}\sqrt{x_2^2 + iy_2^2}\sqrt{x_3^2 + iy_3^2}\cdots\sqrt{x_n^2 + iy_n^2} = \sqrt{a^2 + b^2}$$

Squaring on both sides,

$$(x_1^2 + iy_1^2)(x_2^2 + iy_2^2)(x_3^2 + iy_3^2)\cdots(x_n^2 + iy_n^2) = (a^2 + b^2)$$

3. (ii) Given that

$$(x_1 + iy_1)(x_2 + iy_2)(x_3 + iy_3)\cdots(x_n + iy_n) = a + ib$$

Taking argument on both sides, we get,

$$Arg[(x_1 + iy_1)(x_2 + iy_2)(x_3 + iy_3)\cdots(x_n + iy_n)] = Arg(a + ib)$$

$$Arg(x_1 + iy_1) + Arg(x_2 + iy_2) + Arg(x_3 + iy_3) \cdots + Arg(x_n + iy_n) = Arg(a + ib) + 2k\pi$$

$$\tan^{-1}\left(\frac{y_1}{x_1}\right) + \tan^{-1}\left(\frac{y_2}{x_2}\right) + \tan^{-1}\left(\frac{y_3}{x_3}\right) \cdots + \tan^{-1}\left(\frac{y_n}{x_n}\right) = \tan^{-1}\left(\frac{b}{a}\right) + 2k\pi,\ k \in Z$$

$$\sum_{i=1}^{n}\tan^{-1}\left(\frac{y_i}{x_i}\right)=\tan^{-1}\left(\frac{b}{a}\right)+2k\pi,\ k\in Z$$

4. Given that

$$\frac{1+z}{1-z}=\cos 2\theta+i\sin 2\theta$$

$$1+z=(\cos 2\theta+i\sin 2\theta)(1-z)$$

$$1+z=(\cos 2\theta+i\sin 2\theta)-z(\cos 2\theta+i\sin 2\theta)$$

$$z+z(\cos 2\theta+i\sin 2\theta)=\cos 2\theta+i\sin 2\theta-1$$

$$(1+\cos 2\theta+i\sin 2\theta)z=\cos 2\theta+i\sin 2\theta-1$$

$$z=\frac{\cos 2\theta+i\sin 2\theta-1}{1+\cos 2\theta+i\sin 2\theta}$$

$$z=\frac{\cos 2\theta-1+i\sin 2\theta}{1+\cos 2\theta+i\sin 2\theta}$$

$$z=\frac{-(1-\cos 2\theta)+i\sin 2\theta}{1+\cos 2\theta+i\sin 2\theta}$$

$$=\frac{-2\sin^2\theta+i\sin 2\theta}{2\cos^2\theta+i\sin 2\theta}$$

$$=\frac{2(-1)\sin^2\theta+i\sin 2\theta}{2\cos^2\theta+i\sin 2\theta}$$

$$=\frac{2i^2\sin^2\theta+i2\sin\theta\cos\theta}{2\cos^2\theta+i2\sin\theta\cos\theta}$$

$$=\frac{2i\sin\theta(i\sin\theta+\cos\theta)}{2\cos\theta(\cos\theta+i\sin\theta)}$$

$$=\frac{i\sin\theta(\cos\theta+i\sin\theta)}{\cos\theta(\cos\theta+i\sin\theta)}$$

$$=i\frac{\sin\theta}{\cos\theta}=i\tan\theta$$

5. Let $a=\cos\alpha+i\sin\alpha$; $b=\cos\beta+i\sin\beta$; and $c=\cos\gamma+i\sin\gamma$. Then

$$a+b+c$$

$$=\cos\alpha+i\sin\alpha+\cos\beta+i\sin\beta+\cos\gamma+i\sin\gamma$$

$$=(\cos\alpha+\cos\beta+\cos\gamma)+i(\sin\alpha+\sin\beta+\sin\gamma)$$

$$=0+i0=0$$

(i) Since $a+b+c=0$, from the identity,

$$a^3+b^3+c^3-3abc$$

$$=(a+b+c)(a^2+b^2+c^2-ab-bc-ca)$$

we get, $a^3+b^3+c^3-3abc=0$

and hence $a^3+b^3+c^3=3abc$.

$$(\cos\alpha+i\sin\alpha)^3+(\cos\beta+i\sin\beta)^3+(\cos\gamma+i\sin\gamma)^3$$

$$=3(\cos\alpha+i\sin\alpha)(\cos\beta+i\sin\beta)(\cos\gamma+i\sin\gamma)$$

$$\cos 3\alpha+i\sin 3\alpha+\cos 3\beta+i\sin 3\beta+\cos 3\gamma+i\sin 3\gamma$$

$$=3[\cos(\alpha+\beta+\gamma)+i\sin(\alpha+\beta+\gamma)]$$

$$(\cos 3\alpha+\cos 3\beta+\cos 3\gamma)+i(\sin 3\alpha+\sin 3\beta+\sin 3\gamma)$$

$$=3\cos(\alpha+\beta+\gamma)+i3\sin(\alpha+\beta+\gamma)$$

Comparing the real parts on both sides,

$$\cos 3\alpha+\cos 3\beta+\cos 3\gamma=3\cos(\alpha+\beta+\gamma)$$

Comparing the imaginary parts of (1) we get,

$$\sin 3\alpha+\sin 3\beta+\sin 3\gamma=3\sin(\alpha+\beta+\gamma)$$

6. $$\frac{z-i}{z+2}=\frac{x+iy-i}{x+iy+2}$$

$$=\frac{x+i(y-1)}{(x+2)+iy}$$

$$=\frac{x+i(y-1)}{(x+2)+iy}\times\frac{(x+2)-iy}{(x+2)-iy}$$

$$=\frac{x(x+2)-iyx+i(y-1)(x+2)-i^2(y-1)y}{(x+2)^2+y^2}$$

$$=\frac{x(x+2)+i((y-1)(x+2)-xy)-(-1)(y-1)y}{(x+2)^2+y^2}$$

$$=\frac{x(x+2)+i((y-1)(x+2)-xy)+(y-1)y}{(x+2)^2+y^2}$$

$$= \frac{x(x+2)+(y-1)y}{(x+2)^2+y^2} + i\frac{(y-1)(x+2)-xy}{(x+2)^2+y^2}$$

Given $\arg\left(\frac{z-1}{z+2}\right) = \frac{\pi}{4}$

$$\tan^{-1}\left(\frac{\frac{(y-1)(x+2)-xy}{(x+2)^2+y^2}}{\frac{x(x+2)+(y-1)y}{(x+2)^2+y^2}}\right) = \frac{\pi}{4}$$

$$\frac{(y-1)(x+2)-xy}{x(x+2)+(y-1)y} = \tan\frac{\pi}{4}$$

$$\frac{xy+2y-x-2-xy}{x^2+2x+y^2-y} = 1$$

$$x^2+2x+y^2-y-2y+x+2=0$$

$$x^2+y^2+3x-3y+2=0$$

Exercise 2.8

1. $$\frac{a+b\omega+c\omega^2}{b+c\omega+a\omega^2} + \frac{a+b\omega+c\omega^2}{c+a\omega+b\omega^2}$$

$$= \frac{a+b\omega+c\omega^2}{b+c\omega+a\omega^2}\times\frac{\omega}{\omega} + \frac{a+b\omega+c\omega^2}{c+a\omega+b\omega^2}\times\frac{\omega^2}{\omega^2}$$

$$= \frac{\omega(a+b\omega+c\omega^2)}{b\omega+c\omega^2+a\omega^3} + \frac{\omega^2(a+b\omega+c\omega^2)}{c\omega^2+a\omega^3+b\omega^4}$$

$$= \frac{\omega(a+b\omega+c\omega^2)}{b\omega+c\omega^2+a(1)} + \frac{\omega^2(a+b\omega+c\omega^2)}{c\omega^2+a(1)+b\omega}$$

$$= \frac{\omega(a+b\omega+c\omega^2)}{b\omega+c\omega^2+a} + \frac{\omega^2(a+b\omega+c\omega^2)}{c\omega^2+a+b\omega}$$

$$= \frac{\omega(a+b\omega+c\omega^2)}{a+b\omega+c\omega^2} + \frac{\omega^2(a+b\omega+c\omega^2)}{a+b\omega+c\omega^2}$$

$$= \omega+\omega^2 = -1$$

2. Let $z = \frac{\sqrt{3}}{2} + \frac{1}{2}i$

$$r = |z| = \sqrt{\left(\frac{\sqrt{3}}{2}\right)^2 + \left(\frac{1}{2}\right)^2} = \sqrt{\frac{3}{4}+\frac{1}{4}} = \sqrt{\frac{4}{4}} = 1$$

$$\alpha = \tan^{-1}\left|\frac{y}{x}\right| = \tan^{-1}\left|\frac{1/2}{\sqrt{3}/2}\right| = \tan^{-1}\left(\frac{1}{\sqrt{3}}\right) = \frac{\pi}{6}$$

The complex number z lies in the first quadrant. Hence we have

$$\theta = Arg\, z = \alpha = \frac{\pi}{6}$$

Now $z = r(\cos\theta + i\sin\theta)$

$$\frac{\sqrt{3}}{2}+\frac{1}{2}i = 1\left(\cos\left(\frac{\pi}{6}\right)+i\sin\left(\frac{\pi}{6}\right)\right)$$

$$= \cos\left(\frac{\pi}{6}\right)+i\sin\left(\frac{\pi}{6}\right)$$

$$\left(\frac{\sqrt{3}}{2}+\frac{1}{2}i\right)^5 = \left(\cos\left(\frac{\pi}{6}\right)+i\sin\left(\frac{\pi}{6}\right)\right)^5$$

$$\left(\frac{\sqrt{3}}{2}+\frac{1}{2}i\right)^5 = \cos\left(\frac{5\pi}{6}\right)+i\sin\left(\frac{5\pi}{6}\right)$$

Replacing i by $-i$, we get,

$$\left(\frac{\sqrt{3}}{2}-\frac{1}{2}i\right)^5 = \cos\left(\frac{5\pi}{6}\right)-i\sin\left(\frac{5\pi}{6}\right)$$

$$\left(\frac{\sqrt{3}}{2}+\frac{1}{2}i\right)^5 + \left(\frac{\sqrt{3}}{2}-\frac{1}{2}i\right)^5$$

$$= \cos\left(\frac{5\pi}{6}\right)+i\sin\left(\frac{5\pi}{6}\right)+\cos\left(\frac{5\pi}{6}\right)-i\sin\left(\frac{5\pi}{6}\right)$$

$$= 2\cos\left(\frac{5\pi}{6}\right)$$

$$= 2\cos\left(\pi-\frac{\pi}{6}\right) = -2\cos\frac{\pi}{6}$$

$$= -2 \times \frac{\sqrt{3}}{2} = -\sqrt{3}$$

3. Let $z = \sin\frac{\pi}{10} + i\cos\frac{\pi}{10}$

Then $\bar{z} = \sin\frac{\pi}{10} - i\cos\frac{\pi}{10}$

Now $|z| = \sqrt{\sin^2\frac{\pi}{10} + \cos^2\frac{\pi}{10}} = \sqrt{1} = 1$

Since $|z| = 1$, we have

$$|z|^2 = 1 \Rightarrow z\bar{z} = 1 \Rightarrow \bar{z} = \frac{1}{z}$$

$$\left(\frac{1+\sin\frac{\pi}{10}+i\cos\frac{\pi}{10}}{1+\sin\frac{\pi}{10}-i\cos\frac{\pi}{10}}\right)^{10}$$

$$= \left(\frac{1+z}{1+\bar{z}}\right)^{10} = \left(\frac{1+z}{1+\frac{1}{z}}\right)^{10}$$

$$= \left(\frac{1+z}{\frac{z+1}{z}}\right)^{10} = \left(\frac{1+z}{\frac{1+z}{z}}\right)^{10}$$

$$= \left(\frac{1}{1/z}\right)^{10} = (z)^{10}$$

$$= \left(\sin\frac{\pi}{10} + i\cos\frac{\pi}{10}\right)^{10}$$

$$= \left(\cos\left(\frac{\pi}{2}-\frac{\pi}{10}\right) + i\sin\left(\frac{\pi}{2}-\frac{\pi}{10}\right)\right)^{10}$$

$$= \left(\cos\left(\frac{10\pi-2\pi}{20}\right) + i\sin\left(\frac{10\pi-2\pi}{20}\right)\right)$$

$$= \left(\cos\left(\frac{8\pi}{20}\right) + i\sin\left(\frac{8\pi}{20}\right)\right)^{10}$$

$$= \left(\cos\left(\frac{2\pi}{5}\right) + i\sin\left(\frac{2\pi}{5}\right)\right)^{10}$$

$$= \cos\left(\frac{20\pi}{5}\right) + i\sin\left(\frac{20\pi}{5}\right)$$

$$= \cos(4\pi) + i\sin(4\pi)$$

$$= 1 + i0 = 1$$

4. Let us assume $x = \cos\alpha + i\sin\alpha$

Then $\frac{1}{x} = x^{-1} = (\cos\alpha + i\sin\alpha)^{-1}$

$$= \cos\alpha - i\sin\alpha$$

Now

$$x + \frac{1}{x} = \cos\alpha + i\sin\alpha + \cos\alpha - i\sin\alpha$$

$$x + \frac{1}{x} = 2\cos\alpha$$

So our assumption is true.

Similarly we can assume $y = \cos\beta + i\sin\beta$.

(i) $\frac{x}{y} = \frac{\cos\alpha + i\sin\alpha}{\cos\beta + i\sin\beta}$

$$= \cos(\alpha-\beta) + i\sin(\alpha-\beta)$$

$$\frac{y}{x} = \left(\frac{x}{y}\right)^{-1}$$

$$= [\cos(\alpha-\beta) + i\sin(\alpha-\beta)]^{-1}$$

$$= \cos(\alpha-\beta) - i\sin(\alpha-\beta)$$

$$\frac{x}{y} + \frac{y}{x} = \cos(\alpha-\beta) + i\sin(\alpha-\beta)$$

$$+ \cos(\alpha-n) - i\sin(\alpha-\beta)$$

$$= 2\cos(\alpha-\beta)$$

(ii) $xy = (\cos\alpha + i\sin\alpha)(\cos\beta + i\sin\beta)$

$$= \cos(\alpha+\beta) + i\sin(\alpha+\beta)$$

$$\frac{1}{xy} = (xy)^{-1}$$

$$= [\cos(\alpha+\beta) + i\sin(\alpha+\beta)]^{-1}$$

$$= \cos(\alpha+\beta) - i\sin(\alpha+\beta)$$

$$xy - \frac{1}{xy} = \cos(\alpha+\beta) + i\sin(\alpha+\beta)$$

$$-[\cos(\alpha+\beta) - i\sin(\alpha+\beta)]$$

$$= \cos(\alpha+\beta) + i\sin(\alpha+\beta)$$

$$-\cos(\alpha+\beta) + i\sin(\alpha+\beta)$$

$$= 2i\sin(\alpha+\beta)$$

(iii) $$\frac{x^m}{y^n} = \frac{(\cos\alpha + i\sin\alpha)^m}{(\cos\beta + i\sin\beta)^n}$$

$$= \frac{\cos m\alpha + i\sin m\alpha}{\cos n\beta + i\sin n\beta}$$

$$= \cos(m\alpha - n\beta) + i\sin(m\alpha - n\beta)$$

$$\frac{y^n}{x^m} = \left(\frac{x^m}{y^n}\right)^{-1}$$

$$= [\cos(m\alpha - n\beta) + i\sin(m\alpha - n\beta)]^{-1}$$

$$= \cos(m\alpha - n\beta) - i\sin(m\alpha - n\beta)$$

$$\frac{x^m}{y^n} - \frac{y^n}{x^m} = \cos(m\alpha - n\beta) + i\sin(m\alpha - n\beta)$$

$$-[\cos(m\alpha - n\beta) - i\sin(m\alpha - n\beta)]$$

$$= \cos(m\alpha - n\beta) + i\sin(m\alpha - n\beta)$$

$$-\cos(m\alpha - n\beta) + i\sin(m\alpha - n\beta)$$

$$= 2i\sin(m\alpha - n\beta)$$

(iv) $$x^m y^n = (\cos\alpha + i\sin\alpha)^m (\cos\beta + i\sin\beta)^n$$

$$= (\cos m\alpha + i\sin m\alpha)(\cos n\beta + i\sin n\beta)$$

$$= \cos(m\alpha + n\beta) + i\sin(m\alpha + n\beta)$$

$$\frac{1}{x^m y^n} = \left(x^m y^n\right)^{-1}$$

$$= (\cos(m\alpha + n\beta) + i\sin(m\alpha + n\beta))^{-1}$$

$$= \cos(m\alpha + n\beta) - i\sin(m\alpha + n\beta)$$

$$x^m y^n + \frac{1}{x^m y^n} = \cos(m\alpha + n\beta) - i\sin(m\alpha + n\beta)$$

$$+\cos(m\alpha + n\beta) - i\sin(m\alpha + n\beta)$$

$$= 2\cos(m\alpha + n\beta)$$

5. Given

$$z^3 + 27 = 0$$

$$z^3 = -27$$

$$z^3 = 27(-1)$$

$$z^3 = 27(\cos\pi + i\sin\pi)$$

$$z = [27(\cos\pi + i\sin\pi)]^{1/3}$$

$$= 27^{1/3}(\cos\pi + i\sin\pi)^{1/3}$$

$$= 3(\cos(2k\pi + \pi) + i\sin(2k\pi + \pi))^{1/3}$$

$$= 3(\cos(2k+1)\pi + i\sin(2k+1)\pi)^{1/3}$$

$$= 3\left(\cos(2k+1)\frac{\pi}{3} + i\sin(2k+1)\frac{\pi}{3}\right)$$

where $k = 0, 1, 2$.

$z = 3\left(\cos\frac{\pi}{3} + i\sin\frac{\pi}{3}\right)$ when $k = 0$

$$z = 3\left(\cos\frac{3\pi}{3} + i\sin\frac{3\pi}{3}\right)$$

$$= 3(\cos\pi + i\sin\pi)$$

$= 3(-1 + i0) = -3$ when $k = 1$

$z = 3\left(\cos\frac{5\pi}{3} + i\sin\frac{5\pi}{3}\right)$ when $k = 2$.

6. Given

$$(z-1)^3 + 8 = 0$$

$$(z-1)^3 = -8$$

$$(z-1)^3 = (-2)^3$$

$$\frac{(z-1)^3}{(-2)^3}=1$$

$$\left(\frac{z-1}{-2}\right)^3=1$$

$$\frac{z-1}{-2}=\sqrt[3]{1}$$

$$\frac{z-1}{-2}=1 \text{ (or) } \frac{z-1}{-2}=\omega \text{ (or) } \frac{z-1}{-2}=\omega^2$$

$$z-1=-2 \text{ (or) } z-1=-2\omega \text{ (or) } z-1=-2\omega^2$$

$$z=1-2 \text{ (or) } z=1-2\omega \text{ (or) } z=1-2\omega^2$$

$$z=-1 \text{ (or) } z=1-2\omega \text{ (or) } z=1-2\omega^2$$

7. Let us consider, 9th of roots of unity.

$$x=(1)^{1/9} \text{......(1)}$$

$$x=(\cos 0+i\sin 0)^{1/9}$$

$$=[\cos(2k\pi+0)+i\sin(2k\pi+0)]^{1/9}$$

$$=[\cos 2k\pi+i\sin 2k\pi]^{1/9}$$

$$=\cos\left(\frac{2k\pi}{9}\right)+i\sin\left(\frac{2k\pi}{9}\right)$$

where $k=0, 1, 2, 3, \ldots, 8$

The values are ;

when $k=0$; $\cos 0+i\sin 0=1+i0=1$

when $k=1$

$$\cos\left(\frac{2\pi}{9}\right)+i\sin\left(\frac{2\pi}{9}\right)=\omega \text{ (say)}$$

when $k=2$;

$$\cos\left(\frac{4\pi}{9}\right)+i\sin\left(\frac{4\pi}{9}\right)$$

$$=\left(\cos\left(\frac{2\pi}{9}\right)+i\sin\left(\frac{2\pi}{9}\right)\right)^2=\omega^2$$

when $k=3$

$$\cos\left(\frac{6\pi}{9}\right)+i\sin\left(\frac{6\pi}{9}\right)$$

$$=\left(\cos\left(\frac{2\pi}{9}\right)+i\sin\left(\frac{2\pi}{9}\right)\right)^3=\omega^3$$

when $k=4$

$$\cos\left(\frac{8\pi}{9}\right)+i\sin\left(\frac{8\pi}{9}\right)$$

$$=\left(\cos\left(\frac{2\pi}{9}\right)+i\sin\left(\frac{2\pi}{9}\right)\right)^4=\omega^4$$

when $k=5$

$$\cos\left(\frac{10\pi}{9}\right)+i\sin\left(\frac{10\pi}{9}\right)$$

$$=\left(\cos\left(\frac{2\pi}{9}\right)+i\sin\left(\frac{2\pi}{9}\right)\right)^5=\omega^5$$

when $k=6$

$$\cos\left(\frac{12\pi}{9}\right)+i\sin\left(\frac{12\pi}{9}\right)$$

$$=\left(\cos\left(\frac{2\pi}{9}\right)+i\sin\left(\frac{2\pi}{9}\right)\right)^6=\omega^6$$

when $k=7$

$$\cos\left(\frac{14\pi}{9}\right)+i\sin\left(\frac{14\pi}{9}\right)$$

$$=\left(\cos\left(\frac{2\pi}{9}\right)+i\sin\left(\frac{2\pi}{9}\right)\right)^7=\omega^7$$

when $k=8$

$$\cos\left(\frac{16\pi}{9}\right)+i\sin\left(\frac{16\pi}{9}\right)$$

$$=\left(\cos\left(\frac{2\pi}{9}\right)+i\sin\left(\frac{2\pi}{9}\right)\right)^8=\omega^8$$

If we take, $\omega = \cos\left(\frac{2\pi}{9}\right) + i\sin\left(\frac{2\pi}{9}\right)$, then the 9th roots of unity are $1, \omega, \omega^2, \omega^3, \ldots \omega^8$.

From (1), we get $x^9 = 1$ (or) $x^9 - 1 = 0$. Hence $1, \omega, \omega^2, \omega^3, \ldots \omega^8$ are the roots of the equation $x^9 - 1 = 0$.

By Vieta's formula, we get
sum of all the roots =0

Hence $1 + \omega + \omega^2 + \ldots + \omega^8 = 0$

$$\omega + \omega^2 + \ldots + \omega^8 = -1$$

$$\left(\cos\frac{2\pi}{9} + i\sin\frac{2\pi}{9}\right) + \left(\cos\frac{4\pi}{9} + i\sin\frac{4\pi}{9}\right) + \cdots + \left(\cos\frac{16\pi}{9} + i\sin\frac{16\pi}{9}\right) = -1$$

$$\sum_{k=1}^{8}\left(\cos\frac{2k\pi}{9} + i\sin\frac{2k\pi}{9}\right) = -1$$

8. (i) $\left(1 - \omega + \omega^2\right)^6 + \left(1 + \omega - \omega^2\right)^6$

$$= \left(1 + \omega^2 - \omega\right)^6 + \left(1 + \omega - \omega^2\right)^6$$

$$= (-\omega - \omega)^6 + \left(-\omega^2 - \omega^2\right)^6$$

$$= (-2\omega)^6 + \left(-2\omega^2\right)^6$$

$$= 64\,\omega^6 + 64\,\omega^{12}$$

$$= 64(1) + 64(1)$$

$$= 64 + 64 = 128$$

(ii) $(1+\omega)\left(1+\omega^2\right)\left(1+\omega^4\right)\left(1+\omega^8\right)\cdots\left(1+\omega^{2^{11}}\right)$

$$= (1+\omega)\left(1+\omega^2\right)\left(1+\omega^{2^2}\right)\left(1+\omega^{2^3}\right)\cdots\left(1+\omega^{2^{11}}\right)$$

Hence the given expression contains 12 factors or 12 terms.

Now

$$(1+\omega)\left(1+\omega^2\right)\left(1+\omega^4\right)\left(1+\omega^8\right)\cdots\left(1+\omega^{2^{11}}\right)$$

$$= (1+\omega)\left(1+\omega^2\right)(1+\omega)\left(1+\omega^2\right)\cdots 12\,terms$$

$$= \left[(1+\omega)\left(1+\omega^2\right)\right]\left[(1+\omega)\left(1+\omega^2\right)\right]\cdots 6\ terms$$

$$= \left[(1+\omega)\left(1+\omega^2\right)\right]^6$$

$$= \left[1 + \omega + \omega^2 + \omega^3\right]^6$$

$$= [0+1]^6 = 1$$

9. Given that $z = 2 - 2i$

$$r = |z| = \sqrt{2^2 + (-2)^2} = \sqrt{4+4} = \sqrt{8} = 2\sqrt{2}$$

$$\alpha = \tan^{-1}\left|\frac{y}{x}\right| = \tan^{-1}\left|\frac{-2}{2}\right| = \tan^{-1}(1) = \frac{\pi}{4}$$

Since the complex number z lies in the IV quadrant, we have,

$$\theta = Arg\,z = -\alpha = -\frac{\pi}{4}$$

Hence

$$z = r(\cos\theta + i\sin\theta)$$

$$z = 2\sqrt{2}\left(\cos\left(-\frac{\pi}{4}\right) + i\sin\left(-\frac{\pi}{4}\right)\right)$$

(or) $z = 2\sqrt{2}e^{-i\pi/4}$

We know that the complex number $z e^{i\theta}$ is the rotation of z by θ radians in the counter clockwise direction about the origin.

Hence the number obtained by rotating z by $\theta = \frac{\pi}{3}$ radians is $z e^{i\pi/3}$.

$$z e^{i\pi/3} = 2\sqrt{2}e^{-i\pi/4}\,e^{i\pi/3}$$

$$= 2\sqrt{2}\,e^{i\left(\frac{\pi}{3} - \frac{\pi}{4}\right)} = 2\sqrt{2}\,e^{i\left(\frac{4\pi - 3\pi}{12}\right)}$$

$$= 2\sqrt{2}\,e^{i\pi/12}$$

(ii) The number obtained by rotating z by

$\theta = \frac{2\pi}{3}$ radians is $z\,e^{i2\pi/3}$.

$$z\,e^{i2\pi/3} = 2\sqrt{2}e^{-i\pi/4}\,e^{i2\pi/3}$$

$$= 2\sqrt{2}\,e^{i\left(\frac{2\pi}{3}-\frac{\pi}{4}\right)} = 2\sqrt{2}\,e^{i\left(\frac{8\pi-3\pi}{12}\right)}$$

$$= 2\sqrt{2}\,e^{i5\pi/12}$$

(iii) The number obtained by rotating z by $\theta = \frac{3\pi}{2}$ radians is $z\,e^{i3\pi/2}$.

$$z\,e^{i3\pi/2} = 2\sqrt{2}e^{-i\pi/4}\,e^{i3\pi/2}$$

$$= 2\sqrt{2}\,e^{i\left(\frac{3\pi}{2}-\frac{\pi}{4}\right)} = 2\sqrt{2}\,e^{i\left(\frac{12\pi-2\pi}{8}\right)}$$

$$= 2\sqrt{2}\,e^{i10\pi/8} = 2\sqrt{2}\,e^{i5\pi/4}$$

10. Let $x = \sqrt[4]{-1}$. Then

$$\Rightarrow x = (-1)^{1/4}$$

$$x = [\cos\,\pi + i\,\sin\,\pi]^{1/4}$$

$$= [\cos\,(2k\pi+\pi) + i\,\sin\,(2k\pi+\pi)]^{1/4}$$

$$= [\cos\,(2k+1)\pi + i\,\sin\,(2k+1)\pi]^{1/4}$$

$$= \cos\left(\frac{(2k+1)\pi}{4}\right) + i\sin\left(\frac{(2k+1)\pi}{4}\right)$$

where $k = 0, 1, 2, 3$

The values are ;

when $k = 0$,

$$\cos\left(\frac{\pi}{4}\right) + i\,\sin\left(\frac{\pi}{4}\right) = \frac{1}{\sqrt{2}} + i\frac{1}{\sqrt{2}} = \frac{1}{\sqrt{2}}(1+i)$$

when $k = 1$

$$\cos\left(\frac{3\pi}{4}\right) + i\,\sin\left(\frac{3\pi}{4}\right)$$

$$= \cos\left(\pi - \frac{\pi}{4}\right) + i\,\sin\left(\pi - \frac{\pi}{4}\right)$$

$$= -\cos\left(\frac{\pi}{4}\right) + i\,\sin\left(\frac{\pi}{4}\right)$$

$$= -\frac{1}{\sqrt{2}} + i\frac{1}{\sqrt{2}} = -\frac{1}{\sqrt{2}}(1-i)$$

when $k = 2$

$$\cos\left(\frac{5\pi}{4}\right) + i\,\sin\left(\frac{5\pi}{4}\right)$$

$$= \cos\left(\pi + \frac{\pi}{4}\right) + i\,\sin\left(\pi + \frac{\pi}{4}\right)$$

$$= -\cos\left(\frac{\pi}{4}\right) - i\,\sin\left(\frac{\pi}{4}\right)$$

$$= -\frac{1}{\sqrt{2}} - i\frac{1}{\sqrt{2}} = -\frac{1}{\sqrt{2}}(1+i)$$

when $k = 3$

$$\cos\left(\frac{7\pi}{4}\right) + i\,\sin\left(\frac{7\pi}{4}\right)$$

$$= \cos\left(2\pi - \frac{\pi}{4}\right) + i\,\sin\left(2\pi - \frac{\pi}{4}\right)$$

$$= \cos\left(\frac{\pi}{4}\right) - i\,\sin\left(\frac{\pi}{4}\right)$$

$$= \frac{1}{\sqrt{2}} - i\frac{1}{\sqrt{2}} = \frac{1}{\sqrt{2}}(1-i)$$

Hence the roots are $\pm\frac{1}{\sqrt{2}}(1 \pm i)$.

Aliter :

$$x = \sqrt[4]{-1}$$

$$x^4 = -1$$

$$x^4 + 1 = 0$$

$$x^4 + 2x^2 + 1 - 2x^2 = 0$$

$$(x^2+1)^2 - (\sqrt{2}x)^2 = 0$$

$$(x^2 + 1 + \sqrt{2}x)(x^2 + 1 - \sqrt{2}x) = 0$$

$$\left(x^2+\sqrt{2}x+1\right)\left(x^2-\sqrt{2}x+1\right)=0$$

If $x^2+\sqrt{2}x+1=0$, then

$$x=\frac{-\sqrt{2}\pm\sqrt{\left(\sqrt{2}\right)^2-4(1)(1)}}{2}$$

$$=\frac{-\sqrt{2}\pm\sqrt{2-4}}{2}=\frac{-\sqrt{2}\pm\sqrt{-2}}{2}$$

$$=\frac{-\sqrt{2}\pm\sqrt{2}i}{2}=\frac{-\sqrt{2}(1\mp i)}{2}$$

$$=-\frac{1}{\sqrt{2}}(1\pm i)$$

If $x^2-\sqrt{2}x+1=0$, then

$$x=\frac{-\left(-\sqrt{2}\right)\pm\sqrt{\left(-\sqrt{2}\right)^2-4(1)(1)}}{2}$$

$$=\frac{\sqrt{2}\pm\sqrt{2-4}}{2}=\frac{\sqrt{2}\pm\sqrt{-2}}{2}$$

$$=\frac{\sqrt{2}\pm\sqrt{2}i}{2}=\frac{\sqrt{2}(1\pm i)}{2}$$

$$=\frac{1}{\sqrt{2}}(1\pm i)$$

Hence the roots are $\pm\frac{1}{\sqrt{2}}(1\pm i)$.

Exercise 2.9

1. $i^n+i^{n+1}+i^{n+2}+i^{n+3}=0$.

Sum of four consecutive integral powers of i is always zero. Ans: (1)

2. $\sum_{i=1}^{13}\left(i^n+i^{n-1}\right)=\sum_{i=1}^{13}i^n+\sum_{i=1}^{13}i^{n-1}$

$$=\sum_{i=1}^{12}i^n+i^{13}+\sum_{i=0}^{12}i^n$$

$$=0+i+i^0+\sum_{i=1}^{12}i^n$$

$$=i+1+0=1+i$$

Ans: (1)

3. Let $z=a+ib$. Then $|z|=\sqrt{a^2+b^2}$.

Also

$$iz=i(a+ib)=ia+i^2b=ia+(-1)b=-b+ia$$

Then the area of the traiangle is formed by the vertices (a,b), $(a-b,\ a+b)$, and $(-b,\ a)$ taken in anticlockwise direction.

Area of the triangle

$$=\frac{1}{2}\begin{vmatrix} a & a-b & -b & a \\ b & a+b & a & b\end{vmatrix}$$

$$=\frac{1}{2}\Big(a(a+b)+a(a-b)-b^2$$
$$-\left\{b(a-b)-b(a+b)+a^2\right\}\Big)$$

$$=\frac{1}{2}\Big(a^2+ab+a^2-ab-b^2$$
$$-\left\{ab-b^2-ab-b^2+a^2\right\}\Big)$$

$$=\frac{1}{2}\left(2a^2-b^2-\left\{-2b^2+a^2\right\}\right)$$

$$=\frac{1}{2}\left(2a^2-b^2-\left\{-2b^2+a^2\right\}\right)$$

$$=\frac{1}{2}\left(2a^2-b^2+2b^2-a^2\right)$$

$$=\frac{1}{2}\left(a^2+b^2\right)=\frac{1}{2}|z|^2$$

Ans: (1)

4. Given $\bar{z}=\frac{1}{i-2}=\frac{1}{-2+i}$

$$z=\bar{\bar{z}}=\overline{\left(\frac{1}{-2+i}\right)}=\frac{\bar{1}}{\overline{-2+i}}=\frac{1}{-2-i}=\frac{-1}{2+i}=\frac{-1}{i+2}$$

Ans: (2)

5. Given $z=\dfrac{\left(\sqrt{3}+i\right)^3(3i+4)^2}{(8+6i)^2}$

$$|z|=\frac{\left|\left(\sqrt{3}+i\right)^3\right|\left|(3i+4)^2\right|}{\left|(8+6i)^2\right|}=\frac{\left|\sqrt{3}+i\right|^3\left|4+3i\right|^2}{\left|8+6i\right|^2}$$

$$=\frac{\left(\sqrt{\left(\sqrt{3}\right)^2+1^2}\right)^3\left(\sqrt{4^2+3^2}\right)^2}{\left(\sqrt{8^2+6^2}\right)^2}$$

$$=\frac{\left(\sqrt{3+1}\right)^3\left(\sqrt{16+9}\right)^2}{\left(\sqrt{64+36}\right)^2}$$

$$=\frac{\left(\sqrt{4}\right)^3\left(\sqrt{25}\right)^2}{\left(\sqrt{100}\right)^2}=\frac{(2)^3\,(5)^2}{(10)^2}$$

$$=\frac{8\times 25}{100}=\frac{200}{100}=2$$ Ans : (3)

6. Given that $2iz^2=\bar{z}$

$$\left|2iz^2\right|=\left|\bar{z}\right|$$

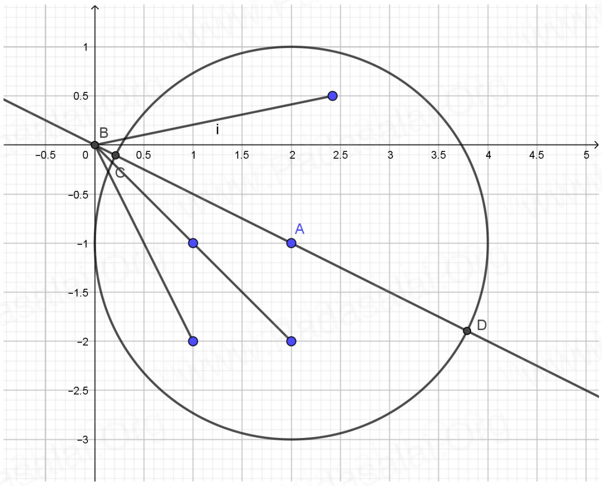
$$|2i|\left|z^2\right|=|z|$$

$$\sqrt{0^2+2^2}\,|z|^2=|z|$$

$$2|z|^2=|z|$$

$$|z|=\frac{1}{2}$$ Ans : (1)

7. Given inequality $|z-2+i|\le 2$ represents all points that lies either inside or on the circel with center at $C(2,-1)$ with radius 2 units $|z|$ is the distance of such points from the origin. To get the maximum value of $|z|$, we have to find the farthest point on this circular region from the origin.



If we draw a line from the origin through the centre of the circle $C(2,-1)$, this line will intersect the circle at two points. Among these two points, one lying between the origin and centre is the nearest and other one is farthest.

Hence the largest distance is given by $OC+r$ where OC is the distance between the orgin and Centre and r is the radius of the circle.

Hence $|z|\le OC+r$

$$|z|\le|2-i|+2$$

$$|z|\le\sqrt{2^2+(-1)^2}+2$$

$$|z|\le\sqrt{4+1}+2$$

$$|z|\le\sqrt{5}+2$$ Ans : (4)

8. Given that $\left|z-\dfrac{3}{z}\right|=2$

$$2=\left|z-\frac{3}{z}\right|\ge\left||z|-\left|\frac{3}{z}\right|\right|$$

$$\left||z|-\left|\frac{3}{z}\right|\right|\le 2$$

$$-2\le|z|-\left|\frac{3}{z}\right|\le 2$$

$$-2 \le |z| - \frac{3}{|z|} \le 2$$

$$-2 \le |z| - \frac{3}{|z|} \le 2$$

$$-2|z| \le |z|^2 - 3 \le 2|z|$$

From $-2|z| \le |z|^2 - 3$, we get,

$$0 \le |z|^2 + 2|z| - 3$$

$$|z|^2 + 2|z| - 3 \ge 0$$

$$(|z|+3)(|z|-1) \ge 0$$

$$|z| \ge 1 \text{ (or) } |z| \le -3$$

As $|z|$ is a non-negative real number, $|z|$ cannot be less than -3. Hence we must have,

$$|z| \ge 1$$ Ans :(1)

9. Since $|z| = 1$, we have

$$|z|^2 = 1 \Rightarrow z\bar{z} = 1 \Rightarrow \bar{z} = \frac{1}{z}$$

$$\frac{1+z}{1+\bar{z}} = \frac{1+z}{1+\frac{1}{z}} = \frac{1+z}{\frac{z+1}{z}} = \frac{1+z}{\frac{1+z}{z}} = \frac{1}{1/z} = z$$ Ans :(1)

10. Let $z = x + iy$.

From the given equation, we get,

$$|z| - z = 1 + 2i$$

$$|x+iy| - (x+iy) = 1 + 2i$$

$$\sqrt{x^2+y^2} - x - iy = 1 + 2i$$

Equating the real and imaginary parts, we get,

$$\sqrt{x^2+y^2} - x = 1 \text{ and } -y = 2$$

From $-y = 2$, we get $y = -2$.

Substituting $y = -2$ in $\sqrt{x^2+y^2} - x = 1$, we get,

$$\sqrt{x^2+(-2)^2} - x = 1$$

$$\sqrt{x^2+4} = 1 + x$$

Squaring on both sides,

$$x^2 + 4 = (1+x)^2$$

$$x^2 + 4 = 1 + 2x + x^2$$

$$2x = 4 - 1$$

$$2x = 3 \Rightarrow x = \frac{3}{2}$$

Hence the required number is

$$z = x + iy = \frac{3}{2} - 2i$$ Ans : (1)

11. Given $|z_1| = 1; |z_2| = 2; |z_3| = 3$. Hence

$$|z_1|^2 = 1; |z_2|^2 = 4; |z_3|^2 = 9$$

$$z_1\overline{z_1} = 1;\ z_2\overline{z_2} = 4;\ z_3\overline{z_3} = 9$$

$$z_1 = \frac{1}{\overline{z_1}};\ z_2 = \frac{4}{\overline{z_2}};\ z_3 = \frac{9}{\overline{z_3}}$$

Then

$$z_1 + z_2 + z_3 = \frac{1}{\overline{z_1}} + \frac{4}{\overline{z_2}} + \frac{9}{\overline{z_3}}$$

$$z_1 + z_2 + z_3 = \frac{\overline{z_2}\,\overline{z_3} + 4\overline{z_1}\,\overline{z_3} + 9\overline{z_1}\,\overline{z_2}}{\overline{z_1}\,\overline{z_2}\,\overline{z_3}}$$

$$= \frac{9\overline{z_1}\,\overline{z_2} + 4\overline{z_1}\,\overline{z_3} + \overline{z_2}\,\overline{z_3}}{\overline{z_1}\,\overline{z_2}\,\overline{z_3}}$$

$$= \frac{\overline{9z_1z_2} + \overline{4z_1z_3} + \overline{z_2z_3}}{\overline{z_1z_2z_3}}$$

$$= \frac{\overline{9z_1z_2 + 4z_1z_3 + z_2z_3}}{\overline{z_1z_2z_3}}$$

$$= \overline{\left(\frac{9z_1z_2 + 4z_1z_3 + z_2z_3}{z_1z_2z_3}\right)}$$

Now

$$|z_1+z_2+z_3| = \left|\overline{\left(\frac{9z_1z_2+4z_1z_3+z_2z_3}{z_1z_2z_3}\right)}\right|$$

$$= \left|\frac{9z_1z_2+4z_1z_3+z_2z_3}{z_1z_2z_3}\right|$$

$$= \frac{|9z_1z_2+4z_1z_3+z_2z_3|}{|z_1z_2z_3|}$$

$$= \frac{|9z_1z_2+4z_1z_3+z_2z_3|}{|z_1||z_2||z_3|}$$

$$= \frac{12}{1\times2\times3} = 2$$

Ans: (2)

12. Let $w = z + \frac{1}{z}$. Since w is a real number, we have

$$w = \overline{w}$$

$$z + \frac{1}{z} = \overline{\left(z + \frac{1}{z}\right)}$$

$$z + \frac{1}{z} = \overline{z} + \frac{1}{\overline{z}}$$

$$z - \frac{1}{\overline{z}} = \overline{z} - \frac{1}{z}$$

$$\frac{z\overline{z}-1}{\overline{z}} = \frac{z\overline{z}-1}{z}$$

$$z(z\overline{z}-1) = \overline{z}(z\overline{z}-1)$$

$$z(z\overline{z}-1) - \overline{z}(z\overline{z}-1) = 0$$

$$(z-\overline{z})(z\overline{z}-1) = 0$$

As $z - \overline{z} \neq 0$, we must have

$$z\overline{z} - 1 = 0$$

$$|z|^2 = 1 \Rightarrow |z| = 1$$

Ans: (2)

13. Given that

$$|z_1| = 1 \Rightarrow |z_1|^2 = 1^2 \Rightarrow z_1\overline{z_1} = 1 \Rightarrow z_1 = \frac{1}{\overline{z_1}}$$

Similarly, $z_2 = \frac{1}{\overline{z_2}}$ and $z_3 = \frac{1}{\overline{z_3}}$

Now $z_1 + z_2 + z_3 = 0$

$$\Rightarrow \frac{1}{\overline{z_1}} + \frac{1}{\overline{z_2}} + \frac{1}{\overline{z_3}} = 0$$

$$\Rightarrow \frac{\overline{z_2}\,\overline{z_3} + \overline{z_1}\,\overline{z_3} + \overline{z_1}\,\overline{z_2}}{\overline{z_1}\,\overline{z_2}\,\overline{z_3}} = 0$$

$$\overline{z_2}\,\overline{z_3} + \overline{z_1}\,\overline{z_3} + \overline{z_1}\,\overline{z_2} = 0$$

$$\overline{z_2z_3} + \overline{z_1z_3} + \overline{z_1z_2} = 0$$

$$\overline{z_2z_3 + z_1z_3 + z_1z_2} = 0$$

$$z_2z_3 + z_1z_3 + z_1z_2 = 0$$

Again from $z_1 + z_2 + z_3 = 0$, we get,

$$(z_1+z_2+z_3)^2 = 0$$

$$z_1^2 + z_2^2 + z_3^2 + 2(z_2z_3 + z_1z_3 + z_1z_2) = 0$$

$$z_1^2 + z_2^2 + z_3^2 + 2(0) = 0$$

$$z_1^2 + z_2^2 + z_3^2 = 0$$

Ans : (4)

14. Since $\frac{z-1}{z+1}$ is purely imaginary, we have

$$\overline{\left(\frac{z-1}{z+1}\right)} = -\frac{z-1}{z+1} \Rightarrow \frac{\overline{z-1}}{\overline{z+1}} = -\frac{z-1}{z+1}$$

$$\frac{\overline{z}-\overline{1}}{\overline{z}+\overline{1}} = -\frac{z-1}{z+1} \Rightarrow \frac{\overline{z}-1}{\overline{z}+1} = -\frac{z-1}{z+1}$$

$$(\overline{z}-1)(z+1) = -(z-1)(\overline{z}+1)$$

$$z\overline{z} + \overline{z} - z - 1 = -(z\overline{z} + z - \overline{z} - 1)$$

$$z\overline{z} + \overline{z} - z - 1 = -z\overline{z} - z + \overline{z} - 1$$

$$2z\overline{z} = 2 \Rightarrow z\overline{z} = 1 \Rightarrow |z|^2 = 1$$

$$|z| = 1$$

Ans : (2)

15. Given that

$$|z+2|=|z-2|$$
$$|x+iy+2|=|x+iy-2|$$
$$|x+2+iy|=|x-2+iy|$$
$$\sqrt{(x+2)^2+y^2}=\sqrt{(x-2)^2+y^2}$$
$$(x+2)^2+y^2=(x-2)^2+y^2$$
$$x^2+2x+4=x^2-2x+4$$
$$4x=0 \Rightarrow x=0$$

Hence the locus is the imaginary axis.

Ans: (2)

16. Let $z_1=3=3+i0$; $z_2=-1+i$

Since the point representing the complex number z_1 lies on the positive real axis, we have $Arg\, z_1=0$.

But for z_2,

$$\alpha=\tan^{-1}\left|\frac{y}{x}\right|=\tan^{-1}\left|\frac{1}{-1}\right|=\tan^{-1}(1)=\frac{\pi}{4}$$

Since the point representing z_2 lies on the second quadrant, we have

$$Arg\, z_2=\pi-\alpha=\pi-\frac{\pi}{4}=\frac{4\pi-\pi}{4}=\frac{3\pi}{4}$$

$$Arg\left(\frac{3}{-1+i}\right)=Arg\left(\frac{z_1}{z_2}\right)$$
$$=Arg\ z_1-Arg\ z_1$$
$$=0-\frac{3\pi}{4}=-\frac{3\pi}{4}$$

Ans: (3)

17. $(\sin 40^\circ+i\cos 40^\circ)^5$

$$=(\sin(90^\circ-50^\circ)+i\cos(90^\circ-50^\circ))^5$$
$$=(\cos 50^\circ+i\sin 50^\circ)^5$$
$$=\cos(5\times 50^\circ)+i\sin(5\times 50^\circ)$$
$$=\cos(250^\circ)+i\sin(250^\circ)$$
$$=\cos(360^\circ-110^\circ)+i\sin(360^\circ-110^\circ)$$
$$=\cos(110^\circ)-i\sin(110^\circ)$$
$$=\cos(-110^\circ)+i\sin(-110^\circ)$$

The principal argument of the given complex number is -110°.

Ans: (1)

18. Given that

$$(1+i)(1+2i)(1+3i)\ldots(1+ni)=x+iy$$

Taking modulus on both side, we get,

$$|(1+i)(1+2i)(1+3i)\ldots(1+ni)|=|x+iy|$$
$$|1+i||1+2i||1+3i|\ldots|1+ni|=|x+iy|$$
$$\sqrt{1^2+1^2}\sqrt{1^2+2^2}\sqrt{1^2+3^2}\ldots\sqrt{1^2+n^2}=\sqrt{x^2+y^2}$$
$$\sqrt{2}\sqrt{5}\sqrt{10}\ldots\sqrt{1+n^2}=\sqrt{x^2+y^2}$$

Squaring on both sides, we get,

$$2\cdot 5\cdot 10\ldots(1+n^2)=x^2+y^2$$

Ans: (3)

19. Given that

$$(1+\omega)^7=A+B\omega$$
$$(-\omega^2)^7=A+B\omega$$
$$-\omega^{14}=A+B\omega$$
$$-\omega^2=A+B\omega$$
$$1+\omega=A+B\omega$$

Equating the corresponding the coefficients, we get, $A=1$; $B=1$.

Ans: (4)

20. Let $z_1=1+i\sqrt{3}$

$$\alpha_1=\tan^{-1}\left|\frac{y}{x}\right|=\tan^{-1}\left|\frac{\sqrt{3}}{1}\right|=\tan^{-1}(\sqrt{3})=\frac{\pi}{3}$$

Since the point representing the complex number z_1 lies in the first quadrant, we have

$$\theta_1=Arg\, z_1=\alpha_1=\frac{\pi}{3}$$

Let $z_2 = 4i$.

Since the point representing the complex number z_2 lies on the positive imaginary axis, we have

$$\theta_2 = Arg\, z_2 = \frac{\pi}{2}$$

Let $z_3 = 1 - i\sqrt{3}$

$$\alpha_3 = \tan^{-1}\left|\frac{y}{x}\right| = \tan^{-1}\left|\frac{-\sqrt{3}}{1}\right| = \tan^{-1}\left(\sqrt{3}\right) = \frac{\pi}{3}$$

Since the point representing the complex number z_3 lies in the fourth quadrant, we have

$$\theta_3 = Arg\, z_3 = -\alpha_3 = -\frac{\pi}{3}$$

Now

$$Arg\left(\frac{\left(1+i\sqrt{3}\right)^2}{4i\left(1-i\sqrt{3}\right)}\right)$$

$$= Arg\left(\frac{z_1^{\,2}}{z_2 z_3}\right)$$

$$= Arg\, z_1^{\,2} - Arg\, z_2 - Arg\, z_3$$

$$= 2Arg\, z_1 - Arg\, z_2 - Arg\, z_3$$

$$= 2\left(\frac{\pi}{3}\right) - \frac{\pi}{2} - \left(-\frac{\pi}{3}\right)$$

$$= \frac{2\pi}{3} - \frac{\pi}{2} + \frac{\pi}{3}$$

$$= \frac{2\pi}{3} + \frac{\pi}{3} - \frac{\pi}{2}$$

$$= \pi - \frac{\pi}{2} = \frac{\pi}{2}$$ Ans: (4)

21. Given that α and β are the roots of the equation $x^2 + x + 1 = 0$.

We know that the roots of the equation $x^2 + x + 1 = 0$ are

$$\alpha = \frac{-1+i\sqrt{3}}{2} = \omega \text{ and } \beta = \frac{-1-i\sqrt{3}}{2} = \omega^2$$

Hence

$$\alpha^{2020} + \beta^{2020} = \omega^{2020} + \left(\omega^2\right)^{2020}$$

$$= \omega^{2020} + \omega^{4040}$$

$$= \omega + \omega^2 = -1$$

22. Let $x = \left(\cos\frac{\pi}{3} + i\sin\frac{\pi}{3}\right)^{3/4}$

$$x = \left(\cos\frac{3\pi}{3} + i\sin\frac{3\pi}{3}\right)^{1/4}$$

$$x = \left(\cos\pi + i\sin\pi\right)^{1/4}$$

$$x = (-1)^{1/4}$$

$$x^4 = -1$$

$$x^4 + 1 = 0$$

By Vieta's formula,

Product of all roots $= +\frac{1}{1} = 1$ Ans: (3)

23. $$\begin{vmatrix} 1 & 1 & 1 \\ 1 & -\omega^2 - 1 & \omega^2 \\ 1 & \omega^2 & \omega^7 \end{vmatrix}$$

$$= \begin{vmatrix} 1 & 1 & 1 \\ 1 & \omega & \omega^2 \\ 1 & \omega^2 & \omega \end{vmatrix}$$

$$= \begin{vmatrix} 3 & 1 & 1 \\ 1+\omega+\omega^2 & \omega & \omega^2 \\ 1+\omega+\omega^2 & \omega^2 & \omega \end{vmatrix} \quad C_1 \to C_1 + C_2 + C_3$$

$$= \begin{vmatrix} 3 & 1 & 1 \\ 0 & \omega & \omega^2 \\ 0 & \omega^2 & \omega \end{vmatrix}$$

$$= 3\left(\omega^2 - \omega\right)$$

Now

$$\omega^2 - \omega = -\frac{1}{2} - i\frac{\sqrt{3}}{2} - \left(-\frac{1}{2} + i\frac{\sqrt{3}}{2}\right)$$

$$= -\frac{1}{2} - i\frac{\sqrt{3}}{2} + \frac{1}{2} - i\frac{\sqrt{3}}{2} = -2i\frac{\sqrt{3}}{2} = -i\sqrt{3}$$

Given that

$$\begin{vmatrix} 1 & 1 & 1 \\ 1 & -\omega^2 - 1 & \omega^2 \\ 1 & \omega^2 & \omega^7 \end{vmatrix} = 3k$$

$$3\left(\omega^2 - \omega\right) = 3k$$

$$k = \omega^2 - \omega$$

$$k = -i\sqrt{3}$$ Ans : (4)

24. Let $z_1 = 1 + i\sqrt{3}$

$$r = |z| = \sqrt{1^2 + \left(\sqrt{3}\right)^2} = \sqrt{1+3} = \sqrt{4} = 2$$

$$\alpha_1 = \tan^{-1}\left|\frac{y}{x}\right| = \tan^{-1}\left|\frac{\sqrt{3}}{1}\right| = \tan^{-1}\left(\sqrt{3}\right) = \frac{\pi}{3}$$

Since the point representing the complex number z_1 lies in the first quadrant, we have

$$\theta_1 = Arg\, z_1 = \alpha_1 = \frac{\pi}{3}$$

Now $z_1 = r\left(\cos\theta_1 + i\sin\theta_1\right)$

$$1 + i\sqrt{3} = 2\left(\cos\frac{\pi}{3} + i\sin\frac{\pi}{3}\right)$$

Let $z_2 = 1 - i\sqrt{3}$

$$\alpha_2 = \tan^{-1}\left|\frac{y}{x}\right| = \tan^{-1}\left|\frac{-\sqrt{3}}{1}\right| = \tan^{-1}\left(\sqrt{3}\right) = \frac{\pi}{3}$$

Since the point representing the complex number z_2 lies in the fourth quadrant, we have

$$\theta_2 = Arg\, z_2 = -\alpha_2 = -\frac{\pi}{3}$$

Now $z_2 = r\left(\cos\theta_2 + i\sin\theta_2\right)$

$$1 - i\sqrt{3} = 2\left(\cos\left(-\frac{\pi}{3}\right) - i\sin\left(-\frac{\pi}{3}\right)\right)$$

Now

$$\frac{1+i\sqrt{3}}{1-i\sqrt{3}} = \frac{2\left(\cos\frac{\pi}{3} + i\sin\frac{\pi}{3}\right)}{2\left(\cos\left(-\frac{\pi}{3}\right) + i\sin\left(-\frac{\pi}{3}\right)\right)}$$

$$= \cos\left(\frac{\pi}{3} - \left(-\frac{\pi}{3}\right)\right) + i\sin\left(\frac{\pi}{3} - \left(-\frac{\pi}{3}\right)\right)$$

$$= \cos\left(\frac{\pi}{3} + \frac{\pi}{3}\right) + i\sin\left(\frac{\pi}{3} + \frac{\pi}{3}\right)$$

$$= \cos\left(\frac{2\pi}{3}\right) + i\sin\left(\frac{2\pi}{3}\right)$$

$$\left(\frac{1+i\sqrt{3}}{1-i\sqrt{3}}\right)^{10} = \left(\cos\left(\frac{2\pi}{3}\right) + i\sin\left(\frac{2\pi}{3}\right)\right)^{10}$$

$$= \cos\left(\frac{20\pi}{3}\right) + i\sin\left(\frac{20\pi}{3}\right)$$

$$= \cos\left(6\pi + \frac{2\pi}{3}\right) + i\sin\left(6\pi + \frac{2\pi}{3}\right)$$

$$= \cos\left(\frac{2\pi}{3}\right) + i\sin\left(\frac{2\pi}{3}\right)$$

$$= cis\left(\frac{2\pi}{3}\right)$$ Ans : (1)

25. $$\begin{vmatrix} z+1 & \omega & \omega^2 \\ \omega & z+\omega^2 & 1 \\ \omega^2 & 1 & z+\omega \end{vmatrix} = 0$$

$$\begin{vmatrix} z+1+\omega+\omega^2 & \omega & \omega^2 \\ z+1+\omega+\omega^2 & z+\omega^2 & 1 \\ z+1+\omega+\omega^2 & 1 & z+\omega \end{vmatrix} = 0$$

$$\begin{vmatrix} z & \omega & \omega^2 \\ z & z+\omega^2 & 1 \\ z & 1 & z+\omega \end{vmatrix} = 0$$

$$z\begin{vmatrix} 1 & \omega & \omega^2 \\ 1 & z+\omega^2 & 1 \\ 1 & 1 & z+\omega \end{vmatrix} = 0$$

$$z = 0 \ \text{(or)} \ \begin{vmatrix} 1 & \omega & \omega^2 \\ 1 & z+\omega^2 & 1 \\ 1 & 1 & z+\omega \end{vmatrix} = 0$$

$z = 0$ is a solution. Other solutions are given by

$$\begin{vmatrix} 1 & \omega & \omega^2 \\ 1 & z+\omega^2 & 1 \\ 1 & 1 & z+\omega \end{vmatrix} = 0$$

$$(z+\omega)(z+\omega^2) - 1 - \omega(z+\omega-1)$$
$$+\omega^2(1-z-\omega^2) = 0$$
$$z^2 + z\omega + z\omega^2 + \omega^3 - 1 - z\omega - \omega^2 + \omega$$
$$+\omega^2 - z\omega^2 - \omega^3 = 0$$
$$z^2 - 1 + \omega = 0$$
$$z^2 = 1 - \omega$$

which has two solutions. Hence the given equation has three distinct solutions. Ans : (3)

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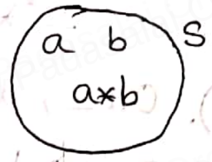
12. DISCRETE MATHEMATICS

Binary Operation (or) closure property

$\forall\, a,b \in S$, $a*b$ is unique and $a*b \in S$

where S is any nonempty set.



Properties

1) Commutative property

$a*b = b*a \quad \forall\, a,b \in S$

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2) Associative property

$a*(b*c) = (a*b)*c \quad \forall\, a,b,c \in S$

3) Existence of identity property

$e \in S$ is an identity element

$a*e = e*a = a \quad \forall\, a \in S$

4) Existence of Inverse property

$b \in S$ is said to be Inverse element of a

$a*b = b*a = e \quad \forall\, a \in S \qquad b = a^{-1}$

Exercise 12.1

1) Determine whether $*$ is a binary operation on the sets given below

(i) $a*b = a.|b|$ on R

Let $a,b \in R$ $\quad |b| \in R$

$a.|b| \in R$

$\Rightarrow a*b \in R \quad \forall\, a,b \in R$

$*$ is the binary operation on R.

(ii) $a*b = \min(a,b)$ on $A = \{1,2,3,4,5\}$

Let $a,b \in A$

$\min(a,b) \in A$

$a*b \in A$

$\forall\, a,b \in A$

For example

$\min(1,2) = 1 \in A$

$\min(1,5) = 1 \in A$

$\min(2,3) = 2 \in A$

$\therefore *$ is the binary operation on A.

(iii) $a*b = a\sqrt{b}$ is binary on R

Let $a, b \in R$

root of negative numbers not in R

$\therefore a\sqrt{b} \notin R$ [$\because$ $2, -2 \in R$, $2\sqrt{-2} \notin R$]

$\Rightarrow a*b \notin R$

$\therefore *$ is not binary on R.

2). On Z, define $\otimes$ by $(m \otimes n) = m^n + n^m \ \forall m, n \in Z$

Is $\otimes$ binary on Z

Let $m, n \in Z$

take $m = 2 \quad n = -2$

$m^n + n^m = 2^{-2} + (-2)^2$

$= \frac{1}{4} + 4 = \frac{17}{4} \notin Z$

$m^n + n^m \notin Z$

$\therefore m \otimes n \notin Z$ $\therefore \otimes$ is not binary on Z.

3) Let $*$ be defined on R by $(a*b) = a+b+ab-7$.

Is $*$ binary on R? If so find $3*\left(\frac{-7}{15}\right)$

Let $a, b \in R$

clearly $a, b, a.b \in R$

$\therefore a+b+ab-7 \in R$

$\Rightarrow a*b \in R$

$\therefore *$ is binary operator on R

$3*\left(\frac{-7}{15}\right) = 3 - \frac{7}{15} + 3\left(\frac{-7}{15}\right) - 7$

$= \frac{45-7-21-105}{15} = \frac{45-133}{15} = \frac{-88}{15}$

4) Let $A = \{a+\sqrt{5}b ; a, b \in Z\}$, check whether the usual multiplication is a binary operation on A

Let $x = a+\sqrt{5}b \quad y = c+d\sqrt{5}$

$x, y \in A \quad , a, b, c, d \in Z$

$xy = (a+\sqrt{5}b)(c+\sqrt{5}d)$

$= ac + 5bd + \sqrt{5}ad + \sqrt{5}bc$

$= (ac+5bd) + \sqrt{5}(ad+bc) \in A$

$\therefore xy \in A$

multiplicatic is binary on R.

5) (i). Define an operation $*$ on Q as follows

$a*b = \frac{a+b}{2}$; $a,b \in Q$. Examine the closure, commutative and associative properties satisfied by $*$ on Q.

① closure property

given $a,b \in Q \Rightarrow a+b \in Q \Rightarrow \frac{a+b}{2} \in Q \Rightarrow a*b \in Q$. $\forall a,b \in Q$.

$*$ is closure on Q.

② commutative property

$$a*b = \frac{a+b}{2}$$

$$= \frac{b+a}{2}$$

$$a*b = b*a \quad \forall a,b \in Q$$

commutative property is true. $\therefore *$ is commutative on Q.

③ associative property

$$a*(b*c) = a*\left(\frac{b+c}{2}\right)$$

$$= \frac{a+\left(\frac{b+c}{2}\right)}{2} = \frac{2a+b+c}{2} \times \frac{1}{2}$$

$$a*(b*c) = \frac{2a+b+c}{4} \quad \text{--- (1)}$$

$$(a*b)*c = \left(\frac{a+b}{2}\right)*c$$

$$= \frac{\left(\frac{a+b}{2}\right)+c}{2} = \frac{a+b+2c}{2} \times \frac{1}{2}$$

$$= \frac{a+b+2c}{4} \quad \text{--- (2)}$$

$\because$ From (1) & (2) $(a*b)*c \neq a*(b*c)$

$*$ is not associative on Q.

(ii) Define an operation $*$ on Q as follows $a*b = \frac{a+b}{2}$ $a,b \in Q$. Examine the existence of identity and the existence of inverse for the operation $*$ on Q.

(i) Existence of Identity

Let $a \in Q$, e be the identity element on Q.

By definition of $*$ $\quad a*e = \frac{a+e}{2}$

By definition of e $\quad a*e = a$.

$$\frac{a+e}{2}=a$$

$$a+e=2a$$

$$e=2a-a$$

$$e=a \quad \forall a \in Q$$

This means every element is a identity element

This is not possible.

$*$ has No Identity element.

(ii) Existence of Inverse

$*$ has no Identity element

∴ cannot defined as $a*a^{-1}=a^{-1}*a=e$.

∴ $*$ has no Inverse.

6) Fill in the following table so that the binary operation $*$ on $A=\{a,b,c\}$ is commutative.

*	a	b	c
a	b		
b	c	b	a
c	a		c

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i) From table. $b*a=c$ [$*$ is commutative]

$\Rightarrow a*b=c$.

ii) $c*a=a \Rightarrow a*c=a$

iii) $b*c=a \Rightarrow c*b=a$

∴ given table is

*	a	b	c
a	b	c	a
b	c	b	a
c	a	a	c

7) consider the binary operation $*$ defined on the set $A=\{a,b,c,d\}$ by the following table.

*	a	b	c	d
a	a	c	b	d
b	d	a	b	c
c	c	d	a	a
d	d	b	a	c

Is it commutative and associative?

(1) <u>commutative property</u>

$a*b=c$ but $b*a=d \Rightarrow a*b \neq b*a$

$a*c=b$ but $c*a=c \Rightarrow a*c \neq c*a$

$\therefore *$ is not commutative.

2) <u>associative Property</u>

$(a*b)*c = c*c = a$

$a*(b*c) = a*b = c$

$\therefore (a*b)*c \neq a*(b*c)$

$*$ is not associative.

8) Let $A=\begin{pmatrix}1&0&1&0\\0&1&0&1\\1&0&0&1\end{pmatrix}$, $B=\begin{pmatrix}0&1&0&1\\1&0&1&0\\1&0&0&1\end{pmatrix}$, $C=\begin{pmatrix}1&1&0&1\\0&1&1&0\\1&1&1&1\end{pmatrix}$

be any three boolean matrices of the same type

Find (i) $A \vee B$ (ii) $A \wedge B$ (iii) $(A \vee B) \wedge C$ (iv) $(A \wedge B) \vee C$.

(i) $A \vee B = \begin{pmatrix}1&0&1&0\\0&1&0&1\\1&0&0&1\end{pmatrix} \vee \begin{pmatrix}0&1&0&1\\1&0&1&0\\1&0&0&1\end{pmatrix} = \begin{pmatrix}1&1&1&1\\1&1&1&1\\1&0&0&1\end{pmatrix}$

(ii) $A \wedge B = \begin{pmatrix}1&0&1&0\\0&1&0&1\\1&0&0&1\end{pmatrix} \wedge \begin{pmatrix}0&1&0&1\\1&0&1&0\\1&0&0&1\end{pmatrix} = \begin{pmatrix}0&0&0&0\\0&0&0&0\\1&0&0&1\end{pmatrix}$

(iii) $(A \vee B) \wedge C = \begin{pmatrix}1&1&1&1\\1&1&1&1\\1&0&0&1\end{pmatrix} \wedge \begin{pmatrix}1&1&0&1\\0&1&1&0\\1&1&1&1\end{pmatrix}$

$= \begin{pmatrix}1&1&0&1\\0&1&1&0\\1&0&0&1\end{pmatrix}$

(iv) $(A \wedge B) \vee C = \begin{pmatrix}0&0&0&0\\0&0&0&0\\1&0&0&1\end{pmatrix} \vee \begin{pmatrix}1&1&0&1\\0&1&1&0\\1&1&1&1\end{pmatrix} =$

$= \begin{pmatrix}1&1&0&1\\0&1&1&0\\1&1&1&1\end{pmatrix}$

9 (i) Let $M = \left\{ \begin{pmatrix} x & x \\ x & x \end{pmatrix} ; x \in R-\{0\} \right\}$ and let $*$ be the matrix multiplication. Determine whether M is closed under $*$. If so examine the commutative and associative properties satisfied by $*$ on M.

Let $A = \begin{pmatrix} x & x \\ x & x \end{pmatrix}$, $B = \begin{pmatrix} y & y \\ y & y \end{pmatrix}$ where $x, y \in R-\{0\}$

$A, B \in M$

1) closure property

$A, B \in M$

$\Rightarrow A*B = \begin{pmatrix} x & x \\ x & x \end{pmatrix}\begin{pmatrix} y & y \\ y & y \end{pmatrix}$

$= \begin{pmatrix} 2xy & 2xy \\ 2xy & 2xy \end{pmatrix} \in M$ $\because x, y \in R-\{0\}$, $xy \in R-\{0\}$

$A*B \in M$

$A, B \in M \Rightarrow A*B \in M$

$\therefore *$ is closed on M

2) commutative property

$A, B \in M$

$A*B = \begin{pmatrix} x & x \\ x & x \end{pmatrix}\begin{pmatrix} y & y \\ y & y \end{pmatrix}$

$= \begin{pmatrix} 2xy & 2xy \\ 2xy & 2xy \end{pmatrix} = \begin{pmatrix} 2yx & 2yx \\ 2yx & 2yx \end{pmatrix}$

$= \begin{pmatrix} y & y \\ y & y \end{pmatrix}\begin{pmatrix} x & x \\ x & x \end{pmatrix}$

$A*B = B*A$

$*$ is commutative on M.

3) associative property

Matrix multiplication is always associative.

i.e $A*(B*C) = (A*B)*C$. $\forall A, B, C \in M$

$*$ is associative on M.

(ii) Let $M = \left\{ \begin{pmatrix} x & x \\ x & x \end{pmatrix} ; x \in R-\{0\} \right\}$ and let $*$ be the matrix multiplication. Determine whether M is closed under $*$. If so examine the existence of identity, inverse property for $*$ on M.

① Closure property

$A, B \in M$ where $A = \begin{pmatrix} x & x \\ x & x \end{pmatrix}$, $B = \begin{pmatrix} y & y \\ y & y \end{pmatrix}$ $x, y \in R - \{0\}$

$$A * B = \begin{pmatrix} 2xy & 2xy \\ 2xy & 2xy \end{pmatrix} \in M$$

$$A * B \in M$$

$A, B \in M \Rightarrow A * B \in M$

$\therefore *$ is closed on M.

② Existence of Identity property

Let $A \in M$, $E = \begin{pmatrix} e & e \\ e & e \end{pmatrix}$ be the Identity element

$$AE = A$$

$$\begin{pmatrix} x & x \\ x & x \end{pmatrix} \begin{pmatrix} e & e \\ e & e \end{pmatrix} = \begin{pmatrix} x & x \\ x & x \end{pmatrix}$$

$$\begin{pmatrix} 2xe & 2xe \\ 2xe & 2xe \end{pmatrix} = \begin{pmatrix} x & x \\ x & x \end{pmatrix}$$

$$2\cancel{x}e = \cancel{x}1$$

$$2e = 1 \quad e = \tfrac{1}{2} \in R - \{0\}$$

Identity element $E = \begin{pmatrix} 1/2 & 1/2 \\ 1/2 & 1/2 \end{pmatrix} \in M$

satisfies $AE = A$

similarly $EA = A$ $\forall A \in M$

$*$ has Identity element on M.

③ Existence of Inverse property.

Let $A \in M$, $A^{-1} = \begin{pmatrix} x^{-1} & x^{-1} \\ x^{-1} & x^{-1} \end{pmatrix}$ be the inverse of A

$$AA^{-1} = E$$

$$\begin{pmatrix} x & x \\ x & x \end{pmatrix} \begin{pmatrix} x^{-1} & x^{-1} \\ x^{-1} & x^{-1} \end{pmatrix} = \begin{pmatrix} 1/2 & 1/2 \\ 1/2 & 1/2 \end{pmatrix}$$

$$\begin{pmatrix} 2xx^{-1} & 2xx^{-1} \\ 2xx^{-1} & 2xx^{-1} \end{pmatrix} = \begin{pmatrix} 1/2 & 1/2 \\ 1/2 & 1/2 \end{pmatrix}$$

$$2xx^{-1} = \tfrac{1}{2}$$

$$x^{-1} = \frac{1}{4x} \in R - \{0\}$$

$A^{-1} = \begin{pmatrix} \frac{1}{4x} & \frac{1}{4x} \\ 1/4x & 1/4x \end{pmatrix} \in M$ is the inverse of A in M

similarly we can find $A^{-1}A = E$ $\forall A \in M$

$*$ is Inverse on M.

—x—

10) (i) Let A be $Q-\{1\}$. Define $*$ on A by $x*y=x+y-xy$. Is $*$ binary on A? If so examine the commutative and associative properties satisfied by $*$ on A.

$A=Q-\{1\}$ $\quad x,y\in A \quad x\neq 1 \quad y\neq 1$

$*$ is defined by $x*y=x+y-xy$

① closure property

Let $x,y\in A$, $\quad x\neq 1, y\neq 1$

$$x-1\neq 0,\ y-1\neq 0$$
$$(x-1)(y-1)\neq 0$$
$$xy-x-y+1\neq 0$$
$$1\neq x+y-xy$$
$$x*y\neq 1 \Rightarrow x*y\in A$$

$x,y\in A\Rightarrow x*y\in A$.

$*$ is closed on A.

② commutative property:-

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Let $x,y\in A$ $\quad x*y=x+y-xy$
$$=y+x-yx$$
$$x*y=y*x$$

$*$ is commutative on A.

③ Associative property:-

$$(x*y)*z=(x+y-xy)*z$$
$$=(x+y-xy)+z-(x+y-xy)z$$
$$(x*y)*z=x+y+z-xy-xz-yz+xyz \quad \text{——①}$$
$$x*(y*z)=x*(y+z-yz)$$
$$=x+(y+z-yz)-x(y+z-yz)$$
$$x*(y*z)=x+y+z-xy-xz-yz+xyz \quad \text{——②}$$

From ①&② $(x*y)*z=x*(y*z)$ $\quad \forall x,y,z\in A$

$*$ is associative on A

(ii) examine the existence of identity, inverse properties for $*$ on A.

① closure property (same as on above)

② existence of Identity property.

Let $x,a\in A$, e be the identity element

By definition of $*$ $x*e = x+e-xe$

By definition of e $x*e = x$

$$\cancel{x}+e-xe=\cancel{x}$$

$$e(1-x)=0$$

$$e=\frac{0}{1-x} \qquad e=0 \in A$$

Identity element $\boxed{e=0 \in A}$

$*$ has Identity element on A

③ Existence of Inverse property :-

Let $x \in A$, x^{-1} be the inverse of x

By definition of $*$ $x*x^{-1} = x+x^{-1}-xx^{-1}$

By definition of x^{-1} $x*x^{-1} = e$

$$x+x^{-1}-xx^{-1}=0$$

$$x^{-1}(1-x)=-x$$

$$\boxed{x^{-1}=\frac{-x}{1-x} \in A}$$

Inverse of x is $x^{-1}=\frac{-x}{1-x} \in A$ $\forall x \in A$

$*$ has Inverse element for $\forall x \in A$.

—x—

Mathematical Logic

Truth tables

① Truth table for NOT

P	¬P
T	F
F	T

② Truth table for AND

P	q	P∧q
T	T	T
T	F	F
F	T	F
F	F	T

③ Truth table for OR

P	q	PVq
T	T	T
T	F	T
F	T	T
F	F	F

④ Truth table for conditional statement.

P	q	P→q
T	T	T
T	F	F
F	T	T
F	F	T

⑤ Truth table for Bi-conditional statement.

p	q	$p \Leftrightarrow q$
T	T	T
T	F	F
F	T	F
F	F	T

⑥ Truth table for Exclusive OR (EOR) $\bar{\vee}$

p	q	$p \bar{\vee} q$
T	T	F
T	F	T
F	T	T
F	F	F

⑦ Tautology

A statement is said to be tautology if its truth values is always T irrespective of the truth values of its compound statements, (denoted by $\mathbb{T}$)

8) contradiction

A statement is said to be contradiction if its truth value is always F irrespective of the truth values of its compound statements. (denoted by $\mathbb{F}$)

9) contingency

A statement which is neither a tautology nor a contradiction is called contingency.

10) Duality

The dual of a statement formula is obtained by replacing $\vee$ by $\wedge$, T by F, $\mathbb{T}$ by $\mathbb{F}$, $\wedge$ by $\vee$, F by T, $\mathbb{F}$ by $\mathbb{T}$.

Exercise 12.2

1) Let p : Jupiter is a planet and q: India is an Island be any two simple statements. Give verbal sentence describing each of the following statements.

p: Jupiter is a planet
q: India is an Island.

(i) $\neg p$: Jupiter is not a planet

(ii) $p \wedge \neg q$: Jupiter is a planet and India is not an Island.

(iii) $\neg p \vee q$: Jupiter is not a planet or India is an Island.

(iv) $p \rightarrow \neg q$: If Jupiter is a planet then India is not an Island.

(v) $p \leftrightarrow q$: Jupiter is a planet if and only if India is an Island.

2) write each of the following sentences in symbolic form using statement variables p and q.

p: 19 is a prime number

q: All the angles of a triangle are equal.

(i) 19 is not a prime number and all the angles of a triangle are equal.

$\neg p \wedge q$

(ii) 19 is a prime number or all the angles of a triangle are not equal.

$p \vee \neg q$

(iii) 19 is a prime number and all the angles of a triangle are equal

$p \wedge q$

(iv) 19 is not a prime number $\neg p$

3) Determine the Truth value of each of the following statements.

(i) If $6+2=5$, then the milk is white.

p: $6+2=5$ F

q: The milk is white T

Symbolic form is

$p \rightarrow q$

$F \rightarrow T$ Truth value is T

(ii) china is in Europe or $\sqrt{3}$ is an integer.

p: china is in Europe F

q: $\sqrt{3}$ is an integer F

Symbolic Form is $p \vee q$

$F \vee F$ Truth value is F

(iii) It is not true that $5+5=9$ or Earth is a planet.

p : It is true that $5+5=9$
q : Earth is a planet.

Symbolic form is $\neg p \vee q$

Truth value T V T

Truth value is T

(iv) 11 is a prime number and all the sides of a rectangle are equal.

p : 11 is a prime number
q : all the sides of a rectangle are equal.

Symbolic form is $p \wedge q$

Truth value $T \wedge F$

Truth value is F

4) Which one of the following sentences is a proposition?

(i) $4+7=12$ proposition
(ii) What are you doing? (not a proposition)
(iii) $3^n \leq 81$, $n \in N$ (proposition)
(iv) Peacock is our national bird. (proposition)
(v) How tall this mountain is! (not a proposition)

∴ (i)(iii)(iv) are propositions.

5) Write the converse, inverse, and contrapositive of each of the following implication

(i) If x and y are numbers such that $x=y$, then $x^2=y^2$

p : x and y are numbers such that $x=y$
q : $x^2=y^2$

given statement ∴ symbolic form is $p \to q$

① converse $q \to p$

If x and y are numbers such that $x^2=y^2$ then $x=y$

② Inverse: $\neg p \rightarrow \neg q$

If x and y are numbers such that $x \neq y$ then $x^2 \neq y^2$

③ contrapositive: $\neg q \rightarrow \neg p$

If x and y are numbers such that $x^2 \neq y^2$ then $x \neq y$

(ii) If a quadrilateral is a square then it is a rectangle.

p: A quadrilateral is a square.

q: A quadrilateral is a rectangle

given statement is $p \rightarrow q$

① converse:- $q \rightarrow p$.

If a quadrilateral is a rectangle then it is a square,

② Inverse:- $\neg p \rightarrow \neg q$

If a quadrilateral is not a square then it is not a rectangle.

③ Contrapositive:-

If a quadrilateral is not a rectangle then it is not a square.

6) construct the truth table for the following

(i) $\neg p \wedge \neg q$

p	q	¬p	¬q	¬p∧¬q
T	T	F	F	F
T	F	F	T	F
F	T	T	F	F
F	F	T	T	T

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(ii) $\neg(p \wedge \neg q)$

p	q	¬q	p∧¬q	¬(p∧¬q)
T	T	F	F	T
T	F	T	T	F
F	T	F	F	T
F	F	T	F	T

(iii) $(p \vee q) \vee \neg q$

p	q	$p \vee q$	$\neg q$	$(p \vee q) \vee (\neg q)$
T	T	T	F	T
T	F	T	T	T
F	T	T	F	T
F	F	F	T	T

(iv) $(\neg p \to r) \wedge (p \leftrightarrow q)$

p	q	r	$\neg p$	$\neg p \to r$	$p \leftrightarrow q$	$(\neg p \to r) \wedge (p \leftrightarrow q)$
T	T	T	F	T	T	T
T	T	F	F	T	T	T
T	F	T	F	T	F	F
T	F	F	F	T	F	F
F	T	T	T	T	F	F
F	T	F	T	F	F	F
F	F	T	T	T	T	T
F	F	F	T	F	T	F

7) Verify whether the following compound propositions are tautologies or contradictions or contingency.

i) $(p \wedge q) \wedge \neg (p \vee q)$

p	q	$p \wedge q$	$p \vee q$	$\neg(p \vee q)$	$(p \wedge q) \wedge \neg(p \vee q)$
T	T	T	T	F	F
T	F	F	T	F	F
F	T	F	T	F	F
F	F	F	F	T	F

Last column contains only F
This is a contradiction

ii) $((p \vee q) \wedge \neg p) \to q$

p	q	$p \vee q$	$\neg p$	$(p \vee q) \wedge \neg p$	$((p \vee q) \wedge \neg p) \to q$
T	T	T	F	F	T
T	F	T	F	F	T
F	T	T	T	T	T
F	F	F	T	F	T

Last column contains only T
This is a Tautology.

(iii) $(p\to q)\leftrightarrow(\neg p\to q)$

p	q	$p\to q$	$\neg p$	$\neg p\to q$	$(p\to q)\leftrightarrow(\neg p\to q)$
T	T	T	F	T	T
T	F	F	F	T	F
F	T	T	T	T	T
F	F	T	T	F	F

Last contains T and F

∴ This is a contingency.

(iv) $((p\to q)\wedge(q\to r))\to(p\to r)$

p	q	r	$p\to q$	$q\to r$	$p\to r$	$(p\to q)\wedge(q\to r)$	$((p\to q)\wedge(q\to r))\to(p\to r)$
T	T	T	T	T	T	T	T
T	T	F	T	F	F	F	T
T	F	T	F	T	T	F	T
T	F	F	F	T	F	F	T
F	T	T	T	T	T	T	T
F	T	F	T	F	T	F	T
F	F	T	T	T	T	T	T
F	F	F	T	T	T	T	T

Last contains only T

∴ This is a tautology.

8) Show that (i) $\neg(p\wedge q)\equiv\neg p\vee\neg q$

TABLE ①

p	q	$p\wedge q$	$\neg(p\wedge q)$
T	T	T	F
T	F	F	T
F	T	F	T
F	F	F	T

TABLE ②

p	q	$\neg p$	$\neg q$	$\neg p\vee\neg q$
T	T	F	F	F
T	F	F	T	T
F	T	T	F	T
F	F	T	T	T

FROM Table ① & ② Last columns are Identical.

∴ $\neg(p\wedge q)\equiv\neg p\vee\neg q$.

8)(ii) $\neg(p\to q)\equiv p\wedge\neg q$

Table ①

p	q	$p\to q$	$\neg(p\to q)$
T	T	T	F
T	F	F	T
F	T	T	F
F	F	T	F

Table ②

p	q	$\neg q$	$p\wedge\neg q$
T	T	F	F
T	F	T	T
F	T	F	F
F	F	T	F

From table ①&②

Last columns are identical ∴ $\neg(p\to q)\equiv p\wedge(\neg q)$

9). prove that $q\to p \equiv \neg p \to \neg q$

TABLE ①

p	q	$q\to p$
T	T	T
T	F	T
F	T	F
F	F	T

TABLE ②

p	q	$\neg p$	$\neg q$	$\neg p\to\neg q$
T	T	F	F	T
T	F	F	T	T
F	T	T	F	F
F	F	T	T	T

From table ①&② The last columns are identical.

∴ $q\to p \equiv \neg p\to\neg q$

10) show that $p\to q$ and $q\to p$ are not equivalent.

TABLE ①

p	q	$p\to q$
T	T	T
T	F	F
F	T	T
F	F	T

TABLE ②

p	q	$q\to p$
T	T	T
T	F	T
F	T	F
F	F	T

From table ①&② Last colum's are not identical ∴ $p\to q \not\equiv q\to p$.

11). show that $\neg(p\leftrightarrow q)\equiv p\leftrightarrow\neg q$

TABLE ①

p	q	$p\leftrightarrow q$	$\neg(p\leftrightarrow q)$
T	T	T	F
T	F	F	T
F	T	F	T
F	F	T	F

TABLE ②

p	q	$\neg q$	$p\leftrightarrow\neg q$
T	T	F	F
T	F	T	T
F	T	F	T
F	F	T	F

From table ①&② Last columns are Identical ∴ $\neg(p\leftrightarrow q)\equiv p\leftrightarrow\neg q$

12) check whether the statement $p\to(q\to p)$ is a tautology or a contradiction without using the truth table.

$$p\to(q\to p)\equiv p\to(\neg q\vee p)$$

$$\equiv \neg p\vee(\neg q\vee p)$$

$$\equiv \neg p\vee(p\vee\neg q)$$ (∵ commutative Law)

$$\equiv(\neg p\vee p)\vee\neg q$$ [∵ associative Law]

$\equiv T \vee \neg q$

$\equiv T$

$\therefore$ $p \to (q \to p)$ is a Tautology.

13) Using truth table check whether the statements $\neg(p \vee q) \vee (\neg p \wedge q)$ and $\neg p$ are logically equivalent.

p	q	$p \vee q$	$\neg(p \vee q)$	$\neg p$	$(\neg p \wedge q)$	$\neg(p \vee q) \vee (\neg p \wedge q)$
T	T	T	F	F	F	F
T	F	T	F	F	F	F
F	T	T	F	T	T	T
F	F	F	T	T	F	T

From the table last column and the Fifth columns are Identical.

$\therefore$ $\neg(p \vee q) \vee (\neg p \wedge q) \equiv \neg p$

14) Prove $p \to (q \to r) \equiv (p \wedge q) \to r$ with out using truth table.

$p \to (q \to r) \equiv p \to (\neg q \vee r)$

$\equiv \neg p \vee (\neg q \vee r)$

$\equiv (\neg p \vee \neg q) \vee r$ [$\because$ associative Law)

$\equiv \neg(p \wedge q) \vee r$

$\equiv (p \wedge q) \to r$

Hence proved.

15) prove that $p \to (\neg q \vee r) \equiv \neg p \vee (\neg q \vee r)$ using Truth table.

TABLE (1) $p \to (\neg q \vee r)$

p	q	r	$\neg q$	$(\neg q \vee r)$	$p \to (\neg q \vee r)$
T	T	T	F	T	T
T	T	F	F	F	F
T	F	T	T	T	T
T	F	F	T	T	T
F	T	T	F	T	T
F	T	F	F	F	T
F	F	T	T	T	T
F	F	F	T	T	T

TABLE (2)

p	q	r	$\neg p$	$\neg q$	$(\neg q \vee r)$	$\neg p \vee (\neg q \vee r)$
T	T	T	F	F	T	T
T	T	F	F	F	F	F
T	F	T	F	T	T	T
T	F	F	F	T	T	T
F	T	T	T	F	T	T
F	T	F	T	F	F	T
F	F	T	T	T	T	T
F	F	F	T	T	T	T

From Table (1) & (2) Last colums are Identical.

$\therefore p \to (\neg q \vee r) \equiv \neg p \vee (\neg q \vee r)$

(Using one table is also good)

——x——

Need suggestions

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