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DEPARTMENT OF MATHEMATICS

SRI RAMAKRISHNA MHSS – ARCOT

VELLORE DT -632503

UNIT - 7

APPLICATIONS OF DIFFERENTIAL CALCULUS

7. Applications of Differential Calculus

Example 7.1For the function $f(x) = x^2, x \in [0,2]$ compute the average rate of changes in the subintervals[0,0.5],[0.5,1],[1,1.5],[1.5,2] and the instantaneous rate of changes at the points x= 0.5, 1, 1.5, 2 .

Solution: Given $f(x) = x^2$

The average rate of changes between the intervals x= a and x= b is $\frac{f(b)-f(a)}{b-a}$

$$= \frac{b^2-a^2}{b-a}$$

$$= \frac{(b-a)(b+a)}{b-a}$$

$$= b + a$$

Hence in the interval $[0, 0.5] = 0.5 + 0 = 0.5$

$[0.5, 1] = 1 + 0.5 = 1.5$

$[1, 1.5] = 1.5 + 1 = 2.5$

$[1.5, 2] = 2 + 1.5 = 3.5$

The instantaneous rate of change $f'(x) = 2x$

Hence the instantaneous rate of changes at the point $x = 0.5$ is $f'(0.5) = 2(0.5) = 1$

$x = 1$ is $f'(1) = 2(1) = 2$

$x = 1.5$ is $f'(1.5) = 2(1.5) = 3$

$x = 2$ is $f'(2) = 2(2) = 4$

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Example 7.2 The temperature in Celsius in a long rod of length 10 m, insulated at both ends, is a function of length x given by $T = x(10 - x)$ Prove that the rate of change of temperature at the midpoint of the rod is zero.

Solution: Length of the rod = 10 m

To prove the rate of change of temperature at the mid point of the rod x = 5 is zero.

Given Temperature $T = x(10 - x)$

$$= 10x - x^2$$

$$T' = 10 - 2x$$

At $x = 5$, $T' = 10 - 2(5)$

$$= 10 - 10$$

$= 0$ Hence proved.

Example 7.3 A person learnt 100 words for an English test. The number of words the person remembers in t days after learning is given by $W(t) = 100 \times (1 - 0.1t)^2, 0 \leq t \leq 10$.
What is the rate at which the person forgets the words 2 days after learning?
Solution: The number of words the person remembers given by

$$W(t) = 100 \times (1 - 0.1t)^2, 0 \leq t \leq 10.$$
$$W^{/}(t) = 100 \times 2(1 - 0.1t)(-0.1)$$
$$= -20 \times (1 - 0.1t)$$

At $t = 2$, $W^{/}(2) = -20 \times (1 - 0.2)$
$$= -20 \times (0.8)$$
$$= -16$$

The rate at which the person forgets the words 2 days after learning is 16 words.
****

Example 7.4 A particle moves so that the distance moved is according to the law $s(t) = \frac{t^3}{3} - t^2 + 3$. At what time the velocity and acceleration are zero respectively?
Solution: Given $s(t) = \frac{t^3}{3} - t^2 + 3$

To fine the time at velocity $s^{/}(t)$ and acceleration $s^{//}(t)$ becomes zero.

$$s(t) = \frac{t^3}{3} - t^2 + 3$$
$$s^{/}(t) = \frac{3t^2}{3} - 2t$$
$$= t^2 - 2t$$
$$s^{//}(t) = 2t - 2$$

(i) *when velocity* $S^{/}(t) = 0$
$$t^2 - 2t = 0$$
$$t(t - 2) = 0$$
Gives, $t = 0, and\ t = 2$

(ii) *when acceleration* $S^{//}(t) = 0$
$$2t - 2 = 0$$
$$2t = 2$$
$$t = 1$$

Example 7.5 A particle is fired straight up from the ground to reach a height of s feet in t seconds, where $s(t) = 128t - 16t^2$.
(1) Compute the maximum height of the particle reached.
(2) What is the velocity when the particle hits the ground?
Solution: Given $s(t) = 128t - 16t^2$

$$Velocity\quad s^{/}(t) = 128 - 32t$$

The particle reaches the maximum height when velocity becomes zero.

$$\therefore 128 - 32t = 0$$
$$32t = 128$$
$$t = \frac{128}{32} = 4$$

(1)The time taken by the particle to reaches the maximum height $t = 4\ seconds$.
Substituting in $t = 4\ in\ S(t)$, we get the maximum height reached

$$s(4) = 128(4) - 16(4)^2$$
$$= 512 - 16(16)$$
$$= 512 - 256$$
$$= 256 \text{ feet.}$$

The time taken to reaches the maximum height is $t = 4\ seconds$, hence the time taken for the downward direction is also $t = 4\ seconds$.
(2)The total time to reach the ground is $t = 8\ s$
Substituting $t = 8, in\ s^{/}(t)$, we the velocity it strikes the ground

$$\therefore s^{/}(8) = 128 - 32(8)$$
$$= 128 - 256$$
$$= -128\ ft/s$$

The velocity when the particle hits the ground is 128 ft/s
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Example 7.6
A particle moves along a horizontal line such that its position at any time $t \geq 0$ is given by $s(t) = t^3 - 6t^2$+9t+1, where s is measured in metres and t in seconds?

(1) At what time the particle is at rest?
(2) At what time the particle changes direction?
(3) Find the total distance travelled by the particle in the first 2 seconds.

Solution: $s(t) = t^3 - 6t^2 + 9t + 1$

$$V(t) = s'(t) = 3t^2 - 12t + 9$$

(1) The time when the particle comes to rest when velocity becomes zero

$$3t^2 - 12t + 9 = 0$$

$$\div 3, \quad t^2 - 4t + 3 = 0$$

$$(t-1)(t-3) = 0$$

$$(t-1) = 0, gives \; t = 1 \text{ and}$$

$$(t-3) = 0, gives \; t = 3$$

Hence at $t = 1, t = 3$ the particle is at rest.

(2) The particle changes direction when $V(t)$ changes its sign.

When t lies between 0 and 1 $V(t) > 0$

When t lies between 1 and 3 $V(t) < 0$

When t lies above 3 $V(t) > 0$

The particle changes direction when $t = 1 \; and \; t = 3$

(3) The total distance travelled by the particle in the first 2 seconds.

$$= |s(0) - s(1)| + |s(1) - s(2)|$$

Now, $s(0) = (0)^3 - 6(0)^2 + 9(0) + 1 = 1$

$$s(1) = (1)^3 - 6(1)^2 + 9(1) + 1$$
$$= 1 - 6 + 9 + 1$$
$$= 11 - 6$$
$$= 5$$

$$s(2) = (2)^3 - 6(2)^2 + 9(2) + 1$$
$$= 8 - 24 + 18 + 1$$
$$= 27 - 24$$
$$= 3$$

$$\therefore |s(0) - s(1)| + |s(1) - s(2)|$$
$$= |1 - 5| + |5 - 3|$$
$$= |-4| + |2|$$
$$= 4 + 2$$
$$= 6 \text{ metres.}$$

The total distance travelled by the particle in the first 2 seconds = 6 metres.

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Example 7.7 If we blow air into a balloon of spherical shape at a rate of 1000 cm^3 per second. At what rate the radius of the baloon changes when the radius is 7cm? Also compute the rate at which the surface area changes.

Solution:

Volume of the spherical balloon $V = \frac{4}{3}\pi r^3$

Given $\frac{dV}{dt} = 1000 cm^3 per\ second$

To find $\frac{dr}{dt}$ and $\frac{dS}{dt}$ when radius r = 7 cm

$$V = \frac{4}{3}\pi r^3$$

$$\frac{dV}{dt} = \frac{4}{3}\pi 3r^2 \times \frac{dr}{dt}$$

$$1000 = 4\pi \times 7 \times 7 \times \frac{dr}{dt}$$

$$\frac{1000}{4\times\pi\times7\times7} = \frac{dr}{dt}$$

$$\frac{dr}{dt} = \frac{250}{49\pi} cm/s$$

(i) The radius changes at the rate of $\frac{250}{49\pi} cm/s$

Surface area of the sphere $S = 4\pi r^2$

$$\frac{dS}{dt} = 4\pi \times 2r \times \frac{dr}{dt}$$
$$= 4\pi \times 2 \times 7 \times \frac{250}{49\pi}$$
$$= 4 \times 2 \times \frac{250}{7}$$
$$\frac{dS}{dt} = \frac{2000}{7} cm^2/s$$

(ii) The Surface area changes at the rate of $\frac{2000}{7} cm^2/s$

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Example 7.8 The price of a product is related to the number of units available (supply) by the equation $Px + 3P - 16x = 234$, where $P$is the price of the product per unit in Rupees and x is the number of units. Find the rate at which the price is changing with respect to time when 90 units are available and the supply is increasing at a rate of 15 units/week.

Solution: The supply and price of the product is x and P. It is related by

$$Px + 3P - 16x = 234$$

To find $\frac{dP}{dt}$, $when\ x = 90\ and\ \frac{dx}{dt} = 15$

Given $\quad Px + 3P - 16x = 234$

$$Px + 3P = 234 + 16x$$

$$P(x+3) = 234 + 16x$$

$$P = \frac{(234+16x)}{(x+3)}$$

$$\frac{dP}{dt} = \frac{(x+3)(234+16x)' - (234+16x)(x+3)'}{(x+3)^2}$$

$$= \frac{(x+3)\left(16\frac{dx}{dt}\right) - (234+16x)\left(\frac{dx}{dt}\right)}{(x+3)^2}$$

$$= \frac{[(x+3)(16) - (234+16x)]\left(\frac{dx}{dt}\right)}{(x+3)^2}$$

$$= \frac{[16x+48-234-16x]\left(\frac{dx}{dt}\right)}{(x+3)^2}$$

$$= \frac{[-186]\left(\frac{dx}{dt}\right)}{(x+3)^2}$$

$$= \frac{[-186](15)}{(90+3)^2}$$

$$= \frac{[-186](15)}{(93)^2}$$

$$= \frac{[-186](15)}{(93)(93)}$$

$$= \frac{(-2)(15)}{(93)}$$

$$= \frac{(-2)(5)}{(31)}$$

$$= \frac{-10}{31}$$

$$\cong -0.32$$

Hence the price decreasing at the rate of Rs. 0.32 per unit.

****

Example 7.9 Salt is poured from a conveyer belt at a rate of 30 cubic meters per minute forming a conical pile with a circular base whose height and diameter of base are always equal. How fast is the height of the pile increasing when the pile is 10 metre high?

Solution:

Given: Diameter = Height

$$2r = h$$

$$r = \frac{h}{2}$$

Rate of change of volume $\frac{dV}{dt} = 30$

To find rate of change of height $\frac{dh}{dt}$ at $h = 10$

We know the volume of the cone

$$V = \frac{1}{3}\pi r^2 h$$

$$= \frac{1}{3}\pi\left(\frac{h}{2}\right)\left(\frac{h}{2}\right)h$$

$$= \frac{\pi}{12}h^3$$

$$\frac{dV}{dt} = \frac{\pi}{12}3h^2\left(\frac{dh}{dt}\right)$$

$$30 = \frac{\pi}{4} \times 10 \times 10 \times \left(\frac{dh}{dt}\right)$$

$$\frac{30\times 4}{10\times 10\times \pi} = \frac{dh}{dt}$$

$$\frac{dh}{dt} = \frac{6}{5\pi}\text{mt/min}$$

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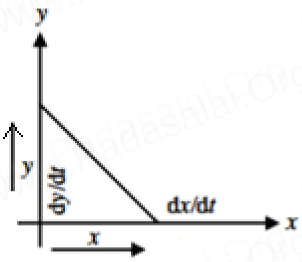
Example 7.10 (Two variable related rate problem) A road running north to south crosses a road going east to west at the point P. Car A is driving north along the first road, and car B is driving east along the second road. At a particular time car A 10 kilometers to the north of P and traveling at 80 km/hr, while car B is 15 kilometers to the east of P and traveling at 100 km/hr. How fast is the distance between the two cars changing?

Solution:



Given $x = 10$ and $y = 15$

To find the change of the distance between the cars $\frac{dZ}{dt}$, when $\frac{dx}{dt} = 80$, and $\frac{dy}{dt} = 100$

By using Pythagoras Theorem,

$$x^2 + y^2 = z^2$$
$$(10)^2 + (15)^2 = z^2$$
$$100 + 225 = z^2$$
$$z^2 = 325$$
$$= 25 \times 13$$
$$z = 5\sqrt{13}$$
$$x^2 + y^2 = z^2$$

Differentiating with respect to t

$$2x\frac{dx}{dt} + 2y\frac{dy}{dt} = 2z\frac{dz}{dt}$$
$$\div 2, \quad x\frac{dx}{dt} + y\frac{dy}{dt} = z\frac{dz}{dt}$$
$$(10)(80) + (15)(100) = (5\sqrt{13})\frac{dz}{dt}$$

Dividing by 5,

$$(2)(80) + (3)(100) = (\sqrt{13})\frac{dz}{dt}$$
$$160 + 300 = (\sqrt{13})\frac{dz}{dt}$$
$$460 = (\sqrt{13})\frac{dz}{dt}$$
$$\frac{dz}{dt} = \frac{460}{\sqrt{13}}$$
$$= \frac{460}{3.65}$$
$$\cong 127.6\ Km/hr$$

The distance between changes at the rate of $127.6\ Km/hr$

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EXERCISE 7.1

1. A point moves along a straight line in such a way that after t seconds its distance from the origin is $s = 2t^2 + 3t$ metres.

(i) Find the average velocity of the points between $t = 3$ and $t = 6$ seconds.

(ii) Find the instantaneous velocities at $t = 3$ and $t = 6$ seconds.

Solution:

Given $s(t) = 2t^2 + 3t$

The average velocity of the points between $t =$ a and $t =$ b is $\frac{S(b)-S(a)}{b-a}$

$$s(a) = s(3) = 2(3)^2 + 3(3)$$
$$= 2(9) + 9$$
$$= 18 + 9$$
$$= 27$$
$$s(b) = s(6) = 2(6)^2 + 3(6)$$
$$= 2(36) + 18$$
$$= 72 + 18$$
$$= 90$$

The average velocity of the points $= \frac{S(6)-S(3)}{6-3}$

$$= \frac{90-27}{3}$$
$$= \frac{63}{3}$$
$$= 21\ m/s$$

(ii) Given $s(t) = 2t^2 + 3t$

Hence instantaneous velocity $= S'(t)$

$$S'(t) = 4t + 3$$

Velocity at $t = 3$ seconds $S'(3) = 4(3) + 3$

$$= 12 + 3$$
$$= 15\ m/s$$

Velocity at $t = 6$ seconds $S'(6) = 4(6) + 3$

$$= 24 + 3$$
$$= 27\ m/s$$

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2. A camera is accidentally knocked off an edge of a cliff 400 ft high. The camera falls a distance of $s = 16t^2$ in t seconds.

(i) How long does the camera fall before it hits the ground?

(ii) What is the average velocity with which the camera falls during the last 2 seconds?

(iii) What is the instantaneous velocity of the camera when it hits the ground?

Solution:

(i) Given $s = 16t^2$. The camera falls down from the height of 400 ft.

∴Time taken to hits the ground is to find t when s=400 feet.

$$400 = 16t^2$$

$$\frac{400}{16} = t^2$$

$$\frac{4\times100}{16} = t^2$$

$$\pm\frac{2\times10}{4} = t$$

$$\pm 5 = t$$

Since $t = -5$, is not possible

$$t = 5$$

∴Time taken for the camera to hit the ground is $t = 5$ seconds.

(ii) To find the average velocity with which the camera falls during the last 2 seconds is to find the average velocity at $t = 3$ and $t = 5$ seconds.

We know that the average velocity of the points between t = a and t = b is $\frac{S(b)-S(a)}{b-a}$

Given $s = 16t^2$

$$s(a) = s(3) = 16(3)^2$$
$$= 16(9)$$
$$= 144$$
$$s(b) = s(5) = 16(5)^2$$
$$= 16(25)$$
$$= 400$$

The average velocity of the points $= \frac{S(5)-S(3)}{5-3}$

$$= \frac{400-144}{2}$$
$$= \frac{256}{2}$$
$$= 128\ ft/s$$

(iii) To find the instantaneous velocity of the camera when it hits the ground is to find the instantaneous velocity at $t = 5$ seconds.

Given $s(t) = 16t^2$

Hence instantaneous velocity $= S^/(t)$

$$S^/(t) = 32t$$

Velocity at $t = 5$ seconds $S^/(5) = 32(5)$

$$= 160$$
$$= 160\ ft/s$$

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3. A particle moves along a line according to the law $s(t) = 2t^3 - 9t^2 + 12t - 4$, +9t+1, where $t \geq 0$.

(i) At what times the particle changes direction?

(ii) Find the total distance travelled by the particle in the first 4 seconds.

(iii) Find the particle's acceleration each time the velocity is zero.

Solution:

(i) To find the times the particle changes direction is to find t when $S^/(t) = 0$

Given $s(t) = 2t^3 - 9t^2 + 12t - 4$

$$s^/(t) = 6t^2 - 18t + 12$$

when $s^/(t) = 0$ gives,

$$6t^2 - 18t + 12 = 0$$

$\div 6,$ $t^2 - 3t + 2 = 0$

$$(t-1)(t-2) = 0$$

$t - 1 = 0, gives\ t = 1$ and

$t - 2 = 0, gives\ t = 2$

∴ The particle changes direction at $t = 1$ and $t = 2$ seconds.

(ii)To find the total distance travelled by the particle in the first 4 seconds

S at $t = 0, 1, 2, 3$ *and* 4 seconds.

Given $s(t) = 2t^3 - 9t^2 + 12t - 4$

$$s(0) = -4$$
$$s(1) = 2 - 9 + 12 - 4$$
$$= 14 - 13$$
$$= 1$$
$$s(2) = 2(2)^3 - 9(2)^2 + 12(2) - 4$$

$= 2(8) - 9(4) + 12(2) - 4$

$= 16 - 36 + 24 - 4$

$= 40 - 40$

$= 0$

$s(3) = 2(3)^3 - 9(3)^2 + 12(3) - 4$

$= 2(27) - 9(9) + 12(3) - 4$

$= 54 - 81 + 36 - 4$

$= 90 - 85$

$= 5$

$s(4) = 2(4)^3 - 9(4)^2 + 12(4) - 4$

$= 2(64) - 9(16) + 12(4) - 4$

$= 128 - 144 + 48 - 4$

$= 176 - 148$

$= 28$

t	0	1	2	3	4
S	−4	1	0	5	28

Total distance travelled in first 4 seconds

$= |s(0) - s(1)| + |s(1) - s(2)|$

$+|s(2) - s(3)| + |s(3) - s(4)|$

$= |-4 - 1| + |1 - 0| + |0 - 5| + |5 - 28|$

$= |-5| + |1| + |-5| + |-23|$

$= 5 + 1 + 5 + 23$

$= 34$ meters.

(iii) To find the particle's acceleration each time the velocity is zero.

Given $s(t) = 2t^3 - 9t^2 + 12t - 4$

Velocity $s'(t) = 6t^2 - 18t + 12$

when $s'(t) = 0$ gives,

$6t^2 - 18t + 12 = 0$

$\div 6,\quad t^2 - 3t + 2 = 0$

$(t-1)(t-2) = 0$

$t - 1 = 0, gives\ t = 1$ and

$t - 2 = 0, gives\ t = 2$

Acceleration $s''(t) = 12t - 18$

When $t = 1$, Acceleration $s''(1) = 12 - 18$

$= -6m/s^2$

When $t = 2$, Acceleration $s''(2) = 12(2) - 18$

$= 24 - 18$

$= 6m/s^2$

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4. If the volume of a cube of side length x is $v = x^3$. Find the rate of change of the volume with respect to x when $x = 5$ units.

Solution: Given volume of the cube $v = x^3$

The rate of change of the volume $= \frac{dv}{dt}$

$\therefore \frac{dv}{dt} = 3x^2$

$\frac{dv}{dt} at\ x = 5 = 3(5)^2$

$= 3(25)$

$= 75$ units.

****

5. If the mass $m(x)$ (in kilograms) of a thin rod of length x (in metres) is given by, $m(x) = \sqrt{3x}$ then what is the rate of change of mass with respect to the length when it is $x = 3$ and $x = 27$ metres.

Solution: Given $m(x) = \sqrt{3x}$

$= \sqrt{3}\sqrt{x}$

Rate of change of mass $= m'(x)$

$m'(x) = \sqrt{3}\frac{1}{2\sqrt{x}}$

(i) Rate of change of mass at $x = 3$

$m'(3) = \sqrt{3}\frac{1}{2\sqrt{3}}$

$= \frac{1}{2}$ Kg/m

(ii) Rate of change of mass at $x = 27$

$m'(27) = \sqrt{3}\frac{1}{2\sqrt{27}}$

$= \sqrt{3}\frac{1}{2\sqrt{9\times 3}}$

$= \sqrt{3}\frac{1}{2\times 3\sqrt{3}}$

$= \frac{1}{6}$ Kg/m

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6. A stone is dropped into a pond causing ripples in the form of concentric circles. The radius r of the outer ripple is increasing at a constant rate at 2 cm per second. When the radius is 5 cm find the rate of changing of the total area of the disturbed water?

Solution:

Given The rate of increasing radius $\frac{dr}{dt} = 2$ cm/s

To find the rate of change of area of the circle $\frac{dA}{dt}$ when $r = 5$

Area of the circle $A = \pi r^2$

$$\frac{dA}{dt} = 2\pi r \times \frac{dr}{dt}$$

$$= 2\pi(5) \times (2)$$

$$= 20\pi r \text{ sq.cm}/s$$

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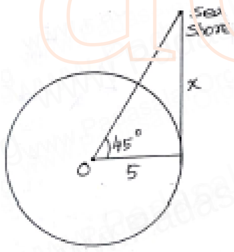
7. A beacon makes one revolution every 10 seconds. It is located on a ship which is anchored 5 km from a straight shore line. How fast is the beam moving along the shore line when it makes an angle of 45° with the shore?

Solution:



Given : one revolution every 10 seconds.

It covers in 10 $sec = 2\pi$

$$\therefore 1\ sec = \frac{2\pi}{10} = \frac{\pi}{5}$$

Revolution in 1 second $\frac{d\theta}{dt} = \frac{\pi}{5}$

From the given data $\tan\theta = \frac{x}{5}$

Gives, $x = 5\tan\theta$

$$\therefore \frac{dx}{d\theta} = 5\,sec^2\theta \times \frac{d\theta}{dt}$$

when $\theta = 45^0$ and $\frac{d\theta}{dt} = \frac{\pi}{5}$

$$\frac{dx}{d\theta} = 5\,sec^2 45^0 \times \frac{\pi}{5}$$

$$= 5\sqrt{2}^2 \times \frac{\pi}{5} = 2 \times \pi$$

The beam moving along the shore line at a rate of $2\pi\ km/s$

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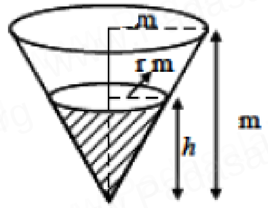
8. A conical water tank with vertex down of 12 metres height has a radius of 5 metres at the top. If water flows into the tank at a rate 10 cubic m/min, how fast is the depth of the water increases when the water is 8 metres deep?

Solution:



Given: Height of the tank $h = 12\ m$

Radius at the top $r = 5\ m$

Hence $\frac{r}{h} = \frac{5}{12} \Rightarrow r = \frac{5}{12}h$

Rate of change of volume $\frac{dV}{dt} = 10$

To find rate of change of height $\frac{dh}{dt}$ at $h = 8$

We know the volume of the cone

$$V = \frac{1}{3}\pi r^2 h$$

$$= \frac{1}{3}\pi\left(\frac{5}{12}h\right)\left(\frac{5}{12}h\right)h$$

$$= \frac{5\times5}{3\times12\times12}\pi h^3$$

$$\frac{dV}{dt} = \frac{5\times5}{3\times12\times12}\pi 3h^2 \times \frac{dh}{dt}$$

$$\frac{dV}{dt} = \frac{25}{144}\pi \times h^2 \times \frac{dh}{dt}$$

Substituting $\frac{dV}{dt} = 10$ and $h = 8$, we get

$$10 = \frac{25}{144}\pi \times 8 \times 8 \times \frac{dh}{dt}$$

$$\frac{10\times144}{25\times8\times8\times\pi} = \frac{dh}{dt}$$

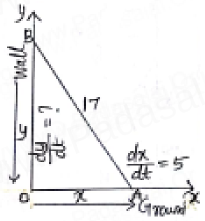
$$\frac{2\times5\times16\times9}{25\times8\times8\times\pi} = \frac{dh}{dt}$$

$$\frac{9}{10\pi} = \frac{dh}{dt}$$

So, the rate of change of height $\frac{dh}{dt} = \frac{9}{10\pi}$ m/min

9. A ladder 17 metre long is leaning against the wall. The base of the ladder is pulled away from the wall at a rate of 5 m/s. When the base of the ladder is 8 metres from the wall.
(i) How fast is the top of the ladder moving down the wall?
(ii) At what rate, the area of the triangle formed by the ladder, wall, and the floor, is changing?

Solution:



Given length of the ladder AB=17 mt.

Let the foot of the ladder is $x\ mt$ away from the wall. That is let $OA = x$

Similarly let the top of the ladder touches the wall $y\ mt$ away from the floor.

That is let $OB = y$

By using Pythagoras Theorem,

$$x^2 + y^2 = 17^2$$

When $x = 8$, $\quad 8^2 + y^2 = 17^2$

$$y^2 = 17^2 - 8^2$$
$$= (17+8)(17-8)$$
$$= (25)(9)$$
$$y = (5)(3)$$
$$y = 15$$

(i) To find the rate at which the ladder slips down the wall $= \frac{dy}{dt}$ when $\frac{dx}{dt} = 5$

From $\quad x^2 + y^2 = 17^2$

Differentiating with respect to t

$$2x\frac{dx}{dt} + 2y\frac{dy}{dt} = 0$$

$$\div 2, \quad x\frac{dx}{dt} + y\frac{dy}{dt} = 0$$

$$(8)(5) + (15)\frac{dy}{dt} = 0$$

$$\frac{dy}{dt} = -\frac{40}{15}$$
$$= -\frac{8}{3}$$

Ladder slips down the wall at the rate of $\frac{8}{3}\ m/s$

(ii) To find the rate at which the area of the triangle OAB changes $= \frac{dA}{dt}$

Area of triangle OAB $A = \frac{1}{2}xy$

Differentiating with respect to t

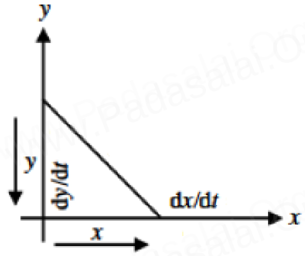
$$\frac{dA}{dt} = \frac{1}{2}\left(x\frac{dy}{dt} + y\frac{dx}{dt}\right)$$
$$= \frac{1}{2}\left[(8)\left(-\frac{8}{3}\right) + (15)(5)\right]$$
$$= \frac{1}{2}\left(\frac{-64}{3} + 75\right)$$
$$= \frac{1}{2}\left(\frac{-64+225}{3}\right)$$
$$= \frac{1}{2}\left(\frac{121}{3}\right)$$
$$= \frac{121}{6}$$
$$= 26.83$$

The area of the triangle OAB changes at the rate $\frac{dA}{dt} = 26.83\ sq.m/s$

****

10, A police jeep, approaching an orthogonal intersection from the northern direction, is chasing a speeding car that has turned and moving straight east. When the jeep is 0.6 km north of the intersection and the car is 0.8 km to the east. The police determine with a radar that the distance between them and the car is increasing at 20 km/hr. If the jeep is moving at 60 km/hr at the instant of measurement, what is the speed of the car?

Solution:



Given $x = 0.8$ and $y = 0.6$

The distance between the car and the jeep changes at the rate $\frac{dZ}{dt} = 20$

To find $\frac{dx}{dt}$, when $\frac{dy}{dt} = -60$

By using Pythagoras Theorem,

$$x^2 + y^2 = z^2$$
$$(0.8)^2 + (0.6)^2 = z^2$$
$$0.64 + 0.36 = z^2$$
$$1 = z^2$$
$$z = 1$$
$$x^2 + y^2 = z^2$$

Differentiating with respect to t

$$2x\frac{dx}{dt} + 2y\frac{dy}{dt} = 2z\frac{dz}{dt}$$
$$\div 2, \quad x\frac{dx}{dt} + y\frac{dy}{dt} = z\frac{dz}{dt}$$
$$(0.8)\left(\frac{dx}{dt}\right) + (0.6)(-60) = (1)(20)$$
$$(0.8)\left(\frac{dx}{dt}\right) - 36 = 20$$
$$\frac{8}{10}\left(\frac{dx}{dt}\right) = 20 + 36$$
$$\frac{dx}{dt} = 56 \times \frac{10}{8}$$
$$\frac{dx}{dt} = 70$$

Hence the speed of the car changes at the rate $\frac{dx}{dt} = 70$ km/hr.

****

Example 7.11 Find the equations of tangent and normal to the curve $y = x^2 + 3x - 2$ at the point (1, 2)

Solution: $y = x^2 + 3x - 2$

$$\frac{dy}{dx} = 2x + 3$$
$$\frac{dy}{dx} \text{ at the point } (1, 2) = 2(1) + 3$$
$$= 2 + 3$$
$$Slope\ m = 5$$

Equation of the tangent with slope m through the point (x_1, y_1) is $y - y_1 = m(x - x_1)$

Substituting $m = 5$, and the point (1, 2)

Tangent equation is $y - 2 = 5(x - 1)$

$$y - 2 = 5x - 5$$
$$5x - y - 5 + 2 = 0$$
$$5x - y - 3 = 0$$

Normal is perpendicular to tangent.

Hence normal is of the form $x + 5y + k = 0$

It passes through the point (1, 2)

Hence, $1 + 5(2) + k = 0$

$$1 + 10 + k = 0$$
$$11 + k = 0$$
$$k = -11$$

Equation of the normal is $x + 5y - 11 = 0$

****

Example 7.12 For what value of x the tangent of the curve $y = x^3 - 3x^2 + x - 2$ is parallel to the line $y = x$.

Solution: Given $y = x^3 - 3x^2 + x - 2$

$$\frac{dy}{dx} = 3x^2 - 6x + 10$$

Slope of the first curve $m_1 = 3x^2 - 6x + 1$

and $y = x$

$$\frac{dy}{dx} = 1$$

Slope of the second curve $m_2 = 1$

Since the tangents are parallel, $m_1 = m_2$

$$\therefore 3x^2 - 6x + 1 = 1$$
$$3x^2 - 6x + 1 - 1 = 0$$
$$3x^2 - 6x = 0$$
$$3x(x - 2) = 0$$

So , $3x = 0 \Rightarrow x = 0$

and $x - 2 = 0 \Rightarrow x = 2$

Substituting $x = 0$ in $y = x^3 - 3x^2 + x - 2$,

we get $y = -2$ and

$x = 2$ in $y = x^3 - 3x^2 + x - 2$,

we get $y = 2^3 - 3(2^2) + 2 - 2$

$$= 8 - 12 + 2 - 2$$
$$= 10 - 14$$
$$y = -4$$

The tangent is parallel to the line $y = x$ at the points $(0, -2)$ *and* $(2, -4)$

****

Example 7.13 Find the equation of the tangent and normal to the Lissajous curve given by $x = 2\cos 3t$ and $y = 3\sin 2t, \quad t \in \mathbb{R}$

Solution: $x = 2\cos 3t$

$$\frac{dx}{dt} = 2 \times 3(-\sin 3t)$$

$$\frac{dx}{dt} = -6\sin 3t$$

$$y = 3\sin 2t$$

$$\frac{dy}{dt} = 3 \times 2(\cos 2t)$$

$$\frac{dy}{dt} = 6\cos 2t$$

Slope of the tangent $m = \frac{dy}{dx} = \frac{6\cos 2t}{-6\sin 3t}$

$$m = -\frac{\cos 2t}{\sin 3t}$$

Equation of the tangent with slope m through the point (x_1, y_1) is $y - y_1 = m(x - x_1)$

Substituting $m = -\frac{\cos 2t}{\sin 3t}$,

and the point $(2\cos 3t, 3\sin 2t)$

Equation of tangent is

$$(y - 3\sin 2t) = -\frac{\cos 2t}{\sin 3t}(x - 2\cos 3t)$$

$$(y - 3\sin 2t)\sin 3t = -\cos 2t(x - 2\cos 3t)$$

$$y\sin 3t - 3\sin 2t\sin 3t = -x\cos 2t + 2\cos 2t\cos 3t$$

$$x\cos 2t + y\sin 3t = 3\sin 2t\sin 3t + 2\cos 2t\cos 3t$$

Normal is perpendicular to tangent.

Hence normal is of the form

$$x\sin 3t - y\cos 2t + k = 0$$

It passes through $(2\cos 3t, 3\sin 2t)$

$$(2\cos 3t)\sin 3t - (3\sin 2t)\cos 2t + k = 0$$

$$2\cos 3t\sin 3t - 3\sin 2t\cos 2t + k = 0$$

$$2\cos 3t\sin 3t - \frac{3}{2}(2\sin 2t\cos 2t) + k = 0$$

$$\sin 6t - \frac{3}{2}\sin 4t + k = 0$$

$$k = \frac{3}{2}\sin 4t - \sin 6t$$

So equation of the normal is

$$x\sin 3t - y\cos 2t + \frac{3}{2}\sin 4t - \sin 6t = 0$$

****

Example 7.14 Find the acute angle between $y = x^2$ and $y = (x-3)^2$.

Solution: Given $y = x^2$ and $y = (x-3)^2$.

So, $x^2 = (x-3)^2$

$$x^2 = x^2 - 6x + 9$$

$$x^2 - 6x + 9 - x^2 = 0$$

$$-6x + 9 = 0$$

$$6x = 9$$

$$x = \frac{9}{6} = \frac{3}{2}$$

When $x = \frac{3}{2}$, $y = x^2$ gives , $y = \frac{9}{4}$

The point of intersection is $\left(\frac{3}{2}, \frac{9}{4}\right)$

From $y = x^2$

$$\frac{dy}{dx} = 2x$$

At the point $\left(\frac{3}{2}, \frac{9}{4}\right)$

$$\frac{dy}{dx} = 2\left(\frac{3}{2}\right) = 3$$

Slope of the first curve $m_1 = 3$

From $y = (x-3)^2$

$$y = x^2 - 6x + 9$$

$$\frac{dy}{dx} = 2x - 6$$

At the point $\left(\frac{3}{2}, \frac{9}{4}\right)$

$$\frac{dy}{dx} = 2\left(\frac{3}{2}\right) - 6 = 3 - 6 = -3$$

Slope of the second curve $m_2 = -3$

If θ is the angle between the curves, then

$$\tan\theta = \left|\frac{m_1 - m_2}{1 + m_1 m_2}\right|$$

$$= \left|\frac{3-(-3)}{1+(3)(-3)}\right|$$

$$= \left|\frac{3+3}{1-9}\right|$$

$$= \left|\frac{6}{-8}\right|$$

$$= \left|\frac{3}{-4}\right|$$

$$= \frac{3}{4}$$

$$\theta = \tan^{-1}\left(\frac{3}{4}\right)$$

The angle between the curves $\theta = \tan^{-1}\left(\frac{3}{4}\right)$

Example 7.15 Find the acute angle between the curves $y = x^2$ and $x = y^2$ at their points of intersection (0, 0), (1, 1).

Solution: From $y = x^2$

$$\frac{dy}{dx} = m_1 = 2x \text{ and}$$

From $x = y^2$

$$1 = 2y\frac{dy}{dx}$$

$$\frac{dy}{dx} = m_2 = \frac{1}{2y}$$

(*i*) At (0, 0)

$$m_1 = 2x = 0 \text{ and}$$

$$m_2 = \frac{1}{2y} = \frac{1}{0} = \infty$$

If θ is the angle between the curves, then

$$\tan\theta = \left|\frac{m_1 - m_2}{1 + m_1 m_2}\right|$$

$$= \left|\frac{0-\infty}{1+\infty}\right| = \infty$$

$$\theta = tan^{-1}(\infty) = \frac{\pi}{2}$$

The angle between the curves $\theta = \frac{\pi}{2}$

(*i*) At (1,1)

$$m_1 = 2x = 2 \text{ and}$$

$$m_2 = \frac{1}{2y} = \frac{1}{2}$$

If θ is the angle between the curves, then

$$\tan\theta = \left|\frac{m_1 - m_2}{1 + m_1 m_2}\right|$$

$$= \left|\frac{2-\left(\frac{1}{2}\right)}{1+(2)\left(\frac{1}{2}\right)}\right|$$

$$= \left|\frac{\frac{4-1}{2}}{1+1}\right|$$

$$= \left(\frac{\frac{3}{2}}{\frac{2}{1}}\right)$$

$$= \left(\frac{3}{2} \times \frac{1}{2}\right)$$

$$= \frac{3}{4}$$

$$\theta = tan^{-1}\left(\frac{3}{4}\right)$$

The angle between the curves $\theta = tan^{-1}\left(\frac{3}{4}\right)$

Example 7.16

Find the angle of intersection of the curve $y = \sin x$ with the positive x-axis.

Solution: Given $y = \sin x$

$$\frac{dy}{dx} = \cos x$$

The given curve intersects with positive x –axis

On x –axis y =0, gives $\sin x = 0$

Hence $x = n\pi$

Slope of the tangent $\frac{dy}{dx} = m = \cos x$ gives

$$m = \cos n\pi = (-1)^n$$

Then $\tan\theta = (-1)^n$

$$\theta = tan^{-1}(-1)^n$$

If n is even, $\theta = tan^{-1}(1) = \frac{\pi}{4}$

If n is odd, $\theta = tan^{-1}(-1) = \frac{3\pi}{4}$

****

Example 7.17 If the curves $ax^2 + by^2 = 1$ and $cx^2 + dy^2 = 1$ intersect each other orthogonally then, $\frac{1}{a} - \frac{1}{b} = \frac{1}{c} - \frac{1}{d}$.

Solution: $ax^2 + by^2 = 1$

$$cx^2 + dy^2 = 1$$

Solving, $\Delta = \begin{vmatrix} a & b \\ c & d \end{vmatrix}$

$$= ad - bc$$

$$\Delta_{x^2} = \begin{vmatrix} 1 & b \\ 1 & d \end{vmatrix}$$

$$= d - b$$

$$\Delta_{y^2} = \begin{vmatrix} a & 1 \\ c & 1 \end{vmatrix}$$

$$= a - c$$

$$\therefore\ x^2 = \frac{d-b}{ad-bc}$$

$$y^2 = \frac{a-c}{ad-bc}$$

Given $ax^2 + by^2 = 1$

$$2ax + 2by\frac{dy}{dx} = 0$$

$$2by\frac{dy}{dx} = -2ax$$

$$\frac{dy}{dx} = -\frac{2ax}{2by}$$

$$m_1 = -\frac{ax}{by} \text{ and}$$

$$cx^2 + dy^2 = 1$$

$$2cx + 2dy\frac{dy}{dx} = 0$$

$$2dy\frac{dy}{dx} = -2cx$$

$$\frac{dy}{dx} = -\frac{2cx}{2dy}$$

$$m_2 = -\frac{cx}{dy}$$

Given the curves cut orthogonally.

$$\therefore m_1 m_2 = -1$$

$$\left(-\frac{ax}{by}\right)\left(-\frac{cx}{dy}\right) = -1$$

$$\frac{acx^2}{bdy^2} = -1$$

$$acx^2 = -bdy^2$$

$$acx^2 + bdy^2 = 0$$

Substituting the values of x^2 and y^2

$$ac\left(\frac{d-b}{ad-bc}\right) + bd\left(\frac{a-c}{ad-bc}\right) = 0$$

$$ac(d-b) + bd(a-c) = 0$$

$$\div \text{ by } abcd,$$

$$\frac{ac}{abcd}(d-b) + \frac{bd}{abcd}(a-c) = 0$$

$$\frac{(d-b)}{bd} + \frac{(a-c)}{ac} = 0$$

$$\frac{d}{bd} - \frac{b}{bd} + \frac{a}{ac} - \frac{c}{ac} = 0$$

$$\frac{1}{b} - \frac{1}{d} + \frac{1}{c} - \frac{1}{a} = 0$$

$$\therefore \frac{1}{a} - \frac{1}{b} = \frac{1}{c} - \frac{1}{d}$$

****

Example 7.18 Prove that the ellipse $x^2 + 4y^2 = 8$ and the hyperbola $x^2 - 2y^2 = 4$ intersect orthogonally.

Solution: $x^2 + 4y^2 = 8 \quad ....(1)$

$$x^2 - 2y^2 = 4 \quad ....(2)$$

$$(1) - (2) \text{ gives}, \; 6y^2 = 4$$

$$y^2 = \frac{4}{6} = \frac{2}{3}$$

Multiplying (2) by 3,

$$3x^2 - 6y^2 = 12$$

Substituting $6y^2 = 4$

$$3x^2 - 4 = 12$$

$$3x^2 = 12 + 4$$

$$3x^2 = 16$$

$$x^2 = \frac{16}{3}$$

Given $x^2 + 4y^2 = 8$

$$2x + 8y\frac{dy}{dx} = 0$$

$$8y\frac{dy}{dx} = -2x$$

$$\frac{dy}{dx} = -\frac{2x}{8y}$$

$$m_1 = -\frac{x}{4y}$$

$$x^2 - 2y^2 = 4$$

$$2x - 4y\frac{dy}{dx} = 0$$

$$4y\frac{dy}{dx} = 2x$$

$$\frac{dy}{dx} = \frac{2x}{4y}$$

$$m_2 = \frac{x}{2y}$$

Now $m_1 \times m_2 = \left(-\frac{x}{4y}\right) \times \left(\frac{x}{2y}\right)$

$$= -\frac{x^2}{8y^2}$$

Substituting the values of x^2 and y^2

$$m_1 \times m_2 = -\frac{16/3}{8(2/3)}$$

$$= -\frac{16/3}{16/3} = -1$$

So, the given curves cut orthogonally.

****

EXERCISE 7.2

1. Find the slope of the tangent to the curves at the respective given points.
   (i) $y = x^4 + 2x^2 - x$ at $x = 1$
   (ii) $x = a\cos^3 t, y = b\sin^3 t$ at $t = \frac{\pi}{2}$.

Solution:

(i) $y = x^4 + 2x^2 - x$

$$\frac{dy}{dx} = 4x^3 + 4x - 1$$

at $x = 1$, $Slope\ \frac{dy}{dx} = 4(1)^3 + 4(1) - 1$

$$= 4 + 4 - 1$$

$$m = 8 - 1 = 7$$

(ii) $x = a\cos^3 t$

$$\frac{dx}{dt} = 3a\cos^2 t(-\sin t)$$

$$y = b\sin^3 t$$

$$\frac{dy}{dt} = 3b\sin^2 t(\cos t)$$

Slope $\frac{dy}{dx} = \frac{3b\sin^2 t(\cos t)}{3a\cos^2 t(-\sin t)}$

$$= -\frac{b\sin t}{a\cos t}$$

at $t = \frac{\pi}{2}$, $\frac{dy}{dx} = -\frac{b\sin\left(\frac{\pi}{2}\right)}{a\cos\left(\frac{\pi}{2}\right)}$

$$= -\frac{b(1)}{a(0)}$$

$$= \infty$$

****

2. Find the point on the curve $y = x^2 - 5x + 4$ at which the tangent is parallel to the line $3x + y = 7$.

<u>Solution</u>: $y = x^2 - 5x + 4$

$$\frac{dy}{dx} = 2x - 5$$

Slope $m_1 = 2x - 5$ and from,

$$3x + y = 7$$

$$y = 7 - 3x$$

$$\frac{dy}{dx} = -3$$

$$m_2 = -3$$

Given the curve is parallel to the line

$$\therefore m_1 = m_2$$

$$2x - 5 = -3$$

$$2x = -3 + 5$$

$2x = 2$ gives

$$x = 1$$

Since to find the point on the curve,

substituting $x = 1$, in

$$y = x^2 - 5x + 4$$

$$= 1 - 5 + 4$$

$$= 5 - 5$$

$$= 0$$

So, at the point $(1, 0)$ the curve is parallel to the given line.

****

3. Find the points on the curve $y = x^3 - 6x^2 + x + 3$ where the normal is parallel to the line $x + y = 1729$.

<u>Solution</u>: Given $y = x^3 - 6x^2 + x + 3$

$$\frac{dy}{dx} = 3x^2 - 12x + 1$$

Slope of the tangent $m = 3x^2 - 12x + 1$

Hence slope of the normal $= -\frac{1}{m}$ and

$$x + y = 1729$$

$$1 + \frac{dy}{dx} = 0$$

$$\frac{dy}{dx} = -1$$

Slope of the line $m = -1$

Since the normal parallel to the line,

$$-\frac{1}{m} = -1$$

$$\frac{1}{m} = 1$$

$$\frac{1}{(3x^2-12x+1)} = 1$$

$$3x^2 - 12x + 1 = 1$$

$$3x^2 - 12x + 1 - 1 = 0$$

$$3x^2 - 12x = 0$$

$$3x(x - 4) = 0$$

So, $3x = 0 \Rightarrow x = 0$

and $x - 4 = 0 \Rightarrow x = 4$

Substituting $x = 0$ in $y = x^3 - 6x^2 + x + 3$,

we get $y = 3$ and

$x = 4$ in $y = x^3 - 6x^2 + x + 3$

we get $y = 4^3 - 6(4^2) + 4 + 3$

$$= 64 - 96 + 4 + 3$$

$$= 71 - 96$$

$$y = -25$$

The normal is parallel to the line at the points $(0, 4)$ *and* $(4, -25)$

****

4. Find the points on the curve $y^2 - 4xy = x^2 + 5$ for which the tangent is horizontal.

Solution: $y^2 - 4xy = x^2 + 5$

$$2y\frac{dy}{dx} - 4\left(x\frac{dy}{dx} + y\right) = 2x$$

$$2y\frac{dy}{dx} - 4x\frac{dy}{dx} - 4y = 2x$$

$$2y\frac{dy}{dx} - 4x\frac{dy}{dx} = 2x + 4y$$

Dividing by 2,

$$\left(y\frac{dy}{dx} - 2x\frac{dy}{dx}\right) = x + 2y$$

$$(y - 2x)\frac{dy}{dx} = x + 2y$$

$$\frac{dy}{dx} = \frac{x+2y}{(y-2x)}$$

When the tangent is horizontal, it is parallel to x axis. So, $\frac{dy}{dx} = 0$

Gives, $\frac{x+2y}{(y-2x)} = 0$

$$x + 2y = 0$$

$$x = -2y$$

Substituting $x = -2y$ in

$y^2 - 4xy = x^2 + 5$ we get

$$y^2 - 4(-2y)y = (-2y)^2 + 5$$

$$y^2 + 8y^2 = 4y^2 + 5$$

$$9y^2 - 4y^2 = 5$$

$$5y^2 = 5$$

$$y^2 = 1$$

$$y = \pm 1$$

When $y = 1, x = -2y$ gives $x = -2$ and

$y = -1, x = -2y$ gives $x = 2$

At the points $(-2,1)$ and $(2,-1)$ the tangent is horizontal.

****

5. Find the tangent and normal to the following curves at the given points on the curve.

(i) $y = x^2 - x^4$ at (1,0)

(ii) $y = x^4 + 2e^x$ at (0, 2)

(iii) $y = x \sin x$ at $\left(\frac{\pi}{2}, \frac{\pi}{2}\right)$

(iv) $x = \cos t, y = 2sin^2 t, at\ t = \frac{\pi}{3}$

Solution:

(i) $y = x^2 - x^4$

$$\frac{dy}{dx} = 2x - 4x^3$$

At (1, 0)

$$\frac{dy}{dx} = 2(1) - 4(1)^3$$

$$= 2 - 4$$

$$m = -2$$

Slope $m = -2$

Equation of the tangent with slope m through the point (x_1, y_1) is $y - y_1 = m(x - x_1)$

Substituting $m = -2$, and the point (1, 0)

Tangent equation is $y - 0 = -2(x - 1)$

$$y = -2x + 2$$

$$2x + y - 2 = 0$$

Normal is perpendicular to tangent.

Hence normal is of the form $x - 2y + k = 0$

It passes through the point (1, 0)

Hence, $1 - 2(0) + k = 0$

$$1 - 0 + k = 0$$

$$1 + k = 0$$

$$k = -1$$

Equation of the normal is $x - 2y - 1 = 0$

(ii) $y = x^4 + 2e^x$

$$\frac{dy}{dx} = 4x^3 + 2e^x$$

at (0, 2)

$$\frac{dy}{dx} = 4(0)^3 + 2e^{(0)}$$

$$= 0 + 2(1)$$

$$m = 2$$

Slope $m = 2$

Equation of the tangent with slope m through the point (x_1, y_1) is $y - y_1 = m(x - x_1)$

Substituting $m = 2$, and the point (0, 2)

Tangent equation is $y-2=2(x-0)$

$$y-2=2x+0$$

$$2x-y+2=0$$

Normal is perpendicular to tangent.

Hence normal is of the form $x+2y+k=0$

It passes through the point $(0,2)$

Hence, $0+2(2)+k=0$

$$0+4+k=0$$

$$4+k=0$$

$$k=-4$$

Equation of the normal is $x+2y-4=0$

(iii) $y=x\sin x$

$$\frac{dy}{dx}=x(\cos x)+\sin x$$

at $\left(\frac{\pi}{2},\frac{\pi}{2}\right)$

$$\frac{dy}{dx}=\frac{\pi}{2}\left(\cos\frac{\pi}{2}\right)+\sin\frac{\pi}{2}$$

$$=\frac{\pi}{2}(0)+1$$

$$=0+1$$

$$m=1$$

Slope $m=1$

Equation of the tangent with slope m through the point (x_1,y_1) is $y-y_1=m(x-x_1)$

Substituting $m=1$, and the point $\left(\frac{\pi}{2},\frac{\pi}{2}\right)$

Tangent equation is $y-\frac{\pi}{2}=1\left(x-\frac{\pi}{2}\right)$

$$y-\frac{\pi}{2}=x-\frac{\pi}{2}$$

$$x-y=0$$

Normal is perpendicular to tangent.

Hence normal is of the form $x+y+k=0$

It passes through the point $\left(\frac{\pi}{2},\frac{\pi}{2}\right)$

Hence, $\frac{\pi}{2}+\frac{\pi}{2}+k=0$

$\pi+k=0$ gives $k=-\pi$

Equation of the normal is $x+y-\pi=0$

(iv) $x=\cos t$, and $y=2\sin^2 t$

$$\frac{dx}{dt}=-\sin t \qquad \frac{dy}{dt}=4\sin t\cos t$$

So, $\frac{dy}{dx}=\frac{4\sin t\cos t}{-\sin t}$

$$=-4\cos t$$

$at\ t=\frac{\pi}{3}$

$$\frac{dy}{dx}=-4\cos t$$

$$=-4\cos\left(\frac{\pi}{3}\right)$$

$$=-4\left(\frac{1}{2}\right)$$

$$m=-2$$

Slope $m=-2$

$$(x_1,y_1)=\left(\cos\frac{\pi}{3},2\sin^2\frac{\pi}{3}\right)$$

$$=\left(\frac{1}{2},2\left(\frac{\sqrt{3}}{2}\right)^2\right)$$

$$=\left(\frac{1}{2},2\left(\frac{3}{4}\right)\right)$$

$$=\left(\frac{1}{2},\frac{3}{2}\right)$$

Equation of the tangent with slope m through the point (x_1,y_1) is $y-y_1=m(x-x_1)$

Substituting $m=-2$, and the point $\left(\frac{1}{2},\frac{3}{2}\right)$

Tangent equation is $y-\frac{3}{2}=-2\left(x-\frac{1}{2}\right)$

Multiplying by 2,

$$2y-3=-4\left(x-\frac{1}{2}\right)$$

$$2y-3=-4x+2$$

$$4x+2y-3-2=0$$

$$4x+2y-5=0$$

Normal is perpendicular to tangent.

Hence normal is of the form $2x-4y+k=0$

It passes through the point $\left(\frac{1}{2},\frac{3}{2}\right)$

Hence, $2\left(\frac{1}{2}\right)-4\left(\frac{3}{2}\right)+k=0$

$$1-6+k=0$$

$$-5+k=0$$

$$k=5$$

Equation of the normal is $2x-4y+5=0$

6. Find the equations of the tangents to the curve $y = 1 + x^3$ for which the tangent is orthogonal with the line $x + 12y = 12$.

Solution: Given $y = 1 + x^3$

$$\frac{dy}{dx} = 3x^2$$

Slope of the tangent $m_1 = 3x^2$

Equation of the given line $x + 12y = 12$

$$1 + 12\frac{dy}{dx} = 0$$

$$12\frac{dy}{dx} = -1$$

$$\frac{dy}{dx} = -\frac{1}{12}$$

Slope of the line $m_2 = -\frac{1}{12}$

Since the tangent is orthogonal to the line,

$$m_1 \times m_2 = -1$$

$$3x^2 \times -\frac{1}{12} = -1$$

$$\frac{x^2}{4} = 1$$

$$x^2 = 4$$

$$x = \pm 2$$

The points lie on the curve $y = 1 + x^3$

When $x = 2$, $y = 1 + 2^3$

$= 1 + 8$

$y = 9$

When $x = -2$, $y = 1 + (-2)^3$

$= 1 - 8$

$y = -7$

Hence $(2, 9)$ and $(-2, -7)$ are the points.

At $x = \pm 2$

Slope of the tangent $m = 3x^2$

$= 3(\pm 2)^2$

$m = 12$

Equation of the tangent with slope m through the point (x_1, y_1) is $y - y_1 = m(x - x_1)$

(i) Substituting $m = 12$, and the point $(2, 9)$

Tangent equation is $y - 9 = 12(x - 2)$

$$y - 9 = 12x - 24$$

$$12x - y - 24 + 9 = 0$$

$$12x - y - 15 = 0$$

(ii) Substituting $m = 12$, and the point $(-2, -7)$

Tangent equation is $y + 7 = 12(x + 2)$

$$y + 7 = 12x + 24$$

$$12x - y + 24 - 7 = 0$$

$$12x - y + 17 = 0$$

****

7. Find the equations of the tangents to the curve $y = \frac{x+1}{x-1}$ which are parallel to the line $x + 2y = 6$.

Solution: Given $y = \frac{x+1}{x-1}$

$$\frac{dy}{dx} = \frac{(x-1)(1)-(x+1)(1)}{(x-1)^2}$$

$$= \frac{x-1-x-1}{(x-1)^2}$$

$$= \frac{-2}{(x-1)^2}$$

Slope of the tangent $m_1 = \frac{-2}{(x-1)^2}$

Equation of the given line $x + 2y = 6$

$$1 + 2\frac{dy}{dx} = 0$$

$$2\frac{dy}{dx} = -1$$

$$\frac{dy}{dx} = -\frac{1}{2}$$

Slope of the line $m_2 = -\frac{1}{2}$

Since the tangent is parallel to the line,

$$m_1 = m_2$$

$$\frac{-2}{(x-1)^2} = -\frac{1}{2}$$

$$\frac{2}{(x-1)^2} = \frac{1}{2}$$

$$4 = (x-1)^2$$

$$(x-1)^2 = 4$$

$$x^2 - 2x + 1 = 4$$

$$x^2 - 2x + 1 - 4 = 0$$

$$x^2 - 2x - 3 = 0$$

$$(x-3)(x+1) = 0$$

$x - 3 = 0$, *gives* $x = 3$ and

$x + 1 = 0, gives \ \ x = -1$

The points lie on the curve $y = \frac{x+1}{x-1}$

When $x = 3, \ \ y = \frac{3+1}{3-1}$

$= \frac{4}{2}$

$y = 2$

When $x = -1, \ y = \frac{-1+1}{-1-1}$

$= 0$

$y = 0$

Hence $(3, 2)$ and $(-1, 0)$ are the points.

(i) At $x = 3$,

Slope of the tangent $m = \frac{-2}{(x-1)^2}$

$= \frac{-2}{(3-1)^2}$

$= \frac{-2}{(2)^2}$

$= \frac{-2}{4}$

$m = -\frac{1}{2}$

Equation of the tangent with slope m through the point (x_1, y_1) is $y - y_1 = m(x - x_1)$

Substituting $m = -\frac{1}{2}$, and the point $(3, 2)$

Tangent equation is $y - 2 = -\frac{1}{2}(x - 3)$

$2(y - 2) = -(x - 3)$

$2y - 4 = -x + 3$

$x + 2y - 4 - 3 = 0$

$x + 2y - 7 = 0$

(ii) At $x = -1$,

Slope of the tangent $m = \frac{-2}{(x-1)^2}$

$= \frac{-2}{(-1-1)^2}$

$= \frac{-2}{(-2)^2}$

$= \frac{-2}{4}$

$m = -\frac{1}{2}$

Equation of the tangent with slope m through the point (x_1, y_1) is $y - y_1 = m(x - x_1)$

Substituting $m = -\frac{1}{2}$, and the point $(-1, 0)$

Tangent equation is $y - 0 = -\frac{1}{2}(x + 1)$

$2y = -(x + 1)$

$2y = -x - 1$

$x + 2y + 1 = 0$

$x + 2y + 1 = 0$

****

8. Find the equation of tangent and normal to the curve given by $x = 7\cos t$ and $y = 2\sin t$, $t \in \mathbb{R}$ at any point on the curve.

<u>Solution</u>: $x = 7\cos t$

$\frac{dx}{dt} = -7\sin t$

$y = 2\sin t$

$\frac{dy}{dt} = 2\cos t$

Slope of the tangent $m = \frac{dy}{dx} = \frac{-2\cos t}{7\sin t}$

$m = -\frac{2\cos t}{7\sin t}$

Equation of the tangent with slope m through the point (x_1, y_1) is $y - y_1 = m(x - x_1)$

Substituting $m = -\frac{2\cos t}{7\sin t}$,

and the point $(7\cos t, 2\sin t)$

Equation of tangent is

$(y - 2\sin t) = -\frac{2\cos t}{7\sin t}(x - 7\cos t)$

$(y - 2\sin t)7\sin t = -2\cos t\,(x - 7\cos t)$

$(7\sin t)y - 14\sin^2 t = -(2\cos t)x + 14\cos^2 t$

$(2\cos t)x + (7\sin t)y - 14\sin^2 t - 14\cos^2 t = 0$

$(2\cos t)x + (7\sin t)y - 14(\sin^2 t + \cos^2 t) = 0$

$(2\cos t)x + (7\sin t)y - 14(1) = 0$

$(2\cos t)x + (7\sin t)y - 14 = 0$

Normal is perpendicular to tangent.

Hence normal is of the form

$(7\sin t)x - (2\cos t)y + k = 0$

It passes through $(7\cos t, 2\sin t)$

$$(7\sin t)(7\cos t)-(2\cos t)(2\sin t)+k=0$$
$$49\sin t\cos t-4\sin t\cos t+k=0$$
$$45\sin t\cos t+k=0$$
$$k=-45\sin t\cos t$$

So equation of the normal is

$$(7\sin t)x-(2\cos t)y-45\sin t\cos t=0$$

****

9. Find the angle between the rectangular hyperbola $xy=2$ and the parabola $x^2+4y=0$.

Solution: Given $xy=2$

Gives $y=\frac{2}{x}$

Substituting $y=\frac{2}{x}$ in $x^2+4y=0$, we get

$$x^2+4\left(\frac{2}{x}\right)=0$$
$$x^2+\frac{8}{x}=0$$

Multiplying by x, $x^3+8=0$

$$x^3=-8=(-2)^3$$

Gives $x=-2$

Substituting $x=-2$, in $xy=2$ we get

$$(-2)y=2$$
$$-2y=2$$

gives, $y=-1$

So, the point of intersection is $(-2,-1)$

Again from $xy=2$

$$x\frac{dy}{dx}+y(1)=0$$
$$x\frac{dy}{dx}+y=0$$
$$x\frac{dy}{dx}=-y$$
$$\frac{dy}{dx}=-\frac{y}{x}$$

Slope of the first curve at $(-2,-1)$

$m_1=-\frac{1}{2}$ and from

$$x^2+4y=0$$
$$2x+4\frac{dy}{dx}=0$$
$$4\frac{dy}{dx}=-2x$$
$$\frac{dy}{dx}=-\frac{2x}{4}$$

Slope of the second curve at $(-2,-1)$

$$m_2=-\frac{x}{2}=\frac{2}{2}=1$$

If θ is the angle between the curves, then

$$\tan\theta=\left|\frac{m_1-m_2}{1+m_1m_2}\right|$$
$$=\left|\frac{\left(-\frac{1}{2}\right)-1}{1+\left(-\frac{1}{2}\right)(1)}\right|$$
$$=\left|\frac{\frac{-1-2}{2}}{1-\frac{1}{2}}\right|$$
$$=\left|\frac{\frac{-3}{2}}{\frac{1}{2}}\right|$$
$$=\left|\frac{-3}{2}\times\frac{2}{1}\right|$$
$$=|-3|$$
$$\tan\theta=3$$
$$\theta=\tan^{-1}(3)$$

****

10. Show that the two curves $x^2-y^2=r^2$ and $xy=c^2$ where c, r are constants, cut orthogonally.

Solution: Given $x^2-y^2=r^2$

$$2x-2y\frac{dy}{dx}=0$$
$$-2y\frac{dy}{dx}=-2x$$
$$2y\frac{dy}{dx}=2x$$
$$\frac{dy}{dx}=\frac{2x}{2y}$$
$$\frac{dy}{dx}=\frac{x}{y}$$

Let (x_1,y_1) be point of intersection.

Then slope of the first curve at (x_1,y_1) $m_1=\frac{x_1}{y_1}$

Again from $xy=c^2$

$$x\frac{dy}{dx}+y(1)=0$$
$$x\frac{dy}{dx}+y=0$$
$$x\frac{dy}{dx}=-y$$
$$\frac{dy}{dx}=-\frac{y}{x}$$

Slope of the second curve at (x_1, y_1) $m_2 = -\frac{y_1}{x_1}$

Product of the slopes $m_1 \times m_2 = \left(\frac{x_1}{y_1}\right) \times \left(-\frac{y_1}{x_1}\right)$

$= -1$

Hence the two given curves cut orthogonally.

****

Example 7.19 Compute the value of 'c' satisfied by the Rolle's theorem for the function $f(x) = x^2(1-x)^2, x \in [0,1]$.

Solution: $f(x)$ is continuous in $[0,1]$ and

$f(x)$ is differentiable in $(0,1)$

Given $f(x) = x^2(1-x)^2, x \in [0,1]$

$f(a) = f(0) = x^2(1-x)^2 = 0$

$f(b) = f(1) = 1^2(1-1)^2 = 0$

Hence $f(a) = f(b)$

$f(x) = x^2(1-x)^2$

$= x^2(x^2 - 2x + 1)$

$= x^4 - 2x^3 + x^2$

$f'(x) = 4x^3 - 6x^2 + 2x$

$f'(c) = 4c^3 - 6c^2 + 2c$

Substituting $f'(c) = 0$, we get

$4c^3 - 6c^2 + 2c = 0$

$c(4c^2 - 6c + 2) = 0$

Gives, $c = 0$ *and* $(4c^2 - 6c + 2) = 0$

When $4c^2 - 6c + 2 = 0$

Dividing by 2, $2c^2 - 3c + 1 = 0$

$(2c-1)(c-1) = 0$

$2c - 1 = 0 \Rightarrow 2c = 1 \Rightarrow c = \frac{1}{2}$ and

$c - 1 = 0 \Rightarrow c = 1$

Hence $c = 0, c = \frac{1}{2}, c = 1$ are the values.

By Rolle's Theorem, when $f'(c) = 0, c \in [a,b]$

$\therefore c = \frac{1}{2} \in [0,1]$

****

Example 7.20 Find the values in the interval $\left(\frac{1}{2}, 2\right)$satisfied by the Rolle's theorem for the function $f(x) = x + \frac{1}{x}, x \in \left[\frac{1}{2}, 2\right]$.

Solution: $f(x)$ is continuous in $\left[\frac{1}{2}, 2\right]$ and

$f(x)$ is differentiable in $\left(\frac{1}{2}, 2\right)$

Given $f(x) = x + \frac{1}{x}, x \in \left[\frac{1}{2}, 2\right]$

$f(a) = f\left(\frac{1}{2}\right) = \frac{1}{2} + 2 = \frac{5}{2}$

$f(b) = f(2) = 2 + \frac{1}{2} = \frac{5}{2}$

Hence $f(a) = f(b)$

$f(x) = x + \frac{1}{x}$

$f'(x) = 1 - \frac{1}{x^2}$

$f'(c) = 1 - \frac{1}{c^2}$

Substituting $f'(c) = 0$, we get

$1 - \frac{1}{c^2} = 0$

$c^2 - 1 = 0$

$c^2 = 1$

$c = \pm 1$ gives

$c = 1$ *and* $c = -1$

By Rolle's Theorem, when $f'(c) = 0, c \in [a,b]$

$\therefore c = 1 \in \left[\frac{1}{2}, 2\right]$

****

Example 7.21 Compute the value of 'c' satisfied by Rolle's theorem for the function $f(x) = log\left(\frac{x^2+6}{5x}\right)$ in the interval $[2,3]$.

Solution: $f(x)$ is continuous in $[2,3]$ and

$f(x)$ is differentiable in $(2,3)$

$f(x) = log\left(\frac{x^2+6}{5x}\right)$

$f(x) = log(x^2+6) - log(5x)$

$f(a) = f(2) = log(4+6) - log(10)$

$= log(10) - log(10)$

$f(a) = 0$

$f(b) = f(3) = log(9+6) - log(15)$

$= log(15) - log(15)$

$f(b) = 0$

Hence, $f(a) = f(b) = 0$

From $f(x) = log\left(\frac{x^2+6}{5x}\right)$

$f'(x) = \frac{1}{x^2+6}(2x) - \frac{1}{5x}(5)$

$= \frac{2x}{x^2+6} - \frac{1}{x}$

$f'(c) = \frac{2c}{c^2+6} - \frac{1}{c}$

When $f'(c) = 0$, gives

$\frac{2c}{c^2+6} - \frac{1}{c} = 0$

$\frac{2c}{c^2+6} = \frac{1}{c}$

$(2c)c = c^2 + 6$

$2c^2 = c^2 + 6$

$2c^2 - c^2 = 6$

$c^2 = 6$

$c = \pm\sqrt{6}$

Since $c = \sqrt{6} \in [2, 3]$, the satisfied value is $c = \sqrt{6}$

****

Example 7.22 Without actually solving show that the equation $x^4 + 2x^3 - 2 = 0$ has only one real root in the interval $(0, 1)$.

Solution: Let $f(x) = x^4 + 2x^3 - 2$

$f(x)$ is continuous in $[0,1]$ and

$f(x)$ is differentiable in $(0,1)$

Now, $f'(x) = 4x^3 + 6x^2$

Substituting $f'(x) = 0$

$4x^3 + 6x^2 = 0$

$2x^2(2x + 3) = 0$

$2x^2 = 0$ *and* $2x + 3 = 0$

$2x^2 = 0 \Rightarrow x = 0\ (twice)$

$2x + 3 = 0 \Rightarrow x = -\frac{3}{2}$

But $x = 0, -\frac{3}{2} \notin (0,1)$

Hence by the Rolle's theorem there do not exist $a, b \in (0,1)$ such that, $f(a) = 0 = f(b)$.

Therefore the equation $f(x) = 0$ cannot have two roots in the interval $(0, 1)$.

But, $f(0) = -2 < 0$ and $f(1) = 1 > 0$ tells us the curve $y = f(x)$ crosses the x -axis between 0 and 1 only once by the Intermediate value theorem. Therefore the equation $x^4 + 2x^3 - 2 = 0$ has only one real root in the interval $(0, 1)$

****

Example 7.23 Prove using the Rolle's theorem that between any two distinct real zeros of the polynomial $a_nx^n + a_{n-1}x^{n-1} + \ldots\ldots.. + a_1x + a_0$ there is a zero of the polynomial $na_nx^{n-1} + (n-1)a_{n-1}x^{n-2} + \ldots\ldots.. + a_1$.

Solution:

$P(x) = a_nx^n + a_{n-1}x^{n-1} + \ldots\ldots.. + a_1x + a_0$

Let α, β be the two zeros of the polynomial.

$\therefore P(x)$ is continuous on $[\alpha, \beta]$

$P(x)$ is differentiable on (α, β)

Hence $P(\alpha) = P(\beta)$

By Rolle's theorem there exists a value $\gamma \in (\alpha, \beta)$ such that $P'(\gamma) = 0$

So,

$P(\gamma) = na_nx^{n-1} + (n-1)a_{n-1}x^{n-2} + \ldots\ldots.. + a_1$

****

Example 7.24 Prove that there is a zero of the polynomial, $2x^3 - 9x^2 - 11x + 12$ in the interval $(2, 7)$ given that 2 and 7 are the zeros of the polynomial $x^4 - 6x^3 - 11x^2 + 24x + 28$

Solution: $f(x) = x^4 - 6x^3 - 11x^2 + 24x + 28$

$f(x)$ is continuous on $[2,7]$

$f(x)$ is differentiable on $(2,7)$

$f(a) = f(2) = (2)^4 - 6(2)^3 - 11(2)^2 + 24(2) + 28$

$= 16 - 6(8) - 11(4) + 48 + 28$

$= 16 - 48 - 44 + 48 + 28$

$= 44 - 48 - 44 + 48$

$= 0$

$f(b) = f(7) = (7)^4 - 6(7)^3 - 11(7)^2 + 24(7) + 28$

$= 2401 - 6(343) - 11(49) + 168 + 28$

$= 2401 - 2058 - 539 + 196$

$= 2597 - 2597 = 0$

Hence $f(a) = f(b) = 0$

By Rolle's theorem there exists a value $c \in (a, b)$ such that $f^{/}(c) = 0$

$f(x) = x^4 - 6x^3 - 11x^2 + 24x + 28$

$f^{/}(x) = 4x^3 - 12x^2 - 22x + 24$

$f^{/}(c) = 4c^3 - 12c^2 - 22c + 24$

When $f^{/}(c) = 0$, gives

$4c^3 - 12c^2 - 22c + 24 = 0$

Dividing by 2,

$2c^3 - 6c^2 - 11c + 12 = 0$

Hence there is a zero of the above polynomial lies in the given internal (2, 7).

****

Example 7.25 Find the values in the interval (1, 2) of the mean value theorem satisfied by the function $f(x) = x - x^2$ for $1 \le x \le 2$.

Solution: Mean value theorem states that

$f(x)$ is continuous on $[a, b]$

$f(x)$ is differentiable on (a, b)

$f(a) \neq f(b)$

There exists a least possible value $c \in (a, b)$ such that $f^{/}(c) = \frac{f(b) - f(a)}{b - a}$

Given $f(x) = x - x^2$ for $1 \le x \le 2$

$f(x)$ is continuous on [1,2]

$f(x)$ is differentiable on (1,2)

$f(a) = f(1) = 1 - 1^2 = 0$

$f(b) = f(2) = 2 - (2)^2$

$= 2 - 4$

$= -2$

Hence, $f(a) \neq f(b)$

Now $f(x) = x - x^2$

$f^{/}(x) = 1 - 2x$

$f^{/}(c) = 1 - 2c$

$f^{/}(c) = \frac{f(b) - f(a)}{b - a}$ gives,

$1 - 2c = \frac{-2 - 0}{2 - 1}$

$1 - 2c = \frac{-2}{1}$

$1 - 2c = -2$

$-2c = -3$

$2c = 3$

$c = \frac{3}{2}$

$\therefore c = \frac{3}{2} \in (1,2)$ is the required value.

****

Example 7.26 A truck travels on a toll road with a speed limit of 80 km/hr. The truck completes a 164 km journey in 2 hours. At the end of the toll road the trucker is issued with a speed violation ticket. Justify this using the Mean Value Theorem.

Solution:

Let $f(t)$ be the distance travelled by the trucker in 't' hours.

This is a continuous function in [0, 2] and differentiable in (0, 2).

Now, $f(0) = 0$ and $f(2)$ =164.

By an application of the Mean Value Theorem, there exists a time c such that,

$f^{/}(c) = \frac{f(b) - f(a)}{b - a}$

$= \frac{164 - 0}{2 - 0}$

$= \frac{164}{2}$

$= 82 > 80$

Therefore at some point of time, during the travel in 2 hours the trucker must have travelled with a speed more than 80 km which justifies the issuance of a speed violation ticket.

****

Example 7.27

Suppose $f(x)$ is a differentiable function for all x with $f^{/}(x) \le 29$ and $f(2)$ =17. What is the maximum value of $f(7)$?

Solution: By an application of the Mean Value Theorem, there exists a time $c \in (2,7)$ such that, $f^{/}(c) = \frac{f(b) - f(a)}{b - a}$

$\frac{f(7) - f(2)}{7 - 2} = f^{/}(c) \le 29$

$$\frac{f(7)-17}{5} \le 29$$

$$f(7) \le 29(5)+17$$

$$\le 145+17$$

$$\le 162$$

Therefore maximum value of $f(7)$ is 162.

****

Example 7.28 Prove, using mean value theorem, that $|\sin\alpha - \sin\beta| \le |\alpha-\beta|, \alpha,\beta \in \mathbb{R}$.

Solution:

Let $f(x) = \sin x$ which is a differentiable function in any open interval. Consider an interval $[\alpha,\beta]$. Applying the mean value theorem there exists $c \in (\alpha,\beta)$ such that,

$$\frac{\sin\beta-\sin\alpha}{\beta-\alpha} = f'(c) = \cos c$$

Therefore $\left|\frac{\sin\beta-\sin\alpha}{\beta-\alpha}\right| = |\cos c| \le 1$

Hence, $|\sin\alpha - \sin\beta| \le |\alpha-\beta|$

****

Example 7.29

A thermometer was taken from a freezer and placed in boiling water. It took 22 seconds for the thermometer to raise from $-10°$C to $100°$C. Show that the rate of change of temperature at some time t is $5°$C per second.

Solution:

Let $f(t)$ be the temperature at time t. By the mean value theorem,

$$f'(c) = \frac{f(b)-f(a)}{b-a}$$

$$= \frac{100-(-10)}{22-0}$$

$$= \frac{100+10}{22}$$

$$= \frac{110}{22}$$

$$= 5^0C \text{ per second.}$$

Hence the instantaneous rate of change of temperature at some time t should be 5^0C per second.

****

EXERCISE 7.3

1. Explain why Rolle's theorem is not applicable to the following functions in the respective intervals.

(i) $f(x) = \left|\frac{1}{x}\right|, x \in [-1,1]$

(ii) $f(x) = \tan x, x \in [0,\pi]$

(iii) $f(x) = x - 2\log x, x \in [2,7]$

Solution:

(i) $f(x) = \left|\frac{1}{x}\right|, x \in [-1,1]$

Given $x \in [-1,1]$.

At $x=0$, $f(x)$ is not continuous, hence Rolle's Theorem is not applicable.

(ii) $f(x) = \tan x, x \in [0,\pi]$

Given $x \in [0,\pi]$.

At $x = \frac{\pi}{2}$, $f(x)$ is not continuous, hence Rolle's Theorem is not applicable.

(iii) $f(x) = x - 2\log x, x \in [2,7]$

$f(x)$ is continuous on $[2,7]$

$f(x)$ is differentiable on $(2,7)$

$f(a) = f(2) = 2 - 2\log 2$ and

$f(b) = f(7) = 7 - 2\log 7$

Hence, $f(a) \ne f(b)$ hence

Rolle's Theorem is not applicable.

****

2. Using the Rolle's theorem, determine the values of x at which the tangent is parallel to the x -axis for the following functions :

(i) $f(x) = x^2 - x, x \in [0,1]$

(ii) $f(x) = \frac{x^2-2x}{x+2}, x \in [-1,6]$

(iii) $f(x) = \sqrt{x} - \frac{x}{3}, x \in [0,9]$

Solution:

(i) $f(x) = x^2 - x, x \in [0,1]$

$f(x)$ is continuous on $[0,1]$

$f(x)$ is differentiable on $(0,1)$

$f(x) = x^2 - x$

$f(a) = f(0) = 0$ and

$f(b) = f(1) = 1 - 1 = 0$

Hence, $f(a) = f(b)$ hence

There exists a least possible value 'c' such that $f'(c) = 0$

From $f(x) = x^2 - x$

$$f'(x) = 2x - 1$$
$$f'(c) = 2c - 1$$

When $f'(c) = 0$, we get

$$2c - 1 = 0$$
$$2c = 1$$
$$c = \frac{1}{2} \in (0,1)$$

At $x = \frac{1}{2}$, $y = f(x) = x^2 - x$

$$y = f\left(\frac{1}{2}\right) = \left(\frac{1}{2}\right)^2 - \frac{1}{2}$$
$$y = \frac{1}{4} - \frac{1}{2}$$
$$= \frac{1-2}{4}$$
$$= -\frac{1}{4}$$

Hence at the point $\left(\frac{1}{2}, -\frac{1}{4}\right)$ the tangent is parallel to x –axis.

(ii) $f(x) = \frac{x^2 - 2x}{x+2}, x \in [-1, 6]$

$f(x)$ is continuous on $[-1, 6]$

$f(x)$ is differentiable on $(-1, 6)$

$$f(x) = \frac{x^2 - 2x}{x+2}$$
$$f(a) = f(-1)$$
$$= \frac{(-1)^2 - 2(-1)}{-1+2}$$
$$= \frac{1+2}{1} = 3$$
$$f(b) = f(6)$$
$$= \frac{(6)^2 - 2(6)}{6+2}$$
$$= \frac{36-12}{8}$$
$$= \frac{24}{8} = 3$$

Hence, $f(a) = f(b)$ hence

There exists a least possible value 'c' such that $f'(c) = 0$

From $f(x) = \frac{x^2 - 2x}{x+2}$

$$f'(x) = \frac{(x+2)(x^2-2x)' - (x^2-2x)(x+2)'}{(x+2)^2}$$
$$= \frac{(x+2)(2x-2) - (x^2-2x)(1)}{(x+2)^2}$$
$$= \frac{(2x^2-2x+4x-4) - (x^2-2x)}{(x+2)^2}$$
$$= \frac{(2x^2+2x-4) - (x^2-2x)}{(x+2)^2}$$
$$= \frac{2x^2+2x-4-x^2+2x}{(x+2)^2}$$
$$f'(x) = \frac{x^2+4x-4}{(x+2)^2}$$
$$f'(c) = \frac{c^2+4c-4}{(c+2)^2}$$

When $f'(c) = 0$

We get $\frac{c^2+4c-4}{(c+2)^2} = 0$

$$c^2 + 4c - 4 = 0$$
$$c = \frac{-4 \pm \sqrt{16+16}}{2}$$
$$= \frac{-4 \pm \sqrt{16 \times 2}}{2}$$
$$= \frac{-4 \pm 4\sqrt{2}}{2}$$
$$= \frac{2(-2 \pm 2\sqrt{2})}{2}$$
$$= -2 \pm 2\sqrt{2}$$
$$c = -2 + 2\sqrt{2} \in (-1, 6)$$

At $x = -2 + 2\sqrt{2}$,

$$y = f(x) = \frac{x^2 - 2x}{x+2}$$
$$y = f(-2 + 2\sqrt{2})$$
$$= \frac{(-2+2\sqrt{2})^2 - 2(-2+2\sqrt{2})}{(-2+2\sqrt{2}) + 2}$$
$$= \frac{4+8-8\sqrt{2}+4-4\sqrt{2}}{2\sqrt{2}}$$
$$= \frac{16-12\sqrt{2}}{2\sqrt{2}} = \frac{2(8-6\sqrt{2})}{2\sqrt{2}}$$
$$= \frac{8-6\sqrt{2}}{\sqrt{2}} \times \frac{\sqrt{2}}{\sqrt{2}}$$
$$= \frac{8\sqrt{2}-6(2)}{2}$$
$$y = \frac{2(4\sqrt{2}-6)}{2} = 4\sqrt{2} - 6$$

Hence at the point $(-2 + 2\sqrt{2}, 4\sqrt{2} - 6)$ the tangent is parallel to x –axis.

(iii) $f(x) = \sqrt{x} - \frac{x}{3}, x \in [0,9]$

$f(x)$ is continuous on $[0,9]$

$f(x)$ is differentiable on $(0,9)$

$f(a) = f(0) = \sqrt{0} - \frac{0}{3} = 0$

$f(b) = f(9) = \sqrt{9} - \frac{9}{3} = 3 - 3 = 0$

Hence, $f(a) = f(b)$ hence

There exists a least possible value 'c' such that $f'(c) = 0$

From $f(x) = \sqrt{x} - \frac{x}{3}$

$f'(x) = \frac{1}{2\sqrt{x}} - \frac{1}{3}$

$f'(c) = \frac{1}{2\sqrt{c}} - \frac{1}{3}$

When $f'(c) = 0$

$\frac{1}{2\sqrt{c}} - \frac{1}{3} = 0$

$\frac{1}{2\sqrt{c}} = \frac{1}{3}$

$3 = 2\sqrt{c}$

Squaring, $4c = 9$

$c = \frac{9}{4} \in (0,9)$

At $x = \frac{9}{4}$,

$y = f(x) = \sqrt{\frac{9}{4}} - \frac{\frac{9}{4}}{3}$

$= \frac{3}{2} - \frac{9}{4} \times \frac{1}{3}$

$= \frac{3}{2} - \frac{3}{4}$

$= \frac{6-3}{4}$

$= \frac{3}{4}$

Hence at the point $\left(\frac{9}{4}, \frac{3}{4}\right)$ the tangent is parallel to x –axis.

****

3. Explain why Lagrange's mean value theorem is not applicable to the following functions in the respective intervals :
(i) $f(x) = \frac{x+1}{x}, x \in [-1, 2]$
(ii) $f(x) = |3x + 1|, x \in [-1, 3]$

<u>Solution</u>:

(i) $f(x) = \frac{x+1}{x}, x \in [-1, 2]$

When $x = 0$,

$f(0) = \frac{0+1}{0}$

$= \frac{1}{0}$

$= \infty$

Hence $f(x)$ is dis continuous, So Lagrange's mean value theorem is not applicable.

(ii) $f(x) = |3x + 1|, x \in [-1, 3]$

$f(x)$ is continuous on $[-1, 3]$

$f(x)$ is not differentiable on $(-1, 3)$

So Lagrange's mean value theorem is not applicable.

****

4. Using the Lagrange's mean value theorem determine the values of x at which the tangent is parallel to the secant line at the end points of the given interval:
(i) $f(x) = x^3 - 3x + 2, x \in [-2, 2]$
(ii) $f(x) = (x-2)(x-7), x \in [3,11]$

<u>Solution</u>:

(i) $f(x) = x^3 - 3x + 2, x \in [-2, 2]$

$f(x)$ is continuous on $[-2, 2]$

$f(x)$ is differentiable on $(-2, 2)$

Now, $f(a) = f(-2) = (-2)^3 - 3(-2) + 2$

$= -8 + 6 + 2$

$= -8 + 8$

$= 0$ and

$f(b) = f(2) = (2)^3 - 3(2) + 2$

$= 8 - 6 + 2$

$= 10 - 6$

$= 4$

Hence, $f(a) \neq f(b)$ hence

By an application of the Mean Value Theorem, there exists a time c such that,

$f'(c) = \frac{f(b) - f(a)}{b - a}$

$f(x) = x^3 - 3x + 2$

$f^/(x) = 3x^2 - 3$

$f^/(c) = 3c^2 - 3$

$\therefore f^/(c) = \frac{f(b) - f(a)}{b-a}$

$3c^2 - 3 = \frac{4-0}{2-(-2)}$

$= \frac{4}{4}$

$3c^2 - 3 = 1$

$3c^2 = 4$

$c^2 = \frac{4}{3}$

$c = \pm\frac{2}{\sqrt{3}}$

$c = \pm\frac{2}{\sqrt{3}} \in [-2, 2]$

At $x = \pm\frac{2}{\sqrt{3}}$ the tangent is parallel to the secant.

(ii) $f(x) = (x-2)(x-7), x \in [3,11]$

$f(x) = x^2 - 9x + 14$

$f(x)$ is continuous on $[3,11]$

$f(x)$ is differentiable on $(3,11)$

Now, $f(a) = f(3) = (3)^2 - 9(3) + 14$

$= 9 - 27 + 14$

$= 23 - 27$

$= -4$ and

$f(b) = f(11) = (11)^2 - 9(11) + 14$

$= 121 - 99 + 14$

$= 135 - 99$

$= 36$

Hence, $f(a) \neq f(b)$ hence

By an application of the Mean Value Theorem, there exists a time c such that,

$f^/(c) = \frac{f(b) - f(a)}{b-a}$

$f(x) = x^2 - 9x + 14$

$f^/(x) = 2x - 9$

$f^/(c) = 2c - 9$

$2c - 9 = \frac{36+4}{11-3}$

$2c - 9 = \frac{40}{8} = 5$

$2c = 5 + 9$

$2c = 14$

$c = \frac{14}{2} = 7 \in (3,11)$

At $x = 7$ the tangent is parallel to the secant.

****

5. Show that the value in the conclusion of the mean value theorem for

(i) $f(x) = \frac{1}{x}$ on a closed interval of positive numbers $[a, b]$ is $\sqrt{ab}$

(ii) $f(x) = Ax^2 + Bx + C$ on any interval $[a, b]$ is $\frac{a+b}{2}$

<u>Solution</u>:

(i) $f(x) = \frac{1}{x},\ x \in [a, b]$

$f(x)$ is continuous on $[a, b]$

$f(x)$ is differentiable on (a, b)

Now, $f(a) = \frac{1}{a}$ and

$f(b) = \frac{1}{b}$

Hence, $f(a) \neq f(b)$ hence

By an application of the Mean Value Theorem, there exists a time c such that,

$f^/(c) = \frac{f(b) - f(a)}{b-a}$

$f(x) = \frac{1}{x}$

$f^/(x) = -\frac{1}{x^2}$

$f^/(c) = -\frac{1}{c^2}$

$-\frac{1}{c^2} = \frac{\frac{1}{b} - \frac{1}{a}}{b-a}$

$= \left(\frac{1}{b} - \frac{1}{a}\right) \times \frac{1}{b-a}$

$= \left(\frac{a-b}{ab}\right) \times \frac{1}{b-a}$

$= -\frac{1}{ab}$

$\therefore c^2 = ab$

$c = \pm\sqrt{ab}$

$c = \sqrt{ab} \in (a, b)$

Hence proved.

(ii) $f(x) = Ax^2 + Bx + C\,, x \in [a,b]$

$f(x)$ is continuous on $[a,b]$

$f(x)$ is differentiable on (a,b)

Now, $f(a) = Aa^2 + Ba + C$ and

$f(b) = Ab^2 + Bb + C$

Hence, $f(a) \neq f(b)$ hence

By an application of the Mean Value Theorem, there exists a time c such that,

$$f'(c) = \frac{f(b) - f(a)}{b-a}$$

$$f(x) = Ax^2 + Bx + C$$

$$f'(x) = 2Ax + B$$

$$f'(c) = 2Ac + B$$

$$2Ac + B = \frac{(Ab^2+Bb+C) - (Aa^2+Ba+C)}{b-a}$$

$$= \frac{Ab^2+Bb+C - Aa^2 - Ba - C}{b-a}$$

$$= \frac{Ab^2+Bb - Aa^2 - Ba}{b-a}$$

$$= \frac{Ab^2 - Aa^2 + Bb - Ba}{b-a}$$

$$= \frac{A(b^2 - a^2) + B(b-a)}{b-a}$$

$$= \frac{A(b-a)(b+a) + B(b-a)}{b-a}$$

$$= \frac{(b-a)[A(b+a)+B]}{b-a}$$

$$2Ac + B = A(b+a) + B$$

$$2Ac = A(b+a)$$

$$2c = (b+a)$$

$$c = \frac{(b+a)}{2}, \in (a,b)$$

Hence proved.

****

6. A race car driver is racing at 20th km. If his speed never exceeds 150 km/hr, what is the maximum distance he can cover in the next two hours.

Solution:

Let $f(x)$ be the distance covered.

Given $x \in [0,2]$

When $x = 0, f(a) = f(0) = 20$ and

$x = 2, f(b) = f(2) = ?$

$f(x)$ is continuous on $[a,b]$

$f(x)$ is differentiable on (a,b)

Now, $f(a) = 20$ and

$f(b) = ?$

Hence, $f(a) \neq f(b)$ hence

By an application of the Mean Value Theorem, there exists a time c such that,

$$f'(c) = \frac{f(b) - f(a)}{b-a}$$

$$\frac{f(b) - 20}{2-0} \leq 150$$

$$\frac{f(b) - 20}{2} \leq 150$$

$$f(b) - 20 \leq 300$$

$$f(b) \leq 300 + 20$$

$$f(b) \leq 320$$

Hence the covers a maximum distance of 320 Km, in next two hours.

****

7. Suppose that for a function $f(x)$, $f'(x) \leq 1$ for all $1 \leq x \leq 4$. Show that $f(4) - f(1) \leq 3$.

Solution:

Given $f'(x) \leq 1$

$f(x)$ is continuous on $[1,4]$

$f(x)$ is differentiable on $(1,4)$

By an application of the Mean Value Theorem, there exists a time c such that,

$$f'(c) = \frac{f(b) - f(a)}{b-a}$$

$$\therefore \frac{f(b) - f(a)}{b-a} = f'(x)$$

$$\frac{f(4) - f(1)}{4-1} = f'(x) \leq 1$$

$$\frac{f(4) - f(1)}{3} \leq 1$$

$f(4) - f(1) \leq 3$. Hence proved

****

8. Does there exist a differentiable function $f(x)$ such that $f(0) = -1$, $f(2) = 4$ and $f'(x) \leq 2$ for all x. Justify your answer.

Solution:

Given $f^{/}(x) \leq 2$

$f(x)$ is continuous on $[0,2]$

$f(x)$ is differentiable on $(0,2)$

By an application of the Mean Value Theorem, there exists a time c such that,

$$f^{/}(c) = \frac{f(b)-f(a)}{b-a}$$

$$= \frac{f(2)-f(0)}{2-0}$$

$$= \frac{4+1}{2}$$

$$c = \frac{5}{2} = 2.5 \notin (0,2)$$

****

9. Show that there lies a point on the curve $f(x) = x(x+3)e^{-\frac{\pi}{2}}, -3 \leq x \leq 0$ where tangent drawn is parallel to the $x - axis$.

Solution:

$$f(x) = x(x+3)e^{-\frac{\pi}{2}}$$

$f(x)$ is continuous on $[-3,0]$

$f(x)$ is differentiable on $(-3,0)$

$$f(x) = (x^2+3x)e^{-\frac{\pi}{2}}$$

$f(a) = f(-3) = 0$ and

$f(b) = f(0) = 0$

Hence, $f(a) = f(b)$ hence

There exists a least possible value 'c' such that $f^{/}(c) = 0$

$$f(x) = (x^2+3x)e^{-\frac{\pi}{2}}$$

$$f^{/}(x) = (2x+3)e^{-\frac{\pi}{2}}$$

$$f^{/}(c) = (2c+3)e^{-\frac{\pi}{2}}$$

When $f^{/}(c) = 0$,

$$(2c+3)e^{-\frac{\pi}{2}} = 0$$

$$2c+3 = 0$$

$$2c = -3$$

$$c = -\frac{3}{2} \in (-3,0)$$

At $x = -\frac{3}{2}, y = f(x) = (x^2+3x)e^{-\frac{\pi}{2}}$

$$= \left(\frac{9}{4} + 3\left(-\frac{3}{2}\right)\right)e^{-\frac{\pi}{2}}$$

$$= \left(\frac{9}{4} - \frac{9}{2}\right)e^{-\frac{\pi}{2}}$$

$$= \left(\frac{9-18}{4}\right)e^{-\frac{\pi}{2}}$$

$$= \left(-\frac{9}{4}\right)e^{-\frac{\pi}{2}}$$

Hence at the point $\left(-\frac{3}{2}, \left(-\frac{9}{4}\right)e^{-\frac{\pi}{2}}\right)$ the tangent is parallel to the $x - axis$.

****

10. Using mean value theorem prove that for, $a > 0, b > 0, |e^{-a} - e^{-b}| < |a-b|$.

Solution:

Given $f(x) = e^{-x}$

$f(x)$ is continuous on $[a,b]$

$f(x)$ is differentiable on (a,b)

By an application of the Mean Value Theorem, there exists a time $c \in (a,b)$ such that,

$$f^{/}(c) = \frac{f(b)-f(a)}{b-a}$$

$$f(x) = e^{-x}$$

$$f^{/}(x) = -e^{-x}$$

$$f^{/}(c) = -e^{-c}$$

$$-e^{-c} = \frac{e^{-b}-e^{-a}}{b-a}$$

$$= \frac{-(e^{-a}-e^{-b})}{-(a-b)}$$

$$-e^{-c} = \frac{(e^{-a}-e^{-b})}{(a-b)}$$

Taking modulus on either sides,

$$|-e^{-c}| = \frac{|e^{-a}-e^{-b}|}{|a-b|}$$

$$\frac{|e^{-a}-e^{-b}|}{|a-b|} < 1$$

So, $|e^{-a} - e^{-b}| < |a-b|$.

****

Example 7.30

Expand $log(1+x)$ as a Maclaurin's series up to 4 non-zero terms for $-1 < x \leq 1$.

Solution:

$$f(x) = f(0) + \frac{x}{1!}f^{/}(0) + \frac{x^2}{2!}f^{//}(0) + \frac{x^3}{3!}f^{///}(0) + \frac{x^4}{4!}f^{/v}(0) + \cdots$$

Function and its derivatives	$log(1+x)$ and its derivatives	Value at $x=0$
$f(x)$	$log(1+x)$	0
$f^{/}(x)$	$\frac{1}{(1+x)}$	1
$f^{//}(x)$	$-\frac{1}{(1+x)^2}$	-1
$f^{///}(x)$	$\frac{2}{(1+x)^3}$	2
$f^{/v}(x)$	$-\frac{6}{(1+x)^4}$	-6

Substituting the values and on simplification we get the required expansion of the function given by

$$log(1+x) = 0+\frac{x}{1}(1)+\frac{x^2}{2}(-1)+\frac{x^3}{6}(2)+\frac{x^4}{24}(-6)+\cdots$$

$$log(1+x) = x-\frac{x^2}{2}+\frac{x^3}{3}-\frac{x^4}{4}+\cdots$$

****

Example 7.31 Expand $\tan x$ in ascending powers of x up to 5th power for $-\frac{\pi}{2}<x<\frac{\pi}{2}$

Function and its derivatives	$\tan x$ and its derivatives	Value at $y=0$
$y=f(x)$	$\tan x$	0
$y_1=f^{/}(x)$	$sec^2x=1+y^2$	1
$y_2=f^{//}(x)$	$2yy_1=2y(1+y^2)$ $=2y+2y^3$	0
$y_3=f^{///}(x)$	$=2y_1+6y^2y_1$ $=2(1+y^2)+6y^2(1+y^2)$ $=2+2y^2+6y^2+6y^4$ $=2+8y^2+6y^4$	2
$y_4=f^{/v}(x)$	$=16yy_1+24y^3y_1$ $=16y(1+y^2)+24y^3(1+y^2)$ $=16y+16y^3+24y^3+24y^5$ $=16y+40y^3+24y^5$	0
$y_5=f^{v}(x)$	$=16y_1+120y^2y_1+120y^4y_1$	16

Substituting the values and on simplification we get the required expansion of the function given by

$$f(x)=f(0)+\frac{x}{1!}f^{/}(0)+\frac{x^2}{2!}f^{//}(0)+\frac{x^3}{3!}f^{///}(0)+\frac{x^4}{4!}f^{/v}(0)+\cdots$$

$$\tan x = 0+\frac{x}{1}(1)+\frac{x^2}{2}(0)+\frac{x^3}{6}(2)+\frac{x^4}{24}(0)+\frac{x^5}{120}(16)..$$

$$\tan x = x+\frac{x^3}{3}+\frac{x^5}{15}(2)+\cdots$$

$$\tan x = x+\frac{x^3}{3}+\frac{2x^5}{15}+\cdots$$

****

Example 7.32

Write the Taylor series expansion of $\frac{1}{x}$ about $x=2$ by finding the first three non-zero terms.

Solution:

Function and its derivatives	$\frac{1}{x}$ and its derivatives	Value at $x=2$
$f(x)$	$\frac{1}{x}$	$\frac{1}{2}$
$f^{/}(x)$	$-\frac{1}{x^2}$	$-\frac{1}{4}$
$f^{//}(x)$	$\frac{2}{x^3}$	$\frac{1}{4}$
$f^{///}(x)$	$-\frac{6}{x^4}$	$-\frac{3}{8}$
$f^{/v}(x)$	$\frac{24}{x^5}$	$\frac{3}{4}$

$$f(x)=f(a)+\frac{(x-a)}{1!}f^{/}(a)+\frac{(x-a)^2}{2!}f^{//}(a)+\frac{(x-a)^3}{3!}f^{///}(a)+\frac{(x-a)^4}{4!}f^{/v}(a)+\cdots$$

$$\frac{1}{x}=f(2)+\frac{(x-2)}{1}f^{/}(2)+\frac{(x-2)^2}{2}f^{//}(2)+\frac{(x-2)^3}{6}f^{///}(2)+\frac{(x-2)^4}{24}f^{/v}(2)+\cdots$$

$$\frac{1}{x}=\frac{1}{2}+\frac{(x-2)}{1}\left(-\frac{1}{4}\right)+\frac{(x-2)^2}{2}\left(\frac{1}{4}\right)+\frac{(x-2)^3}{6}\left(-\frac{3}{8}\right)+\frac{(x-2)^4}{24}\left(\frac{3}{4}\right)+\cdots$$

$$\frac{1}{x}=\frac{1}{2}-\frac{(x-2)}{4}+\frac{(x-2)^2}{8}-\frac{(x-2)^3}{16}+\frac{(x-2)^4}{32}+...$$

****

EXERCISE 7.4

1. Write the Maclaurin series expansion of the following functions:

(i) e^x (ii) $\sin x$ (iii) $\cos x$
(iv) $log(1-x)\ ; -1 \le x < 1$
(v) $tan^{-1}(x); -1 \le x \le 1$
(vi) $\cos 2x$

Solution:

(i) e^x

Function and its derivatives	e^x and its derivatives	Value at $x = 0$
$f(x)$	e^x	1
$f^{/}(x)$	e^x	1
$f^{//}(x)$	e^x	1
$f^{///}(x)$	e^x	1
$f^{iv}(x)$	e^x	1

Substituting the values and on simplification we get the required expansion of the function given by

$$f(x) = f(0) + \frac{x}{1!}f^{/}(0) + \frac{x^2}{2!}f^{//}(0) + \frac{x^3}{3!}f^{///}(0) + \frac{x^4}{4!}f^{iv}(0) + \cdots$$

$$e^x = 1 + \frac{x}{1}(1) + \frac{x^2}{2}(1) + \frac{x^3}{6}(1) + \cdots$$

(ii) $\sin x$

Function and its derivatives	$\sin x$ and its derivatives	Value at $x = 0$
$f(x)$	$\sin x$	0
$f^{/}(x)$	$\cos x$	1
$f^{//}(x)$	$-\sin x$	0
$f^{///}(x)$	$-\cos x$	-1
$f^{iv}(x)$	$\sin x$	0

Substituting the values and on simplification we get the required expansion of the function given by

$$f(x) = f(0) + \frac{x}{1!}f^{/}(0) + \frac{x^2}{2!}f^{//}(0) + \frac{x^3}{3!}f^{///}(0) + \frac{x^4}{4!}f^{iv}(0) + \cdots$$

$$\sin x = 0 + \frac{x}{1!}(1) + \frac{x^2}{2!}(0) + \frac{x^3}{3!}(-1) + \cdots$$

$$\sin x = \frac{x}{1!} - \frac{x^3}{3!} + \cdots$$

(iii) $\cos x$

Function and its derivatives	$\cos x$ and its derivatives	Value at $x = 0$
$f(x)$	$\cos x$	1
$f^{/}(x)$	$-\sin x$	0
$f^{//}(x)$	$-\cos x$	-1
$f^{///}(x)$	$\sin x$	0
$f^{iv}(x)$	$\cos x$	1

Substituting the values and on simplification we get the required expansion of the function given by

$$f(x) = f(0) + \frac{x}{1!}f^{/}(0) + \frac{x^2}{2!}f^{//}(0) + \frac{x^3}{3!}f^{///}(0) + \frac{x^4}{4!}f^{iv}(0) + \cdots$$

$$\cos x = 1 + \frac{x}{1!}(0) + \frac{x^2}{2!}(-1) + \frac{x^3}{3!}(0) + \cdots$$

$$\cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} \ldots$$

(iv) $log(1-x)$

Function and its derivatives	$log(1-x)$ and its derivatives	Value at $x = 0$
$f(x)$	$log(1-x)$	0
$f^{/}(x)$	$\frac{-1}{(1-x)}$	-1
$f^{//}(x)$	$-\frac{1}{(1-x)^2}$	-1
$f^{///}(x)$	$-\frac{2}{(1-x)^3}$	-2
$f^{iv}(x)$	$-\frac{6}{(1-x)^4}$	-6

Substituting the values and on simplification we get the required expansion of the function given by

$$f(x) = f(0) + \frac{x}{1!}f^{/}(0) + \frac{x^2}{2!}f^{//}(0) + \frac{x^3}{3!}f^{///}(0) + \frac{x^4}{4!}f^{iv}(0) + \cdots$$

$$log(1-x) = 0 + \frac{x}{1}(-1) + \frac{x^2}{2}(-1)$$

$$+\frac{x^3}{6}(-2)+\frac{x^4}{24}(-6)+\cdots$$

$$log(1-x)=-x-\frac{x^2}{2}-\frac{x^3}{3}-\frac{x^4}{4}+\cdots$$

(v) $tan^{-1}(x)$

Function and its derivatives	$tan^{-1}(x)$ and its derivatives	Value at $x=0$
$f(x)$	$tan^{-1}(x)$	0
$f^{/}(x)$	$\frac{1}{1+x^2}$ $=1-x^2+x^4-x^6$	1
$f^{//}(x)$	$-2x+4x^3-6x^5$	0
$f^{///}(x)$	$-2+12x^2-30x^4$	-2
$f^{/v}(x)$	$24x-120x^3$	0

Substituting the values and on simplification we get the required expansion of the function given by

$$f(x)=f(0)+\frac{x}{1!}f^{/}(0)+\frac{x^2}{2!}f^{//}(0)$$

$$+\frac{x^3}{3!}f^{///}(0)+\frac{x^4}{4!}f^{/v}(0)+\cdots$$

$$tan^{-1}(x)=0+\frac{x}{1!}(1)+\frac{x^2}{2!}(0)+\frac{x^3}{3!}(-2)+\cdots$$

$$tan^{-1}(x)=x-\frac{2x^3}{3!}+\frac{24x^5}{5!}\ldots$$

$$tan^{-1}(x)=x-\frac{x^3}{3}+\frac{x^5}{5}\ldots$$

(vi) cos^2x

Function and its derivatives	cos^2x and its derivatives	Value at $x=0$
$f(x)$	cos^2x	1
$f^{/}(x)$	$-2cosx\sin x$ $=-\sin 2x$	0
$f^{//}(x)$	$-2\cos 2x$	-2
$f^{///}(x)$	$4\sin 2x$	0
$f^{/v}(x)$	$8\cos 2x$	8

Substituting the values and on simplification we get the required expansion of the function given by

$$f(x)=f(0)+\frac{x}{1!}f^{/}(0)+\frac{x^2}{2!}f^{//}(0)$$

$$+\frac{x^3}{3!}f^{///}(0)+\frac{x^4}{4!}f^{/v}(0)+\cdots$$

$$cos^2x=1+\frac{x}{1!}(0)+\frac{x^2}{2!}(-2)$$

$$+\frac{x^3}{3!}(0)+\frac{x^4}{4!}(8)+\cdots$$

$$cos^2x=1+\frac{x^2}{2!}(-2)+\frac{x^4}{4!}(8)+\cdots$$

****

2. Write down the Taylor series expansion, of the function $\log x$ about $x=1$ up to three non-zero terms for $x>0$.

Solution:

Function and its derivatives	$\log x$ and its derivatives	Value at $x=1$
$f(x)$	$\log x$	0
$f^{/}(x)$	$\frac{1}{x}$	1
$f^{//}(x)$	$-\frac{1}{x^2}$	-1
$f^{///}(x)$	$\frac{2}{x^3}$	2
$f^{/v}(x)$	$-\frac{6}{x^4}$	-6

$$f(x)=f(a)+\frac{(x-a)}{1!}f^{/}(a)+\frac{(x-a)^2}{2!}f^{//}(a)$$

$$+\frac{(x-a)^3}{3!}f^{///}(a)+\frac{(x-a)^4}{4!}f^{/v}(a)+\cdots$$

$$\log x=0+\frac{(x-2)}{1}(1)+\frac{(x-2)^2}{2}(-1)$$

$$+\frac{(x-2)^3}{6}(2)+\frac{(x-2)^4}{24}(-6)+\cdots$$

$$\log x=\frac{(x-2)}{1}-\frac{(x-2)^2}{2}+\frac{(x-2)^3}{3}-\frac{(x-2)^4}{4}+\cdots$$

****

3. Expand $\sin x$ in ascending powers $x-\frac{\pi}{4}$ up to three non-zero terms.

Solution:

Function and its derivatives	$\sin x$ and its derivatives	Value at $x=\frac{\pi}{4}$
$f(x)$	$\sin x$	$\frac{1}{\sqrt{2}}$
$f^{/}(x)$	$\cos x$	$\frac{1}{\sqrt{2}}$
$f^{//}(x)$	$-\sin x$	$-\frac{1}{\sqrt{2}}$
$f^{///}(x)$	$-\cos x$	$-\frac{1}{\sqrt{2}}$
$f^{/v}(x)$	$\sin x$	$\frac{1}{\sqrt{2}}$

$$f(x) = f(a) + \frac{(x-a)}{1!}f^{/}(a) + \frac{(x-a)^2}{2!}f^{//}(a) + \frac{(x-a)^3}{3!}f^{///}(a) + \frac{(x-a)^4}{4!}f^{/v}(a) + \cdots$$

$$\sin x = \frac{1}{\sqrt{2}} + \frac{\left(x-\frac{\pi}{4}\right)}{1}\left(\frac{1}{\sqrt{2}}\right) + \frac{\left(x-\frac{\pi}{4}\right)^2}{2}\left(-\frac{1}{\sqrt{2}}\right) + \frac{\left(x-\frac{\pi}{4}\right)^3}{6}\left(\frac{1}{\sqrt{2}}\right) + \frac{\left(x-\frac{\pi}{4}\right)^4}{24}\left(\frac{1}{\sqrt{2}}\right) + \cdots$$

****

4. Expand the polynomial $f(x) = x^2 - 3x + 2$ in powers of $x - 1$.

Solution:

Function and its derivatives	$f(x)$ and its derivatives	Value at $x = 1$
$f(x)$	$x^2 - 3x + 2$	0
$f^{/}(x)$	$2x - 3$	-1
$f^{//}(x)$	2	2
$f^{///}(x)$	0	0

$$f(x) = f(a) + \frac{(x-a)}{1!}f^{/}(a) + \frac{(x-a)^2}{2!}f^{//}(a) + \frac{(x-a)^3}{3!}f^{///}(a) + \frac{(x-a)^4}{4!}f^{/v}(a) + \cdots$$

$$x^2 - 3x + 2 = 0 + \frac{(x-1)}{1}(-1) + \frac{(x-1)^2}{2}(2) + \frac{(x-1)^3}{6}(0)$$

$$x^2 - 3x + 2 = -(x - 1) + (x - 1)^2$$

****

Example 7.33 Evaluate: $\lim_{x\to 1}\left(\frac{x^2-3x+2}{x^2-4x+3}\right)$.

Solution: $\lim_{x\to 1}\left(\frac{x^2-3x+2}{x^2-4x+3}\right)$

$$= \frac{(1)^2-3(1)+2}{(1)^2-4(1)+3}$$

$$= \frac{1-3+2}{1-4+3} = \frac{3-3}{4-4} = \frac{0}{0} \text{ Indeterminate form.}$$

Applying l'hopital's Rule,

$$\lim_{x\to 1}\left(\frac{x^2-3x+2}{x^2-4x+3}\right) = \lim_{x\to 1}\left(\frac{2x-3}{2x-4}\right)$$

$$= \frac{2(1)-3}{2(1)-4} = \frac{2-3}{2-4} = \frac{-1}{-2} = \frac{1}{2}$$

$$\therefore \lim_{x\to 1}\left(\frac{x^2-3x+2}{x^2-4x+3}\right) = \frac{1}{2}$$

****

Example 7.34

Compute the limit: $\lim_{x\to a}\left(\frac{x^n-a^n}{x-a}\right)$.

Solution: $\lim_{x\to a}\left(\frac{x^n-a^n}{x-a}\right)$

$$= \frac{a^n-a^n}{a-a} = \frac{0}{0} \text{ Indeterminate form.}$$

Applying l'hopital's Rule,

$$\lim_{x\to a}\left(\frac{x^n-a^n}{x-a}\right) = \lim_{x\to a}\left(\frac{nx^{n-1}-0}{1-0}\right)$$

$$= \frac{na^{n-1}}{1}$$

$$\therefore \lim_{x\to a}\left(\frac{x^n-a^n}{x-a}\right) = na^{n-1}$$

****

Example 7.35

Evaluate the limit: $\lim_{x\to 0}\left(\frac{\sin mx}{x}\right)$.

Solution: $\lim_{x\to 0}\left(\frac{\sin mx}{x}\right)$

$$= \frac{\sin m(0)}{(0)}$$

$$= \frac{\sin 0}{(0)} = \frac{0}{0} \text{ Indeterminate form.}$$

Applying l'hopital's Rule,

$$\lim_{x\to 0}\left(\frac{\sin mx}{x}\right) = \lim_{x\to 0}\left(\frac{m\cos mx}{1}\right)$$

$$= \frac{m\cos m(0)}{1}$$

$$= \frac{m\cos 0}{1} = m$$

$$\therefore \lim_{x\to 0}\left(\frac{\sin mx}{x}\right) = m$$

****

Example 7.36 Evaluate the limit: $\lim_{x\to 0}\left(\frac{\sin x}{x^2}\right)$.

Solution: $\lim_{x\to 0}\left(\frac{\sin x}{x^2}\right)$

$$= \frac{\sin(0)}{(0)^2}$$

$$= \frac{\sin 0}{(0)} = \frac{0}{0} \text{ Indeterminate form.}$$

Applying l'hopital's Rule,

$$\lim_{x\to 0}\left(\frac{\sin x}{x^2}\right) = \lim_{x\to 0^+}\left(\frac{\cos x}{2x}\right)$$

$$= \frac{\cos(0)}{2(0)} = \frac{1}{0} = \infty \text{ and also,}$$

$$\lim_{x\to 0}\left(\frac{\sin x}{x^2}\right) = \lim_{x\to 0^-}\left(\frac{\cos x}{2x}\right)$$

$$= \frac{\cos(-0)}{2(-0)} = \frac{1}{-0} = -\infty$$

As the left limit and the right limit are not the same we conclude that the limit does not exist.

Example 7.37 If $\lim_{\theta\to0}\left(\frac{1-\cos m\theta}{1-\cos n\theta}\right)=1,$

then prove that $m=\pm n$

Solution: $\lim_{\theta\to0}\left(\frac{1-\cos m\theta}{1-\cos n\theta}\right)$

$$=\frac{1-\cos m(0)}{1-\cos n(0)}$$

$$=\frac{1-\cos 0}{1-\cos 0}$$

$$=\frac{1-1}{1-1}=\frac{0}{0}\text{ Indeterminate form.}$$

Applying l'hopital's Rule,

$$\lim_{\theta\to0}\left(\frac{1-\cos m\theta}{1-\cos n\theta}\right)$$

$$=\lim_{\theta\to0}\left[\frac{0-(-m\sin m\theta)}{0-(-n\sin n\theta)}\right]$$

$$=\lim_{\theta\to0}\left[\frac{m\sin m\theta}{n\sin n\theta}\right]$$

$$=\lim_{\theta\to0}\frac{m}{n}\left(\frac{\sin m\theta}{\sin n\theta}\right)$$

$$=\frac{m}{n}\left(\frac{\sin m(0)}{\sin n(0)}\right)$$

$$=\frac{m}{n}\left(\frac{\sin 0}{\sin 0}\right)=\frac{0}{0}\text{ Indeterminate form.}$$

Applying l'hopital's Rule,

$$\lim_{\theta\to0}\frac{m}{n}\left(\frac{\sin m\theta}{\sin n\theta}\right)=\lim_{\theta\to0}\frac{m}{n}\left(\frac{m\cos m\theta}{n\cos n\theta}\right)$$

$$=\lim_{\theta\to0}\frac{m^2}{n^2}\left(\frac{\cos m\theta}{\cos n\theta}\right)$$

$$=\frac{m^2}{n^2}\left(\frac{\cos m(0)}{\cos n(0)}\right)$$

$$=\frac{m^2}{n^2}\left(\frac{\cos 0}{\cos 0}\right)$$

$$=\frac{m^2}{n^2}\left(\frac{1}{1}\right)$$

$$=\frac{m^2}{n^2}$$

$$\therefore\lim_{\theta\to0}\left(\frac{1-\cos m\theta}{1-\cos n\theta}\right)=\frac{m^2}{n^2}$$

Given $\lim_{\theta\to0}\left(\frac{1-\cos m\theta}{1-\cos n\theta}\right)=1$

Hence, $\frac{m^2}{n^2}=1$, gives

$$m^2=n^2$$

So, $m=\pm n$ Hence proved.

****

Example 7.38 Evaluate: $\lim_{x\to1^-}\left[\frac{\log(1-x)}{\cot(\pi x)}\right]$.

Solution: $\lim_{x\to1^-}\left[\frac{\log(1-x)}{\cot(\pi x)}\right]$

$$=\left[\frac{\log(1-1)}{\cot(\pi)}\right]$$

$$=\left[\frac{\log(0)}{\cot(\pi)}\right]$$

$$=\frac{-(-\infty)}{\infty}=\frac{\infty}{\infty}\text{ Indeterminate form}$$

Applying l'hopital's Rule,

$$\lim_{x\to1^-}\left[\frac{\log(1-x)}{\cot(\pi x)}\right]=\lim_{x\to1^-}\left[\frac{\frac{1}{(1-x)}(-1)}{-\pi\,cosec^2(\pi x)}\right]$$

$$=\lim_{x\to1^-}\left[\frac{\frac{1}{(1-x)}}{\pi\,cosec^2(\pi x)}\right]$$

$$=\frac{\frac{1}{(1-1)}}{\pi\,cosec^2(\pi)}$$

$$=\frac{\frac{1}{0}}{\pi\left(\frac{1}{0}\right)}$$

$$=\frac{\infty}{\infty}\text{ Indeterminate form}$$

Applying l'hopital's Rule,

$$\lim_{x\to1^-}\left[\frac{\frac{1}{(1-x)}}{\pi\,cosec^2(\pi x)}\right]$$

$$=\lim_{x\to1^-}\left[\frac{\frac{1}{(1-x)}}{\pi\frac{1}{\sin^2(\pi x)}}\right]$$

$$=\lim_{x\to1^-}\left[\frac{1}{(1-x)}\times\frac{\sin^2(\pi x)}{\pi}\right]$$

$$=\lim_{x\to1^-}\left[\frac{\sin^2(\pi x)}{\pi-\pi x}\right]$$

$$=\frac{\sin^2(\pi)}{\pi-\pi}=\frac{0}{0}\text{ Indeterminate form.}$$

Applying l'hopital's Rule,

$$\lim_{x\to1^-}\left[\frac{\sin^2(\pi x)}{\pi-\pi x}\right]=\lim_{x\to1^-}\left[\frac{2\sin(\pi x)\cos(\pi x)\pi}{-\pi}\right]$$

$$=\lim_{x\to1^-}[-2\sin(\pi x)\cos(\pi x)]$$

$$=-2\sin(-\pi)\cos(-\pi)$$

$$=2\sin(\pi)\cos(\pi)$$

$$=2(0)(-1)$$

$$=0$$

$$\therefore\lim_{x\to1^-}\left[\frac{\log(1-x)}{\cot(\pi x)}\right]=0$$

****

Example 7.39 Evaluate: $\lim_{x\to 0^+}\left(\frac{1}{x}-\frac{1}{e^x-1}\right)$

Solution: $\lim_{x\to 0^+}\left(\frac{1}{x}-\frac{1}{e^x-1}\right)$

$=\left(\frac{1}{0}-\frac{1}{e^0-1}\right)$

$=\left(\frac{1}{0}-\frac{1}{1-1}\right)$

$=\left(\frac{1}{0}-\frac{1}{0}\right)$

$=(\infty-\infty)$ Indeterminate form

Applying l'hopital's Rule,

$\lim_{x\to 0^+}\left(\frac{1}{x}-\frac{1}{e^x-1}\right)=\lim_{x\to 0^+}\left[\frac{e^x-1-x}{x(e^x-1)}\right]$

$=\left[\frac{e^0-1-0}{0(e^0-1)}\right]$

$=\frac{1-1}{0(1-1)}$

$=\frac{0}{0}$ Indeterminate form

Applying l'hopital's Rule,

$\lim_{x\to 0^+}\left[\frac{e^x-1-x}{x(e^x-1)}\right]=\lim_{x\to 0^+}\left[\frac{e^x-1}{x(e^x)+(e^x-1)}\right]$

$=\left[\frac{e^0-1}{0(e^0)+(e^0-1)}\right]$

$=\left[\frac{1-1}{0(1)+(1-1)}\right]$

$=\frac{0}{0}$ Indeterminate form

Applying l'hopital's Rule, $\lim_{x\to 0^+}\left[\frac{e^x-1}{x(e^x)+(e^x-1)}\right]=$

$\lim_{x\to 0^+}\left[\frac{e^x}{x(e^x)+e^x+(e^x)}\right]$

$=\left[\frac{e^0}{0(e^0)+e^0+(e^0)}\right]$

$=\frac{1}{0(1)+1+1}$

$=\frac{1}{2}$

$\therefore \lim_{x\to 0^+}\left(\frac{1}{x}-\frac{1}{e^x-1}\right)=\frac{1}{2}$

****

Example 7.40 Evaluate: $\lim_{x\to 0^+} x\log x$

Solution: $\lim_{x\to 0^+} x\log x$

$=0(-\infty)$ Indeterminate form.

Applying l'hopital's Rule,

$\lim_{x\to 0^+} x\log x=\lim_{x\to 0^+}\frac{\log x}{\frac{1}{x}}$

$=\left[\frac{\log 0}{\frac{1}{0}}\right]$

$=\frac{\infty}{\infty}$ Indeterminate form

Applying l'hopital's Rule,

$\lim_{x\to 0^+}\frac{\log x}{\frac{1}{x}}=\lim_{x\to 0^+}\frac{\frac{1}{x}}{-\left(\frac{1}{x^2}\right)}$

$=\lim_{x\to 0^+}\left(\frac{1}{x}\right)\times\left(\frac{x^2}{-1}\right)$

$=\lim_{x\to 0^+}\left(\frac{x}{-1}\right)$

$=\left(\frac{0}{-1}\right)$

$=0$

$\therefore \lim_{x\to 0^+} x\log x=0$

****

Example 7.41Evaluate: $\lim_{x\to\infty}\left(\frac{x^2+17x+29}{x^4}\right)$

Solution: $\lim_{x\to\infty}\left(\frac{x^2+17x+29}{x^4}\right)$

$=\frac{\infty}{\infty}$ Indeterminate form

Applying l'hopital's Rule,

$\lim_{x\to\infty}\left(\frac{x^2+17x+29}{x^4}\right)=\lim_{x\to\infty}\left(\frac{2x+17}{4x^3}\right)$

$=\frac{\infty}{\infty}$ Indeterminate form

Applying l'hopital's Rule,

$\lim_{x\to\infty}\left(\frac{2x+17}{4x^3}\right)=\lim_{x\to\infty}\left(\frac{2}{12x^2}\right)$

$=\frac{2}{\infty}$

$=0$

$\therefore \lim_{x\to\infty}\left(\frac{x^2+17x+29}{x^4}\right)=0$

****

Example 7.42Evaluate: $\lim_{x\to\infty}\left(\frac{e^x}{x^m}\right), m\in N$

Solution: $\lim_{x\to\infty}\left(\frac{e^x}{x^m}\right)$

$=\frac{e^\infty}{\infty^m}$

$=\frac{\infty}{\infty}$ Indeterminate form

Applying l'hopital's Rule,

$\lim_{x\to\infty}\left(\frac{e^x}{x^m}\right)=\lim_{x\to\infty}\left(\frac{e^x}{m!}\right)$

$$= \frac{e^{\infty}}{m!}$$

$$= \infty$$

$$\therefore \lim_{x\to\infty} \left(\frac{e^x}{x^m}\right) = \infty$$

****

Example 7.43 Using the l' hospital Rule prove that: $\lim_{x\to0^+} (1+x)^{\frac{1}{x}} = e$

Solution: $\lim_{x\to0^+} (1+x)^{\frac{1}{x}}$

$$= (1+0)^{\frac{1}{0}}$$

$$= (1)^{\infty} \text{ Indeterminate form}$$

Let $A = (1+x)^{\frac{1}{x}}$

Taking log on either side,

$$\log A = log(1+x)^{\frac{1}{x}}$$

$$\lim_{x\to0^+} \log A = \lim_{x\to0^+} \frac{1}{x} log(1+x)$$

$$= \lim_{x\to0^+} \frac{\log(1+x)}{x}$$

$$= \frac{\log(1+0)}{0}$$

$$= \frac{\log(1)}{0}$$

$$= \frac{0}{0} \text{ Indeterminate form}$$

Applying l'hopital's Rule,

$$\lim_{x\to0^+} \log A = \lim_{x\to0^+} \frac{\log(1+x)}{x}$$

$$= \lim_{x\to0^+} \frac{\frac{1}{(1+x)}}{1}$$

$$= \lim_{x\to0^+} \frac{1}{(1+x)}$$

$$= \frac{1}{(1+0)}$$

$$= \frac{1}{1}$$

$$\lim_{x\to0^+} \log A = 1$$

By Composite function theorem,

$$\log \lim_{x\to0^+} A = 1$$

Taking exponential on both side,

$$\lim_{x\to0^+} A = e$$

$$\lim_{x\to0^+} (1+x)^{\frac{1}{x}} = e$$

****

Example 7.44Evaluate: $\lim_{x\to\infty} (1+2x)^{\frac{1}{2\log x}}$

Solution: $\lim_{x\to\infty} (1+2x)^{\frac{1}{2\log x}}$

$$= (1+\infty)^{\frac{1}{2\log\infty}}$$

$$= (\infty)^{\frac{1}{\infty}}$$

$$= (\infty)^0 \text{ Indeterminate form}$$

Let $A = (1+2x)^{\frac{1}{2\log x}}$

Taking log on either side,

$$\log A = \log (1+2x)^{\frac{1}{2\log x}}$$

$$\lim_{x\to\infty} \log A = \lim_{x\to\infty} \frac{1}{2\log x} log(1+2x)$$

$$= \lim_{x\to\infty} \frac{\log(1+2x)}{2\log x}$$

$$= \frac{\log(1+\infty)}{2\log\infty}$$

$$= \frac{\log(\infty)}{\infty}$$

$$= \frac{\infty}{\infty} \text{ Indeterminate form}$$

Applying l'hopital's Rule,

$$\lim_{x\to\infty} \log A = \lim_{x\to\infty} \frac{\log(1+2x)}{2\log x}$$

$$= \lim_{x\to\infty} \frac{\frac{1}{(1+2x)}\times 2}{\frac{2}{x}}$$

$$= \lim_{x\to\infty} \frac{2}{(1+2x)} \times \frac{x}{2}$$

$$= \lim_{x\to\infty} \frac{x}{(1+2x)}$$

$$= \frac{\infty}{\infty} \text{ Indeterminate form}$$

Applying l'hopital's Rule

$$\lim_{x\to\infty} \log A = \lim_{x\to\infty} \frac{x}{(1+2x)}$$

$$= \lim_{x\to\infty} \frac{1}{(2)}$$

$$= \frac{1}{2}$$

By Composite function theorem,

$$\log \lim_{x\to\infty} A = \frac{1}{2}$$

Taking exponential on both side,

$\lim_{x\to\infty} A = e^{\frac{1}{2}}$

$\lim_{x\to\infty}(1+2x)^{\frac{1}{2\log x}} = e^{\frac{1}{2}} = \sqrt{e}$

****

Example 7.45Evaluate: $\lim_{x\to 1} x^{\frac{1}{1-x}}$

<u>Solution</u>: $\lim_{x\to 1} x^{\frac{1}{1-x}}$

$= x^{\frac{1}{1-x}}$

$= 1^{\frac{1}{1-1}}$

$= 1^{\frac{1}{0}}$

$= 1^{\infty}$ Indeterminate form

Let $A = x^{\frac{1}{1-x}}$

Taking log on either side,

$\log A = \log x^{\frac{1}{1-x}}$

$\lim_{x\to 1}\log A = \lim_{x\to 1}\frac{1}{1-x}\log(x)$

$= \lim_{x\to 1}\frac{\log x}{1-x}$

$= \frac{\log(1)}{1-1}$

$= \frac{\log(1)}{0}$

$= \frac{0}{0}$ Indeterminate form

Applying l'hopital's Rule,

$\lim_{x\to 1}\log A = \lim_{x\to 1}\frac{\log x}{1-x}$

$= \lim_{x\to 1}\frac{\frac{1}{x}}{-1}$

$= \lim_{x\to 1}\frac{-1}{x}$

$= \frac{-1}{1}$

$= -1$

By Composite function theorem,

$\log\lim_{x\to 1} A = -1$

Taking exponential on both side,

$\lim_{x\to 1} A = e^{-1} = \frac{1}{e}$

$\lim_{x\to 1} x^{\frac{1}{1-x}} = \frac{1}{e}$

****

EXERCISE 7.5

Evaluate the following limits, if necessary use l'Hôpital Rule :

1. $\lim_{x\to 0}\left(\frac{1-\cos x}{x^2}\right)$

<u>Solution</u>: $\lim_{x\to 0}\left(\frac{1-\cos x}{x^2}\right)$

$= \frac{1-\cos(0)}{(0)^2}$

$= \frac{1-1}{0} = \frac{0}{0}$ Indeterminate form

Applying l'hopital's Rule,

$\lim_{x\to 0}\left(\frac{1-\cos x}{x^2}\right) = \lim_{x\to 0}\left(\frac{0-(-\sin x)}{2x}\right)$

$= \lim_{x\to 0}\left(\frac{0+\sin x}{2x}\right)$

$= \frac{0+\sin 0}{0}$

$= \frac{0}{0}$ Indeterminate form

Applying l'hopital's Rule,

$\lim_{x\to 0}\left(\frac{\sin x}{2x}\right) = \lim_{x\to 0}\left(\frac{\cos x}{2}\right)$

$= \frac{\cos(0)}{2}$

$= \frac{1}{2}$

$\therefore \lim_{x\to 0}\left(\frac{1-\cos x}{x^2}\right) = \frac{1}{2}$

****

2. $\lim_{x\to\infty}\frac{2x^2-3}{x^2-5x+3}$

<u>Solution</u>: $\lim_{x\to\infty}\frac{2x^2-3}{x^2-5x+3}$

$= \frac{\infty}{\infty}$ Indeterminate form

Applying l'hopital's Rule

$\lim_{x\to\infty}\frac{2x^2-3}{x^2-5x+3} = \lim_{x\to\infty}\frac{4x}{2x-5}$

$= \frac{\infty}{\infty}$ Indeterminate form

Applying l'hopital's Rule

$\lim_{x\to\infty}\frac{4x}{2x-5} = \lim_{x\to\infty}\frac{4}{2}$

$= \frac{4}{2} = 2$

$\therefore \lim_{x\to\infty}\frac{2x^2-3}{x^2-5x+3} = 2$

****

3. $\lim_{x\to\infty} \frac{x}{\log x}$

Solution: $\lim_{x\to\infty} \frac{x}{\log x}$

$= \frac{\infty}{\log \infty}$

$= \frac{\infty}{\infty}$ Indeterminate form

Applying l'hopital's Rule

$\lim_{x\to\infty} \frac{x}{\log x} = \lim_{x\to\infty} \frac{1}{\frac{1}{x}}$

$= \lim_{x\to\infty} x$

$= \infty$

$\therefore \lim_{x\to\infty} \frac{x}{\log x} = \infty$

4. $\lim_{x\to\frac{\pi}{2}} \frac{\sec x}{\tan x}$

Solution: $\lim_{x\to\frac{\pi}{2}} \frac{\sec x}{\tan x}$

$= \frac{\sec\frac{\pi}{2}}{\tan\frac{\pi}{2}}$

$= \frac{\infty}{\infty}$ Indeterminate form

Applying l'hopital's Rule

$\lim_{x\to\frac{\pi}{2}} \frac{\sec x}{\tan x} = \lim_{x\to\frac{\pi}{2}} \frac{1/\cos x}{\sin x/\cos x}$

$= \lim_{x\to\frac{\pi}{2}} \left(\frac{1}{\cos x} \times \frac{\cos x}{\sin x}\right)$

$= \lim_{x\to\frac{\pi}{2}} \frac{1}{\sin x}$

$= \frac{1}{\sin\frac{\pi}{2}}$

$= \frac{1}{1} = 1$

$\therefore \lim_{x\to\frac{\pi}{2}} \frac{\sec x}{\tan x} = 1$

****

5. $\lim_{x\to\infty} e^{-x}\sqrt{x}$

Solution:

$\lim_{x\to\infty} e^{-x}\sqrt{x} = \lim_{x\to\infty} \frac{\sqrt{x}}{e^x}$

$= \frac{\sqrt{\infty}}{e^\infty}$

$= \frac{\infty}{\infty}$ Indeterminate form

Applying l'hopital's Rule,

$\lim_{x\to\infty} \frac{\sqrt{x}}{e^x} = \lim_{x\to\infty} \frac{\frac{1}{2\sqrt{x}}}{e^x}$

$= \lim_{x\to\infty} \frac{1}{2\sqrt{x}e^x}$

$= \frac{1}{2\sqrt{\infty}e^\infty}$

$= \frac{1}{\infty}$

$= 0$

$\therefore \lim_{x\to\infty} e^{-x}\sqrt{x} = 0$

****

6. $\lim_{x\to0} \left(\frac{1}{\sin x} - \frac{1}{x}\right)$

Solution: $\lim_{x\to0} \left(\frac{1}{\sin x} - \frac{1}{x}\right)$

$= \frac{1}{\sin 0} - \frac{1}{0}$

$= \frac{1}{0} - \frac{1}{0}$

$= \infty - \infty$ Indeterminate form.

Applying l'hopital's Rule,

$\lim_{x\to0} \left(\frac{1}{\sin x} - \frac{1}{x}\right) = \lim_{x\to0} \left[\frac{x-\sin x}{(\sin x)x}\right]$

$= \left[\frac{0-\sin 0}{(\sin 0)0}\right]$

$= \left[\frac{0-0}{(0)0}\right]$

$= \frac{0}{0}$ Indeterminate form.

Applying l'hopital's Rule,

$\lim_{x\to0} \left[\frac{x-\sin x}{(\sin x)x}\right] = \lim_{x\to0} \left[\frac{1-\cos x}{(\cos x)x+(\sin x)1}\right]$

$= \left[\frac{1-\cos 0}{(\cos 0)0+(\sin 0)1}\right]$

$= \left[\frac{1-1}{(1)0+(0)1}\right]$

$= \frac{0}{0}$ Indeterminate form.

Applying l'hopital's Rule, $\lim_{x\to0} \left[\frac{1-\cos x}{(\cos x)x+(\sin x)1}\right] = \lim_{x\to0} \left[\frac{-(-\sin x)}{(-\sin x)x+(\cos x)+\cos x}\right]$

$= \lim_{x\to0} \left[\frac{\sin x}{(-\sin x)x+2\cos x}\right]$

$= \left[\frac{\sin 0}{(-\sin 0)0+2\cos 0}\right]$

$= \left[\frac{0}{(0)0+2(1)}\right]$

$= \frac{0}{2}$

$= 0$

$\therefore \lim_{x\to 0}\left(\frac{1}{\sin x} - \frac{1}{x}\right) = 0$

****

7. $\lim_{x\to 1^+}\left(\frac{2}{x^2-1} - \frac{x}{x-1}\right)$

Solution: $\lim_{x\to 1^+}\left(\frac{2}{x^2-1} - \frac{x}{x-1}\right)$

$= \frac{2}{1^2-1} - \frac{1}{1-1}$

$= \frac{2}{0} - \frac{1}{0}$

$= \infty - \infty$ Indeterminate form.

Applying l'hopital's Rule,

$\lim_{x\to 1^+}\left(\frac{2}{x^2-1} - \frac{x}{x-1}\right) = \lim_{x\to 1^+}\left(\frac{2}{x^2-1} - \frac{x}{x-1}\right)$

$= \lim_{x\to 1^+}\left(\frac{2}{(x-1)(x+1)} - \frac{x}{x-1}\right)$

$= \lim_{x\to 1^+}\left(\frac{2-x(x+1)}{x^2-1}\right)$

$= \lim_{x\to 1^+}\left(\frac{2-x^2-x}{x^2-1}\right)$

$= \frac{2-1^2-1}{1^2-1}$

$= \frac{2-2}{1-1}$

$= \frac{0}{0}$ Indeterminate form.

Applying l'hopital's Rule,

$\lim_{x\to 1^+}\left(\frac{2-x^2-x}{x^2-1}\right) = \lim_{x\to 1^+}\left(\frac{-2x-1}{2x}\right)$

$= \frac{-2(1)-1}{2(1)}$

$= \frac{-3}{2}$

$\lim_{x\to 1^+}\left(\frac{2}{x^2-1} - \frac{x}{x-1}\right) = \frac{-3}{2}$

****

8. $\lim_{x\to 0^+} x^x$

Solution: $\lim_{x\to 0^+} x^x$

$= 0^0$ Indeterminate form.

Let $A = x^x$

Taking log on either side,

$\log A = \log x^x$

$\lim_{x\to 0^+} \log A = \lim_{x\to 0^+} x\log(x)$

$= \lim_{x\to 0^+} \frac{\log x}{\frac{1}{x}}$

$= \frac{\log(0)}{\frac{1}{0}}$

$= \frac{-\infty}{\infty}$ Indeterminate form

Applying l'hopital's Rule,

$\lim_{x\to 0^+} \log A = \lim_{x\to 0^+} \frac{\log x}{\frac{1}{x}}$

$= \lim_{x\to 0^+} \frac{\frac{1}{x}}{\left(-\frac{1}{x^2}\right)}$

$= \lim_{x\to 0^+} \frac{1}{x} \times \frac{x^2}{-1}$

$= \lim_{x\to 0^+} \frac{x}{-1}$

$= \frac{0}{-1}$

$= 0$

By Composite function theorem,

$\log \lim_{x\to 0^+} A = 0$

Taking exponential on both side,

$\lim_{x\to 0^+} A = e^0 = 1$

$\therefore \lim_{x\to 0^+} x^x = 1$

****

9. $\lim_{x\to\infty}\left(1+\frac{1}{x}\right)^x$

Solution: $\lim_{x\to\infty}\left(1+\frac{1}{x}\right)^x$

$= \left(1+\frac{1}{\infty}\right)^\infty$

$= (1+0)^\infty$

$= 1^\infty$ Indeterminate form.

Let $A = \left(1+\frac{1}{x}\right)^x$

Taking log on either side,

$\log A = \log\left(1+\frac{1}{x}\right)^x$

$\lim_{x\to\infty} \log A = \lim_{x\to\infty} x\log\left(1+\frac{1}{x}\right)$

$= \lim_{x\to\infty} \frac{\log\left(1+\frac{1}{x}\right)}{\frac{1}{x}}$

$= \frac{\log\left(1+\frac{1}{\infty}\right)}{\frac{1}{\infty}}$

$$= \frac{\log(1+0)}{0}$$

$$= \frac{\log(1)}{0}$$

$$= \frac{0}{0} \text{ Indeterminate form}$$

Applying l'hopital's Rule,

$$\lim_{x\to\infty} \log A = \lim_{x\to\infty} \frac{\log\left(1+\frac{1}{x}\right)}{\frac{1}{x}}$$

$$= \lim_{x\to\infty} \frac{\left(1+\frac{1}{x}\right)\left(\frac{-1}{x^2}\right)}{\left(-\frac{1}{x^2}\right)}$$

$$= \lim_{x\to\infty} \left(1+\frac{1}{x}\right)$$

$$= \left(1+\frac{1}{\infty}\right)$$

$$= (1+0)$$

$$= 1$$

By Composite function theorem,

$$\log \lim_{x\to\infty} A = 1$$

Taking exponential on both side,

$$\lim_{x\to\infty} A = e^1 = e$$

$$\therefore \lim_{x\to\infty} \left(1+\frac{1}{x}\right)^x = e$$

****

10. $\lim_{x\to\frac{\pi}{2}} (\sin x)^{\tan x}$

Solution: $\lim_{x\to\frac{\pi}{2}} (\sin x)^{\tan x}$

$$= \left(\sin\frac{\pi}{2}\right)^{\tan\frac{\pi}{2}}$$

$$= (1)^{\infty}$$

$$= 1^{\infty} \text{ Indeterminate form.}$$

Let $A = (\sin x)^{\tan x}$

Taking log on either side,

$$\log A = \log (\sin x)^{\tan x}$$

$$= \tan x \log (\sin x)$$

$$= \frac{\log(\sin x)}{\frac{1}{\tan x}}$$

$$= \frac{\log(\sin x)}{\cot x}$$

$$\lim_{x\to\frac{\pi}{2}} \log A = \lim_{x\to\frac{\pi}{2}} \frac{\log(\sin x)}{\cot x}$$

$$= \frac{\log\left(\sin\frac{\pi}{2}\right)}{\cot\frac{\pi}{2}}$$

$$= \frac{\log(1)}{0}$$

$$= \frac{0}{0} \text{ Indeterminate form}$$

Applying l'hopital's Rule,

$$\lim_{x\to\frac{\pi}{2}} \log A = \lim_{x\to\frac{\pi}{2}} \frac{\frac{1}{\sin x}(\cos x)}{-\operatorname{cosec}^2 x}$$

$$= \lim_{x\to\frac{\pi}{2}} \frac{\cos x}{\sin x} \times \frac{\sin^2 x}{-1}$$

$$= \lim_{x\to\frac{\pi}{2}} \frac{\cos x \sin x}{-1}$$

$$= \frac{\cos\left(\frac{\pi}{2}\right)\sin\left(\frac{\pi}{2}\right)}{-1}$$

$$= \frac{(0)(1)}{-1}$$

$$= 0$$

Taking exponential on both side,

$$\lim_{x\to\frac{\pi}{2}} A = e^0 = 1$$

$$\therefore \lim_{x\to\frac{\pi}{2}} (\sin x)^{\tan x} = 1$$

****

11. $\lim_{x\to 0^+} (\cos x)^{\frac{1}{x^2}}$

Solution: $\lim_{x\to 0^+} (\cos x)^{\frac{1}{x^2}}$

$$= (\cos 0)^{\frac{1}{0}}$$

$$= (1)^{\infty}$$

$$= 1^{\infty} \text{ Indeterminate form.}$$

Let $A = (\cos x)^{\frac{1}{x^2}}$

Taking log on either side,

$$\log A = \log (\cos x)^{\frac{1}{x^2}}$$

$$= \frac{1}{x^2} \log (\cos x)$$

$$= \frac{\log (\cos x)}{x^2}$$

$$\lim_{x\to 0^+} \log A = \lim_{x\to 0^+} \frac{\log (\cos x)}{x^2}$$

$$= \frac{\log(\cos 0)}{0}$$

$$= \frac{\log(1)}{0}$$

$$= \frac{0}{0} \text{ Indeterminate form}$$

Applying l'hopital's Rule,

$$\lim_{x\to 0^+} logA = \lim_{x\to 0^+} \frac{\frac{1}{\cos x}(-\sin x)}{2x}$$

$$= \lim_{x\to 0^+} \frac{-\sin x}{(\cos x)2x}$$

$$= \frac{-\sin 0}{(\cos 0)(0)}$$

$$= \frac{0}{0} \text{ Indeterminate form}$$

Applying l'hopital's Rule,

$$\lim_{x\to 0^+} \frac{-\sin x}{(\cos x)2x} = \lim_{x\to 0^+} \frac{-(\cos x)}{(\cos x)(2)+2x(-\sin x)}$$

$$= \frac{-(\cos 0)}{(\cos 0)(2)+0(-\sin 0)}$$

$$= \frac{-(1)}{(1)(2)+0(0)}$$

$$= \frac{-1}{2}$$

By Composite function theorem,

$$\log \lim_{x\to 0^+} A = -\frac{1}{2}$$

Taking exponential on both side,

$$\lim_{x\to 0^+} A = e^{-\frac{1}{2}} = \frac{1}{\sqrt{e}}$$

****

12. If an initial amount A_0 of money is invested at an interest rate r compounded n times a year, the value of the investment after t years is $A = A_0\left(1+\frac{r}{n}\right)^{nt}$. If the interest is compounded continuously, (that is as $n \to \infty$), show that the amount after t years is $A = A_0e^{rt}$

Solution: $\log_{n\to\infty} A_0\left(1+\frac{r}{n}\right)^{nt}$

$$= A_0\left(1+\frac{r}{\infty}\right)^{\infty t}$$

$$= A_0(1+0)^{\infty}$$

$$= A_0(1)^{\infty} \text{Indeterminate form}$$

$$\text{Let } Y = \left(1+\frac{r}{n}\right)^{nt}$$

Taking log on either side,

$$\log Y = log\left(1+\frac{r}{n}\right)^{nt}$$

$$= (nt)log\left(1+\frac{r}{n}\right)$$

$$= (t)\frac{log\left(1+\frac{r}{n}\right)}{\frac{1}{n}}$$

$$\log_{n\to\infty} Y = \log_{n\to\infty}(t)\frac{log\left(1+\frac{r}{n}\right)}{\frac{1}{n}}$$

$$= (t)\log_{n\to\infty}\frac{log\left(1+\frac{r}{n}\right)}{\frac{1}{n}}$$

$$= (t)\log_{n\to\infty}\left[\frac{\frac{1}{\left(1+\frac{r}{n}\right)}\left(-\frac{r}{n^2}\right)}{\left(-\frac{1}{n^2}\right)}\right]$$

$$= (t)\log_{n\to\infty}\left[\frac{1}{\left(1+\frac{r}{n}\right)}\left(-\frac{r}{n^2}\right)\times\left(-\frac{n^2}{1}\right)\right]$$

$$= (t)\log_{n\to\infty}\frac{1}{\left(1+\frac{r}{n}\right)}(r)$$

$$= (rt)\log_{n\to\infty}\frac{1}{\left(1+\frac{r}{n}\right)}$$

$$= (rt)\frac{1}{\left(1+\frac{r}{\infty}\right)}$$

$$= (rt)\frac{1}{(1+0)}$$

$$= (rt)(1)$$

$$= (rt)$$

By Composite function theorem,

$$\log \lim_{x\to\infty} Y = rt$$

Taking exponential on both side,

$$\lim_{x\to\infty} Y = e^{rt}$$

Hence, $\log_{n\to\infty} A_0\left(1+\frac{r}{n}\right)^{nt} = A_0e^{rt}$

****

Example 7.46

Prove that the function $f(x) = x^2 + 2$ is strictly increasing in the interval $(2, 7)$ and strictly decreasing in the interval $(-2, 0)$.

Solution: Given $f(x) = x^2 + 2$

$$f'(x) = 2x$$

In the interval $(2, 7)$ let $x = 3$

$$f'(x) = 2(3) = 6 > 0$$

So, $f(x)$ is strictly increasing function.

In the interval $(-2, 0)$ let $x = -1$

$$f'(x) = 2(-1) = -2 < 0$$

So, $f(x)$ is strictly decreasing function.

****

Example 7.47

Prove that the function $f(x) = x^2 - 2x - 3$ is strictly increasing in $(2, \infty)$.

Solution: $f(x) = x^2 - 2x - 3$

$$f'(x) = 2x - 2$$

In the interval $(2, \infty)$ let $x = 3$

$$f'(x) = 2(3) - 2$$
$$= 6 - 2 = 4 > 0$$

So, $f(x)$ is strictly increasing function.

Example 7.48

Find the absolute maximum and absolute minimum values of the function $f(x) = 2x^3 + 3x^2 - 12x$ on $[-3, 2]$

Solution: $f(x) = 2x^3 + 3x^2 - 12x$

$$f'(x) = 6x^2 + 6x - 12$$

When, $f'(x) = 0$

$$6x^2 + 6x - 12 = 0$$

Dividing by 6,

$$x^2 + x - 2 = 0$$
$$(x + 2)(x - 1) = 0$$
$$x = -2 \text{ and } x = 1$$

$since\ x = -2,1 \in [-3, 2]$

The critical numbers are $x = -2,1$and

given the end points are $x = -3,2$

At $x = -2$

$$f(-2) = 2(-2)^3 + 3(-2)^2 - 12(-2)$$
$$= 2(-8) + 3(4) + 24$$
$$= -16 + 12 + 24$$
$$= -16 + 36$$
$$f(-2) = 20$$

At $x = 1$

$$f(1) = 2(1)^3 + 3(1)^2 - 12(1)$$
$$= 2 + 3 - 12$$
$$= 5 - 12$$
$$f(1) = -7$$

At $x = -3$

$$f(-3) = 2(-3)^3 + 3(-3)^2 - 12(-3)$$
$$= 2(-27) + 3(9) + 36$$
$$= -54 + 27 + 36$$
$$= -54 + 63$$
$$f(-3) = 9$$

At $x = 2$

$$f(2) = 2(2)^3 + 3(2)^2 - 12(2)$$
$$= 2(8) + 3(4) - 24$$
$$= 16 + 12 - 24$$
$$= 28 - 24$$
$$f(2) = 4$$

Hence, absolute maximum value of $f(x) = 20$

and absolute minimum value of $f(x) = -7$

Example 7.49

Find the absolute extrema of the function $f(x) = 3\cos x$ on the closed interval $[0, 2\pi]$.

Solution: $f(x) = 3\cos x$

$$f'(x) = 3(-\sin x)$$

When, $f'(x) = 0$

$$-3\sin x = 0$$
$$\sin x = 0$$

Since $\sin \pi = 0, x = \pi \in [0, 2\pi]$

The critical number is $x = \pi$ and

given the end points are $x = 0, 2\pi$

At $x = \pi$

$$f(x) = 3\cos(\pi)$$
$$= 3(-1)$$
$$= -3$$

At $x = 0$

$$f(x) = 3\cos(0)$$
$$= 3(1)$$
$$= 3$$

At $x = 2\pi$

$$f(x) = 3\cos(2\pi)$$
$$= 3(1)$$
$$= 3$$

Hence, absolute maximum value of $f(x) = 3$ and absolute minimum value of $f(x) = -3$

****

Example 7.50

Find the intervals of monotonicity and hence find the local extrema for the function $f(x) = x^2 - 4x + 4$.

Solution: $f(x) = x^2 - 4x + 4$

$f^{/}(x) = 2x - 4$

When, $f^{/}(x) = 0$

$2x - 4 = 0$

$2x = 4$

$x = 2$ is the critical number.

$\therefore$ The interval are $(-\infty, 2)$ and $(2, \infty)$

In the interval $(-\infty, 2)$ let $x = 0$

$f^{/}(x) = 2x - 4$

$= 2(0) - 4$

$= -4 < 0$

So, $f(x)$is strictly decreasing function.

In the interval $(2,\infty)$ let $x = 3$

$f^{/}(x) = 2x - 4$

$f^{/}(x) = 2(3) - 4 = 6 - 4 = 2 > 0$

So, $f(x)$is strictly increasing function.

Because $f^{/}(x)$ changes its sign from negative to positive when passing through $x = 2$ for the function $f(x)$, it has a local minimum at $x = 2$.

The local minimum value is

$f(2) = (2)^2 - 4(2) + 4$

$= 4 - 8 + 4$

$= 8 - 8$

$f(2) = 0$

****

Example 7.51Find the intervals of monotonicity and hence find the local extrema for the function $f(x) = x^{\frac{2}{3}}$

Solution: $f(x) = x^{\frac{2}{3}}$

$f^{/}(x) = \frac{2}{3}x^{\frac{2}{3}-1}$

$= \frac{2}{3}x^{\frac{2-3}{3}}$

$= \frac{2}{3}x^{\frac{-1}{3}}$

$= \frac{2}{3x^{\frac{1}{3}}}$

$f^{/}(x)$ can't be equal to zero for $x \in \mathbb{R}$ hence

$f^{/}(x)$ does not exists at $x = 0$.

$\therefore$ The interval are $(-\infty, 0)$ and $(0, \infty)$

In the interval $(-\infty, 0)$ $f^{/}(x) < 0$

So, $f(x)$is strictly decreasing function.

In the interval $(0,\infty)$ $f^{/}(x) > 0$

So, $f(x)$is strictly increasing function.

Because $f^{/}(x)$ changes its sign from negative to positive when passing through $x = 0$ for the function $f(x)$, it has a local minimum at $x = 0$

The local minimum value is $f(0) = 0$. Local minimum occurs at $x = 0$ which is not a stationary point.

****

Example 7.52

Prove that the function $f(x) = x - \sin x$ is increasing on the real line. Also discuss for the existence of local extrema.

Solution: $f(x) = x - \sin x$

$f^{/}(x) = 1 - \cos x$

When $f^{/}(x) = 0$, we get

$1 - \cos x = 0$

$\cos x = 1$

So, $x = 2n\pi, n \in \mathbb{Z}$

$\therefore$ $f(x)$ is increasing on the real line. Since there is no change $f^{/}(x)$ passing through $x = 2n\pi$, by the first derivative test there is no local extrema.

****

Example 7.53

Discuss the monotonicity and local extrema of the function $f(x) = log(1 + x) - \frac{x}{1+x}, x > -1$ and hence find the domain where, $log(1 + x) > \frac{x}{1+x}$.

Solution: $f(x) = log(1 + x) - \frac{x}{1+x}$

$f^{/}(x) = \frac{1}{(1+x)} - \left[\frac{(1+x)(1)-x(1)}{(1+x)^2}\right]$

$$= \frac{1}{(1+x)} - \left[\frac{1+x-x}{(1+x)^2}\right]$$

$$= \frac{1}{(1+x)} - \left[\frac{1}{(1+x)^2}\right]$$

$$= \frac{(1+x)-1}{(1+x)^2}$$

$$= \frac{1+x-1}{(1+x)^2}$$

$$f'(x) = \frac{x}{(1+x)^2}$$

When $f'(x) = 0$,

$\frac{x}{(1+x)^2} = 0$ gives

$$x = 0$$

∴ The interval are $(-\infty, 0)$ and $(0, \infty)$

In the interval $(-\infty, 0)$ $f'(x) < 0$

So, $f(x)$ is strictly decreasing function.

In the interval $(0,\infty)$ $f'(x) > 0$

So, $f(x)$ is strictly increasing function.

Because $f'(x)$ changes its sign from negative to positive when passing through $x = 0$ for the function $f(x)$, it has a local minimum at $x = 0$

The local minimum value is

$$f(0) = log(1+0) - \frac{0}{1+0}$$

$$= 0$$

Further when $x > 0$,

$$f(x) > f(0)$$

$$log(1+x) - \frac{x}{1+x} > 0$$

$log(1+x) > \frac{x}{1+x}$ Proved.

****

Example 7.54

Find the intervals of monotonicity and local extrema of the function $f(x) = x\log x + 3x$.

<u>Solution</u>: $f(x) = x\log x + 3x$

$$f'(x) = x\left(\frac{1}{x}\right) + \log x\,(1) + 3$$

$$= 1 + \log x + 3$$

$$= 4 + \log x$$

When $f'(x) = 0$,

$$4 + \log x = 0$$

$$\log x = -4$$

$$x = e^{-4}$$

∴ The interval are $(-\infty, e^{-4})$ and (e^{-4}, ∞)

In the interval $(-\infty, e^{-4})$ $f'(x) < 0$

So, $f(x)$ is strictly decreasing function.

In the interval (e^{-4},∞) $f'(x) > 0$

So, $f(x)$ is strictly increasing function.

Because $f'(x)$ changes its sign from negative to positive when passing through $x = e^{-4}$ for the function $f(x)$, it has a local minimum at $x = e^{-4}$, $f(x) = x\log x + 3x$

$$f(e^{-4}) = e^{-4}\log e^{-4} + 3e^{-4}$$

$$= e^{-4}(-4)\log e + 3e^{-4}$$

$$= e^{-4}(-4)(1) + 3e^{-4}$$

$$= -4\,e^{-4} + 3e^{-4}$$

$$= -e^{-4}$$

Local minimum is $-e^{-4}$

****

Example 7.55

Find the intervals of monotonicity and local extrema of the function $f(x) = \frac{1}{1+x^2}$

<u>Solution</u>: $f(x) = \frac{1}{1+x^2}$

$$f'(x) = \frac{-2x}{(1+x^2)^2}$$

When $f'(x) = 0$,

$\frac{-2x}{(1+x^2)^2} = 0$ gives

$$x = 0$$

∴ The interval are $(-\infty, 0)$ and $(0, \infty)$

In the interval $(-\infty, 0)$ $f'(x) > 0$

So, $f(x)$ is strictly increasing function.

In the interval $(0,\infty)$ $f'(x) < 0$

So, $f(x)$ is strictly decreasing function.

Because $f'(x)$ changes its sign from positive to negaitive when passing through $x = 0$ for the function $f(x)$, it has a local maximum at $x = 0$,

$f(0) = \frac{1}{1+0} = 1$

****

Example 7.56
Find the intervals of monotonicity and local extrema of the function $f(x) = \frac{x}{1+x^2}$

Solution: $f(x) = \frac{x}{1+x^2}$

$$f^{/}(x) = \frac{(1+x^2)(1) - x(2x)}{(1+x^2)^2}$$

$$= \frac{1+x^2-2x^2}{(1+x^2)^2}$$

$$= \frac{1-x^2}{(1+x^2)^2}$$

When $f^{/}(x) = 0$,

$\frac{1-x^2}{(1+x^2)^2} = 0$ gives

$$1 - x^2 = 0$$

$$x^2 = 1$$

$$x = \pm 1$$

Hence stationary points are $x = 1, -1$

$\therefore$ The interval are $(-\infty, -1)$, $(-1, 1)$ and $(1, \infty)$

In the interval $(-\infty, -1)$ $f^{/}(x) < 0$

So, $f(x)$is strictly decreasing function.

In the interval $(-1,1)$ $f^{/}(x) > 0$

So, $f(x)$is strictly increasing function and

In the interval $(1, \infty)$ $f^{/}(x) < 0$

So, $f(x)$is strictly decreasing function

Because $f^{/}(x)$ changes its sign from negative to positive when passing through $x = -1$ for the function $f(x)$, it has a local minimum at $x = -1$, $f(x) = \frac{(-1)}{1+(-1)^2}$

$$= \frac{-1}{1+1}$$

$$= -\frac{1}{2}$$

Local minimum is $-\frac{1}{2}$

Because $f^{/}(x)$ changes its sign from positive to negative when passing through $x = 1$ for the function $f(x)$, it has a local maximum at $x = 1$,

$f(x) = \frac{(1)}{1+(1)^2}$

$$= \frac{1}{1+1}$$

$$= \frac{1}{2}$$

Local maximum is $-\frac{1}{2}$

****

EXERCISE 7.6

1. Find the absolute extrema of the following functions on the given closed interval.
   (i) $f(x) = x^2 - 12x + 10$; $[1, 2]$
   (ii) $f(x) = 3x^4 - 4x^3$; $[-1, 2]$
   (iii) $f(x) = 6x^{\frac{4}{3}} - 3x^{\frac{1}{3}}$; $[-1,1]$
   (iv) $f(x) = 2\cos x + \sin 2x$; $\left[0, \frac{\pi}{2}\right]$

Solution: (i) $f(x) = x^2 - 12x + 10$; $[1, 2]$

$$f(x) = x^2 - 12x + 10$$

$$f^{/}(x) = 2x - 12$$

When, $f^{/}(x) = 0$

$$2x - 12 = 0$$

$$2x = 12$$

$$x = 6$$

The critical number $x = 6$ and

given the end points are $x = 1,2$

At $x = 6$

$f(6) = (6)^2 - 12(6) + 10$
$= 36 - 72 + 10$
$= 46 - 72$
$= -26$

$f(6) = -26$

At $x = 1$

$f(1) = (1)^2 - 12(1) + 10$
$= 1 - 12 + 10$
$= 11 - 12$

$f(1) = -1$

At $x = 2$

$f(2) = (2)^2 - 12(2) + 10$
$= 4 - 24 + 10$
$= 14 - 24$

$= -10$

Hence, absolute maximum value of $f(x) = -1$
and absolute minimum value of $f(x) = -26$

(ii) $f(x) = 3x^4 - 4x^3; [-1, 2]$

$$f(x) = 3x^4 - 4x^3$$

$$f'(x) = 12x^3 - 12x^2$$

When, $f'(x) = 0$

$$12x^3 - 12x^2 = 0$$

$$12x^2(1 - x) = 0$$

$12x^2 = 0$, gives $x = 0$

$1 - x = 0$, gives $x = 1$

The critical numbers $x = 0,1$ and

given the end points are $x = -1,2$

At $x = 0$

$$f(0) = 3(0)^4 - 4(0)^3 = 0$$

$$f(0) = 0$$

At $x = 1$

$$f(1) = 3(1)^4 - 4(1)^3 = 3 - 4$$

$$f(1) = -1$$

At $x = -1$

$$f(-1) = 3(-1)^4 - 4(-1)^3 = 3(1) - 4(-1) = 3 + 4$$

$$f(-1) = 7$$

At $x = 2$

$$f(2) = 3(2)^4 - 4(2)^3 = 3(16) - 4(8) = 48 - 32 = 16$$

Hence, absolute maximum value of $f(x) = 16$
and absolute minimum value of $f(x) = -1$

(iii) $f(x) = 6x^{\frac{4}{3}} - 3x^{\frac{1}{3}}; [-1,1]$

$$f(x) = 6x^{\frac{4}{3}} - 3x^{\frac{1}{3}}$$

$$f'(x) = 6\left(\frac{4}{3}\right)x^{\frac{4}{3}-1} - 3\left(\frac{1}{3}\right)x^{\frac{1}{3}-1}$$

$$= 8x^{\frac{1}{3}} - x^{\frac{1}{3}-1}$$

$$= x^{\frac{1}{3}}(8 - x^{-1})$$

$$= x^{\frac{1}{3}}\left(8 - \frac{1}{x}\right)$$

When $f'(x) = 0$, we get

$$x^{\frac{1}{3}}\left(8 - \frac{1}{x}\right) = 0$$

Hence $x = 0$ and $8 - \frac{1}{x} = 0$ gives $x = \frac{1}{8}$

So, critical numbers are $x = \frac{1}{8}, 0$ and the end points are $x = -1, 1$

At $x = 0$,

$$f(0) = 6(0)^{\frac{4}{3}} - 3(0)^{\frac{1}{3}} = 0$$

At $x = \frac{1}{8}$,

$$f\left(\frac{1}{8}\right) = 6\left(\frac{1}{8}\right)^{\frac{4}{3}} - 3\left(\frac{1}{8}\right)^{\frac{1}{3}}$$

$$= 6\left[\left(\frac{1}{2}\right)^3\right]^{\frac{4}{3}} - 3\left[\left(\frac{1}{2}\right)^3\right]^{\frac{1}{3}}$$

$$= 6\left(\frac{1}{2}\right)^4 - 3\left(\frac{1}{2}\right)^1$$

$$= 6\left(\frac{1}{16}\right) - 3\left(\frac{1}{2}\right)$$

$$= \frac{3}{8} - \frac{3}{2}$$

$$= \frac{3-12}{8}$$

$$= -\frac{9}{8}$$

At $x = -1$,

$$f(-1) = 6(-1)^{\frac{4}{3}} - 3(-1)^{\frac{1}{3}}$$

$$= 6[(-1)^3]^{\frac{4}{3}} - 3[(-1)^3]^{\frac{1}{3}}$$

$$= 6(-1)^4 - 3(-1)^1$$

$$= 6(1) - 3(-1)$$

$$= 6 + 3$$

$$= 9$$

At $x = 1$,

$$f(1) = 6(1)^{\frac{4}{3}} - 3(1)^{\frac{1}{3}}$$

$$= 6(1) - 3(1)$$

$$= 6 - 3$$

$$= 3$$

Hence, absolute maximum value of $f(x) = 9$

and absolute minimum value of $f(x) = -\frac{9}{8}$

(iv) $f(x) = 2\cos x + \sin 2x\,;\left[0, \frac{\pi}{2}\right]$

$f(x) = 2\cos x + \sin 2x$

$f'(x) = 2(-\sin x) + 2\cos 2x$

When $f'(x) = 0$,

$-2\sin x + 2\cos 2x = 0$

$2\cos 2x = 2\sin x$

$\cos 2x = \sin x$

$1 - 2sin^2 x = \sin x$

$2sin^2 x + \sin x - 1 = 0$

$(2\sin x - 1)(\sin x + 1) = 0$

$2\sin x - 1 = 0$, gives

$2\sin x = 1$

$\sin x = \frac{1}{2}$

$x = \frac{\pi}{6}$ and

$\sin x + 1 = 0$, gives

$\sin x = -1$

$x = -\frac{\pi}{2}, \frac{3\pi}{2}$ they does not belong to the given interval . So the critical number is $x = \frac{\pi}{6}$ and the end points are $x = 0, \frac{\pi}{2}$

At $x = \frac{\pi}{6}$

$f\left(\frac{\pi}{6}\right) = 2\cos\frac{\pi}{6} + \sin 2\left(\frac{\pi}{6}\right)$

$= 2\cos\frac{\pi}{6} + \sin\frac{\pi}{3}$

$= 2\left(\frac{\sqrt{3}}{2}\right) + \frac{\sqrt{3}}{2}$

$= \frac{3\sqrt{3}}{2}$

At $x = 0$

$f(0) = 2\cos 0 + \sin 2(0)$

$= 2\cos 0 + \sin 0$

$= 2(1) + 0$

$= 2$

At $x = \frac{\pi}{2}$

$f\left(\frac{\pi}{2}\right) = 2\cos\frac{\pi}{2} + \sin 2\left(\frac{\pi}{2}\right)$

$= 2\cos\frac{\pi}{2} + \sin\pi$

$= 2(0) + 0$

$= 0$

Hence, absolute maximum value of $f(x) = \frac{3\sqrt{3}}{2}$

and absolute minimum value of $f(x) = 0$

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2. Find the intervals of monotonicities and hence find the local extremum for the following functions:

(i) $f(x) = 2x^3 + 3x^2 - 12x$

(ii) $f(x) = \frac{x}{x-5}$ (iii) $f(x) = \frac{e^x}{1-e^x}$

(iv) $f(x) = \frac{x^3}{3} - \log x$

(v) $f(x) = \sin x\cos x + 5, x \in (0, 2\pi)$

Solution:

(i) $f(x) = 2x^3 + 3x^2 - 12x$

$f'(x) = 6x^2 + 6x - 12$

When, $f'(x) = 0$

$6x^2 + 6x - 12 = 0$

Dividing by 6,

$x^2 + x - 2 = 0$

$(x + 2)(x - 1) = 0$

$x + 2 = 0$, gives $x = -2$ and

$x - 1 = 0$, gives $x = 1$

The critical numbers $x = -2, 1$

$\therefore$ The interval are $(-\infty, -2)$, $(-2, 1)$ and $(1, \infty)$

In the interval $(-\infty, -2)$let $x = -3$

$f'(-3) = 6(-3)^2 + 6(-3) - 12$

$= 6(9) - 18 - 12$

$= 54 - 30$

$= 24 > 0$

So, $f(x)$is strictly increasing function.

In the interval $(-2, 1)$let $x = 0$

$f'(0) = 6(0)^2 + 6(0) - 12$

$= 0 + 0 - 12$

$= -12 < 0$

So, $f(x)$is strictly decreasing function.

Because $f'(x)$ changes its sign from positive to negative when passing through $x = -2$ for the function $f(x)$, has a local maximum at $x = -2$

$f(-2) = 2(-2)^3 + 3(-2)^2 - 12(-2)$

$= 2(-8) + 3(4) - 12(-2)$

$= -16 + 12 + 24$

$= -16 + 36$

$= 20$

Local maximum is 20

In the interval $(1, \infty)$ let $x = 2$

$f'(2) = 6(2)^2 + 6(2) - 12$

$= 6(4) + 12 - 12$

$= 24$

$= 24 > 0$

So, $f(x)$is strictly increasing function. Because $f'(x)$ changes its sign from negative to positive when passing through $x = 1$ for the function $f(x)$, has a local minimum at $x = 1$

$f(1) = 2(1)^3 + 3(1)^2 - 12(1)$

$= 2(1) + 3(1) - 12(1)$

$= 2 + 3 - 12$

$= 5 - 12$

$= -7$

Local minimum is -7

(ii) $f(x) = \frac{x}{x-5}$

$f'(x) = \frac{(x-5)(1) - x(1)}{(x-5)^2}$

$= \frac{x-5-x}{(x-5)^2}$

$= \frac{-5}{(x-5)^2}$

$f'(x)$ cannot be equal to zero. And $f'(x)$ does not exists at $x = 5$.

Hence the critical number is $x = 5$. ∴ The interval are $(-\infty, 5)$and $(5, \infty)$

In the interval $(-\infty, 5)$ $f'(x) < 0$

So, $f(x)$is strictly decreasing function.

In the interval $(5,\infty)$ $f'(x) < 0$

So, $f(x)$is strictly decreasing function.

Since there is no change $f'(x)$ passing through $x = 5$, by the first derivative test there is no local extrema.

(iii) $f(x) = \frac{e^x}{1-e^x}$

$f'(x) = \frac{(1-e^x)(e^x) - e^x(e^x)}{(1-e^x)^2}$

$= \frac{e^x - e^{2x} + e^{2x}}{(1-e^x)^2}$

$= \frac{e^x}{(1-e^x)^2}$

$f'(x)$ cannot be equal to zero. And $f'(x)$ does not exists at $x = 0$.

Hence the critical number is $x = 0$.

∴ The interval are $(-\infty, 0)$and $(0, \infty)$

In the interval $(-\infty, 0)$ $f'(x) > 0$

So, $f(x)$is strictly increasing function.

In the interval $(0, \infty)$ $f'(x) > 0$

So, $f(x)$is strictly increasing function.

Since there is no change $f'(x)$ passing through $x = 0$, by the first derivative test there is no local extrema.

(iv) $f(x) = \frac{x^3}{3} - \log x$

$f'(x) = \frac{3x^2}{3} - \frac{1}{x}$

$= x^2 - \frac{1}{x}$

$= \frac{x^3-1}{x}$

When $f'(x) = 0$,

$x^3 - 1 = 0$

$(x-1)(x^2 + x + 1) = 0$

$x - 1 = 0$, Gives $x = 1$.

Hence there is a critical point at $x = 1$.

∴ The interval are $(-\infty, 1)$and $(1, \infty)$

In the interval $(-\infty, 1)$ $f^{/}(x) < 0$

So, $f(x)$is strictly decreasing function.

In the interval $(1, \infty)$ $f^{/}(x) > 0$

So, $f(x)$is strictly increasing function.

Because $f^{/}(x)$ changes its sign from negative to positive when passing through $x = 1$ for the function $f(x)$, it has a local minimum at $x = 1$,

$$f(1) = \frac{1}{3} - \log 1$$
$$= \frac{1}{3} - 0$$
$$= \frac{1}{3},\ \text{So local minimum is } \frac{1}{3}$$

(v) $f(x) = \sin x \cos x + 5, x \in (0, 2\pi)$

$$f(x) = \sin x \cos x + 5$$
$$f^{/}(x) = \sin x(-\sin x) + \cos x(\cos x)$$
$$= -\sin^2 x + cos^2 x$$
$$= cos^2 x - \sin^2 x$$
$$= \cos 2x$$

When $f^{/}(x) = 0$,

$\cos 2x = 0$

We know that $\cos\frac{\pi}{2} = 0$, hence

$2x = \frac{\pi}{2}$ gives, $x = \frac{\pi}{4}$

The values $\frac{\pi}{4}, \frac{3\pi}{4}, \frac{5\pi}{4}$ and $\frac{7\pi}{4} \in (0, 2\pi)$

$\therefore$ The interval are $\left(0, \frac{\pi}{4}\right), \left(\frac{\pi}{4}, \frac{3\pi}{4}\right), \left(\frac{3\pi}{4}, \frac{5\pi}{4}\right)$, $\left(\frac{5\pi}{4}, \frac{7\pi}{4}\right)$ and $\left(\frac{7\pi}{4}, 2\pi\right)$

In the interval $\left(0, \frac{\pi}{4}\right)$ $f^{/}(x) > 0$

So, $f(x)$is strictly increasing function.

In the interval $\left(\frac{\pi}{4}, \frac{3\pi}{4}\right)$ $f^{/}(x) < 0$

So, $f(x)$is strictly decreasing function.

In the interval $\left(\frac{3\pi}{4}, \frac{5\pi}{4}\right)$ $f^{/}(x) > 0$

So, $f(x)$is strictly increasing function.

In the interval $\left(\frac{5\pi}{4}, \frac{7\pi}{4}\right)$ $f^{/}(x) < 0$

So, $f(x)$is strictly decreasing function.

In the interval $\left(\frac{7\pi}{4}, 2\pi\right)$ $f^{/}(x) > 0$

So, $f(x)$is strictly increasing function.

Because $f^{/}(x)$ changes its sign from positive to negative when passing through $x = \frac{\pi}{4}, \frac{5\pi}{4}$ for the function $f(x)$, it has a local maximum at $x = \frac{\pi}{4}, \frac{5\pi}{4}$

$$f(x) = \sin x \cos x + 5$$
$$f\left(\frac{\pi}{4}\right) = \sin\frac{\pi}{4}\cos\frac{\pi}{4} + 5$$
$$= \left(\frac{1}{\sqrt{2}}\right)\left(\frac{1}{\sqrt{2}}\right) + 5$$
$$= \frac{1}{2} + 5$$
$$= \frac{1+10}{2} = \frac{11}{2}$$

$$f\left(\frac{5\pi}{4}\right) = \sin\left(\frac{5\pi}{4}\right)\cos\left(\frac{5\pi}{4}\right) + 5$$
$$= \sin\left(\pi + \frac{\pi}{4}\right)\cos\left(\pi + \frac{\pi}{4}\right) + 5$$
$$= \left[-\sin\left(\frac{\pi}{4}\right)\right]\left[-\cos\left(\frac{\pi}{4}\right)\right] + 5$$
$$= \left(\frac{1}{\sqrt{2}}\right)\left(\frac{1}{\sqrt{2}}\right) + 5$$
$$= \frac{1}{2} + 5$$
$$= \frac{1+10}{2} = \frac{11}{2}$$

Because $f^{/}(x)$ changes its sign from negative to positive when passing through $x = \frac{3\pi}{4}, \frac{7\pi}{4}$ for the function $f(x)$, it has a local maximum at $x = \frac{3\pi}{4}, \frac{7\pi}{4}$

$$f(x) = \sin x \cos x + 5$$
$$f\left(\frac{3\pi}{4}\right) = \sin\left(\frac{3\pi}{4}\right)\cos\left(\frac{3\pi}{4}\right) + 5$$
$$= \sin\left(\frac{\pi}{2} + \frac{\pi}{4}\right)\cos\left(\frac{\pi}{2} + \frac{\pi}{4}\right) + 5$$
$$= \left[\cos\left(\frac{\pi}{4}\right)\right]\left[-\sin\left(\frac{\pi}{4}\right)\right] + 5$$
$$= \left(\frac{1}{\sqrt{2}}\right)\left(-\frac{1}{\sqrt{2}}\right) + 5$$
$$= -\frac{1}{2} + 5$$
$$= \frac{-1+10}{2} = \frac{9}{2}$$

$f\left(\frac{7\pi}{4}\right) = \sin\left(\frac{7\pi}{4}\right)\cos\left(\frac{7\pi}{4}\right) + 5$

$= \sin\left(\frac{3\pi}{2}+\frac{\pi}{4}\right)\cos\left(\frac{3\pi}{2}+\frac{\pi}{4}\right) + 5$

$= \left[-\cos\left(\frac{\pi}{4}\right)\right]\left[\sin\left(\frac{\pi}{4}\right)\right] + 5$

$= \left(-\frac{1}{\sqrt{2}}\right)\left(\frac{1}{\sqrt{2}}\right) + 5$

$= -\frac{1}{2} + 5$

$= \frac{-1+10}{2} = \frac{9}{2}$

So local minimum is $\frac{9}{2}$ and local maximum is $\frac{11}{2}$

****

Example 7.57

Determine the intervals of concavity of the curve $f(x) = (x-1)^3.(x-5), x \in \mathbb{R}$ and, points of inflection if any.

Solution: $f(x) = (x-1)^3.(x-5)$

$f(x) = (x^3 - 3x^2 + 3x - 1).(x-5)$

$= x^4 - 3x^3 + 3x^2 - x - 5x^3 + 15x^2 - 15x + 5$

$f(x) = x^4 - 8x^3 + 18x^2 - 16x + 5$

$f^{/}(x) = 4x^3 - 24x^2 + 36x - 16$

$f^{//}(x) = 12x^2 - 48x + 36$

When $f^{//}(x) = 0$,

$12x^2 - 48x + 36 = 0$

Dividing by 12,

$x^2 - 4x + 3 = 0$

$(x-3)(x-1) = 0$

$x - 3 = 0$, gives $x = 3$ and

$x - 1 = 0$, gives $x = 1$

∴ The interval are $(-\infty, 1), (1,3)$ and $(3, \infty)$

In the interval $(-\infty, 1)$ let $x = 0$

$f^{//}(0) = 12(0)^2 - 48(0) + 36$

$= 0 - 0 + 36$

$= 36 > 0$

Hence $f(x)$ is concave upward in the $(-\infty, 1)$

In the interval $(1,3)$ let $x = 2$

$f^{//}(2) = 12(2)^2 - 48(2) + 36$

$= 48 - 96 + 36$

$= 84 - 96$

$= -12 < 0$

Hence $f(x)$ is concave downward in the $(3, \infty)$

In the interval $(3, \infty)$ let $x = 4$

$f^{//}(2) = 12(4)^2 - 48(4) + 36$

$= 192 - 192 + 36$

$= 36 > 0$

Hence $f(x)$ is concave upward in the $(3, \infty)$

$f^{//}(x)$ changes the direction at $x = 1, 3$

We get the points of inflection at $x = 1, 3$

(i) at $x = 1$

$f(x) = (x-1)^3.(x-5)$

$f(1) = (1-1)^3.(1-5)$

$= 0$

(ii) at $x = 3$

$f(x) = (x-1)^3.(x-5)$

$f(3) = (3-1)^3.(3-5)$

$= (2)^3.(-2)$

$= 8.(-2)$

$= -16$

The points of inflections are $(1, 0), (3, -16)$

****

Example 7.58 Determine the intervals of concavity of the curve $y = 3 + \sin x$.

Solution:

$y = 3 + \sin x$

$y^{/} = \frac{dy}{dx} = \cos x$

$y^{//} = -\sin x$

Substituting $y^{//} = 0$, we get

$\sin x = 0$, gives $x = 0, n\pi$

As the limit $(-\pi, \pi)$ we get $(-\pi, 0)$ and $(0, \pi)$ as intervals.

In the interval $(-\pi, 0)$, let $x = -\frac{\pi}{2}$

$y^{//} = -\sin x$

$= -\sin\left(-\frac{\pi}{2}\right)$

$= \sin\left(\frac{\pi}{2}\right) = 1 > 0$

Hence $f(x)$ is concave upward in the $(-\pi, 0)$

In the interval $(0, \pi)$, let $x = \frac{\pi}{2}$

$y^{//} = -\sin x$

$= -\sin\left(\frac{\pi}{2}\right)$

$= -1 < 0$

Hence $f(x)$ is concave downward in the $(0, \pi)$

As $y^{//}$ changes the sign as it passes through $x = 0$, we get a point of inflection at $x = 0$

Substituting $x = 0$ in the given function,

$y = 3 + \sin x$ we get

$y = 3 + \sin(0)$

$y = 3 + 0$

$y = 3$

Hence the point of inflection is (0,3)

In general the intervals are $(n\pi, (n+1)\pi), n \in \mathbb{Z}$

Hence $(n\pi, 3)$ is also the point of inflection.

****

Example 7.59 Find the local extremum of the function $f(x) = x^4 + 32x$.

Solution:

$f(x) = x^4 + 32x$

$f^{/}(x) = 4x^3 + 32$

$f^{//}(x) = 12x^2$

When $f^{/}(x) = 0$, we get

$4x^3 + 32 = 0$

$4x^3 = -32$

$x^3 = -8$

$x = -2$

At $x = -2$, $f^{//}(x) = 12x^2$

$f^{//}(-2) = 12(-2)^2$

$= 12(4)$

$= 48 > 0$

So, the function has local minima at $x = -2$

at $x = -2$, $f(x) = x^4 + 32x$ has

$f(-2) = (-2)^4 + 32(-2)$

$= 16 - 64$

$= -48$

So the local minimum is -48 and the extreme point is $(-2, -48)$

****

Example 7.60

Find the local extrema of the function $f(x) = 4x^6 - 6x^4$.

<u>Solution</u>: $f(x) = 4x^6 - 6x^4$

$f^{/}(x) = 24x^5 - 24x^3$

$f^{//}(x) = 120x^4 - 72x^2$

When $f^{/}(x) = 0$, we get

$24x^5 - 24x^3 = 0$

$24x^3(x^2 - 1) = 0$

$24x^3 = 0$ gives $x = 0$

$x^2 - 1 = 0$ gives $x = \pm 1$

The critical points are $x = -1, 0, 1$

Substituting $x = -1$ in

$f^{//}(x) = 120x^4 - 72x^2$ we get,

$f^{//}(-1) = 120(-1)^4 - 72(-1)^2$

$= 120(1) - 72(1)$

$= 120 - 72$

$= 48 > 0$

Hence $f(x)$ has local minimum at $x = -1$

Substituting $x = 1$ in

$f^{//}(x) = 120x^4 - 72x^2$ we get,

$f^{//}(1) = 120(1)^4 - 72(1)^2$

$= 120(1) - 72(1)$

$= 120 - 72$

$= 48 > 0$

Hence $f(x)$ has local minimum at $x = 1$

Substituting $x = 0$ in

$f^{//}(x) = 120x^4 - 72x^2$ we get,

$f^{//}(-1) = 120(0) - 72(0)$

$= 0$

Hence the second derivative test does not give any information about local extrema at $x = 0$

So, to try by first derivative test,

Intervals are $(-\infty, -1), (-1, 0), (0,1), (1, \infty)$

In $(-\infty, -1)$ $f^{/}(x) < 0$

So, $f(x)$ is strictly decreasing function.

In $(-1, 0)$ $f^{/}(x) > 0$

So, $f(x)$ is strictly increasing function.

In $(0,1)$ $f^{/}(x) < 0$

So, $f(x)$ is strictly decreasing function.

In $(1, \infty)$ $f^{/}(x) > 0$

So, $f(x)$ is strictly increasing function.

Because $f^{/}(x)$ changes its sign from negative to positive when passing through $x = -1$ for the function $f(x)$, it has a local minimum at $x = -1$ and the local minimum value of the function is

$f(-1) = 4(-1)^6 - 6(-1)^4$

$= 4(1) - 6(1)$

$= 4 - 6$

$= -2$

Also $f^{/}(x)$ changes its sign from positive to negative when passing through $x = 0$ for the function $f(x)$, it has a local maximum is

$f(0) = 4(0)^6 - 6(0)^4$

$= 0$

And $f^{/}(x)$ changes its sign from negative to positive when passing through $x = 1$ for the function $f(x)$, it has a local minimum at $x = 1$ and the local minimum value of the function is

$f(1) = 4(1)^6 - 6(1)^4$

$= 4(1) - 6(1)$

$= 4 - 6$

$= -2$

Hence the local minimum is -2 and the local maximum is 0.

****

Example 7.61

Find the local maximum and minimum of the function x^2y^2 on the line $x + y = 10$

Solution: Given $x + y = 10$ gives

$y = 10 - x$

Substituting the value of y in x^2y^2

$f(x) = x^2(10 - x)^2$

$= x^2(100 + x^2 - 20x)$

$= 100x^2 + x^4 - 20x^3$

$f^{/}(x) = 200x + 4x^3 - 60x^2$

$f^{//}(x) = 200 + 12x^2 - 120x$

When $f^{/}(x) = 0$, we get

$200x + 4x^3 - 60x^2 = 0$

$4x(50 + x^2 - 15x) = 0$

$4x = 0$ gives $x = 0$

$x^2 - 15x + 50 = 0$ gives

$(x - 5)(x - 10) = 0$

Hence $x = 5, 10$

So, the critical points are $x = 0, 5, 10$

Substituting $x = 0$ in $f^{//}(x)$

$f^{//}(0) = 200 + 12(0)^2 - 120(0)$

$= 200 > 0$

So, has local minimum at $x = 0$ and the local minimum value of the function is $f(0)$ and

$f(0) = (0)^2(10 - 0)^2$

$= 0$

Substituting $x = 5$ in $f^{//}(x)$

$f^{//}(5) = 200 + 12(5)^2 - 120(5)$

$= 200 + 12(25) - 120(5)$

$= 200 + 300 - 600$

$= 500 - 600$

$= -100 < 0$

So, has local maximum at $x = 5$ and the local maximum value of the function is $f(5)$ and

$f(5) = (5)^2(10 - 5)^2$

$= (5)^2(5)^2$

$= 25 \times 25$

$= 625$

Substituting $x = 10$ in $f''(x)$

$f''(10) = 200 + 12(10)^2 - 120(10)$

$= 200 + 12(100) - 120(10)$

$= 200 + 1200 - 1200$

$= 200 > 0$

So, has local minimum at $x = 10$ and the local minimum value of the function is $f(10)$ and

$f(10) = (10)^2(10-10)^2$

$= 0$

Hence the local minimum is 0 and the local maximum is 625

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EXERCISE 7.7

1. Find intervals of concavity and points of inflexion for the following functions:
(i) $f(x) = x(x-4)^3$
(ii) $f(x) = \sin x + \cos x,\ 0 < x < 2\pi$
(iii) $f(x) = \frac{1}{2}(e^x - e^{-x})$

Solution:

(i) $f(x) = x(x-4)^3$

$= x(x^3 - 12x^2 + 48x - 64)$

$= x^4 - 12x^3 + 48x^2 - 64x$

$f'(x) = 4x^3 - 36x^2 + 96x - 64$

$f''(x) = 12x^2 - 72x + 96$

When $f''(x) = 0$, we get

$12x^2 - 72x + 96 = 0$

Dividing by 12,

$x^2 - 6x + 8 = 0$

$(x-2)(x-4) = 0$

Hence $x = 2, 4$

$\therefore$ The interval are $(-\infty, 2), (2,4)$and $(4, \infty)$

In the interval $(-\infty, 2)$ let $x = 0$

$f''(0) = 12(0)^2 - 72(0) + 96$

$= 96 > 0$

Hence $f(x)$ is concave upward in the $(-\infty, 2)$

In the interval $(2,4)$ let $x = 3$

$f''(3) = 12(3)^2 - 72(3) + 96$

$= 12(9) - 216 + 96$

$= 108 - 216 + 96$

$= 204 - 216$

$= -12 < 0$

Hence $f(x)$ is concave downward in the $(2,4)$

In the interval $(4, \infty)$ let $x = 5$

$f''(5) = 12(5)^2 - 72(5) + 96$

$= 12(25) - 360 + 96$

$= 300 - 360 + 96$

$= 396 - 360$

$= 36 > 0$

Hence $f(x)$ is concave upward in the $(2,4)$

$f''(x)$ changes the direction at $x = 2, 4$

When $x = 2$, $f(x) = x(x-4)^3$ gives

$f(2) = 2(2-4)^3$

$= 2(-2)^3$

$= 2(-8)$

$= -16$

When $x = 4$, $f(x) = x(x-4)^3$ gives

$f(4) = 2(4-4)^3$

$= 2(0)^3$

$= 0$

The points of inflections are $(2, -16)$, $(4, 0)$

(ii) $f(x) = \sin x + \cos x,\ 0 < x < 2\pi$

$f'(x) = \cos x - \sin x$

$f''(x) = -\sin x - \cos x$

When $f''(x) = 0$,

$-\sin x - \cos x = 0$

$\sin x = -\cos x$

$\frac{\sin x}{\cos x} = -1$

$\tan x = -1$ gives

$$x = \frac{3\pi}{4}, \frac{7\pi}{4} \in (0,2\pi)$$

Hence the intervals are $\left(0,\frac{3\pi}{4}\right),\left(\frac{3\pi}{4},\frac{7\pi}{4}\right),\left(\frac{7\pi}{4},2\pi\right)$

In $\left(0,\frac{3\pi}{4}\right) f^{//}(x) < 0$

Hence $f(x)$ is concave downward in the $\left(0,\frac{3\pi}{4}\right)$

In $\left(\frac{3\pi}{4},\frac{7\pi}{4}\right) f^{//}(x) > 0$

Hence $f(x)$ is concave upward in the $\left(\frac{3\pi}{4},\frac{7\pi}{4}\right)$

In $\left(\frac{7\pi}{4},2\pi\right) f^{//}(x) < 0$

Hence $f(x)$ is concave downward in $\left(\frac{7\pi}{4},2\pi\right)$

$f^{//}(x)$ changes the direction at $x = \frac{3\pi}{4}, \frac{7\pi}{4}$

At $x = \frac{3\pi}{4}$, $f(x) = \sin x + \cos x$,

$$f\left(\frac{3\pi}{4}\right) = \sin\left(\frac{3\pi}{4}\right) + \cos\left(\frac{3\pi}{4}\right)$$
$$= \sin\left(\frac{\pi}{2}+\frac{\pi}{4}\right) + \cos\left(\frac{\pi}{2}+\frac{\pi}{4}\right)$$
$$= \cos\left(\frac{\pi}{4}\right) - \sin\left(\frac{\pi}{4}\right)$$
$$= \frac{1}{\sqrt{2}} - \frac{1}{\sqrt{2}}$$
$$= 0$$

At $x = \frac{7\pi}{4}$, $f(x) = \sin x + \cos x$,

$$f\left(\frac{7\pi}{4}\right) = \sin\left(\frac{7\pi}{4}\right) + \cos\left(\frac{7\pi}{4}\right)$$
$$= \sin\left(2\pi-\frac{\pi}{4}\right) + \cos\left(2\pi-\frac{\pi}{4}\right)$$
$$= \sin\left(-\frac{\pi}{4}\right) + \cos\left(-\frac{\pi}{4}\right)$$
$$= -\sin\left(\frac{\pi}{4}\right) + \cos\left(\frac{\pi}{4}\right)$$
$$= -\frac{1}{\sqrt{2}} + \frac{1}{\sqrt{2}}$$
$$= 0$$

The points of inflections are $\left(\frac{3\pi}{4},0\right)$, $\left(\frac{7\pi}{4},0\right)$

(iii) $f(x) = \frac{1}{2}(e^x - e^{-x})$

$$f^{/}(x) = \frac{1}{2}[e^x - (-e^{-x})]$$
$$= \frac{1}{2}(e^x + e^{-x})$$
$$f^{//}(x) = \frac{1}{2}(e^x - e^{-x})$$

When $f^{//}(x) = 0$,

$\frac{1}{2}(e^x - e^{-x}) = 0$ gives

$$e^x - e^{-x} = 0$$
$$e^x = e^{-x}$$
$$e^x = \frac{1}{e^x}$$
$$e^x e^x = 1$$
$$e^{2x} = e^0 \text{ gives}$$
$$2x = 0 \Rightarrow x = 0$$

$\therefore$ The interval are $(-\infty,0)$ and $(0,\infty)$

In $(-\infty,0) f^{//}(x) < 0$

Hence $f(x)$ is concave downward in $(-\infty,0)$

In $(0,\infty) f^{//}(x) > 0$

Hence $f(x)$ is concave upward in the $(0,\infty)$

$f^{//}(x)$ changes the direction at $x = 0$

At $x = 0$, $f(x) = \frac{1}{2}(e^x - e^{-x})$

$$f(0) = \frac{1}{2}(e^0 - e^{-0})$$
$$= \frac{1}{2}(1-1)$$
$$= 0$$

So, the point of inflection is $(0,0)$

****

2. Find the local extrema for the following functions using second derivative test:

(i) $f(x) = -3x^5 + 5x^3$ (ii) $f(x) = x \log x$

(iii) $f(x) = x^2 e^{-2x}$

<u>Solution:</u> $f(x) = -3x^5 + 5x^3$

$$f^{/}(x) = -15x^4 + 15x^2$$
$$f^{//}(x) = -60x^3 + 30x$$

When $f^{/}(x) = 0$

$$-15x^4 + 15x^2 = 0$$
$$-15x^2(x^2-1) = 0$$

$-15x^2 = 0$, gives $x = 0$ and

$x^2 - 1 = 0$, gives $x = \pm 1$

Hence the critical points are $x = 0, -1, 1$

At $x = 0$, $f^{//}(x) = -60x^3 + 30x$

$$f^{//}(0) = -60(0)^3 + 30(0) = 0$$

At $x = -1$, $f^{//}(x) = -60x^3 + 30x$

$$f^{//}(-1) = -60(-1)^3 + 30(-1)$$
$$= -60(-1) + 30(-1)$$

$= 60 - 30$

$= 30 > 0$

So, $f(x)$ has local minimum at $x = -1$

Local minimum value is $f(-1)$

$f(-1) = -3(-1)^5 + 5(-1)^3$

$= -3(-1) + 5(-1)$

$= 3 - 5$

$= -2$

At $x = 1$, $f^{//}(x) = -60x^3 + 30x$

$f^{//}(1) = -60(1)^3 + 30(1)$

$= -60 + 30$

$= -30 < 0$

So, $f(x)$ has local maximum at $x = 1$

Local maximum value is $f(1)$

$f(1) = -3(1)^5 + 5(1)^3$

$= -3 + 5 = 2$

Hence the local minimum is -2 and the local maximum is 2

(ii) $f(x) = x \log x$

$f^{/}(x) = x\left(\frac{1}{x}\right) + \log x \ (1)$

$= 1 + \log x$

$f^{//}(x) = \frac{1}{x}$

When $f^{/}(x) = 0$

$1 + \log x = 0$

$\log_e x = -1$

$x = e^{-1} = \frac{1}{e}$ is the critical point

Substituting $x = \frac{1}{e}$ in $f^{//}(x) = \frac{1}{x}$

$f^{//}\left(\frac{1}{e}\right) = \frac{1}{\frac{1}{e}} = e > 0$

Hence, $f(x)$ has local minimum at $x = \frac{1}{e}$. The local minimum value of the function is $f\left(\frac{1}{e}\right)$

$f\left(\frac{1}{e}\right) = \left(\frac{1}{e}\right)\log\left(\frac{1}{e}\right)$

$= \left(\frac{1}{e}\right)[\log(1) - \log(e)]$

$= \left(\frac{1}{e}\right)[0 - 1]$

$= -\frac{1}{e}$

Hence the local minimum is $-\frac{1}{e}$

(iii) $f(x) = x^2 e^{-2x}$

$f^{/}(x) = x^2(-2e^{-2x}) + e^{-2x}(2x)$

$= -2x^2(e^{-2x}) + e^{-2x}(2x)$

$f^{//}(x) = -2[x^2(-2e^{-2x}) + e^{-2x}(2x)]$

$+e^{-2x}(2) + (2x)(-2e^{-2x})$

$= 4x^2e^{-2x} - 4xe^{-2x} + 2e^{-2x} - 4xe^{-2x}$

$= 4x^2e^{-2x} - 8xe^{-2x} + 2e^{-2x}$

$= 2e^{-2x}(2x^2 - 4x + 1)$

When $f^{/}(x) = 0$

$-2x^2(e^{-2x}) + e^{-2x}(2x) = 0$

$2x(e^{-2x})(-x + 1) = 0$ we get

$2x = 0$, gives $x = 0$ and

$-x + 1 = 0$, gives $x = 1$ and $e^{-2x} \neq 0$

The critical points are $x = 0, 1$

At $x = 0$, $f^{//}(x) = 2e^{-2x}(2x^2 - 4x + 1)$

$f^{//}(0) = 2e^{-2(0)}(2(0)^2 - 4(0) + 1)$

$= 2e^0(0 - 0 + 1)$

$= 2(1)(1)$

$= 2 > 0$

So, $f(x)$ has local minimum at $x = 0$

Local maximum value is $f(0) = 0$

At $x = 1$, $f^{//}(x) = 2e^{-2x}(2x^2 - 4x + 1)$

$f^{//}(1) = 2e^{-2(1)}(2(1)^2 - 4(1) + 1)$

$= 2e^{-2}(2 - 4 + 1)$

$= 2e^{-2}(3 - 4)$

$= 2e^{-2}(-1) < 0$

So, $f(x)$ has local maximum at $x = 1$

Local maximum value is $f(1)$

$f(1) = (1)^2e^{-2(1)}$

$= e^{-2}$

Hence the local minimum is 0 and

the local maximum is e^{-2}

****

3. For the function $f(x) = 4x^3 + 3x^2 - 6x + 1$ find the intervals of monotonicity, local extrema, intervals of concavity and points of inflection.

Solution: $f(x) = 4x^3 + 3x^2 - 6x + 1$

$$f'(x) = 12x^2 + 6x - 6$$

Substituting $f'(x) = 0$, we get

$$12x^2 + 6x - 6 = 0$$

Dividing by 6,

$$2x^2 + x - 1 = 0$$

$$(2x - 1)(x + 1) = 0$$

$2x - 1 = 0$ gives $2x = 1$ which gives $x = \frac{1}{2}$

$x + 1 = 0$ gives $x = -1$

The critical points are $x = -1, \ \frac{1}{2}$

Intervals are $(-\infty, -1), \left(-1, \frac{1}{2}\right), \left(\frac{1}{2}, \infty\right)$

In $(-\infty, -1)$ let $x = -2$

Then $f'(x) = 12x^2 + 6x - 6$ gives

$$f'(-2) = 12(-2)^2 + 6(-2) - 6$$
$$= 12(4) - 12 - 6$$
$$= 48 - 18$$
$$= 30 > 0$$

$f(x)$ is strictly increasing function in $(-\infty, -1)$

In $\left(-1, \frac{1}{2}\right)$ let $x = 0$

Then $f'(x) = 12x^2 + 6x - 6$ gives

$$f'(0) = 12(0)^2 + 6(0) - 6$$
$$= -6 < 0$$

$f(x)$ is strictly decreasing function in $\left(-1, \frac{1}{2}\right)$

In $\left(\frac{1}{2}, \infty\right)$ let $x = 1$

Then $f'(x) = 12x^2 + 6x - 6$ gives

$$f'(1) = 12(1)^2 + 6(1) - 6$$
$$= 12 + 6 - 6$$
$$= 18 - 6$$
$$= 12 > 0$$

$f(x)$ is strictly increasing function in $\left(\frac{1}{2}, \infty\right)$

Also $f'(x)$ changes its sign from positive to negative when passing through $x = -1$ for the function $f(x)$, it has a local maximum is

$$f(-1) = 4(-1)^3 + 3(-1)^2 - 6(-1) + 1$$
$$= 4(-1) + 3(1) - 6(-1) + 1$$
$$= -4 + 3 + 6 + 1$$
$$= -4 + 10$$
$$= 6$$

And $f'(x)$ changes its sign from negative to positive when passing through $x = \frac{1}{2}$ for the function $f(x)$, it has a local minimum at $x = \frac{1}{2}$ and the local minimum value of the function is

$$f\left(\frac{1}{2}\right) = 4\left(\frac{1}{2}\right)^3 + 3\left(\frac{1}{2}\right)^2 - 6\left(\frac{1}{2}\right) + 1$$
$$= 4\left(\frac{1}{8}\right) + 3\left(\frac{1}{4}\right) - 3 + 1$$
$$= \frac{1}{2} + \frac{3}{4} - 2$$
$$= \frac{2 + 3 - 8}{4}$$
$$= \frac{5 - 8}{4}$$
$$= \frac{-3}{4}$$

Hence the local minimum is $\frac{-3}{4}$ and

the local maximum is 6

And from $f'(x) = 12x^2 + 6x - 6$

$$f''(x) = 24x + 6$$

Substituting $f''(x) = 0$, we get

$$24x = -6$$

$$x = -\frac{6}{24} = -\frac{1}{4}$$

Intervals are $\left(-\infty, -\frac{1}{4}\right), \left(-\frac{1}{4}, \infty\right)$

In $\left(-\infty, -\frac{1}{4}\right)$ let $x = -1$

Then $f''(x) = 24x + 6$ gives

$$f^{//}(-1) = 24(-1) + 6$$
$$= -24 + 6$$
$$= -18 < 0$$

Hence $f(x)$ is concave downward in $\left(-\frac{1}{4}, \infty\right)$

In $\left(-\frac{1}{4}, \infty\right)$ let $x = 0$

Then $f^{//}(x) = 24x + 6$ gives

$$f^{//}(0) = 24(0) + 6$$
$$= 6 > 0$$

Hence $f(x)$ is concave upward in $\left(-\frac{1}{4}, \infty\right)$

$f^{//}(x)$ changes the direction at $x = -\frac{1}{4}$

$$f\left(-\frac{1}{4}\right) = 4\left(-\frac{1}{4}\right)^3 + 3\left(-\frac{1}{4}\right)^2 - 6\left(-\frac{1}{4}\right) + 1$$
$$= 4\left(-\frac{1}{64}\right) + 3\left(\frac{1}{16}\right) + \frac{6}{4} + 1$$
$$= -\frac{1}{16} + \frac{3}{16} + \frac{6}{4} + 1$$
$$= \frac{-1+3+24+16}{16}$$
$$= \frac{-1+43}{16}$$
$$= \frac{42}{16}$$
$$= \frac{21}{8}$$

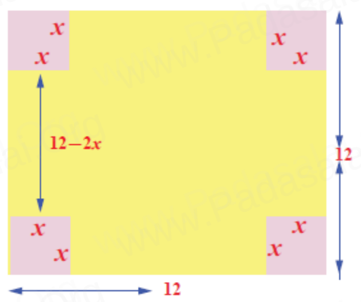
So, the point of inflection is $\left(-\frac{1}{4}, \frac{21}{8}\right)$

****

Example 7.62

We have a 12 square unit piece of thin material and want to make an open box by cutting small squares from the corners of our material and folding the sides up. The question is, which cut produces the box of maximum volume?

Solution:



Let x = length of the cut on each side of the little squares.

V = the volume of the folded box.

The length of the base after two cuts along each edge of size x is $(12 - x)$. The depth of the box after folding is x , so the volume is

$$V = (12 - 2x) \times (12 - 2x) \times x$$

At $x = 0, 6$ volume becomes zero.

Hence $x \in (0, 6)$

$$V = (144 + 4x^2 - 48x)x$$
$$= 144x + 4x^3 - 48x^2$$
$$V^{/} = 144 + 12x^2 - 96x$$
$$V^{//} = 24x - 96$$

Substituting $V^{/} = 0$, we get

$$12x^2 - 96x + 144 = 0$$

Dividing by 12,

$$x^2 - 8x + 12 = 0$$
$$(x - 2)(x - 6) = 0$$

$x - 2 = 0$, gives $x = 2$

$x - 6 = 0$, gives $x = 6$ but $x \neq 6$

So, $x = 2$ is the only stationary point.

When $x = 2$, $V^{//} = 24x - 96$ becomes

$$V^{//} = 24(2) - 96$$
$$= 48 - 96$$
$$= -48 < 0$$

Hence volume of the box is maximum at $x = 2$

Hence the maximum volume of the box is $V(2)$

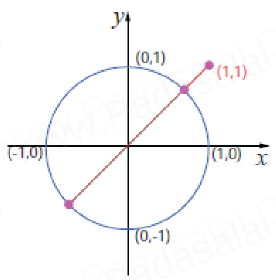
$$V(2) = [(12 - 2(2)) \times (12 - 2(2))] \times (2)$$
$$= [(12 - 4) \times (12 - 4)] \times (2)$$
$$= (8)(8)(2)$$
$$= (64)(2)$$
$$= 128 \text{ units.}$$

So the maximum cut can be 2 units.

****

Example 7.63 Find the points on the unit circle $x^2 + y^2 = 1$ nearest and farthest from $(1, 1)$.

Solution:



Given point is (1, 1). Let other point be (x, y).

Then distance $d = \sqrt{(x-1)^2 + (y-1)^2}$

$$\text{Let } D = (x-1)^2 + (y-1)^2$$
$$= x^2 + 1 - 2x + y^2 + 1 - 2y$$
$$= x^2 + y^2 - 2x - 2y + 2$$
$$D' = 2x + 2yy' - 2 - 2y'$$

From $x^2 + y^2 = 1$

$$2x + 2yy' = 0$$
$$2yy' = -2x$$
$$yy' = -x$$
$$y' = -\frac{x}{y}$$

Substituting the value in D' we get

$$D' = 2x + 2yy' - 2 - 2y'$$
$$= 2x + 2y\left(-\frac{x}{y}\right) - 2 - 2\left(-\frac{x}{y}\right)$$
$$= 2x - 2x - 2 + \frac{2x}{y}$$
$$= -2 + \frac{2x}{y}$$

Substituting $D' = 0$, we get

$$-2 + \frac{2x}{y} = 0$$
$$\frac{2x}{y} = 2$$
$$2x = 2y \text{ gives } x = y$$

Since the points lie on the circle $x^2 + y^2 = 1$

we get $x^2 + x^2 = 1$

$$2x^2 = 1$$
$$x^2 = \frac{1}{2}$$
$$x = \pm\frac{1}{\sqrt{2}}$$

So,the required points are $\left(\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}\right), \left(-\frac{1}{\sqrt{2}}, -\frac{1}{\sqrt{2}}\right)$

****

Example 7.64 A steel plant is capable of producing x tones per day of a low-grade steel and y tones per day of a high-grade steel, where $y = \frac{40-5x}{10-x}$. If the fixed market price of low-grade steel is half that of high-grade steel, then what should be optimal productions in low-grade steel and high-grade steel in order to have maximum receipts.

Solution:

Per day the low grade steel produced= x tones

Let the market price per ton be Rupees p

Then the cost $= px$

Per day the high grade steel produced= y tones

Let the market price per ton be Rupees $2p$

Then the cost $= 2py$

Then total receipt $R = px + 2py$

Substituting the given value $y = \frac{40-5x}{10-x}$, we get

$$R = px + 2p\left(\frac{40-5x}{10-x}\right)$$
$$= \frac{px(10-x)+2p(40-5x)}{10-x}$$
$$= \frac{10\,px - px^2 + 80p - 10px}{10-x}$$
$$R = \frac{-px^2 + 80p}{10-x}$$
$$R' = \frac{(10-x)(-2px) - (-px^2 + 80p)(-1)}{(10-x)^2}$$
$$= \frac{-20px + 2px^2 - px^2 + 80p}{(10-x)^2}$$
$$= \frac{-20px + px^2 + 80p}{(10-x)^2}$$
$$= \frac{p(x^2 - 20x + 80)}{(10-x)^2}$$

Substituting $R' = 0$, we get

$$\frac{p(x^2-20x+80)}{(10-x)^2} = 0$$

$$x^2 - 20x + 80 = 0$$

$$x = \frac{-b\pm\sqrt{b^2-4ac}}{2a}$$

$$= \frac{20\pm\sqrt{400-320}}{2}$$

$$= \frac{20\pm\sqrt{80}}{2}$$

$$= \frac{20\pm\sqrt{16\times5}}{2}$$

$$= \frac{20\pm4\sqrt{5}}{2}$$

$$= \frac{2(10\pm2\sqrt{5})}{2}$$

$$x = 10 \pm 2\sqrt{5}$$

$$R^{//} = p\left[\frac{(10-x)^2(2x-20) - (x^2-20x+80)2(10-x)(-1)}{(10-x)^4}\right]$$

$$= p\frac{[(10-x)^2(2x-20)+(x^2-20x+80)2(10-x)]}{(10-x)^4}$$

$$= p\frac{(10-x)[(10-x)(2x-20)+(x^2-20x+80)2]}{(10-x)^4}$$

$$= \frac{p[(10-x)(2x-20)+(x^2-20x+80)2]}{(10-x)^3}$$

$$= \frac{p[20x-2x^2-200+20x+2x^2-40x+160]}{(10-x)^3}$$

$$= \frac{-40p}{(10-x)^3}$$

At $x = 10 - 2\sqrt{5}$, $R^{//} < 0$

Hence Receipt R is maximum at $x = 10 - 2\sqrt{5}$

When $x = 10 - 2\sqrt{5}$, then $y = \frac{40-5x}{10-x}$

$$y = \frac{40-5(10-2\sqrt{5})}{10-(10-2\sqrt{5})}$$

$$= \frac{40-50+10\sqrt{5}}{10-10+2\sqrt{5}}$$

$$= \frac{-10+10\sqrt{5}}{2\sqrt{5}}$$

$$= \frac{10(\sqrt{5}-1)}{2\sqrt{5}} = \frac{2\times\sqrt{5}\times\sqrt{5}(\sqrt{5}-1)}{2\sqrt{5}}$$

$$= \sqrt{5}(\sqrt{5}-1)$$

$$= 5 - \sqrt{5}$$

So, the plant should produce low grade steels $10 - 2\sqrt{5}$ tones and produce high grade $5 - \sqrt{5}$ tones.

****

Example 7.65 Prove that among all the rectangles of the given area square has the least perimeter.

Solution:

Let the dimensions of the rectangle be x, y

Then its area $A = xy$

Perimeter of the rectangle $P = 2x + 2y$

Substituting the value $y = \frac{A}{x}$ in P

$$P = 2x + 2\left(\frac{A}{x}\right)$$

$$P^{/} = 2 - \frac{2A}{x^2}$$

$$P^{//} = \frac{4A}{x^3}$$

Substituting $P^{/} = 0$, we get

$$2 - \frac{2A}{x^2} = 0$$

$$2 = \frac{2A}{x^2}$$

$2x^2 = 2A$ gives

$$x^2 = A \Longrightarrow x = \sqrt{A}$$

Substituting $x = \sqrt{A}$ in $P^{//}$,

$P^{//} > 0$. Hence perimeter P is minimum at $x = \sqrt{A}$. Then $y = \frac{A}{x}$ becomes

$$y = \frac{A}{\sqrt{A}} = \frac{\sqrt{A}\sqrt{A}}{\sqrt{A}} = \sqrt{A}$$

Since $x = \sqrt{A}, y = \sqrt{A}$ the given rectangle is a square when the perimeter is minimum.

****

EXERCISE 7.8

1. Find two positive numbers whose sum is 12 and their product is maximum.

Solution:

Let x, y be the two numbers then the sum

$x + y = 12$ gives

$$y = 12 - x$$

Product of the numbers $P = xy$

$$= x(12 - x)$$

$$P = 12x - x^2$$

$$P^{/} = 12 - 2x$$

$$P^{//} = -2$$

Substituting $P^{/} = 0$, we get

$$12 - 2x = 0$$

$$2x = 12 \text{ gives}$$

$$x = 6$$

Since $P^{//} = -2 < 0$, Product P is maximum at $x = 6$. Then $y = 12 - x$, gives $y = 12 - 6 = 6$.

The required two numbers are 6, 6.

****

2. Find two positive numbers whose product is 20 and their sum is minimum.

Solution: Let x, y be the two numbers then the product $x\,y = 20$ gives

$$y = \frac{20}{x}$$

Sum of the numbers $S = x + y$

$$= x + \left(\frac{20}{x}\right)$$

$$S^{/} = 1 - \frac{20}{x^2}$$

$$S^{//} = \frac{40}{x^3}$$

Substituting $S^{/} = 0$, we get

$$1 - \frac{20}{x^2} = 0$$

$$1 = \frac{20}{x^2}$$

$$x^2 = 20 \text{ gives}$$

$$x = \pm\sqrt{20}$$

$$x = \pm\sqrt{4 \times 5}$$

Since the number is positive,

$$x = 2\sqrt{5}$$

Since $S^{//} = \frac{40}{x^3} > 0$, when $x = 2\sqrt{5}$ sum S is minimum at $x = 2\sqrt{5}$ .Then $y = \frac{20}{x}$, gives

$$y = \frac{20}{2\sqrt{5}} = \frac{10}{\sqrt{5}} = \frac{10\times\sqrt{5}}{\sqrt{5\times\sqrt{5}}} = \frac{10\times\sqrt{5}}{5} = 2\sqrt{5}$$

The required two numbers are $2\sqrt{5}, 2\sqrt{5}$

****

3. Find the smallest possible value of $x^2 + y^2$ given that $x + y = 10$.

Solution:

Given $x + y = 10$ gives

$$y = 10 - x$$

Let $f(x) = x^2 + y^2$ gives

$$= x^2 + (10 - x)^2$$

$$= x^2 + 100 + x^2 - 20x$$

$$f(x) = 2x^2 - 20x + 100$$

$$f^{/}(x) = 4x - 20$$

$$f^{//}(x) = 4$$

Substituting $f^{/}(x)$, we get

$$4x - 20 = 0$$

$$4x = 20 \text{ gives}$$

$$x = 5$$

Since $f^{//}(x) = 4 > 0, f(x) = x^2 + y^2$ is minimum at $x = 5$.

When $x = 5, y = 10 - x$ gives

$$y = 10 - 5 = 5$$

The required two values are $x = 5, y = 5$ and the smallest possible value of $x^2 + y^2$ is $(5)^2 + (5)^2$= 25+25=50.

****

4. A garden is to be laid out in a rectangular area and protected by wire fence. What is the largest possible area of the fenced garden with 40 metres of wire.

Solution:

Let the dimensions of the rectangle be x, y

Then its area $A = xy$

Perimeter of the rectangle $40 = 2x + 2y$

Then, $x + y = 20$ gives

$$y = 20 - x$$

Then its area $A = x(20 - x)$

$$= 20x - x^2$$

$$A' = 20 - 2x$$

$$A'' = -2$$

Substituting $A' = 0$, we get

$$20 - 2x = 0 \text{ gives}$$

$$2x = 20$$

$$x = 10$$

Since $A'' = -2 < 0$, area is maximum

At $x = 10$, then $y = 20 - x$ gives $y = 10$

Hence the maximum area is

$$A = (10)(10) = 100 \text{ sq. meters.}$$

****

5. A rectangular page is to contain 24 cm^2 of print. The margins at the top and bottom of the page are 1.5 cm and the margins at other sides of the page are 1 cm. What should be the dimensions of the page so that the area of the paper used is minimum ?

Solution:

Let x and y be dimension of the printed area.

Given $xy = 24$ gives

$$y = \frac{24}{x}$$

From by the given data, dimensions of the paper are $x + 3, y + 2$

Hence area of the paper

$$A = (x+3)(y+2)$$

$$= xy + 2x + 3y + 6$$

$$= 24 + 2x + 3\left(\frac{24}{x}\right) + 6$$

$$A = 2x + \frac{72}{x} + 30$$

$$A' = 2 - \frac{72}{x^2}$$

$$A'' = \frac{144}{x^3}$$

Substituting $A' = 0$, we get

$$2 - \frac{72}{x^2} = 0$$

$$2x^2 = 72$$

$$x^2 = 36 \text{ gives}$$

$$x = 6$$

At $x = 6$, $A'' = \frac{144}{x^3} = \frac{144}{6^3} > 0$

Hence printed area of the paper is minimum at $x = 6$, then $y = \frac{24}{x} = \frac{24}{6} = 4$

The dimensions of the paper are

$$x + 3 = 6 + 3 = 9cm, \text{ and}$$

$$y + 2 = 4 + 2 = 6cm$$

****

6. A farmer plans to fence a rectangular pasture adjacent to a river. The pasture must contain 1,80,000 sq.mtrs in order to provide enough grass for herds. No fencing is needed along the river. What is the length of the minimum needed fencing material?

Solution:

Let x and y be dimension of the pasture area.

Given $xy = 180000$ gives

$$y = \frac{180000}{x}$$

Fencing required $= x + 2y$

$$F = x + 2\left(\frac{180000}{x}\right)$$

$$F = x + \frac{360000}{x}$$

$$F' = 1 - \frac{360000}{x^2}$$

$$F'' = \frac{720000}{x^3}$$

Substituting $F' = 0$, we get

$$1 - \frac{360000}{x^2} = 0$$

$$x^2 = 360000 \text{ gives}$$

$$x = 600$$

At $x = 600$, $F'' = \frac{720000}{x^3} = \frac{720000}{600^3} > 0$

Hence pasture area is minimum at $x = 600$, then $y = \frac{180000}{600} = \frac{1800}{6} = 300$

Hence minimum length of the fence needed $= x + 2y$

$$= 600 + 2(300)$$

$$= 600 + 600 = 1200 \text{ meters.}$$

****

7. Find the dimensions of the rectangle with maximum area that can be inscribed in a circle of radius 10 cm.

Solution:

Let us take the circle to be a circle with centre (0, 0) and radius r 10 units and PQRS be the rectangle inscribed in it. Let P(x, y) be the vertex of the rectangle that lies on the first quadrant. Let θ be the angle made by OP with the x- axis.

then $x = 10\cos\theta$ and $y = 10\sin\theta$

Now the dimensions of the rectangle are

$2x = 20\cos\theta$; $2y = 20\sin\theta$, $0 \le \theta \le \frac{\pi}{2}$

Area of the rectangle $A = (2x)(2y)$

$= (20\cos\theta)(20\sin\theta)$

$= 200\,(2\sin\theta\cos\theta)$

$= 200\,(\sin 2\theta)$

Area is maximum when $\sin 2\theta$ is maximum.

We know the maximum value of $\sin 2\theta = 1$

It gives $2\theta = \frac{\pi}{2}$ that is $\theta = \frac{\pi}{4}$, the area of the rectangle is maximum.

When $\theta = \frac{\pi}{4}$, $2x = 20\cos\theta$

$= 20\cos\frac{\pi}{4}$

$= 20 \times \frac{1}{\sqrt{2}}$

$= \frac{20}{\sqrt{2}} \times \frac{\sqrt{2}}{\sqrt{2}}$

$= \frac{20\sqrt{2}}{2} = 10\sqrt{2}$ and

$2y = 20\sin\theta$

$= 20\sin\frac{\pi}{4}$

$= 20 \times \frac{1}{\sqrt{2}}$

$= \frac{20}{\sqrt{2}} \times \frac{\sqrt{2}}{\sqrt{2}}$

$= \frac{20\sqrt{2}}{2} = 10\sqrt{2}$

So, dimensions of the rectangle are $10\sqrt{2}, 10\sqrt{2}$

****

8. Prove that among all the rectangles of the given perimeter, the square has the maximum area.

Solution:

Let the dimensions of the rectangle be x, y

Perimeter of the rectangle $P = 2x + 2y$

$\frac{P}{2} = x + y$

$y = \frac{P}{2} - x$

Then its area $A = xy$

$= x\left(\frac{P}{2} - x\right)$

$A = \frac{P}{2}x - x^2$

$A' = \frac{P}{2} - 2x$

$A'' = -2$

Substituting $A' = 0$, we get

$\frac{P}{2} - 2x = 0$

$\frac{P}{2} = 2x$

$x = \frac{P}{4}$

Since $A'' = -2 < 0$, area is maximum,

at $x = \frac{P}{4}$, then $y = \frac{P}{2} - x$ gives

$y = \frac{P}{2} - \frac{P}{4}$

$= \frac{2P - P}{4}$

$y = \frac{P}{4}$

Since, $x = y = \frac{P}{4}$,

the given rectangle is a square when the area is maximum.

****

9. Find the dimensions of the largest rectangle that can be inscribed in a semi circle of radius r cm.

Solution:

Let us take the circle to be a circle with centre (0, 0) and radius r cm and PQRS be the rectangle inscribed in the semi circle.

Let P(x, y) be the vertex of the rectangle that lies on the first quadrant. Let θ be the angle made by OP with the x- axis.

then $x = r\cos\theta$ and $y = r\sin\theta$

Now the dimensions of the rectangle are

$x = r\cos\theta$; $2y = 2r\sin\theta$, $0 \le \theta \le \frac{\pi}{2}$

Area of the rectangle $A = (x)(2y)$

$= (r\cos\theta)(2r\sin\theta)$

$= r^2\,(2\sin\theta\cos\theta)$

$= r^2\,(\sin 2\theta)$

Area is maximum when $\sin 2\theta$ is maximum.

We know the maximum value of $\sin 2\theta = 1$

It gives $2\theta = \frac{\pi}{2}$ that is $\theta = \frac{\pi}{4}$, the area of the rectangle is maximum. Hence maximum area of the rectangle is $A = r^2\,(1) = r^2$ sq. cm

****

10. A manufacturer wants to design an open box having a square base and a surface area of 108 sq.cm. Determine the dimensions of the box for the maximum volume.

Solution: Given the box base is square.

So, let length, breadth and height of the box be $l = x, b = x$ *and* $h = y$.

Then box base area $= x^2$

Area of the 4 sides of the box $= 4xy$

Surface area of the open box $S = x^2 \times 4xy$

Given $x^2 \times 4xy = 108$

$4xy = 108 - x^2$

$y = \frac{108-x^2}{4x}$

Volume of the box $V = x \times x \times y$

$= x^2 \times \frac{108-x^2}{4x}$

$V = x \times \frac{108-x^2}{4}$

$V = \frac{1}{4}(108x - x^3)$

$V' = \frac{1}{4}(108 - 3x^2)$

$V'' = \frac{1}{4}(-6x)$

Substituting $V' = 0$, we get

$\frac{1}{4}(108 - 3x^2) = 0$

$108 - 3x^2 = 0$

$3x^2 = 108$

$x^2 = \frac{108}{3}$

$x^2 = 36$ gives

$x = \pm 6$

Since, $x \neq -6$, we get $x = 6$.

When $x = 6$, $V'' = \frac{1}{4}(-6x)$

$V'' = \frac{1}{4}(-36) < 0$

So, volume is maximum at $x = 6$

When $x = 6$, then $y = \frac{108-x^2}{4x} = \frac{108-36}{4(6)} = \frac{72}{24} = 3$

Hence dimensions of the box are 6cm,6cm,3cm.

****

11. The volume of a cylinder is given by the formula $V = \pi r^2 h$. Find the greatest and least values of V if $r + h = 6$.

Solution: Given $r + h = 6$ which gives

$h = 6 - r$

Hence, volume of the cylinder $V = \pi r^2 h$

$V = \pi r^2(6 - r)$

$= \pi(6r^2 - r^3)$

$V' = \pi(12r - 3r^2)$

$V'' = \pi(12 - 6r)$

Substituting $V' = 0$, we get

$\pi(12r - 3r^2) = 0$

$12r - 3r^2 = 0$

$3r(4 - r) = 0$

$3r = 0$, gives $r = 0$ and

$4 - r = 0$, gives $r = 4$

When $r = 0$, $V'' = \pi(12 - 6r) = 12\pi > 0$

Hence volume is minimum at $r = 0$.

Hence the least value of V is 0

When $r = 4, V^{//} = \pi(12 - 24) = -12\pi < 0$

Hence volume is maximum at $r = 4$.

Then $h = 6 - r$, gives $h = 6 - 4 = 2$

Hence the greatest value of $V = \pi r^2 h$

$$V = \pi(4^2)(2)$$

$$= \pi(16)(2)$$

$$V = 32\pi$$

Hence the least value of V is 0 and

Hence the greatest value of V is 32π

****

12. A hollow cone with base radius a cm and height b cm is placed on a table. Show that the volume of the largest cylinder that can hide underneath is $\frac{4}{9}$ times volume of the cone.

Solution: Given radius of cone $= a$

Height $= b$

Hence volume of cone $= \frac{1}{3}\pi a^2 b$

Let the radius of cylinder $= x$ and

height of the cylinder be $= y$

Then the volume of cylinder $V = \pi x^2 y$

To prove the maximum volume $= \frac{4}{9}\left(\frac{1}{3}\pi a^2 b\right)$

Let the height of cone OC $= b$ and

radius CE $= a$

Radius of cylinder AB $=$ CE $= x$

Height of cylinder AC $= y$

Triangle OAB and OCD are similar, hence the corresponding sides are proportional.

Hence, $\frac{OA}{OC} = \frac{AB}{CD}$

$$\frac{b-y}{b} = \frac{x}{a}$$

$$b - y = \frac{b}{a}x$$

$$y = b - \frac{b}{a}x$$

Substituting the value in Volume of cylinder,

$V = \pi x^2 y$ becomes

$$V = \pi x^2\left(b - \frac{b}{a}x\right)$$

$$= \pi b\left(x^2 - \frac{x^3}{a}\right)$$

$$V^/ = \pi b\left(2x - \frac{3x^2}{a}\right)$$

$$V^{//} = \pi b\left(2 - \frac{6x}{a}\right)$$

Substituting $V^/ = 0$, we get

$$\pi b\left(2x - \frac{3x^2}{a}\right) = 0 \text{ gives}$$

$$2x - \frac{3x^2}{a} = 0$$

$$x\left(2 - \frac{3x}{a}\right) = 0$$

$$x = 0, \text{ and } 2 - \frac{3x}{a} = 0 \text{ gives}$$

$$2 = \frac{3x}{a}$$

$$2a = 3x$$

$$x = \frac{2a}{3}$$

When $x = \frac{2a}{3}$, $V^{//} = \pi b\left(2 - \frac{6x}{a}\right)$ gives

$$V^{//} = \pi b\left[2 - \left(\frac{6}{a}\right)\left(\frac{2a}{3}\right)\right]$$

$$= \pi b(2 - 4)$$

$$V^{//} = \pi b(-2) < 0$$

So, volume of cylinder is maximum when $x = \frac{2a}{3}$ then $y = b - \frac{b}{a}x$ gives

$$y = b - \frac{b}{a}\left(\frac{2a}{3}\right)$$

$$= b - \frac{2b}{3}$$

$$= \frac{3b-2b}{3}$$

$$y = \frac{b}{3}$$

Maximum volume of cylinder $V = \pi x^2 y$

$$V = \pi\left(\frac{2a}{3}\right)^2\left(\frac{b}{3}\right)$$

$$= \pi\left(\frac{4a^2}{9}\right)\left(\frac{b}{3}\right)$$

$V = \frac{4}{9}\left(\frac{1}{3}\pi a^2 b\right)$ Hence proved.

****

Example 7.66
Find the asymptotes of the function $f(x) = \frac{1}{x}$

Example 7.67 Find the slant (oblique) asymptote for the function $f(x) = \frac{x^2 - 6x + 7}{x + 5}$

Example 7.68
Find the asymptotes of the curve $f(x) = \frac{2x^2 - 8}{x^2 - 16}$

Example 7.69
Sketch the curve $y = f(x) = x^2 - x - 6$

Example 7.70
Sketch the curve $y = f(x) = x^3 - 6x + 9$

Example 7.71
Sketch the curve $y = \frac{x^2 - 3x}{(x - 1)}$

Example 7.72
Sketch the graph of the function $y = \frac{3x}{x^2 - 1}$

****

EXERCISE 7.9

1. Find the asymptotes of the following curves:
   (i) $f(x) = \frac{x^2}{x^2 - 1}$ (ii) $f(x) = \frac{x^2}{x + 1}$
   (iii) $f(x) = \frac{3x}{\sqrt{x^2 + 1}}$ (iv) $f(x) = \frac{x^2 - 6x - 1}{x + 3}$
   (v) $f(x) = \frac{x^2 + 6x - 4}{3x - 6}$

2. Sketch the graphs of the following functions:
   (i) $y = -\frac{1}{3}(x^3 - 3x + 2)$ (ii) $y = x\sqrt{4 - x}$
   (iii) $y = \frac{x^2 + 1}{x^2 - 4}$ (iv) $y = \frac{1}{1 + e^{-x}}$
   (v) $y = \frac{x^3}{24} - \log x$

****

EXERCISE 7.10

Choose the correct or the most suitable answer from the given four alternatives:

1. The volume of a sphere is increasing in volume at the rate of $3\pi cm^3/sec$. The rate of change of its radius when radius is $\frac{1}{2}$ cm
   (1) $3\ cm/s$ (2) $2\ cm/s$
   (3) $1\ cm/s$ (4) $\frac{1}{2}$ cm/s

2. A balloon rises straight up at 10 m/s. An observer is 40 m away from the spot where the balloon left the ground. Find the rate of change of the balloon's angle of elevation in radian per second when the balloon is 30 metres above the ground.
   (1) $\frac{3}{25}$ radians/sec **(2) $\frac{4}{25}$ radians/sec**
   (3) $\frac{1}{5}$ radians/sec (4) $\frac{1}{3}$ radians/sec

3. The position of a particle moving along a horizontal line of any time t is given by $s(t) = 3t^2 - 2t - 8$. The time at which the particle is at rest is
   (1) $t = 0$ **(2) $t = \frac{1}{3}$**
   (3) $t = 1$ (4) $t = 3$

4. A stone is thrown up vertically. The height it reaches at time t seconds is given by $x = 80t - 16t^2$. The stone reaches the maximum height in time t seconds is given by
   (1) 2 **(2) 2.5** (3) 3 (4) 3.5

5. Find the point on the curve $6y = x^3 + 2$ at which $y - coordinate$ changes 8 times as fast as $x - coordinate$ is
   (1) (4, 11) (2) (4, −11)
   (3) (−4, 11) (4) (−4, −11)

6. The abscissa of the point on the curve $f(x) = \sqrt{8 - 2x}$ at which the slope of the tangent is −0.25 ?
   (1) −8 **(2) −4** (3) −2 (4) 0

7. The slope of the line normal to the curve $f(x) = 2\cos 4x$ at $x = \frac{\pi}{2}$ is
   (1) $-4\sqrt{3}$ (2) −4 **(3) $\frac{\sqrt{3}}{12}$** (4) $4\sqrt{3}$

8. The tangent to the curve $y^2 - xy + 9 = 0$ is vertical when
   (1) $y = 0$ (2) $y = \pm\sqrt{3}$
   (3) $y = \frac{1}{2}$ **(4) $y = \pm 3$**

9. Angle between $y^2 = x$ and $x^2 = y$ at the origin is
(1) $tan^{-1}\left(\frac{3}{4}\right)$ (2) $tan^{-1}\left(\frac{4}{3}\right)$
(3) $\frac{\pi}{2}$ (4) $\frac{\pi}{4}$

10. What is the value of the limit $\lim_{x\to 0}\left(\cot x - \frac{1}{x}\right)$?
(1) 0 (2) 1 (3) 2 **(4) $\leq$**

11. The function $sin^4 x + cos^4 x$ is increasing in the interval
(1) $\left[\frac{5\pi}{8}, \frac{3\pi}{4}\right]$ (2) $\left[\frac{\pi}{2}, \frac{5\pi}{8}\right]$
(3) $\left[\frac{\pi}{4}, \frac{\pi}{2}\right]$ (4) $\left[0, \frac{\pi}{4}\right]$

12. The number given by the Rolle's theorem for the function $x^3 - 3x^2, x \in [0, 3]$is
(1) 1 (2) $\sqrt{2}$ (3) $\frac{3}{2}$ **(4) 2**

13. The number given by the Mean value theorem for the function $\frac{1}{x}$, $x \in [1, 9]$ is
(1) 2 (2) 2.5 **(3) 3** (4) 3.5

14. The minimum value of the function $|3 - x| + 9$ is
(1) 0 (2) 3 (3) 6 **(4) 9**

15. The maximum slope of the tangent to the curve $y = e^x \sin x$, $x \in [0, 2\pi]$ is at
(1) $x = \frac{\pi}{4}$ **(2) $x = \frac{\pi}{2}$**
(3) $x = \pi$ (4) $x = \frac{3\pi}{2}$

16. The maximum value of the function $x^2 e^{-2x}$, $x > 0$ is
(1) $\frac{1}{e}$ (2) $\frac{1}{2e}$ **(3) $\frac{1}{e^2}$** (4) $\frac{4}{e^4}$

17. One of the closest points on the curve $x^2 - y^2 = 4$ to the point $(6, 0)$ is
(1) $(2, 0)$ (2) $(\sqrt{5}, 1)$
(3) $(3, \sqrt{5})$ (4) $(\sqrt{13}, -\sqrt{3})$

18. The maximum product of two positive numbers, when their sum of the squares is 200, is
(1) 100 (2) $25\sqrt{7}$
(3) 28 (4) $24\sqrt{14}$

19. The curve $y = ax^4 + bx^2$ with $ab > 0$
(1) has no horizontal tangent
(2) is concave up
(3) is concave down
(4) has no points of inflection

20. The point of inflection of the curve $y = (x - 1)^3$ is
(1) $(0, 0)$ (2) $(0, 1)$
(3) $(1, 0)$ (4) $(1, 1)$

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